Fast Computation of Univariate And Multivariate Memory Auto-callable Notes

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Abstract. In this paper we consider the valuation of a special type of equity-linked notes, so-called Memory Autocallable Notes. In contrast to the common Autocallable Notes these notes contain a memory feature that allows the recovery of any missed coupon payment. In this context, our first result is a general mathematical formulation of the cash flows of such Memory Autocallable Notes.
Since the Monte Carlo simulation is an appropriate tool for the valuation of Memory Autocallable Notes, the reduction of computational time for a given accuracy is of practical interest. We show that this could be reached with either the Antithetic Sampling technique or the Importance Sampling method. This is also true for the multivariate case.
Furthermore, we demonstrate how to calculate the sensitivities of Univariate Memory Autocallable Notes more effectively.

1 Introduction

In contrast to common Autocallable Notes a Memory Autocallable Note (which is also known as a Memory Phoenix Autocallable Note or Memory Express Note) includes a Memory feature that can pay any missed coupon on a subsequent payment date. Table 1 shows an example of a Memory Autocallable Note.

Underlying	Allianz
Exchange	XETRA
Start Date	01.09.2011
Maturity Date	01.09.2014
Set Date	30.08.2011
Valuation Date	30.08.2014
Payment Dates	Annually on 01.09.
Observation Dates	5 business days before the payment date
Initial Price S_0	The price of the underlying on the close of the relevant exchange on the set date
Final Price S_{final}	The price of the underlying on the close of the relevant exchange on the valuation date
S_i	The price of the underlying on the close of the relevant exchange on the i-th observation date
Barrier Level B	$\mathbf{b \cdot S_0}$ (percentage of initial price) with $0<b<1$
Memory Coupon	1) If $S_i \geq$ barrier level, then $coupon_i = \mathbf{i \cdot c[\%] - \Sigma_{k<i} coupon_k}$ 2) If $S_i <$ barrier level, then $coupon_i=0$
Final Flow	▪ If $S_{final} \geq$ barrier level, then $100\% \cdot$ Notional Amount ▪ If $S_{final} <$ barrier level, then $\text{Notional Amount} \cdot \frac{S_{final}}{S_{initial}}$
Early Termination	If $S_i \geq S_0$, then the note shall terminate and 100% of the notional amount is paid back plus a final coupon payment subject to the memory coupon above.

Table 1: Terms of a 3-year Memory Autocallable Note

At each observation date, it is checked whether a reference asset (i.e. a single stock or index) closes at or above a predetermined level (i.e. a percentage of the initial price). In this case, the investor receives a coupon payment c. Otherwise the coupon payment is missed, but the investor has a chance of a deferred coupon payment, if the underlying equals or exceeds the barrier level at a later observation date.

Additionally, at each observation date the reference asset is compared with its initial price at the set date. If the reference asset is at or above the initial price S_0 the note will be redeemed at par prior to maturity.

An additional component of a Memory Autocallable Note is the uncertain redemption price at maturity (partial capital guarantee). The investor will suffer a percentage loss of the notional amount depending on the overall performance of the reference asset, if the underlying closes below the barrier level at the valuation date. This means that the investor is short a put option at maturity.

Figure 1 gives a comprehensive overview of the payoff profile as described in Table 1.

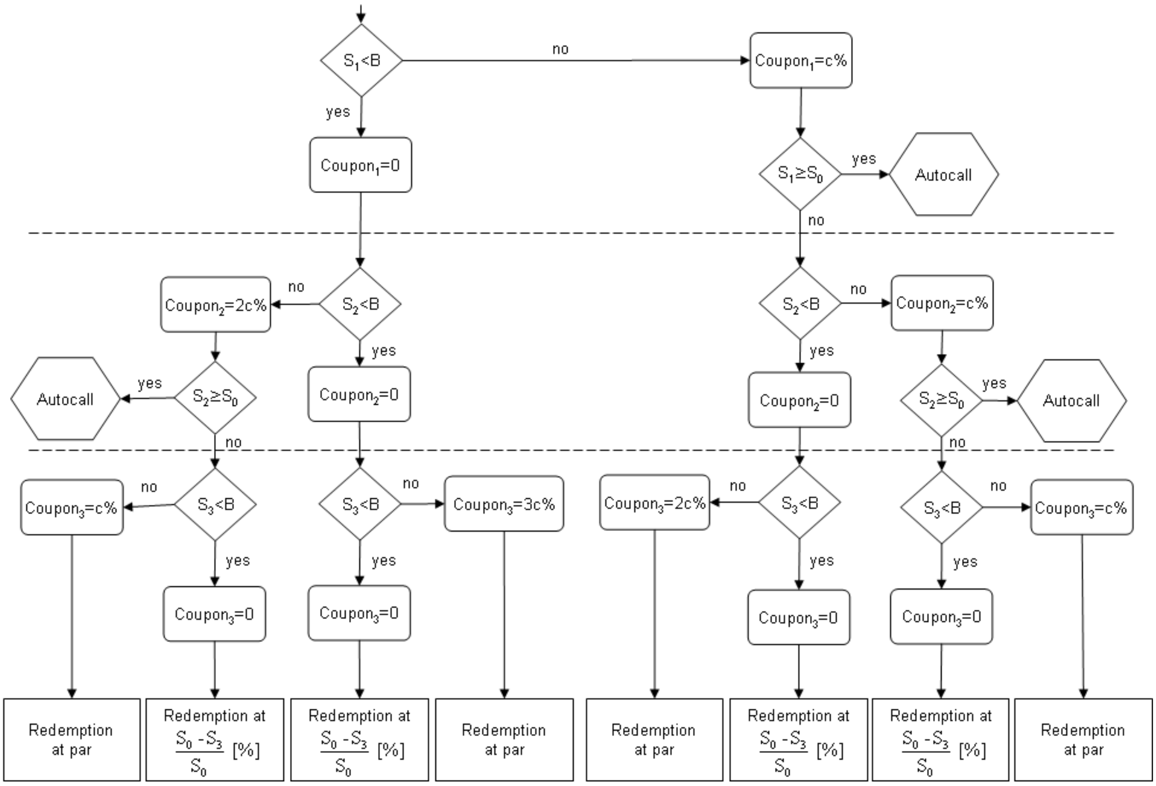

Figure 1: Payoff profile of a 3-year Memory Autocallable Note

As we can see, going to the next coupon period, which is equivalent to crossing one of the horizontal dotted lines, leads to a doubling of the number of possible states due to the memory feature and the autocall mechanism.[1] This holds for all periods except in the case of the final coupon period, for which the early redemption event is omitted. Hence, a N-year Memory Autocallable Note has $3 \cdot 2^{N-1}-1$ different paths for consideration. This is also the reason, why a Memory Autocallable Note is called an exotic (structured) product.

The valuation of Standard Autocallable Notes is well known. For example, the general decomposition of Autocallable Notes can be found in Bouzoubaa and Osseiran (2010) (§12) or in Weert (2008) (§16). See also Deng et al (2011) for modelling Autocallable Notes within a Partial Differential Framework. Finally, Alm

[1] The possible states are given by “coupon payment and autocall”, “coupon payment and no autocall” and “no coupon payment and no autocall”.

et al (2013) and Fries and Joshi (2008) show how to use the Importance Sampling method for valuation of Autocallable Notes hence improving the accuracy of sensitivity calculations. However, to the knowledge of the author, this is the first time that the coupon cash flows of a Memory Autocallable Note are explicitly written down and are being adapted to the variance reduction techniques of pricing exotic options via Monte Carlo simulations.

This paper is organized as follows. First, we show a closed-form formula of the coupon cash flows. We proceed to present two well-known variance reduction techniques (Antithetic Sampling and Importance Sampling) and show with the help of the coupon cash flow formula that the speed of computation valuing Memory Autocallable Notes can be significantly increased without loss of accuracy. Then, we will use the same approach for the multivariate case. The results are verified by some numerical examples for the described methods. We finish this paper with some concluding remarks.

2 Formulation of the cash flows

We obtain the mathematical formulation of any coupon payment (C_i) and the final flow (δ) as follows. Extracting the last summand from the sum, using its recursive composition and factoring ($1-\mathbb{1}_{\{S_m/S_0\geq b\}}$) for each step provides us

$$\begin{aligned}
C_i &= \left[i \cdot c - \sum_{k=1}^{i-1} C_k \right] \cdot M_i = \left[i \cdot c - C_{i-1} - \sum_{k=1}^{i-2} C_k \right] M_i \\
&= \left[i \cdot c - (i-1) \cdot c \cdot \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}} - \left(1 - \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}}\right) \cdot \sum_{k=1}^{i-2} C_k \right] M_i \\
&= \left[i \cdot c - (i-1) \cdot c \cdot \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}} - \left(1 - \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}}\right) \cdot C_{i-2} - \left(1 - \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}}\right) \cdot \sum_{k=1}^{i-3} C_k \right] M_i \\
&= \left[i \cdot c - (i-1) \cdot c \cdot \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}} - (i-2) \cdot c \cdot \mathbb{1}_{\left\{\frac{S_{i-2}}{S_0}\geq b\right\}} \left(1 - \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}}\right) \right] M_i \\
&\qquad - \left(1 - \mathbb{1}_{\left\{\frac{S_{i-2}}{S_0}\geq b\right\}}\right) \cdot \left(1 - \mathbb{1}_{\left\{\frac{S_{i-1}}{S_0}\geq b\right\}}\right) \cdot \sum_{k=1}^{i-3} C_k \cdot M_i \\
&\;\;\vdots \\
&= \left[i \cdot c - \sum_{k=1}^{i-1} k \cdot c \cdot \mathbb{1}_{\left\{\frac{S_k}{S_0}\geq b\right\}} \prod_{j=k+1}^{i-1} \left(1 - \mathbb{1}_{\left\{\frac{S_j}{S_0}\geq b\right\}}\right) \right] \cdot M_i
\end{aligned}$$

with the usual convention, that we set

$$\sum_{i=m}^{n} a_i := 0 \text{ and } \prod_{i=m}^{n} a_i \coloneqq 1.$$

for $n,m \in \mathbb{N}$ with $m>n$.

