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Research Article

A Generic Decomposition Formula for Pricing Vanilla Options under Stochastic Volatility Models

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We obtain a decomposition of the call option price for a very general stochastic volatility diffusion model, extending a previous decomposition formula for the Heston model. We realize that a new term arises when the stock price does not follow an exponential model. The techniques used for this purpose are nonanticipative. In particular, we also see that equivalent results can be obtained by using Functional Itô Calculus. Using the same generalizing ideas, we also extend to nonexponential models the alternative call option price decomposition formula written in terms of the Malliavin derivative of the volatility process. Finally, we give a general expression for the derivative of the implied volatility under both the anticipative and the nonanticipative cases.

1. Introduction

Stochastic volatility models are a natural extension of the Black-Scholes model in order to manage the skew and the smile observed in real data. It is well known that in these models the average of future volatilities is a relevant quantity. See, for example, [1, Chapter 2]. Unfortunately, adding a stochastic volatility structure makes pricing and calibration more complicated, as closed formulas do not always exist. See, for example, [2], for a first reference on this topic. Moreover, even when these formulas exist, like for the Heston model (see [3]), in general, they do not allow a fast calibration of the parameters.

During last years, different developments for finding approximations to the closed-form option pricing formulas have been published. Malliavin techniques are naturally used to solve this problem in [4, 5] as the average future volatility is an anticipative quantity. Otherwise, a nonanticipative method to obtain an approximation of the pricing formula is developed for the Heston model in [6]. The method is based on the use of the adapted projection of the average future volatility. As a result, the model allows obtaining a decomposition of the call option price in terms of such future volatility.

In the present paper, we generalize [6] to general stochastic volatility diffusion models. Similarly, following the same kind of ideas, we extend the expansion based on Malliavin calculus obtained in [4, 5]. This is important because Heston model is not the unique stochastic volatility model currently used in practice, and some of them, like SABR model, are not of exponential type. For a general discussion about stochastic volatility models in practice, see [7].

The main ideas developed in this paper are the following:

(i) A generic call option price decomposition is found without having to specify the volatility structure.

(ii) A new term emerges when the stock option prices do not follow an exponential model, as, for example, in the SABR case.

(iii) The Feynman-Kac formula is a key element in the decomposition. It allows expressing the new terms that emerge under the new framework (i.e., stochastic volatility) as corrections of the Black-Scholes formula.

(iv) The decomposition found using Functional Itô calculus appears to be the same as the decomposition obtained through our techniques.

(v) A general expression of the derivative of the implied volatility, both for nonanticipative and anticipative cases, is given.

2. Notations

Let $S = \{S(t),\ t \in [0,T]\}$ be a strictly positive price process under a market chosen risk neutral probability that follows the model

$$dS(t) = \mu(t,S(t))\,dt + \theta(t,S(t),\sigma(t))\left(\rho dW(t) + \sqrt{1-\rho^2}dB(t)\right), \tag{1}$$

where W and B are independent Brownian motions, $\rho \in (-1,1)$, $\mu : [0,T]\times\mathbb{R}_+ \to \mathbb{R}$, $\theta : [0,T]\times\mathbb{R}_+^2 \to \mathbb{R}_+$, and $\sigma(t)$ is a positive square-integrable process adapted to the filtration of W. We assume on μ and σ sufficient conditions to ensure the existence and uniqueness of the solution of (1). Notice that we do not assume any concrete volatility structure. Thus, our decompositions can be adapted to many different models. In particular, we cover the following models:

(i) Black-Scholes model: $\mu(t,S(t)) := rS(t)$, $\theta(t,S(t),\sigma(t)) := \sigma S(t)$, $\rho = 0$, $r > 0$, and $\sigma > 0$.

(ii) CEV model: $\mu(t,S(t)) := rS(t)$, $\theta(t,S(t),\sigma(t)) := \sigma S(t)^{\beta}$ with $\beta \in (0,1]$, $\rho = 0$, $r > 0$, and $\sigma > 0$.

