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On the first passage problem for correlated Brownian motion

Adam Metzler *

Department of Statistics and Actuarial Science, Mathematics and Computer Building, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada, N2L 3G1

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ABSTRACT

Suppose that $X = (X_1, X_2)$ is two-dimensional correlated Brownian motion. Let τ_i denote the first passage time of X_i to a fixed level, and τ the minimum of τ_1, τ_2. When X has zero drift, several distributions of interest are available in closed form, including the joint density of the passage times and the distribution of $X(\tau)$. Unfortunately these published formulae contain errors, and the corresponding distributions in the presence of drift are not expressible in closed form. The purpose of this paper is to address these issues by presenting corrected formulae and outlining a Monte Carlo algorithm for approximating quantities of interest in the presence of drift.

1. Introduction

The problem we are interested in here involves the first passage times

$$\tau_i = \inf\{t \geq 0 : X_i(t) = 0\},$$

where $X = (X_1, \ldots, X_N)$ is N-dimensional correlated Brownian motion,

$$dX(t) = \mu dt + \sigma dW(t) \quad X(0) = x_0.$$

Here $\mu, x_0 \in \mathbb{R}^N$ with all components of x_0 strictly positive, σ is a non-singular square matrix and W is a standard N-dimensional Brownian motion. We define $\tau = \min(\tau_1, \ldots, \tau_N)$ as the time at which $X(t)$ first exits the (interior of the) positive orthant. In general this problem is analytically intractable, however significant recent progress has been made with regard to perturbation approximations to the distribution of τ (Bhansali and Wise, 2008), as well as Monte Carlo methods for various financial applications (Huh and Kolkiewicz, 2008; Shevchenko, 2003).

When $N = 2$ and $\mu = 0$, Iyengar (1985) demonstrates that several distributions of interest are available in closed form. These include the joint density of the passage times (τ_1, τ_2), the survivor function of the exit time τ, the distribution of the exit location $X(\tau)$, the joint distribution of exit time and location $(\tau, X(\tau))$, and the transition density of the absorbed process $X(\tau \wedge t)$. Unfortunately several of the formulae presented by Iyengar (1985) contain errors, and we have been unable to locate correct expressions elsewhere in the literature. Moreover when $\mu \neq 0$ these distributions are not expressible in closed form, and the computational effort required to approximate quantities of interest numerically can be prohibitive (Zhou, 2001).

The purpose of this paper is to address these issues by presenting corrected formulae in the zero-drift case and outlining a simulation method for approximating various quantities of interest in the presence of drift. The paper is organized as follows. In the remainder of this section we reformulate the problem in terms of planar Brownian motion (i.e. two-dimensional Brownian motion with independent components) inside a wedge and review several known results. In Section 2 we derive the correct distribution of the exit location $X(\tau)$ (Eqs. (2.4) and (2.5)), Section 3 presents correct formulae for the joint density of the passage times (Eqs. (3.2) and (3.3)), while Section 4 introduces non-zero drift and outlines the Monte Carlo algorithm.

* Present address: Department of Applied Mathematics, 255 Middlesex College, The University of Western Ontario, 1151 Richmond Street North, London, Ontario, Canada, N6A 5B7. Tel.: +1 519 661 2111x88786.

E-mail address: ametzle@uwo.ca.

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1.1. Transformation

The problem may be fruitfully reformulated via a transformation to independence. To this end we parametrize the covariance matrix $\Sigma = \sigma\sigma^T$ via

$$\sigma = \begin{bmatrix} \sigma_1\sqrt{1-\rho^2} & \sigma_1\rho \\ 0 & \sigma_2 \end{bmatrix},$$

and consider the process $Z = \sigma^{-1}X$, which has independent components. Note that we have assumed σ non-singular, thus $|\rho| < 1$ and $\sigma_i \neq 0$. It is easily verified that the horizontal axis is invariant under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \sigma^{-1}\mathbf{x}$, while the vertical axis is mapped to the line $z_1 = -\frac{\rho}{\sqrt{1-\rho^2}}z_2$. As such we obtain the following

- τ_2 is the first passage time of $Z(t)$ to the horizontal axis.
- τ_1 is the first passage time of $Z(t)$ to the line $z_2 = z_1 \tan\alpha$, where

$$\alpha = \begin{cases} \pi + \tan^{-1}\left(-\dfrac{\sqrt{1-\rho^2}}{\rho}\right) & \rho > 0 \\ \dfrac{\pi}{2} & \rho = 0 \\ \tan^{-1}\left(-\dfrac{\sqrt{1-\rho^2}}{\rho}\right) & \rho < 0. \end{cases}$$

It is easily verified that $0 < \alpha < \pi$, and we note that the line is interpreted as the vertical axis when $\alpha = \pi/2$.