Here $\mathbb{1}_A$ stands for the indicator function, which has a value of one at points of a set A and zero otherwise, and M_i is defined as $\mathbb{1}_{\{S_i/S_0 \geq b\}} \cdot \prod_{j=1,\dots,i-1} \mathbb{1}_{\{S_j/S_0 \leq 1\}}$. Remember, that

$$1 - \mathbb{1}_{\left\{\frac{S_i}{S_0} \geq b\right\}} = \mathbb{1}_{\left\{\frac{S_i}{S_0} < b\right\}} = \mathbb{1}_{\left\{\frac{S_i}{S_0} < b\right\}} \cdot \mathbb{1}_{\left\{\frac{S_i}{S_0} \leq 1\right\}}$$

for b<1 holds. This leads to

$$C_i = \left[i \cdot c - \sum_{k=1}^{i-1} k \cdot c \cdot \mathbb{1}_{\left\{\frac{S_k}{S_0} \geq b\right\}} \prod_{j=k+1}^{i-1} \mathbb{1}_{\left\{\frac{S_j}{S_0} < b\right\}} \right] \cdot \mathbb{1}_{\left\{\frac{S_i}{S_0} \geq b\right\}} \cdot \prod_{m=1}^{i} \mathbb{1}_{\left\{\frac{S_m}{S_0} \leq 1\right\}}$$

$$\delta = \frac{S_I}{S_0} \cdot \mathbb{1}_{\left\{\frac{S_I}{S_0} < b\right\}} \cdot \prod_{k=1}^{I-1} \mathbb{1}_{\left\{\frac{S_k}{S_0} \leq 1\right\}} \qquad \textbf{(1)}$$

for the coupon periods i=1,…,I, where c denotes the coupon rate and b indicates the factor, which is set for the barrier level.

Because of the several if-then-conditions (path dependency) as shown in Figure 1, the Monte Carlo simulation is the easiest way to evaluate a Memory Autocallable Note. As usual, if the underlying asset follows a Geometric Brownian Motion, one can use Itô's Lemma and the Euler Scheme to obtain the following discretization for the Monte Carlo generation of paths for discrete time slots $0=t_0<t_1<\dots$[2]

$$S_{t_{k+1}} = S_{t_k} \cdot \exp\left(\left[\mathrm{r}(\mathrm{t}_k) - \mathrm{d}(\mathrm{t}_k) - \frac{1}{2}\sigma^2(S_{t_k}, \Delta\mathrm{t}) \right] \cdot \Delta t + \sigma(S_{t_k}, \Delta\mathrm{t}) \cdot \sqrt{\Delta t} \cdot \mathrm{Z}_{k+1} \right) \qquad \textbf{(2)}$$

with $\Delta t = t_{k+1} - t_k$ and where the $Z_1, Z_2, \dots$ are independent standard normally distributed random variables. Therefore, the function

$$G(Z_1, \dots, Z_I) = \sum_{i=1}^{I} e^{-r \cdot t_i} \cdot C_i - e^{-r \cdot t_i} \cdot \delta \qquad \textbf{(3)}$$

with the terms of (1) describes the present value of a Memory Autocallable Note, which should be evaluated by the Monte Carlo simulation. More precisely, taking the average over n sample paths provides us the following Monte Carlo estimator for (3)

$$X = \frac{1}{N} \sum_{i=1}^{I} G\left(z_1^j, \dots, z_I^j\right)$$

The accuracy of a standard Monte Carlo simulation algorithm is measured by the standard error, which is defined by

$$\varepsilon = \frac{\sigma_{simulation}}{\sqrt{\#simulations}}$$

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[2] See for example Glasserman (2004) §3.2, Hull (2008) §19.6 or Wilmott (2007) §29.

Thus, reducing the variance is equivalent to reducing the number of simulations for a given accuracy. Therefore, we present two variance reduction techniques for the fast computation of Memory Autocallable Notes.

3 Antithetic Sampling

This method aims to reduce the variance using the negative correlation of individual pairs of samples. With reference to the value function $G(Z_1,\ldots,Z_I)$ as in (3), the Gaussian random vectors $(-Z_1,\ldots,-Z_I)$ and $(Z_1,\ldots,Z_I)$ are obviously antithetic variables. [3] Hence, we consider

$$\frac{G\left(z_1^j,\ldots,z_I^j\right)+G\left(-z_1^j,\ldots,-z_I^j\right)}{2}$$

as an individual sample and the Monte Carlo estimator implied by antithetic sampling is given by

$$X_{AV}=\frac{1}{N}\sum_{i=1}^{I}\frac{G\left(z_1^j,\ldots,z_I^j\right)+G\left(-z_1^j,\ldots,-z_I^j\right)}{2}$$

There are two points to mention. First, the expectation values of $G(Z_1,\ldots,Z_I)$ and $G(-Z_1,\ldots,-Z_I)$ coincide because of the symmetry of the standard normal probability density; thus, $\varphi(z)=\varphi(-z)$.

Second, the antithetic sampling technique provides a variance reduction if

$$\mathrm{Cov}(G\left(Z_1^j,\ldots,Z_I^j\right);G\left(-Z_1^j,\ldots,-Z_I^j\right))\leq 0 \tag{4}$$

Since the random variables Z_i are independent and identically distributed, this yields

$$\begin{aligned}\mathrm{Var}[X_{AV}]&=\frac{1}{4}(\mathrm{Var}\left[G\left(Z_1^j,\ldots,Z_I^j\right)\right]+\mathrm{Var}[G\left(-Z_1^j,\ldots,-Z_I^j\right)])\\&\qquad+\frac{1}{2}\mathrm{Cov}[G\left(Z_1^j,\ldots,Z_I^j\right);G\left(-Z_1^j,\ldots,-Z_I^j\right)]\\&=\frac{1}{2}(\mathrm{Var}\left[G\left(Z_1^j,\ldots,Z_I^j\right)\right]+\mathrm{Cov}[G\left(Z_1^j,\ldots,Z_I^j\right);G\left(-Z_1^j,\ldots,-Z_I^j\right)])\end{aligned}$$

with $\mathrm{Var}[G(Z_1,\ldots,Z_I)]=\mathrm{Var}[G(-Z_1,\ldots,-Z_I)]$. In this case, the variance is reduced by not less than a half. To validate (4), it is sufficient to show that the function $G(Z_1,\ldots,Z_I)$ as in (3) is monotone in each component Z_i.[4]

In the sense of the present value of future cash flows, it is always better to receive a coupon immediately at the coupon date rather than recover a coupon

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[3] See Glasserman (2004) §4.2 or Jäckel (2002) §10.1.
[4] See Glasserman (2004), p. 207.

payment later. Hence, each coupon payoff C_i is monotone in Z_i even if the missed coupons are recovered at these dates (see Figure 2).

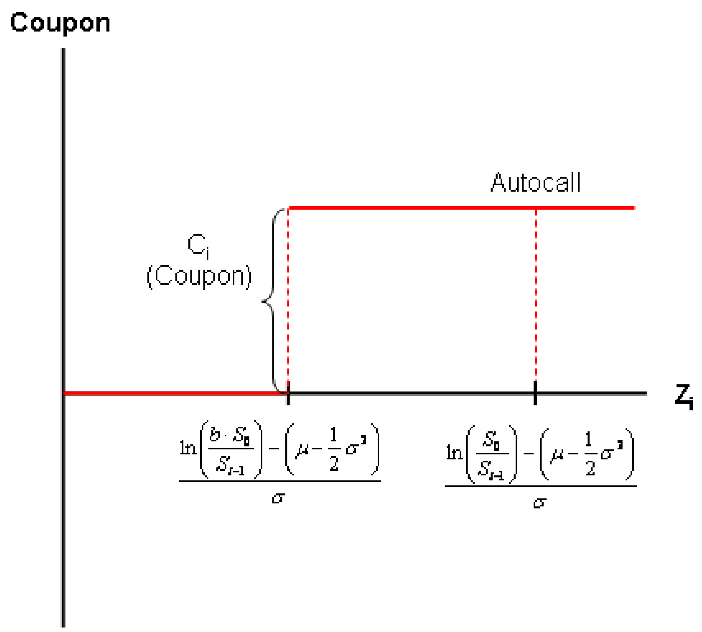

Figure 2: Monotony of the coupon payments

This is also valid in the case of an early redemption event at time i*, because all subsequent payments $C_{i^*+1},\dots,C_I$ and δ are automatically set to zero, with the consequence that $G(\cdot)$ remains constant.

Finally, Figure 3 shows the monotony of the final flow, which is given by the potential loss if the price of the underlying falls below the coupon barrier $b \cdot S_0$.

