

# The option on $n$ assets with exchange rate and exercise price risk

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## Abstract

Solutions to the call option on the maximum or the minimum of  $n$  assets are explicitly provided when the exercise price is stochastic, and all assets carry both asset price and exchange rate risk in a  $n + 1$  country model with  $2(n + 1)$  state variables. The model can be seen as an extension of Johnson (Johnson, H., 1987. Options on the maximum or the minimum of several assets. *Journal of Financial and Quantitative Analysis* 22, 227–283), Margrabe (Margrabe, W., 1978. The value of an option to exchange one asset for another. *Journal of Finance* 33, 177–186), and Reiner (Reiner, E., 1992. Quanto mechanics, *RISK*, March, 59–63), and it is useful for valuation of both financial and real options. As an application, a contract is valued that allows a portfolio manager to participate in the out-performance of the returns of international assets, portfolios or stock indexes. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Valuation models for options with one underlying risky asset and a stochastic exercise price have already appeared in the literature. They include the European option of Margrabe (1978) to exchange one asset for another, the European option

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of Fischer (1978) when the exercise price is uncertain, the investment model of McDonald and Siegel (1985) with a shutdown option, the model of McDonald and Siegel (1986) of the value of waiting to invest, the option of Myers and Majd (1990) to abandon an existing investment for its salvage value, and the empirical testing of Quigg (1993) of the option to wait to develop vacant land.

Contingent-claim models on  $n$  underlying risky assets also exist in the financial economics literature. The valuation methodology usually requires a high degree of computational intensity, but the constant increase in computational power available to financial researchers has considerably enhanced the attractiveness of such models. Such work includes Broadie and Detemple (1997), Barraquand and Martineau (1995), Boyle and Tse (1990), Boyle et al. (1989), Johnson (1987), and Kamrad and Ritchken (1991). These contributions provide a general valuation methodology, numerical or analytic, and their applicability is not limited to traded options alone, an area that during the last decade has shown a tremendous proliferation in both exchange-traded and over-the-counter instruments. Instead, they are directly extendible to decision making and valuation of real investments under uncertainty and irreversibility — see Pindyck (1991), Dixit and Pindyck (1994), or Trigeorgis (1996) — the area usually referred to as real options. These results allow the treatment of real options with multiple, mutually-exclusive investment alternatives.

Contingent claim valuation with exchange rate risk is established in Garman and Kohlhagen (1983), and Grabbe (1983). They value options on foreign currency and demonstrate that the risk-neutral drift of the underlying asset equals the home riskless rate of interest minus the foreign riskless rate of interest. The foreign rate of interest can be interpreted as a dividend yield that the option holder would capture only if a call option is exercised. Real options applications with exchange rate risk appear in Mello et al. (1995). They consider a multinational firm that can shift production back and forth between the home and a foreign country. They use a partial irreversibility model as opposed to the complete irreversibility that is implied by the real options in the context of the applications. Applications in financial options have appeared in Reiner (1992), Jamshidian (1993), and Kat and Roozen (1994) where options on foreign stocks are valued when the exchange rate relevant to option exercise is predetermined (the ‘quanto’ models).

In the case of multivariate contingent-claim models on the best of  $n$  underlying assets, the usual assumption is that the exercise price is constant. Overall, this is a somewhat restrictive assumption. In this paper the valuation methodology of options on  $n$  underlying assets is extended when the exercise price is stochastic, and there is exchange rate risk present for all assets in a  $n + 1$  country model. In effect, the case of  $2(n + 1)$  state variables that follow correlated Brownian motion processes are treated. Some of the applications provided nest the models known as quantos as special cases. The results can be applied for valuation of both financial and real (investment) options. The implementation for the European option on  $n$  assets, extending the Johnson (1987) model and nesting the Stulz (1982) model are provided as a special case. The results allow the multinomial lattice and multivariate Monte Carlo simulation techniques to be similarly extended.

For clarity of presentation, the extension to Johnson's model for a stochastic exercise price in the absence of exchange rate risk is first provided. Then the model is extended when all assets (including the exercise price) are in foreign countries but the exchange rates relevant to option exercise are predetermined in the quanto context; and then when all assets are in foreign countries without any exchange rate protection. These results extend a previous work (Martzoukos, 1994, 1995). They are demonstrated (with both the analytic and a multivariate lattice technique) in the valuation of a contract that gives a portfolio manager the option to exchange the proceeds from a foreign (index) investment for the best of other foreign (index) investments.