(iii) Heston model: $\mu(t,S(t)) := rS(t)$, $\theta(t,S(t),\sigma(t)) := \sigma(t)S(t)$, $r > 0$, $\sigma > 0$, and

$$d\sigma^2(t) = k\left(\theta - \sigma^2(t)\right)dt + \nu\sqrt{\sigma^2(t)}dW(t), \tag{2}$$

where k, θ, and ν are positive constants satisfying the Feller condition $2k\theta > \nu^2$.

(iv) SABR model: $\mu(t,S(t)) := rS(t)$, $\theta(t,S(t),\sigma(t)) := \sigma(t)S(t)^{\beta}$ with $\beta \in (0,1]$, $r > 0$, $\sigma > 0$, and

$$d\sigma(t) = \alpha\sigma(t)\,dW(t) \tag{3}$$

with $\alpha > 0$.

For existence and unicity of the solution in the Heston case, see, for example, [8, Section 2.2]. For the CEV and SABR models, see [9] and the references therein.

The following notation will be used in all the paper:

(i) We will denote by $\mathrm{BS}(t,S,\sigma)$ the price of a plain vanilla European call option under the classical Black-Scholes model with constant volatility σ, current stock price S, time to maturity $\tau = T-t$, strike price K, and interest rate r. In this case,

$$\mathrm{BS}(t,S,\sigma) = S\Phi(d_+) - Ke^{-r\tau}\Phi(d_-), \tag{4}$$

where $\Phi(\cdot)$ denotes the cumulative probability function of the standard normal law and

$$d_\pm = \frac{\ln(S/K) + \left(r \pm \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}. \tag{5}$$

(ii) We use in all the paper the notation $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot \mid \mathscr{F}_t]$, where $\{\mathscr{F}_t,\ t \geq 0\}$ is the natural filtration of S.

(iii) In our setting, the call option price is given by

$$V(t) = e^{-r\tau}\mathbb{E}_t\left[(S(T)-K)^+\right]. \tag{6}$$

(iv) Recalling that from the Feynman-Kac formula, the operator

$$\mathscr{L}_\theta := \partial_t + \frac{1}{2}\theta(t,S(t),\sigma(t))^2\partial_S^2 + \mu(t,S(t))\partial_S - r \tag{7}$$

satisfies $\mathscr{L}_\theta \mathrm{BS}(t,S(t),\theta(t,S(t),\sigma(t))) = 0$.

(v) We will also use the following definitions for $y \geq 0$:

$$\begin{aligned} G(t,S(t),y) &:= S^2(t)\partial_S^2\mathrm{BS}(t,S(t),y), \\ H(t,S(t),y) &:= S(t)\partial_S G(t,S(t),y), \\ K(t,S(t),y) &:= S^2(t)\partial_S^2 G(t,S(t),y), \\ L(t,S(t),y) &:= \frac{\theta(t,S(t),y)}{S(t)}. \end{aligned} \tag{8}$$

3. A Decomposition Formula Using Itô Calculus

In this section, following the ideas in [6], we extend the decomposition formula to a generic stochastic volatility diffusion process. We note that the new formula can be extended without having to specify the underlying volatility process, obtaining a more flexible decomposition formula. When the stock price does not follow an exponential process, a new term emerges. The formula proved in [6] is a particular case.

It is well known that if the stochastic volatility process is independent from the price process, then the pricing formula of a plain vanilla European call is given by

$$V(t) = \mathbb{E}_t\left[\mathrm{BS}(t,S(t),\overline{\sigma}(t))\right], \tag{9}$$

where $\overline{\sigma}^2(t)$ is the so-called average future variance and it is defined by

$$\overline{\sigma}^2(t) := \frac{1}{T-t}\int_t^T \sigma^2(s)\,ds. \tag{10}$$

Naturally, $\overline{\sigma}(t)$ is called the average future volatility. See [1, page 51].

The idea used in [6] consists in considering the adapted projection of the average future variance

$$\nu^2(t) := \mathbb{E}_t\left(\overline{\sigma}^2(t)\right) = \frac{1}{T-t}\int_t^T \mathbb{E}_t\left[\sigma^2(s)\right]ds \tag{11}$$

to obtain a decomposition of $V(t)$ in terms of $\nu(t)$. This idea switches an anticipative problem related with the anticipative process $\overline{\sigma}(t)$ into a nonanticipative one with the adapted process $\nu(t)$. We apply this technique to our generic stochastic differential equation (1).

