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S.I.: RISK MANAGEMENT DECISIONS AND VALUE UNDER UNCERTAINTY

Closed form valuation of barrier options with stochastic barriers

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Abstract
This article deals with the computation of the probability, for a GBM (geometric Brownian motion) process, to hit sequences of one-sided stochastic boundaries defined as GBM processes, over a closed time interval. Explicit formulae are obtained, allowing the analytical valuation of all the main kinds of barrier options in a much more general setting than the usual one assuming constant or time-dependent, deterministic barriers. The numerical implementation of all stated formulae is shown to be easy, fast and accurate. The practical applications are potentially substantial, since barrier options play a major role in quantitative finance, not only as intensively traded contracts on their own, but also as the building blocks of a large variety of structured products. Barrier options are also an important tool in financial modelling, used to measure default risk in the so-called "structural" models.

Keywords Boundary crossing probability · First passage time probability · Barrier option · Stochastic barrier · Option valuation

1 Introduction

The question of the crossing of a boundary by a Brownian motion or a function of Brownian motion is of central importance in many mathematical sciences. Almost all the research articles published on this topic deal with non-random boundaries, whether constant or deterministic functions of time. The vast majority of the contributions either focus on numerical algorithms for general classes of boundaries, or they seek to obtain approximate solutions, typically substituting the initial boundary with another one for which computations are easier, and then deriving a bound for the error entailed by using the approximating boundary. Much attention has also been paid to asymptotic estimates. However, exact analytical results are scarce, not to mention closed form formulae.

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Focusing on one-sided boundaries, i.e. boundaries that cannot be crossed both from below and above but only from one side, the main exact results related to Brownian motion that have been published can be quickly listed in chronological order:

- The basic building block is the distribution of the first passage time of Brownian motion to a boundary defined as a straight line over a closed time interval. This is a very classical result that can be traced back to Schrödinger (1915), or Levy (1939) for a more modern approach based on the "reflection principle".
- The so-called Bachelier–Levy formula (Levy 1948), provides the first-passage time density of Brownian motion to a linear boundary
- Breiman (1966), Sato (1977), Salminen (1988) and Novikov et al. (1999) study various sorts of square root boundaries
- The first passage time density of Brownian motion to a quadratic boundary is obtained independently by Salminen (1988) and Groeneboom (1989)

There are also a few other exact results for special parametric families of one-sided boundaries (Daniels 1969, 1982; Di Nardo et al. 2001). However, they involve specific forms of the boundary, determined not by real life applications but by their ability to suit analytical techniques of computation, and thus have restrictive use in practice, although they are quite valuable to test numerical algorithms.

A few exact results have also been published with regard to multidimensional Brownian motion. In particular, Iyengar (1985) provides the joint density function of the first hitting times of a constant boundary by a pair of correlated standard Brownian motions, while He et al. (1998) and Patras (2005) extend the analysis to a Brownian motion with drift and provide applications to mathematical finance.

Of all the results thus far mentioned, only those of Levy (1939, 1948) lend themselves to elementary numerical evaluation. The formulae by Breiman (1966), Satto (1977) and Salminen (1988) are only semi-analytical, as they are expressed via Laplace transforms and require numerical inversion. The formulae by Salminen (1988), Groeneboom (1989) and Novikov et al. (1999) involve infinite sums of combinations of Airy functions or confluent hypergeometric functions and the computation of their roots. Similar numerical complexities are entailed by the previously mentioned results on multidimensional Brownian motion, in the form of multiple integration of infinite sums of modified Bessel functions.

A common feature of all the previously mentioned references is that they deal with boundaries moving in a non-random manner. Very few analytical results have been published for stochastic boundaries. Park and Beekman (1983) examine general procedures for finding the first-passage time density of Brownian motion to a boundary defined as a compound Poisson process. The same question is tackled by Che and Dassios (2012) with jumps in the boundary that are either exponentially distributed or constant. More recently, Guillaume (2018b) derives an explicit formula for the probability of hitting a two-sided boundary defined as a pair of simultaneously monitored geometric Brownian motions. The present article extends this field of research by computing the probability of hitting a sequence of one-sided stochastic boundaries defined as GBM (geometric Brownian motion) processes. More precisely, if S_1, S_2, S_3 and S_4 are four correlated GBM processes, with S_1 and S_2 defined as lower or upper boundaries with respect to S_3 on two successive time intervals $\left[t_0, t_1\right]$ and $\left[t_1, t_2\right]$, then a closed form formula is provided for the joint probability that S_3 will not cross S_1 between t_0 and t_1, and that S_3 will not cross S_2 between t_1 and t_2, and that S_4 will stand above or below a prespecified constant at a time $t_3 \geq t_2$. The reason for the choice of this sequence of events is that it allows the analytical valuation of all the main kinds of the so-called "barrier options", which are a fundamental category of contingent claims in mathematical finance, in a much more general setting than the usual one assuming constant or time-dependent, deterministic barriers. Other distributions are also analytically computed, in view of their relevance for practical purposes. It is shown that exact results can be derived using a straightforward probabilistic approach based on the analytical integration of products of conditional density functions. Moreover, the obtained formulae allow easy, fast and accurate numerical implementation, making them a valuable practical tool. These two attractive features stand in contrast to the problem raised by two-sided stochastic diffusive boundaries (Guillaume 2018), which requires a partial differential equation framework leading to a more complicated numerical implementation.

The practical applications of these analytical results are potentially substantial, since barrier options play a major role in quantitative finance, not only as intensively traded contracts on their own, but also as the building blocks of a large variety of structured products (Wystup 2006; Bouzoubaa and Osseiran 2010). Barrier options are also an important tool in financial modelling to measure default risk in the so-called "structural" models (Bielecki and Rutkowski 2004).

This paper is organized as follows: Sect. 2 states the main result and develops its proof; Sect. 3 provides applications in mathematical finance along with additional analytical results; Sect. 4 deals with the numerical implementation of the stated formulae.

2 Main result

Let S_1, S_2, S_3 and S_4 be four correlated geometric Brownian motions, the instantaneous variations of which, under a given probability measure P, are given by:

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dB_i(t), \forall i \in \{1, 2, 3, 4\} \tag{1}$$

where $\mu_i \in \mathbb{R}$, $\sigma_i \in \mathbb{R}_+$, and each B_i is a standard Brownian motion with $d\left[B_i, B_j\right](t) = \rho_{i,j}dt, \forall (i,j) \in \mathbb{N}^2, i < j$

The objective is to compute $\mathrm{P}_{[.]}$, defined as one of the four following joint cumulative distribution functions:

$$\mathrm{P}_{[\mathrm{UOUO}]} = P\left(\left(S_3(t) < S_1(t), 0 \le t \le t_1\right) \cap \left(S_3(t) < S_2(t), t_1 \le t \le t_2\right) \cap \left(S_4\left(t_3\right) < K\right)\right) \tag{2}$$

$$\mathrm{P}_{[\mathrm{DODO}]} = P\left(\left(S_3(t) > S_1(t), 0 \le t \le t_1\right) \cap \left(S_3(t) > S_2(t), t_1 \le t \le t_2\right) \cap \left(S_4\left(t_3\right) > K\right)\right) \tag{3}$$

$$\mathrm{P}_{[\mathrm{DOUO}]} = P\left(\left(S_3(t) > S_1(t), 0 \le t \le t_1\right) \cap \left(S_3(t) < S_2(t), t_1 \le t \le t_2\right) \cap \left(S_4\left(t_3\right) < K\right)\right) \tag{4}$$

$$\mathrm{P}_{[\mathrm{UODO}]} = P\left(\left(S_3(t) < S_1(t), 0 \le t \le t_1\right) \cap \left(S_3(t) > S_2(t), t_1 \le t \le t_2\right) \cap \left(S_4\left(t_3\right) > K\right)\right) \tag{5}$$

where $K > 0$, $t_3 \ge t_2 \ge t_1 \ge t_0 = 0$ and the acronyms UO and DO stand for Up-and-Out and Down-and-Out, respectively, in reference to the terminology used for barrier options in mathematical finance

Let us introduce the following notations, $\forall (i, j, k, l) \in \mathbb{N}^4$:

$$\sigma_{j|i} = \sqrt{1-\rho_{i.j}^2} \tag{6}$$

$$\rho_{j.k|i} = \left(\rho_{j.k} - \rho_{i.j}\rho_{i.k}\right)/\sigma_{j|i} \tag{7}$$

$$\sigma_{k|i.j} = \sqrt{1-\rho_{i.k}^2-\rho_{j.k|i}^2} \tag{8}$$

$$\rho_{k.l|i.j} = \left(\rho_{k.l} - \rho_{i.k}\rho_{i.l} - \rho_{j.k|i}\,\rho_{j.l|i}\right)/\sigma_{k|i.j} \tag{9}$$

Let $\boldsymbol{\Phi}$ be the function defined by:

$$\begin{aligned}&\boldsymbol{\Phi}\left[b_1, b_2, b_3, b_4; \rho_{1.2}, \rho_{1.3}, \rho_{1.4}, \rho_{2.3}, \rho_{3.4}\right]\\ &= \int_{y_1=-\infty}^{\frac{b_1-\rho_{1.2}y_2}{\sigma_{2|1}}} \int_{y_2=-\infty}^{b_2} \int_{y_3=-\infty}^{\frac{b_3-\rho_{2.3}y_2}{\sigma_{3|2}}} \frac{1}{\sqrt{8\pi^3}} \exp\left(-\frac{1}{2}\left(y_1^2+y_2^2+y_3^2\right)\right) N\left[\frac{b_4-a_1y_1-a_2y_2-a_3y_3}{\sigma_{4|1.3}}\right] dy_3dy_2dy_1\end{aligned} \tag{10}$$

where $b_i \in \mathbb{R}$, $\rho_{i.j} \in [-1, 1]$ and:

$$a_1 = \rho_{1.4}\sigma_{2|1} - \frac{\rho_{1.3}\rho_{3.4|1}\,\sigma_{2|1}}{\sigma_{3|1}}, a_2 = \rho_{1.4}\rho_{1.2} + \frac{\rho_{3.4|1}\left(\rho_{2.3}-\rho_{1.3}\rho_{1.2}\right)}{\sigma_{3|1}}, a_3 = \frac{\rho_{3.4|1}\,\sigma_{3|2}}{\sigma_{3|1}} \tag{11}$$

The main result can now be stated.

Proposition 1 *Under the previous assumptions, we have:*

$$\mathrm{P}_{[.]} = \boldsymbol{\Phi}\left[\lambda_1\left(\frac{\alpha_1-\nu_1t_1}{\beta_1\sqrt{t_1}}\right), \lambda_2\left(\frac{\alpha_2-\nu_2t_1}{\beta_2\sqrt{t_1}}\right), \lambda_2\left(\frac{\alpha_2-\nu_2t_2}{\beta_2\sqrt{t_2}}\right), \lambda_2\left(\frac{k-\nu_3t_3}{\sigma_4\sqrt{t_3}}\right); \Theta_1\right] \tag{12}$$

$$-\exp\left(\frac{2\nu_1\alpha_1}{\beta_1^2}\right)\boldsymbol{\Phi}\left[\begin{array}{l}\lambda_1\left(\frac{-\alpha_1-\nu_1t_1}{\beta_1\sqrt{t_1}}\right), \lambda_2\left(\frac{\alpha_2-\nu_2t_1}{\beta_2\sqrt{t_1}} - \theta_{1.2}\frac{2\alpha_1}{\beta_1\sqrt{t_1}}\right), \lambda_2\left(\frac{\alpha_2-\nu_2t_2}{\beta_2\sqrt{t_2}} - \theta_{1.2}\frac{2\alpha_1}{\beta_1\sqrt{t_2}}\right),\\ \lambda_2\left(\frac{k-\nu_3t_3}{\sigma_4\sqrt{t_3}} - \theta_{1.4}\frac{2\alpha_1}{\beta_1\sqrt{t_1}} - \theta_{3.4|1}\left(\theta_{1.2}\frac{2\alpha_1}{\beta_1\sqrt{t_2}} - \theta_{1.3}\frac{2\alpha_1}{\beta_1\sqrt{t_1}}\right)\right); \Theta_1\end{array}\right] \tag{13}$$

$$-\exp\left(\frac{2\nu_2\alpha_2}{\beta_2^2}\right)\boldsymbol{\Phi}\left[\begin{array}{l}\lambda_1\left(\frac{\alpha_1-\nu_1t_1}{\beta_1\sqrt{t_1}} - \theta_{1.2}\frac{2\nu_2t_1}{\beta_2\sqrt{t_1}}\right), \lambda_2\left(\frac{\alpha_2+\nu_2t_1}{\beta_2\sqrt{t_1}}\right), \lambda_2\left(\frac{-\alpha_2-\nu_2t_2}{\beta_2\sqrt{t_2}}\right),\\ \lambda_2\left(\frac{k-\nu_3t_3}{\sigma_4\sqrt{t_3}} - \theta_{3.4|1}\left(\frac{2\alpha_2}{\beta_2\sqrt{t_2}} + \theta_{1.3}\theta_{1.2}\frac{2\nu_2t_1}{\beta_2\sqrt{t_1}}\right) + \theta_{1.4}\theta_{1.2}\frac{2\nu_2t_1}{\beta_2\sqrt{t_1}}\right); \Theta_2\end{array}\right] \tag{14}$$

$$+\exp\left(\left(\frac{2\nu_1}{\beta_1^2} - \frac{4\nu_2\theta_{1.2}}{\beta_1\beta_2}\right)\alpha_1 + \frac{2\nu_2\alpha_2}{\beta_2^2}\right) \tag{15}$$