## 2. Derivation of the $n$ -dimensional model

There are  $n + 1$  risky asset prices. These are the prices of the option's  $n$  underlying assets, and the stochastic exercise price. Each follows geometric Brownian motion of the form

$$\frac{dS}{S} = (\mu_S - \delta_S)dt + \sigma_S dz_S, \quad (1a)$$

where the  $dz$  terms denote the increment of standard Wiener processes with instantaneous correlation coefficients  $\rho$ . Each asset pays a continuous payoff  $\delta_S$ . The drift  $\mu_S - \delta_S$  and volatility  $\sigma_S$  terms can be either constant or deterministic functions of time, and standard regularity conditions are assumed to hold. The dividend yields may represent actual cash flow, as in the case of stock options, or may be equivalent to one, like in the case of foreign exchange. McDonald and Siegel (1985, 1986) provide real options applications where the dividend yields represent the difference between the equilibrium total expected rate of return and the actual expected growth rate on the underlying assets, as shown by McDonald and Siegel (1984) drawing on Constantinides (1978). Brennan (1991) provides a convenience yield interpretation for the dividend yield. In general, a continuous time capital asset pricing model like in Merton (1973a) or Breeden (1979) is assumed to hold. Under the risk-neutral measure total required returns equal the riskless rate of interest  $r$ , and the assets' law of motion

$$\frac{dS}{S} = (r - \delta_S)dt + \sigma_S dz_S. \quad (1b)$$

The price  $P$  of a multivariate contingent claim on several assets is described (see for example, Cox et al., 1985) by a partial differential equation (PDE) of the form

$$\begin{aligned} \partial P / \partial t = & rP - \sum_I [(r - \delta_I)I' \partial P / \partial I'] \\ & - 0.5 \left\{ \sum_I [\sigma_I^2 I'^2 \partial^2 P / \partial I'^2] + \sum_{I', J', I'' \neq J''} [\sigma_{I', J'} I' J' \partial^2 P / (\partial I' \partial J')] \right\}, \end{aligned} \quad (2)$$

where  $I'$  and  $J'$  denote all pairs of the stochastic variables, with instantaneous covariance  $\sigma_{I',J'}$ . Summation is always over all stochastic assets. When the exercise price  $X$  is similarly stochastic, the explicit dependence is shown by rewriting PDE (2) as

$$\begin{aligned} \partial P / \partial t = & rP - \sum_I [(r - \delta_I) I' \partial P / \partial I'] - (r - \delta_X) X \partial P / \partial X \\ & - 0.5 \left\{ \sum_I [\sigma_I^2 I'^2 \partial^2 P / \partial I'^2] + \sigma_X^2 X^2 \partial^2 P / \partial X^2 \right. \\ & \left. + \sum_{I', J', I' \neq J'} [\sigma_{I', J'} I' J' \partial^2 P / (\partial I' \partial J')] + 2 \sum_I [\sigma_{IX} I' X \partial^2 P / (\partial I' \partial X)] \right\}, \end{aligned} \quad (2a)$$

where asset  $X$  pays a constant dividend yield  $\delta_X$ .

To remove the dependence on the stochastic variable  $X$ , one must reduce the dimensionality of the contingent claim with  $n + 1$  stochastic variables to one with  $n$  stochastic variables. To achieve this a change of variables is employed from all pairs of  $I'$  and  $J'$  to  $I = I'/X$  and  $J = J'/X$ , where the exercise price  $X$  is used as a numeraire. PDE (3) that follows gives the solution (see Appendix A for the proof) with the variable  $f$  defined as  $P/X$

$$\begin{aligned} \partial f / \partial t = & \delta_X f - \sum_I [(\delta_X - \delta_I) I \partial f / \partial I] \\ & - 0.5 \left\{ \sum_I [\sigma_I^2 I^2 \partial^2 f / \partial I^2] + \sum_{I, J, I \neq J} [\sigma_{I, J} I J \partial^2 f / (\partial I \partial J)] \right\}, \end{aligned} \quad (3)$$

with

$$\sigma_I^2 = \sigma_X^2 + \sigma_{I'}^2 - 2\rho_{I', X} \sigma_{I'} \sigma_X, \quad (4a)$$

and

$$\sigma_{I, J} = \sigma_X^2 + \sigma_{I', J'} - \sigma_{I', X} - \sigma_{J', X}. \quad (4b)$$

Note the similarity of the derived PDE in (3) with (2). Two features are notable. First, the variance of the underlying assets and the covariances are adjusted to account for the uncertain exercise price used as a numeraire. Second, the riskless rate of interest,  $r$ , does not appear in the model, and is replaced by the dividend yield on the exercise price,  $\delta_X$ . These results are sufficient to allow contingent claims models on the best of  $n$  assets to be extended to the case of a stochastic exercise price. Simply replace the riskless rate with  $\delta_X$ , all variances  $\sigma_I^2$  with  $\sigma_I^2$ , and all covariances  $\sigma_{I', J'}$  with  $\sigma_{I, J}$ .

