Autocallable Structured Products

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In this article, a general, flexible form of autocallable note is analytically valued, and its payoff profile and risk-management properties are discussed. The general autocallable structure under consideration includes the following features: regular coupons, reverse-convertible provision, down-and-out American barrier, best-of mechanism, and snowball effect. These features are more or less fully addressed according to the entailed valuation difficulties. Simpler notes are easily designed and priced on the basis of this general structure. The formulas provided in this article can be expected to be a valuable tool for both buyers and issuers in terms of risk management. Indeed, they enable investors to assess their chances of early redemption as well as their expected return on investment as a function of the contract's specifications, and they allow issuers to accurately and efficiently analyze and compute their various risk exposures.

Autocallables, also known as auto-trigger structures or kick-out plans, are very popular in the world of structured products. They have captured a large part of the market share in recent years. Product providers use them to offer higher payoffs than those on structured products that automatically run to a full term. In its standard form, an autocallable is a note that is linked to an underlying risky asset (usually a single stock, a basket of stocks, or an equity index) and that has no fixed maturity. What is referred to as the maturity of the autocallable is actually the maximum duration this product can stay alive, usually ranging from two to five years.

Several observation dates within the product's life are prespecified in the contract, typically on an annual or semiannual basis. At each observation date, if the value of the underlying is at or above a prespecified level, usually called the autocall trigger level or autocall barrier, then the principal amount is paid back by the issuer to the holder of the note, along with a coupon rate. It is said then that the note autocalls. The prespecified autocall trigger level is often defined as the level of the underlying asset at the contract's inception, but it does not have to be. It may also vary in time. If there is no early redemption, the note proceeds to the next observation date, where there is again the possibility of early redemption. A lot of plans kick out in year one or two, leaving investors with the choice to reinvest in rollover substitutes from the same provider, switch to another offer, or opt back into the markets.

Another level can be prespecified for each observation date, below the autocall trigger level, such that if the note does not autocall but the underlying is above that lower level, usually called the coupon barrier, then the note pays a coupon rate. Some autocallable notes have a memory function embedded—also called a snowball effect. With a memory function, the product will pay any coupons that have not been paid on previous observation dates, if on a subsequent observation date all prerequisites are met.

Several variants of the standard structure are traded on the markets. One way to allow for yield enhancement is to include a reverse-convertible mechanism: if the note has not been autocalled and if the underlying asset has fallen below a prespecified conversion level at maturity, which is a kind of safety threshold from the point of view of the investor, then the principal amount is not redeemed and shares (or their equivalent cash amount) are paid back instead, which entails a capital loss for the investor. This amounts to the sale of an out-of-the-money put by the investor to the issuer and is a suitable feature in the current market environment. Indeed, low interest rates leave little room for yield enhancement if investors want full capital protection, since providers must use more of the initial capital to guarantee its complete return. Moreover, in high-volatility markets, volatility has to be sold in order to generate a higher income. The answer is then to introduce capital-at-risk structures, typically losing money if the underlying index has fallen 50% or further from its initial level.

Another way to allow for yield enhancement is to make the coupon payments at a given observation date contingent on the underlying asset not having crossed a prespecified lower level since the latest observation date. This amounts to the introduction of a down-and-out American barrier (that is, a continuous barrier) in a time interval between two observation dates.

Investors may also want to be given the opportunity to participate in potential increases in the value of the underlying instead of receiving fixed coupons. More specifically, some contracts define a prespecified upper level such that if the underlying is above that level at a given observation date, then the note autocalls and the coupon paid to the holder of the note is a percentage of the spot value of the underlying. Finally, some autocallable notes include best-of (worst-of) features, which usually consist in providing investors with a percentage of the maximum (minimum) between two or more assets in case the contract allows for participation in the market's upward potential.

Despite the widespread use of autocallable products in the markets, there are not many academic studies on them. Though Bouzoubaa and Osseiran [2010] describe a variety of payoffs and analyze risk-management issues, all the other contributions focus on numerical pricing schemes. Fries and Joshi [2008] study a product-specific variance reduction scheme for Monte Carlo simulation purposes. Deng et al. [2011] discuss finite difference methods in a numerical partial differential equation framework. Kamtchueng [2011] discusses smoothing algorithms for the Monte Carlo simulation of the Greeks. Alm et al. [2013] develop a Monte Carlo algorithm that allows stability with respect to differentiation. No research article has yet come up with an analytical solution to the valuation problem raised by the autocallable structured products traded in the markets, that is, autocallables with discrete observation dates.

The main purpose of this article is to provide an analytical formula for a flexible autocallable structure with discrete observation dates that includes the following features: regular coupons, reverse-convertible provision, down-and-out American barrier, best-of mechanism, and snowball effect. The way in which all these features can be embedded into an analytical formula is more or less restrictive according to the entailed valuation difficulties: although regular coupons, the reverse-convertible provision, and the snowball effect can be fully tackled, the American barrier and best-of features are only partially covered, in the sense that the American barrier is not alive during the entire product life and the best-of mechanism is limited to two assets. To achieve an analytical solution, the structure is priced as a whole in the form of an option-valuation formula that is a function of the contract's specifications and of fixed levels of the riskless rate and of volatility. This formula can be expected to be a valuable tool both for investors and issuers in terms of risk management, since it allows them to analyze the influence of each variable and to study the global properties of the structure in an accurate and efficient manner. It also provides a sensible, fast approximation of the structure's fair price before more general numerical schemes may be tested, allowing for more stochastic factors.

In this article, we describe in detail the general autocallable structure under consideration, analyze the risk-management issues, and state the formulas (the numerical implementation of which is discussed in Appendix A). Mathematical proofs of the formulas are not reported because they are cumbersome, but they are available from the author upon request.

A FLEXIBLE AUTOCALLABLE STRUCTURE

Several observation dates $t_1, t_2, \ldots, t_n = T$ are set within the product life $[t_0 = 0, T]$, where T is the maximum duration of the product (its maturity). Two risky assets, S_1 and S_2, are picked. At each observation date t_m, $1 \le m < n$, prior to expiration, autocall may occur in two ways :

1. If the value of S_1 at time t_m, denoted by $S_1(t_m)$, lies within the range $[D_m, U_m]$, $D_m < U_m$, then the investor's initial capital or notional N is redeemed at time t_m, along with a coupon rate y_m. Both the range $[D_m, U_m]$ and the coupon rate y_m are prespecified in the contract. The autocall trigger level D_m is typically the value of the underlying at inception, that is, $S_1(t_0)$.
2. If $S_1(t_m)$ is greater than the level U_m, the note autocalls too, but instead of yielding a prespecified coupon rate, it provides the investor with a percentage λ_1 of the return ratio $S_1(t_m)/S_1(t_0)$ if this ratio is greater than the return ratio $S_2(t_m)/S_2(t_0)$ or with a percentage λ_2 of $S_2(t_m)/S_2(t_0)$ if the latter is greater than $S_1(t_m)/S_1(t_0)$. Thus, the asset S_2 was introduced in the first place to allow for a best-of mechanism in the event of early redemption with a participation rate.

Besides, if $S_1(t_m)$ lies within a range $[C_m, D_m]$, $C_m < D_m$, then the note does not autocall but a prespecified coupon rate z_m is paid out to the investor; furthermore, all prior coupons that may have been lost because the coupon barriers C_k, $1 \le k < m$, had not been crossed, are also paid out to the investor at time t_m. This amounts to embedding a memory function, or snowball effect, into the note.

Thus, there are three different barriers at each observation date t_m: the autocall trigger level without participation denoted by D_m, the autocall trigger level with participation denoted by U_m, and the coupon barrier without autocall denoted by C_m. The relative positions of these barriers are as follows: $C_m < D_m < U_m$, with D_m being defined as $S_1(t_0)$ in most traded contracts. Exhibit 1 recapitulates all possible payoffs at any observation date prior to expiration.

Let us now turn to the profile of the note at expiration or maximum duration $t_n = T$. The safety barrier, denoted by B, is the European-type barrier under which the notional N will not be redeemed at par and the reverse-convertible mechanism will be activated. For the investor to receive a final coupon Y_n, two conditions must be met: $S_1(t_n)$ must lie in the range $[D_n, U_n]$, $D_n < U_n$, and the minimum value reached by S_1 at any time between t_{n-1} and t_n must not be smaller than a prespecified American-type down-and-out barrier H. The way in which the investor may participate in the growth of assets S_1 and S_2 also differs from previous observation dates, since potential participation in the growth of S_2 is not contingent on $S_1(t_n)$ being above level U_n. Rather, a European-type up-and-in barrier W is specifically attached to S_2 and may trigger participation in the growth of S_2 even if $S_1(t_n)$ lies below U_n. The best-of mechanism is thus less restrictive at expiration than at the previous observation dates. Finally, a European-type barrier C_n determines whether the memory function, or snowball effect, can be activated.

Thus, there are six different barriers at expiration or maximum duration $t_n = T$:

EXHIBIT 1
Possible Payoffs at Each Observation Date t_m, $1 \le m < n$ Prior to Expiration

Event	Outcome
$S_1(t_m) < C_m$	$\Rightarrow$ the note proceeds to the next observation date
$D_m > S_1(t_m) \ge C_m$	$\Rightarrow$ the note pays out a prespecified, coupon $N \times Z_m$, foregoing missed coupons are recovered, and the note proceeds to the next observation date
$U_m > S_1(t_m) \ge D_m$	$\Rightarrow$ early exit with coupon: the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a final prespecified coupon $N \times (1 + y_m)$
$S_1(t_m) \ge U_m$ and $S_1(t_m) > S_2(t_m)$	$\Rightarrow$ early exit with participation in S_1: the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a final coupon equal to a prespecified percentage λ_1 of the ratio $S_1(t_m)/S_1(t_0)$
$S_1(t_m) \ge U_m$ and $S_1(t_m) < S_2(t_m)$	$\Rightarrow$ early exit with participation in S_2: the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a final coupon equal to a prespecified percentage λ_2 of the ratio $S_2(t_m)/S_2(t_0)$

Note: This exhibit shows the possible payoffs provided by the autocallable structure under consideration at an observation date prior to expiration.

