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An analytical approximation method for pricing barrier options under the double Heston model

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ABSTRACT
The purpose of the paper is to provide an efficient pricing method for single barrier options under the double Heston model. By rewriting the model as a singular and regular perturbed BS model, the double Heston model can separately mimic a fast time-scale and a slow time-scale. With the singular and regular perturbation techniques, we analytically derive the first-order asymptotic expansion of the solution to a barrier option pricing partial differential equation. The convergence and efficiency of the approximate method is verified by Monte Carlo simulation. Numerical results show that the presented asymptotic pricing method is fast and accurate.

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Asymptotic expansion; barrier option; option pricing; perturbation technique; stochastic volatility

1. Introduction

Barrier options, which are one of the most popular instruments for risk management, are extinguished (knock out) or activated (knock in) if the underlying asset prices hit the barriers throughout the lifetime of the options. The BS model (Black and Scholes 1973) which assumes the volatility is constant is inconsistent with the volatility smile and skew present in real markets. The diffusion-based single-factor stochastic volatility models have made many advances in fitting option prices and implied volatilities. A great deal of research evaluates barrier options under these models of which the most popular is the Heston model (Heston 1993). However, the evidence from Bergomi (2004), Gatheral (2006), Da Fonseca et al. (2008), Christoffersen et al. (2009), Fouque et al. (2011b) indicates that single-factor stochastic volatility models are poor in fitting short-maturity implied volatilities and the observed options prices. Moreover, they cannot generate steep smirks and flat smirks at a given volatility level. Da Fonseca et al. (2008) empirically show that the two-factor stochastic volatility models can offer more consistent European option prices and better fit of the smile-skew effect at short maturities compared to single-factor models. The double Heston model, a two-factor stochastic volatility model proposed by Christoffersen et al. (2009), can separately control the level and slope of the smile, which is consistent with the empirical finding. Li and Zhang (2010) verify that the double Heston model is adequate for pricing the index options and fitting the data in both time series and

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cross sections. Recently, Zhang and Feng (2019) verify that the double Heston model can capture volatility persistence which is a stylized fact regarding implied volatility surface (Gatheral et al. 2018). Besides, the double Heston model which consists of two uncorrelated processes remains computational tractability of the Heston model.

The main numerical methods for pricing barrier options under single-factor stochastic volatility models are the partial differential equation (PDE) methods, including Winkler et al. (2002), Yousuf (2007), Yousuf (2009), Hout and Foulon (2010), Chiarella et al. (2012), Seydel (2012), Haentjens (2013) and Monte Carlo simulations, including Glasserman (2004), Tian et al. (2012), Achtsis et al. (2013), Shevchenko and Moral (2017). In the context of a two-factor stochastic volatility model, valuating a barrier option leads to a three-dimensional system of stochastic partial differential equations which make the PDE-based methods quite complex and potentially prone to accuracy and stability problems. Monte Carlo simulations are easier to be extended to this case. Göncü and G. Ökten (2014) and Zhang and Zhao (2017) provide quasi-Monte Carlo and Monte Carlo solutions to single barrier options pricing under a two-factor stochastic volatility model, respectively. However, Monte Carlo simulations require substantially computing time to attain the specified accuracy which are not suitable for practical use.

An alternative strategy for pricing barrier options is to analytically derive approximate solutions based on perturbation techniques. The methods which only involve analytic terms are very fast. Besides, a competitive advantage of the perturbation methods is that they can reduce the model parameters to less group market parameters which are the ones needed to evaluate path-dependent options. In recent years many researchers use perturbation techniques to evaluate barrier options under single-factor stochastic volatility models. Based on a singular perturbation technique, Fouque et al. (2000), Ilhan et al. (2004), Park et al. (2010) and Fouque et al. (2011a) develop the first-order asymptotic expansion for single barrier options under a fast mean-reverting stochastic volatility model, respectively. Kato et al. (2013) develop the first-order asymptotic expansion for up-and-out barrier options under the SABR model with slow factor. Sousa et al. (2018) develop the similar expansion for down-and-out barrier options under 2-hypergeometric stochastic volatility model with slow factor. By combining singular and regular perturbation techniques, Fouque et al. (2011b) first evaluate down-and-out barrier options under a two-factor stochastic volatility model with fast and slow factors.

Motivated by the superior feature of the double Heston model and the power of perturbation techniques, this paper slightly modifies the double Heston model by rewriting the model as a singular and regular perturbed BS model. The obtained model which runs a fast-time scale and a slow-time scale can be used for pricing barrier options in the perturbation analytical framework. The main contribution of the paper is an analytical asymptotic formula for single barrier options pricing under the double Heston model framework. The paper is organized as follows. Section 2 presents the pricing model and problem statement. Section 3 details the pricing method. Section 4 verifies the convergence and efficiency of the proposed method by some numerical experiments. Section 5 concludes.

2. The model and problem statement

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\}$ be a complete probability space with a filtration continuous on the right, where P is a risk-neutral probability measure. Let $S(t)$ denote the asset price process, $W_1^s(t), W_2^s(t), W_1^v(t)$ and $W_2^v(t)$ be all Wiener process. Moreover, any two processes are uncorrelated each other except $Cov(dW_1^s(t), dW_1^v(t)) = \rho_1 dt, Cov(dW_2^s(t), dW_2^v(t)) = \rho_2 dt$. The double Heston model assumes the following stochastic differential equations

$$\begin{aligned} \frac{\mathrm{d}S(t)}{S(t)} &= r\mathrm{d}t + \sqrt{V_1(t)}\mathrm{d}W_1^s(t) + \sqrt{V_2(t)}\mathrm{d}W_2^s(t) \\ \mathrm{d}V_1(t) &= k_1(\theta_1 - V_1(t))\mathrm{d}t + \sigma_1\sqrt{V_1(t)}\mathrm{d}W_1^v(t) \\ \mathrm{d}V_2(t) &= k_2(\theta_2 - V_2(t))\mathrm{d}t + \sigma_2\sqrt{V_2(t)}\mathrm{d}W_2^v(t) \end{aligned} \tag{2.1}$$

where r is constant interest rate, k_j, θ_j and σ_j $(j=1,\ 2)$ denote the mean-reversion speeds, long-run volatility means and instantaneous volatilities of processes $V_j(t)$, respectively. To make $V_j(t)$ be positive processes, the Feller conditions (Feller 1951) $2k_j\theta_j \geq \sigma_j^2$ need to be satisfied. Suppose $V_1(0) = V_1, V_2(0) = V_2, S(0) = S$.