### 2.1. The european call option with stochastic exercise price

The solution to the European call option on the maximum or minimum (max/min) of  $n$  assets with a stochastic exercise price  $X$  is given by integration of the risk-neutral density  $\Phi[\cdot]^+$

$$\exp(-rT) \iiint_{n+1} \Phi[\max/\min(S_1, S_2, \dots, S_n) - X]^+ dS_1 dS_2 \dots dS_n dX$$

discounted at the riskless rate of interest, where  $T$  is the option's maturity. The results are extended in Johnson (1987), finding that the price of a European call option on the maximum  $c_{\max}$  is a function of the asset prices  $S_1, \dots, S_n$ ,  $X$ , time to maturity  $T$ , the dividend yields  $\delta_r$ ,  $\delta_x$ , and the instantaneous variances and covariances of the transformed assets. Its price equals

$$\begin{aligned} c_{\max} = & S_1 \exp(-\delta_{S_1} T) \\ & N_n[d_1(S_1, X, \sigma_{S_1}^2), d'_1(S_1, S_2, \sigma_{S_1/S_2}^2), \dots, d'_1(S_1, S_n, \sigma_{S_1/S_n}^2), \rho_{112}, \rho_{113}, \dots] \\ & + S_2 \exp(-\delta_{S_2} T) \\ & N_n[d_1(S_2, X, \sigma_{S_2}^2), d'_1(S_2, S_1, \sigma_{S_1/S_2}^2), \dots, d'_1(S_2, S_n, \sigma_{S_2/S_n}^2), \rho_{212}, \rho_{223}, \dots] \\ & + \dots + S_n \exp(-\delta_{S_n} T) \\ & N_n[d_1(S_n, X, \sigma_{S_n}^2), d'_1(S_n, S_1, \sigma_{S_1/S_n}^2), \dots, d'_1(S_n, S_{n-1}, \sigma_{S_{n-1}/S_n}^2) \\ & , \rho_{n1n}, \rho_{n2n}, \dots] \\ & - X \exp(-\delta_{S_X} T) \\ & \{1 - N_n[-d_2(S_1, X, \sigma_{S_1}^2), -d_2(S_2, X, \sigma_{S_2}^2), \dots, \\ & -d_2(S_n, X, \sigma_{S_n}^2), \rho_{1,2}, \rho_{1,3}, \dots]\} \end{aligned} \quad (5)$$

with

$$\sigma_I^2 = \sigma_X^2 + \sigma_r^2 - 2\rho_{r,X} \sigma_r \sigma_X, \quad (6a)$$

$$\sigma_{I/J}^2 = \sigma_I^2 + \sigma_J^2 - 2\rho_{I,J} \sigma_I \sigma_J, \quad (6b)$$

$$\rho_{I,J} = (\sigma_I \sigma_J \rho_{r,r} - \sigma_X \sigma_I \rho_{r,X} - \sigma_X \sigma_J \rho_{r,X} + \sigma_X^2) / (\sigma_I \sigma_J). \quad (6c)$$

where  $I'$  and  $J'$  represent all pairs of the  $n$  assets, and  $I$  and  $J$  their transformed counterparts.

The following also hold

$$d'_1(S_1, S_2, \sigma_{S_1/S_2}^2) = [\ln(S_1/S_2) + (\delta_{S_2} - \delta_{S_1} + 0.5\sigma_{S_1/S_2}^2)T] / (\sigma_{S_1/S_2} \sqrt{T}), \quad (6d)$$

$$d_2(S_1, X, \sigma_{S_1}^2) = [\ln(S_1/X) + (\delta_{S_X} - \delta_{S_1} - 0.5\sigma_{S_1}^2)T] / (\sigma_{S_1} \sqrt{T}), \quad (6e)$$

$$d_1(S_1, X, \sigma_{S_1}^2) = d_2 + \sigma_{S_1} \sqrt{T}, \quad (6f)$$

where  $\sigma_{S_1}^2$ , etc. are defined as in Eq. (6a), and the triple indexed correlation coefficients (like in Johnson) are

$$\rho_{IIJ} = (\sigma_I - \rho_{I,J} \sigma_J) / \sigma_{I,J}, \quad (6g)$$

$$\rho_{IJK} = (\sigma_I^2 - \rho_{I,J} \sigma_I \sigma_J - \rho_{I,K} \sigma_I \sigma_K + \rho_{J,K} \sigma_J \sigma_K) / (\sigma_{I,J} \sigma_{I,K}). \quad (6h)$$

The European call option on the minimum  $c_{\min}$ , equals

$$\begin{aligned}
c_{\min} = & S_1 \exp(-\delta_{S_1} T) \\
& N_n[d_1(S_1, X, \sigma_{S_1}^2), -d'_1(S_1, S_2, \sigma_{S_1/S_2}^2), \dots, -d'_1(S_1, S_n, \sigma_{S_1/S_n}^2), -\rho_{112}, \\
& -\rho_{113}, \dots] \\
& + S_2 \exp(-\delta_{S_2} T) \\
& N_n[d_1(S_2, X, \sigma_{S_2}^2), -d'_1(S_2, S_1, \sigma_{S_1/S_2}^2), \dots, -d'_1(S_2, S_n, \sigma_{S_2/S_n}^2), -\rho_{212}, \\
& -\rho_{223}, \dots] + \dots \\
& + S_n \exp(-\delta_{S_n} T) \\
& N_n[d_1(S_n, X, \sigma_{S_n}^2), -d'_1(S_n, S_1, \sigma_{S_1/S_n}^2), \dots, -d'_1(S_n, S_{n-1}, \sigma_{S_{n-1}/S_n}^2), \\
& -\rho_{n1n}, \rho_{n2n}, \dots] \\
& - X \exp(-\delta_{S_X} T) \\
& N_n[d_2(S_1, X, \sigma_{S_1}^2), d_2(S_2, X, \sigma_{S_2}^2), \dots, d_2(S_n, X, \sigma_{S_n}^2), \rho_{1,2}, \rho_{1,3}, \dots] \quad (7)
\end{aligned}$$

