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ON THE MAXIMUM OF THE GENERALIZED BROWNIAN BRIDGE

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Abstract. We present some extensions of the distributions of the maximum of the Brownian bridge in $[0, t]$ when the conditioning event is placed at a future time $u > t$ or at an intermediate time $u < t$. The standard distributions of Brownian motion and Brownian bridge are obtained as limiting cases. These results permit us to derive also the distribution of the first-passage time of the Brownian bridge. Similar generalizations are carried out for the Brownian bridge with drift μ; in this case, it is shown that the maximal distribution is independent of μ (when $u \geqslant t$). Finally, the case of the two-sided maximal distribution of Brownian motion in $[0, t]$, conditioned on $B(u) = \eta$ (for both $u > t$ and $u < t$), is considered.

Key words: Brownian bridge, maximal distribution, two-sided maximal distribution, Rayleigh distribution, first-passage time, empirical process.

1. INTRODUCTION

The distributions derived over the years for Brownian motion (B.M.) have recently been collected in the book by Borodin and Salminen [1]; these authors, however, do not consider the conditional cases. In this paper, we present some extensions of the distributions of the maximum of the Brownian bridge in $[0, t]$ conditional on the event $B(u) = \eta$, where $u > t$ or $u < t$. This means that we are concerned with

$$\Pr\left\{ \max_{0 \leqslant s \leqslant t} B(s) \geqslant \beta \mid B(u) = \eta \right\} \tag{1.1}$$

in two substantially different cases, namely, where $u > t$ and where $u < t$, for arbitrary values of η.

The problem of evaluating this distribution for $u > t$ emerged in the analysis of randomly vibrating strings when an initial white noise disturbance is assumed: in this case, (1.1) can be interpreted as the probability that level β is exceeded in the subinterval $[0, t] \subset [0, u]$. The maximum distribution of the Brownian bridge is also a precious tool in the investigation of the limiting behavior of empirical processes; in particular, the distribution (1.1) for $u > t$ is relevant in the case of samples with random size (cf. Nikitin [2] and Wellner and van der Vaart [6], Chap. 3.5).

For $u > t$, we will show that, when $u \to \infty$, the usual distribution of the unconditional maximum of Brownian motion is obtained and η does not exert any influence in the limit; furthermore, when $u \to t^+$, the usual well-known one-sided distribution of Brownian bridge emerges.

For $u < t$, we obtain a result that yields the maximum of the Brownian bridge, when $u \to t^-$, and the maximum of B.M., when $u \to 0$.

All these results permit us to obtain the distribution of the first-passage time of the Brownian bridge.

We have also examined the distribution of

$$\Pr\left\{ \max_{0 \leqslant s \leqslant t} B_\mu(s) \geqslant \beta \mid B_\mu(u) = \eta \right\}, \tag{1.2}$$

where B_μ is a Brownian motion with drift. It is relevant to point out that the distribution (1.2) is not affected by the drift as long as $u \geqslant t$. This is because the constraint on the sample paths to pass through the point η at

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time u cancels the drift. This is not the case for $u < t$, since, in $[u, t]$, they are again subject to the swarming effect of the drift.

We are also able to study the conditional distribution of the maximum for the two-sided case (when no drift is assumed). In this case, we are also able to derive the expected limiting result that, for $u \to \infty$, the usual distribution

$$\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta\right\} = \frac{1}{\sqrt{2\pi}}\sum_{h=-\infty}^{\infty}(-1)^h \int_{(-\beta+2\beta h)/\sqrt{t}}^{(\beta+2\beta h)/\sqrt{t}} e^{-y^2/2}\,dy \tag{1.3}$$

holds. Finally, for $u \to t$, we obtain from (1.3) the well-known distribution of the maximum for the two-sided Brownian bridge (see, for instance, Shorack and Wellner [4], p. 34).

Many of the results presented here can also be obtained by means of the conditional Feynman–Kac functional, although, in this case, many lengthy and clumsy calculations would be needed.

2. MAXIMAL DISTRIBUTION FOR THE ONE-SIDED CASE

The cases where $u > t$ and $u < t$ are qualitatively different and the corresponding distributions are presented in two different theorems.