1. the safety barrier, denoted by B, under which the reverse-convertible mechanism is activated
2. an exit barrier denoted by D_n triggering a final coupon payment
3. an American-type down-and-out barrier denoted by H and monitored during time interval $[t_{n-1}, t_n]$, that determines the possibility of receiving a final coupon payment but not the possibility of receiving a final payment through equity growth participation
4. an exit barrier denoted by U_n triggering a final payment contingent on the value of $S_1(t_n)$
5. an exit barrier denoted by W triggering a final payment contingent on the value of $S_2(t_n)$
6. a coupon barrier C_n triggering the memory function of the note

The following order holds: $B < C_n < D_n < U_n$. The barrier W has to stand above B, but its position can be freely chosen relative to barriers C_n, D_n, and U_n, depending on how large the influence of the best-of mechanism should be. The barrier H has to lie below D_n, but its position can be freely chosen relative to barriers B and C_n.

Exhibit 2 recapitulates all possible payoffs at expiration.

These multiple features are introduced in order to span a large variety of possible contracts. Despite its rather complex payoff function, the fair value of the general autocallable structure under consideration can be analytically obtained. Forthcoming Proposition 1 provides a formula for any number of observation dates without snowball effect, whereas Proposition 2 provides a formula for four observation dates. The reason a formula is written down in the specific case of four observation dates is threefold: First, it is useful to provide a fully explicit example of how Proposition 1 is expanded, in case the latter might look somewhat terse to the reader at first sight. Second, an autocallable note with four observation dates will serve as the basis for subsequent numerical examples. Last but not least, Proposition 2 includes the pricing of the snowball effect. The reason the latter is not included in Proposition 1 is that it seems impossible to come up with a compact formula for it in general, that is, without specifying the number of observation dates.

RISK-MANAGEMENT ISSUES

Simpler notes can be designed on the basis of the above general autocallable structure. Proposition 1 and Proposition 2 nest the valuation of all kinds of simpler structures by putting the suitable inputs into the formulas:

- notes that do not allow exit through equity participation are valued by setting the parameter A_1 equal to one and the parameters A_2 and A_3 equal to zero
- notes that allow exit through participation in asset S_1 only are valued by setting the parameter A_2 equal to one and the parameters A_1 and A_3 equal to zero
- the best-of provision is activated by setting the parameter A_3 equal to one and the parameters A_1 and A_2 equal to zero
- the snowball effect is activated by setting the parameter A_4 equal to one in Proposition 2.

All other adjustments are fairly obvious. For example, if you do not want an American barrier to condition exit with a coupon rate at expiration, just let H tend to zero; if you do not want coupon payments prior to expiration, just set all the z_i parameters equal to zero.

Investors need to be aware of the cost of benefiting from additional opportunities, compared with a standard contract. Introducing intermediary coupon payments before expiration increases the note's value in a relatively straightforward manner, as long as they are fixed coupon payments. It is less easy to anticipate precisely the effect of exit through equity participation, or the effect of a memory function, or that of a best-of provision. In Exhibit 3, the prices of four different types of contracts with a growing number of options are compared. The first one is a plain single-asset note, with no autocall through participation in the growth of S_1, and no memory function. The second one adds to the first one the possibility of autocall through participation in the growth of S_1. The third one adds to the second one a memory function. The fourth one adds to the third one a best-of provision.

In all three volatility settings of Exhibit 3, the sharpest increase in value comes from adding the possibility of autocall through participation in the growth

EXHIBIT 2
Possible Payoffs at Expiration or Maximum Duration $t_n = T$

Event	Outcome
$S_1(t_n) < B$	$\Rightarrow$ the reverse convertible mechanism is triggered: only a fraction $S_1(t_n) / S_1(t_0)$ of the investor's initial capital is redeemed, foregoing missed coupons are definitely lost, and no final coupon payment is received
$S_1(t_n) \in [B, C_n]$ and $S_2(t_n) < W$	$\Rightarrow$ the investor's initial capital is fully redeemed, but foregoing missed coupons are definitely lost and no final coupon is received
$S_1(t_n) \in [B, C_n]$ and $S_2(t_n) \geq W$	$\Rightarrow$ foregoing missed coupons are definitely lost, but the investor's initial capital is fully redeemed and the note pays out a final coupon equal to a prespecified percentage λ_2 of the ratio $S_2(t_n)/S_2(t_0)$
$S_1(t_n) \in [C_n, D_n]$ and $S_2(t_n) < W$	$\Rightarrow$ the investor's initial capital is fully redeemed, foregoing missed coupons are recovered but no final coupon payment is received
$S_1(t_n) \in [C_n, D_n]$ and $S_2(t_n) \geq W$	$\Rightarrow$ the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a final coupon equal to a prespecified percentage λ_2 of the ratio $S_2(t_n)/S_2(t_0)$
$S_1(t_n) \in [D_n, U_n]$ and $\inf_{t_n-1 \leq t \leq t_n} S_1(t) > H$	$\Rightarrow$ the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a prespecified final coupon $N \times y_n$
$S_1(t_n) \in [D_n, U_n]$ and $\inf_{t_n-1 \leq t \leq t_n} S_1(t) \leq H$	$\Rightarrow$ the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, but no final coupon is received since the American down-and-out barrier has been breached
$S_1(t_n) \geq U_n$ and $S_1(t_n) \geq S_2(t_n)$	$\Rightarrow$ the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a final coupon equal to a prespecified percentage λ_1 of the ratio $S_1(t_n)/S_1(t_0)$
$S_1(t_n) \geq U_n$ and $S_1(t_n) < S_2(t_n)$	$\Rightarrow$ the investor's initial capital is fully redeemed, foregoing missed coupons are recovered, and the note pays out a final coupon equal to a prespecified percentage λ_2 of the ratio $S_2(t_n)/S_2(t_0)$

Note: This exhibit shows the possible payoffs provided by the autocallable structure under consideration at expiration or maximum duration.

of asset S_1. This effect is more and more pronounced as volatility rises. The smallest increase in value results from adding a memory function, but it is not negligible and it would be greater if the coupon barriers were higher. The impact of the best-of provision is substantial, especially in the high-volatility setting. Overall, the introduction of nonstandard features thus increases the value of the autocallable note to a large extent: averaging across all three volatility settings, the price of contract 4 is 23.25% higher than that of contract 1. There is a major difference between contracts 1 and 2 on the one hand, and contracts 3 and 4, on the other hand, as regards the effect of volatility: the value of the former is a decreasing function of volatility, whereas the value of the latter grows with volatility. We will come back in more detail to the issue of price sensitivity later in this section.

Whatever the number of options available in an autocallable contract, the issue of the expected autocall date is fundamental. Indeed, potential automatic early

Exhibit 3

Prices of Various Types of Four-Year Maturity Autocallable Notes

	Low Volatility	Medium Volatility	High Volatility
	$\sigma_1 = 12\%, \sigma_2 = 15\%$	$\sigma_1 = 28\%, \sigma_2 = 35\%$	$\sigma_1 = 45\%, \sigma_2 = 60\%$
Value of contract 1	102.138322	90.7335053	80.5106173
Value of contract 2	104.916379	102.215849	101.203114
Value of contract 3	106.44384	108.306387	113.688814
Value of contract 4	107.733706	109.940314	115.210895

Notes: This exhibit compares the values of four types of autocallable notes with maximum expiration four years and annual observation dates. Contract 1 is a single-asset note linked to S_1, with no early exit through participation, and no memory function. Contract 2 includes the possibility of early exit through participation in the growth of S_1. Contract 3 includes a memory function. Contract 4 includes a best-of provision. All reported prices are obtained using forthcoming Proposition 2 with the following inputs:

$S_1(0) = S_2(0) = 100$, $r = 2.5\%$, $\delta_1 = \delta_1 = 0$, $\rho = 35\%$, $D_1 = D_2 = D_3 = D_4 = 100$, $U_1 = U_2 = U_3 = U_4 = 115$

$C_1 = C_2 = C_3 = C_4 = 90$, $B = 75$, $W = 100$, $H = 80$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 8\%$, $z_1 = z_2 = z_3 = z_4 = 5\%$, $\lambda_1 = \lambda_2 = 1$

Contract 1 is valued by setting $A_1 = 1$ and $A_2 = A_3 = A_4 = 0$. Contract 2 is valued by setting $A_2 = 1$ and $A_1 = A_3 = A_4 = 0$. Contract 3 is valued by setting $A_2 = A_4 = 1$, $A_1 = A_3 = 0$. Contract 4 is valued by setting $A_1 = A_2 = 0$, $A_3 = A_4 = 1$.

redemption is the specific property of these products, relative to other index-linked notes, and most investors who choose to put their money in autocallable structures hope to get their capital back at an early stage. The invested notional can be redeemed as soon as on the first observation date, t_1. In this respect, moderately bullish investors would be well advised to set a high value for γ_1, the coupon rate offered upon exit without participation in either S_1 or S_2 at time t_1. They will thus be in a position to benefit from higher returns than if they invest in standard fixed income. Moreover, strongly bullish investors had better trade off a high value of γ_1 for high levels of participation in the growth of assets S_1 and S_2. They will thus be able to benefit from an amplification of upside market movements, relative to a long position in the stocks themselves, while being protected from downside risk.

Overall, in terms of contract specifications, the chances to benefit from autocall as soon as on the first or maybe the second observation date depend mostly on the choice of the autocall trigger level. Investors who contemplate lowering that level must be aware that this will cause the cost of the note to rise. For a given price, this means that the coupon rates and the participation rates will be lower. The autocall trigger levels may increase or decrease during the note's life. The former case is often referred to in the markets as a step-up autocall, and the latter is called a step-down autocall. When autocall levels vary, they usually do so in a step-down pattern, so that the investor is more and more likely to kick out as time passes.

When dealing with notes that provide possible early exit with participation in the growth of risky assets, it is obvious that lowering the U_i 's will increase the likelihood of early redemption with participation, which may be called the probability effect. But investors should be aware that this positive probability effect may be offset by the lower expected return received when the note autocalls, which may be called the return effect, so that the overall effect on the value of the note is ambiguous. Roughly speaking, the higher the volatility, the more the return effect tends to prevail over the probability effect, as the likelihood of reaching relatively high U_i's increases.