We exemplarily consider an up-and-out call (UOC) option. Other types of single barrier options can be treated similarly. Let $C_{UO}(S, V_1, V_2, t)$ denote the price of an UOC option with strike price K, barrier price B and maturity T. Assume that $C_{UO}(S, V_1, V_2, t)$ is at least twice differentiable with respect to underlying state variables. By the Feynman-Kac theorem, $C_{UO}(S, V_1, V_2, t)$ is the solution to the following three-dimensional PDE with terminal and boundary value problem

$$\begin{aligned} &\mathcal{L}C_{UO} = 0 \\ &C_{UO}(S, V_1, V_2, T) = \max(S-K, 0), \quad 0<S<B \\ &C_{UO}(B, V_1, V_2, t) = 0 \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{L} = {} & k_1(\theta_1 - V_1)\frac{\partial}{\partial V_1} + \frac{1}{2}\sigma_1^2 V_1 \frac{\partial^2}{\partial V_1^2} + \rho_1\sigma_1 S V_1 \frac{\partial^2}{\partial S \partial V_1} + \frac{\partial}{\partial t} + rS\frac{\partial}{\partial S} \\ & + \frac{1}{2}(V_1 + V_2)S^2\frac{\partial^2}{\partial S^2} - r - \lambda_1 V_1 \frac{\partial}{\partial V_1} - \lambda_2 V_2 \frac{\partial}{\partial V_2} + \rho_2\sigma_2 S V_2 \frac{\partial^2}{\partial S \partial V_2} \\ & + k_2(\theta_2 - V_2)\frac{\partial}{\partial V_2} + \frac{1}{2}\sigma_2^2 V_2 \frac{\partial^2}{\partial V_2^2} \end{aligned}$$

$\lambda_1 V_1$ and $\lambda_2 V_2$ are risk premiums produced by process $V_1(t)$ and $V_2(t)$, respectively. One can choose the values of λ_1 and λ_2 as in Canhanga et al. (2018). However, we set $\lambda_1 = \lambda_2 = 0$ for simplicity throughout the paper. The state variables are defined in the domains $0<V_1(t), V_2(t)<\infty, 0<S(t)<B$.

3. Approximating barrier options prices by perturbation techniques

Our method is based on singular and regular perturbation techniques. To this end, we introduce two small parameters ϵ, δ $(0<\epsilon, \delta \ll 1)$ and rewrite model (Equation 2.1) as

$$
\begin{aligned}
\frac{\mathrm{d}S^{\epsilon,\delta}(t)}{S^{\epsilon,\delta}(t)} &= r\mathrm{d}t + \sqrt{V_1^{\epsilon}(t)}\mathrm{d}W_1^s(t) + \sqrt{V_2^{\delta}(t)}\mathrm{d}W_2^s(t) \\
\mathrm{d}V_1^{\epsilon}(t) &= \frac{1}{\epsilon}\left(\theta_1 - V_1^{\epsilon}(t)\right)\mathrm{d}t + \frac{\sigma_1}{\sqrt{\epsilon}}\sqrt{V_1^{\epsilon}(t)}\mathrm{d}W_1^v(t) \\
\mathrm{d}V_2^{\delta}(t) &= \delta\Big(\theta_2 - V_2^{\delta}(t)\Big)\mathrm{d}t + \sqrt{\delta}\sigma_2\sqrt{V_2^{\delta}(t)}\mathrm{d}W_2^v(t)
\end{aligned}
\tag{3.1}
$$

thus volatility process $V_1^{\epsilon}(t)$ and $V_2^{\delta}(t)$ capture the effects of high and low frequency trading factors, respectively which accordingly produce a singular perturbation and a regular perturbation. To make $V_1^{\epsilon}(t)$ and $V_2^{\delta}(t)$ be both positive processes, the Feller conditions $2\theta_j \geq \sigma_j^2$ need to be satisfied.

Remark. The similar modified model was used for pricing European options in Canhanga et al. (2018).

With the modified model (Equation 3.1), Equation (2.2) can be accordingly written as

$$
\begin{aligned}
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{U}_1 + \delta\mathcal{U}_2\right)C_{UO}^{\epsilon,\delta} = 0 \\
&C_{UO}^{\epsilon,\delta}(S, V_1, V_2, T) = \max(S-K, 0), \quad 0<S<B \\
&C_{UO}^{\epsilon,\delta}(B, V_1, V_2, t) = 0
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
\mathcal{L}_0 &= (\theta_1 - V_1)\frac{\partial}{\partial V_1} + \frac{1}{2}\sigma_1^2 V_1 \frac{\partial^2}{\partial V_1^2} \\
\mathcal{L}_1 &= \rho_1\sigma_1 V_1 S \frac{\partial^2}{\partial S \partial V_1} \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + rS\frac{\partial}{\partial S} + \frac{1}{2}(V_1 + V_2)S^2\frac{\partial^2}{\partial S^2} - r \\
\mathcal{U}_1 &= \rho_2\sigma_2 S V_2 \frac{\partial^2}{\partial S \partial V_2} \\
\mathcal{U}_2 &= (\theta_2 - V_2)\frac{\partial}{\partial V_2} + \frac{1}{2}\sigma_2^2 V_2 \frac{\partial^2}{\partial V_2^2}
\end{aligned}
$$