- $Z(t)$ begins at the point z_0 whose polar coordinates are given by

$$r_0 = \sqrt{\frac{a_1^2 + a_2^2 - 2\rho a_1 a_2}{1-\rho^2}}$$

$$\theta_0 = \begin{cases} \pi + \tan^{-1}\left(\dfrac{a_2\sqrt{1-\rho^2}}{a_1 - \rho a_2}\right) & a_1 < \rho a_2 \\ \dfrac{\pi}{2} & a_1 = \rho a_2 \\ \tan^{-1}\left(\dfrac{a_2\sqrt{1-\rho^2}}{a_1 - \rho a_2}\right) & a_1 > \rho a_2 \end{cases}$$

where $a_i = x_0^i/\sigma_i$. It is easily verified that $0 < \theta_0 < \alpha$.

This setup is illustrated in Fig. 1. We note that $\tau = \min(\tau_1, \tau_2)$ is the first exit time of Z from the wedge

$$C_\alpha = \{(r\cos\theta, r\sin\theta) : \; r > 0, \; 0 < \theta < \alpha\} \subset \mathbb{R}^2,$$

and that $Z(\tau)$ lives on the boundary of this wedge

$$\partial C_\alpha = \{(r\cos\theta, r\sin\theta) : \; r \geq 0, \; \theta \in \{0, \alpha\}\} \subset \mathbb{R}^2.$$

1.2. Analytic formulae

In an elegant paper, Iyengar (1985) studies the following quantities via partial differential equations

(a) $P^{z_0}(\tau > t, \; Z(t) \in dz)$ (b) $P^{z_0}(\tau \in dt, \; Z(\tau) \in dz)$
(c) $P^{z_0}(\tau > t)$ (d) $P^{z_0}(\tau_1 \in ds, \; \tau_2 \in dt)$
(e) $P^{z_0}(Z(\tau) \in dz)$.

In both (b) and (e) z is taken to be a point on the boundary of the wedge. The formulae for (a) and (b) given by Iyengar (1985) are correct, the expressions for (c) and (d) contain small errors, while the expression for (e) contains a more serious error. We discuss (a)–(c) briefly here, in Section 2 we derive the exit location distribution (e) and in Section 3 we present the correct joint density of the first passage times (d).

Sub-density $P^{z_0}(Z(t) \in dz, \; \tau > t)$

If $z = (r\cos\theta, r\sin\theta)$ is a point in C_α we have

$$P^{z_0}(\tau > t, \; Z(t) \in dz) = \frac{2r}{t\alpha} e^{-(r^2+r_0^2)/2t} \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) \, dr \, d\theta \tag{1.1}$$

where I_ν denotes the modified Bessel function of the first kind of order ν. Iyengar (1985) derives this expression using the method of images. We note that (1.1) has been derived by several authors, including Buckholtz and Wasan (1979) and Rebholz (1994). Each of (b) and (c) may be derived from (a) as follows.

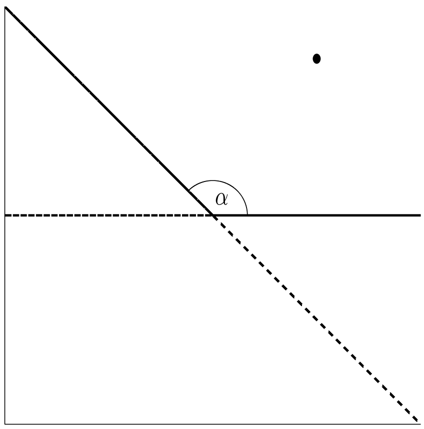

Fig. 1. $z_0 = \bullet$, τ_1 is FPT to line, τ_2 is FPT to horizontal axis, $\tau = \min(\tau_1, \tau_2)$ is FPT to solid line (∂C_α). Note also that $\tau' = \max(\tau_1, \tau_2)$ is not the FPT to the dashed line.

Survivor function $P^{z_0}(\tau > t)$

To obtain (b) one simply integrates (1.1) over the wedge. The computation is facilitated by a conversion to polar coordinates, integration by parts and the identities[1] (Abramowitz and Stegun, 1967)

$$\int_0^\infty \mathrm{e}^{-\beta t^2} I_\nu(\alpha t)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} \exp\left(\frac{\alpha^2}{8\beta}\right) I_{\nu/2}\left(\frac{\alpha^2}{8\beta}\right), \tag{1.2}$$

$$2I'_\nu(x) = I_{\nu-1}(x) + I_{\nu+1}(x). \tag{1.3}$$

The end result of this computation is

$$P^{z_0}(\tau > t) = \frac{2r_0}{\sqrt{2\pi t}} \mathrm{e}^{-r_0^2/4t} \sum_{n \text{ odd}} \frac{1}{n} \sin\frac{n\pi\theta_0}{\alpha} \left[I_{(\nu_n-1)/2}\left(r_0^2/4t\right) + I_{(\nu_n+1)/2}\left(r_0^2/4t\right)\right] \tag{1.4}$$

where $\nu_n = n\pi/\alpha$. In the finance literature this result is often attributed to Zhou (2001), whose proof resembles that given in an earlier paper of Buckholtz and Wasan (1979) (the ultimate formula given in the latter paper contains two small errors).