$$\Phi\left[\begin{array}{l}\lambda_1\left(\frac{-\alpha_1-\nu_1 t_1}{\beta_1\sqrt{t_1}}-\theta_{1.2}\frac{2\nu_2 t_1}{\beta_2\sqrt{t_1}}\right),\lambda_2\left(\frac{\alpha_2+\nu_2 t_1}{\beta_2\sqrt{t_1}}-\theta_{1.2}\frac{2\alpha_1}{\beta_1\sqrt{t_1}}\right),\lambda_2\left(\frac{-\alpha_2-\nu_2 t_2}{\beta_2\sqrt{t_2}}+\theta_{1.2}\frac{2\alpha_1}{\beta_1\sqrt{t_2}}\right),\\ \lambda_2\left(\begin{array}{l}\frac{k-\nu_3 t_3}{\sigma_4\sqrt{t_3}}+\theta_{1.4}\left(\theta_{1.2}\frac{2\nu_2\sqrt{t_1}}{\beta_2}-\frac{2\alpha_1}{\beta_1\sqrt{t_1}}\right)\\ -\theta_{3.4|1}\left(\frac{2\alpha_2}{\beta_2\sqrt{t_2}}-\theta_{1.2}\frac{2\alpha_1}{\beta_1\sqrt{t_2}}+\theta_{1.3}\left(\theta_{1.2}\frac{2\nu_2\sqrt{t_1}}{\beta_2}-\frac{2\alpha_1}{\beta_1\sqrt{t_1}}\right)\right)\end{array}\right);\Theta_2\end{array}\right]$$

where

$$\begin{array}{l}\lambda_1=1\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UOUO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UODO}]}\\ \lambda_1=-1\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DODO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DOUO}]}\\ \lambda_2=1\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UOUO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DOUO}]}\\ \lambda_2=-1\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DODO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UODO}]}\end{array}\tag{16}$$

$$\alpha_1=\ln\left(\frac{S_1(0)}{S_3(0)}\right),\alpha_2=\ln\left(\frac{S_2(0)}{S_3(0)}\right),k=\ln\left(\frac{K}{S_4(0)}\right)\tag{17}$$

$$\nu_1=\mu_3-\mu_1-\frac{\sigma_3^2-\sigma_1^2}{2},\nu_2=\mu_3-\mu_2-\frac{\sigma_3^2-\sigma_2^2}{2},\nu_3=\mu_4-\frac{\sigma_4^2}{2}\tag{18}$$

$$\beta_1=\sqrt{\sigma_1^2-2\sigma_1\sigma_3\rho_{1.3}+\sigma_3^2},\beta_2=\sqrt{\sigma_2^2-2\sigma_2\sigma_3\rho_{2.3}+\sigma_3^2}\tag{19}$$

$$\theta_{1.2}=\frac{\sigma_3^2-\sigma_3\sigma_1\rho_{1.3}-\sigma_3\sigma_2\rho_{2.3}+\sigma_1\sigma_2\rho_{1.2}}{\beta_1\beta_2},\theta_{1.3}=\theta_{1.2}\sqrt{\frac{t_1}{t_2}}\tag{20}$$

$$\theta_{1.4}=\sqrt{\frac{t_1}{t_3}}\frac{\left(\sigma_3\rho_{1.3}-\sigma_1\right)\rho_{1.4}+\sigma_3\rho_{2.3|1}\,\rho_{2.4|1}\,+\sigma_3\sigma_{3|1.2}\,\rho_{3.4|1.2}}{\beta_1}\tag{21}$$

$$\theta_{2.3}=\sqrt{\frac{t_1}{t_2}},\theta_{3.4}=\sqrt{\frac{t_2}{t_3}}\frac{\left(\sigma_3\rho_{1.3}-\sigma_2\rho_{1.2}\right)\rho_{1.4}+\left(\sigma_3\rho_{2.3|1}\,-\sigma_2\sigma_{2|1}\,\right)\rho_{2.4|1}\,+\sigma_3\sigma_{3|1.2}\,\rho_{3.4|1.2}}{\beta_2}\tag{22}$$

$$\theta_{3.4|1}\,=\frac{\theta_{3.4}-\theta_{1.3}\theta_{1.4}}{\sqrt{1-\theta_{1.3}^2}}\tag{23}$$

$$\begin{array}{l}\Theta_1=\left\{\theta_{1.2},\theta_{1.3},\theta_{1.4},\theta_{2.3},\theta_{3.4}\right\}\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UOUO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DODO}]}\\ \Theta_1=\left\{-\theta_{1.2},-\theta_{1.3},-\theta_{1.4},\theta_{2.3},\theta_{3.4}\right\}\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UODO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DOUO}]}\\ \Theta_2=\left\{\theta_{1.2},\theta_{1.3},\theta_{1.4},-\theta_{2.3},\theta_{3.4}\right\}\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UOUO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DODO}]}\\ \Theta_2=\left\{-\theta_{1.2},-\theta_{1.3},-\theta_{1.4},-\theta_{2.3},\theta_{3.4}\right\}\quad\text{if }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UODO}]}\text{ or }\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DOUO}]}\end{array}\tag{24}$$

Proof of Proposition 1 Let us define the following new processes X_1, X_2 and X_3:

$$
\begin{aligned}
X_1(t) &\triangleq \left\{ \ln\left(\frac{S_3(t)S_1(0)}{S_1(t)S_3(0)}\right), t \geq 0 \right\} \\
X_2(t) &\triangleq \left\{ \ln\left(\frac{S_3(t)S_2(0)}{S_2(t)S_3(0)}\right), t \geq 0 \right\} \\
X_3(t) &\triangleq \left\{ \ln\left(\frac{S_4(t)}{S_4(0)}\right), t \geq 0 \right\}
\end{aligned}
\tag{25}
$$

The joint distributions denoted as $\mathrm{P}_{[.]}$ can be written in terms of X_1, X_2 and X_3 as follows:

$$
\mathrm{P}_{[\mathrm{UOUO}]} \triangleq p_1 = P\left(\sup_{0\leq t\leq t_1} X_1(t) < \alpha_1, \sup_{t_1\leq t\leq t_2} X_2(t) < \alpha_2, X_3(t_3) < k\right) \tag{26}
$$

$$
\mathrm{P}_{[\mathrm{DODO}]} \triangleq p_2 = P\left(\inf_{0\leq t\leq t_1} X_1(t) > \alpha_1, \inf_{t_1\leq t\leq t_2} X_2(t) > \alpha_2, X_3(t_3) > k\right) \tag{27}
$$

$$
\mathrm{P}_{[\mathrm{DOUO}]} \triangleq p_3 = P\left(\inf_{0\leq t\leq t_1} X_1(t) > \alpha_1, \sup_{t_1\leq t\leq t_2} X_2(t) < \alpha_2, X_3(t_3) < k\right) \tag{28}
$$

$$
\mathrm{P}_{[\mathrm{UODO}]} \triangleq p_4 = P\left(\sup_{0\leq t\leq t_1} X_1(t) < \alpha_1, \inf_{t_1\leq t\leq t_2} X_2(t) > \alpha_2, X_3(t_3) > k\right) \tag{29}
$$

At each $t \geq 0$, $X_1(t)$, $X_2(t)$ and $X_3(t)$ are absolutely continuous random variables. By conditioning with respect to $X_1(t_1)$, $X_2(t_1)$, $X_2(t_2)$ and $X_3(t_3)$, and by using the Markov property of the processes X_1, X_2 and X_3, the distributions under consideration can thus be written as the following multiple integrals:

$$
\begin{aligned}
p_1 = &\int_{x_1=-\infty}^{\alpha_1} \int_{x_2=-\infty}^{\alpha_2} \int_{x_3=-\infty}^{\alpha_2} \int_{x_4=-\infty}^{k} P\left(X_1(t_1) \in dx_1, \sup_{0\leq t\leq t_1} X_1(t) < \alpha_1\right) P\left(X_2(t_1) \in dx_2 \middle| X_1(t_1) \in dx_1\right) \\
&P\left(X_2(t_2) \in dx_3, \sup_{t_1\leq t\leq t_2} X_2(t) < \alpha_2 \middle| X_2(t_1) \in dx_2\right) P\left(X_3(t_3) \in dx_4 \middle| X_1(t_1) \in dx_1, X_2(t_2) \in dx_3\right) \\
&dx_4 dx_3 dx_2 dx_1
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
p_2 = &\int_{x_1=\alpha_1}^{\infty} \int_{x_2=\alpha_2}^{\infty} \int_{x_3=\alpha_2}^{\infty} \int_{x_4=k}^{\infty} P\left(X_1(t_1) \in dx_1, \inf_{0\leq t\leq t_1} X_1(t) > \alpha_1\right) P\left(X_2(t_1) \in dx_2 \middle| X_1(t_1) \in dx_1\right) \\
&P\left(X_2(t_2) \in dx_3, \inf_{t_1\leq t\leq t_2} X_2(t) > \alpha_2 \middle| X_2(t_1) \in dx_2\right) P\left(X_3(t_3) \in dx_4 \middle| X_1(t_1) \in dx_1, X_2(t_2) \in dx_3\right) \\
&dx_4 dx_3 dx_2 dx_1
\end{aligned}
\tag{31}
$$

$$p_3 = \int\limits_{x_1=\alpha_1}^{\infty} \int\limits_{x_2=-\infty}^{\alpha_2} \int\limits_{x_3=-\infty}^{\alpha_2} \int\limits_{x_4=-\infty}^{k} P\left(X_1(t_1) \in dx_1, \inf_{0\le t\le t_1} X_1(t) > \alpha_1\right) P\left(X_2(t_1) \in dx_2 \middle| X_1(t_1) \in dx_1\right)$$
$$P\left(X_2(t_2) \in dx_3, \sup_{t_1\le t\le t_2} X_2(t) < \alpha_2 \middle| X_2(t_1) \in dx_2\right) P\left(X_3(t_3) \in dx_4 \middle| X_1(t_1) \in dx_1, X_2(t_2) \in dx_3\right)$$
$$dx_4 dx_3 dx_2 dx_1 \tag{32}$$

$$p_4 = \int\limits_{x_1=-\infty}^{\alpha_1} \int\limits_{x_2=\alpha_2}^{\infty} \int\limits_{x_3=\alpha_2}^{\infty} \int\limits_{x_4=k}^{\infty} P\left(X_1(t_1) \in dx_1, \sup_{0\le t\le t_1} X_1(t) < \alpha_1\right) P\left(X_2(t_1) \in dx_2 \middle| X_1(t_1) \in dx_1\right)$$
$$P\left(X_2(t_2) \in dx_3, \inf_{t_1\le t\le t_2} X_2(t) > \alpha_2 \middle| X_2(t_1) \in dx_2\right) P\left(X_3(t_3) \in dx_4 \middle| X_1(t_1) \in dx_1, X_2(t_2) \in dx_3\right)$$
$$dx_4 dx_3 dx_2 dx_1 \tag{33}$$

By performing the orthogonal decomposition of the Brownian motions B_1, B_2, B_3 and B_4 into the basis comprised of the four pairwise independent Brownian motions B_1, $\tilde{B}_2$, $\tilde{B}_3$ and $\tilde{B}_4$ defined on the same probability space as the B_i' s [Guillaume 2018a], it can be shown that the processes X_1, X_2 and X_3 admit the following expansions under P at any given $t \ge 0$:

$$X_1(t) = \left(\mu_3 - \mu_1 - \frac{\sigma_3^2 - \sigma_1^2}{2}\right)t + \left(\sigma_3\rho_{1.3} - \sigma_1\right)B_1(t) + \sigma_3\rho_{2.3|1}\tilde{B}_2(t) + \sigma_3\sigma_{3|1.2}\tilde{B}_3(t) \tag{34}$$

$$\begin{aligned} X_2(t) =& \left(\mu_3 - \mu_2 - \frac{\sigma_3^2 - \sigma_2^2}{2}\right)t + \left(\sigma_3\rho_{1.3} - \sigma_2\rho_{1.2}\right)B_1(t) \\ &+ \left(\sigma_3\rho_{2.3|1} - \sigma_2\sigma_{2|1}\right)\tilde{B}_2(t) + \sigma_3\sigma_{3|1.2}\tilde{B}_3(t) \end{aligned} \tag{35}$$

$$X_3(t) = \left(\mu_4 - \frac{\sigma_4^2}{2}\right)t + \sigma_4\left(\rho_{1.4}B_1(t) + \rho_{2.4|1}\tilde{B}_2(t) + \rho_{3.4|1.2}\tilde{B}_3(t) + \sigma_{4|1.2.3}\tilde{B}_4(t)\right) \tag{36}$$

where

- $\rho_{j.k|i}$ is the correlation between $B_j(t)$ and $B_k(t)$ conditional on $B_i(t)$
- $\rho_{k.l|i.j}$ is the correlation between $B_k(t)$ and $B_l(t)$ conditional on $B_i(t)$ and $B_j(t)$
- $\sigma_{k|i.j}$ is the standard deviation of $B_k(t)$ conditional on $B_i(t)$ and $B_j(t)$.
- $\sigma_{l|i.j.k}$ is the standard deviation of $B_l(t)$ conditional on $B_i(t)$, $B_j(t)$ and $B_k(t)$.