We first expand $C_{UO}^{\epsilon,\delta}$ in powers of δ followed as

$$
C_{UO}^{\epsilon,\delta} = C_0^{\epsilon} + \sqrt{\delta}C_1^{\epsilon} + \delta C_2^{\epsilon} + \delta\sqrt{\delta}C_3^{\epsilon} + \cdots \tag{3.3}
$$

then expand C_i^{ϵ} $(i = 0, 1, 2, \ldots)$ in powers of ϵ followed as

$$
C_i^{\epsilon} = C_{0,i} + \sqrt{\epsilon}C_{1,i} + \epsilon C_{2,i} + \epsilon\sqrt{\epsilon}C_{3,i} + \cdots \tag{3.4}
$$

The approximate solution to the UOC option pricing PDE Equation (3.2) can be obtained by combining singular perturbation expansion Equation (3.4) and regular perturbation expansion Equation (3.3) as

$$
C_{UO}^{\epsilon,\delta} = C_{0,0} + \sqrt{\epsilon}C_{1,0} + \sqrt{\delta}C_{0,1} + \epsilon C_{2,0} + \delta C_{0,2} + \sqrt{\epsilon\delta}C_{1,1} + \cdots \tag{3.5}
$$

The first-order approximation of the solution is given by

$$C_{UO}^{\epsilon,\delta} \approx C_{0,0} + \sqrt{\epsilon}C_{1,0} + \sqrt{\delta}C_{0,1} \tag{3.6}$$

3.1. Computing $C_{0,0}$

Let $\pi(v_1)$ be the invariant density of the process $V_1^\epsilon(t)$. According to Canhanga et al. (2018),

$$\pi(v_1) = \left(\frac{\alpha}{\theta_1}\right)^\alpha \frac{1}{\Gamma(\alpha)} v_1^{\alpha-1} e^{-\alpha v_1/\theta_1} \tag{3.7}$$

where $\alpha = \frac{2\theta_1}{\sigma_1^2}$.

Lemma 4.1. *If the solution to the Poisson equation*

$$\mathcal{L}_0 g(v) + \varphi(v) = 0 \tag{3.8}$$

exits, then the centering condition $<\varphi(v)> = \int_R \varphi(v_1)\pi(v_1)dv_1 = 0$ *must be satisfied, where* $< \cdot >$ *denotes the expectation with respect to the invariant distribution of volatility process* $V_1^\epsilon(t)$.

Remark. See Fouque et al. (2000).

Lemma 4.2. *Let* $\sigma^2(V_1, V_2)$ *be the volatility of asset process* S(t) *which assumes model Equation (3.1) and* $\bar{\sigma}$ *be the square root of the expectation of* $\sigma^2(V_1, V_2)$, *then*

$$\bar{\sigma} = \sqrt{\theta_1 + V_2} \tag{3.9}$$

Proof. Model (Equation 3.1) implies that $\sigma^2(V_1, V_2) = V_1 + V_2$. Since $V_1^\epsilon(t)$ and $V_2^\delta(t)$ is uncorrelated and the risk premiums produced by $V_1^\epsilon(t)$ and $V_2^\delta(t)$ are set zero, we have

$$<V_2> = \int_R V_2 \pi(v) \mathrm{d}v = V_2$$

According to the mean ergodic theorem, that is, the long-run time average of a bounded function of an ergodic process is close to it's expectation, we have $<V_1> = \theta_1$, then $<\sigma^2(V_1, V_2)> = \theta_1 + V_2$, thus we obtain Lemma 4.2.

Theorem 4.1.

$$\begin{aligned} C_{0,0} = {} & C_{BS}(S,K) - C_{BS}(S,B) - (B-K)e^{-r(T-t)}\Phi(d_1) - \left(\tfrac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1} \\ & \left[C_{BS}\left(\frac{B^2}{S}, K\right) - C_{BS}\left(\frac{B^2}{S}, B\right) - (B-K)e^{-r(T-t)}\Phi(d_2)\right] \end{aligned} \tag{3.10}$$

where $C_{BS}(\cdot)$ *denotes the European call option price under the BS model with the constant volatility* $\bar{\sigma}, \Phi(\cdot)$ *denotes the standard normal cumulative distribution function,* $d_1 = \frac{\log\left(\frac{S}{B}\right) + (r - 0.5\bar{\sigma}^2)(T-t)}{\bar{\sigma}} \sqrt{T-t}$, $d_2 = \frac{\log\left(\frac{B}{S}\right) + (r - 0.5\bar{\sigma}^2)(T-t)}{\bar{\sigma}} \sqrt{T-t}$.

Proof. Putting expansion Equation (3.3) into Equation (3.2) and collecting the terms for $\sqrt{\delta}$ and the terms with constant coefficients, we get

$$\begin{aligned}\left(\frac{1}{\epsilon}\mathcal{L}_0+\frac{1}{\sqrt{\epsilon}}\mathcal{L}_1+\mathcal{L}_2\right)C_0^\epsilon &= 0\\ C_0^\epsilon(S,V_1,V_2,T) &= (S-K)^+,\quad 0<S<B\\ C_0^\epsilon(B,V_1,V_2,t) &= 0\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}\left(\frac{1}{\epsilon}\mathcal{L}_0+\frac{1}{\sqrt{\epsilon}}\mathcal{L}_1+\mathcal{L}_2\right)C_1^\epsilon+\mathcal{U}_1C_0^\epsilon &= 0\\ C_1^\epsilon(S,V_1,V_2,T) &= 0,\quad 0<S<B\\ C_1^\epsilon(B,V_1,V_2,t) &= 0\end{aligned}\tag{3.12}$$