Theorem 1 (decomposition formula). *For all $t \in [0,T)$, we have*

$$V(t) = BS(t,S(t),v(t)) + \frac{1}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u,S(u),v(u)) \cdot \left(L^2(u,S(u),\sigma(u)) - \sigma^2(u)\right) du\right] + \frac{1}{8} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u,S(u),v(u))\, d[M,M](u)\right] + \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} L(u,S(u),\sigma(u)) \cdot H(u,S(u),v(u))\, d[W,M](u)\right], \tag{12}$$

where $M(t) := \int_0^T \mathbb{E}_t[\sigma^2(s)]ds = \int_0^t \sigma^2(s)ds + (T-t)v(t)^2$.

Proof. Notice that $e^{-rT}\mathrm{BS}(T,S(T),v(T)) = e^{-rT}V(T)$. As $e^{-rt}V(t)$ is a martingale, we can write

$$e^{-rt}V(t) = \mathbb{E}_t\left(e^{-rT}V(T)\right) = \mathbb{E}_t\left(e^{-rT}\mathrm{BS}(T,S(T),v(T))\right). \tag{13}$$

Our idea is to apply the Itô formula to the process $e^{-rt}\mathrm{BS}(t, S(t), v(t))$.

As the derivatives of BS are not bounded, we have to use an approximation to the identity changing $\mathrm{BS}(t,S,\sigma)$ by

$$\mathrm{BS}_n(t,S,\sigma) := \mathrm{BS}(t,S,\sigma)\,\psi_n(S), \tag{14}$$

where $\psi_n(S) = \phi((1/n)S)$ for some $\phi \in \mathscr{C}_b^2$ such that $\phi(S) = 1$ for all $|S| < 1$ and $\phi(S) = 0$ for all $|S| > 2$, and $v(t)$ by $v^{\varepsilon}(t) = \sqrt{(1/(T-t))(\delta + \int_t^T \mathbb{E}[\sigma^2(s)ds])}$, where $\varepsilon > 0$, and finally apply the dominated convergence theorem. For simplicity, we skip this mollifying argument across the paper.

So, applying Itô formula, using the fact that

$$\partial_\sigma \mathrm{BS}(t,S,\sigma) = S^2\sigma\tau\partial_S^2\mathrm{BS}(t,S,\sigma) \tag{15}$$

and the Feynman-Kac operator (7), we deduce

$$e^{-rT}\mathrm{BS}(T,S(T),v(T)) - e^{-rt}\mathrm{BS}(t,S(t),v(t)) = \int_t^T e^{-ru}\mathscr{L}_{vS}\mathrm{BS}(u,S(u),v(u))\,du + \int_t^T e^{-ru}\partial_S\mathrm{BS}(u,S(u),v(u))\,\theta(u,S(u),\sigma(u)) \cdot \left(\rho dW(u) + \sqrt{1-\rho^2}dB(u)\right) + \frac{1}{2}\int_t^T e^{-ru}S^2(u) \cdot \partial_S^2\mathrm{BS}(u,S(u),v(u))\,dM(u) + \frac{1}{2}\int_t^T e^{-ru}S^2(u) \cdot \partial_S^2\mathrm{BS}(u,S(u),v(u))\left[L^2(u,S(u),\sigma(u))\,du - \sigma^2(u)\,du\right] + \frac{1}{8}\int_t^T e^{-ru}\left(S^2(u) \cdot \partial_S^2\left(S^2(u)\,\partial_S^2\mathrm{BS}(u,S(u),v(u))\right)\right) d[M,M](u) + \frac{\rho}{2}\int_t^T e^{-ru}\theta(u,S(u),\sigma(u)) \cdot \left(\partial_S\left(S^2(u)\,\partial_S^2\mathrm{BS}(u,S(u),v(u))\right)\right) d[W,M](u). \tag{16}$$