We start with the following result:

THEOREM 2.1. *Let B be the standard Brownian motion and $u > t$; then, for any positive β, the following distribution holds*:

$$\begin{aligned}\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(u) = \eta\right\} &= \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\}\int_{-\infty}^{(2\beta t-\eta t-\beta u)/\sqrt{ut(u-t)}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy \\ &\quad + \int_{(\beta u-\eta t)/\sqrt{ut(u-t)}}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy, \quad \beta > 0,\ u \geqslant t,\ \eta \in R.\end{aligned} \tag{2.1}$$

Proof. Since

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\ B(t) \in dz\right\} = \begin{cases} \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(2\beta-z)^2}{2t}\right\}dz, & z < \beta, \\ \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{z^2}{2t}\right\}dz, & z > \beta, \end{cases} \tag{2.2}$$

we obtain

$$\begin{aligned}&\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\, B(u) \in d\eta\right\} \\ &= \int_{-\infty}^{+\infty} \Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\, B(t) \in dz\right\}\Pr\{B(u) \in d\eta \mid B(t) = z\} \\ &= \int_{-\infty}^{\beta} \Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\, B(t) \in dz\right\}\frac{1}{\sqrt{2\pi(u-t)}}\exp\left\{-\frac{(\eta-z)^2}{2(u-t)}\right\}d\eta \\ &\quad + \int_{\beta}^{+\infty} \Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\, B(t) \in dz\right\}\frac{1}{\sqrt{2\pi(u-t)}}\exp\left\{-\frac{(\eta-z)^2}{2(u-t)}\right\}d\eta\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(2\beta - z)^2}{2t}\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{(\eta - z)^2}{2(u-t)}\right\} \mathrm{d}z\,\mathrm{d}\eta \\
&+ \int_{\beta}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{z^2}{2t}\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{(\eta - z)^2}{2(u-t)}\right\} \mathrm{d}z\,\mathrm{d}\eta.
\end{aligned} \tag{2.3}
$$

The first integral of (2.3) can be developed as follows:

$$
\begin{aligned}
&\int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(2\beta - z)^2}{2t}\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{(\eta - z)^2}{2(u-t)}\right\} \mathrm{d}z \\
&= \frac{1}{2\pi\sqrt{t(u-t)}} \exp\left\{-\frac{4\beta^2 - 4\beta\eta + \eta^2}{2u}\right\} \int_{-\infty}^{\beta} \exp\left\{-\frac{[zu - 2\beta(u-t) - \eta t]^2}{2ut(u-t)}\right\} \mathrm{d}z \\
&= \frac{1}{2\pi\sqrt{u}} \exp\left\{-\frac{4\beta^2 - 4\beta\eta + \eta^2}{2u}\right\} \int_{-\infty}^{(2\beta t - \eta t - \beta u)/\sqrt{ut(u-t)}} \mathrm{e}^{-y^2/2}\,\mathrm{d}y.
\end{aligned} \tag{2.4}
$$

Similarly, the second term of (2.3) can be rewritten as follows:

$$
\begin{aligned}
&\int_{\beta}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{z^2}{2t}\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{(\eta - z)^2}{2(u-t)}\right\} \mathrm{d}z \\
&= \frac{1}{2\pi\sqrt{u}} \exp\left\{-\frac{\eta^2}{2u}\right\} \int_{(\beta u - \eta t)/\sqrt{ut(u-t)}}^{+\infty} \mathrm{e}^{-y^2/2}\,\mathrm{d}y.
\end{aligned} \tag{2.5}
$$

With (2.4) and (2.5) at hand, it is a simple matter to obtain

$$
\begin{aligned}
\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\, B(u) \in \mathrm{d}\eta\right\} &= \frac{\mathrm{d}\eta}{2\pi\sqrt{u}} \exp\left\{-\frac{(2\beta - \eta)^2}{2u}\right\} \int_{-\infty}^{(2\beta t - \eta t - \beta u)/\sqrt{ut(u-t)}} \mathrm{e}^{-y^2/2}\,\mathrm{d}y \\
&+ \frac{\mathrm{d}\eta}{2\pi\sqrt{u}} \exp\left\{-\frac{\eta^2}{2u}\right\} \int_{(\beta u - \eta t)/\sqrt{ut(u-t)}}^{+\infty} \mathrm{e}^{-y^2/2}\,\mathrm{d}y.
\end{aligned} \tag{2.6}
$$

Passing from (2.6) to (2.1) is now straightforward.

Remark 2.1. From (2.1), it is easy to derive the well-known distribution of the maximum of the Brownian bridge:

$$
\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(t) = \eta\right\} = \begin{cases} \exp\left\{-\frac{2\beta(\beta-\eta)}{t}\right\}, & \beta > \eta, \\ 1, & \beta < \eta. \end{cases} \tag{2.7}
$$

The case $\eta = 0$ gives the Rayleigh distribution in $(0,\infty)$.