Exit time and location $P^{z_0}(Z(\tau) \in \mathrm{d}z,\ \tau \in \mathrm{d}t)$

In order to obtain (c), it is claimed in Iyengar (1985) and verified in Blanchet-Scalliet and Patras (2008) that

$$P^{z_0}(\tau \in \mathrm{d}t,\ Z(\tau) \in \mathrm{d}z) = \frac{1}{2}\frac{\partial}{\partial n} P^{z_0}(Z(t) \in \mathrm{d}z,\ \tau > t),$$

where $\frac{\partial}{\partial n}$ denotes derivative in the direction of the (unit) inward normal to the boundary ∂C_α, at the point z. Note that this unit normal is either $(0, 1)$ or $(\sin\alpha, -\cos\alpha)$, according as z lies on the polar axis or the ray. For $z = (r, 0)$ on the polar axis this yields

$$P^{z_0}(\tau \in \mathrm{d}t,\ Z(\tau) \in \mathrm{d}z) = \frac{\pi}{\alpha^2 tr} \mathrm{e}^{-(r^2+r_0^2)/2t} \sum_{n=1}^\infty n \sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) \mathrm{d}r\ \mathrm{d}t, \tag{1.5}$$

while for a point $z = r(\cos\alpha, \sin\alpha)$ on the ray we obtain

$$P^{z_0}(\tau \in \mathrm{d}t,\ Z(\tau) \in \mathrm{d}z) = \frac{\pi}{\alpha^2 tr} \mathrm{e}^{-(r^2+r_0^2)/2t} \sum_{n=1}^\infty n \sin\left(\frac{n\pi\tilde{\theta}_0}{\alpha}\right) I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) \mathrm{d}r\ \mathrm{d}t, \tag{1.6}$$

where $\tilde{\theta}_0 = \alpha - \theta_0$. This expression may be derived via direct calculation, or by using (1.5) and the fact that the process $\tilde{Z}$, obtained by reflecting Z about the line $z_2 = \tan(\alpha/2)\, z_1$, is also a planar Brownian motion. Note that in principle one could obtain the distribution of $Z(\tau)$ by integrating (1.5) or (1.6) over t, however direct evaluation of the resulting integral using known identities does not appear possible.

[1] In Iyengar (1985) an incorrect version of (1.2) was used, leading to a small error in the ultimate formula given there.

2. Distribution of $Z(\tau)$

Our goal here is to derive the distribution of the exit location $Z(\tau)$. That is, we seek an expression for probabilities of the form

$$P^{z_0}\left(R(\tau)\in \mathrm{d}r, \Theta(\tau)=\delta\right) \quad (r,\delta)\in(0,\infty)\times\{0,\alpha\},$$

where $R(t)$, $\Theta(t)$ denote the radial and angular parts of $Z(t)$, respectively. Before proceeding we briefly review the relation between this distribution and the Dirichlet problem in the wedge.

Recall that if g is a continuous real-valued function on ∂C_α, the Dirichlet problem is to find a function u, harmonic in C_α, which agrees with g on ∂C_α. Under certain conditions (Karatzas and Shreve, 1991) it is known that u has the stochastic representation

$$u(z)=E^z\left[g\left(Z\left(\tau\right)\right)\right] \quad z\in\overline{C_\alpha},$$

where $\overline{C_\alpha}$ is the closure of the wedge. Thus the distribution of $Z(\tau)$ is the harmonic measure for the wedge,[2] and in Iyengar (1985) it is claimed that this function is easily obtained. However the ultimate expression given there is incorrect (it does not integrate to unity), and no details are provided regarding its derivation. In the remainder of this section we present a probabilistic solution to this problem, using a combination of the reflection principle and conformal local martingales (see Appendix A for definitions and basic results).