Exhibit 4 reports the maturity breakdown on a four-year autocallable note including equity participation and a best-of provision, as a function of volatility; for simplicity, volatility is supposed to be flat, that is, the skew and the term structure are not taken into account. Exhibit 4 shows a couple of noticeable results. First, the most likely outcome is to autocall on the first observation date, whether volatility is low or high. The probability of autocall occurring as early as on the first or the second observation date increases with volatility. This increase is not linear; it accelerates when volatility is shifted from the "medium volatility" category to the "high volatility" one. When volatility is low, the chances of autocall occurring at the fourth and final observation date are quite substantial, but they go down as volatility gets higher. Again, this decrease is not linear; it considerably accelerates as volatility moves to the "high volatility" regime, at a faster pace than the observed increase in the probability of autocall at the first observation date. Interestingly, autocall at the third observation date is always the least likely outcome, whatever the volatility input, and its probability of occurrence changes more or less linearly, unlike the probability of autocall at the first or at the last observation date.

For risk-management purposes, the probabilities of autocall need to be computed under the physical measure. In Exhibit 5, the risky assets S_1 and S_2 include risk

Exhibit 4

Maturity Breakdown of a Four-Year Autocallable Note as a Function of Volatility under Risk Neutrality

	Low Volatility	Medium Volatility	High Volatility
	$\sigma_1 = 12\%, \sigma_2 = 15\%$	$\sigma_1 = 28\%, \sigma_2 = 35\%$	$\sigma_1 = 45\%, \sigma_2 = 60\%$
Maturity = 1 year	0.59292711	0.60704343	0.66483994
Maturity = 2 years	0.14079933	0.14795149	0.16333123
Maturity = 3 years	0.06899081	0.0738965	0.08193993
Maturity = 4 years	0.19728274	0.17110857	0.0898889

Notes: This exhibit reports the risk-neutral probabilities that the actual maturity on a four-year autocallable note providing the payoff described in Exhibit 1 and Exhibit 2 will be either one, two, three, or four years. For simplicity, the volatilities of S_1 and S_2 are assumed to be flat. The reported probabilities are obtained using forthcoming Proposition 2 with the following inputs: $A_1 = A_2 = 0$, $A_3 = 1$, $S_1(0) = S_2(0) = 100$, $r = 2.5\%$, $\delta_1 = \delta_2 = 0$, $\rho = 35\%$, $D_1 = D_2 = D_3 = D_4 = 100$, $U_1 = U_2 = U_3 = U_4 = 115$. Other inputs to Proposition 2 have no relevance here. The probability that the actual maturity is one year is obtained by adding up $\mathbb{P}_3 + \mathbb{P}_5 + \mathbb{P}_6$. The probability that the actual maturity is two years is obtained by adding up $\mathbb{P}_9 + \mathbb{P}_{11} + \mathbb{P}_{12}$. The probability that the actual maturity is three years is obtained by adding up $\mathbb{P}_{16} + \mathbb{P}_{18} + \mathbb{P}_{19}$. The probability that the actual maturity is four years is equal to one minus the probability that the actual maturity is one, two, or three years.

Exhibit 5

Maturity Breakdown of a Four-Year Autocallable Note as a Function of Volatility under Risk Aversion

	Low Volatility	Medium Volatility	High Volatility
	$\sigma_1 = 12\%, \sigma_2 = 15\%$	$\sigma_1 = 28\%, \sigma_2 = 35\%$	$\sigma_1 = 45\%, \sigma_2 = 60\%$
Maturity = 1 year	0.663790051	0.641161764	0.687729344
Maturity = 2 years	0.147111224	0.153577508	0.167175316
Maturity = 3 years	0.068736885	0.076020303	0.083416916
Maturity = 4 years	0.120361840	0.129240425	0.061678424

Notes: This table reports the probabilities that the actual maturity on a four-year autocallable note providing the payoff described in Exhibits 1 and 2 will be either one, two, three, or four years. The difference from Exhibit 4 is that the returns on assets S_1 and S_2 now include risk premiums of 2% and 3%, respectively.

premiums. Overall, the structure of the maturity breakdown remains the same as under the risk-neutral measure. There is little difference between the probabilities of early exit in year two and in year three in Exhibits 4 and 5. However, under the risk-averse, physical measure, the likelihood that the note will extend to its maximum maturity of four years is even lower, whereas the probability that autocall will occur as early as in year one is even higher. Under the high-volatility regime, exit at maximum expiration becomes the least likely outcome in Exhibit 5. Interestingly, the probability of autocall in year one does not increase monotonically with volatility, since it is lower in the medium-volatility regime than in the low-volatility one. This observation points out the need to accurately compute the sensitivity of the probability of early exit at any observation date with respect to volatility. This is easily done by mere differentiation, thanks to the formulas provided in this article.

Finally, two other features should be borne in mind by investors. First, it must be noticed that the best-of provision plays a more important role at expiration t_n than at previous observation dates, since it is not contingent on S_1 being above the U_n barrier. Indeed, as long as $S_2(t_n)$ trades above the independent barrier W, the best-of provision can be activated even if $S_1(t_n)$ trades at levels below D_n, unless the reverse-convertible mechanism is activated (that is, the reverse-convertible provision dominates the best-of one). One should also pay attention to the fact that the main role of the American down-and-out barrier is to determine whether a final coupon payment is received in case the note does not autocall before expiration, so that this barrier will not have a significant impact on the value of the note unless a sufficiently high value for γ_n is agreed on in the contract, which makes sense as a compensation to the investor when early redemption has not taken place.

Now, from the perspective of traders who sell these products, the risks associated with the various components of the autocallable structure are very similar to those associated with digitals. These components have positive delta, and sellers of the option will have to buy delta of the underlying, meaning that they will be long dividends, short interest rates, and long borrow costs of the underlying. To tackle the discontinuity risk around the coupon barriers, a barrier shift will usually be applied so that the resulting shifted payoff over-replicates the initial payoff by the least amount, making the Greeks of the new payoff manageable near the barrier. The position in volatility depends on the coupon barrier levels and on the forward price of the underlying asset S_1. The vega of the autocallable's components is positive if the underlying's forward price is lower than the trigger level; otherwise, vega is negative. Hence, the vega of the general autocallable structure under consideration in this article will depend on the respective weights of the downside and upside components within the overall structure.

Typically, with the D_i's equal to the starting spot price $S_1(0)$ and the C_i's being smaller than the D_i's, the components of the autocallable structure providing fixed coupon payments will have a negative vega. The reverse holds for the components that provide participating payments. In view of the potentially offsetting effects from the various components, one should always check whether the vega of the overall structure is positive or negative. Once endowed with the formulas provided in this article, this becomes an easy task using analytical or numerical differentiation.

One of the difficulties associated with hedging the sale of an autocallable note is that this product is sensitive to the skew. Traders can hedge the components providing fixed payments by implementing a simple roll-over strategy as follows: at each observation date t_i, if there is no autocall and the note proceeds to the next observation date t_{i+1}, they will take a long position in call spreads expiring at time t_{i+1}. These call spreads will be more or less tight or wide depending on how conservative the trader is, bearing in mind that the tighter the call spread, the larger gamma can become near the coupon barrier. The catch is that the moneyness of the involved future call options cannot be known at inception. As a result, the forward implied volatility inputs that might be used to price the structure at time t_0 will generally be wrong. Moreover, all components with expiration greater than t_1 are conditional on not having autocalled at a previous observation date, which introduces a path-dependent element in the valuation problem.

In this respect, the usefulness of the formulas provided in this article is twofold: First, they are a function of the skew at time t_0, which, by definition, is observable, that is, they do not involve pricing a combination of forward start digitals that would inevitably require the input of the forward implied volatility surface. Second, by giving the risk-neutral conditional probabilities of autocalling at each observation date, these formulas provide traders with a simple way to accurately weight the call spreads put in place for each maturity.

Exhibit 6 compares the prices of four-year maturity autocallable notes with and without the possibility of exit through participation under three stylized regimes of skew. For the sake of simplicity, the at-the-money volatility remains flat until maximum duration. In line with most observed data on equity, the slope of the upper part of the skew curve (that is, the part to the left of the at-the-money level) is greater than the slope of the lower part of the skew curve (that is, the part to the right of the at-the-money level); besides, the "small" skew decreases more or less linearly with time, whereas the decrease in the "large" skew with respect to time displays more curvature. Since the downside components of the autocallable structure have negative vega and the upside components have positive vega, the effect of the skew is to lower the value of the autocallable structure, since the skew increases the volatility of downside components and decreases the volatility of the upside components with respect to the at-the-money line. The effect of the skew on nonparticipating autocallable notes is moderate but nonnegligible and would obviously be bigger under more extreme skew curves than those chosen here. The effect of the skew on participating autocallable notes is of great magnitude, so that pricing the autocallable note using flat at-the-money volatility will largely overestimate the product's value.

EXHIBIT 6
Prices of Two Different Kinds of Autocallable Notes under Different Regimes of Skew

	No Skew	Small Skew	Large Skew
Contract 1: no possibility of exit through participation	87.4125025	86.76425608	85.425556
Contract 2: possibility of exit through participation, including a best-of provision	111.102677	108.869329	105.180181

Notes: This exhibit compares the prices of four-year maturity autocallable notes with and without the possibility of early exit through participation under three stylized regimes of skew. The reported prices are computed using Proposition 2. The "small skew" and "large skew" regimes are defined by the implied volatility surface displayed in Exhibit 7. The "no skew" regime assumes an identically flat volatility at $\sigma_1 = 36\%$ and $\sigma_2 = 42\%$. The term-structure of the risk-free rate is given by: one-year at 2.5%, two-year at 2.75%, three-year at 3%, and four-year at 3.25%. The contract specifications are as follows: $S_1(0) = S_2 = (0) = 100$, $\rho = 35\%$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 4$, $C_1 = C_2 = C_3 = C_4 = 90$, $D_1 = D_2 = D_3 = D_4 = 100$, $U_1 = U_2 = U_3 = U_4 = 115$, $B = 75$, $W = 100$, $H = 80$ $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 8\%$, $z_1 = z_2 = z_3 = 5\%$, $A_1 = A_4 = 1$, $A_2 = A_3 = 0$ for contract 1, $A_1 = A_2 = 0$, $A_3 = A_4 = 1$ for contract 2.