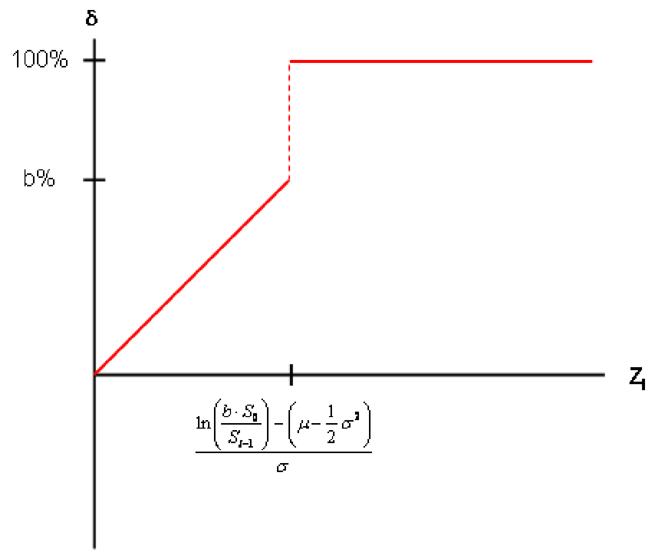

Figure 3: Monotony of the final flow

4 Importance Sampling

The importance sampling technique is characterized by the change of probability measures.[5] Expanding the integral

$$\mathrm{E}_\varphi\left[G\left(Z_1^j,\dots,Z_I^j\right)\right]=\int G\left(z_1^j,\dots,z_I^j\right)\cdot\varphi\left(z_1^j,\dots,z_I^j\right)d\left(z_1^j,\dots,z_I^j\right)$$

with the probability density function $\eta(\cdot)$ leads to the importance sampling estimator

$$X_{IS}=\frac{1}{N}\sum_{i=1}^{I}G\left(z_1^j,\dots,z_I^j\right)\cdot\frac{\varphi\left(z_1^j,\dots,z_I^j\right)}{\eta\left(z_1^j,\dots,z_I^j\right)} \tag{5}$$

to evaluate

$$\mathrm{E}_\eta\left[G\left(Z_1^j,\dots,Z_I^j\right)\right]=\int G\left(z_1^j,\dots,z_I^j\right)\cdot\frac{\varphi\left(z_1^j,\dots,z_I^j\right)}{\eta\left(z_1^j,\dots,z_I^j\right)}d\left(z_1^j,\dots,z_I^j\right)$$

Since X_{IS} as in (5) is an unbiased estimator ($E_\eta[X_{IS}]=X$), the Monte Carlo simulation can be applied. The sufficient condition of a successful variance reduction can be expressed as the comparison of the second moments[6] which is given by

$$\boxed{\text{If } \varphi(\cdot)/\eta(\cdot)<1 \text{ then } E_\varphi[G^2(\cdot)]> E_\varphi[G^2(\cdot)\cdot\varphi(\cdot)/\eta(\cdot)] \text{ holds.} \qquad \text{(C)}}$$

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[5] See Glasserman (2004) §4.6.1 for the theoretical approach.

[6] The variances can be written as

$$\begin{aligned}\mathrm{Var}_\varphi[X]&=\mathrm{Var}_\varphi\left[\frac{1}{N}\sum_{j=1}^{N}G\left(Z_1^j,\dots,Z_I^j\right)\right]=\frac{1}{N^2}\sum_{j=1}^{N}\mathrm{Var}_\varphi\left[G\left(Z_1^j,\dots,Z_I^j\right)\right]\\&=\frac{1}{N}\mathrm{Var}_\varphi\left[G\left(Z_1^j,\dots,Z_I^j\right)\right]=\frac{1}{N}\left(\mathrm{E}_\varphi\left[G^2\left(Z_1^j,\dots,Z_I^j\right)\right]-\mathrm{E}_\varphi^2\left[G\left(Z_1^j,\dots,Z_I^j\right)\right]\right)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Var}_\eta[X_{IS}]\\&\quad=\mathrm{Var}_\eta\left[\frac{1}{N}\sum_{j=1}^{N}G\left(Z_1^j,\dots,Z_I^j\right)\cdot\frac{\varphi\left(Z_1^j,\dots,Z_I^j\right)}{\eta\left(Z_1^j,\dots,Z_I^j\right)}\right]\\&\quad=\frac{1}{N^2}\sum_{j=1}^{N}\mathrm{Var}_\eta\left[G\left(Z_1^j,\dots,Z_I^j\right)\frac{\varphi\left(Z_1^j,\dots,Z_I^j\right)}{\eta\left(Z_1^j,\dots,Z_I^j\right)}\right]\\&\quad=\frac{1}{N}\mathrm{Var}_\eta\left[G\left(Z_1^j,\dots,Z_I^j\right)\frac{\varphi\left(Z_1^j,\dots,Z_I^j\right)}{\eta\left(Z_1^j,\dots,Z_I^j\right)}\right]\\&\quad=\frac{1}{N}\left(\mathrm{E}_\eta\left[G^2\left(Z_1^j,\dots,Z_I^j\right)\frac{\varphi^2\left(Z_1^j,\dots,Z_I^j\right)}{\eta^2\left(Z_1^j,\dots,Z_I^j\right)}\right]-\mathrm{E}_\eta^2\left[G\left(Z_1^j,\dots,Z_I^j\right)\frac{\varphi\left(Z_1^j,\dots,Z_I^j\right)}{\eta\left(Z_1^j,\dots,Z_I^j\right)}\right]\right)\\&\quad=\frac{1}{N}\left(\mathrm{E}_\varphi\left[G^2\left(Z_1^j,\dots,Z_I^j\right)\frac{\varphi\left(Z_1^j,\dots,Z_I^j\right)}{\eta\left(Z_1^j,\dots,Z_I^j\right)}\right]-\mathrm{E}_\varphi^2\left[G\left(Z_1^j,\dots,Z_I^j\right)\right]\right).\end{aligned}$$

Or in other words, it strongly depends on the choice of the probability density function $\eta(\cdot)$, if X_{IS} has a smaller variance than X.

A common way to find a suitable probability density function is to restrict the sampling path of the reference asset on intervals, which are important for the valuation of the note. Intuitively, the important paths of the underlying lie above the coupon barrier $b \cdot S_0$ and below the autocall barrier S_0.[7]

As a consequence, Z_i must lie in the interval

$$\left[\frac{\ln\left(b \cdot \frac{S_0}{S_{i-1}}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sqrt{\Delta t} \cdot \sigma}, \frac{\ln\left(\frac{S_0}{S_{i-1}}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sqrt{\Delta t} \cdot \sigma}\right]$$

with $0<b<1$ and $\Delta t = t_{i+1} - t_i$ for all $i \in \mathbb{N}$. Let u_i be an uniformly distributed random variable over [0,1] and let

$$p_i = \mathbb{P}(S_i < S_0 | S_{i-1} = s_{i-1}) \text{ resp. } q_i = \mathbb{P}(b < S_i/S_0 | S_{i-1} = s_{i-1})$$

be probabilities that the price of the underlying will be observed below the autocall barrier resp. above the coupon barrier in the next time step, then the actual simulation is described by $\Phi^{-1}(p_i u_i - (1 - q_i) u_i)$ with

$$p_i = \Phi\left(\frac{\ln\left(\frac{S_0}{S_{i-1}}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sqrt{\Delta t} \cdot \sigma}\right), q_i = 1 - \Phi\left(\frac{\ln\left(b \cdot \frac{S_0}{S_{i-1}}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sqrt{\Delta t} \cdot \sigma}\right).$$

Since this result is biased, the missing barrier hits must be included. Hence, we use the modified Monte Carlo estimator

$$\hat{X}_{IS} = \frac{1}{N} \sum_{i=1}^{I} \tilde{G}\left(z_1^j, \dots, z_I^j\right)$$

with

$$\begin{aligned}\tilde{G}\left(z_1^j, \dots, z_I^j\right) = &\sum_{i=1}^{I} e^{-rt_i} \cdot i \cdot c \cdot q_i \cdot \prod_{m=1}^{i-1} p_m \\ &- \sum_{i=1}^{I} \sum_{k=1}^{i-1} e^{-rt_i} k \cdot c \cdot [p_k - (1 - q_k)] \cdot q_i \prod_{j=k+1}^{i-1} \left(1 - q_j\right) \prod_{m=1}^{k-1} p_m \\ &- e^{-rt_I} \cdot b \cdot (1 - \nu) \cdot (1 - q_I) \cdot \prod_{m=1}^{I-1} p_m\end{aligned} \quad \textbf{(6)}$$

[7] This is the same approach as in Alm et al (2013) for Autocallable Notes, which is finally based on Glasserman and Staum (2001).

as well as $\nu \in [0,1]$, since the probabilities p_i and q_i depend explicitly on the output $z_1,\ldots,z_I$. Splitting the integrals of $E_\varphi[G(Z_1, \ldots,Z_I)]$ into different parts and taking into account the independence of $Z_1,\ldots,Z_I$ and the relating product form of the joint probability density function[8], this yields

$$
\begin{aligned}
&\mathrm{E}_\varphi\left[G\left(Z_1^j,\ldots,Z_I^j\right)\right] \\
&= \int\cdots\int\left(\sum_{i=1}^{I} e^{-rt_i}C_i - e^{-rt_I}\delta\right)\varphi_1(z_1^j)\cdot\ldots\cdot\varphi_I(z_I^j)dz_1^j\ldots dz_I^j \\
&= \int\cdots\int\sum_{i=1}^{I} e^{-rt_i}\cdot i\cdot c\cdot \mathbb{1}_{\left\{\frac{S_i}{S_0}\geq b\right\}}\cdot\prod_{m=1}^{i}\mathbb{1}_{\left\{\frac{S_m}{S_0}\leq 1\right\}}\varphi_1(z_1^j)\cdot\ldots\cdot\varphi_I(z_I^j)dz_1^j\ldots dz_I^j \\
&\quad -\int\cdots\int\sum_{i=1}^{I} e^{-rt_i}\sum_{k=1}^{i-1} k\, c\, \mathbb{1}_{\left\{\frac{S_k}{S_0}\geq b\right\}}\mathbb{1}_{\left\{\frac{S_i}{S_0}\geq b\right\}}\prod_{j=k+1}^{i-1}\mathbb{1}_{\left\{\frac{S_j}{S_0}< b\right\}}\prod_{m=1}^{i}\mathbb{1}_{\left\{\frac{S_m}{S_0}\leq 1\right\}}\varphi_1(z_1^j)\cdot\ldots\cdot\varphi_I(z_I^j)dz_1^j\ldots dz_I^j \\
&\quad -e^{-rt_I}\int\cdots\int\frac{S_I}{S_0}\cdot\mathbb{1}_{\left\{\frac{S_I}{S_0}< b\right\}}\cdot\prod_{k=1}^{I-1}\mathbb{1}_{\left\{\frac{S_k}{S_0}\leq 1\right\}}\varphi_1(z_1^j)\cdot\ldots\cdot\varphi_I(z_I^j)dz_1^j\ldots dz_I^j
\end{aligned}
$$