Positive quadrant. To begin we consider the simplest case $\alpha=\pi/2$, in which case τ_i is simply the first passage time of Z_i to zero. It is well known (Karatzas and Shreve (1991)) that for $z_0=(x_0,y_0)\in C_{\pi/2}$ and $r>0$ we have

$$P^{z_0}\left(Z_1\left(t\right)\in \mathrm{d}r,\ \tau_1>t\right)=\frac{1}{\sqrt{2\pi t}}\left[\mathrm{e}^{-(r-x_0)^2/2t}-\mathrm{e}^{-(r+x_0)^2/2t}\right]\mathrm{d}r, \tag{2.1}$$

which follows directly from the reflection principle. Since τ_2 is independent of both Z_1 and τ_1 here, we may condition on τ_2 and integrate (2.1) against the density $P^{z_0}\left(\tau_2\in \mathrm{d}t\right)=\frac{y_0}{\sqrt{2\pi t^3}}\mathrm{e}^{-y_0^2/2t}\ \mathrm{d}t$ to obtain

$$P^{z_0}\left(Z_1\left(\tau_2\right)\in \mathrm{d}r,\tau_1>\tau_2\right)=\left[\frac{1}{\pi}\frac{y_0}{(r-x_0)^2+y_0^2}-\frac{1}{\pi}\frac{y_0}{(r+x_0)^2+y_0^2}\right]\mathrm{d}r.$$

An analogous argument may be used to obtain $P^{z_0}\left(Z_2\left(\tau_1\right)\in \mathrm{d}r,\ \tau_2>\tau_1\right)$. Expressing these quantities in terms of polar coordinates we obtain

$$P^{z_0}\left(R(\tau)\in \mathrm{d}r,\Theta(\tau)=0\right) = \left[\frac{2}{\pi r_0}\frac{(r/r_0)\sin(2\theta_0)}{\sin^2(2\theta_0)+[(r/r_0)^2-\cos(2\theta_0)]^2}\right]\mathrm{d}r, \tag{2.2}$$

$$P^{z_0}\left(R(\tau)\in \mathrm{d}r,\Theta(\tau)=\frac{\pi}{2}\right) = \left[\frac{2}{\pi r_0}\frac{(r/r_0)\sin(2\theta_0)}{\sin^2(2\theta_0)+[(r/r_0)^2+\cos(2\theta_0)]^2}\right]\mathrm{d}r, \tag{2.3}$$

where $z_0=(x_0,y_0)=r_0\left(\cos\theta_0,\sin\theta_0\right)$.

Arbitrary α. Turning now to the case of arbitrary $\alpha\in(0,\pi)$, we recall that for complex z and real a we may define z^a as $r^a\mathrm{e}^{\mathrm{i}a\theta}$, where (r,θ) are the radial and angular parts of z, respectively. Hence the mapping $z\mapsto z^{\pi/2\alpha}$ "folds the wedge up" into the positive quadrant, while the mapping $z\mapsto z^{2\alpha/\pi}$ "unfolds" the positive quadrant into the wedge. This suggests that in order to study the exit location of our planar Brownian motion Z, we consider the transformed process $Z^{\pi/2\alpha}$ (here we are viewing Z as a $\mathbb{C}$-valued process) and exploit the distribution obtained in the previous paragraph. This connection is made clear in Proposition 2.1, whose proof hinges on the fact that $Z^{\pi/2\alpha}$ is a conformal local martingale, hence a time-changed Brownian motion. As our boundary is independent of time, this facilitates an explicit representation of the exit location in terms of planar Brownian motion in the positive quadrant.

Proposition 2.1. *Let Z be a planar Brownian motion beginning at $z_0\in C_\alpha$. Then there exists a planar Brownian motion B, beginning at $b_0=z_0^{\pi/2\alpha}\in C_{\pi/2}$, such that*

$$Z(\tau)=[B(\tau_B)]^{2\alpha/\pi}$$

where

- $\tau=\inf\{t\geq 0:\ Z(t)\in\partial C_\alpha\}$
- $\tau_B=\inf\left\{t\geq 0:\ B(t)\in\partial C_{\pi/2}\right\}$.

Proof. With R, Θ denoting the radial and angular parts of Z, we view $Z(t)=R(t)\mathrm{e}^{\mathrm{i}\Theta(t)}$ as a $\mathbb{C}$-valued process, and define $V=Z^{\pi/2\alpha}$ via

$$V(t)=(R(t))^{\pi/2\alpha}\,\mathrm{e}^{\mathrm{i}\pi\Theta(t)/2\alpha}.$$

[2] In an earlier version of this paper we mistakenly referred to this distribution as the Green's function for this problem. The author wishes to thank an anonymous referee for carefully illustrating the difference.