where again Eqs. (6a), (6b), (6c), (6d), (6e), (6f), (6g) and (6h) hold. For numerical solutions to the cumulative multivariate normal  $N_n(\dots)$  see the references in Johnson (1987) and in Boyle and Tse (1990).

The American option on the maximum or the minimum of  $n$  assets can be handled like in Boyle et al. (1989), and Kamrad and Ritchken (1991). One only needs to replace the riskless rate of interest,  $r$ , with the dividend yield on the exercise price,  $\delta_X$ , and use equations (Eqs. (6a), (6b) and (6c)) to adjust the asset variances and correlations. Barraquand and Martineau (1995) give simulation methods for both European and American-type multivariate claims that can be similarly extended for a stochastic exercise price.

### 3. The model with exchange rate risk

Reiner (1992), Jamshidian (1993) and Kat and Roozen (1994) value ‘quanto’, contracts that involve both foreign exchange and stock price risk. Note that quanto refers to the case where the option payoff is in home currency, so the option is effectively protected against foreign exchange risk. As also discussed in Hull (1997) (pp. 298–301), the risk-neutral drift of a foreign stock (quanto), if viewed from the perspective of the home country, equals the foreign risk-neutral drift minus the covariance between the underlying asset and the exchange rate, a result called ‘Siegel’s paradox’. In all the relations that follow, the home country’s exchange rate convention is considered, where home is the country of the option holder. The sign of the correlation between the exchange rate and the foreign asset would be reversed if the foreign convention was used to define exchange rates.

Two general classes of options are priced. First the fully protected option where all payoffs are in currency of the home country were priced. This is the ‘exchange

quanto', that offers the right to exchange one foreign asset for the best of other foreign assets, and offers protection against exchange rate risk. Second the option to exchange a foreign asset for the best of other foreign assets was priced, but without the exchange rate protection. In the following relations,  $SI'$  denotes the  $n$  foreign assets, with  $SI'$ ,  $SJ'$  each pairwise combination, and  $SX$  denotes a foreign asset that serves as the exercise price. Similarly,  $EI'$ ,  $EJ'$ , and  $EX$  denote the exchange rates, and all state variables follow correlated geometric Brownian motion processes.

One *first* prices the exchange quanto. Again one needs to solve PDE (3). All foreign assets  $SI'$  (and likewise the one used as an exercise price) follow from the perspective the risk-neutral process

$$\frac{dSI'}{SI'} = (r_I - \delta_{SI'} - \sigma_{SI'} \sigma_{EI'} \rho_{SI',EI'}) dt + \sigma_{SI'} dz_{SI'},$$

where the rate  $r_I$  denotes the foreign riskless rate for country  $I$ ,  $\delta_{SI'}$  denotes the dividend yield on the foreign asset from the perspective of the foreign country, and  $\sigma_{SI'}$  the instantaneous S.D. of the foreign asset in the respective foreign currency. One can now value the exchange quanto option given that the dividend yields of the underlying assets  $\delta_{I'}$ , and of the exercise price  $\delta_X$  are known from the author's perspective

$$\delta_{I'} = r - r_I + \delta_{SI'} + \sigma_{SI'} \sigma_{EI'} \rho_{SI',EI'}, \quad (8a)$$

$$\delta_X = r - r_X + \delta_{SX} + \sigma_{SX} \sigma_{EX} \rho_{SX,EX} \quad (8b)$$

with  $\sigma_{EI'}$ , and  $\rho_{SI',EI'}$  the instantaneous S.D. of the exchange rate, and the instantaneous correlation between the foreign asset and the exchange rate. This correlation is only needed to evaluate each dividend yield. The S.D.s of the underlying assets  $\sigma_{I'}$ , and of the exercise price  $\sigma_X$ , and their correlations  $\rho_{I',J'}$ ,  $\rho_{I',X}$  are still needed in order to use the call option on the maximum or the minimum that were derived earlier. These are simply

$$\sigma_{I'} = \sigma_{SI'}, \quad \sigma_X = \sigma_{SX}, \quad (8c)$$

$$\rho_{I',J'} = \rho_{SI',SJ'}, \quad \rho_{I',X} = \rho_{SI',SX}, \quad (8d)$$

Eqs. (8a), (8b), (8c) and (8d) provide the dividend yields, standard deviations and correlations of the  $n$  normalized foreign (quanto) assets. They allow, together with Eqs. (5), (6a), (6b), (6c), (6d), (6e), (6f), (6g), (6h) and (7), pricing of the call option on the maximum or minimum of  $n$  foreign assets in exchange for another foreign asset. All foreign assets are protected against exchange rates' movements (in the quanto context), since each is translated in home currency using a predetermined (fixed) exchange rate.