Exhibit 7

Implied Volatility Surface Used to Compute the Prices in Exhibit 6

Coordinates	Small Skew S_1 – Volatility Surface	Small Skew S_2 – Volatility Surface	Large Skew S_1 – Volatility Surface	Large Skew S_2 – Volatility Surface
(C_1, t_1)	0.38		0.41	
(C_2, t_2)	0.375		0.40	
(C_3, t_3)	0.37		0.39	
(C_4, t_4)	0.365		0.38	
(D_1, t_1)	0.36		0.36	
(D_2, t_2)	0.36		0.36	
(D_3, t_3)	0.36		0.36	
(D_4, t_4)	0.36		0.36	
(U_1, t_1)	0.35	0.39	0.32	0.38
(U_2, t_2)	0.35	0.39	0.33	0.39
(U_3, t_3)	0.355	0.395	0.34	0.40
(U_4, t_4)	0.355	0.395	0.35	0.41
(B, t_4)	0.40		0.48	
(W, t_4)		0.42		0.42

Notes: This exhibit reports the implied volatility inputs used to define the small skew and the large skew in Exhibit 6. Only the points that are needed to compute the autocallable note's value are reported.

It must be noticed that, for some components of the autocallable structure, it is not obvious to know which point of the implied volatility surface to use, due to the path-dependent nature of the product and to the many different coupon and early-redemption barriers. The rule followed in this article is, for every component involved, to pick the point in the implied volatility surface that matches the last barrier and the observation date in the digital; for example, the digital providing a coupon equal to γ_3 upon early exit at time t_3 when $S_1(t_3) \in [D_3, U_3]$, conditional on autocall not having occurred prior to t_3, is priced using the point with coordinates (D_3, t_3) in the S_1–volatility surface.

For implied volatility curves that "smile," that is, that display higher implied volatilities to the left but also to the right of the at-the-money level, the previous analysis does not hold anymore. Indeed, the higher volatility on the out-of-the-money components of the autocallable structure raises their prices since they are vega positive, so that the net effect of the smile is ambiguous and depends on the weight of the upside digitals with respect to the downside digitals, "weight" meaning their respective contribution to the total note's value. Clearly, notes without the possibility to autocall through upward market participation are skew negative, whether the implied volatility curve is skew- or smile-shaped.

Finally, the impact of the correlation between equity and interest rate should be briefly discussed. Indeed, when the stock market goes up, the duration of the autocallable structure goes down. If there is a positive correlation between equity and interest rate, sellers of the note make losses while hedging their interest rate exposure, whether equity increases (since they have to sell longer-dated zero-coupon bonds and buy more short-term zero-coupon bonds under higher interest rates) or decreases (since they need to sell short-term bonds and buy more longer-dated bonds under lower interest rates). Conversely, if there is a negative correlation between equity and interest rate, sellers of the notes make a net profit while hedging their interest rate exposure, whether equity goes up and down because of the opposite directions of equity and interest rate.

As a consequence, pricing models that do not take the correlation between equity and interest rate into account will underprice the structure when this correlation is positive and overprice it when that correlation is negative. However, as emphasized by De Weert [2008], that correlation is notoriously hard to measure. Moreover, for short-term maturities, it is near zero. Over very long periods, equity and interest rate have negative interest rate. But over a two-year period, equity and interest rate can very well be positively correlated. That is why a number of traders may prefer to price the autocallable using a simpler model and increase or lower the coupon depending on their view on the interest/equity correlation during the term of the note.

VALUATION FORMULAS

This section includes the two formulas denoted by Proposition 1 and Proposition 2.

Proposition 1

Let $S_1(t)$ and $S_2(t)$ be two geometric Brownian motions driven by standard Brownian motions $B_1(t)$ and $B_2(t)$ with correlation coefficient ρ. Under the risk-neutral measure denoted by $\mathbb{P}^{(RN)}$, the dynamics of $S_1(t)$ and $S_2(t)$ are given by

$$dS_1(t) = \left(r - \delta_1\right) S_1(t) + \sigma_1 S_1(t) dB_1(t) \tag{1}$$

$$dS_2(t) = (r - \delta_2) S_2(t) + \sigma_2 S_2(t) dB_2(t) \tag{2}$$

where r is the risk-free rate, δ_1 and δ_2 are two constant continuous dividend rates, and $\sigma_1 > 0, \sigma_2 > 0$ are two volatility inputs assumed to be extracted from an implied volatility surface across a continuous range of strikes and maturities for both assets S_1 and S_2.

Let the function $\Phi_n(\alpha_1, \alpha_2, ..., \alpha_n; \beta_1, \beta_2, ..., \beta_{n-1})$, $\alpha_i \in \mathbb{R}, \beta_i \in [-1, 1], i \in \mathbb{N}$, be defined as follows: for $n = 1$ and $n = 2$, Φ_n is the standard Gaussian cumulative distribution function; for $n > 2$, Φ_n is given by the following special case of the multivariate standard Gaussian cumulative distribution function:

$$\Phi_n = \int_{x_1=-\infty}^{\alpha_1} \int_{x_2=-\infty}^{\alpha_2} \cdots \int_{x_n=-\infty}^{\alpha_n} \frac{\exp\left(-\frac{x_1^2}{2} - \sum_{i=1}^{n-1} \frac{(x_{i+1} - \beta_i x_i)^2}{2(1-\beta_i^2)}\right)}{(2\pi)^{n/2} \prod_{i=1}^{n-1} \sqrt{(1-\beta_i^2)}} \quad dx_1 dx_2 \ldots dx_n \tag{3}$$

The numerical computation of the function Φ_n is explained in Appendix A.

Let the following notations hold:

$$d_i = \ln\left(\frac{D_i}{S_1(0)}\right), \quad u_i = \ln\left(\frac{U_i}{S_1(0)}\right), \quad c_i = \ln\left(\frac{C_i}{S_1(0)}\right), \quad i \in \{1, ..., n\} \tag{4}$$

$$b = \ln\left(\frac{B}{S_1(0)}\right), \quad h = \ln\left(\frac{H}{S_2(0)}\right), \quad w = \ln\left(\frac{W}{S_2(0)}\right), \quad x_{12} = \ln\left(\frac{S_1(0)}{S_2(0)}\right) \tag{5}$$

$$\tilde{\mu}_1 = r - \delta_1 - \sigma_1^2/2, \quad \bar{\mu}_1 = r - \delta_1 + \sigma_1^2/2, \quad \bar{\mu}_{12} = -(\delta_1 - \delta_2) + (\sigma_1^2 + \sigma_2^2)/2 - \rho\sigma_1\sigma_2 \tag{6}$$

$$\sigma_{12} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \quad \theta = \frac{\sigma_1 - \rho\sigma_2}{\sigma_{12}} \tag{7}$$

$$\hat{\mu}_{12} = (\delta_2 - \delta_1) - (\sigma_1^2 + \sigma_2^2)/2 + \rho\sigma_1\sigma_2, \quad \hat{\mu}_1 = r - \delta_1 + \rho\sigma_1\sigma_2 - \sigma_1^2/2, \quad \hat{\mu}_2 = r - \delta_2 + \sigma_2^2/2 \tag{8}$$

$\mathbb{I}_{\{.\}}$ is the indicator function, taking value 1 if the argument inside the braces is true and value zero otherwise

A_1 is a parameter that takes value 1 if the autocall note does not allow exit through participation in equity growth, and zero otherwise

A_2 is a parameter that takes value 1 if the autocall note allows exit through participation in asset S_1 and S_1 only, and zero otherwise

A_3 is a parameter that takes value 1 if the autocall note allows exit through participation in asset S_1 or in asset S_2, that is, if it includes a best-of provision, and zero otherwise

the symbol $\langle a_i; i = j \ldots k\rangle$ denotes the sequence of real numbers a_i, where i ranges from j to k

Then, the no-arbitrage value, $V_{AUTOCALL}$, of an autocallable note with n observation dates until maximum expiration t_n, $n \in \mathbb{N}$, including all the features defined in Exhibits 1 and 2 *except* for the snowball effect, that is, except for the possibility of recovering foregoing missed coupons at any observation date, is given by the following formula:

$$
\begin{aligned}
V_{AUTOCALL} = \sum_{i=1}^{n-1} & \Big\{\exp(-r \times t_i) \times N \times z_i \times \mathbb{P}_1 + \exp(-r \times t_i) \times N \times (1+\gamma_i) \times \big(\mathrm{A}_1 \times \mathbb{P}_2 + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_3\big) \\
& + \mathrm{A}_2 \times \exp(-\delta_1 \times t_i) \times \lambda_1 \times N \times \mathbb{P}_4 + \mathrm{A}_3 \times \exp(-\delta_1 \times t_i) \times \lambda_1 \times N \times \mathbb{P}_5 + \mathrm{A}_3 \times \exp(-\delta_2 \times t_i) \times \lambda_2 \times N \times \mathbb{P}_6\Big\} \\
& + N \times \mathbb{P}_7 + (\mathrm{A}_1 + \mathrm{A}_2) \times \exp(-r \times t_n) \times N \times \mathbb{P}_8 + \mathrm{A}_3 \times \exp(-r \times t_n) \times N \times \mathbb{P}_9 + \mathrm{A}_3 \times \exp(-\delta_2 \times t_n) \times \lambda_2 \times N \times \mathbb{P}_{10} \\
& + \exp(-r \times t_n) \times N \times (1+\gamma_n) \times \big(\mathrm{A}_1 \times \mathbb{P}_{11} + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_{12}\big) + \exp(-r \times t_n) \times N \times \big(\mathrm{A}_1 \times \mathbb{P}_{13} + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_{14}\big) \\
& + \mathrm{A}_2 \times \exp(-\delta_1 \times t_n) \times \lambda_1 \times N \times \mathbb{P}_{15} + \mathrm{A}_3 \times \exp(-\delta_1 \times t_n) \times \lambda_1 \times N \times \mathbb{P}_{16} + \mathrm{A}_3 \times \exp(-\delta_2 \times t_n) \times \lambda_2 \times N \times \mathbb{P}_{17}
\end{aligned} \tag{9}
$$

where

$\mathbb{P}_1$ is the probability, under the money market numeraire, that the coupon $\mathrm{N} \times z_i$ is paid out to the investor at time t_i. It is given by

$$
\mathbb{P}_1 = \Phi_i \begin{bmatrix} \left\langle \dfrac{d_j - \tilde{\mu}_1 t_j}{\sigma_1 \sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \dfrac{-c_i + \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; \\ \left\langle \sqrt{\dfrac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}}, -\sqrt{\dfrac{t_{i-1}}{t_i}}\, \mathbb{I}_{\{i>1\}} \end{bmatrix} - \Phi_i \begin{bmatrix} \left\langle \dfrac{d_j - \tilde{\mu}_1 t_j}{\sigma_1 \sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \dfrac{-d_i + \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; \\ \left\langle \sqrt{\dfrac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}}, -\sqrt{\dfrac{t_{i-1}}{t_i}}\, \mathbb{I}_{\{i>1\}} \end{bmatrix} \tag{10}
$$