The processes X_1, X_2 and X_3 are three correlated arithmetic Brownian motions and the three-dimensional process $\left(X_1, X_2, X_3\right)$ is trivariate Gaussian. Denoting as $\mathcal{N}(a, b)$ the normal distribution with expectation a and standard deviation b, we have, at any given $t \ge 0$, the marginal distributions $X_1(t) \sim \mathcal{N}\left(\nu_1 t, \beta_1\sqrt{t}\right)$, $X_2(t) \sim \mathcal{N}\left(\nu_2 t, \beta_2\sqrt{t}\right)$ and $X_3(t) \sim \mathcal{N}\left(\nu_3 t, \sigma_4\sqrt{t}\right)$ under the probability measure P. Thanks to Eqs. (34)–(36), the required correlation coefficients can be easily computed. Writing $X_1(t_1) \triangleq Z_1$, $X_2(t_1) \triangleq Z_2$,

$X_2(t_2) = Z_3$ and $X_3(t_3) = Z_4$, each $\theta_{i,j}$ in (20)–(22) is thus the correlation between Z_i and Z_j.

The terms $P\left(X_1(t_1) \in dx_1, \sup_{0\le t\le t_1} X_1(t) < \alpha_1\right)$ and $P\left(X_1(t_1) \in dx_1, \inf_{0\le t\le t_1} X_1(t) > \alpha_1\right)$ are obtained by differentiating the classical formula for the joint cumulative distribution of the extremum of a Brownian motion with drift and its endpoint over a closed time interval. The term $P\left(X_2(t_1) \in dx_2 \middle| X_1(t_1) \in dx_1\right)$ is derived from the bivariate normality of the pair $(X_1(t_1), X_2(t_1))$. The term $P\left(X_3(t_3) \in dx_4 \middle| X_1(t_1) \in dx_1, X_2(t_2) \in dx_3\right)$ can be derived from the trivariate normality of the triple $(X_1(t_1), X_2(t_2), X_3(t_3))$, with $\theta_{3.4|1}$ in Eq. (15) being the correlation between $X_2(t_2)$ and $X_3(t_3)$ conditional on $X_1(t_1)$. The terms $P\left(X_2(t_2) \in dx_3, \sup_{t_1\le t\le t_2} X_2(t) < \alpha_2 \middle| X_2(t_1) \in dx_2\right)$ and $P\left(X_2(t_2) \in dx_3, \inf_{t_1\le t\le t_2} X_2(t) > \alpha_2 \middle| X_2(t_1) \in dx_2\right)$ can be obtained by using the following simple lemma.

Lemma 1 *Let $\{S(t), t \ge 0\}$ be a geometric Brownian motion whose instantaneous variations under a given probability measure P are driven by:*

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{37}$$

where $B(t)$ *is a standard Brownian motion, and* $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$.

Let K and H be two positive real numbers such that $H > S(0)$ and $K \le H$. Let T be a finite positive real number. Then, we have

$$\begin{aligned}&P\left(\sup_{0<t\le u\le T} S(u) \le H, S(T) \le K | S(t) = S(0)e^x\right)\\&= N\left[\frac{k - x - v(T-t)}{\sigma\sqrt{T-t}}\right] - \exp\left(\frac{2v}{\sigma^2}(h-x)\right)N\left[\frac{k - 2h + x - v(T-t)}{\sigma\sqrt{T-t}}\right]\end{aligned} \tag{38}$$

$$\begin{aligned}&P\left(\inf_{0<t\le u\le T} S(u) \ge H, S(T) \ge K | S(t) = S(0)e^x\right)\\&= N\left[\frac{-k + x + v(T-t)}{\sigma\sqrt{T-t}}\right] - \exp\left(\frac{2v}{\sigma^2}(h-x)\right)N\left[\frac{-k + 2h - x + v(T-t)}{\sigma\sqrt{T-t}}\right]\end{aligned} \tag{39}$$

where $k = \ln\left(\frac{K}{S(0)}\right), h = \ln\left(\frac{H}{S(0)}\right)$ *and* $v = \mu - \sigma^2/2$.

Proof of Lemma 1 It is a corollary of Levy (1939) that:

$$\begin{aligned}&P\left(\sup_{t\le u\le T} S(u) \le H, S(T) \le K | S(t)\right)\\&= N\left[\frac{\ln\left(\frac{K}{S(t)}\right) - v(T-t)}{\sigma\sqrt{T-t}}\right] - \left(\frac{H}{S(t)}\right)^{\frac{2v}{\sigma^2}} N\left[\frac{\ln\left(\frac{K}{S(t)}\right) - 2\ln\left(\frac{H}{S(t)}\right) - v(T-t)}{\sigma\sqrt{T-t}}\right]\end{aligned} \tag{40}$$

which can be rewritten as:

$$
\begin{aligned}
&P\left(\sup_{t\le u\le T} S(u)\le H, S(T)\le K|S(t)\right)\\
&=N\left[\frac{\ln\left(\frac{K}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)-\nu(T-t)}{\sigma\sqrt{T-t}}\right]\\
&\quad-\exp\left(\frac{2\nu}{\sigma^2}\left(\ln\left(\frac{H}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)\right)\right)\\
&\quad N\left[\frac{\ln\left(\frac{K}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)-2\left(\ln\left(\frac{H}{S(0)}\right)-\ln\left(\frac{S(t)}{S(0)}\right)\right)-\nu(T-t)}{\sigma\sqrt{T-t}}\right]
\end{aligned}
\tag{41}
$$

Therefore, by conditioning with respect to $\ln\left(\frac{S(t)}{S(0)}\right)$, we obtain:

$$
\begin{aligned}
&E_P\left[\mathbb{1}_{\left\{\sup\limits_{0<t\le u\le T} S(u)\le H, S(T)\le K\right\}}|S(0)\right]\\
&=\int_{-\infty}^{h} P\left(\ln\left(\frac{S(t)}{S(0)}\right)\in dx\right)P\left(\sup_{0<t\le u\le T}\ln\left(\frac{S(u)}{S(t)}\right)\le\ln\left(\frac{H}{S(0)}\right),\ln\left(\frac{S(T)}{S(t)}\right)\le\ln\left(\frac{K}{S(0)}\right)\middle|\ln\left(\frac{S(t)}{S(0)}\right)\in dx\right)\\
&=\int_{-\infty}^{h}\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(x-\nu t)^2}{2\sigma^2 t}\right)\\
&\quad\left\{N\left[\frac{k-x-\nu(T-t)}{\sigma\sqrt{T-t}}\right]-\exp\left(\frac{2\nu}{\sigma^2}(h-x)\right)N\left[\frac{k-x-2(h-x)-\nu(T-t)}{\sigma\sqrt{T-t}}\right]\right\}dx
\end{aligned}
\tag{42}
$$

The case of $P\left(\inf\limits_{0<t\le u\le T} S(u)\ge H, S(T)\ge K|S(t)\in dx\right)$ is dealt with similarly.□

Thus, one can come up with the following expansion of the integrals in (30)–(33):

$$
p_1=\int_{x_1=-\infty}^{\alpha_1}\int_{x_2=-\infty}^{\alpha_2}\int_{x_3=-\infty}^{\alpha_2}\int_{x_4=-\infty}^{k} f_1(x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_1,x_3,x_4)dx_4dx_3dx_2dx_1 \tag{43}
$$

$$
p_2=\int_{x_1=\alpha_1}^{\infty}\int_{x_2=\alpha_2}^{\infty}\int_{x_3=\alpha_2}^{\infty}\int_{x_4=k}^{\infty} g_1(x_1)g_2(x_1,x_2)g_3(x_2,x_3)g_4(x_1,x_3,x_4)dx_4dx_3dx_2dx_1 \tag{44}
$$

$$
p_3=\int_{x_1=\alpha_1}^{\infty}\int_{x_2=-\infty}^{\alpha_2}\int_{x_3=-\infty}^{\alpha_2}\int_{x_4=-\infty}^{k} g_1(x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_1,x_3,x_4)dx_4dx_3dx_2dx_1 \tag{45}
$$

$$p_4 = \int_{x_1=-\infty}^{\alpha_1} \int_{x_2=\alpha_2}^{\infty} \int_{x_3=\alpha_2}^{\infty} \int_{x_4=k}^{\infty} f_1(x_1) g_2(x_1, x_2) g_3(x_2, x_3) g_4(x_1, x_3, x_4) dx_4 dx_3 dx_2 dx_1 \quad (46)$$

where

$$f_1(x_1) = \frac{\exp\left(-\frac{1}{2}\left(\frac{x_1-\nu_1 t_1}{\beta_1\sqrt{t_1}}\right)^2\right)}{\beta_1\sqrt{2\pi t_1}} - \exp\left(\frac{2\nu_1}{\beta_1^2}\alpha_1\right)\frac{\exp\left(-\frac{1}{2}\left(\frac{x_1-2\alpha_1-\nu_1 t_1}{\beta_1\sqrt{t_1}}\right)^2\right)}{\beta_1\sqrt{2\pi t_1}} \quad (47)$$

$$g_1(x_1) = \frac{\exp\left(-\frac{1}{2}\left(\frac{-x_1+\nu_1 t_1}{\beta_1\sqrt{t_1}}\right)^2\right)}{\beta_1\sqrt{2\pi t_1}} - \exp\left(\frac{2\nu_1}{\beta_1^2}\alpha_1\right)\frac{\exp\left(-\frac{1}{2}\left(\frac{-x_1+2\alpha_1+\nu_1 t_1}{\beta_1\sqrt{t_1}}\right)^2\right)}{\beta_1\sqrt{2\pi t_1}} \quad (48)$$

$$f_2(x_1, x_2) = \frac{\exp\left(-\frac{1}{2(1-\theta_{1.2}^2)}\left(\frac{x_2-\nu_2 t_1}{\beta_2\sqrt{t_1}} - \theta_{1.2}\frac{x_1-\nu_1 t_1}{\beta_1\sqrt{t_1}}\right)^2\right)}{\beta_2\sqrt{2\pi t_1}\sqrt{1-\theta_{1.2}^2}} \quad (49)$$

$$g_2(x_1, x_2) = \frac{\exp\left(-\frac{1}{2(1-\theta_{1.2}^2)}\left(\frac{-x_2+\nu_2 t_1}{\beta_2\sqrt{t_1}} + \theta_{1.2}\frac{x_1-\nu_1 t_1}{\beta_1\sqrt{t_1}}\right)^2\right)}{\beta_2\sqrt{2\pi t_1}\sqrt{1-\theta_{1.2}^2}} \quad (50)$$

$$\begin{aligned} f_3(x_2, x_3) = & \frac{\exp\left(-\frac{1}{2}\left(\frac{x_3-x_2-\nu_2(t_2-t_1)}{\beta_2\sqrt{t_2-t_1}}\right)^2\right)}{\beta_2\sqrt{2\pi(t_2-t_1)}} \\ & - \exp\left(\frac{2\nu_2}{\beta_2^2}(\alpha_2-x_2)\right)\frac{\exp\left(-\frac{1}{2}\left(\frac{x_3+x_2-2\alpha_2-\nu_2(t_2-t_1)}{\beta_2\sqrt{t_2-t_1}}\right)^2\right)}{\beta_2\sqrt{2\pi(t_2-t_1)}} \end{aligned} \quad (51)$$

$$\begin{aligned} g_3(x_2, x_3) = & \frac{\exp\left(-\frac{1}{2}\left(\frac{-x_3+x_2+\nu_2(t_2-t_1)}{\beta_2\sqrt{t_2-t_1}}\right)^2\right)}{\beta_2\sqrt{2\pi(t_2-t_1)}} \\ & - \exp\left(\frac{2\nu_2}{\beta_2^2}(\alpha_2-x_2)\right)\frac{\exp\left(-\frac{1}{2}\left(\frac{-x_3-x_2+2\alpha_2+\nu_2(t_2-t_1)}{\beta_2\sqrt{t_2-t_1}}\right)^2\right)}{\beta_2\sqrt{2\pi(t_2-t_1)}} \end{aligned} \quad (52)$$

$$f_4(x_1, x_3, x_4) =$$

$$\frac{\exp\left(-\frac{1}{2\phi_{4|1.3}^2}\left(\frac{x_4-v_3t_3}{\sigma_4\sqrt{t_3}}-\theta_{1.4}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}-\theta_{3.4|1}\left(\frac{x_3-v_2t_2}{\beta_2\sqrt{t_2}}-\theta_{1.3}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)\right)^2\right)}{\sigma_4\sqrt{2\pi t_3}\phi_{4|1.3}} \tag{53}$$

$$g_4\left(x_1,x_3,x_4\right)=\frac{\exp\left(-\frac{1}{2\phi_{4|1.3}^2}\left(\frac{-x_4+v_3t_3}{\sigma_4\sqrt{t_3}}+\theta_{1.4}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}+\theta_{3.4|1}\left(\frac{x_3-v_2t_2}{\beta_2\sqrt{t_2}}-\theta_{1.3}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)\right)^2\right)}{\sigma_4\sqrt{2\pi t_3}\phi_{4|1.3}} \tag{54}$$

where $\phi_{4|1.3}$ is the standard deviation of $X_3\left(t_3\right)$ conditional on $X_1\left(t_1\right)$ and $X_2\left(t_2\right)$

By performing the necessary calculations, one can then obtain Proposition 1.