Letting $u \to \infty$ in (2.1), we get

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(\infty) = \eta\right\} = \int_{-\infty}^{-\beta/\sqrt{t}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y + \int_{\beta/\sqrt{t}}^{+\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y = 2\Pr\{B(t) \geqslant \beta\}. \tag{2.8}$$

This proves that the classical (unconditional) law of the maximum of Brownian motion is obtained as a particular case of (2.1).

When $u \to t^+$, from formula (2.6) it is easy to derive the joint distribution (2.2).

Remark 2.2. A particular case of formula (2.1) has been obtained by Orsingher [3] within the framework of the analysis of vibrations excited by random noise.

We now consider the maximum of B.M. in $[0, t]$ when its position at some intermediate time u is known. This can be viewed as a process which, in $[0, u]$, is a Brownian bridge and, in $[u, t]$, is again a free B.M. The following result holds:

THEOREM 2.2. *If $u < t$, then*

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(u) = \eta\right\} \tag{2.9}$$

$$= \begin{cases} 1, & \text{for } \eta > \beta, \\ \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\} + 2\left\{1 - \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\}\right\} \int_{(\beta-\eta)/\sqrt{t-u}}^{+\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y, & \text{for } \eta < \beta. \end{cases}$$

Proof. We restrict ourselves to the case where $\eta < \beta$, since the other one is obvious.

For the case $0 \leqslant u \leqslant t$, we clearly have, bearing in mind formula (2.2),

$$\begin{aligned}
&\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta, B(u) \in \mathrm{d}\eta\right\} \\
&= \Pr\left\{\max_{0\leqslant s\leqslant u} B(s) \geqslant \beta, B(u) \in \mathrm{d}\eta\right\} + \Pr\left\{\max_{u\leqslant s\leqslant t} B(s) \geqslant \beta, B(u) \in \mathrm{d}\eta\right\} \\
&\quad - \Pr\left\{\max_{0\leqslant s\leqslant u} B(s) \geqslant \beta, B(u) \in \mathrm{d}\eta, \max_{u\leqslant s\leqslant t} B(s) \geqslant \beta\right\} \\
&= \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{-\frac{(2\beta-\eta)^2}{2u}\right\} + \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{-\frac{\eta^2}{2u}\right\} \Pr\left\{\max_{u\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(u) = \eta\right\} \\
&\quad - \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{-\frac{(2\beta-\eta)^2}{2u}\right\} \Pr\left\{\max_{u\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(u) = \eta\right\} \\
&= \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{-\frac{(2\beta-\eta)^2}{2u}\right\} + \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} 2\Pr\{B(t-u) \geqslant \beta - \eta\} \exp\left\{-\frac{\eta^2}{2u}\right\} \\
&\quad - \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{-\frac{(2\beta-\eta)^2}{2u}\right\} 2\Pr\{B(t-u) \geqslant \beta - \eta\}.
\end{aligned} \tag{2.10}$$

In the last step, the equality

$$\Pr\left\{\max_{u\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(u) = \eta\right\} = 2\Pr\{B(t-u) \geqslant \beta - \eta\} \tag{2.11}$$

is used.

From (2.10) result (2.9) swiftly follows.

Remark 2.3. Letting $u \to t^-$ formula (2.9) yields the distribution (2.7) and, for $u \to 0^+$, it yields the classical distribution of the maximum of Brownian motion emanating from the point $x = \eta$.

Remark 2.4. From Theorems 2.1 and 2.2, it is possible to derive the explicit distribution of the first passage time of the Brownian bridge.

Let $T_\beta = \inf\{s < t : B(s) = \beta\}$. Then, for $\eta < \beta$, we have

$$\Pr\left\{T_\beta \leqslant t \mid B(u) = \eta\right\} = \Pr\left\{\max_{0 \leqslant s \leqslant t} B(s) \geqslant \beta \mid B(u) = \eta\right\} \tag{2.12}$$

$$= \begin{cases} \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\} \int\limits_{-\infty}^{(2\beta t - \eta t - \beta u)/\sqrt{ut(u-t)}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\ \quad + \int\limits_{(\beta u - \eta t)/\sqrt{ut(u-t)}}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy, & \text{for } t < u, \\ \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\} \\ \quad + \frac{2}{\sqrt{2\pi}} \left\{1 - \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\}\right\} \int\limits_{(\beta-\eta)/\sqrt{t-u}}^{\infty} e^{-y^2/2} dy, & \text{for } t > u. \end{cases}$$