Putting the singular expansion for C_0^ϵ into Equation (3.11) and collecting the terms for $\frac{1}{\epsilon},\frac{1}{\sqrt{\epsilon}},\sqrt{\epsilon}$ and the terms with constant coefficients, we have

$$\begin{aligned}\mathcal{L}_0C_{0,0} &= 0\\ C_{0,0}(S,V_1,V_2,T) &= (S-K)^+,\quad 0<S<B\\ C_{0,0}(B,V_1,V_2,t) &= 0\end{aligned}\tag{3.13}$$

$$\begin{aligned}\mathcal{L}_0C_{1,0}+\mathcal{L}_1C_{0,0} &= 0\\ C_{1,0}(S,V_1,V_2,T) &= 0,\quad 0<S<B\\ C_{1,0}(B,V_1,V_2,t) &= 0\end{aligned}\tag{3.14}$$

$$\begin{aligned}\mathcal{L}_0C_{2,0}+\mathcal{L}_1C_{1,0}+\mathcal{L}_2C_{0,0} &= 0\\ C_{2,0}(S,V_1,V_2,T) &= 0,\quad 0<S<B\\ C_{2,0}(B,V_1,V_2,t) &= 0\end{aligned}\tag{3.15}$$

$$\begin{aligned}\mathcal{L}_0C_{3,0}+\mathcal{L}_1C_{2,0}+\mathcal{L}_2C_{1,0} &= 0\\ C_{3,0}(S,V_1,V_2,T) &= 0,\quad 0<S<B\\ C_{3,0}(B,V_1,V_2,t) &= 0\end{aligned}\tag{3.16}$$

Equations (3.13) and (3.14) show that $C_{0,0}$ and $C_{1,0}$ are both constants in V_1 and Equation (3.15) is a Poisson equation with source term $-\mathcal{L}_2C_{0,0}$. From Lemma 4.1, we have

$$<\mathcal{L}_2C_{0,0}> = 0\tag{3.17}$$

that is,

$$\mathcal{L}_{BS}(\bar{\sigma})C_{0,0} = 0\tag{3.18}$$

Therefore, $C_{0,0}$ is the solution to the UOC option pricing PDE under the BS model with the constant volatility $\bar{\sigma}$. According to Zhang (1998) and Lemma 4.2, we have Theorem 4.1.

3.2. Computing $C_{1,0}$

Lemma 4.3. *Let* $\phi(V_1) = \mathcal{L}_0^{-1}(V_1 - \theta_1)$, *then*

$$\frac{\partial \phi(V_1)}{\partial V_1} = -1 \tag{3.19}$$

Proof. See Canhanga et al. (2018).

Lemma 4.4. *Under the measure P, the density of the first passage time from zero to the transferred barrier point* $a = \ln \frac{S}{B}$ *for an up-barrier option is given by*

$$f(t|a<0) = \frac{-a}{\bar{\sigma}\sqrt{2\pi}t^{3/2}} \exp\left(-\frac{a^2}{2\bar{\sigma}^2 t}\right) \tag{3.20}$$

Proof. See Zhang (1998).

Lemma 4.5. *Let* $D_1 = S\frac{\partial}{\partial S}, D_2 = S^2\frac{\partial^2}{\partial S^2}$, *then*

$$\begin{aligned}
D_1 D_2 C_{00} = {} & 2S^2\Gamma_{BS}(S,K) + S^3\Gamma'_{BS}(S,K) - 2S^2\Gamma_{BS}(S,B) - S^3\Gamma'_{BS}(S,B) \\
& + \frac{(1 - d_1\bar{\sigma}\sqrt{\tau} - \bar{\sigma}^2\tau - d_1^2)(B-K)e^{-r\tau}\phi(d_1)}{\bar{\sigma}^3\tau^{3/2}} - \left(3 - \frac{2r}{\bar{\sigma}^2}\right)S^2 \\
& \left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\left[\Gamma_{BS}\left(\frac{B^2}{S},K\right) - \Gamma_{BS}\left(\frac{B^2}{S},B\right) - (B-K)e^{-r\tau}\phi(d_2)\right. \\
& \left.\frac{\bar{\sigma}\sqrt{\tau} - d_2}{S^2\bar{\sigma}^2\tau}\right] - S^3\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\left[\Gamma'_{BS}\left(\frac{B^2}{S},K\right) - \Gamma'_{BS}\left(\frac{B^2}{S},B\right)\right. \\
& \left. -(B-K)e^{-r\tau}\phi(d_2)\frac{1 - d_2^2 - 2\bar{\sigma}^2\tau + 3d_2\bar{\sigma}\sqrt{\tau}}{S^3\bar{\sigma}^3\tau^{3/2}}\right] + S\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1} \\
& \frac{8(\bar{\sigma}^2 - r)(r - 0.5\bar{\sigma}^2)}{\bar{\sigma}^4}\left[\Delta_{BS}\left(\frac{B^2}{S},K\right) - \Delta_{BS}\left(\frac{B^2}{S},B\right) + (B-K)\right. \\
& \left.\frac{e^{-r\tau}\phi(d_2)}{S\bar{\sigma}\sqrt{\tau}}\right] + \frac{4S^2(r - 0.5\bar{\sigma}^2)}{\bar{\sigma}^2}\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\left[\Gamma_{BS}\left(\frac{B^2}{S},K\right)\right. \\
& \left. -\Gamma_{BS}\left(\frac{B^2}{S},B\right) - (B-K)e^{-r\tau}\phi(d_2)\frac{\bar{\sigma}\sqrt{\tau} - d_2}{S^2\bar{\sigma}^2\tau}\right] + \frac{8r(r - 0.5\bar{\sigma}^2)^2}{\bar{\sigma}^6} \\
& \left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\left[C_{BS}\left(\frac{B^2}{S},K\right) - C_{BS}\left(\frac{B^2}{S},B\right) - (B-K)e^{-r\tau}\Phi(d_2)\right] \\
& - \frac{4r(r - 0.5\bar{\sigma}^2)}{\bar{\sigma}^4}\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\left[\Delta_{BS}\left(\frac{B^2}{S},K\right) - \Delta_{BS}\left(\frac{B^2}{S},B\right)\right. \\
& \left. + (B-K)e^{-r\tau}\frac{\phi(d_2)}{S\bar{\sigma}\sqrt{\tau}}\right]
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
\Gamma_{BS}(S,\cdot) &= \frac{\phi(d_3)}{S\bar{\sigma}\sqrt{\tau}}\\
\Gamma'_{BS}(S,\cdot) &= -\frac{\phi(d_3)}{S^2\bar{\sigma}\sqrt{\tau}}\left(1+\frac{d_3}{\bar{\sigma}\sqrt{\tau}}\right)\\
\Delta_{BS}\left(\frac{B^2}{S},\cdot\right) &= -\frac{B^2}{S^2}\Phi(d_4)\\
\Gamma_{BS}\left(\frac{B^2}{S},\cdot\right) &= \frac{2B^2}{S^3}\Phi(d_4)+\frac{B^2\phi(d_4)}{S^3\bar{\sigma}\sqrt{\tau}}\\
\Gamma'_{BS}\left(\frac{B^2}{S},\cdot\right) &= -\frac{6B^2\Phi(d_4)}{S^4}-\frac{5B^2\phi(d_4)}{S^4\bar{\sigma}\sqrt{\tau}}+\frac{B^2d_4\phi(d_4)}{S^4\bar{\sigma}^2\tau}
\end{aligned}$$