*Second* the exchange option will be demonstrated without such exchange rate protection. Each underlying asset and the exercise price are the product of a foreign asset  $SI'$  with the relevant exchange rate  $EI'$ . Each foreign asset (before translation to home currency) has a risk-neutral drift  $r_I - \delta_{SI'}$  from the perspective of the foreign country, and a risk-neutral drift  $r_I - \delta_{SI'} - \sigma_{SI'} \sigma_{EI'} \rho_{SI',EI'}$  from the perspec-

tive of the home country (as discussed earlier in the quanto context). The foreign currency from the home country perspective has a risk-neutral drift of  $r - r_f$ . Using Ito's lemma it was found that under risk-neutrality all underlying (foreign) assets  $I'$ , and similarly the exercise price  $X$ , follow (from the author's perspective) the law of motion

$$\frac{dI'}{I'} = (r - \delta_{SI'}) dt + \sigma_{I'} dz_{I'}.$$

Thus, the dividend yields  $\delta_{I'}$ ,  $\delta_X$  needed in the option model are simply

$$\delta_{I'} = \delta_{SI'}, \text{ and } \delta_X = \delta_{SX}, \quad (9)$$

The instantaneous S.D. of each underlying foreign asset, and of the exercise price  $X$  equal

$$\sigma_{I'} = \sqrt{(\sigma_{SI'}^2 + \sigma_{EI'}^2 + 2\sigma_{SI'}\sigma_{EI'}\rho_{SI',EI'})}, \quad (10a)$$

$$\sigma_X = \sqrt{(\sigma_{SX}^2 + \sigma_{EX}^2 + 2\sigma_{SX}\sigma_{EX}\rho_{SX,EX})}. \quad (10b)$$

Again and in order to solve the problem one starts from PDE (2) and work the way to PDE (3). To use Eq. (2) one needs to get the covariance among all foreign assets  $\sigma_{I'J'}$ , and between the foreign assets and the exercise price  $\sigma_{I'X}$ , as functions of the S.D.s  $\sigma_{SI'}$ ,  $\sigma_{SJ'}$ ,  $\sigma_{SX}$ ,  $\sigma_{EI'}$ ,  $\sigma_{EJ'}$ , and  $\sigma_{EX}$ , and the correlations between the foreign assets and the exchange rates  $\rho_{SI',EI'}, \dots, \rho_{SI',SJ'}, \dots, \rho_{EI',EJ'}, \dots$ . With the use of standard stochastic calculus one gets

$$\sigma_{I',J'} = (\sigma_{SI'}\sigma_{SJ'}\rho_{SI',SJ'} + \sigma_{SI'}\sigma_{EJ'}\rho_{SI',EJ'} + \sigma_{EI'}\sigma_{SJ'}\rho_{EI',SJ'} + \sigma_{EI'}\sigma_{EJ'}\rho_{EI',EJ'}), \quad (11a)$$

and

$$\sigma_{I',X} = (\sigma_{SI'}\sigma_{SX}\rho_{SI',SX} + \sigma_{SI'}\sigma_{EX}\rho_{SI',EX} + \sigma_{EI'}\sigma_{SX}\rho_{EI',SX} + \sigma_{EI'}\sigma_{EX}\rho_{EI',EX}). \quad (11b)$$

PDE (3) with the usual boundary conditions again defines the solution for the price of American or European options on  $n$  assets with a stochastic exercise price  $X$  (when all foreign assets are affected by exchange rate risk). The dividend yields, the S.D.s, and the covariances/correlations are given from Eqs. (9), (10a), (10b), (11a) and (11b). These equations, together with Eqs. (5), (6a), (6b), (6c), (6d), (6e), (6f), (6g), (6h) and (7) define the European call option on the maximum and the minimum in exchange for foreign asset  $X$ . Similarly, finite difference methods, multinomial lattice, or Monte Carlo simulations can also be used with the use of Eqs. (9), (10a), (10b), (11a) and (11b).