$\mathbb{P}_2$ is the probability, under the money market numeraire, of autocall at time t_i when the note does not allow exit through equity participation. It is given by

$$
\mathbb{P}_2 = \Phi_i \begin{bmatrix} \left\langle \dfrac{d_j - \tilde{\mu}_1 t_j}{\sigma_1 \sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \dfrac{-d_i + \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; \\ \left\langle \sqrt{\dfrac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}}, -\sqrt{\dfrac{t_{i-1}}{t_i}}\, \mathbb{I}_{\{i>1\}} \end{bmatrix} \tag{11}
$$

$\mathbb{P}_3$ is the probability, under the money market numeraire, of autocall at time t_i without equity participation when the note allows equity participation. It is given by

$$
\mathbb{P}_3 = \Phi_i \begin{bmatrix} \left\langle \dfrac{d_j - \tilde{\mu}_1 t_j}{\sigma_1 \sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \dfrac{-d_i + \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; \\ \left\langle \sqrt{\dfrac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}},\ -\sqrt{\dfrac{t_{i-1}}{t_i}}\, \mathbb{I}_{\{i>1\}} \end{bmatrix} - \Phi_i \begin{bmatrix} \left\langle \dfrac{d_j - \tilde{\mu}_1 t_j}{\sigma_1 \sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \dfrac{-u_i + \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; \\ \left\langle \sqrt{\dfrac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}},\ -\sqrt{\dfrac{t_{i-1}}{t_i}}\, \mathbb{I}_{\{i>1\}} \end{bmatrix} \tag{12}
$$

$\mathbb{P}_4$ is the probability, under the S_1 numeraire, of autocall at time t_i with participation in the growth of S_1 in the absence of a best-of provision. It is given by

$$
\mathbb{P}_4 = \Phi_i \begin{bmatrix} \left\langle \dfrac{d_j - \bar{\mu}_1 t_j}{\sigma_1 \sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \dfrac{-u_i + \bar{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; \\ \left\langle \sqrt{\dfrac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}},\ -\sqrt{\dfrac{t_{i-1}}{t_i}}\, \mathbb{I}_{\{i>1\}} \end{bmatrix} \tag{13}
$$

$\mathbb{P}_5$ is the probability, under the S_1 numeraire, of autocall at time t_i with participation in the growth of S_1 when the contract includes a best-of provision. It is given by

$$\mathbb{P}_5 = \Phi_{i+1}\left[\begin{array}{l}\left\langle \frac{d_j - \bar{\mu}_1 t_j}{\sigma_1\sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \frac{-u_i + \bar{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}, \ \frac{-x_{12} + \bar{\mu}_{12} t_i}{\sigma_{12}\sqrt{t_i}}; \\ \left\langle \sqrt{\frac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}}, -\sqrt{\frac{t_{i-1}}{t_i}}\,\mathbb{I}_{\{i>1\}}, \theta\end{array}\right] \tag{14}$$

$\mathbb{P}_6$ is the probability, under the S_2 numeraire, of autocall at time t_i with participation in the growth of S_2 when the contract includes a best-of provision. It is given by

$$\mathbb{P}_6 = \Phi_{i+1}\left[\begin{array}{l}\left\langle \frac{d_j - \hat{\mu}_1 t_j}{\sigma_1\sqrt{t_j}}; j = 1, \ldots, i-1 \right\rangle \mathbb{I}_{\{i>1\}}, \frac{-u_i + \hat{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}, \frac{x_{12} - \hat{\mu}_{12} t_i}{\sigma_{12}\sqrt{t_i}}; \\ \left\langle \sqrt{\frac{t_j}{t_{j+1}}}; j = 1, \ldots, i-2 \right\rangle \mathbb{I}_{\{i>2\}}, -\sqrt{\frac{t_{i-1}}{t_i}}\,\mathbb{I}_{\{i>1\}}, -\theta\end{array}\right] \tag{15}$$

$\mathbb{P}_7$ is the probability, under the S_1 numeraire, of exit at maximum expiration t_n under the reverse-convertible provision. It is given by

$$\mathbb{P}_7 = \Phi_n\left[\left\langle \frac{d_i - \bar{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-1 \right\rangle, \quad \frac{b - \bar{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}; \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-1 \right\rangle\right] \tag{16}$$

$\mathbb{P}_8$ is the probability, under the money market numeraire, of exit at maximum expiration t_n without a final coupon because the autocall barrier has not been reached, in the absence of a best-of provision. It is given by

$$\begin{aligned}\mathbb{P}_8 = {} & \Phi_n\left[\begin{array}{l}\left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-b + \tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right] \\ & - \Phi_n\left[\begin{array}{l}\left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-d_n + \tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]\end{aligned} \tag{17}$$

$\mathbb{P}_9$ is the probability, under the money market numeraire, of exit at maximum expiration t_n without a final coupon because the autocall barrier has not been reached, when the contract includes a best-of provision. It is given by

$$\mathbb{P}_9 = \Phi_{n+1}\left[\begin{array}{l}\left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \quad \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-b + \tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}, \frac{w - \tilde{\mu}_2 t_n}{\sigma_2\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \quad -\sqrt{\frac{t_{n-1}}{t_n}}, -\rho \end{array}\right]$$
$$- \Phi_{n+1}\left[\begin{array}{l}\left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \quad \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-d_n + \tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}, \frac{w - \tilde{\mu}_2 t_n}{\sigma_2\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \quad -\sqrt{\frac{t_{n-1}}{t_n}}, -\rho \end{array}\right] \tag{18}$$

$\mathbb{P}_{10}$ is the probability, under the S_2 numeraire, of exit at maximum expiration t_n with participation in the growth of S_2, when the contract includes a best-of provision and when S_1 is below the participation barrier U_n at time t_n. It is given by

$$\mathbb{P}_{10} = \Phi_{n+1}\left[\begin{array}{l}\left\langle \frac{d_i - \hat{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \hat{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-b + \hat{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}, \frac{-w + \hat{\mu}_2 t_n}{\sigma_2\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}}, \rho \end{array}\right]$$
$$- \Phi_{n+1}\left[\begin{array}{l}\left\langle \frac{d_i - \hat{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \hat{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-d_n + \hat{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}, \frac{-w + \hat{\mu}_2 t_n}{\sigma_2\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}}, \rho \end{array}\right] \tag{19}$$

$\mathbb{P}_{11}$ is the probability, under the money market numeraire, of exit at maximum expiration t_n with a final coupon $N \times \gamma_n$ when the note does not allow exit through equity participation. It is given by

$$\mathbb{P}_{11} = \left\{ \Phi_n\left[\begin{array}{l}\left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{-h + \tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-d_n + \tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-3 \right\rangle \mathbb{I}_{\{n>3\}}, -\sqrt{\frac{t_{n-2}}{t_{n-1}}}\, \mathbb{I}_{\{n>2\}}, \sqrt{\frac{t_{n-1}}{t_n}} \end{array}\right]\right.$$
$$- e^{\frac{2\tilde{\mu}_1}{\sigma_1^2}h} \Phi_n\left[\begin{array}{l}\left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{-h - \tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-d_n + 2h + \tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-3 \right\rangle \mathbb{I}_{\{n>3\}}, -\sqrt{\frac{t_{n-2}}{t_{n-1}}}\, \mathbb{I}_{\{n>2\}}, \sqrt{\frac{t_{n-1}}{t_n}} \end{array}\right] \tag{20a}$$

$$
-\Phi_n\left[\begin{array}{l}\left\langle\frac{d_i-\tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}};i=1,\ldots,n-2\right\rangle\mathbb{I}_{\{n>2\}},\frac{-d_{n-1}+\tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}},\frac{-d_n+\tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}};\\ \left\langle\sqrt{\frac{t_i}{t_{i+1}}};i=1,\ldots,n-3\right\rangle\mathbb{I}_{\{n>3\}},-\sqrt{\frac{t_{n-2}}{t_{n-1}}}\,\mathbb{I}_{\{n>2\}},\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]
$$

$$
\left.+e^{\frac{2\tilde{\mu}_1}{\sigma_1^2}h}\Phi_n\left[\begin{array}{l}\left\langle\frac{d_i-\tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}};i=1,\ldots,n-2\right\rangle\mathbb{I}_{\{n>2\}},\frac{-d_{n-1}-\tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}},\frac{-d_n+2h+\tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}\\ \left\langle\sqrt{\frac{t_i}{t_{i+1}}};i=1,\ldots,n-3\right\rangle\mathbb{I}_{\{n>3\}},-\sqrt{\frac{t_{n-2}}{t_{n-1}}}\,\mathbb{I}_{\{n>2\}},-\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]\right\} \tag{20b}
$$