Taking conditional expectation and multiplying by e^{rt}, we have

$$\mathbb{E}_t\left[e^{-r(T-t)}\mathrm{BS}(T,S(T),v(T))\right] = \mathrm{BS}(t,S(t),v(t)) + \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}S^2(u)\,\partial_S^2\mathrm{BS}(u,S(u),v(u)) \cdot \left(L^2(u,S(u),\sigma(u))\,du - \sigma^2(u)\right)du\right] + \frac{1}{8} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\left(S^2(u) \cdot \partial_S^2\left(S^2(u)\,\partial_S^2\mathrm{BS}(u,S(u),v(u))\right)\right) d[M,M](u)\right] + \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\theta(u,S(u),\sigma(u))\,\partial_S\left(S^2(u) \cdot \partial_S^2\mathrm{BS}(u,S(u),v(u))\right) d[W,M](u)\right]. \tag{17}$$

□

Remark 2. In [6], the following operators are defined for $X(t) = \log S(t)$:

(i) $\widetilde{G}(t,X(t),\sigma(t)) := (\partial_x^2 - \partial_x)\mathrm{BS}(t,X(t),\sigma(t))$.

(ii) $\widetilde{H}(t,X(t),\sigma(t)) := (\partial_x^3 - \partial_x^2)\mathrm{BS}(t,X(t),\sigma(t))$.

(iii) $\widetilde{K}(t,X(t),\sigma(t)) := (\partial_x^4 - 2\partial_x^3 + \partial_x^2)\mathrm{BS}(t,X(t),\sigma(t))$.

We observe the following:

(i) $\widetilde{G}(t,X(t),\sigma(t)) = G(t,S(t),\sigma(t))$.

(ii) $\widetilde{K}(t,X(t),\sigma(t)) = K(t,S(t),\sigma(t))$.

(iii) $\widetilde{H}(t,X(t),\sigma(t)) = H(t,S(t),\sigma(t))$.

Remark 3. We have extended the decomposition formula in [6] to the generic SDE (1). When we apply Itô Calculus, we realize that Feynman-Kac formula absorbs some of the terms that emerge. It is important to note that this technique works for any payoff or any diffusion model satisfying Feynman-Kac formula.

Remark 4. Note that when $\theta(t, S(T), \sigma(t)) = \sigma(t)S(t)$ (i.e., the stock price follows an exponential process), then

$$\begin{aligned} V(t) &= \mathrm{BS}(t, S(t), v(t)) + \frac{1}{8} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u, S(u), v(u))\, d[M,M](u)\right] \\ &+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} \sigma(u) H(u, S(u), v(u))\, d[W,M] \right. \\ &\left. \cdot (u)\right], \end{aligned} \tag{18}$$

and the term

$$\begin{aligned} &\frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} S^2(u)\, \partial_S^2 \mathrm{BS}(u, S(u), v(u)) \right. \\ &\left. \cdot \left(L^2(u, S(u), \sigma(u)) - \sigma^2(u)\right) du\right] \end{aligned} \tag{19}$$

vanishes.

Indeed, we show that, due to the use of Feynman-Kac formula,

$$\begin{aligned} &\text{Movement of the asset} \\ &+ \text{Movement of the volatility} = \frac{1}{2}\theta(u, S_u)^2 \\ &\cdot \partial_S^2 \mathrm{BS}(u, S(u), v(u))\, du \\ &+ \partial_\sigma \mathrm{BS}(u, S(u), v(u))\, dv(u) = \frac{1}{2}\sigma^2(u) S^2(u) \\ &\cdot \partial_S^2 \mathrm{BS}(u, S(u), v(u))\, du + \frac{1}{2}S^2(u) \\ &\cdot \partial_S^2 \mathrm{BS}(u, S(u), v(u)) \left(dM + v^2 du - \sigma^2 du\right) = \frac{1}{2} \\ &\cdot S^2(u)\, \partial_S^2 \mathrm{BS}(u, S(u), v(u)) \left(dM + v^2 du\right), \end{aligned} \tag{20}$$

where $(1/2)S^2(u)\partial_S^2 \mathrm{BS}(u, S(u), v(u))v^2$ is used into the Feynman-Kac formula and

$$\mathbb{E}_t\left[\int_t^T \frac{1}{2} S^2 \partial_S^2 \mathrm{BS}(u, S(u), v(u))\, dM\right] = 0. \tag{21}$$

4. Basic Elements of Functional Itô Calculus

In this section, we give the insights of the Functional Itô Calculus developed in [10–13].