$d_3=\frac{\ln\frac{S}{\cdot}+(r+0.5\bar{\sigma}^2)\tau}{\bar{\sigma}\sqrt{\tau}}$, $d_4=\frac{\ln\frac{B^2}{S\cdot}+(r+0.5\bar{\sigma}^2)\tau}{\bar{\sigma}\sqrt{\tau}}$, $\tau=T-t$, $\phi(\cdot)$ denotes the standard normal density.

Remark. Direct computing can obtain Lemma 4.5.

Theorem 4.2.

$$C_{1,0}=-(T-t)\mathcal{A}_1C_{0,0}+S^{\frac{0.5\bar{\sigma}^2-r}{\bar{\sigma}^2}}e^{\left(r+\frac{(r-0.5\bar{\sigma}^2)^2}{2\bar{\sigma}^2}\right)(T-t)}v_1\left(\ln\frac{S}{B},t\right) \tag{3.22}$$

where

$$\begin{aligned}
\mathcal{A}_1 &= -0.5\rho_1\sigma_1\theta_1D_1D_2\\
v_1\left(\ln\frac{S}{B},t\right) &= \frac{1}{\bar{\sigma}\sqrt{2\pi}}\int_t^T\frac{\ln S/B}{(s-t)^{3/2}}e^{-\frac{(\ln S/B)^2}{2\bar{\sigma}^2(s-t)}}\bar{g}_1(s)ds\\
\bar{g}_1(t) &= e^{\left(r+\frac{(r-0.5\bar{\sigma}^2)^2}{2\bar{\sigma}^2}\right)(T-t)}B^{-\frac{1}{2}+\frac{r}{\bar{\sigma}^2}}(T-t)\mathcal{A}_1C_{0,0}|_{S=B^-}
\end{aligned}$$

Proof. From Equation (3.17), we have

$$\mathcal{L}_2C_{0,0}=\mathcal{L}_2C_{0,0}-<\mathcal{L}_2>C_{0,0}=\frac{1}{2}(V_1-\theta_1)D_2C_{0,0}$$

From Equation (3.15), we have

$$\begin{aligned}
C_{2,0} &= -\mathcal{L}_0^{-1}\mathcal{L}_2C_{0,0}\\
&= -\mathcal{L}_0^{-1}[\mathcal{L}_2-<\mathcal{L}_2>]C_{0,0}+h(S,V_2,t)
\end{aligned}$$

thus $\mathcal{L}_1C_{2,0}=-\mathcal{L}_1\mathcal{L}_0^{-1}[\mathcal{L}_2-<\mathcal{L}_2>]C_{0,0}$.

Equation (3.16) is a Poisson equation with source term $-\mathcal{L}_1C_{2,0}-\mathcal{L}_2C_{1,0}$. From Lemma 4.1, we have $<\mathcal{L}_1C_{2,0}+\mathcal{L}_2C_{1,0}>=0$, that is

$$
\begin{aligned}
<\mathcal{L}_2 C_{1,0}> &= -<\mathcal{L}_1 C_{2,0}> \\
&= <\mathcal{L}_1 \mathcal{L}_0^{-1}[\mathcal{L}_2 - <\mathcal{L}_2>]> C_{0,0} \\
&= <\mathcal{L}_1 \mathcal{L}_0^{-1}\left[\frac{1}{2}\mathcal{L}_0 \phi(V_1) D_2\right]> C_{0,0} \\
&= \frac{1}{2}\rho_1\sigma_1 <V_1 \frac{\partial \phi(V_1)}{\partial V_1}> D_1 D_2 C_{0,0}
\end{aligned}
$$

From Lemma 4.3, we have

$$
<\mathcal{L}_2 C_{1,0}> = -\frac{1}{2}\rho_1\sigma_1\theta_1 D_1 D_2 C_{0,0}.
$$

Let $\mathcal{A}_1 = -\frac{1}{2}\rho_1\sigma_1\theta_1 D_1 D_2$, then $C_{1,0}$ is the solution to the following PDE with terminal and boundary problem:

$$
\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}) C_{1,0} &= \mathcal{A}_1 C_{0,0} \\
C_{1,0}(S, V_1, V_2, T) &= 0, \quad 0<S<B \\
C_{1,0}(B, V_1, V_2, t) &= 0
\end{aligned}
\tag{3.23}
$$

According to Fouque et al. (2000), the solution to Equation (3.23) without the boundary condition is given by $(T-t)\mathcal{A}_1 C_{0,0}$. To obtain $C_{1,0}$ we set $U(S, V_2, t) = C_{1,0} + (T-t)\mathcal{A}_1 C_{0,0}$, then Equation (3.23) can be written as

$$
\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma}) U &= 0 \\
U(S, V_2, T) &= 0, \quad 0<S<B \\
U(B, V_2, t) &= g_1(t)
\end{aligned}
\tag{3.24}
$$

where $g_1(t) = (T-t)\mathcal{A}_1 C_{0,0}|_{S=B^-}$.