Alternatively, the distributions under consideration can be expressed as:

$$p_1=\int_{x_1=-\infty}^{\alpha_1}\int_{x_2=-\infty}^{\alpha_2}\int_{x_3=-\infty}^{\alpha_2}\int_{x_4=-\infty}^{k} P\left(X_1\left(t_1\right)\in dx_1,X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3,X_3\left(t_3\right)\in dx_4\right)$$
$$P\left(\sup_{0\le t\le t_1} X_1(t)<\alpha_1\middle|X_1\left(t_1\right)\in dx_1\right)P\left(\sup_{t_1\le t\le t_2} X_2(t)<\alpha_2\middle|X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3\right)$$
$$dx_4dx_3dx_2dx_1 \tag{55}$$

$$p_2=\int_{x_1=\alpha_1}^{\infty}\int_{x_2=\alpha_2}^{\infty}\int_{x_3=\alpha_2}^{\infty}\int_{x_4=k}^{\infty} P\left(X_1\left(t_1\right)\in dx_1,X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3,X_3\left(t_3\right)\in dx_4\right)$$
$$P\left(\inf_{0\le t\le t_1} X_1(t)>\alpha_1\middle|X_1\left(t_1\right)\in dx_1\right)P\left(\inf_{t_1\le t\le t_2} X_2(t)>\alpha_2\middle|X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3\right)$$
$$dx_4dx_3dx_2dx_1 \tag{56}$$

$$p_3=\int_{x_1=\alpha_1}^{\infty}\int_{x_2=-\infty}^{\alpha_2}\int_{x_3=-\infty}^{\alpha_2}\int_{x_4=-\infty}^{k} P\left(X_1\left(t_1\right)\in dx_1,X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3,X_3\left(t_3\right)\in dx_4\right)$$
$$P\left(\inf_{0\le t\le t_1} X_1(t)>\alpha_1\middle|X_1\left(t_1\right)\in dx_1\right)P\left(\sup_{t_1\le t\le t_2} X_2(t)<\alpha_2\middle|X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3\right)$$
$$dx_4dx_3dx_2dx_1 \tag{57}$$

$$p_4=\int_{x_1=-\infty}^{\alpha_1}\int_{x_2=\alpha_2}^{\infty}\int_{x_3=\alpha_2}^{\infty}\int_{x_4=k}^{\infty} P\left(X_1\left(t_1\right)\in dx_1,X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3,X_3\left(t_3\right)\in dx_4\right)$$
$$P\left(\sup_{0\le t\le t_1} X_1(t)<\alpha_1\middle|X_1\left(t_1\right)\in dx_1\right)P\left(\inf_{t_1\le t\le t_2} X_2(t)>\alpha_2\middle|X_2\left(t_1\right)\in dx_2,X_2\left(t_2\right)\in dx_3\right)$$
$$dx_4dx_3dx_2dx_1 \tag{58}$$

The term $P\left(X_1\left(t_1\right) \in dx_1, X_2\left(t_1\right) \in dx_2, X_2\left(t_2\right) \in dx_3, X_3\left(t_3\right) \in dx_4\right)$ is a quadrivariate standard normal density function. Following Guillaume (2018a), it can be expressed as follows:

$$h_1\left(x_1,x_2,x_3,x_4\right) \triangleq P\left(X_1\left(t_1\right) \in dx_1, X_2\left(t_1\right) \in dx_2, X_2\left(t_2\right) \in dx_3, X_3\left(t_3\right) \in dx_4\right) = \frac{1}{4\pi^2\phi_{2|1}\,\phi_{3|1.2}\,\phi_{4|1.2.3}\,\beta_1\sqrt{t_1}\beta_2\sqrt{t_1}\beta_2\sqrt{t_2}\sigma_4\sqrt{t_3}} \tag{59}$$

$$\exp\left(-\frac{1}{2}\left(\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)^2 - \frac{1}{2\phi_{2|1}^2}\left(\frac{x_2-v_2t_1}{\beta_2\sqrt{t_1}} - \theta_{1.2}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)^2\right. \tag{60}$$

$$-\frac{1}{2\phi_{3|1.2}^2}\left(\frac{x_3-v_2t_2}{\beta_2\sqrt{t_2}} - \theta_{1.3}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}} - \frac{\theta_{2.3|1}}{\phi_{2|1}}\left(\frac{x_2-v_2t_1}{\beta_2\sqrt{t_1}} - \theta_{1.2}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)\right)^2 \tag{61}$$

$$\left.-\frac{1}{2\phi_{4|1.2.3}^2}\left(\begin{array}{r}\frac{x_4-v_3t_3}{\sigma_4\sqrt{t_3}} - \theta_{1.4}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}} - \frac{\theta_{2.4|1}}{\phi_{2|1}}\left(\frac{x_2-v_2t_1}{\beta_2\sqrt{t_1}} - \theta_{1.2}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)\\ -\frac{\theta_{3.4|1.2}}{\phi_{3|1.2}}\left(\frac{x_3-v_2t_2}{\beta_2\sqrt{t_2}} - \theta_{1.3}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}} - \frac{\theta_{2.3|1}}{\phi_{2|1}}\left(\frac{x_2-v_2t_1}{\beta_2\sqrt{t_1}} - \theta_{1.2}\frac{x_1-v_1t_1}{\beta_1\sqrt{t_1}}\right)\right)\end{array}\right)^2\right) \tag{62}$$

where

$$\phi_{j|i} = \sqrt{1-\theta_{i.j}^2}, \quad \theta_{j.k|i} = \left(\theta_{j.k} - \theta_{i.j}\theta_{i.k}\right)/\phi_{j|i}, \quad \phi_{k|i.j} = \sqrt{1-\theta_{i.k}^2-\theta_{j.k|i}^2} \tag{63}$$

$$\theta_{k.l|i.j} = \left(\theta_{k.l} - \theta_{i.k}\theta_{i.l} - \theta_{j.k|i}\theta_{j.l|i}\right)/\phi_{k|i.j}, \quad \phi_{l|i.j.k} = \sqrt{1-\theta_{i.l}^2-\theta_{j.l|i}^2-\theta_{k.l|i.j}^2} \tag{64}$$

A new correlation coefficient arises, $\theta_{2.4}$, which is the correlation between $X_2\left(t_1\right)$ and $X_3\left(t_3\right)$, the value of which can be easily computed using (34)–(36):

$$\theta_{2.4} = \sqrt{\frac{t_1}{t_3}}\frac{\left(\sigma_3\rho_{1.3} - \sigma_2\rho_{1.2}\right)\rho_{1.4} + \left(\sigma_3\rho_{2.3|1} - \sigma_2\sigma_{2|1}\right)\rho_{2.4|1} + \sigma_3\sigma_{3|1.2}\,\rho_{3.4|1.2}}{\beta_2} \tag{65}$$

The other terms inside the integrals (55)–(58) are derived from the distribution of a Brownian bridge over a time interval starting forward in time and can be found, e.g., in Pötzelberger and Wang (2001).

One can thus come up with the following expansion of the integrals (55)–(58):

$$p_1 = \int_{x_1=-\infty}^{\alpha_1}\int_{x_2=-\infty}^{\alpha_2}\int_{x_3=-\infty}^{\alpha_2}\int_{x_4=-\infty}^{k} h_1\left(x_1,x_2,x_3,x_4\right)h_2\left(x_1\right)h_3\left(x_2,x_3\right)dx_4dx_3dx_2dx_1 \tag{66}$$

$$p_2 = \int_{x_1=\alpha_1}^{\infty} \int_{x_2=\alpha_2}^{\infty} \int_{x_3=\alpha_2}^{\infty} \int_{x_4=k}^{\infty} h_1\left(x_1, x_2, x_3, x_4\right) h_2\left(x_1\right) h_3\left(x_2, x_3\right) dx_4 dx_3 dx_2 dx_1 \tag{67}$$

$$p_3 = \int_{x_1=\alpha_1}^{\infty} \int_{x_2=-\infty}^{\alpha_2} \int_{x_3=-\infty}^{\alpha_2} \int_{x_4=-\infty}^{k} h_1\left(x_1, x_2, x_3, x_4\right) h_2\left(x_1\right) h_3\left(x_2, x_3\right) dx_4 dx_3 dx_2 dx_1 \tag{68}$$

$$p_4 = \int_{x_1=-\infty}^{\alpha_1} \int_{x_2=\alpha_2}^{\infty} \int_{x_3=\alpha_2}^{\infty} \int_{x_4=k}^{\infty} h_1\left(x_1, x_2, x_3, x_4\right) h_2\left(x_1\right) h_3\left(x_2, x_3\right) dx_4 dx_3 dx_2 dx_1 \tag{69}$$

where

$$h_2\left(x_1\right) \triangleq 1 - \exp\left(\frac{2\alpha_1\left(x_1 - \alpha_1\right)}{\beta_1^2 t_1}\right) \mathbb{I}_A \tag{70}$$

$$h_3\left(x_2, x_3\right) \triangleq 1 - \exp\left(\frac{2\left(\alpha_2 - x_2\right)\left(x_3 - \alpha_2\right)}{\beta_2^2\left(t_2 - t_1\right)}\right) \mathbb{I}_B \tag{71}$$

$$A = \left\{x_1 < \alpha_1\right\} \quad \text{if } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{UOUO}]} \text{ or } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{UODO}]} \tag{72}$$

$$A = \left\{x_1 > \alpha_1\right\} \quad \text{if } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{DODO}]} \text{ or } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{DOUO}]} \tag{73}$$

$$B = \left\{x_2 < \alpha_2, x_3 < \alpha_2\right\} \quad \text{if } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{UOUO}]} \text{ or } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{DOUO}]} \tag{74}$$

$$B = \left\{x_2 > \alpha_2, x_3 > \alpha_2\right\} \text{ if } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{DODO}]} \text{ or } \mathrm{P}_{[.]} = \mathrm{P}_{[\mathrm{UODO}]} \tag{75}$$

The symbol $\mathbb{I}$ denotes the indicator function.

By performing the necessary calculations, one can then obtain a linear combination of four quadrivariate standard normal cumulative distributive functions. The resulting formula is slightly more cumbersome than Proposition 1 and is not easier to evaluate numerically, therefore it is omitted. However, the expression of the valuation problem in (55)–(58) is useful inasmuch as it provides the foundation for a powerful conditional Monte Carlo approximation scheme, as will be elaborated on in Sect. 4.

3 Applications

In general, barriers are introduced into contingent claims as a powerful and clever way of reducing the cost of an option position by tailoring contracts to the specific constraints and anticipations of each investor. The impact on the option premium varies with the level of the barrier (s) relative to the spot as well as with the level of volatility, but it is usually quite large, typically ranging between a 10% and a 35% discount according to market conditions and to the degree of risk aversion of the option holder. Figures outside this range are possible, but, for most investors, they would point either to excessive caution or to excessive risk-taking. Provided that the location and the nature of the barriers are set correctly, which is the role of the financial engineer, the probability of the option holder's position ending up out-of-the-money should not be significantly raised. This is because barriers enable investors to choose the adverse scenarios against which they want to be protected, while leaving them the opportunity to waive their insurance against market movements that are favourable to them. For example, in a high volatility environment, a substantial reduction of hedging costs can be obtained by adding an up-and-out barrier to a long position in a put. On the upside, it will not matter much to the option holder if they have been knocked-out, since this would reflect that the market has moved in the right direction for them. On the downside, an automatic rehedging provision triggered by a carefully selected downward crossing point can protect them against the risk of a short lived spike followed by a persistent downward trend in the underlying asset. Symmetrically, it can be recommended to an investor faced with the risk of an increase in the underlying asset value to add a down-and-out barrier to their long position in a call. The level of upward crossing at which a rehedging order is triggered can then be adjusted according to their risk aversion. Relative to standard hedging, barrier-based hedging strategies are thus designed to achieve a "win or don't lose" outcome: option holders end up winning either when the market has moved adversely (with regard to their spot position) and their barrier-based hedging has worked out just as well as a standard hedging but at a reduced cost, or when the market has moved favourably and their barrier-based hedging has turned out to be just as unnecessary as a standard hedging but at least it has been less expensive; while investors do not lose when the direction of the market is eventually adverse after it has been favourable in the first place, as long as the global cost of their barrier-based hedging, including the initial knock-out option and the additional one purchased when rehedging occurs, does not exceed the cost of a standard hedging strategy. These ideas, as well as several other reasons for the great popularity of barrier options, are elaborated on in Das (2005) and Kat (2001).

In particular, when it comes to structured products, barriers are a way of enhancing return for a given level of capital protection, since they allow to construct portfolios with more participation in the growth of equity markets. This is especially profitable in an environment of low yield on risk-free bonds and high stock market volatility such as the current one. Indeed, low interest rates leave little room for investing in risky assets, as a larger part of the investor's capital has to be allocated in bonds, while high levels of volatility prompt financial engineers to acquire instruments strongly sensitive to volatility such as barrier options, since the differential between realized volatility and implicit volatility can be fruitfully exploited on vega exposed portfolios. Knock-in barrier options have particularly high vega values as the usual impact of volatility on the probability of being in-the-money at expiry is magnified by the strong impact of volatility on the probability of knocking-in before expiry.

Barrier options are thus embedded in a large variety of popular structured products. Sometimes, the barrier feature is at the core of the payoff structure. This is so for range-type instruments, e.g. the "range accrual note", that pays out a coupon based on the number of days in which the underlying asset's price has remained within a predetermined corridor, or the so-called "wedding cake", that pays a variable coupon dependent upon the price of an underlying asset moving within pre-specified ranges, or the so-called "twin twin", that offers capped or uncapped participation in the upside of the underlying asset and limited participation in its absolute price drop, provided that throughout the lifespan of the product the underlying asset never breaches a barrier. In other cases, barriers are added to the initial payoff structure in order to make it more flexible and appealing to investors. This applies to the classical capital protected funds which guarantee that investors will receive at least 100% of their initial investment, as well as to the classical reverse convertible notes which provide yield enhancement if the underlying asset closes at or above its initial fixing at maturity, or downside participation in the market otherwise. Barriers are also often embedded into structured products that feature baskets as their underlyings, such as the mountain-type instruments. For instance, the so-called “altiplano” pays out a predetermined high yield at maturity on condition that none of the basket constituents has breached a predefined barrier below the initial fixings; while in the “annapurna” note, the level of the fixed coupon and participation rate depends on whether and when the worst-performing basket constituent reaches a predefined barrier below its initial fixing—the later a barrier breach occurs, the higher the coupon/participation rate.