Let $X : [0,T] \times \Omega \mapsto \mathbb{R}$ be an Itô process, that is, a continuous semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which admits the stochastic integral representation

$$X(t) = x_0 + \int_0^t \mu(u)\, du + \int_0^t \sigma(u)\, dW(u), \tag{22}$$

where W is a Brownian motion and $\mu(t)$ and $\sigma(t)$ are continuous processes, respectively, in $L^1(\Omega \times [0,T])$ and $L^2(\Omega \times [0,T])$.

We define $D([0,T], \mathbb{R})$ the space of cadlag functions. Given a path $x \in D([0,T], \mathbb{R})$, we will denote by x_t its restriction to $[0,t]$. For $h \geq 0$, the horizontal extension $x_{t,h}$ is defined as

$$\begin{aligned} x_{t,h}(u) &= x_t(u) = x(u), \quad u \in [0,t[; \\ x_{t,h}(u) &= x(t), \quad u \in (t, t+h] \end{aligned} \tag{23}$$

and the vertical extension is defined as

$$\begin{aligned} x_t^h(u) &= x_t(u) = x(u), \quad u \in [0,t[; \\ x_t^h(t) &= x(t) + h, \quad \text{that is } x_t^h(u) = x(u) + h\mathbb{1}_{\{t=u\}}. \end{aligned} \tag{24}$$

A process $Y : [0,T] \times \mathbb{R} \to \mathbb{R}$, progressively measurable with respect to the natural filtration of X, may be represented as

$$Y(t) = F(t, \{X(s),\ 0 \leq s \leq t\}) = F(t, X_t) \tag{25}$$

for a certain measurable functional $F : [0,t] \times D([0,t], \mathbb{R}) \to \mathbb{R}$. Let $\mathbb{F}^\infty$ be the space of local Lipschitz functionals with respect to the norm of the supremum on $D([0,t+h], \mathbb{R})$; that is, there exists a constant $C > 0$ such that for any compact K and for any $x \in D([0,t], K)$ and $y \in D([0,t+h], K)$ we have

$$\left|F(t, x_t) - F(t+h, y_{t+h})\right| \leq C \left\|x_{t,h} - y_{t+h}\right\|_\infty. \tag{26}$$

Under this framework, we have the next definitions of derivative.

Definition 5 (horizontal derivative). The horizontal derivative of a functional $F \in \mathbb{F}^\infty$ at t is defined as

$$\mathcal{D}_t F(t, x_t) = \lim_{h \to 0^+} \frac{F(t+h, x_{t,h}) - F(t, x_t)}{h}. \tag{27}$$

Definition 6 (vertical derivative). The vertical derivative of a functional $F \in \mathbb{F}^\infty$ at t is defined as

$$\nabla_x F(t, x_t) = \lim_{h \to 0^+} \frac{F(t, x_t^h) - F(t, x_t)}{h}. \tag{28}$$

Of course we can consider iterated derivatives as ∇_{xx}.

We also have the following Itô formula that works for nonanticipative functionals:

Theorem 7 (Functional Itô Formula). *For any nonanticipative functional $F \in \mathbb{F}^\infty$ and any $t \in [0,T]$, we have*

$$\begin{aligned} &F(t, X_t) - F(0, X_0) \\ &\quad = \int_0^t \mathcal{D}_u F(u, X_u)\, du + \int_0^t \nabla_x F(u, X_u)\, dX(u) \\ &\quad + \frac{1}{2}\int_0^t \nabla_{xx} F(u, X_u)\, d[X,X](u), \end{aligned} \tag{29}$$

provided that $\mathcal{D}_t F$, $\nabla_x F$, and $\nabla_{xx} F$ belong to $\mathbb{F}^\infty$.

Proof. See [11, 12]. □

5. A General Decomposition Using Functional Itô Calculus

In this section, we apply Functional Itô Calculus to the problem of finding a decomposition for the call option price. The decomposition problem is an anticipative path-dependent problem. Using a smart choice of the volatility process into the Black-Scholes formula, we can convert it into a nonanticipative one. It is natural to wonder whether the Functional Itô Calculus brings some new insides into the problem.