Replacing the marginal probability density function $\varphi_i(z_i^j)$ with

$$
\eta_i\left(z_i^j\right) = \frac{\varphi_i\left(z_i^j\right)}{p_i-(1-q_i)}\cdot\mathbb{1}_{\left\{\frac{S_i\left(z_i^j\right)}{S_0}\leq 1\right\}}\cdot\mathbb{1}_{\left\{\frac{S_i\left(z_i^j\right)}{S_0}> b\right\}}
$$

only for those integrals, which contain the condition $S_i/S_0>b$, the whole term can be simply integrated piece by piece, so that we get

$$
\begin{aligned}
&\sum_{i=1}^{I} e^{-rt_i}\cdot i\cdot c\cdot q_i\cdot\prod_{m=1}^{i-1}p_m \\
&\quad -\sum_{i=1}^{I}\sum_{k=1}^{i-1} e^{-rt_i}\cdot k\cdot c\cdot \mathrm{E}_\varphi\left[\mathbb{1}_{\left\{\frac{S_k}{S_0}\geq b\right\}}\mathbb{1}_{\left\{\frac{S_i}{S_0}\geq b\right\}}\prod_{j=k+1}^{i-1}\mathbb{1}_{\left\{\frac{S_j}{S_0}< b\right\}}\prod_{m=1}^{k}\mathbb{1}_{\left\{\frac{S_m}{S_0}\leq 1\right\}}\right] \\
&\quad\quad -e^{-rt_I}\cdot b\cdot(1-\nu)\cdot(1-q_I)\cdot\prod_{m=1}^{I-1}p_m
\end{aligned}
$$

for $E_\varphi[G(Z_1, \ldots,Z_I)]$ and finally

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[8] The joint probability density function of a standard normal distribution takes the form

$$
\begin{aligned}
\frac{1}{(2\pi)^{d/2}}\exp\left(-\frac{1}{2}\left[x_1^2+x_2^2+\cdots+x_d^2\right]\right) &= \frac{1}{\left(\sqrt{2\pi}\right)^d}\cdot\exp\left(-\frac{1}{2}x_1^2\right)\cdot\ldots\cdot\exp\left(-\frac{1}{2}x_d^2\right) \\
&= \prod_{i=1}^{d}\frac{1}{\sqrt{2\pi}}\cdot\exp\left(-\frac{1}{2}x_i^2\right)
\end{aligned}
$$

which is equivalent to its product of marginal densities.

$$\sum_{i=1}^{I} e^{-rt_i} \cdot i \cdot c \cdot q_i \cdot \prod_{m=1}^{i-1} p_m$$

$$-\sum_{i=1}^{I}\sum_{k=1}^{i-1} e^{-rt_i} k\, c[p_k - (1-q_k)]\mathrm{E}_{\varphi_1,\dots,\eta_k,\dots,\varphi_i}\left[\mathbb{1}_{\left\{\frac{S_i}{S_0}\geq b\right\}} \frac{\mathbb{1}_{\left\{\frac{S_k}{S_0}\geq b\right\}}\mathbb{1}_{\left\{\frac{S_k}{S_0}\leq 1\right\}}}{p_k-(1-q_k)} \prod_{j=k+1}^{i-1} \mathbb{1}_{\left\{\frac{S_j}{S_0}<b\right\}} \prod_{m=1}^{k-1} \mathbb{1}_{\left\{\frac{S_m}{S_0}\leq 1\right\}}\right]$$

$$-e^{-rt_I} \cdot b \cdot (1-\nu) \cdot (1-q_I) \cdot \prod_{m=1}^{I-1} p_m$$

which equals $\widetilde{G}(Z_1, \dots, Z_I)$ as of (6). The final payoff expression results from solving the inner integral of

$$e^{-rt_I} \int \cdots \int \frac{S_I}{S_0} \cdot \mathbb{1}_{\left\{\frac{S_I}{S_0}<b\right\}} \cdot \prod_{k=1}^{I-1} \mathbb{1}_{\left\{\frac{S_k}{S_0}\leq 1\right\}} \varphi_1(z_1^j) \cdot \ldots \cdot \varphi_I(z_I^j) dz_1^j \ldots dz_I^j$$

with the integration by parts formula and using the mean value theorem for integrals afterwards, more precisely

$$\frac{e^{-rt_I}}{S_0}\left[S_0 \cdot b \cdot (1-q_I) - \int_0^{S_0 b} \Phi_I(z_I^j) dz_I^j\right] \prod_{m=1}^{I-1} p_m = e^{-rt_I} \cdot b \cdot (1-q_I)(1-\nu) \prod_{m=1}^{I-1} p_m$$

where the parameter $\nu \in [0,1]$ should be properly estimated.[9]

It remains to verify, how big the impact of the variance reduction will be. Based on the results above, we obtain for the case that the denominator is not 0

$$\frac{\varphi\left(z_1^j, \dots, z_I^j\right)}{\eta\left(z_1^j, \dots, z_I^j\right)} = \prod_{i=1}^{I} \frac{\varphi\left(z_i^j\right)}{\eta\left(z_i^j\right)} = \prod_{i=1}^{I} p_i (1-q_i) \left(\mathbb{1}_{\left\{\frac{S_i}{S_0}\leq 1\right\}} \cdot \mathbb{1}_{\left\{\frac{S_i}{S_0}>b\right\}}\right)^{-1} \leq \prod_{i=1}^{I} p_i < 1$$

if at least one $i \in \{1, \dots I\}$ exists with $p_i < 1$. Thus, equation (C) holds and the variance reduction is most effective if at least one p_i is close to zero, or in other words, if the exotic call option is deep-in-the-money.

[9] Matching the prices of other counterparties the parameter ν can typically set to a value between 0.2 and 0.3.

5 Sensitivities

Various techniques of calculating sensitivities with Monte Carlo including the **Finite-Difference method** and the **Pathwise Derivative method** are discussed in Glasserman (2004) (§7). Although the latter one avoids the generation of an additional random variable and thus saves computational time, we still show the problems applying this method to a Memory Autocallable Note and give reasons why the Finite-Difference method is preferable. Consequently, the Finite Difference method in combination with the Importance Sampling technique as it is seen in Alm et al (2013) is a powerful tool for calculating sensitivities.

The basic idea of the Pathwise Derivative method is the following functional dependence for a given payoff function $G(Z_1,\ldots,Z_I)$ as in (3) and a parameter ψ[10]

$$\mathrm{E}\left[\frac{dG(Z_1,\ldots,Z_I,\psi)}{d\psi}\right]=\frac{d\mathrm{E}[G(Z_1,\ldots,Z_I,\psi)]}{d\psi},$$

if the interchange of integration and differentiation is justified. In this case, the left side of the equation indicates an estimator for the sensitivity of the structure.

However, since the payoff function has discontinuities over the Zi's (see Figure 2 and Figure 3), the Pathwise Derivative method is not directly applicable. This problem can be avoided by using the Importance Sampling technique due to the fact that the payoff function is Lipschitz continuous on

$$\left[\frac{\ln\left(b\cdot\frac{S_0}{S_{i-1}}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t}{\sqrt{\Delta t}\cdot\sigma},\frac{\ln\left(\frac{S_0}{S_{i-1}}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t}{\sqrt{\Delta t}\cdot\sigma}\right]$$

with $0<b<1$, which enables us the interchange of integration and differentiation. Let p'_i and q'_i be the pathwise derivatives of the probabilities p_i resp. q_i with respect to the parameter ψ, the use of elementary differentiation rules provides us the following Monte Carlo estimator for the requested sensitivity

$$\begin{aligned}&\tilde{G}'\left(z_1^j,\ldots,z_I^j,\psi\right)\\&=\sum_{i=1}^{I}e^{-rt_i}\cdot i\cdot c\cdot\left(q'_i\cdot\prod_{m=0}^{i-1}p_m+q_i\cdot\sum_{m=0}^{i-1}p'_m\prod_{\substack{k=0\\k\neq m}}^{i-1}p_k\right)\\&\quad-\sum_{i=1}^{I}\sum_{k=1}^{i-1}e^{-rt_i}\cdot k\cdot c\cdot\left(q'_i\cdot[p_k-(1-q_k)]+q_i\cdot[p'_k+q'_k]\right)\cdot\prod_{m=0}^{k-1}p_m\prod_{j=k+1}^{i-1}(1-q_j)\end{aligned}$$

[10] If ψ is the initial price of the underlying, then the sensitivity reflects the option's delta. If ψ stands for the volatility, then the option's vega is calculated.

$$-\sum_{i=1}^{I}\sum_{k=1}^{i-1} e^{-rt_i}\cdot k\cdot c\cdot q_i\cdot[p_k-(1-q_k)]\cdot\left(\sum_{m=0}^{k-1} p'_m\prod_{\substack{l=0\\l\neq m}}^{k-1} p_l\right)\cdot\prod_{j=k+1}^{i-1}(1-q_j)$$

$$+\sum_{i=1}^{I}\sum_{k=1}^{i-1} e^{-rt_i}\cdot k\cdot c\cdot q_i\cdot[p_k-(1-q_k)]\cdot\prod_{m=0}^{k-1} p_m\cdot\left(\sum_{j=k+1}^{i-1} q'_j\prod_{\substack{l=k+1\\l\neq j}}^{i-1}(1-q_l)\right)$$

$$+e^{-rt_I}\cdot(1-\nu)\cdot\left(q'_I\cdot\prod_{m=0}^{I-1} p_m+(q_I-1)\cdot\sum_{m=0}^{I-1} p'_m\prod_{\substack{k=0\\k\neq m}}^{I-1} p_k\right)$$

Clearly, an increasing number of valuation dates leads to a relatively large number of operations. The dependence of Z_i on S_{i-1} is another disadvantage, because each calculation of the partial derivatives p'_i and q'_i with respect to S_i contributes a term, which increases the complexity. In conclusion, the practical value of the Pathwise Derivative method is rather doubtful for our purposes.