Note that $V = Z^{\pi/2\alpha}$ begins at the point $v_0 = z_0^{\pi/2\alpha} \in C_{\pi/2}$, and that V strikes the boundary of the positive quadrant at precisely the same moment (namely τ) as Z strikes the boundary of the wedge. Moreover $Z(\tau)$ may be recovered from $V(\tau)$ via $Z(\tau) = [V(\tau)]^{2\alpha/\pi}$.

Since $V(t) = \exp\left(\frac{\pi}{2\alpha}\log Z(t)\right)$ is an analytic function of the conformal local martingale $\log Z$, we see that V itself is a conformal local martingale. As such there exists a planar Brownian motion B, beginning at v_0, with the property that $V(t) = B([V](t))$. Here $[V]$ denotes the common quadratic variation of the real and imaginary parts of V, and we note that B is a Brownian motion relative to the filtration generated by $A(t) = \inf\{s \geq 0 : [V](s) > t\}$. The key observation here is that V and B will trace out exactly the same path in the plane, indeed they will simply move along this path at different speeds. As such they will strike the boundary of the wedge at exactly the same point, that is $V(\tau) = B(\tau_B)$, where τ_B is the time at which B first exits the (interior of the) positive quadrant. □

In light of Proposition 2.1 and Eqs. (2.2)–(2.3), obtaining the distribution of $Z(\tau)$ for general α is now possible via an elementary change of variables.

Corollary 2.2. *Let R, Θ denote the radial and angular parts of a planar Brownian motion Z, and let*

$$\tau = \inf\{t \geq 0 : Z(t) \notin C_\alpha\}$$

denote the first exit time of Z from the interior of C_α. Then for $z_0 \in C_\alpha$ we have

$$P^{z_0}\left(R(\tau) \in \mathrm{d}r, \Theta(\tau) = 0\right) = \frac{\mathrm{d}r}{\alpha r_0} \frac{(r/r_0)^{(\pi/\alpha)-1}\sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha) + \left[(r/r_0)^{\pi/\alpha} - \cos(\pi\theta_0/\alpha)\right]^2}, \tag{2.4}$$

$$P^{z_0}\left(R(\tau) \in \mathrm{d}r, \Theta(\tau) = \alpha\right) = \frac{\mathrm{d}r}{\alpha r_0} \frac{(r/r_0)^{(\pi/\alpha)-1}\sin(\pi\theta_0/\alpha)}{\sin^2(\pi\theta_0/\alpha) + \left[(r/r_0)^{\pi/\alpha} + \cos(\pi\theta_0/\alpha)\right]^2}. \tag{2.5}$$

3. Joint density of the first passage times

In this section we provide a corrected formula for $f(s,t)$, the joint density of the passage times (τ_1, τ_2)

$$P^{z_0}(\tau_1 \in \mathrm{d}s,\ \tau_2 \in \mathrm{d}t) = f(s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

Our analysis follows that given in Iyengar (1985) closely, where the only error was the use of an incorrect version of (1.2).

To begin we recall that $\tau = \min(\tau_1, \tau_2)$ and define $\tau' = \max(\tau_1, \tau_2)$. Next we consider Fig. 2, where we begin our Brownian motion at $z_0 = \bullet$ and are told that $Z(\tau) = \circ$; the process strikes the boundary of the wedge for the first time at the point $\circ = (r\cos\alpha, r\sin\alpha)$. By the strong Markov property, we observe that, conditional upon $Z(\tau) = \circ$

- $\tau' - \tau = \tau_2 - \tau_1$ is independent of τ_1; the excess time required to travel from $\circ$ to the horizontal axis is independent of the amount of time required to reach $\circ$ from $\bullet$
- $\tau' - \tau = \tau_2 - \tau_1$ is the time required for a linear (one-dimensional) Brownian motion beginning at $r\sin\alpha$ to reach the origin.

In the case where $\circ$ lies on the horizontal axis a counter-clockwise rotation by $\pi - \alpha$, coupled with the rotational invariance of Brownian motion, leads to the same conclusion. Thus we obtain

$$\begin{aligned} P^{z_0}\left(\tau' - \tau \in \mathrm{d}t | \tau,\ Z(\tau)\right) &= P^{z_0}\left(\tau' - \tau \in \mathrm{d}t | Z(\tau)\right) \\ &= \frac{R(\tau)\sin\alpha}{\sqrt{2\pi t^3}}\, \mathrm{e}^{-(R(\tau)\sin\alpha)^2/2t}\,\mathrm{d}t. \end{aligned} \tag{3.1}$$