Let $z = \ln\frac{S}{B}$, $U = e^{(r+\frac{(r-0.5\bar{\sigma}^2)^2}{2\bar{\sigma}^2})(t-T)-\frac{(r-0.5\bar{\sigma}^2)z}{\bar{\sigma}^2}} v_1$, Equation (3.24) can be written as

$$
\begin{aligned}
\frac{\partial v_1}{\partial t} + \frac{1}{2}\bar{\sigma}^2\frac{\partial^2 v_1}{\partial z^2} &= 0, \\
v_1(z, V_2, T) &= 0, \quad z<0 \\
v_1(0, V_2, t) &= \bar{g}_1(t)
\end{aligned}
\tag{3.25}
$$

where $\bar{g}_1(t) = e^{(r+\frac{(r-0.5\bar{\sigma}^2)^2}{2\bar{\sigma}^2})(T-t)} B^{-\frac{1}{2}+\frac{r}{\bar{\sigma}^2}} g_1(t)$. From Lemma 4.4, the solution to Equation (3.25) is given by

$$
v_1(z, V_2, t) = \frac{1}{\bar{\sigma}\sqrt{2\pi}}\int_t^T \frac{z}{(s-t)^{3/2}} e^{-\frac{z^2}{2\bar{\sigma}^2(s-t)}} \bar{g}_1(s) ds
\tag{3.26}
$$

Combining Lemma 4.5 we obtain Theorem 4.2.

3.3. Computing $C_{0,1}$

Lemma 4.6.

$$
\begin{aligned}
D_1\frac{\partial C_{00}}{\partial V_2} = {} & S\frac{\partial \nu_{BS}(S,K)}{\partial S} - S\frac{\partial \nu_{BS}(S,B)}{\partial S} - S(B-K)e^{-r\tau}\phi(d_1)\\
& \left[\frac{d_1\left((\bar{\sigma}^2+2r)\tau+2\ln\frac{S}{B}\right)}{4S\bar{\sigma}^3\tau\sqrt{\theta_1+V_2}} - \frac{1}{\bar{\sigma}S\sqrt{V_2\tau}}\right] - S\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\\
& \left\{\frac{\partial \nu_{BS}\left(\frac{B^2}{S},K\right)}{\partial S} - \frac{\partial \nu_{BS}\left(\frac{B^2}{S},B\right)}{\partial S} - S(B-K)e^{-r\tau}\phi(d_2)\right.\\
& \left.\left[\frac{1}{\bar{\sigma}S\sqrt{V_2\tau}} - \frac{d_2\left((\bar{\sigma}^2+2r)\tau+2\ln\frac{B}{S}\right)}{4S\bar{\sigma}^3\tau\sqrt{\theta_1+V_2}}\right]\right\} + \frac{2(r-0.5\bar{\sigma}^2)}{\bar{\sigma}^2}\\
& \left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\left[\nu_{BS}\left(\frac{B^2}{S},K\right) - \nu_{BS}\left(\frac{B^2}{S},B\right) + (B-K)e^{-r\tau}\right.\\
& \left.\frac{\phi(d_2)}{2\sqrt{\theta_1+V_2}}\left(\frac{\sqrt{\tau}}{2} + \frac{\ln\frac{B}{S}+r\tau}{\bar{\sigma}^2\sqrt{\tau}}\right)\right] + \frac{2r}{\bar{\sigma}^3\sqrt{\theta_1+V_2}}\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\\
& \left[\left(1-\frac{2r}{\bar{\sigma}^2}\right)\ln\frac{B}{S} - 1\right]\left[C_{BS}\left(\frac{B^2}{S},K\right) - C_{BS}\left(\frac{B^2}{S},B\right)\right.\\
& \left. -(B-K)e^{-r\tau}\Phi(d_2)\right] + \frac{2rS}{\bar{\sigma}^3\sqrt{\theta_1+V_2}}\ln\frac{B}{S}\left(\frac{B}{S}\right)^{\frac{2r}{\bar{\sigma}^2}-1}\\
& \left[\Delta_{BS}\left(\frac{B^2}{S},K\right) - \Delta_{BS}\left(\frac{B^2}{S},B\right) + \frac{(B-K)e^{-r\tau}\phi(d_2)}{S\bar{\sigma}\sqrt{\tau}}\right]
\end{aligned}
\tag{3.27}
$$

where

$$
\begin{aligned}
\frac{\partial \nu_{BS}(S,\cdot)}{\partial S} &= \sqrt{\tau}\phi(d_3)\left(1-\frac{d_3}{\bar{\sigma}\sqrt{\tau}}\right)\\
\nu_{BS}\left(\frac{B^2}{S},\cdot\right) &= \frac{B^2\sqrt{\tau}\phi(d_4)}{2\bar{\sigma}S}\\
\frac{\partial \nu_{BS}\left(\frac{B^2}{S},\cdot\right)}{\partial S} &= \frac{B^2\sqrt{\tau}\phi(d_4)}{2\bar{\sigma}S^2}\left(\frac{d_4}{\bar{\sigma}\sqrt{\tau}}-1\right)
\end{aligned}
$$

Remark. Direct computing can obtain Lemma 4.6.

Theorem 4.3.

$$
C_{0,1} = (T-t)\mathcal{A}_2C_{0,0} + e^{\left(r+\frac{\left(r-0.5\bar{\sigma}^2\right)^2}{2\bar{\sigma}^2}\right)(t-T)-\frac{\left(r-0.5\bar{\sigma}^2\right)z}{\bar{\sigma}^2}}v_2\left(\ln\frac{S}{B},t\right) \tag{3.28}
$$

where

Table 1. Comparisons of the first-order asymptotic expansion (AE) solution and Monte Carlo (MC) solutions with 1,000,000 simulation trials and 10,000 time steps for evaluating UOC options under the double Heston model with fast factor ϵ and slow factor δ.