#### 4. An application in international portfolio management

The model is useful for valuation of either financial, or real (investment) options. Here, it will be applied to price a financial contract that allows a portfolio manager to exchange assets in foreign country  $X$  for the best performing among assets in two other countries,  $I$  and  $J$ . The manager can actually have an exposure in country  $X$ ,



thereby wishing the option to give it up for assets in country  $I$ , or  $J$ ; or simply wishing an option on the outperformance between the best of (assets) of  $I$  or  $J$ , and those of country  $X$ . A call option on the maximum is obviously more valuable than a call option on the minimum. This option can either be of the quanto type for all three assets, or they are all unprotected against exchange rate risk (and of course it is straightforward to modify the results for the case where some assets are protected and some are not). The first contract type will offer participation in the outperformance of the assets in the respective foreign countries regardless of the movements of the exchange rates; the second's performance will in addition depend on exchange rate risk and is expected to be more valuable (and more costly). The numerical results for the call the option on the maximum and on the minimum appear in Table 1 for the quanto contract, and Table 2 for the unprotected contract. In both tables computations are presented using the analytic results (1st and 4th columns for a European option), and a 200-steps 2-dimensional lattice framework (for both the European and the American options). The comparison between the analytic and the (European) lattice confirms the accuracy of the lattice implementation. For the base case it was assumed that all foreign assets pay dividend yields of 3%, all riskless rates are 5%, all instantaneous standard deviations are 10%, and all correlations are 25%. All underlying assets (as well as the asset serving as the exercise price) are currently priced at 100 in the respective currencies, with the exchange rates for the quanto contract fixed to the ones prevailing at time zero (which for simplicity are assumed to equal unity). Thus, any differences between the two contract types depend on the option dynamics alone.

From the earlier analysis in Stulz (1982) and Johnson (1987) the impact of an increase in the riskless rate (of the *home* country) or the dividend yields of the underlying assets is known. An increase in the riskless rate, would increase the option value for both the call on the minimum and the call on the maximum; and an increase in dividend yields of the underlying assets would effectively decrease their value and the value of call option prices. In this case the dynamics are more complex. When the contract is not of the quanto type, the riskless rates of all four countries (home and the three foreign) do not appear in the model and subsequently do not affect valuation. The riskless rate has been replaced by the dividend yield of the asset serving as an exercise price, so an increase of that yield will also increase the value of the options on the maximum and the option on the minimum (see last line of Table 2). When the contract is of the quanto type, the riskless rate is replaced by the *effective* dividend yield of the exercise price as in Eqs. (8a) and (8b). An increase in that yield — everything else staying the same — will increase option values. But that yield is an increasing function of the home riskless rate, and a decreasing function of the foreign riskless rate. Thus, an increase in the foreign riskless rate in country  $X$  would decrease option values (bottom line, Table 1). Similarly, an increase in the dividend yield in country  $X$  would increase the option values (2nd line from the bottom, Table 1). An increase in the home riskless rate would increase the dividend yields in all three countries, having thus an ambiguous effect. An increase in the riskless rate of any of the countries  $I$  or  $J$  would decrease

Table 1  
Call option on  $n$  ( $= 2$ ) foreign assets in exchange for another foreign (the  $n$ th + 1) asset<sup>a</sup>

	Call on the maximum			Call on the minimum		
	European (analytic)	European (lattice)	American (lattice)	European (analytic)	European (lattice)	American (lattice)
<b>Base case</b>	7.19	7.18	7.21	2.26	2.26	2.80
$\rho_{S_I, S_J} = 0.50$	6.66	6.65	6.69	2.68	2.68	3.12
$\rho_{S_I, E_I} = \rho_{S_I, E_J}$	7.10	7.09	7.13	2.24	2.23	2.78
$= 0.50$						
$\rho_{S_X, E_X} = 0.50$	7.27	7.26	7.29	2.32	2.31	2.84
$\delta_{S_I} = \delta_{S_J} = 0.06$	7.12	7.11	7.15	2.24	2.24	2.78
$r_I = r_J = 0.07$	9.35	9.34	9.34	3.36	3.35	3.72
$\delta_{S_X} = 0.06$	7.82	7.81	7.81	2.58	2.58	3.06
$r_X = 0.07$	6.00	5.99	6.18	1.72	1.72	2.36

<sup>a</sup> All with exchange rate protection (Quanto type). For the base case, the underlying foreign assets  $SI'$ ,  $SJ'$ , and the exercise price  $SX'$ , all equal 100 (in the respective foreign currency units), the applicable exchange rates  $EI'$ ,  $EJ'$ , and  $EX$  are *fixed* (quanto contract) to unity (for simplicity of exposition), the local riskless interest rate  $r$  and the foreign riskless rates  $r_f$ ,  $r_x$  and  $r_x$  all equal 0.05, the time to maturity of the option is 1 year, the dividend yields of the assets in the respective countries  $\delta_{SI'}$ ,  $\delta_{SJ'}$ , and  $\delta_{SX}$  equal 0.03, all S.D.s of the foreign assets and the exchange rates  $\sigma_{SI'}$ ,  $\sigma_{SJ'}$ ,  $\sigma_{SX}$ , and  $\sigma_{EX}$ , equal 0.10, and all correlations  $\rho_{SI',EI'}$ ,  $\rho_{SI',EX}$ ,  $\rho_{SJ',EI'}$ ,  $\rho_{SJ',EX}$ ,  $\rho_{SX,EI'}$ ,  $\rho_{SX,EX}$ ,  $\rho_{SI',SJ'}$ ,  $\rho_{SI',SX}$ ,  $\rho_{SJ',SX}$ ,  $\rho_{EI',EI'}$ ,  $\rho_{EI',EX}$ , equal 0.25. For the lattice, the Boyle et al. (1989) method was implemented with 200 steps, augmented with the approach in order to reduce the dimensionality from three to two dimensions (three *quanto protected* foreign assets  $\rightarrow$  two effective dimensions).