In addition to their widespread use as traded contracts in the financial markets, barrier options also play an important role as analytical tools in the modelling of default risk (Bielecki and Rutkowski 2004). In the structural approach, an issuer defaults when the value of their assets falls below a default barrier. One can thus calculate the probability of default of a firm, based on the dynamics of its assets and liabilities. One can also improve the valuation of a contingent claim by taking into consideration the risk of its writer becoming in default—an approach known as the “vulnerable” option pricing (Klein and Inglis 2001; Liao and Huang 2005). Other applications of barrier options in financial modelling deal with the valuation of investment projects. Traditional methods for evaluating investment decisions, such as the net present value, do not properly account for the flexibility inherent in many investment projects. The attempt to value such flexibility is known as real options analysis, as explained, e.g., in Koussis et al. (2007). Barrier options are involved whenever the opportunity cost of continuing or abandoning a project is significant (Mun 2002), e.g. when the investment project may get cancelled due to profitability falling below a critical level—a form of down-and-out option, or when the investment project may be expanded as a result of profitability rising above a certain threshold—a form of up-and-in option.

As far as barrier option pricing is concerned, an overwhelming majority of exact analytical formulae assume a classical Black–Scholes model (Black and Scholes 1973) with a geometric Brownian motion for the underlying asset. To cope with alternative models for the dynamics of the underlying asset, a wide variety of numerical methods have been developed. The number of variations on the basic barrier option payoffs has become so large over time that, today, nobody can claim to know all the payoffs that have been traded and priced in the markets since the first time barrier options were introduced. Even in a standard geometric Brownian motion modelling framework, a lot of payoffs are complicated enough to entail serious valuation difficulties, so that approximate pricing by Monte Carlo is very often resorted to by practitioners. For a survey of a few of the best known closed form formulae, one can refer to Haug (2007). For recent developments on the numerical pricing of barrier options under stochastic volatility, one can refer to Funahashi and Higushi (2018).

A common feature of almost all published results on barrier option pricing so far, whether by analytical or by numerical approaches, is that they deal with non-random barriers. To the best of our knowledge, there are no known exact formulae for single barrier options under the assumption that the barrier is itself a geometric Brownian motion, correlated with the process(es) driving the underlying asset(s). Yet, in real life, it is often practically impossible to set ex ante an optimal level for the barrier (s), especially one that is supposed to remain fixed or to evolve according to a deterministic pattern until expiry. In most of the first passage models used in the applied mathematical sciences, knock-out barriers stand for risk factors. As such, they inevitably involve some degree of randomness as a consequence of their interactions with a constantly changing environment, whether the latter may be driven by natural forces (in the case of physical and biological applications) or social forces (in the case of economic and financial applications). Symmetrically, knock-in barriers usually model growth opportunities, which are always subject to some dynamics if they cannot be kept isolated, as in the real world, in contrast to the artificially constrained conditions of a controlled experiment. The first passage of a state variable to a barrier is thus more effectively modelled as the point of intersection of two random processes, rather than as the time at which one process hits a predetermined value, whenever both the time and the space coordinates of the encounter cannot be known ex ante with certainty. In other words, first passage models that involve randomness both in time and space are more general and have a wider scope than first passage time models only; as such, they can provide a better fit to real data and describe actual phenomena more precisely than abstract models in which the location of the barrier (s) is assumed to be known (Redner et al. 2001; Metzler et al. 2014). For instance, it is notoriously spurious to assume that the market value of a firm's assets at which bankruptcy of the firm will occur in the future can be known exactly at the present time and that it will never be affected by non-predictable events; yet, this is the way default risk is modelled in existing structural models, in which the default barrier is assumed to be constant or deterministic in order to make computations easier. It makes more sense to allow a certain amount of unpredictability in the variations of the default barrier, which should be driven by a source of randomness distinct from the one affecting the value of the issuer's assets, in order to avoid the issues associated with defining default time as a predictable stopping time, as explained in Jeanblanc and Rutkowski (2000). Similarly, the performance of barrier-based structured products can be improved if the barrier levels are set in a way that adapts to future market conditions, e.g. in terms of a fraction of the future underlying asset price, rather than according to current market conditions. This, in turn, implies non-deterministic barriers, driven by a source of randomness more or less closely correlated with that of the underlying asset. Barriers themselves thus become stochastic processes designed to reset automatically according to market fluctuations. When markets rise, up-and-out barriers go up too if they are positively correlated with the underlying; when markets fall, down-and-out barriers go down too for the same reason. The increase in cost entailed by this automatic protection mechanism is offset by the savings made on the option premium as a consequence of the set of possible paths in which market conditions move favourably with regard to the knock-out risk, i.e. those in which markets fall in the case of an up-and-out barrier and those in which markets rise in the case of a down-and-out barrier.

To actually put in place these improvements in the way risk and return can be managed, new results on the valuation of barrier options are required, based on a model in which barriers themselves include randomness. In this Section, we proceed to show that all the main kinds of barrier options can be analytically valued by means of Proposition 1, under a much more general framework than the standard one assuming constant or time-dependent deterministic barriers, in which barriers are modelled as geometric Brownian motions just as the underlying asset. In financial terminology, the precise denomination for the general payoff tackled by Proposition 1 would be an outside, early-ending, two-step, knock-out option. This vocabulary can be explained as follows:

- “outside” refers to the fact that the process with respect to which boundary crossing is monitored is not the same as the process whose value is compared with the strike price at expiry
- “early-ending” refers to the fact that the barriers cease to be monitored before expiry
- “two-step” means that there are two different barriers in the time intervals $\left[0, t_1\right]$ and $\left[t_1, t_2\right]$
- “knock-out” indicates that the option expires worthless if the barriers are hit at any moment during the monitoring period

No valuation formula is currently known for such an option, even when the barriers are constant. When the latter may move randomly in time as geometric Brownian motions, as modelled in Sect. 2, the no-arbitrage value of such a contingent claim in a Black–Scholes setting is given by the following Proposition 2.

Proposition 2 *Let* $\mathrm{V(OEETS)}_{[.]}$ *denote the no-arbitrage value, in a Black–Scholes model, of an option with a strike price* K*, written on an asset* S_4*, with two successive stochastic barriers* S_1 *in* $\left[0, t_1\right]$ *and* S_2 *in* $\left[t_1, t_2\right]$ *monitored with respect to an asset* S_3*, that can take on one of the four following forms,* $\forall t_3 \geq t_2 \geq t_1 \geq t_0 = 0$:

(i) $\mathrm{V(OEETS)}_{[\mathrm{UOUOP}]} \triangleq$ *value of an outside early-ending up-and-up knock-out put, whose payoff formula at expiry* t_3 *is given by:*

$$\left(K - S_4\left(t_3\right)\right)\mathbb{1}_{\left\{S_3(t)<S_1(t), 0\leq t<t_1\right\}}\mathbb{1}_{\left\{S_3(t)<S_2(t), t_1\leq t<t_2\right\}}\mathbb{1}_{\left\{S_4\left(t_3\right)<K\right\}}, \quad S_1(0) > S_3(0) \tag{76}$$

(ii) $\mathrm{V(OEETS)}_{[\mathrm{DODOC}]} \triangleq$ *value of an outside early-ending down-and-down knock-out call, whose payoff formula at expiry* t_3 *is given by:*

$$\left(S_4\left(t_3\right) - K\right)\mathbb{1}_{\left\{S_3(t)>S_1(t), 0\leq t<t_1\right\}}\mathbb{1}_{\left\{S_3(t)>S_2(t), t_1\leq t<t_2\right\}}\mathbb{1}_{\left\{S_4\left(t_3\right)>K\right\}}, \quad S_1(0) < S_3(0) \tag{77}$$

(iii) $\mathrm{V(OEETS)}_{[\mathrm{DOUOP}]} \triangleq$ *value of an outside early-ending down-and-up knock-out put, whose payoff formula at expiry* t_3 *is given by:*

$$\left(K - S_4\left(t_3\right)\right)\mathbb{1}_{\left\{S_3(t)>S_1(t), 0\leq t<t_1\right\}}\mathbb{1}_{\left\{S_3(t)<S_2(t), t_1\leq t<t_2\right\}}\mathbb{1}_{\left\{S_4\left(t_3\right)<K\right\}}, \quad S_1(0) < S_3(0) \tag{78}$$

(iv) $\mathrm{V(OEETS)}_{[\mathrm{UODOC}]} \triangleq$ *value of an outside early-ending up-and-down knock-out call, whose payoff formula at expiry* t_3 *is given by:*

$$\left(S_4\left(t_3\right) - K\right)\mathbb{1}_{\left\{S_3(t)<S_1(t), 0\leq t<t_1\right\}}\mathbb{1}_{\left\{S_3(t)>S_2(t), t_1\leq t<t_2\right\}}\mathbb{1}_{\left\{S_4\left(t_3\right)>K\right\}}, \quad S_1(0) > S_3(0) \tag{79}$$

Then,

$$\mathrm{V(OEETS)}_{[\mathrm{UOUOP}]} \triangleq \exp\left(-rt_3\right)K \times \hat{\mathrm{P}}_{[\mathrm{UOUO}]} - S_4(0) \times \bar{\mathrm{P}}_{[\mathrm{UOUO}]} \tag{80}$$

$$\mathrm{V(OEETS)}_{[\mathrm{DODOC}]} \triangleq S_4(0) \times \bar{\mathrm{P}}_{[\mathrm{DODO}]} - \exp\left(-rt_3\right)K \times \hat{\mathrm{P}}_{[\mathrm{DODO}]} \tag{81}$$

$$\mathrm{V(OEETS)}_{[\mathrm{DOUOP}]} \triangleq \exp\left(-rt_3\right)K \times \hat{\mathrm{P}}_{[\mathrm{DOUO}]} - S_4(0) \times \bar{\mathrm{P}}_{[\mathrm{DOUO}]} \tag{82}$$

$$\mathrm{V(OEETS)}_{[\mathrm{UODOC}]} \triangleq S_4(0) \times \bar{\mathrm{P}}_{[\mathrm{UODO}]} - \exp\left(-rt_3\right)K \times \hat{\mathrm{P}}_{[\mathrm{UODO}]} \tag{83}$$

where

- *r is the riskless interest rate, assumed to be constant*
- $\hat{\mathrm{P}}_{[.]}$ *is identical to* $\mathrm{P}_{[.]}$ *as given by Proposition 1 except for the parameters* ν_1, ν_2 *and* ν_3, *which should be replaced by*

$$\nu_1 = -\frac{\sigma_3^2 - \sigma_1^2}{2}, \quad \nu_2 = -\frac{\sigma_3^2 - \sigma_2^2}{2}, \quad \nu_3 = r - \frac{\sigma_4^2}{2} \tag{84}$$

-
- $\bar{\mathrm{P}}_{[.]}$ *is identical to* $\mathrm{P}_{[.]}$ *as given by Proposition 1 except for the parameters* ν_1, ν_2 *and* ν_3, *which should be replaced by*

$$\nu_1 = -\frac{\sigma_3^2 - \sigma_1^2}{2} + \sigma_3\sigma_4\rho_{3.4} - \sigma_1\sigma_4\rho_{1.4}, \ \nu_2 = -\frac{\sigma_3^2 - \sigma_2^2}{2} + \sigma_3\sigma_4\rho_{3.4} - \sigma_2\sigma_4\rho_{2.4}, \ \nu_3 = r + \frac{\sigma_4^2}{2} \tag{85}$$

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Remarks The acronym OEETS stands for Outside Early-Ending Two-Step.

Proof of Proposition 2 Proposition 2 is an immediate application of Proposition 1 combined with a straightforward change of measure. Using standard no-arbitrage pricing arguments, the value of $\mathrm{V(OEETS)}_{[\mathrm{UOUOP}]}$ is given by:

$$\exp\left(-rt_3\right)\left\{K \times E_{\hat{P}}\left[\mathbb{1}_{\{S_3(t)<S_1(t),0\le t<t_1\}}\mathbb{1}_{\{S_3(t)<S_2(t),t_1\le t<t_2\}}\mathbb{1}_{\{S_4(t_3)<K\}}\right]\right. \tag{86}$$

$$\left.-E_{\hat{P}}\left[S_4\left(t_3\right)\mathbb{1}_{\{S_3(t)<S_1(t),0\le t<t_1\}}\mathbb{1}_{\{S_3(t)<S_2(t),t_1\le t<t_2\}}\mathbb{1}_{\{S_4(t_3)<K\}}\right]\right\} \tag{87}$$

where $\hat{P}$ is the unique equivalent martingale measure in the Black–Scholes model

The expectation in (86) is given by Proposition 1 with $\mu_1 = \mu_2 = \mu_3 = \mu_4 = r$. By applying Girsanov's theorem, the expectation in (87) can be turned into the following new expectation:

$$S_4(0)\exp\left(rt_3\right)E_{\bar{P}}\left[\mathbb{1}_{\{S_3(t)<S_1(t),0\le t<t_1\}}\mathbb{1}_{\{S_3(t)<S_2(t),t_1\le t<t_2\}}\mathbb{1}_{\{S_4(t_3)<K\}}\right] \tag{88}$$

where $\bar{P}$ is a probability measure equivalent to $\hat{P}$ under which:

$$W_1(t) = B_1(t) - \sigma_4\rho_{1.4}t \tag{89}$$

$$\tilde{W}_2(t) = \tilde{B}_2(t) - \sigma_4\rho_{2.4|1}\,t \tag{90}$$

$$\tilde{W}_3(t) = \tilde{B}_3(t) - \sigma_4\rho_{3.4|1.2}\,t \tag{91}$$

$$\tilde{W}_4(t) = \tilde{B}_4(t) - \sigma_4\sigma_{4|1.2.3}\,t \tag{92}$$

are standard independent Brownian motions

The expectation in (88) is therefore given by Proposition 1 using the parameters in (85).□

In terms of financial modeling, Proposition 2 allows to handle a fairly general formal structure in which the processes S_1 and S_2 may represent risk factors, the paths of which should not intersect with the path of a state variable S_3 for an optional insurance to be exercisable at time t_3. The process S_4 is another state variable that is not subject to a continuous risk such as S_3, although it is contingent on the path followed by S_3 between times t_0 and t_2 in order to come into play subsequently at time t_3. That formal framework is general enough to model a large number of real problems where default risk or real options are involved.