It can easily be seen from (2.12) that

$$\lim_{t \to u^-} \Pr\left\{T_\beta \leqslant t \mid B(u) = \eta\right\} = \lim_{t \to u^+} \Pr\left\{T_\beta \leqslant t \mid B(u) = \eta\right\} = \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\} \tag{2.13}$$

and that

$$\lim_{t \to +\infty} \Pr\left\{T_\beta \leqslant t \mid B(u) = \eta\right\} = 1. \tag{2.14}$$

The density function of T_β is

$$f_\beta^{(u,\eta)}(t) = \frac{\partial}{\partial t} \Pr\left\{T_\beta \leqslant t \mid B(u) = \eta\right\} \tag{2.15}$$

$$= \begin{cases} \beta \sqrt{\frac{u}{(u-t)2\pi t^3}} \exp\left\{-\frac{(u\beta - \eta t)^2}{2ut(u-t)}\right\}, & \text{for } 0 < t < u, \\ \frac{\beta - \eta}{\sqrt{2\pi(t-u)^3}} \exp\left\{-\frac{(\beta-\eta)^2}{2(t-u)}\right\} \left\{1 - \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\}\right\}, & \text{for } t > u. \end{cases}$$

As $u \to +\infty$, (2.12) converges to the first passage time density of B.M.:

$$\lim_{u \to +\infty} f_\beta^{(u,\eta)}(t) = \frac{\beta}{\sqrt{2\pi t^3}} \exp\left\{-\frac{\beta^2}{2t}\right\}, \quad \beta > 0,\ t > 0. \tag{2.16}$$

3. MAXIMAL DISTRIBUTION FOR THE ONE-SIDED CASE WHEN A DRIFT IS ASSUMED

In this section, we present the extensions of Theorems 2.1 and 2.2 to the case where a drift μ is assumed. We begin our analysis by considering the counterpart of formula (2.2) for a Brownian motion endowed with drift μ (denoted hereafter by B_μ):

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s) \geqslant \beta,\, B_\mu(t)\in \mathrm{d}z\right\} = \begin{cases} \frac{1}{\sqrt{2\pi t}}\exp\{2\mu(z-\beta)\}\exp\left\{-\frac{(2\beta-z-\mu t)^2}{2t}\right\}\mathrm{d}z, & z<\beta,\\ \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(z-\mu t)^2}{2t}\right\}\mathrm{d}z, & z>\beta,\end{cases}$$
$$= \begin{cases} \frac{1}{\sqrt{2\pi t}}\exp\{\mu z\}\exp\left\{-\frac{(2\beta-z)^2}{2t}\right\}\exp\left\{-\frac{\mu^2 t}{2}\right\}\mathrm{d}z, & z<\beta,\\ \frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(z-\mu t)^2}{2t}\right\}\mathrm{d}z, & z>\beta.\end{cases} \tag{3.1}$$

Formula (3.1) can be easily obtained by using the law of an absorbing B.M. with drift. By integrating (3.1) with respect to z, we derive the distribution of the maximum of Brownian motion with drift

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s)\geqslant\beta\right\} = \int_{(\beta-\mu t)/\sqrt{t}}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2}\mathrm{d}y + \mathrm{e}^{2\mu\beta}\int_{(\beta+\mu t)/\sqrt{t}}^{+\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2}\mathrm{d}y \tag{3.2}$$

(cf. Borodin and Salminen [1], 2.1.1.4).

For $\mu=0$, this gives the well-known distribution of the maximum of standard B.M.

The most important result of this section is that the drift does not affect the maximal distribution of the Brownian bridge when conditioned on $B(u)=\eta$, $u>t$. It can be shown, by simple manipulations, that the same is true for the standard Brownian bridge (i.e., when $u=t$) endowed with drift μ; more precisely,

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s)\geqslant\beta \mid B_\mu(t)=\eta\right\} = \exp\left\{-\frac{2\beta(\beta-\eta)}{t}\right\} \tag{3.3}$$

for any value of η and for $\beta>\eta$ (independently of μ).

On the other hand, in the unconditional distribution of the maximum (3.2) (as well as in (3.1)) the drift, as expected, appears explicitly.