Taking into account the already developed cash flow formulation of a Memory Autocallable Note, the Finite-Difference method as described in Glasserman (2004), Glasserman and Staum (2001) or Jäckel (2002) seems to be a more adequate way to evaluate the sensitivities. Here, taking the average of independent sample paths $\tilde{G}(z_1^j,\dots,z_I^j,\psi)$ and $\tilde{G}(z_1^j,\dots,z_I^j,\psi+h)$ with h>0, say

$$\hat{X}_{IS}^{\psi}=\frac{1}{N}\sum_{j=1}^{N}\tilde{G}(z_1^j,\dots,z_I^j,\psi)\text{ and }\hat{X}_{IS}^{\psi+h}=\frac{1}{N}\sum_{j=1}^{N}\tilde{G}(z_1^j,\dots,z_I^j,\psi+h),$$

we get the forward-difference estimator by

$$\Delta_\psi=\frac{\hat{X}_{IS}^{\psi+h}-\hat{X}_{IS}^{\psi}}{h}$$

Note that the sample paths are generated by the Importance Sampling method to save computational time.

Instead of using the forward-differencing approach, we can apply the centre-differencing method to obtain an approximation of the Gamma

$$\Delta_\psi=\frac{\hat{X}_{IS}^{\psi+h}-2\cdot\hat{X}_{IS}^{\psi}+\hat{X}_{IS}^{\psi-h}}{h}.$$

6 Multivariate Memory Autocallable Notes

A Multivariate Memory Autocallable Note depends on at least two underlyings. Similar to the univariate case, a coupon will be paid if each underlying closes at or above the (percentage) barrier at an observation date. Otherwise, the coupon is missed but could be paid at a subsequent coupon date. More precisely, let $M \in \mathbb{N}$ be the number of underlyings and $S_{t_i}^{(j)}$ be the closing price of the j-th underlying at an observation date t_i, a coupon is paid at $t_i \leq t_I$ if

$$\min\left\{\frac{S_{t_i}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_i}^{(M)}}{S_0^{(M)}}\right\} \geq b$$

holds. The closing prices can also be used to check, whether all underlyings reaches or exceeds its initial price. If this is the case, the note will be redeemed at pari. The early termination event is as follows:

$$\min\left\{\frac{S_{t_i}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_i}^{(M)}}{S_0^{(M)}}\right\} \geq 1.$$

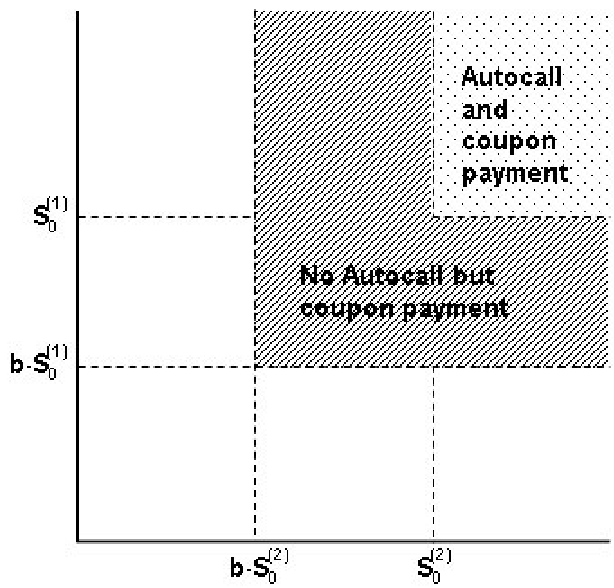

Figure 4: Trigger areas of a Duo Memory Autocallable Note for t_i<t_I

The final payoff at t_I is adapted to the worst performance of the underlyings, if at least one underlying falls below its barrier. It is described by

$$\min\left\{\frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}}\right\} \cdot \mathbb{1}_{\left\{\min\left\{\frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}}\right\} < b\right\}}.$$

Considering a Duo Memory Autocallable Note as a special case of a Multivariate Memory Autocallable Note with M=2,[11] its trigger areas defined by the "min-functions" above are illustrated in Figure 4 resp. Figure 5.

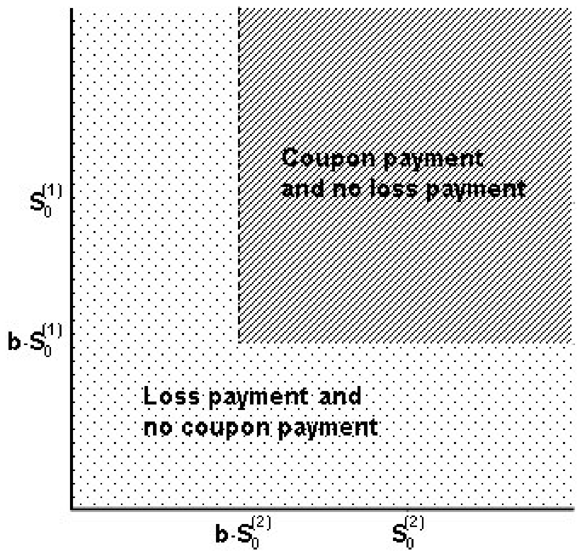

Figure 5: Trigger areas of a Duo Memory Autocallable Note at maturity t_I

Clearly, a Multivariate Memory Autocallable Note is valuated with Monte Carlo as this is also true for the univariate case. We require for this some standard properties of the multivariate normal distribution and explicit cash flow formulas, which we obtain in §6.1 with some slightly modifications as in §2. Not surprisingly, we finally find out that the Importance Sampling method can also be used to make the valuation of Multivariate Memory Autocallable Notes more effective. Concrete formulas of a Duo Memory Autocallable Note contribute to a better understanding of the results.

6.1 Mathematical Formulation of the Multivariate Cash Flows

Before the mathematical formulas of the coupon payments C_i and the final cash flow δ are derived, we show some helpful results concerning the calculus with indicator functions first, which can be easily verified.

1. For any $b \in \mathbb{R}_+$ the following equation holds:

$$\mathbb{1}_{\left\{\min\left\{\frac{S_{t_i}^{(1)}}{S_0^{(1)}},\ldots,\frac{S_{t_i}^{(M)}}{S_0^{(M)}}\right\}<b\right\}} = 1 - \mathbb{1}_{\left\{\min\left\{\frac{S_{t_i}^{(1)}}{S_0^{(1)}},\ldots,\frac{S_{t_i}^{(M)}}{S_0^{(M)}}\right\}\geq b\right\}}. \quad \textbf{(7)}$$

[11] This is the most common version in the German market.

2. Let M_i be a function defined by

$$M_i := \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_i}^{(m)}}{S_0^{(m)}} \geq b\right\}} \cdot \prod_{n=1}^{i-1} \left(1 - \mathbb{1}_{\left\{\min\left\{\frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}}\right\} \geq 1\right\}} \right), \tag{8}$$

then with $t_2 \leq t_{i'} < t_i$

$$M_i \cdot M_{i'} = M_i \cdot \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{i'}}^{(m)}}{S_0^{(m)}} \geq b\right\}}. \tag{9}$$

For any arbitrary $i \in \mathbb{N}$ with $t_i \leq t_l$ we can deduce from the iterative form of the coupon payments

$$C_i = \left(i \cdot c - \sum_{k=1}^{i-1} C_k \right) \cdot \mathbb{1}_{\left\{\min\left\{\frac{S_{t_i}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_i}^{(M)}}{S_0^{(M)}}\right\} \geq b\right\}} \cdot \prod_{n=1}^{i-1} \mathbb{1}_{\left\{\min\left\{\frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}}\right\} < 1\right\}}$$

$$\stackrel{(7)}{=} \left(i \cdot c - \sum_{k=1}^{i-1} C_k \right) \cdot \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_i}^{(m)}}{S_0^{(m)}} \geq b\right\}} \cdot \prod_{n=1}^{i-1} \left(1 - \mathbb{1}_{\left\{\min\left\{\frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}}\right\} \geq 1\right\}} \right)$$

$$\stackrel{(8)}{=} \left(i \cdot c - C_{i-1} - \sum_{k=1}^{i-2} C_k \right) \cdot M_i$$

$$= \left(i \cdot c - \left[(i-1) \cdot c - \sum_{k=1}^{i-2} C_k \right] \cdot M_{i-1} - \sum_{k=1}^{i-2} C_k \right) \cdot M_i$$

$$\stackrel{(9)}{=} \left(i \cdot c - \left[(i-1) \cdot c - \sum_{k=1}^{i-2} C_k \right] \cdot \prod_{\boldsymbol{m=1}}^{\boldsymbol{M}} \mathbb{1}_{\left\{\frac{S_{t_{i-1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} - \sum_{k=1}^{i-2} C_k \right) \cdot M_i$$

$$= \left(i \cdot c - (i-1) \cdot c \cdot \prod_{\boldsymbol{m=1}}^{\boldsymbol{M}} \mathbb{1}_{\left\{\frac{S_{t_{i-1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} - \left[1 - \prod_{\boldsymbol{m=1}}^{\boldsymbol{M}} \mathbb{1}_{\left\{\frac{S_{t_{i-1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} \right] \cdot \sum_{k=1}^{i-2} C_k \right) \cdot M_i = \cdots,$$