$\mathbb{P}_{12}$ is the probability, under the money market numeraire, of exit at maximum expiration t_n with a final coupon $N\times\gamma_n$ when the note allows equity participation. It is given by

$$
\begin{aligned}
\mathbb{P}_{12}=\mathbb{P}_{11}&-\Phi_n\left[\begin{array}{l}\left\langle\frac{d_i-\tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}};i=1,\ldots,n-2\right\rangle\mathbb{I}_{\{n>2\}},\frac{-h+\tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}},\frac{-u_n+\tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}};\\ \left\langle\sqrt{\frac{t_i}{t_{i+1}}};i=1,\ldots,n-3\right\rangle\mathbb{I}_{\{n>3\}},-\sqrt{\frac{t_{n-2}}{t_{n-1}}}\,\mathbb{I}_{\{n>2\}},\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]\\
&+e^{\frac{2\tilde{\mu}_1}{\sigma_1^2}h}\Phi_n\left[\begin{array}{l}\left\langle\frac{d_i-\tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}};i=1,\ldots,n-2\right\rangle\mathbb{I}_{\{n>2\}},\frac{-h-\tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}},\frac{-u_n+2h+\tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}};\\ \left\langle\sqrt{\frac{t_i}{t_{i+1}}};i=1,\ldots,n-3\right\rangle\mathbb{I}_{\{n>3\}},-\sqrt{\frac{t_{n-2}}{t_{n-1}}}\,\mathbb{I}_{\{n>2\}},-\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]\\
&+\Phi_n\left[\begin{array}{l}\left\langle\frac{d_i-\tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}};i=1,\ldots,n-2\right\rangle\mathbb{I}_{\{n>2\}},\frac{-d_{n-1}+\tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}},\frac{-u_n+\tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}};\\ \left\langle\sqrt{\frac{t_i}{t_{i+1}}};i=1,\ldots,n-3\right\rangle\mathbb{I}_{\{n>3\}},-\sqrt{\frac{t_{n-2}}{t_{n-1}}}\,\mathbb{I}_{\{n>2\}},\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]\\
&-e^{\frac{2\tilde{\mu}_1}{\sigma_1^2}h}\Phi_n\left[\begin{array}{l}\left\langle\frac{d_i-\tilde{\mu}_1 t_i}{\sigma_1\sqrt{t_i}};i=1,\ldots,n-2\right\rangle\mathbb{I}_{\{n>2\}},\frac{-d_{n-1}-\tilde{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}},\frac{-u_n+2h+\tilde{\mu}_1 t_n}{\sigma_1\sqrt{t_n}};\\ \left\langle\sqrt{\frac{t_i}{t_{i+1}}};i=1,\ldots,n-3\right\rangle\mathbb{I}_{\{n>3\}},-\sqrt{\frac{t_{n-2}}{t_{n-1}}}\,\mathbb{I}_{\{n>2\}},-\sqrt{\frac{t_{n-1}}{t_n}}\end{array}\right]
\end{aligned} \tag{21}
$$

$\mathbb{P}_{13}$ is the probability, under the money market numeraire, of exit at maximum expiration t_n without receiving a final coupon $N\times\gamma_n$ because the down-and-out barrier has been crossed in the time interval $[t_{n-1}, t_n]$, when the note does not allow exit through equity participation. It is given by

$$\mathbb{P}_{13} = \boldsymbol{\Phi}_n \begin{bmatrix} \left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1 \sqrt{t_{n-1}}}, \frac{-d_n + \tilde{\mu}_1 t_n}{\sigma_1 \sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}} \end{bmatrix} - \mathbb{P}_{11} \tag{22}$$

$\mathbb{P}_{14}$ is the probability, under the money market numeraire, of exit at maximum expiration t_n without receiving a final coupon $N \times y_n$ because the down-and-out barrier has been crossed in the time interval $[t_{n-1}, t_n]$, when the note allows exit through equity participation. It is given by

$$\begin{aligned} \mathbb{P}_{14} = \boldsymbol{\Phi}_n &\begin{bmatrix} \left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1 \sqrt{t_{n-1}}}, \frac{-d_n + \tilde{\mu}_1 t_n}{\sigma_1 \sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}} \end{bmatrix} \\ - \boldsymbol{\Phi}_n &\begin{bmatrix} \left\langle \frac{d_i - \tilde{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \tilde{\mu}_1 t_{n-1}}{\sigma_1 \sqrt{t_{n-1}}}, \frac{-u_n + \tilde{\mu}_1 t_n}{\sigma_1 \sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}} \end{bmatrix} - \mathbb{P}_{12} \end{aligned} \tag{23}$$

$\mathbb{P}_{15}$ is the probability, under the S_1 numeraire, of exit at maximum expiration t_n with participation in the growth of S_1 in the absence of a best-of provision. It is given by

$$\mathbb{P}_{15} = \boldsymbol{\Phi}_n \begin{bmatrix} \left\langle \frac{d_i - \bar{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \bar{\mu}_1 t_{n-1}}{\sigma_1 \sqrt{t_{n-1}}}, \frac{-u_n + \bar{\mu}_1 t_n}{\sigma_1 \sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}} \end{bmatrix} \tag{24}$$

$\mathbb{P}_{16}$ is the probability, under the S_1 numeraire, of exit at maximum expiration t_n with participation in the growth of S_1, when the contract includes a best-of provision. It is given by

$$\mathbb{P}_{16} = \boldsymbol{\Phi}_{n+1} \begin{bmatrix} \left\langle \frac{d_i - \bar{\mu}_1 t_i}{\sigma_1 \sqrt{t_i}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \bar{\mu}_1 t_{n-1}}{\sigma_1 \sqrt{t_{n-1}}}, \frac{-u_n + \bar{\mu}_1 t_n}{\sigma_1 \sqrt{t_n}}, \frac{-x_{12} + \bar{\mu}_{12} t_n}{\sigma_{12} \sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, \ldots, n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}}, \theta \end{bmatrix} \tag{25}$$

$\mathbb{P}_{17}$ is the probability, under the S_2 numeraire, of exit at maximum expiration t_n with participation in the growth of S_2, when the contract includes a best-of provision and when S_1 is above the participation barrier U_n at time t_n. It is given by

$$\mathbb{P}_{17} = \Phi_{n+1}\left[\begin{array}{l}\left\langle \frac{d_i - \hat{\mu}_1 t_i}{\sigma_1\sqrt{t_i}}; i = 1, ..., n-2 \right\rangle \mathbb{I}_{\{n>2\}}, \frac{d_{n-1} - \hat{\mu}_1 t_{n-1}}{\sigma_1\sqrt{t_{n-1}}}, \frac{-u_n + \hat{\mu}_1 t_n}{\sigma_1\sqrt{t_n}}, \frac{x_{12} - \hat{\mu}_{12} t_n}{\sigma_{12}\sqrt{t_n}}; \\ \left\langle \sqrt{\frac{t_i}{t_{i+1}}}; i = 1, ..., n-2 \right\rangle \mathbb{I}_{\{n>2\}}, -\sqrt{\frac{t_{n-1}}{t_n}}, -\theta \end{array}\right] \tag{26}$$

Although Proposition 1 looks bulky, it may actually be regarded as quite compact, given that it accommodates any number n of observation dates and that it nests a large variety of possible payoffs. However, as mentioned earlier, Proposition 1 does not tackle the memory function embedded into some autocallable notes. So let us now introduce Proposition 2, which expands Proposition 1 in the special case $n = 4$, thus providing an easy way for readers to make sure they fully understand how to expand Proposition 1 for any integer n; furthermore, Proposition 2 prices the snowball effect.

Proposition 2

Let the assumptions of Proposition 1 hold. Then, the no-arbitrage value, $V_{AUTOCALL}$, of an autocallable note with four observation dates until expiration t_4, endowed with all the features described in Exhibits 1 and 2, *including* the snowball effect, is given by the following formula:

$$\begin{aligned} V_{AUTOCALL} = {} & \exp(-r \times t_1) \times N \times z_1 \times \mathbb{P}_1 + \exp(-r \times t_1) \times N \times (1+\gamma_1) \times (\mathrm{A}_1 \times \mathbb{P}_2 + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_3) \\ & + \mathrm{A}_2 \times \exp(-\delta_1 \times t_1) \times \lambda_1 \times N \times \mathbb{P}_4 + \mathrm{A}_3 \times \exp(-\delta_1 \times t_1) \times \lambda_1 \times N \times \mathbb{P}_5 + \mathrm{A}_3 \times \exp(-\delta_2 \times t_1) \times \lambda_2 \times N \times \mathbb{P}_6 \\ & + \exp(-r \times t_2) \times N \times z_2 \times \mathbb{P}_7 + \exp(-r \times t_2) \times N \times (1+\gamma_2) \times (\mathrm{A}_1 \times \mathbb{P}_8 + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_9) \\ & + \mathrm{A}_2 \times \exp(-\delta_1 \times t_2) \times \lambda_1 \times N \times \mathbb{P}_{10} + \mathrm{A}_3 \times \exp(-\delta_1 \times t_2) \times \lambda_1 \times N \times \mathbb{P}_{11} \\ & + \mathrm{A}_3 \times \exp(-\delta_2 \times t_2) \times \lambda_2 \times N \times \mathbb{P}_{12} + \mathrm{A}_4 \times \exp(-r \times t_2) \times N \times z_1 \times \mathbb{P}_{13} \\ & + \exp(-r \times t_3) \times N \times z_3 \times \mathbb{P}_{14} + \exp(-r \times t_3) \times N \times (1+\gamma_3) \times (\mathrm{A}_1 \times \mathbb{P}_{15} + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_{16}) \\ & + \mathrm{A}_2 \times \exp(-\delta_1 \times t_3) \times \lambda_1 \times N \times \mathbb{P}_{17} + \mathrm{A}_3 \times \exp(-\delta_1 \times t_3) \times \lambda_1 \times N \times \mathbb{P}_{18} \\ & + \mathrm{A}_3 \times \exp(-\delta_2 \times t_3) \times \lambda_2 \times N \times \mathbb{P}_{19} + \mathrm{A}_4 \times \exp(-r \times t_3) \times N \times (z_1 + z_2) \times \mathbb{P}_{20} + \mathrm{A}_4 \times \exp(-r \times t_3) \times N \times z_2 \times \mathbb{P}_{21} \\ & + N \times \mathbb{P}_{22} + (\mathrm{A}_1 + \mathrm{A}_2) \times \exp(-r \times t_4) \times N \times \mathbb{P}_{23} + \mathrm{A}_3 \times \exp(-r \times t_4) \times N \times \mathbb{P}_{24} \\ & + \mathrm{A}_3 \times \exp(-\delta_2 \times t_4) \times \lambda_2 \times N \times \mathbb{P}_{25} + \exp(-r \times t_4) \times N \times (1+\gamma_4) \times (\mathrm{A}_1 \times \mathbb{P}_{26} + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_{27}) \\ & + \exp(-r \times t_4) \times N \times (\mathrm{A}_1 \times \mathbb{P}_{28} + (\mathrm{A}_2 + \mathrm{A}_3) \times \mathbb{P}_{29}) + \mathrm{A}_2 \times \exp(-\delta_1 \times t_4) \times \lambda_1 \times N \times \mathbb{P}_{30} \\ & + \mathrm{A}_3 \times \exp(-\delta_1 \times t_4) \times \lambda_1 \times N \times \mathbb{P}_{31} + \mathrm{A}_3 \times \exp(-\delta_2 \times t_4) \times \lambda_2 \times N \times \mathbb{P}_{32} \\ & + \mathrm{A}_4 \times \exp(-r \times t_4) \times N \times (z_1 + z_2 + z_3) \times \mathbb{P}_{33} + \mathrm{A}_4 \times \exp(-r \times t_4) \times N \times (z_2 + z_3) \times \mathbb{P}_{34} \\ & + \mathrm{A}_4 \times \exp(-r \times t_4) \times N \times z_3 \times \mathbb{P}_{35} \end{aligned} \tag{27}$$