THEOREM 3.1. *If $u>t$, then, for any positive β, the following distribution holds for a Brownian motion with drift μ:*

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s)\geqslant\beta \mid B_\mu(u)=\eta\right\}$$
$$= \exp\left\{-\frac{2\beta(\beta-\eta)}{u}\right\}\int_{-\infty}^{(2\beta t-\eta t-\beta u)/\sqrt{ut(u-t)}}\frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y + \int_{(\beta u-\eta t)/\sqrt{ut(u-t)}}^{\infty}\frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\mathrm{d}y,\quad \eta\in R. \tag{3.4}$$

Proof. Following the same argument as in Theorem 2.2, we can write

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s)\geqslant\beta,\, B_\mu(u)\in\mathrm{d}\eta\right\}$$
$$= \int_{-\infty}^{+\infty}\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s)\geqslant\beta,\, B_\mu(t)\in\mathrm{d}z\right\}\Pr\left\{B_\mu(u)\in\mathrm{d}\eta \mid B_\mu(t)=z\right\}. \tag{3.5}$$

Then we obtain the analogue of (2.6) by inserting formula (3.1) into (3.5):

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s) \geqslant \beta,\, B_\mu(u) \in \mathrm{d}\eta\right\}$$

$$= \mathrm{d}\eta \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi t}} \exp\{\mu z\} \exp\left\{-\frac{(2\beta - z)^2}{2t}\right\} \exp\left\{-\frac{\mu^2 t}{2}\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{[\eta - z - \mu(u-t)]^2}{2(u-t)}\right\} \mathrm{d}z$$

$$+ \mathrm{d}\eta \int_{\beta}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(z-\mu t)^2}{2t}\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{[\eta - z - \mu(u-t)]^2}{2(u-t)}\right\} \mathrm{d}z$$

$$= \frac{d\eta}{2\pi\sqrt{u}} \exp\left\{-\frac{1}{2u}\left[4\beta^2 - 4\beta\eta + (\eta - \mu u)^2\right]\right\} \int_{-\infty}^{(2\beta t - \eta t - \beta u)/\sqrt{ut(u-t)}} \mathrm{e}^{-y^2/2}\mathrm{d}y$$

$$+ \frac{\mathrm{d}\eta}{2\pi\sqrt{u}} \exp\left\{-\frac{(\eta - \mu u)^2}{2u}\right\} \int_{(\beta u - \eta t)/\sqrt{ut(u-t)}}^{+\infty} \mathrm{e}^{-y^2/2}\mathrm{d}y. \tag{3.6}$$

By dividing formula (3.6) by $\frac{1}{2\pi\sqrt{u}}\exp\{-\frac{(\eta-\mu u)^2}{2u}\}$ we obtain (3.4).

For $u \to t^+$, the joint distribution (3.6) coincides with (3.1) when $z = 0$. The scrupulous reader can also check that, when $\mu = 0$, formula (3.6) coincides with (2.6).

Finally, (3.4) tends to the Rayleigh distribution when $u \to t^+$.

We also derive an analogue of Theorem 2.2 for the case where a drift is assumed:

THEOREM 3.2. *If $u < t$, we have, for $\eta < \beta$,*

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s) \geqslant \beta \mid B_\mu(u) = \eta\right\}$$

$$= \mathrm{e}^{-2\beta(\beta-\eta)/u} + \left\{1 - \mathrm{e}^{-2\beta(\beta-\eta)/u}\right\} \left\{ \int_{\frac{\beta-\eta}{\sqrt{t-u}} - \mu\sqrt{t-u}}^{+\infty} \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\mathrm{d}z + \mathrm{e}^{2\mu(\beta-\eta)} \int_{\frac{\beta-\eta}{\sqrt{t-u}} + \mu\sqrt{t-u}}^{+\infty} \frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\mathrm{d}z \right\} \tag{3.7}$$

while, for $\eta > \beta$, the above probability is, obviously, equal to one.

Proof. As in Theorem 2.2, we can write for the B.M. with drift μ

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B_\mu(s) \geqslant \beta,\, B_\mu(u) \in \mathrm{d}\eta\right\}$$

$$= \Pr\left\{\max_{0\leqslant s\leqslant u} B_\mu(s) \geqslant \beta,\, B_\mu(u) \in \mathrm{d}\eta\right\} + \Pr\left\{\max_{u\leqslant s\leqslant t} B_\mu(s) \geqslant \beta,\, B_\mu(u) \in \mathrm{d}\eta\right\}$$

$$- \Pr\left\{\max_{0\leqslant s\leqslant u} B_\mu(s) \geqslant \beta,\, B_\mu(u) \in \mathrm{d}\eta,\, \max_{u\leqslant s\leqslant t} B_\mu(s) \geqslant \beta\right\}$$