(ϵ, δ)	AE	MC ± standard deviation	Relative error
$T=0.25$			
$(0.1, 0.001)$	12.4702	11.5337 ± 0.0110	0.0812
$(0.01, 0.001)$	11.8717	11.5130 ± 0.0110	0.0312
$(0.001, 0.1)$	11.6528	11.5517 ± 0.0110	0.0088
$(0.001, 0.01)$	11.5983	11.5154 ± 0.0110	0.0072
$(0.001, 0.001)$	11.5867	11.5095 ± 0.0110	0.0067
$T=0.5$			
$(0.1, 0.001)$	9.2208	8.5212 ± 0.0107	0.0821
$(0.01, 0.001)$	8.7842	8.5151 ± 0.0107	0.0316
$(0.001, 0.1)$	8.6645	8.5864 ± 0.0107	0.0091
$(0.001, 0.01)$	8.5968	8.5269 ± 0.0107	0.0082
$(0.001, 0.001)$	8.5792	8.5204 ± 0.0107	0.0069
$T=1$			
$(0.1, 0.001)$	6.5697	6.0696 ± 0.0097	0.0824
$(0.01, 0.001)$	6.2730	6.0803 ± 0.0098	0.0317
$(0.001, 0.1)$	6.2203	6.1654 ± 0.0097	0.0089
$(0.001, 0.01)$	6.1276	6.0784 ± 0.0097	0.0081
$(0.001, 0.001)$	6.1130	6.0699 ± 0.0097	0.0071

$$\mathcal{A}_2 = 0.5\rho_2\sigma_2 V_2 D_1 \frac{\partial}{\partial V_2}$$
$$v_2\left(\ln\frac{S}{B}, t\right) = \frac{1}{\bar{\sigma}\sqrt{2\pi}}\int_t^T \frac{\ln S/B}{(s-t)^{3/2}} e^{-\frac{(\ln S/B)^2}{2\bar{\sigma}^2(s-t)}} \bar{g}_2(s)ds$$
$$\bar{g}_2(t) = e^{\left(r+\frac{(r-0.5\bar{\sigma}^2)^2}{2\bar{\sigma}^2}\right)(T-t)} B^{-\frac{1}{2}+\frac{r}{\bar{\sigma}^2}}(T-t)\mathcal{A}_2 C_{0,0}|_{S=B^-}$$

Proof. Putting the singular expansions for C_1^ϵ and C_0^ϵ into Equation (3.12) and collecting the terms for $\frac{1}{\epsilon}, \frac{1}{\sqrt{\epsilon}}$ and the terms with constant coefficients, we have

$$\begin{aligned} \mathcal{L}_0 C_{0,1} &= 0, \\ C_{0,1}(S, V_1, V_2, T) &= 0, \quad 0<S<B, \\ C_{0,1}(B, V_1, V_2, t) &= 0, \end{aligned} \tag{3.29}$$

$$\begin{aligned} \mathcal{L}_0 C_{1,1} + \mathcal{L}_1 C_{0,1} &= 0, \\ C_{1,1}(S, V_1, V_2, T) &= 0, \quad 0<S<B, \\ C_{1,1}(B, V_1, V_2, t) &= 0, \end{aligned} \tag{3.30}$$

$$\begin{aligned} \mathcal{L}_0 C_{2,1} + \mathcal{L}_1 C_{1,1} + \mathcal{L}_2 C_{0,1} + \mathcal{U}_1 C_{0,0} &= 0, \\ C_{2,1}(S, V_1, V_2, T) &= 0, \quad 0<S<B, \\ C_{2,1}(B, V_1, V_2, t) &= 0. \end{aligned} \tag{3.31}$$

Equation (3.29) and (3.30) show that both $C_{0,1}$ and $C_{1,1}$ are constant in V_1 and Equation (3.31) is a Poisson equation with source $-\mathcal{L}_2 C_{0,1} - \mathcal{U}_1 C_{0,0}$. By Lemma 4.1, we have

$$<\mathcal{L}_2 C_{0,1} + \mathcal{U}_1 C_{0,0}> = 0, \tag{3.32}$$

that is,

$$\mathcal{L}_{BS}(\bar{\sigma})C_{0,1} = -\rho_2\sigma_2 V_2 D_1 \frac{\partial}{\partial V_2} C_{0,0} \tag{3.33}$$

Let $\mathcal{A}_2 = 0.5\rho_2\sigma_2 V_2 D_1 \frac{\partial}{\partial V_2}$, then $C_{0,1}$ is the solution to the following PDE with terminal and boundary problem:

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})C_{0,1} &= -2\mathcal{A}_2 C_{0,0} \\ C_{0,1}(S, V_1, V_2, T) &= 0, \quad 0<S<B \\ C_{0,1}(B, V_1, V_2, t) &= 0 \end{aligned} \tag{3.34}$$

Let $V(S, V_2, t) = C_{0,1} - (T-t)\mathcal{A}_2 C_{0,0}$, then Equation (3.34) can be written as

$$\begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})V &= 0 \\ V(S, V_2, T) &= 0, \quad 0<S<B \\ V(B, V_2, t) &= g_2(t) \end{aligned} \tag{3.35}$$

where $g_2(t) = -(T-t)\mathcal{A}_2 C_{0,0}|_{S=B^-}$. Using the same reasoning for Theorem 4.2 and combining Lemma 4.6 we obtain Theorem 4.3.

4. Numerical experiments

This section verifies the convergence and efficiency of the asymptotic method for UOC options. To this end, we compare the first-order asymptotic expansion solution obtained in Section 3 with the solution obtained by Monte Carlo simulation. All numerical experiments are performed in Matlab 9.0.0 on an Intel(R) Xeon(R) CPU E5-1650 v4 @3.60 GHz and 32GB RAM. The CPU times for all experiments are determined by averaging over 20 repeated tests.

We first implement the asymptotic expansion solution (Equation 3.6). To this end, we approximate $v_1(\ln \frac{S}{B}, t)$ and $v_2(\ln \frac{S}{B}, t)$ in Theorem 4.2 and 4.3 by numerical integration with the Simpson's rule. The numerical results not reported in this paper show that the Simpson's rule converges very fast with only a small number of grid points and the convergence is of fourth order.