Table 2  
Call option on  $n$  ( $= 2$ ) foreign assets in exchange for another foreign  $(n\text{th} + 1)$  asset<sup>a</sup>

	Call on the maximum			Call on the minimum		
	European (analytic)	European (lattice)	American (lattice)	European (analytic)	European (lattice)	American (lattice)
Base case	10.25	10.23	10.27	3.15	3.15	3.92
$\rho_{ST,ST} = 0.50$	8.50	8.48	8.55	4.50	4.50	4.96
$\rho_{ET,ET} = 0.50$	8.95	8.93	9.00	4.05	4.05	4.62
$\rho_{ST,ET} = \rho_{ST,ET}$	9.32	9.30	9.38	3.68	3.68	4.33
$= 0.50$						
$\delta_{ST} = \delta_{ST} = 0.06$	9.94	9.93	10.02	3.06	3.05	3.85
$\delta_{SX} = 0.06$	12.16	12.14	12.14	4.11	4.10	4.72

<sup>a</sup> All without exchange rate protection. For the base case, the underlying foreign assets  $ST'$ ,  $SJ'$ , and the exercise price  $SX$ , all equal 100 (in the respective foreign currency units), the applicable exchange rates  $ET'$ ,  $EJ'$ , and  $EX$  will fluctuate randomly and at time zero are equal to unity (for simplicity of exposition), the local riskless interest rate  $r$  and the foreign riskless rates  $r_F$ ,  $r_J$ , and  $r_X$  all equal 0.05, the time to maturity of the option is 1 year, the dividend yields of the assets in the respective countries  $\delta_{ST}$ ,  $\delta_{SJ}$ , and  $\delta_{SX}$  equal 0.03, all S.D.s of the foreign assets and the exchange rates  $\sigma_{ST}$ ,  $\sigma_{ET}$ ,  $\sigma_{SJ}$ ,  $\sigma_{EJ}$ ,  $\sigma_{SX}$ ,  $\sigma_{EX}$ , equal 0.10, and all correlations  $\rho_{ST,ET}$ ,  $\rho_{ST,EJ}$ ,  $\rho_{ST,EX}$ ,  $\rho_{SJ,ET}$ ,  $\rho_{SJ,EJ}$ ,  $\rho_{SJ,EX}$ ,  $\rho_{ET,EJ}$ ,  $\rho_{ET,EX}$ ,  $\rho_{EJ,EX}$ , equal 0.25. For the lattice, the Boyle et al. (1989) was implemented method with 200 steps, augmented with the approach in order to reduce the dimensionality from six to two dimensions (six state variables  $\rightarrow$  three foreign assets  $\rightarrow$  two effective dimensions).

*effective* dividend yields in those countries and would increase the quanto protected option values (3rd line from the bottom, Table 1). An increase in the dividend yields of any of the assets in countries  $I$  or  $J$  would decrease all option values in both the quanto and the unprotected contracts (5th line, Table 1, and 2nd from the bottom, Table 2). Also, the quanto contracts are affected by the correlation between a foreign asset value and the exchange rate of that country (seen from the author's perspective). An increase in that correlation increases the *effective* dividend yield of the assets in the respective country. In the case of the underlying assets such an increase results in a decrease in option values (3rd line, Table 1); and in the case of the asset in country  $X$  results in an increase in option values (4th line, Table 1).

The correlations among several variables can affect option values significantly. In the absence of exercise price and exchange rate risk, a higher correlation between the two underlying assets will increase the value of the option on the minimum, and will decrease the value of the option on the maximum. In this case the dynamics are again more complex. For the case of the quanto option, the correlation among the underlying (foreign) assets will increase the *effective* correlation in Eq. (4b), or equivalently Eq. (6c). Subsequently, the option values on the minimum increase, and the option values on the maximum decrease (2nd line, Table 1). For the case of the unprotected exchange option, Eqs. (4b) and (6c) are affected through Eq. (11a). The *effective* correlation between the asset in country  $I$  and that in country  $J$  increases with an increase in correlations between the two underlying foreign assets (2nd line, Table 2). Also between the two exchange rates (3th line, Table 2); and between asset in country  $I$  and exchange rate in country  $J$ , and between asset in country  $J$  and exchange rate in country  $I$  (4th line, Table 2). The correlation between each underlying asset and the relevant exchange rate simply increases the volatility of that asset, as seen in Eq. (10a). Such a change though, as well as any change in the volatility of each asset has an ambiguous effect, since the *effective* volatility of each underlying asset as seen in Eq. (4a) is not a linear function of the individual volatilities. All other variables (instantaneous correlations and volatilities) also have ambiguous effects.