In terms of barrier options, it can be noticed that Proposition 1 nests all the probability distributions required to tackle the main payoffs traded in the markets. To begin with a particularly simple instance, it suffices to set $t_3 = t_2$ in Proposition 1 to tackle outside two-step knock-out options without the early-ending feature. This, in turn, allows to value standard outside early-ending barrier options, i.e. options with one outside early-ending barrier only instead of two. Focusing on the up-and-out put case, the value of $P\left(\left(S_3(t) < S_1(t), 0 \le t \le t_1\right) \cap \left(S_4\left(t_2\right) < K\right)\right)$ is theoretically obtained by plugging in $\mathrm{P}_{[\mathrm{UOUO}]}$ both $t_3 = t_2$ and a "very high" value of $S_2(0)$, i.e. such that the probability that the process S_3 crosses the process S_2 at any moment in $\left[t_1, t_2\right]$ goes to zero. Likewise, an outside forward start knock-out option can, in principle, be dealt with using Proposition 1. This is a knock-out option, the barrier of which begins to be monitored after the contract's inception. The value of $P\left(\left(S_3(t) < S_2(t), t_1 \le t \le t_2\right) \cap \left(S_4\left(t_2\right) < K\right)\right)$ is theoretically obtained by plugging in $\mathrm{P}_{[\mathrm{UOUO}]}$ both $t_3 = t_2$ and a "very high" value of $S_1(0)$, i.e. such that the probability that the process S_3 crosses the process S_1 at any moment in $\left[0, t_1\right]$ goes to zero.

However, taking limits in Proposition 1 may create numerical instability. It is therefore useful to calculate specific formulae for early-ending and forward-start instruments. Besides, the integration problem involved by the latter is of lower dimension, so that the resulting formulae should be easier to evaluate numerically and may thus provide a benchmark to assess the accuracy of the implementation of Proposition 1. The following Proposition 3 provides a formula for the probability distribution involved in the valuation of an outside early-ending knock-out option.

Proposition 3 *Let the processes S_1, S_2 and S_3 be defined as in Proposition 1.*

Let $\mathrm{P(OEE)}_{[.]}$ *be defined as one of the two following joint cumulative distribution functions:*

$$\mathrm{P(OEE)}_{[\mathrm{UOP}]} \triangleq P\left(\left(S_2(t) < S_1(t), 0 \le t \le t_1\right) \cap \left(S_3\left(t_2\right) < K\right)\right) \tag{93}$$

$$\mathrm{P(OEE)}_{[\mathrm{DOC}]} \triangleq P\left(\left(S_2(t) > S_1(t), 0 \le t \le t_1\right) \cap \left(S_3\left(t_2\right) > K\right)\right) \tag{94}$$

where $K > 0$ *and* $t_2 \ge t_1 \ge t_0 = 0$

Then, we have:

$$\begin{aligned} \mathrm{P(OEE)}_{[.]} = {} & N_2\left[\lambda\left(\frac{\alpha - v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{k - v_2 t_2}{\sigma_3\sqrt{t_2}}\right); \sqrt{\frac{t_1}{t_2}}\theta\right] \\ & - \exp\left(\frac{2 v_1 \alpha}{\beta^2}\right) N_2\left[\lambda\left(\frac{-\alpha - v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{k - v_2 t_2}{\sigma_3\sqrt{t_2}} - \frac{2\theta\alpha}{\beta\sqrt{t_2}}\right); \sqrt{\frac{t_1}{t_2}}\theta\right] \end{aligned} \tag{95}$$

where $N_2\left[b_1, b_2; \xi\right]$ *is the bivariate standard normal cumulative distribution function with upper bounds* b_1 *and* b_2, *and correlation coefficient* ξ, *and:*

$$\lambda = 1 \text{ if } \mathrm{P(OEE)}_{[.]} = \mathrm{P(OEE)}_{[\mathrm{UOP}]}, \lambda = -1 \text{ if } \mathrm{P(OEE)}_{[.]} = \mathrm{P(OEE)}_{[\mathrm{DOC}]} \tag{96}$$

$$\nu_1 = \mu_2 - \mu_1 - \frac{\sigma_2^2 - \sigma_1^2}{2}, \nu_2 = \mu_3 - \frac{\sigma_3^2}{2}, \alpha = \ln\left(\frac{S_1(0)}{S_2(0)}\right), \beta = \sqrt{\sigma_2^2 - 2\sigma_1\sigma_2\rho_{1.2} + \sigma_1^2} \tag{97}$$

$$k = \ln\left(\frac{K}{S_3(0)}\right), \theta = \frac{\left(\sigma_2\rho_{1.2} - \sigma_1\right)\rho_{1.3} + \sigma_2\sigma_{2|1}\,\rho_{2.3|1}}{\beta} \tag{98}$$

Remark 1 The acronym OEE stands for Outside Early-Ending

Remark 2 In terms of option valuation, the process S_1 in Proposition 3 plays the part of the stochastic knock-out barrier, while S_2 is the process subject to a boundary crossing condition and S_3 is the process defining the moneyness of the option at expiry.

Sketch of proof of Proposition 3 Following steps similar to those in the proof of Proposition 1, $\mathrm{P(OEE)}_{[\mathrm{UOP}]}$ can be expressed as the solution of the following integration problem:

$$\mathrm{P(OEE)}_{[\mathrm{UOP}]} = \int_{x_1=-\infty}^{\alpha} \int_{x_2=-\infty}^{k} P\left(\ln\left(\frac{S_2\left(t_1\right)S_1(0)}{S_1\left(t_1\right)S_2(0)}\right) \in dx_1, \sup_{0\le t\le t_1} \ln\left(\frac{S_2(t)S_1(0)}{S_1(t)S_2(0)}\right) < \alpha\right)$$
$$P\left(\ln\left(\frac{S_3\left(t_2\right)}{S_3(0)}\right) \in dx_2 \middle| \ln\left(\frac{S_2\left(t_1\right)S_1(0)}{S_1\left(t_1\right)S_2(0)}\right) \in dx_1\right) dx_2 dx_1 \tag{99}$$

Expansion of this double integral can be carried out similarly as in the proof of Proposition 1.

$\mathrm{P(OEE)}_{[\mathrm{DOC}]}$ is obtained in the same manner. □

The following Proposition 4 provides a formula for the probability distribution involved in the valuation of an outside forward-start knock-out option.

Proposition 4 *Let the processes S_1, S_2 and S_3 be defined as in Proposition 1.*

Let $\mathrm{P(OFS)}_{[.]}$ be defined as one of the two following joint cumulative distribution functions:

$$\mathrm{P(OFS)}_{[\mathrm{UOP}]} \triangleq P\left(\left(S_2(t) < S_1(t), t_1 \le t \le t_2\right) \cap \left(S_3\left(t_2\right) < K\right)\right) \tag{100}$$

$$\mathrm{P(OFS)}_{[\mathrm{DOC}]} \triangleq P\left(\left(S_2(t) > S_1(t), t_1 \le t \le t_2\right) \cap \left(S_3\left(t_2\right) > K\right)\right) \tag{101}$$

where $K > 0$ and $t_2 \ge t_1 \ge t_0 = 0$

Let Ψ_3 be the function defined by:

$$\Psi_3\left[b_1, b_2, b_3; \theta_1, \theta_2\right] = \int_{-\infty}^{b_2} \frac{\exp\left(-\frac{x_2^2}{2}\right)}{\sqrt{2\pi}} N\left[\frac{b_1 - \theta_1 x_2}{\sqrt{1-\theta_1^2}}\right] N\left[\frac{b_3 - \theta_2 x_2}{\sqrt{1-\theta_2^2}}\right] dx_2 \tag{102}$$

where $b_i \in \mathbb{R}$, $\theta_i \in [-1, 1]$, $i \in \mathbb{N}$, and $N[.]$ is the univariate standard normal cumulative distribution function

Then, we have:

$$\mathrm{P(OFS)}_{[.]} = \Psi_3\left[\lambda\left(\frac{\alpha - v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_2}{\sigma_3\sqrt{t_2}}\right); \sqrt{\frac{t_1}{t_2}}, \theta\right]$$
$$- \exp\left(\frac{2v_1\alpha}{\beta^2}\right)\Psi_3\left[\lambda\left(\frac{\alpha + v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{-\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_2}{\sigma_3\sqrt{t_2}} - \frac{2\theta\alpha}{\beta\sqrt{t_2}}\right); -\sqrt{\frac{t_1}{t_2}}, \theta\right] \tag{103}$$

where

$$\lambda = 1 \text{ if } \mathrm{P(OFS)}_{[.]} = \mathrm{P(OFS)}_{[\mathrm{UOP}]}, \lambda = -1 \text{ if } \mathrm{P(OFS)}_{[.]} = \mathrm{P(OFS)}_{[\mathrm{DOC}]} \tag{104}$$

$$v_1 = \mu_2 - \mu_1 - \frac{\sigma_2^2 - \sigma_1^2}{2}, v_2 = \mu_3 - \frac{\sigma_3^2}{2}, \alpha = \ln\left(\frac{S_1(0)}{S_2(0)}\right), \beta = \sqrt{\sigma_2^2 - 2\sigma_1\sigma_2\rho_{1.2} + \sigma_1^2} \tag{105}$$

$$k = \ln\left(\frac{K}{S_3(0)}\right), \theta = \frac{\left(\sigma_2\rho_{1.2} - \sigma_1\right)\rho_{1.3} + \sigma_2\sigma_{2|1}\,\rho_{2.3|1}}{\beta} \tag{106}$$

Equivalently, we have:

$$\mathrm{P(OFS)}_{[.]} = N_3\left[\lambda\left(\frac{\alpha - v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_2}{\sigma_3\sqrt{t_2}}\right); \sqrt{\frac{t_1}{t_2}}, \theta\sqrt{\frac{t_1}{t_2}}, \theta\right]$$
$$- \exp\left(\frac{2v_1\alpha}{\beta^2}\right)N_3\left[\lambda\left(\frac{\alpha + v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{-\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_2}{\sigma_3\sqrt{t_2}} - \frac{2\theta\alpha}{\beta\sqrt{t_2}}\right); -\sqrt{\frac{t_1}{t_2}}, -\theta\sqrt{\frac{t_1}{t_2}}, \theta\right] \tag{107}$$

where $N_3\left[b_1, b_2, b_3; \xi_{1.2}, \xi_{1.3}, \xi_{2.3}\right]$ is the trivariate standard normal cumulative distribution function; denoting by Z_1, Z_2, Z_3 the variables with upper bounds b_1, b_2 and b_3, respectively, each $\xi_{i.j}$ is the correlation coefficient between Z_i and Z_j.

Remark The acronym OFS stands for Outside Forward-Start.

Sketch of proof of Proposition 4 Let us consider the process X_1 defined as $X_1(t) \triangleq \left\{\ln\left(\frac{S_2(t)S_1(0)}{S_1(t)S_2(0)}\right), t \geq 0\right\}$ and let us define X_2 as $X_2(t) \triangleq \left\{\ln\left(\frac{S_3(t)}{S_3(0)}\right), t \geq 0\right\}$. Then, $\mathrm{P(OFS)}_{[\mathrm{UOP}]}$ can be expressed as the solution of the following integration problem:

$$\mathrm{P(OFS)}_{[\mathrm{UOP}]} = \int_{x_1=-\infty}^{\alpha}\int_{x_2=-\infty}^{\alpha}\int_{x_3=-\infty}^{k} P\left(X_1\left(t_1\right) \in dx_1\right)$$
$$P\left(X_1\left(t_2\right) \in dx_2, \sup_{t_1 \leq t \leq t_2} X_1(t) < \alpha \middle| X_1\left(t_1\right) \in dx_1\right) P\left(X_2\left(t_2\right) \in dx_3 \middle| X_1\left(t_2\right) \in dx_2\right) dx_3 dx_2 dx_1 \tag{108}$$

Expanding the functions inside this triple integral along the lines of the proof of Proposition 1 and performing the necessary calculations, one can obtain $\mathrm{P(OFS)}_{[\mathrm{UOP}]}$ as given by (103).