To obtain the Monte Carlo solution, we discretize the stochastic differential Equation (3.1) by the QE scheme proposed by Andersen (2008) and then adapted to the double Heston model by Gauthier and Possamai (2011). Suppose that Z_j $(j=1, 2)$ are independent standard normal random variables and U_j $(j=1, 2)$ are independent uniform random numbers. Given a positive threshold Ψ_c, the QE scheme for $S^{\epsilon,\delta}(t)$ under the modified model (Equation 3.1) can be written as

$$\begin{aligned} \ln\frac{S^{\epsilon,\delta}(t+\Delta)}{S^{\epsilon,\delta}(t)} &= r\Delta + K_1^0 + K_1^1 V_1^\epsilon(t) + K_1^2 V_1^\epsilon(t+\Delta) + \sqrt{K_1^3 V_1^\epsilon(t) + K_1^4 V_1^\epsilon(t+\Delta)}Z_1 \\ &\quad + K_2^0 + K_2^1 V_2^\delta(t) + K_2^2 V_2^\delta(t+\Delta) + \sqrt{K_2^3 V_2^\delta(t) + K_2^4 V_2^\delta(t+\Delta)}Z_2, \\ V_1^\epsilon(t+\Delta) &= 1_{\Psi_1\le\Psi_C}\left[a_1(b_1+Z_1)^2\right] + 1_{\Psi_1>\Psi_C}\left(1_{p_1<U_1\le 1}\chi_1^{-1}\ln\frac{1-p_1}{1-U_1}\right), \\ V_2^\delta(t+\Delta) &= 1_{\Psi_2\le\Psi_C}\left[a_2(b_2+Z_2)^2\right] + 1_{\Psi_2>\Psi_C}\left(1_{p_2<U_2\le 1}\chi_2^{-1}\ln\frac{1-p_2}{1-U_2}\right), \end{aligned}$$

where $K_1^0 = -\rho_1\theta_1\Delta/\sigma_1\sqrt{\epsilon}, K_2^0 = -\rho_2\theta_2\Delta\sqrt{\delta}/\sigma_2$, $K_1^1 = \beta\Delta(\rho_1/\sqrt{\epsilon}\sigma_1 - 0.5) - \rho_1\sigma_1/\sqrt{\epsilon}, K_2^1 = \beta\Delta(\rho_2\sqrt{\delta}/\sigma_2 - 0.5) - \rho_2\sigma_2\sqrt{\delta}$, $K_1^2 = (1-\beta)\Delta(\rho_1/\sqrt{\epsilon}\sigma_1 - 0.5) + \rho_1\sigma_1/\sqrt{\epsilon}$, $K_2^2 = (1-\beta)\Delta(\rho_2\sqrt{\delta}/\sigma_2 - 0.5) + \rho_2\sqrt{\delta}\sigma_2$. For $j = 1,\ 2$, $K_j^3 = \beta\Delta(1-\rho_j^2), K_j^4 = (1-\beta)\Delta(1-\rho_j^2), \beta \in [0,1], a_j = m_j/(1+b_j^2), b_j = 2\Psi_j^{-1} - 1 + \sqrt{2\Psi_j^{-1}(2\Psi_j^{-1}-1)}, \Psi_j = s_j^2/m_j^2, \chi_j = (1-p_j)/m_j, p_j = (\Psi_j - 1)/(\Psi_j + 1), m_1 = \theta_1 + (V_1^\epsilon(t) - \theta_1)e^{-\Delta/\epsilon}, m_2 = \theta_2 + (V_2^\delta(t) - \theta_2)e^{-\delta\Delta}, s_1^2 = V_1^\epsilon(t)\sigma_1^2 e^{-\Delta/\epsilon}(1 - e^{-\Delta/\epsilon}) + \theta_1\sigma_1^2(1 - e^{-\Delta/\epsilon})^2/2, s_2^2 = V_2^\delta(t)\sigma_2^2 e^{-\Delta\delta}(1 - e^{-\Delta\delta}) + \theta_2\sigma_2^2(1 - e^{-\delta\Delta})^2/2$.

To obtain exact option price, we use 1, 000, 000 sample paths and 10, 000 time steps each path in our Monte Carlo simulation experiments. We set $\Psi_C = 1.5, \beta = 0.5$. For our asymptotic expansion method, we use 2^6 grid points to discretize $v_1(\ln\frac{S}{B}, t)$ and $v_2(\ln\frac{S}{B}, t)$. In all numerical experiments we use the parameters taken from Canhanga et al. (2018), that is, $\theta_1 = 0.04, \sigma_1 = 0.1, \rho_1 = -0.5, \theta_2 = 0.01, \sigma_2 = 0.1, \rho_2 = -0.5$, $V_1 = 0.04, V_2 = 0.09$. We set $S = 100$, $K = 80$, $t = 0$, $r = 0.03$, $B = 115$. To test the convergence of the asymptotic method, we set three maturities: $T = 0.25, 0.5, 1$ and five groups (ϵ, δ): $(0.1, 0.001), (0.01, 0.001), (0.001, 0.1)$, $(0.001, 0.01), (0.001, 0.001)$ for each maturity. Table 1 compares the results computed by the above two methods.

Table 1 shows that the first-order asymptotic expansion solutions at all maturities converge fast with the decreasing of ϵ and δ. When $\epsilon = 0.001, \delta = 0.001$, the largest relativity error is no more than 0.0071. The accuracy of the asymptotic method is almost same for different maturity. The effect of ϵ on the accuracy of the asymptotic method is more evident than δ. The accuracy of the asymptotic method can be significantly improved by decreasing properly the value of ϵ. Besides, the asymptotic method is very fast for pricing UOC options. To evaluate single option the asymptotic method only needs 0.1203 seconds, while Monte Carlo simulation needs 4786.93 seconds. We conclude that the first-order asymptotic expansion method is fast and accurate for pricing single barrier options under the double Heston model framwork.

5. Conclusion

The paper analytically derives the first-order asymptotic expansion solution to single barrier options pricing PDE under the double Heston model with a fast factor and a slow factor. Although there are two integrate terms, the asymptotic expansion solution is still very fast implemented with the simpson's rule. The convergence and efficiency of the asymptotic method is verified by Monte Carlo simulation. Numerical results show that our asymptotic method with small ϵ and δ is very efficient which is suitable for practical use. The provided approach can be extended to other type of barrier options. The pricing model can also be extended by introducing the correlation of the two volatility process. These extensions are left for future research.

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