The parameter A_4 takes value one if the autocall note includes a memory function and value zero otherwise. The full expansion of all $\mathbb{P}_m$ terms, $m \in \{1, 2, ..., 35\}$, is given in Appendix B. There are 6 out of the 35 $\mathbb{P}_m$ terms in this formula that are not a direct expansion of Proposition 1 for $n = 4$, the role of which is to value the coupon-related memory function; they are $\mathbb{P}_{13}$, $\mathbb{P}_{20}$, $\mathbb{P}_{21}$, $\mathbb{P}_{33}$, $\mathbb{P}_{34}$, and $\mathbb{P}_{35}$.

The meaning of these six terms can be intuitively defined as follows:

1. $\exp\left(-r \times t_2\right) \times N \times z_1 \times \mathbb{P}_{13}$ is the risk-neutral value of the digital allowing to recover at time t_2 the coupon that was not received at time t_1.
2. $\exp\left(-r \times t_3\right) \times N \times \left(z_1 + z_2\right) \times \mathbb{P}_{20}$ is the risk-neutral value of the digital allowing to recover at time t_3 the coupon that was not received at time t_1 and the coupon that was not received at time t_2.
3. $\exp\left(-r \times t_3\right) \times N \times z_2 \times \mathbb{P}_{21}$ is the risk-neutral value of the digital allowing to recover at time t_3 the coupon that was not received at time t_2.
4. $\exp\left(-r \times t_4\right) \times N \times \left(z_1 + z_2 + z_3\right) \times \mathbb{P}_{33}$ is the risk-neutral value of the digital allowing to recover at time t_4 the coupon that was not received at time t_1 and the coupon that was not received at time t_2 and the coupon that was not received at time t_3.
5. $\exp\left(-r \times t_4\right) \times N \times \left(z_2 + z_3\right) \times \mathbb{P}_{34}$ is the risk-neutral value of the digital allowing to recover at time t_4 the coupon that was not received at time t_2 and the coupon that was not received at time t_3.
6. $\exp\left(-r \times t_4\right) \times N \times z_3 \times \mathbb{P}_{35}$ is the risk-neutral value of the digital allowing to recover at time t_4 the coupon that was not received at time t_3.

APPENDIX A

From a numerical point of view, the valuation formulas presented in this article raise the question of the computation of the function Φ_n. As the number n of possible exit dates increases, so does the dimension of numerical integration. The latter can be reduced by using the following identities:

1. when there are three observation dates, the actual numerical dimension of the function Φ_3 can be brought down from three to one by using

$$\Phi_3\left[\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2\right] = \int_{x_2=-\infty}^{\alpha_2} \frac{\exp\left(-x_2^2/2\right)}{\sqrt{2\pi}} \Phi_1\left[\frac{\alpha_1 - \beta_1 x_2}{\sqrt{1-\beta_1^2}}\right] \Phi_1\left[\frac{\alpha_3 - \beta_2 x_2}{\sqrt{1-\beta_2^2}}\right] dx_2 \tag{A-1}$$

2. when there are four observation dates, the actual numerical dimension of the function Φ_4 can be reduced by a factor of two by using

$$\Phi_4\left[\alpha_1, \alpha_2, \alpha_3, \alpha_4; \beta_1, \beta_2, \beta_3\right] = \int_{x_2=-\infty}^{\alpha_2} \int_{x_3=-\infty}^{\frac{\alpha_3-\beta_2 x_2}{\sqrt{1-\beta_2^2}}} \frac{\exp\left(-\frac{\left(x_2^2 + x_3^2\right)}{2}\right)}{2\pi} \Phi_1\left[\frac{\alpha_1 - \beta_1 x_2}{\sqrt{1-\beta_1^2}}\right] \Phi_1\left[\frac{\alpha_4 - \beta_3\sqrt{1-\beta_2^2}\,x_3 - \beta_3\beta_2 x_2}{\sqrt{1-\beta_3^2}}\right] dx_2 dx_3 \tag{A-2}$$

3. more generally, when there are more than four observation dates, the actual numerical dimension of the function Φ_n can always be reduced by a factor of two by using

$$\Phi_n\left[\alpha_1, \alpha_2, ..., \alpha_{n-1}, \alpha_n; \beta_1, ..., \beta_{n-2}, \beta_{n-1}\right] = \int_{x_2=-\infty}^{\alpha_2} \int_{x_3=-\infty}^{\alpha_3} \cdots \int_{x_{n-1}=-\infty}^{\alpha_{n-1}}$$

$$\frac{\exp\left(-\frac{x_2^2}{2} - \frac{1}{2\left(1-\beta_2^2\right)}\left(x_3 - \beta_2 x_2\right)^2 ... - \frac{1}{2\left(1-\beta_{n-2}^2\right)}\left(x_{n-1} - \beta_{n-2}x_{n-2}\right)^2\right)}{\prod_{i=2}^{n-2}\sqrt{\left(1-\beta_i^2\right)}\,(2\pi)^{\frac{n-2}{2}}} \quad \Phi_1\left[\frac{\alpha_1 - \beta_1 x_2}{\sqrt{1-\beta_1^2}}\right] \Phi_1\left[\frac{\alpha_n - \beta_{n-1}x_{n-1}}{\sqrt{1-\beta_{n-1}^2}}\right] dx_2\ dx_3\ ...\ dx_{n-1} \tag{A-3}$$

The quality of numerical integration using Equations (A-1)–(A-3) was tested up to six observation dates by applying a classical 16-point Gauss–Legendre quadrature rule, which is very easy to implement. A first set of tests was performed using analytical benchmarks that consisted of products of univariate normal cumulative distribution functions, by assuming that all correlation coefficients were equal to zero. In every dimension from three to six, some 500 integral convolutions were computed by means of Proposition 1 and Proposition 2, with randomly drawn parameters. The results always matched the analytical benchmarks to at least 10^{-8} accuracy. Then, option prices with non-zero, arbitrary values of the correlation coefficients were computed and compared with numerical values obtained by Monte Carlo simulation using antithetic variates and the Mersenne Twister random number generator (Glasserman [2004]).

In terms of efficiency, it always took less than one second to compute prices of four-year autocallable notes with yearly observation dates by means of Proposition 2 on a mainstream commercial PC, which is very fast in absolute terms and dramatically more efficient than simulation techniques, which required from four to ten minutes to yield accurate approximations in dimensions five and six. Moreover, the efficiency gains are even greater when it comes to the computation of the option sensitivities or Greeks. A clear pattern of convergence of the Monte Carlo estimates to the analytical values was noticed, as more and more simulations were performed. For autocallable notes with four observation dates, 400,000 simulations along with a time discretization factor of eight quotations per business day in the time interval $[t_{n-1}, t_n]$ were necessary to achieve 10^{-4} convergence, and it took 1,000,000 simulations to achieve 10^{-5} convergence.

Overall, the observed high quality of numerical integration of Proposition 1 and Proposition 2 is not surprising, given the smoothness of the integrands in every dimension. When $|\beta_2|$ is close to one, though, the upper bound of the inner integral in (A-2) becomes very 'large'. To avoid numerical errors, it is safe to prespecify a maximum value for all possible integral endpoints. Given the standard normal nature of the integrands, bounding at an absolute value of five will entail negligible loss of accuracy. The fact that the arguments in the $\Phi_1[x]$ functions in (A-1)–(A-3) may rise sharply when the β_i coefficients, $i \in \{1, \ldots, n-1\}$, are close to ± 1 is not a cause for concern since these functions are, by definition, bounded at one and have already reached their maximum to 10^{-9} accuracy at $x = 5$.

Thus, in moderate dimension, the analytical formulas provided in this article give very efficient and accurate numerical results, whereas the computational time required for a Monte Carlo simulation to return reasonably precise approximations is clearly not satisfactory for practical purposes. It must be stressed that the valuation of the majority of autocallable notes does not involve more than moderate dimension, as most traded products include between four and eight possible exit dates, being typically annual observation dates on four- to eight-year maturity notes.

In higher dimension, as more and more exit dates might be added, the quality of a plain Gauss–Legendre implementation of (A-3) will deteriorate. One solution, then, would be to implement an adaptive Gauss–Legendre quadrature, in order to control the error resulting from numerical integration. Combining this with a Kronrod rule will reduce the number of required iterations (Davis and Rabinowitz [2007]). There is plenty of scientific software available for that matter so that it is not necessary to know the technical details. Alternatively, another solution is to notice that the function Φ that needs to be computed in increasing dimension has the attractive feature that it matches the special structure of Gaussian convolutions handled by the efficient Broadie–Yamamoto algorithm (Broadie and Yamamoto [2005]).