Alternatively, $\mathrm{P(OFS)}_{[\mathrm{UOP}]}$ can also be expressed as:

$$\mathrm{P(OFS)}_{[\mathrm{UOP}]} = \int_{x_1=-\infty}^{\alpha}\int_{x_2=-\infty}^{\alpha}\int_{x_3=-\infty}^{k} P\left(X_1\left(t_1\right) \in dx_1, X_1\left(t_2\right) \in dx_2, X_2\left(t_2\right) \in dx_3\right)$$
$$P\left(\sup_{t_1 \le t \le t_2} X_1(t) < \alpha \middle| X_1\left(t_1\right) \in dx_1, X_1\left(t_2\right) \in dx_2\right) dx_3 dx_2 dx_1 \tag{109}$$

Expanding this triple integral and solving it yields $\mathrm{P(OFS)}_{[\mathrm{UOP}]}$ as given by (107).□

Early-ending and forward-start barrier options are a subset of a larger class of instruments known as window barrier options, characterized by the fact that the monitoring of the barrier both starts and terminates before expiry, i.e. the barrier is monitored during the time interval $\left[t_1, t_2\right]$ and the expiry is t_3. A formula for the underlying probability distribution is provided in the following Proposition 5.

Proposition 5 *Let the processes S_1, S_2 and S_3 be defined as in Proposition 1.*

Let $\mathrm{P(OW)}_{[.]}$ be defined as one of the two following joint cumulative distribution functions:

$$\begin{aligned} \mathrm{P(OW)}_{[\mathrm{UOP}]} &\triangleq P\left(\left(S_2(t) < S_1(t), t_1 \le t \le t_2\right) \cap \left(S_3\left(t_3\right) < K\right)\right) \\ \mathrm{P(OW)}_{[\mathrm{DOC}]} &\triangleq P\left(\left(S_2(t) > S_1(t), t_1 \le t \le t_2\right) \cap \left(S_3\left(t_3\right) > K\right)\right) \end{aligned} \tag{110}$$

where $K > 0$ and $t_3 \ge t_2 \ge t_1 \ge t_0 = 0$

Then, we have:

$$\begin{aligned} \mathrm{P(OW)}_{[.]} =& \Psi_3\left[\lambda\left(\frac{\alpha - v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_3}{\sigma_3\sqrt{t_3}}\right); \sqrt{\frac{t_1}{t_2}}, \theta\sqrt{\frac{t_2}{t_3}}\right] \\ & - \exp\left(\frac{2v_1\alpha}{\beta^2}\right)\Psi_3\left[\lambda\left(\frac{\alpha + v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{-\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_3}{\sigma_3\sqrt{t_3}} - \frac{2\theta\alpha}{\beta\sqrt{t_3}}\right); -\sqrt{\frac{t_1}{t_2}}, \theta\sqrt{\frac{t_2}{t_3}}\right] \end{aligned} \tag{111}$$

Equivalently,

$$\begin{aligned} \mathrm{P(OW)}_{[.]} =& N_3\left[\lambda\left(\frac{\alpha - v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_3}{\sigma_3\sqrt{t_3}}\right); \sqrt{\frac{t_1}{t_2}}, \theta\sqrt{\frac{t_1}{t_3}}, \theta\sqrt{\frac{t_2}{t_3}}\right] \\ & - \exp\left(\frac{2v_1\alpha}{\beta^2}\right) N_3\left[\lambda\left(\frac{\alpha + v_1 t_1}{\beta\sqrt{t_1}}\right), \lambda\left(\frac{-\alpha - v_1 t_2}{\beta\sqrt{t_2}}\right), \lambda\left(\frac{k - v_2 t_3}{\sigma_3\sqrt{t_3}} - \frac{2\theta\alpha}{\beta\sqrt{t_3}}\right); \right. \\ & \left. -\sqrt{\frac{t_1}{t_2}}, -\theta\sqrt{\frac{t_1}{t_3}}, \theta\sqrt{\frac{t_2}{t_3}}\right] \end{aligned} \tag{112}$$

where the same definitions hold as in Proposition 4.

Remark 1 The acronym OW stands for Outside Window.

Remark 2 For the sake of generality, all the payoffs thus far considered have been assumed to be of "outside" type.

In order to value their non-outside versions, all that needs to be done is to set $S_3(0) = S_2(0)$, $\mu_3 \to \mu_2 \sigma_3 \to \sigma_2$ and $\rho_{2,3} \to 1$ in the formulae for outside payoffs.

Sketch of proof of Proposition 5 Replace $X_2\left(t_2\right)$ by $X_2\left(t_3\right)$ in the outline of proof of Proposition 4.

4 Numerical results

The simplest implementation of Proposition 1 consists in selecting an appropriate cutoff value for the negative infinity lower bounds and then applying a fixed-degree quadrature rule. Given the smoothness of the rapidly decaying exponential functions in the integrands, even a low-degree rule can be expected to perform well. The nature of the integrands makes them good candidates for a Gauss–Legendre rule. A modified Gauss-Hermite rule can also be applied after an elementary transformation of the integrals, but it proved to be slightly less accurate in our testing so it was discarded. Another, more sophisticated, form of implementation of Proposition 1 consists in replacing the fixed-degree quadrature rule by a subregion adaptive algorithm, as explained by Berntsen et al. (1991). This second approach adapts the number of integrand evaluations in each subregion according to the rate of change of the integrand, thus concentrating the computational effort where it is most needed. The subdivision of the integration domain stops when the sum of the local error deterministic estimates becomes smaller than some prespecified requested accuracy. Adaptive integration is more accurate than fixed-degree rules but it can also be more time-consuming.

In Table 1, three different implementations of Proposition 1 are carried out:

- A fixed-degree 16-point Gauss–Legendre quadrature, denoted by Imp1.
- A fixed-degree 96-point Gauss–Legendre quadrature, denoted by Imp2.
- The Cuhre adaptive integration algorithm as implemented by Hahn (2005), based on Berntsen et al. (1991), denoted by Imp3, with a level of requested accuracy equal to 10^{-6}

The numerical results reported in Table 1 come from a sample of 1000 randomly drawn sets of parameters, with a check for non-singularity of each matrix of covariances. The benchmark for the reported average and maximum absolute errors of Imp1 and Imp2 is

Table 1 Comparison of the relative levels of accuracy and efficiency for three different numerical implementations of Proposition 1 using 1000 sets of randomly drawn parameters

	Imp1	Imp2	Imp3 10^{-6} requested accuracy
Average absolute error	3.10^{-5}	4.10^{-6}	
Maximum absolute error	6.10^{-4}	8.10^{-5}	
Proportion of errors > 9.10^{-5}	22.6%	0	
Computational time	< 0.01	4.08	[2.7;8.3]

Imp3. The reported computational time for the Imp3 method is a range with observed minimum and maximum, as the number of iterations necessary to achieve a requested level of accuracy depends on the integration problem under consideration. Computational time is measured in seconds using a computer equipped with an Intel Core i-9 8950HK processor.

Since its maximum absolute error recorded was only 6.10^{-4}, and given its unmatched speed of execution, the Imp1 method can be considered a good choice for most practical purposes. In particular, it is well suited to financial applications. Although slightly more precise than Imp1, the Imp2 method does not provide any clear advantage as it is much slower than Imp1 while it does not guarantee ex ante a given degree of precision like Imp3. The latter will be chosen if high accuracy is necessary but it is much less efficient than Imp1.

In Table 2, Proposition 1 is implemented with the following list of parameters denoted as List 1:

$$S_3(0) = S_4(0) = K = 100$$
$$S_1(0) = 110\left(\mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UOUOP}]}\}} + \mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UODOC}]}\}}\right) + 90\left(\mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DODOC}]}\}} + \mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DOUOP}]}\}}\right)$$
$$S_2(0) = 115\left(\mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UOUOP}]}\}} + \mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DOUOP}]}\}}\right) + 85\left(\mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{DODOC}]}\}} + \mathbb{1}_{\{\mathrm{P}_{[.]}=\mathrm{P}_{[\mathrm{UODOC}]}\}}\right)$$
$$\mu_1 = -0.02, \mu_2 = 0.035, \mu_3 = 0.03, \mu_4 = 0.025$$
$$\sigma_1 = 0.46, \sigma_2 = 0.42, \sigma_3 = 0.28, \sigma_4 = 0.36$$
$$\rho_{1.2} = 0.34, \rho_{1.3} = -0.28, \rho_{1.4} = 0.18, \rho_{2.3} = -0.24, \rho_{3.4} = -0.32$$
$$t_1 = 0.2, t_2 = 0.5, t_3 = 1$$

Table 3 reports numerical values obtained by Monte Carlo simulation using the Mersenne Twister pseudo-random number generator. The inputs are the same as in Table 2. Two kinds of simulation techniques are applied. The first one is standard Monte Carlo, denoted as SMC. A total of 500,000 process paths are drawn. Two discretization steps per unit time are selected: 1/8 and 1/32. The results are very poor, both in terms of accuracy and efficiency. The second approach, called Conditional Monte Carlo, denoted as CMC, is much more effective. It consists in approximating the integrals in (55)–(58) by stochastic simulation. This involves drawing at most the random variables $S_1(t_1)$, $S_2(t_1)$, $S_3(t_1)$, $S_4(t_1)$, $S_2(t_2)$, $S_3(t_2)$ and $S_4(t_3)$ at each run, which dramatically reduces the computational burden compared with a standard Monte Carlo procedure. Moreover, no discretization bias is entailed with regard to the boundary crossing probabilities within the time intervals $[t_0, t_1]$ and $[t_1, t_2]$. A detailed algorithm is provided at the end of Sect. 4. Convergence with the values obtained by implementation of Proposition 1 is achieved to at least 3 digits using 10,000,000 simulations, and the error on the fourth digit clearly diminishes as the number

Table 2 Numerical implementation of Proposition 1 using List 1 of parameters

	Proposition 1 Imp 1	Proposition 1 Imp 2	Proposition 1 Imp 3
$\mathrm{P}_{[\mathrm{UOUOP}]}$	0.062622667	0.062542248	0.062596144
$\mathrm{P}_{[\mathrm{DODOC}]}$	0.072330579	0.072241677	0.072200632
$\mathrm{P}_{[\mathrm{DOUOP}]}$	0, 030514506	0.030475775	0.030414768
$\mathrm{P}_{[\mathrm{UODOC}]}$	0.023330478	0.023298896	0.023291025
Computational time	< 0.01	4.08	[3.8;4.3]

Table 3 Numerical approximation of Proposition 1 through Monte Carlo simulation using List 1 of parameters

	CMC 100,000,000	CMC 10,000,000	SMC 500,000 1/8	SMC 500,000 1/32
$P_{[UOUOP]}$	0.062431845	0.062389341	0.06945173	0.0667192
$P_{[DODOC]}$	0.072163167	0.072102457	0.07712166	0.0758349
$P_{[DOUOP]}$	0.030533894	0.030570406	0.03554157	0.03428615
$P_{[UODOC]}$	0.023367425	0.023408512	0.02873846	0.02581643
Computational time	164.77	17.12	187.82	745.36

of simulations is multiplied by ten. However, convergence is very slow. Moreover, it relies on the quality of the underlying random number generator.

Proposition 2 is not implemented, as its numerical evaluation is identical to that of Proposition 1.

Proposition 3 is not implemented either, because bivariate normal distribution functions are easy to evaluate numerically.

In Table 4, Proposition 4 is implemented with the following list of parameters denoted as List 2:

$$
\begin{aligned}
&S_2(0) = S_3(0) = K = 100\\
&S_1(0) = 105\ \mathbb{1}_{\{P(OFS)_{[.]}=P(OFS)_{[UOP]}\}} + 95\ \mathbb{1}_{\{P(OFS)_{[.]}=P(OFS)_{[DOC]}\}}\\
&\mu_1 = -0.02, \mu_2 = 0.035, \mu_3 = 0.03\\
&\sigma_1 = 0.46, \sigma_2 = 0.42, \sigma_3 = 0.28\\
&\rho_{1.2} = -0.34, \rho_{1.3} = 0.28, \rho_{2.3} = -0.24\\
&t_1 = 0.2, t_2 = 0.5
\end{aligned}
$$

Two different implementations of Proposition 4 are reported in Table 4. The first one uses Eq. (103), with the Ψ_3 function defined in (102) and numerically evaluated through a fixed-degree 16-point Gauss–Legendre quadrature. The second one uses Eq. (107), with the N_3 function evaluated by means of the Genz algorithm (Genz 2004). One can notice convergence to at least 9 digits between the two methods. Conditional Monte Carlo approximations are also reported. More comprehensive tests are not performed, since the computation of the N_3 function is a known subject; the reader is referred to Genz (2004) for more detail.