$$
\begin{aligned}
&= \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{2\mu(\eta-\beta) - \frac{(2\beta-\eta-\mu u)^2}{2u}\right\} + \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp^{-\frac{(\eta-\mu u)^2}{2u}} \\
&\times \frac{1}{\sqrt{2\pi}} \left\{ \int_{\frac{\beta-\eta}{\sqrt{t-u}}-\mu\sqrt{t-u}}^{+\infty} \mathrm{e}^{-z^2/2}\mathrm{d}z + \mathrm{e}^{2\mu(\beta-\eta)} \int_{\frac{\beta-\eta}{\sqrt{t-u}}+\mu\sqrt{t-u}}^{+\infty} \mathrm{e}^{-z^2/2}\mathrm{d}z \right\} \\
&- \frac{\mathrm{d}\eta}{\sqrt{2\pi u}} \exp\left\{2\mu(\eta-\beta) - \frac{(2\beta-\eta-\mu u)^2}{2u}\right\} \\
&\times \frac{1}{\sqrt{2\pi}} \left\{ \int_{\frac{\beta-\eta}{\sqrt{t-u}}-\mu\sqrt{t-u}}^{+\infty} \mathrm{e}^{-z^2/2}\mathrm{d}z + \exp\{2\mu(\beta-\eta)\} \int_{\frac{\beta-\eta}{\sqrt{t-u}}+\mu\sqrt{t-u}}^{+\infty} \mathrm{e}^{-z^2/2}\mathrm{d}z \right\}.
\end{aligned} \tag{3.8}
$$

Dividing (3.8) by $\frac{1}{\sqrt{2\pi u}}\exp\{-\frac{(\eta-\mu u)^2}{2u}\}$, we readily obtain distribution (3.7).

As can be easily seen, the distribution (3.7) reduces to (2.9) for $\mu = 0$. In the case $u < t$, the influence of the drift cannot be eliminated by any means.

For $u \to t^-$, we obtain the Rayleigh distribution (3.3), while, for $u \to 0^+$, we have the unconditional maximal distribution (3.2).

4. MAXIMAL DISTRIBUTION FOR THE TWO-SIDED CASE

In this section, we derive the conditional distribution

$$
\Pr\left\{ \max_{0\leqslant s\leqslant t} |B(s)| < \beta \mid B(u) = \eta \right\} \tag{4.1}
$$

in the two cases $u > t$ and $u < t$. As expected, the analysis of the two-sided maximal distribution involves much more entangled formulas.

For the derivation of (4.1), we need some formulas that we report here for the sake of completeness. We start with the following one (obtainable by a repeated application of the reflection principle):

$$
\begin{aligned}
&\Pr\left\{\alpha < \min_{0\leqslant s\leqslant t} B(s) < \max_{0\leqslant s\leqslant t} B(s) < \beta,\, B(t) \in \mathrm{d}y \mid B(0) = \eta\right\} \\
&= \frac{\mathrm{d}y}{\sqrt{2\pi t}} \sum_{h=-\infty}^{+\infty} \left\{ \exp\left\{-\frac{(y-\eta-2h(\beta-\alpha))^2}{2t}\right\} - \exp\left\{-\frac{(2\alpha-y-\eta-2h(\beta-\alpha))^2}{2t}\right\}\right\}.
\end{aligned} \tag{4.2}
$$

A considerable number of other results can be extracted from (4.2). For example, the joint distribution of maximum and minimum of B.M.:

$$
\begin{aligned}
&\Pr\left\{\alpha < \min_{0\leqslant s\leqslant t} B(s) < \max_{0\leqslant s\leqslant t} B(s) < \beta \mid B(0) = \eta\right\} \\
&= \frac{1}{\sqrt{2\pi t}} \sum_{h=-\infty}^{+\infty} \int_{\alpha}^{\beta} \left\{ \exp\left\{-\frac{(y-\eta-2h(\beta-\alpha))^2}{2t}\right\} - \exp\left\{-\frac{(2\alpha-y-\eta-2h(\beta-\alpha))^2}{2t}\right\}\right\} \mathrm{d}y.
\end{aligned} \tag{4.3}
$$

In particular, for $\alpha = -\beta$, formula (4.3) yields

$$\Pr\left\{\max_{0\leqslant s\leqslant t} |B(s)| < \beta \mid B(0) = \eta\right\} = \frac{1}{\sqrt{2\pi t}} \sum_{h=-\infty}^{+\infty} \int_{-\beta}^{\beta} \left\{\exp\left\{-\frac{(y-\eta-4h\beta)^2}{2t}\right\} - \exp\left\{-\frac{(-2\beta-y-\eta+4h\beta)^2}{2t}\right\}\right\} \mathrm{d}y. \tag{4.4}$$