Appendix B

This appendix provides the explicit formulas for the 35 $\mathbb{P}_m$ terms in Proposition 2:

$$\mathbb{P}_1 = \Phi_1\left[\frac{-c_1 + \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}\right] - \Phi_1\left[\frac{-d_1 + \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}\right] \quad \mathbb{P}_2 = \Phi_1\left[\frac{-d_1 + \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}\right]$$

$$\mathbb{P}_3 = \Phi_1\left[\frac{-d_1 + \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}\right] - \Phi_1\left[\frac{-u_1 + \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}\right] \quad \mathbb{P}_4 = \Phi_1\left[\frac{-u_1 + \bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}\right]$$

$$\mathbb{P}_5 = \Phi_2\left[\frac{-u_1 + \bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-x_{12} + \bar{\mu}_{12} t_1}{\sigma_{12}\sqrt{t_1}}; \theta\right] \quad \mathbb{P}_6 = \Phi_2\left[\frac{-u_1 + \hat{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{x_{12} - \hat{\mu}_{12} t_1}{\sigma_{12}\sqrt{t_1}}; -\theta\right]$$

$$\mathbb{P}_7 = \Phi_2\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-c_2 + \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right] - \Phi_2\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-d_2 + \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right]$$

$$\mathbb{P}_8 = \Phi_2\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-d_2 + \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right]$$

$$\mathbb{P}_9 = \Phi_2\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-d_2 + \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right] - \Phi_2\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-u_2 + \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right]$$

$$\mathbb{P}_{10} = \Phi_2\left[\frac{d_1 - \bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-u_2 + \bar{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right]$$

$$\mathbb{P}_{11} = \Phi_3\left[\frac{d_1 - \bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-u_2 + \bar{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-x_{12} + \bar{\mu}_{12} t_2}{\sigma_{12}\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}, \theta\right]$$

$$\mathbb{P}_{12} = \Phi_3\left[\frac{d_1 - \hat{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-u_2 + \hat{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{x_{12} - \hat{\mu}_{12} t_2}{\sigma_{12}\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}, -\theta\right]$$

$$\mathbb{P}_{13} = \Phi_2\left[\frac{c_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-c_2 + \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right]$$

$$\mathbb{P}_{14} = \left\{\Phi_3\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-c_3 + \tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right] - \Phi_3\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-d_3 + \tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]\right\}$$

$$\mathbb{P}_{15} = \Phi_3\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-d_3 + \tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]$$

$$\mathbb{P}_{16} = \left\{\Phi_3\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-d_3 + \tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right] - \Phi_3\left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-u_3 + \tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]\right\}$$

$$\mathbb{P}_{17} = \Phi_3\left[\frac{d_1 - \bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \bar{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-u_3 + \bar{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]$$

$$\mathbb{P}_{18} = \Phi_4\left[\frac{d_1 - \bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \bar{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-u_3 + \bar{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-x_{12} + \bar{\mu}_{12} t_3}{\sigma_{12}\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, \theta\right]$$

$$\mathbb{P}_{19} = \Phi_4\left[\frac{d_1 - \hat{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2 - \hat{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-u_3 + \hat{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{x_{12} - \hat{\mu}_{12} t_3}{\sigma_{12}\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, -\theta\right]$$

$$\mathbb{P}_{20} = \Phi_3\left[\frac{c_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{c_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-c_3+\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]$$

$$\mathbb{P}_{21} = \left\{\Phi_3\left[\frac{-c_1+\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{c_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-c_3+\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; -\sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]\right.$$
$$\left. -\Phi_3\left[\frac{-d_1+\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{c_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-c_3+\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}; -\sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}\right]\right\}$$

$$\mathbb{P}_{22} = \Phi_4\left[\frac{d_1-\bar{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\bar{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\bar{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{b-\bar{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, \sqrt{\frac{t_3}{t_4}}\right]$$

$$\mathbb{P}_{23} = \left\{\Phi_4\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-b+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}\right]\right.$$
$$\left. -\Phi_4\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}\right]\right\}$$

$$\mathbb{P}_{24} = \left\{\Phi_5\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-b+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}, \frac{w-\tilde{\mu}_2 t_4}{\sigma_2\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}, -\rho\right]\right.$$
$$\left. -\Phi_5\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}, \frac{w-\tilde{\mu}_2 t_4}{\sigma_2\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}, -\rho\right]\right\}$$

$$\mathbb{P}_{25} = \left\{\Phi_5\left[\frac{d_1-\hat{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\hat{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\hat{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-b+\hat{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}, \frac{-w+\hat{\mu}_2 t_4}{\sigma_2\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}, \rho\right]\right.$$
$$\left. -\Phi_5\left[\frac{d_1-\hat{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\hat{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{d_3-\hat{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+\hat{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}, \frac{-w+\hat{\mu}_2 t_4}{\sigma_2\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}, \rho\right]\right\}$$

$$\mathbb{P}_{26} = \left\{\Phi_4\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-h+\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, \sqrt{\frac{t_3}{t_4}}\right]\right.$$
$$-e^{\frac{2\tilde{\mu}_1}{\sigma_1^2}h}\Phi_4\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-h-\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+2h+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}\right]$$
$$-\Phi_4\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-d_3+\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, \sqrt{\frac{t_3}{t_4}}\right]$$
$$\left. +e^{\frac{2\tilde{\mu}_1}{\sigma_1^2}h}\Phi_4\left[\frac{d_1-\tilde{\mu}_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{d_2-\tilde{\mu}_1 t_2}{\sigma_1\sqrt{t_2}}, \frac{-d_3-\tilde{\mu}_1 t_3}{\sigma_1\sqrt{t_3}}, \frac{-d_4+2h+\tilde{\mu}_1 t_4}{\sigma_1\sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}\right]\right\}$$

$$\mathbb{P}_{27} = \left\{ \mathbb{P}_{26} - \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{-h + \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, \sqrt{\frac{t_3}{t_4}} \right] \right.$$

$$+ e^{\frac{2\tilde{\mu}_1}{\sigma_1^2} h} \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{-h - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + 2h + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right]$$

$$+ \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{-d_3 + \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, \sqrt{\frac{t_3}{t_4}} \right]$$

$$\left. - e^{\frac{2\tilde{\mu}_1}{\sigma_1^2} h} \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{-d_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + 2h + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right] \right\}$$

$$\mathbb{P}_{28} = \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{d_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-d_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right] - \mathbb{P}_{26}$$

$$\mathbb{P}_{29} = \left\{ \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{d_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-d_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right] \right.$$

$$\left. - \boldsymbol{\Phi}_4 \left[\frac{d_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{d_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right] - \mathbb{P}_{27} \right\}$$

$$\mathbb{P}_{30} = \boldsymbol{\Phi}_4 \left[\frac{d_1 - \bar{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \bar{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{d_3 - \bar{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + \bar{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right]$$

$$\mathbb{P}_{31} = \boldsymbol{\Phi}_5 \left[\frac{d_1 - \bar{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \bar{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{d_3 - \bar{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + \bar{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}, \frac{-x_{12} + \bar{\mu}_{12} t_4}{\sigma_{12} \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}, \theta \right]$$

$$\mathbb{P}_{32} = \boldsymbol{\Phi}_5 \left[\frac{d_1 - \hat{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{d_2 - \hat{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{d_3 - \hat{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-u_4 + \hat{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}, \frac{x_{12} - \hat{\mu}_{12} t_4}{\sigma_{12} \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}}, -\theta \right]$$

$$\mathbb{P}_{33} = \boldsymbol{\Phi}_4 \left[\frac{c_1 - \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{c_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{c_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-c_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right]$$

$$\mathbb{P}_{34} = \left\{ \boldsymbol{\Phi}_4 \left[\frac{-c_1 + \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{c_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{c_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-c_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; -\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right] \right.$$

$$\left. - \boldsymbol{\Phi}_4 \left[\frac{-d_1 + \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{c_2 - \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{c_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-c_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; -\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right] \right\}$$

$$\mathbb{P}_{35} = \boldsymbol{\Phi}_4 \left[\left(\frac{-c_1 + \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{-d_1 + \tilde{\mu}_1 t_1}{\sigma_1 \sqrt{t_1}} \right), \left(\frac{-c_2 + \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}}, \frac{-d_2 + \tilde{\mu}_1 t_2}{\sigma_1 \sqrt{t_2}} \right), \frac{c_3 - \tilde{\mu}_1 t_3}{\sigma_1 \sqrt{t_3}}, \frac{-c_4 + \tilde{\mu}_1 t_4}{\sigma_1 \sqrt{t_4}}; \sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_2}{t_3}}, -\sqrt{\frac{t_3}{t_4}} \right]$$

where the function $\Phi[(\alpha_{11,} \alpha_{12}), (\alpha_{21,} \alpha_{22})_{,} \alpha_3, \alpha_4; \beta_1, \beta_2, \beta_3)]$ is the same as in Equation (3) except for the bounds of the x_1 integral which are changed from $x_1 = -\infty$ (lower bound) and $x_1 = \alpha_1$ (upper bound) to $x_1 = \alpha_{11}$ and $x_1 = \alpha_{12}$ respectively, and for the bounds of the x_2 integral which are changed from $x_2 = -\infty$ (lower bound) and $x_2 = \alpha_2$ (upper bound) to $x_2 = \alpha_{21}$ and $x_2 = \alpha_{22}$ respectively.

REFERENCES

Alm, T., B. Harrach, D. Harrach, and M. Keller. "A Monte Carlo Pricing Algorithm for Autocallables That Allows for Stable Differentiation." *Journal of Computational Finance*, Vol. 17, No. 1, 2013.

Bouzoubaa, M., and A. Osseiran. *Exotic Options and Hybrids*. New York: Wiley Finance, 2010.

Broadie, M., and Y. Yamamoto. "A Double-Exponential Fast Gauss Transform Algorithm for Pricing Discrete Path-Dependent Options." *Operations Research*, Vol. 53, No. 5 (2005), pp. 764-779.

Davis, P.J., and P. Rabinowitz. *Methods of Numerical Integration*. Dover Publications, 2007.

De Weert, F. *Exotic Options Trading*. New York: Wiley Finance, 2008.

Deng, G., J. Mallett, and C. McCann. "Modeling Autocallable Structured Products." *Journal of Derivatives and Hedge Funds*, 17 (2011), pp. 326-340.

Fries, C.P., and M.S. Joshi. "Perturbation Stable Conditional Analytic Monte-Carlo Pricing Scheme for Autocallable Products." *International Journal of Theoretical and Applied Finance*, Vol. 14, No. 2 (2011), pp. 197-219.

Glasserman, P. *Monte Carlo Methods in Financial Engineering*. New York: Springer, 2004.

Kamtchueng, C. "Autocallable: More American than European?" Working paper, Barclays Capital, 2011.