It is interesting to note that the distribution of $\tau' - \tau$ is an inverse Gaussian mixture, where the mixing variable is $R(\tau)$. In order to compute $P^{z_0}(\tau_1 \in \mathrm{d}s,\ \tau_2 \in \mathrm{d}t)$ for $s < t$, then, we need only compute

$$\begin{aligned} P^{z_0}(\tau_1 \in \mathrm{d}s,\ \tau_2 \in \mathrm{d}t) &= P^{z_0}\left(\tau' - \tau \in \mathrm{d}(t-s),\ \tau \in \mathrm{d}s,\ \Theta(\tau) = \alpha\right) \\ &= \int_0^\infty P^{z_0}\left(\tau' - \tau \in \mathrm{d}(t-s),\ \tau \in \mathrm{d}s,\ \Theta(\tau) = \alpha,\ R(\tau) \in \mathrm{d}r\right). \end{aligned}$$

The integrand above may be factored as

$$P^{z_0}\left(\tau' - \tau \in \mathrm{d}(t-s) | R(\tau) \in \mathrm{d}r\right) P^{z_0}\left(\tau \in \mathrm{d}s,\ \Theta(\tau) = \alpha,\ R(\tau) \in \mathrm{d}r\right).$$

The first term here is obtained from (3.1), while the second is given by (1.6). Simplifying the resulting expression, we see that we must compute integrals of the form

$$\int_0^\infty \mathrm{e}^{-\beta r^2} I_{n\pi/\alpha}(rr_0/s)\,\mathrm{d}r \quad \beta = \frac{t - s\cos^2\alpha}{2s(t-s)}.$$

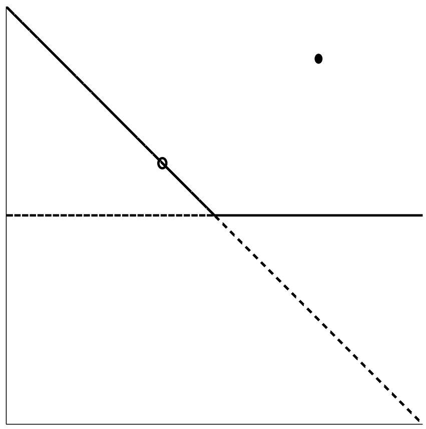

Fig. 2. $z_0 = \bullet$ and $Z(\tau) = \circ$. τ is the time required to reach $\circ$ from the initial position $\bullet$, while $\tau' - \tau$ is the time required to travel from $\circ$ to the horizontal axis.

Using identity (1.2) and simplifying, we obtain that for $s < t$ we have

$$f(s,t) = \frac{\mathrm{d}s\,\mathrm{d}t\;\pi\,\sin\alpha}{2\alpha^2\sqrt{s(t-s\cos^2\alpha)}(t-s)}\exp\left(-\frac{r_0^2}{2s}\frac{t-s\cos 2\alpha}{(t-s)+(t-s\cos 2\alpha)}\right)$$
$$\times\sum_{n=1}^{\infty} n\sin\left(\frac{n\pi(\alpha-\theta_0)}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r_0^2}{2s}\frac{t-s}{(t-s)+(t-s\cos 2\alpha)}\right). \tag{3.2}$$

An analogous argument may be used for $s > t$, leading to

$$f(s,t) = \frac{\mathrm{d}s\,\mathrm{d}t\;\pi\,\sin\alpha}{2\alpha^2\sqrt{t(s-t\cos^2\alpha)}(s-t)}\exp\left(-\frac{r_0^2}{2t}\frac{s-t\cos 2\alpha}{(s-t)+(s-t\cos 2\alpha)}\right)$$
$$\times\sum_{n=1}^{\infty} n\sin\left(\frac{n\pi\theta_0}{\alpha}\right) I_{n\pi/2\alpha}\left(\frac{r_0^2}{2t}\frac{s-t}{(s-t)+(s-t\cos 2\alpha)}\right). \tag{3.3}$$

An interesting feature of this density is that it has a singularity along the line $s = t$ when the original process has positive correlation (in which case $\alpha > \pi/2$), that is $f(s,t) \to \infty$ as $|s-t| \to 0$. This can be checked using the well-known asymptotic relation $I_\nu(x) \sim (x/2)^\nu/\Gamma(\nu+1)$ as $x \to 0$.

4. The presence of drift and Monte Carlo simulation

Thus far we have assumed the original process has zero drift, however in both physical and financial applications this is typically inappropriate. In this section we briefly consider measures $P_\gamma^{z_0}$ under which Z has drift $\gamma \in \mathbb{R}^2$, and set $P^{z_0} = P_0^{z_0}$.