In Table 5, Proposition 4 is implemented with the following list of parameters denoted as List 3:

Table 4 Numerical evaluation of Proposition 4 using List 2 of parameters vs Conditional Monte Carlo approximation

	Proposition 4 Psi Function	Proposition 4 N3 Function	CMC 100,000,000	CMC 10,000,000
$P(OFS)_{[UOP]}$	0.087024643	0.087024643	0.087137184	0.087141651
$P(OFS)_{[DOC]}$	0.100471148	0.100471148	0.10092942	0.10151073
Computational time	< 0.01	< 0.2	123.26	12.84

$$S_2(0) = S_3(0) = K = 100$$
$$S_1(0) = 105\ \mathbb{1}_{\{P(OFS)_{[.]}=P(OFS)_{[UOP]}\}} + 95\ \mathbb{1}_{\{P(OFS)_{[.]}=P(OFS)_{[DOC]}\}}$$
$$\mu_1 = 0, \mu_2 = 0.035, \mu_3 = 0.03$$
$$\sigma_1 = 0.0001, \sigma_2 = 0.42, \sigma_3 = 0.28$$
$$\rho_{1.2} = 0, \rho_{1.3} = 0, \rho_{2.3} = -0.24$$
$$t_1 = 0.2, t_2 = 0.5$$

The purpose of Table 5 is to give a quick illustration of the observed gap in the probabilities $P(OFS)_{[UOP]}$ and $P(OFS)_{[DOC]}$ when the boundary is random and when it is a constant equal to $S_1(0)$ Notice that $\sigma_1 = 0$ cannot be substituted in Proposition 4, so that the limit is taken as σ_1 approaches zero. The probability for the process S_2 to remain below the boundary is quite higher when the boundary is constant. Several factors concur to produce this outcome. First, the standard deviation parameter σ_1 of the random boundary S_1 is rather large, meaning that the sensitivity of S_1 to the risk factor represented by Brownian motion B_1 is quite high. If σ_1 is divided by two and all other parameters in List 2 remain unchanged, then $P(OFS)_{[UOP]}$ increases from 0.087024643 to 0.104353727. Moreover, the drift coefficient μ_1 of S_1 is negative in Table 4, entailing that S_1 has a tendency to decrease and thus to get nearer to S_2, even if this effect is played down by the fact that the absolute value of μ_1 is quite small. If μ_1 is equal to 0.02 instead of −0.02, then $P(OFS)_{[UOP]}$ becomes equal to 0.091669378, which is a small but non-negligible increase. Since σ_1 is quite high, raising the value of μ_1 will only slowly increase the probability of remaining below the boundary, as the deterministic growth factor represented by μ_1 is offset by the random variations in B_1, which have a probability of one half of being negative. Thus, increasing μ_1 to as much as 0.25 results in $P(OFS)_{[UOP]}$ "merely" changing to 0.121074998, which is still less than the value of $P(OFS)_{[UOP]}$ when the boundary is kept at the constant level $S_1(0)$ Lastly, one can observe that the effect of the negative correlation between S_1 and S_2 on $P(OFS)_{[UOP]}$ is ambiguous: when S_2 increases, the upper boundary S_1 tends to decrease, thus magnifying the risk of hitting the boundary, but the reverse effect holds when S_2 decreases. Whichever effect prevails depends on the sign of μ_2 and on the absolute value of the ratio μ_2/σ_2. Roughly speaking, the higher the value of μ_2, the more negative correlation between S_1 and S_2 will compound, on average, the risk of hitting an upper boundary.

Turning to $P(OFS)_{[DOC]}$, Table 5 shows that the effect of maintaining the boundary at the constant level $S_1(0)$ on the probability that the processes S_1 and S_2 will not cross, is much less pronounced than in the up-and-out case, although the distance between $S_1(0)$ and $S_2(0)$ is the same whether $S_1(0) < S_2(0)$ or $S_1(0) > S_2(0)$. This has to do with the fact that μ_1 is negative in Table 4, but only partly. Indeed, taking $\mu_1 = 0.02$ results in $P(OFS)_{[DOC]}$ going from 0.100471148 to 0.095640137, so that the gap between $P(OFS)_{[DOC]}$ in Table 4 and $P(OFS)_{[DOC]}$ in Table 5 is still significantly smaller than the gap between $P(OFS)_{[UOP]}$

Table 5 Numerical evaluation of Proposition 4 using List 3 of parameters vs Conditional Monte Carlo approximation

	Proposition 4 Psi Function	Proposition 4 N3 Function	CMC 100,000,000
$P(OFS)_{[UOP]}$	0.131071868	0.131071868	0.131117638
$P(OFS)_{[DOC]}$	0.103169541	0.103169541	0.103202476

in Table 4 and $\mathrm{P(OFS)}_{[\mathrm{UOP}]}$ in Table 5. This suggests that other dynamics also interact here. In particular, one should investigate the consequences of S_2 being a submartingale, as well the consequences of the boundary starting forward in time, in order to account for the non-symmetric probability of hitting the boundary from below to from above. It must be emphasized that many different factors come into play depending on the set of parameters used, resulting in a complex overall outcome. This is all the more true when it comes to the more general Proposition 1. That is why it is useful to have exact formulae for the probabilities under consideration, as they allow to conduct accurate risk analysis by mere differentiation of the formulae with respect to the relevant parameters.

Finally, by taking $\mu_1 \neq 0$ and keeping all other inputs as in Table 5, one can compute the probability of not hitting a boundary defined as a deterministic exponential function of time. If we take $\mu_1 = -0.02$ in order to make meaningful comparisons with Table 4 and Table 5, then the up-and-out boundary begins at time $t_1 = 0.2$ at a level approximately equal to 104.58 and terminates at time $t_1 = 0.5$ at a level approximately equal to 103.95, while it begins at 94.62 and terminates at 94.05 when it is defined as down-and-out. Its precise location at any moment t is given by $S_1(0)\exp(-0.02t)$. Obviously, the value of $\mathrm{P(OFS)}_{[\mathrm{UOP}]}$ should be lower than in Table 5, while the value of $\mathrm{P(OFS)}_{[\mathrm{DOC}]}$ should be higher than in Table 5. What is less obvious is where the values should stand relative to Table 4. The implementation of Proposition 4 yields $\mathrm{P(OFS)}_{[\mathrm{UOP}]} = 0.126218936$ and $\mathrm{P(OFS)}_{[\mathrm{DOC}]} = 0.107459695$ This illustrates clearly, at least in the up-and-out case, how significant the impact of introducing randomness in S_1 is.

Before closing this Section by the Conditional Monte Carlo algorithm used in Tables 3, 4 and 5, let us briefly mention that Proposition 5 is not implemented because its numerical evaluation is identical to that of Proposition 4.

ALGORITHM

****

The purpose of this algorithm is to compute a Conditional Monte Carlo approximation of Proposition 1. This algorithm does not provide coding details such as the declaration of variables according to their types or the precomputation of constants iteratively used in the central loop

****

INPUTS = $\{S_1(0), S_2(0), S_3(0), S_4(0), K, \mu_1, \mu_2, \mu_3, \mu_4, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \rho_{1.2}, \rho_{1.3}, \rho_{1.4}, \rho_{2.3}, \rho_{3.4},$
$t_1, t_2, t_3, \text{type}, \text{nsim}\}$

REMARK 1 : *the variable* nsim, *which is an integer, is the required number of simulations*

REMARK 2 : *the variable* type, *which is a string of characters, defines the required probability among the four cumulative distribution functions covered by Proposition 1, with "UOUO" returning* $\mathrm{P}_{[\mathrm{UOUO}]}$, *"DODO" returning* $\mathrm{P}_{[\mathrm{DODO}]}$, *"DOUO" returning* $\mathrm{P}_{[\mathrm{DOUO}]}$ *and "UODO" returning* $\mathrm{P}_{[\mathrm{UODO}]}$

REMARK 3 : *all subsequent random numbers from a standard normal distribution* $\mathcal{N}(0,1)$ *are independently drawn*

ASSIGN $\upsilon_1 \leftarrow \mu_1 - \sigma_1^2/2; \upsilon_2 \leftarrow \mu_2 - \sigma_2^2/2; \upsilon_3 \leftarrow \mu_3 - \sigma_3^2/2; \upsilon_4 \leftarrow \mu_4 - \sigma_4^2/2$

ASSIGN cumvar $\leftarrow 0$

FOR c = 1 TO nsim

ASSIGN $\varepsilon_1 \leftarrow \mathcal{N}(0,1); S_1(t_1) \leftarrow S_1(0)\exp\left(\upsilon_1 t_1 + \sigma_1\sqrt{t_1}\varepsilon_1\right)$

ASSIGN $\varepsilon_2 \leftarrow \mathcal{N}(0,1); \bar{\varepsilon}_2 \leftarrow \rho_{1.2}\varepsilon_1 + \sigma_{2|1}\varepsilon_2; S_2(t_1) \leftarrow S_2(0)\exp\left(\upsilon_2 t_1 + \sigma_2\sqrt{t_1}\bar{\varepsilon}_2\right)$

ASSIGN $\varepsilon_3 \leftarrow \mathcal{N}(0,1); \bar{\varepsilon}_3 \leftarrow \rho_{1.3}\varepsilon_1 + \rho_{2.3|1}\varepsilon_2 + \sigma_{3|1.2}\varepsilon_3; S_3(t_1) \leftarrow S_3(0)\exp\left(\upsilon_3 t_1 + \sigma_3\sqrt{t_1}\bar{\varepsilon}_3\right)$

IF type = 'UOUO' THEN

  IF $S_3(t_1) > S_1(t_1)$ OR $S_3(t_1) > S_2(t_1)$ THEN GOTO Next_Sim

END IF

IF type = 'UODO' THEN

  IF $S_3(t_1) > S_1(t_1)$ OR $S_3(t_1) < S_2(t_1)$ THEN GOTO Next_Sim

END IF

IF type = 'DODO' THEN

  IF $S_3(t_1) < S_1(t_1)$ OR $S_3(t_1) < S_2(t_1)$ THEN GOTO Next_Sim

END IF

IF type = 'DOUO' THEN

  IF $S_3(t_1) < S_1(t_1)$ OR $S_3(t_1) > S_2(t_1)$ THEN GOTO Next_Sim

END IF

ASSIGN $\varepsilon_4 \leftarrow \mathcal{N}(0,1); \bar{\varepsilon}_4 \leftarrow \rho_{1.4}\varepsilon_1 + \rho_{2.4|1}\varepsilon_2 + \rho_{3.4|1.2}\varepsilon_3 + \sigma_{4|1.2.3}\varepsilon_4; S_4(t_1) \leftarrow S_4(0)\exp\left(\upsilon_4 t_1 + \sigma_4\sqrt{t_1}\bar{\varepsilon}_4\right)$

$\varepsilon_5 \leftarrow \mathcal{N}(0,1); S_2(t_2) \leftarrow S_2(t_1)\exp\left(\upsilon_2(t_2 - t_1) + \sigma_1\sqrt{t_2 - t_1}\varepsilon_5\right)$

$\varepsilon_6 \leftarrow \mathcal{N}(0,1); \bar{\varepsilon}_6 \leftarrow \rho_{2.3}\varepsilon_5 + \sigma_{3|2}\varepsilon_6; S_3(t_2) \leftarrow S_3(t_1)\exp\left(\upsilon_3(t_2 - t_1) + \sigma_3\sqrt{t_2 - t_1}\bar{\varepsilon}_6\right)$

IF type = 'UOUO' OR type = 'DOUO' THEN

  IF $S_3(t_2) > S_2(t_2)$ THEN GOTO Next_Sim

END IF

IF type = 'DODO' OR type = 'UODO' THEN

  IF $S_3(t_2) < S_2(t_2)$ THEN GOTO Next_Sim

END IF

ASSIGN $\varepsilon_7 \leftarrow \mathcal{N}(0,1); \bar{\varepsilon}_7 \leftarrow \rho_{2.4}\varepsilon_5 + \rho_{3.4|2}\varepsilon_6 + \sigma_{4|2.3}\varepsilon_7; S_4(t_3) \leftarrow S_4(t_1)\exp\left(\upsilon_4(t_3 - t_1) + \sigma_4\sqrt{t_3 - t_1}\bar{\varepsilon}_7\right)$

IF $S_4(t_3) > K$ THEN

  IF type = 'UOUO' OR type = 'DOUO' THEN GOTO Next_Sim

END IF

ASSIGN $l_1 \leftarrow \ln\left(\dfrac{S_3(t_1)S_1(0)}{S_3(0)S_1(t_1)}\right); l_2 \leftarrow \ln\left(\dfrac{S_3(t_1)S_2(0)}{S_3(0)S_2(t_1)}\right); l_3 \leftarrow \ln\left(\dfrac{S_3(t_2)S_2(0)}{S_3(0)S_2(t_2)}\right)$

ASSIGN cumvar $\leftarrow$ cumvar $+\left(1 - \exp\left(\dfrac{2\alpha_1(l_1 - \alpha_1)}{\beta_1^2 t_1}\right)\right)\left(1 - \exp\left(\dfrac{2(\alpha_2 - l_2)(l_3 - \alpha_2)}{\beta_2^2(t_2 - t_1)}\right)\right)$

Next_Sim :

NEXT c

OUTPUT = cumvar / nsim

5 Conclusion

In this paper, it was shown how to compute the exact values of all the main kinds of single barrier options accurately and efficiently in a framework where barriers themselves are diffusive random processes correlated with one another and with the underlying asset(s). This is a much richer setting than the one dealt with so far in the literature, i.e. featuring barriers that are constant or that can change only in a deterministic manner. Moreover, analytical tractability is preserved through a multidimensional geometric Brownian motion model for the involved stochastic dynamics. An easy extension would be to introduce independent exponentially-distributed times at which monitoring of the barriers begins and ends. From a modelling perspective, it may make more sense to model monitoring times as random variables correlated with the state variables, but that would not be a trivial extension from a computational point of view. Another direction in which the results shown in this article could be extended would be to introduce a multidimensional jump diffusion framework, in order to account for the discontinuous variations displayed by financial data, as well as for some forms of "smile" and "skew" in the equity and currency markets. It might be possible to preserve analytical tractability under the assumption that jumps follow a double exponential distribution using the approach developed by Kou and Wang (2003) for constant barriers, but the computations involved are certainly challenging and thus left for future research. Even when models more realistic and more sophisticated than the Black–Scholes one are needed, e.g. featuring stochastic volatility, closed form formulae such as the ones provided in this paper are still valuable for practioners due to their accuracy and efficiency, and because they allow to derive key risk management parameters and to speed up calibration procedures significantly.

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