by factoring $1 - \Pi^{M}_{m=1} \mathbb{1}_{\{S_{t_{i'}}^{(m)}/S_0^{(m)} \geq b\}}$ at each step and using (9) and we obtain at the end of the calculation

$$C_i = \left(i \cdot c - \sum_{k=1}^{i-1} k \cdot c \cdot \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_k}^{(m)}}{S_0^{(m)}} \geq b\right\}} \prod_{n=k+1}^{i-1} \left[1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq b\right\}} \right] \right) \cdot M_i$$

$$\delta = \min\left\{\frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}}\right\} \cdot \mathbb{1}_{\left\{\min\left\{\frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}}\right\} < b\right\}} \cdot \prod_{n=1}^{I-1}\left(1 - \mathbb{1}_{\left\{\min\left\{\frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}}\right\} \geq 1\right\}}\right) \quad \textbf{(10)}$$

with the usual convention, that we set

$$\sum_{i=m}^{n} a_i := 0 \text{ and } \prod_{i=m}^{n} a_i \coloneqq 1$$

for $n,m \in \mathbb{N}$ with $m>n$. If the multidimensional process of M underlyings $(S_t^{(1)},\ldots,S_t^{(M)})$ follows a **multidimensional geometric Brownian motion** with a drift vector $(\mu_1,\ldots,\mu_M)$ and a covariance matrix, then each component can be characterized by a stochastic differential equation of the form

$$\frac{dS_t^{(m)}}{S_t^{(m)}} = \mu_m dt + \sigma_m dX_m(t), \quad m = 1, \ldots, M,$$

where each X_m denotes a one-dimensional geometric Brownian motion and ρ_{mn} indicates the correlation between $X_m(t)$ and $X_n(t)$.[12] The (componentwise) solution is given by

$$S_t^{(m)} = S_0^{(m)} \cdot \exp\left(\left[\mu_m - \tfrac{1}{2}\sigma_m^2\right] t + \sigma_m X_m(t)\right), \quad m = 1, \ldots, M.$$

Finally, the Euler's method provides us

$$S_{t+\Delta t}^{(m)} = S_t^{(m)} \cdot \exp\left(\left[\mu_m - \frac{1}{2}\sigma_m^2\right] \cdot \Delta\, t + \sigma_m \cdot \sqrt{\Delta\, t} \cdot [X_m(t + \Delta\, \mathrm{t}) - X_m(t)]\right) \quad \textbf{(11)}$$

for $m=1,\ldots,M$. Since the increments $X_m(t+\Delta t) - X_m(t)$ are not independent and not standard normally distributed for $m=1,\ldots,M$, we can achieve this by a linear transformation.

6.2 *Linear Transformation of the Multivariate Normal Distribution*

A vector-valued (d-dimensional) random variable $Z \in \mathbb{R}^d$ is said to have a multivariate normal distribution with mean μ and covariance matrix Σ, if its probability density function is given by[13]

$$f_Z(z) = \frac{1}{(2\pi)^{d/2} \cdot \sqrt{\det \Sigma}} \cdot \exp\left\{-\frac{1}{2}(z-\mu)^T \cdot \Sigma^{-1} \cdot (z-\mu)\right\}$$

We write this as $Z \sim N_d(\mu,\Sigma)$ and proceed with the **linear transformation property**.

---

[12] Vgl. [1], §3.2.3
[13] Vgl. [1], §2.3.1

Let Z ~ $N_d(\mu,\Sigma)$. Furthermore, assume that B is a real q x d matrix with q ≤ d and a is a q-dimensional scalar vector. Then Y = a + BZ is a q-dimensional random variable with Y ~ $N_q(a+B\mu,B\Sigma B^T)$.

Hence, we are able to generate a q-dimensional non-standard normally distributed random variable by a d-dimensional standard normally distributed random variable Z with Z ~ $N_d(0,\mathrm{I})$, where I denotes the unit matrix. This yields

$$Y = a + B\cdot Z \rightarrow Y \sim N_q(a,B\cdot B^T)$$

and the aim is to find a covariance matrix of the form $B\cdot B^T$.

By considering the important case of a **Duo Memory Autocallable Note** (d=q=2)[14], a solution of the covariance matrix

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\cdot\sigma_1\cdot\sigma_2 \\ \rho\cdot\sigma_1\cdot\sigma_2 & \sigma_2^2 \end{pmatrix}$$

is given by

$$\mathrm{B} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\cdot\sigma_2 & \sqrt{1-\rho^2}\cdot\sigma_2 \end{pmatrix}, \mathrm{B}^\mathrm{T} = \begin{pmatrix} \sigma_1 & \rho\cdot\sigma_2 \\ 0 & \sqrt{1-\rho^2}\cdot\sigma_2 \end{pmatrix}$$

Therefore, a two-dimensional random variable $Y=(Y_1,Y_2)$, which is $N_2(a,BB^T)$ distributed, can be generated componentwise by

$$\begin{aligned} Y_1 &= a_1 + \sigma_1\cdot Y_1^* \\ Y_2 &= a_2 + \rho\cdot\sigma_2\cdot Y_1^* + \sqrt{1-\rho^2}\cdot\sigma_2\cdot Y_1^* \end{aligned}$$

where Y^*_1 and Y^*_2 are independent standard normally distributed random variables.

6.3 Standard Monte Carlo Simulation

Let $0=t_0<t_1<\ldots<t_l$ be a set of observation dates. Thus, from §6.2 we can deduce for (11)

$$S_{t_{k+1}}^{(m)}(\mathbf{Z}_{k+1}) = S_{t_k}^{(m)}\cdot\exp\left(\left[\mu_m - \frac{1}{2}\sigma_m^2\right](t_{k+1}-t_k) + \sqrt{(t_{k+1}-t_k)}\sum_{n=1}^{M} B_{m,n}\cdot Z_{k+1,n}\right), \qquad \textbf{(12)}$$

where $B_{m,n}$ stands for the entry in the m-th row and n-th column of the matrix B according to the Cholesky decomposition and $\mathbf{Z_{k+1}}=(Z_{k+1,1},\ldots,Z_{k+1,M})$ denotes an independent sequence of standard normally distributed random vectors. We will sometimes use the term $S_{t_{k+1}}^{(m)}$ later in this work.

[14] Vgl. [1], §2.3.3

The paths of a **Duo Memory Autocallable Note** are generated by

$$
\begin{aligned}
S_{t_{k+1}}^{(1)} &= S_{t_k}^{(1)} \cdot \exp\left(\left[\mu_1 - \frac{1}{2}\sigma_1^2\right](t_{k+1} - t_k) + \sqrt{(t_{k+1} - t_k)} \cdot \sigma_1 \cdot Z_{k+1,1}\right) \\
S_{t_{k+1}}^{(2)} &= S_{t_k}^{(2)} \cdot \exp\left(\left[\mu_2 - \frac{1}{2}\sigma_2^2\right](t_{k+1} - t_k)\right. \\
&\qquad \left. + \sqrt{(t_{k+1} - t_k)}\left[\rho \cdot \sigma_2 \cdot Z_{k+1,1} + \sqrt{1-\rho} \cdot \sigma_2 \cdot Z_{k+1,2}\right]\right).
\end{aligned}
$$

The cashflow formulation (10) and the discretization procedure of the underlying's stochastic behavior as in (12) provide us the following Monte Carlo estimator

$$
X = \frac{1}{N}\sum_{j=1}^{N}\left(\sum_{i=1}^{I} e^{-r\cdot t_i} \cdot C_i\left(z_1^j, \dots, z_i^j\right) - e^{-r\cdot t_I} \cdot \delta\left(z_1^j, \dots, z_I^j\right)\right), \tag{13}
$$

where $\mathbf{z}^{\mathbf{j}}_{\mathbf{i}}$ denotes the j-th realisation of $\mathbf{Z}_{\mathbf{i}}$.

6.4 Variance Reduction via Importance Sampling Technique

In this section we describe the implementation of the Importance Sampling technique, since we are interested in reducing the computational effort without loss of accuracy in the multivariate case as well. Due to its big effectiveness we will reduce ourselves to this approach.

Note first that we consider only these paths, which do not violate any coupon barrier ($b\cdot S_0^{(1)}$, $b\cdot S_0^{(2)}$,…, $b\cdot S_0^{(M)}$) as it is shown in Figure 4. Therefore, every $Z_{k+1,m}$ satisfying (12) is an element of

$$
\left[\frac{\ln\left(b\frac{S_0^{(m)}}{S_{t_k}^{(m)}}\right) - \left(\mu_m - \frac{1}{2}\sigma_m^2\right)\Delta t_{k+1}}{\sqrt{\Delta t_{k+1}} \cdot B_{m,m}} - \frac{1}{B_{m,m}} \cdot \sum_{\substack{n=1,\\ n\neq m}}^{M} B_{m,n} \cdot Z_{k+1,n}\, , \infty\right)
$$

with $\Delta t_{k+1} = t_{k+1} - t_k$. Suppose that

$$
\begin{aligned}
&q_{t_{k+1}}^{(m)}(\boldsymbol{z_{k+1}}, \dots, \boldsymbol{z_1}) \\
&\quad = \mathbb{P}\left(\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq b \,\middle|\, S_{t_k}^{(m)} = s_{t_k}^{(m)}\right) \\
&\quad = 1 - \Phi\left(\frac{\ln\left(b\frac{S_0^{(m)}}{S_{t_k}^{(m)}}\right) - \left(\mu_m - \frac{1}{2}\sigma_m^2\right)\Delta t_{k+1}}{\sqrt{\Delta t_{k+1}} \cdot B_{m,m}} - \frac{1}{B_{m,m}} \cdot \sum_{\substack{n=1,\\ n\neq m}}^{M} B_{m,n} \cdot z_{k+1,n}\right)
\end{aligned}
$$