In the special case where $\eta = 0$, (4.4) can be rewritten as

$$\Pr\left\{\max_{0\leqslant s\leqslant t} |B(s)| < \beta \mid B(0) = 0\right\} = \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{+\infty} (-1)^h \int_{(-\beta+2h\beta)/\sqrt{t}}^{(\beta+2h\beta)/\sqrt{t}} \mathrm{e}^{-y^2/2}\mathrm{d}y. \tag{4.5}$$

Remark 4.1. When $\alpha \to -\infty$, formula (4.2) yields the one-sided distribution

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) < \beta,\, B(t) \in \mathrm{d}y \mid B(0) = \eta\right\} = \frac{\mathrm{d}y}{\sqrt{2\pi t}} \left\{\exp\left\{-\frac{(y-\eta)^2}{2t}\right\} - \exp\left\{-\frac{(y+\eta-2\beta)^2}{2t}\right\}\right\} \tag{4.6}$$

for $\max(\eta, y) \leqslant \beta$.

From this formula, we obtain, in particular,

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta,\, B(t) \in \mathrm{d}y \mid B(0) = \eta\right\} = \frac{\mathrm{d}y}{\sqrt{2\pi t}} \exp\left\{-\frac{(y+\eta-2\beta)^2}{2t}\right\}, \quad \max(\eta, y) \leqslant \beta, \tag{4.7}$$

and, by integrating with respect to y,

$$\Pr\left\{\max_{0\leqslant s\leqslant t} B(s) \geqslant \beta \mid B(0) = \eta\right\} = \frac{2}{\sqrt{2\pi}} \int_{(\beta-\eta)/\sqrt{t}}^{+\infty} \mathrm{e}^{-y^2/2}\mathrm{d}y. \tag{4.8}$$

We now present the main result of this section:

THEOREM 4.1. *Let B be the standard Brownian motion and $u > t$. Then we have*

$$\Pr\left\{\max_{0\leqslant s\leqslant t} |B(s)| < \beta \mid B(u) = \eta\right\} = \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{+\infty} (-1)^h \exp\left\{-\frac{2h\beta(h\beta-\eta)}{u}\right\} \int_{[-\beta u-2h\beta(u-t)-\eta t]/\sqrt{ut(u-t)}}^{[\beta u-2h\beta(u-t)-\eta t]/\sqrt{ut(u-t)}} \mathrm{e}^{-y^2/2}\mathrm{d}y. \tag{4.9}$$

Proof. To derive the previous result, we use the following joint distribution which, as is well known, holds for the standard B.M. for $-\beta < z < \beta$ (cf. (4.2) for $\eta = 0$):

$$\Pr\left\{\max_{0\leqslant s\leqslant t} |B(s)| < \beta,\, B(t) \in \mathrm{d}z\right\} = \frac{\mathrm{d}z}{\sqrt{2\pi t}} \sum_{k=-\infty}^{+\infty} \left[\exp\left\{-\frac{(z-4k\beta)^2}{2t}\right\} - \exp\left\{-\frac{(-2\beta-z+4k\beta)^2}{2t}\right\}\right]. \tag{4.10}$$

Formula (4.9) is then derived through the following steps:

$$
\begin{aligned}
&\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta \mid B(u)=\eta\right\} \\
&= \sqrt{2\pi u}\, e^{\eta^2/(2u)} \int_{-\beta}^{\beta} \Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta,\, B(t)\in dz\right\} \frac{1}{\sqrt{2\pi(u-t)}} \exp\left\{-\frac{(z-\eta)^2}{2(u-t)}\right\} \\
&= \sqrt{\frac{u}{u-t}}\, e^{\eta^2/(2u)} \int_{-\beta}^{\beta} \sum_{k=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi t}} \left[\exp\left\{-\frac{(z-4k\beta)^2}{2t}\right\}\right. \\
&\quad \left. - \exp\left\{-\frac{(-2\beta-z+4k\beta)^2}{2t}\right\}\right] \exp\left\{-\frac{(z-\eta)^2}{2(u-t)}\right\} dz \\
&= \sqrt{\frac{u}{u-t}}\, e^{\eta^2/(2u)} \int_{-\beta}^{\beta} \sum_{k=-\infty}^{+\infty} (-1)^k \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(z-2k\beta)^2}{2t}\right\} \exp\left\{-\frac{(z-\eta)^2}{2(u-t)}\right\} dz.
\end{aligned}
\tag{4.11}
$$