As noted in Iyengar (1985), an appeal to Girsanov's theorem provides quantities analogous to (a)–(b), for example

$$P_\gamma^{z_0}\left(Z(t) \in \mathrm{d}z,\ \tau > t\right) = \exp\left(\gamma'(z-z_0) - |\gamma|^2 t/2\right) P^{z_0}\left(Z(t) \in \mathrm{d}z,\ \tau > t\right), \tag{4.1}$$

where γ' and $|\gamma|$ denote transpose and norm, respectively. Thus we obtain the semi-analytic expression

$$P_\gamma^{z_0}(\tau > t) = \int_{C_\alpha} \mathrm{e}^{\gamma'(z-z_0)-|\gamma|^2 t/2} P^{z_0}\left(Z(t) \in \mathrm{d}z,\ \tau > t\right), \tag{4.2}$$

which becomes a series whose terms are double integrals of Bessel functions; the approximation of (4.2) is computationally expensive to the point of being prohibitive (Zhou, 2001).

Turning now to the computation of the joint density in the presence of drift

$$P_\gamma^{z_0}\left(\tau_1 \in \mathrm{d}s,\ \tau_2 \in \mathrm{d}t\right),$$

an argument similar to that used in Section 3 yields that for $s < t$ this quantity is given by

$$\int_0^\infty P_\gamma^{z_0}\left(\tau \in \mathrm{d}s,\ R(\tau) \in \mathrm{d}r,\ \Theta(\tau) = \alpha\right) \frac{r\sin\alpha}{\sqrt{2\pi(t-s)^3}} \mathrm{e}^{-(r\sin\alpha+\gamma_2(t-s))^2/2(t-s)}, \tag{4.3}$$

where the integral is taken with respect to r and the first term appearing in the integrand may be obtained via a Girsanov transformation

$$P^{z_0}_{\gamma}\left(\tau \in \mathrm{d}t,\ Z(\tau) \in \mathrm{d}z\right) = \exp\left(\gamma'(z - z_0) - |\gamma|^2\, t/2\right) P^{z_0}\left(\tau \in \mathrm{d}t,\ Z(\tau) \in \mathrm{d}z\right). \tag{4.4}$$

Again, the resulting integral is no longer analytically tractable. As a final note we mention that when $\gamma \neq 0$ the process $\log Z$ is no longer a conformal local martingale, and our technique for deriving the distribution of $Z(\tau)$ under $P^{z_0}_{\gamma}$ no longer applies.

4.1. Monte Carlo simulation

We motivate the material presented in this section by considering the problem of estimating $E^{z_0}[g(\tau_1, \tau_2)]$. Numerical instabilities near the singularity in the joint density, as well as truncation error, make this a challenging problem. These problems are exacerbated when non-zero drift is introduced, or are interested in more general quantities such as Kendall's tau. In such circumstances Monte Carlo simulation provides an attractive alternative, either to obtain outright approximations or provide simple checks on the accuracy of numerical methods.

To this end, consider the problem of simulating (τ_1, τ_2) under P^{z_0}. Since the joint density of the pair is available, in principle one could sample (τ_1, τ_2) as follows

- Sample τ_2 from its marginal density

  $$P^{z_0}(\tau_2 \in \mathrm{d}t) = \frac{r_0 \sin\theta_0}{\sqrt{2\pi t^3}}\, \mathrm{e}^{-(r_0 \sin\theta_0)^2/2t}\, \mathrm{d}t.$$

  This is easily accomplished using the inverse transform method.
- Having generated $\tau_2 = t$, sample τ_1 from its conditional density

  $$P^{z_0}(\tau_1 \in \mathrm{d}s | \tau_2 = t) = \frac{\sqrt{2\pi t^3}}{r_0 \sin\theta_0}\, \mathrm{e}^{(r_0 \sin\theta_0)^2/2t}\, f(s, t)\, \mathrm{d}s$$

  where $f(s, t)$ is given by (3.2) and (3.3). As $f(s, t)$ is rather unwieldy, the conditional density in the second step is not amenable to common simulation techniques. For example using inverse transform would require numerical integration of f and inversion of this integral; an imposing endeavor. In addition when $\alpha > \pi/2$ the conditional density has a singularity at the point $s = t$, which creates difficulties for acceptance–rejection.

The simulation problem becomes tractable by introducing the "auxiliary" variable $Z(\tau)$, and noting that (τ_1, τ_2) is uniquely determined by the triplet $(\tau, \tau', Z(\tau))$. Moreover, this triplet may be simulated as follows

- Simulate $Z(\tau)$. In light of Proposition 2.1 this only requires simulating from the distribution given by (2.2)–(2.3) (details can be found in Metzler (2008)). By the strong Markov property we may now simulate $\tau' - \tau$ and τ *independently*.
- Conditional simulation of τ requires the distribution of $Z(\tau)$, as well as the joint density of $(\tau, Z(\tau))$ given by Iyengar (1985). As shown by Metzler (2008) this is a manageable problem using acceptance–rejection.
- In light of (3.1) conditional simulation of $\tau' - \tau$ only requires generating an inverse Gaussian variate, which is easily accomplished using inverse transform.