is the conditional probability, that the price of the m-th underlying will be observed above the coupon barrier at t_{k+1}, a realization of $Z_{k+1,m}$ is given recursively by

$$z_{k+1,m} = \Phi^{-1}\left(q_{t_{k+1}}^{(m)}(\boldsymbol{z_{k+1}}, \dots, \boldsymbol{z_1}) \cdot u_{k+1,m}\right),$$

where $u_{k+1,m}$ is an uniformly distributed random variable on (0,1). Furthermore, we define the conditional probability, that the price of the m-th underlying will exceed its initial fixing at t_{k+1}, say

$$\begin{aligned}
&p_{t_{k+1}}^{(m)}(\boldsymbol{z_{k+1}}, \dots, \boldsymbol{z_1}) \\
&\quad = \mathbb{P}\left(\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq 1 \middle| S_{t_k}^{(m)} = s_{t_k}^{(m)}\right) \\
&\quad = q_{t_{k+1}|b=1}^{(m)}(\boldsymbol{z_{k+1}}, \dots, \boldsymbol{z_1}) \\
&\quad = 1 - \Phi\left(\frac{\ln\left(\frac{S_0^{(m)}}{S_{t_k}^{(m)}}\right) - \left(\mu_m - \frac{1}{2}\sigma_m^2\right)\Delta t_{k+1}}{\sqrt{\Delta t_{k+1}} \cdot B_{m,m}} - \frac{1}{B_{m,m}} \cdot \sum_{\substack{n=1,\\ n\neq m}}^{M} B_{m,n} \cdot z_{k+1,n}\right).
\end{aligned}$$

From here, will use the shorter notation $q_{t_{k+1}}^{(m)}$ resp. $p_{t_{k+1}}^{(m)}$ instead of $q_{t_{k+1}}^{(m)}(\boldsymbol{z_{k+1}}, \dots, \boldsymbol{z_1})$ resp. $p_{t_{k+1}}^{(m)}(\boldsymbol{z_{k+1}}, \dots, \boldsymbol{z_1})$. If the marginal distribution function $\varphi_{k+1}^{(m)}$ is replaced by $\eta_{k+1}^{(m)}$ via

$$\eta_{k+1}^{(m)}(z_{k+1,m}) \cdot q_{t_{k+1}}^{(m)} = \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} \cdot \varphi_{k+1}^{(m)}(z_{k+1,m}), \tag{14}$$

we will obtain the modified Monte Carlo estimator for the valuation of a multivariate Memory Autocallable Note using the Importance Sampling technique

$$\widetilde{X} = \frac{1}{N}\sum_{j=1}^{N}\left(\sum_{i=1}^{I} e^{-r\cdot t_i} \cdot \left[C_i^1\left(z_1^j, \dots, z_i^j\right) - C_i^2\left(z_1^j, \dots, z_i^j\right)\right] - e^{-r\cdot t_I} \cdot \delta\left(z_1^j, \dots, z_I^j\right)\right)$$

with

$$C_i^1\left(z_1^j, \dots, z_i^j\right) = i \cdot c \prod_{m=1}^{M} q_{t_i}^{(m)} \prod_{n=1}^{i-1}\left(1 - \prod_{m=1}^{M} p_{t_n}^{(m)}\right)$$

$$\begin{aligned}
&C_i^2\left(z_1^j, \dots, z_i^j\right) \\
&= \sum_{k=1}^{i-1} k \cdot c \prod_{n=k+1}^{i-1}\left(1 - \prod_{m=1}^{M} p_{t_n}^{(m)}\right) \prod_{p=1}^{k-1}\left(1 - \prod_{m=1}^{M} p_{t_p}^{(m)}\right)\left(\prod_{m=1}^{M} q_{t_k}^{(m)} - \prod_{m=1}^{M} p_{t_k}^{(m)}\right) \prod_{m=1}^{M} q_{t_i}^{(m)}
\end{aligned}$$

$$\delta\left(z_1^j, \dots, z_I^j\right) \approx \nu \cdot \left(1 - \prod_{m=1}^{M} q_{t_I}^{(m)}\right) \prod_{n=1}^{I-1}\left(1 - \prod_{m=1}^{M} p_{t_n}^{(m)}\right) \text{ with } \nu \epsilon (0, b),$$

which is derived as follows. Since all indicator functions of the coupon payment formula C_i, with the exception of M_i, depend on the maximum highest date t_{i-1}, we simply require for an arbitrary k∈{1,…,i-1}

$$\prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} \left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq 1\right\}}\right) = \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq 1\right\}} \tag{15}$$

$$\left(\mathbf{1} - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq b\right\}}\right)\left(\mathbf{1} - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq 1\right\}}\right) = \mathbf{1} - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_{k+1}}^{(m)}}{S_0^{(m)}} \geq b\right\}} \tag{16}$$

for the multiplication with M_i. The rest of the derivations follow from the change of probability measures as described in (14). Decomposing the undiscounted expected value $E[C_i]$ with respect to the $(i \cdot M)$-dimensional density function

$$\varphi_i = \prod_{n=1}^{i} \prod_{m=1}^{M} \varphi_n^{(m)}$$

leads to

$$\mathrm{E}_{\varphi_i}[C_i] = \mathrm{E}_{\varphi_i}[i \cdot c \cdot M_i] - \mathrm{E}_{\varphi_i}\left[\sum_{k=1}^{i-1} k \cdot c \cdot \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_k}^{(m)}}{S_0^{(m)}} \geq b\right\}} \prod_{n=k+1}^{i-1}\left[1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq b\right\}}\right] \cdot M_i\right]$$

$$= \mathrm{E}_{\varphi_i}\left[i \cdot c \cdot \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_i}^{(m)}}{S_0^{(m)}} \geq b\right\}} \cdot \prod_{n=1}^{i-1}\left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq 1\right\}}\right)\right]$$

$$-\mathrm{E}_{\varphi_i}\left[\sum_{k=1}^{i-1} k \cdot c \cdot \underbrace{\prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_k}^{(m)}}{S_0^{(m)}} \geq b\right\}}\left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_k}^{(m)}}{S_0^{(m)}} \geq 1\right\}}\right)}_{\text{s. (15)}} \cdot\right.$$

$$\left.\underbrace{\prod_{n=k+1}^{i-1}\left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq b\right\}}\right)\left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq 1\right\}}\right)}_{\text{s. (16)}} \cdot \prod_{p=1}^{k-1}\left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_p}^{(m)}}{S_0^{(m)}} \geq 1\right\}}\right) \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_i}^{(m)}}{S_0^{(m)}} \geq b\right\}}\right]$$

$$= \mathrm{E}_{\varphi_i}\left[i \cdot c \cdot \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_i}^{(m)}}{S_0^{(m)}} \geq b\right\}} \cdot \prod_{n=1}^{i-1} \left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq 1\right\}} \right) \right]$$

$$- \mathrm{E}_{\varphi_i}\left[\sum_{k=1}^{i-1} k \cdot c \left(\prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_k}^{(m)}}{S_0^{(m)}} \geq b\right\}} - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_k}^{(m)}}{S_0^{(m)}} \geq 1\right\}} \right) \prod_{n=k+1}^{i-1} \left(1 - \prod_{m=1}^{M} \mathbb{1}_{\left\{\frac{S_{t_n}^{(m)}}{S_0^{(m)}} \geq b\right\}} \right) \cdot \ \dots \right]$$

and finally from (14)

$$\mathrm{E}_{\eta_i}\left[i \cdot c \prod_{m=1}^{M} q_{t_i}^{(m)} \prod_{n=1}^{i-1} \left(1 - \prod_{m=1}^{M} p_{t_n}^{(m)} \right) \right]$$

$$- \mathrm{E}_{\eta_i}\left[\sum_{k=1}^{i-1} k \cdot c \left(\prod_{m=1}^{M} q_{t_k}^{(m)} - \prod_{m=1}^{M} p_{t_k}^{(m)} \right) \prod_{n=k+1}^{i-1} \left(1 - \prod_{m=1}^{M} p_{t_n}^{(m)} \right) \prod_{p=1}^{k-1} \left(1 - \prod_{m=1}^{M} p_{t_p}^{(m)} \right) \prod_{m=1}^{M} q_{t_i}^{(m)} \right].$$

Clearly,

$$\eta_i = \prod_{n=1}^{i} \prod_{m=1}^{M} \eta_n^{(m)}$$

stands for the joint distribution function on $\mathbb{R}^{i \cdot M}$.[15] To prove the modified Monte Carlo estimator for the terminal payoff δ, we get

$$\mathrm{E}_{\varphi_I}[\delta] = \mathrm{E}_{\varphi_I}\left[\min\left\{ \frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}} \right\} \cdot \mathbb{1}_{\left\{ \min\left\{ \frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}} \right\} < b \right\}} \cdot \prod_{n=1}^{I-1} \left(1 - \mathbb{1}_{\left\{ \min\left\{ \frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}} \right\} \geq 1 \right\}} \right) \right]$$

$$= \int \left(\int_0^b u \cdot f_U(u) du \right) \prod_{n=1}^{I-1} \left(1 - \mathbb{1}_{\left\{ \min\left\{ \frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}} \right\} \geq 1 \right\}} \right) \varphi_{I-1} \, d(\boldsymbol{z_1}, \dots, \boldsymbol{z_{I-1}})$$

---

[15] For a clear presentation of the mathematical equations, we do not apply the iterated integral formulation, however it is satisfied by the independence of the $Z_{i,m}$'s.