## 5. Summary and conclusions

This article shows how to extend multivariate contingent claim models on the best of  $n$  underlying assets to the case of a stochastic exercise price when all  $n + 1$  assets carry foreign exchange risk. In the most general case one works with a  $n + 1$  country model each represented by assets like stocks, stock indexes, or real assets. The problem is reduced from 1 with  $2(n + 1)$  state variables to a problem with  $n$  variables and a solution isomorphic to Johnson (1987) is derived. In the reduced problem, the riskless rate of interest is replaced with the dividend yield on the exercise price. Also the variance of each underlying asset is replaced with the variance of a transformed variable (namely the ratio of the underlying asset over the exercise price), and the covariance between pairs of the underlying assets is replaced by the covariance of the transformed variables. In the presence of

exchange rate risk, two interesting classes of models are solved for: *first* the exchange quantos (where the exchange rates are predetermined), and *then* exchange options on foreign assets without exchange rate protection. An application in international portfolio management is given where the sensitivity of the models in some important parameters is demonstrated and discussed. Beyond the analytic solution, the results allow numerical and simulation techniques to be similarly extended with the use of the risk-neutral processes transformed for the exercise price as the numeraire. The above results apply also to the case of real (investment) options in the presence of multiple uncertainties with exercise price and exchange rate risks.

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### Appendix A. Proof of PDE (3)

The homogeneity property of the option value is used (see Merton, 1973b),  $P(I', X) = Xf(I'/X) = Xf(I)$ . The notation  $P(I', X)$  implies that  $P$  is a function of  $n$  risky assets and of the stochastic exercise price  $X$ ;  $f(I)$  implies that  $f$  is only a function of the  $n$  normalized asset prices (using  $X$  as a numeraire). Parentheses are dropped for ease of notation. The following relations are useful

$$\partial P / \partial I' = \partial f / \partial I, \quad (\text{A1a})$$

$$\partial P / \partial X = f - \sum_I (I \partial f / \partial I), \quad (\text{A1b})$$

$$\partial^2 P / \partial I'^2 = (\partial^2 f / \partial I^2) / X, \quad (\text{A1c})$$

$$\partial^2 P / (\partial I' \partial X) = -(\partial^2 f / \partial I^2) I / X - \sum_{J, J \neq I} [J \partial^2 f / (\partial I \partial J)] / X, \quad (\text{A1d})$$

$$\partial^2 P / \partial X^2 = \sum_I I \{ (\partial^2 f / \partial I^2) I / X + \sum_{J, J \neq I} [J \partial^2 f / (\partial I \partial J)] / X \}, \quad (\text{A1e})$$

$$\partial^2 P / (\partial I' \partial J') = [\partial^2 f / (\partial I \partial J)] / X, \quad (\text{A1f})$$

where the summations (unless noted otherwise) apply over the entire set of assets. Substituting Eqs. (A1a), (A1b), (A1c), (A1d), (A1e) and (A1f) into PDE (2a),

$$\begin{aligned}
\partial P / \partial t = & rXf - \sum_I [(r - \delta_r)I' \partial f / \partial I] - (r - \delta_x) \left[ f - \sum_I (I \partial f / \partial I) \right] X \\
& - 0.5 \left\{ \sum_I (I'^2 \sigma_r^2 \partial^2 f / \partial I^2) + X^2 \sigma_x^2 \sum_{I, J \neq J} I \left[ I \partial^2 f / \partial I^2 + \sum_J (J \partial^2 f / (\partial I \partial J)) \right] \right. \\
& + \sum_{I, J, I \neq J} [I' J' \sigma_{r, J} \partial^2 f / (\partial I \partial J)] \\
& \left. + 2 \sum_{I, I \neq J} I' X \sigma_{rX} \left[ -I \partial^2 f / \partial I^2 - \sum_J J \partial^2 f / (\partial I \partial J) \right] \right\} / X,
\end{aligned}$$

and using

$$\partial P / \partial t = X \partial f / \partial t,$$

removing  $X$  from both sides, and some algebra, one gets

$$\begin{aligned}
\partial f / \partial t = & \delta_x f - \sum_I [(\delta_x - \delta_r)I \partial f / \partial I] \\
& - 0.5 \left\{ \sum_I [(\sigma_x^2 + \sigma_r^2 - 2\rho_{r,x} \sigma_r \sigma_x) I^2 \partial^2 f / \partial I^2] \right. \\
& \left. + \sum_{I, J, I \neq J} [(\sigma_x^2 + \sigma_{r,J} - \sigma_{r,x} - \sigma_{J,x}) IJ \partial^2 f / (\partial I \partial J)] \right\}, \tag{A2}
\end{aligned}$$

QED.

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