The output of this algorithm may also be used to estimate expectations of the form $E^{z_0}_{\gamma}[g(\tau, Z(\tau))]$ for $\gamma \neq 0$. To see this we note that, provided $f(\tau, Z(\tau)) = 0$ on the event $\{\tau = \infty\}$, we have by the Girsanov theorem

$$E^{z_0}_{\gamma}[g(\tau, Z(\tau))] = E^{z_0}\left[\mathrm{e}^{\gamma'(Z(\tau)-z_0)-|\gamma|^2\tau/2} g(\tau, Z(\tau))\right].$$

More generally, if $g(\tau_1, \tau_2, Z(\tau_1), Z(\tau_2)) = 0$ on the event $\{\tau_1 = \infty \text{ or } \tau_2 = \infty\}$, we have

$$E^{z_0}_{\gamma}[g(\tau_1, \tau_2, Z(\tau_1), Z(\tau_2))] = E^{z_0}\left[\mathrm{e}^{\gamma'(Z(\tau')-z_0)-|\gamma|^2\tau'/2} g(\tau_1, \tau_2, Z(\tau_1), Z(\tau_2))\right],$$

which requires simulation of $Z(\tau')$, and where $\tau' = \max(\tau_1, \tau_2)$. Conditional upon $(\tau, \tau', Z(\tau))$, simulation of $Z(\tau')$ is an easy task. If it turns out that $\Theta(\tau) = \alpha$, then $Z_2(\tau') = 0$ and $Z_1(\tau')$ is Gaussian with mean $R(\tau)\cos\alpha$ and variance $\tau' - \tau$. If it turns out that $\Theta(\tau) = 0$, we have that $\tilde{Z}_2(\tau') = 0$ and $\tilde{Z}_1(\tau')$ is Gaussian with mean $-R(\tau)\cos\alpha$ and variance $\tau' - \tau$, where $\tilde{Z}$ is obtained from Z via a counter-clockwise rotation of $\pi - \alpha$. Thus in this case we may recover $Z(\tau') = -(\cos\alpha \tilde{Z}_1(\tau'), \sin\alpha \tilde{Z}_1(\tau'))$, which is easily simulated.

5. Conclusion

This paper addresses several issues concerning the first passage problem for two-dimensional correlated Brownian motion to fixed levels. To begin, when the underlying process has zero drift, we present corrected formulae for the joint density of the passage times and the distribution of the location of first exit from the positive quadrant. Through its relation to Dirichlet problem in the wedge, we are led to believe that this is a relatively straightforward problem from a PDE perspective. Here we present a probabilistic solution which exploits the geometry of the problem, using a combination of the reflection principle and conformal local martingales. Finally, we exploit the strong Markov property and conformal invariance (modulo a time change) of planar Brownian motion to obtain an "indirect" method for simulating the passage times which makes no use of their unwieldy joint density. This method may be used to approximate quantities of interest when the underlying process has non-zero drift, in which case the aforementioned distributions are not expressible in closed form.

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Appendix. Conformal local martingales

Here we collect several known facts concerning conformal local martingales, each of which may be found in Rogers and Williams (2000). Given a probability space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $\{\mathcal{F}_t : t \geq 0\}$, we recall that a $\mathbb{C}$-valued process $Y = U + iV$ is said to be a conformal local martingale (CLM) relative to $\mathcal{F}_t$ if U and V are local martingales relative to $\mathcal{F}_t$, such that $[U] = [V]$ and $[U, V] = 0$, where $[\cdot]$, $[\cdot, \cdot]$ denote quadratic variation and quadratic covariation, respectively. The results used in this paper are as follows

(C1) *Log-Brownian motion is a CLM. If Z is a complex Brownian motion, then* $\log(Z(t)) := \log(R(t)) + i\Theta(t)$ *is a CLM, where R and Θ denote the radial and angular parts of Z, respectively.*

(C2) *Analytic transformations preserve the CLM property. If Y is a CLM and $f : \mathbb{C} \to \mathbb{C}$ is analytic, then $f(Y)$ is a CLM.*

(C3) *CLMs are time-changed Brownian motion. If $Y = U + iV$ is a CLM, then there exists a complex Brownian motion B such that* $Y(t) = B([U](t))$. Note that B is not necessarily a Brownian motion with respect to $\mathcal{F}_t$, rather it is a Brownian motion relative to $\mathcal{G}_t = \mathcal{F}_{A_t}$, where

$$A_t = \inf\{s \geq 0 : [U](s) > t\}.$$

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