$$= \int \left[\nu \cdot \mathbb{P}\left(\min\left\{ \frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}} \right\} < b \,\middle|\, S_{t_{I-1}}^{(m)} = s_{t_{I-1}}^{(m)} \right) \right.$$

$$\left. \cdot \prod_{n=1}^{I-1} \left(1 - \mathbb{1}_{\left\{ \min\left\{ \frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}} \right\} \geq 1 \right\}} \right) \right] \varphi_{I-1} \, d(\boldsymbol{z_1}, \dots, \boldsymbol{z_{I-1}})$$

$$= \int \left[\nu \cdot \left(1 - \mathbb{P}\left(\min\left\{ \frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}} \right\} \geq b \,\middle|\, S_{t_{I-1}}^{(m)} = s_{t_{I-1}}^{(m)} \right) \right) \right.$$

$$\left. \cdot \prod_{n=1}^{I-1} \left(1 - \mathbb{1}_{\left\{ \min\left\{ \frac{S_{t_n}^{(1)}}{S_0^{(1)}}, \cdots, \frac{S_{t_n}^{(M)}}{S_0^{(M)}} \right\} \geq 1 \right\}} \right) \right] \varphi_{I-1} \, d(\boldsymbol{z_1}, \dots, \boldsymbol{z_{I-1}})$$

$$= \mathrm{E}_{\eta_I} \left[\nu \cdot \left(1 - \prod_{m=1}^{M} q_{t_I}^{(m)} \right) \cdot \prod_{n=1}^{I-1} \left(1 - \prod_{m=1}^{M} p_{t_n}^{(m)} \right) \right] \text{ with } \nu \epsilon (0, b),$$

where

$$U = \min\left\{ \frac{S_{t_I}^{(1)}}{S_0^{(1)}}, \dots, \frac{S_{t_I}^{(M)}}{S_0^{(M)}} \right\}$$

is a real random variable with density function $f_U(u)$ and the integral $\int U \cdot f_U(u)$ is solved by the mean value theorem for integrals analogous to the univariate case.

The modified Monte Carlo estimator for the undiscounted cash flows of a **3-year-Duo Memory Autocallable Note** at t_0 contains of

$$C_1\left(z_1^j, z_2^j, z_3^j\right) = c \cdot q_{t_1}^{(1)} \cdot q_{t_1}^{(2)}$$

$$C_2\left(z_1^j, z_2^j, z_3^j\right) = 2 \cdot c \cdot q_{t_2}^{(1)} q_{t_2}^{(2)} \left(1 - p_{t_1}^{(1)} p_{t_1}^{(2)}\right) - c \cdot q_{t_2}^{(1)} q_{t_2}^{(2)} \left(q_{t_1}^{(1)} q_{t_1}^{(2)} - p_{t_1}^{(1)} p_{t_1}^{(2)}\right)$$

$$\begin{aligned} C_3\left(z_1^j, z_2^j, z_3^j\right) = {} & 3 \cdot c \cdot q_{t_3}^{(1)} q_{t_3}^{(2)} \left(1 - p_{t_2}^{(1)} p_{t_2}^{(2)}\right)\left(1 - p_{t_1}^{(1)} p_{t_1}^{(2)}\right) \\ & -2 \cdot c \cdot q_{t_3}^{(1)} q_{t_3}^{(2)} \left(q_{t_2}^{(1)} q_{t_2}^{(2)} - p_{t_2}^{(1)} p_{t_2}^{(2)}\right)\left(1 - p_{t_1}^{(1)} p_{t_1}^{(2)}\right) \\ & -c \cdot q_{t_3}^{(1)} q_{t_3}^{(2)} \left(1 - p_{t_2}^{(1)} p_{t_2}^{(2)}\right)\left(q_{t_1}^{(1)} q_{t_1}^{(2)} - p_{t_1}^{(1)} p_{t_1}^{(2)}\right). \end{aligned}$$

$$\delta\left(z_1^j, z_2^j, z_3^j\right) \approx \nu \cdot \left(1 - q_{t_3}^{(1)} q_{t_3}^{(2)}\right)\left(1 - p_{t_2}^{(1)} p_{t_2}^{(2)}\right)\left(1 - p_{t_1}^{(1)} p_{t_1}^{(2)}\right) \text{ with } \nu \epsilon (0, b).$$

7 Numerical Results

In this section, we present numerical results using the above described simulation algorithms for some selected Memory Autocallable Notes. The examined Notes differ with respect to maturity or initial spot price, whereas the global parameters are the same for all (as given in Table 2).

Underlying	Allianz
Exchange	XETRA
Payment Dates	Annually
Initial Price S_0	125 EUR
Barrier Level B	75 EUR ($b \cdot S_0$ with b=0.6)
Memory Coupon	1) If $S_i \geq$ barrier level, then $coupon_i = \mathbf{i \cdot 8[\%] - \Sigma_{k<i} coupon_k}$ 2) If $S_i <$ barrier level, then $coupon_i=0$

Table 2: Global parameters of the numerical examples

To assess the general computational efforts of Memory Autocallable Notes, we consider a 2-year Memory Autocallable Note (E1) and 3-year Memory Autocallable Note (E2) in the standard scenario ($b \cdot S_0 \leq S_t \leq S_0$), where S_t denotes the current spot price of the underlying asset. As mentioned in Section 1, the valuation of (E2) requires 6 ($=3 \cdot 2^2 - 3 \cdot 2$) additional paths for consideration compared to the valuation of (E1).

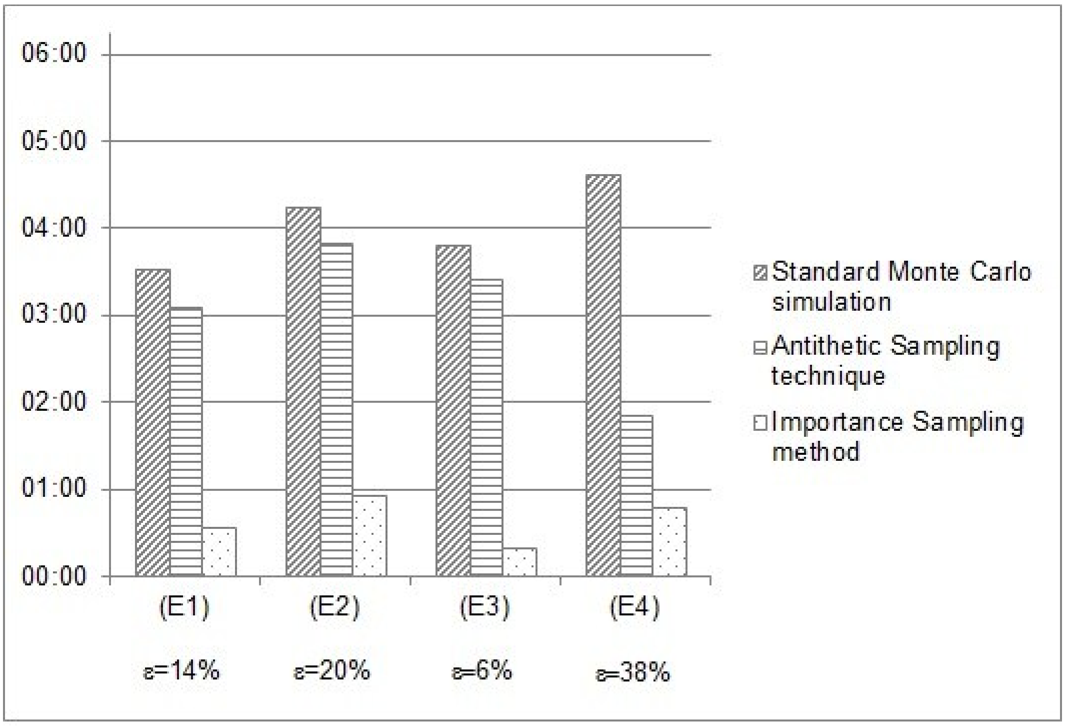

Figure 6: Study of the variance reduction effect concerning the computational time

Since the moneyness of the call option influences the standard error, we also consider a deep-in-the-money 3-year-Memory Autocallable Note with S_t=160 (E3) and a deep-out-of-the-money 3-year-Memory Autocallable Note with S_t=80 (E4). Obviously, (E3) indicates a high call probability and (E4) is characterized by a significantly high probability that the coupon will be missed (and the note will continue). It is clear, that (E3) will have the lowest standard error because of the low fluctuation of the present value.

Each of our examples is evaluated with the standard Monte Carlo simulation algorithm, the Antithetic Sampling technique and the Importance Sampling method as well. Figure 6 shows the computational times for comparison.

It is not surprising that the standard Monte Carlo simulation is beaten by both of the variance reduction techniques, but the impact differs according to the simulation method. In summary, the antithetic sampling method can reduce the computational time on average up to 20% with the cost of an additional computation of the function $G(-z_1,\ldots,-z_l)$, whereas the Importance Sampling method is more effective with a saving of computational time of more than 80 percent compared to the standard Monte Carlo simulation. This is also the case if we compute sensitivities with the Finite-Difference method and the Importance Sampling technique instead of using Standard Monte Carlo samples.

Finally, we note that we get a similar result of the variance reduction effect for the multivariate Memory Autocallable Note as in the univariate case.

8 Concluding Remarks and Outlook

We presented a closed formula of the cash flows of a univariate and multivariate Memory Autocallable Note. For the valuation of a univariate Memory Autocallable Note with an Euler scheme of a Geometric Brownian Motion we showed how to save computational time without loss of accuracy, using either the Antithetic Sampling technique or the Importance Sampling method, whereas we restrict ourselves to the Importance Sampling method in the multivariate case. Numerical examples demonstrated the higher performance of the Importance Sampling method. Furthermore, we demonstrated that the Finite-Difference method in combination with the Importance Sampling technique is the easiest and fastest way to compute sensitivities at least in the univariate case.

The results presented in this paper provide a basis for further research on Memory Autocallable Notes. One aspect of future research could be the integration of an accurate (dynamic) estimator for the parameter ν.

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