Remark 4.2. By letting $u \to \infty$ in formula (4.9), we again obtain the classical two-sided maximal distribution for B.M. (4.5), while, for $u = t$, formula (4.9) provides the two-sided maximal distribution for the Brownian bridge (cf. Shorack and Wellner [4], p. 34):

$$
\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta \mid B(t)=\eta\right\} = \sum_{h=-\infty}^{+\infty} (-1)^h \exp\left\{-\frac{2h\beta(h\beta-\eta)}{t}\right\}. \tag{4.12}
$$

For the case where $u < t$, we have, instead, the following:

THEOREM 4.2. *For $-\beta < \eta < \beta$ and for $u < t$, we have*

$$
\begin{aligned}
\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta \mid B(u)=\eta\right\} &= \frac{1}{\sqrt{2\pi}} \sum_{h=-\infty}^{+\infty} (-1)^h \exp\left\{-\frac{2h\beta(h\beta-\eta)}{u}\right\} \\
&\times \left\{ \sum_{\substack{k=-\infty \\ k\ \text{even}}}^{+\infty} \int_{[-(1+2k)\beta-\eta]/\sqrt{t-u}}^{[(1-2k)\beta-\eta]/\sqrt{t-u}} e^{-y^2/2} dy - \sum_{\substack{k=-\infty \\ k\ \text{odd}}}^{+\infty} \int_{[-(1+2k)\beta+\eta]/\sqrt{t-u}}^{[(1-2k)\beta+\eta]/\sqrt{t-u}} e^{-y^2/2} dy \right\}.
\end{aligned}
\tag{4.13}
$$

Proof. We can rewrite (4.13) as

$$
\begin{aligned}
\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta \mid B(u)=\eta\right\} &= \Pr\left\{\max_{0\leqslant s\leqslant u}|B(s)| < \beta,\, \max_{u\leqslant s\leqslant t}|B(s)| < \beta \mid B(u)=\eta\right\} \\
&= \Pr\left\{\max_{0\leqslant s\leqslant u}|B(s)| < \beta \mid B(u)=\eta\right\} \Pr\left\{\max_{u\leqslant s\leqslant t}|B(s)| < \beta \mid B(u)=\eta\right\}.
\end{aligned}
\tag{4.14}
$$

The first term is again the two-sided maximal distribution for the Brownian bridge (4.12).

The second term can be calculated by using formula (4.4). Result (4.13) now easily follows.

When $\eta = 0$, formula (4.13) can be rewritten in the more compact form

$$\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta \mid B(u) = 0\right\} = \sum_{h=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}(-1)^{h+k}\exp\left\{-\frac{2h^2\beta^2}{u}\right\}\int_{[-(1+2k)\beta]/\sqrt{t-u}}^{[(1-2k)\beta]/\sqrt{t-u}}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy. \tag{4.15}$$

Remark 4.3. By letting $u \to 0^+$, formula (4.13) reduces to

$$\Pr\left\{\max_{0\leqslant s\leqslant t}|B(s)| < \beta \mid B(0) = \eta\right\} = \frac{1}{\sqrt{2\pi}}\left\{\sum_{\substack{k=-\infty \\ k\ \text{even}}}^{+\infty}(-1)^k\int_{[-(1+2k)\beta-\eta]/\sqrt{t}}^{[(1-2k)\beta-\eta]/\sqrt{t}}e^{-y^2/2}dy - \sum_{\substack{k=-\infty \\ k\ \text{odd}}}^{+\infty}\int_{[-(1+2k)\beta+\eta]/\sqrt{t}}^{[(1-2k)\beta+\eta]/\sqrt{t}}e^{-y^2/2}dy\right\}, \tag{4.16}$$

since, in the sum with respect to h in (4.13), only the term corresponding to $h = 0$ does not vanish in the limit.
By letting $u \to t^-$, formula (4.13) reduces to (4.12), since

$$\lim_{u\to t^-}\int_{[-(1+2k)\beta-\eta]/\sqrt{t-u}}^{[(1-2k)\beta-\eta]/\sqrt{t-u}}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \lim_{u\to t^-}\int_{[-(1+2k)\beta+\eta]/\sqrt{t-u}}^{[(1-2k)\beta+\eta]/\sqrt{t-u}}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy = \begin{cases}1, & k=0,\\ 0, & k\neq 0.\end{cases} \tag{4.17}$$

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