We consider the functional

$$F\left(t,S(t),\sigma_t^2\right)=e^{-rt}\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right), \tag{30}$$

where σ^2 is the path-dependent process and $f\in\mathbb{F}^{\infty}$ is a nonanticipative functional.

Under this framework, we calculate the derivatives using Functional Itô Calculus with respect to the variance. Then we write them in terms of the classical Black-Scholes derivatives. We must realize that, for simplicity, the new derivatives are calculated with respect to the variance instead of the volatility of the process.

Remark 8. If ∂ denotes the classical derivative, we have

(i) Alternative Vega:

$$\nabla_{\sigma^2}F=e^{-rt}\partial_f\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\nabla_{\sigma^2}f\left(t,\sigma_t^2\right). \tag{31}$$

(ii) Alternative Vanna:

$$\nabla_{\sigma^2,S}F=\partial_{f,S}\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\nabla_{\sigma^2}f\left(t,\sigma_t^2\right). \tag{32}$$

(iii) Alternative Vomma:

$$\begin{aligned}\nabla_{\sigma^2,\sigma^2}F&=e^{-rt}\partial_{f,f}\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\left(\nabla_{\sigma^2}f\left(t,\sigma_t^2\right)\right)^2\\&\quad-e^{-rt}\partial_f\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\nabla_{\sigma^2}^2f\left(t,\sigma_t^2\right).\end{aligned} \tag{33}$$

(iv) Alternative Theta:

$$\begin{aligned}\mathscr{D}_tF&=-re^{-rt}\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\\&\quad+e^{-rt}\partial_t\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\\&\quad+e^{-rt}\partial_f\mathrm{BS}\left(t,S(t),f\left(t,\sigma_t^2\right)\right)\mathscr{D}_tf\left(t,\sigma_t^2\right).\end{aligned} \tag{34}$$

Theorem 9 (decomposition formula). *For all $t\in[0,T)$, $S(t)$, and $f(t,\sigma_t^2)>0$, we have*

$$\begin{aligned}V(t)&=BS\left(t,S(t),f\left(u,\sigma_t^2\right)\right)\\&\quad+\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}f\left(u,\sigma_u^2\right)\tau G\left(u,S(u),f\left(u,\sigma_u^2\right)\right)\right.\\&\quad\left.\cdot\mathscr{D}_uf\left(u,\sigma_u^2\right)du\right]+\frac{1}{2}\\&\quad\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}G\left(u,S(u),f\left(u,\sigma_u^2\right)\right)\right.\\&\quad\left.\cdot\left(L^2\left(u,S(u),\sigma_u\right)-f^2\left(u,\sigma_u^2\right)\right)du\right]+\frac{1}{2}\\&\quad\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}f\left(u,\sigma_u^2\right)^2\right.\\&\quad\left.\cdot\tau^2K\left(u,S(u),f\left(u,\sigma_u^2\right)\right)d\left[f\left(u,\sigma_u^2\right),f\left(u,\sigma_u^2\right)\right]\right]\\&\quad+\rho\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}L^2\left(u,S(u),\sigma(u)\right)\right.\\&\quad\cdot H\left(u,S(u),f\left(u,\sigma_u^2\right)\right)f\left(u,\sigma_u^2\right)\\&\quad\left.\cdot\tau d\left[W(u),f\left(t,\sigma_u^2\right)\right]\right].\end{aligned} \tag{35}$$

Proof. Notice that $F(T,X(T),\sigma_T^2)=e^{-rT}\mathrm{BS}(T,S(T),f(T,\sigma_T^2))=e^{-rT}V_T$. As $e^{-rt}V(t)$ is a martingale, we can write

$$\begin{aligned}e^{-rt}V(t)&=\mathbb{E}_t\left(e^{-rT}V(T)\right)\\&=\mathbb{E}_t\left(e^{-rT}\mathrm{BS}\left(T,S(T),f\left(T,\sigma_T^2\right)\right)\right)\\&=\mathbb{E}_t\left(F\left(T,S(T),\sigma_T^2\right)\right).\end{aligned} \tag{36}$$

Our idea is to apply an approximation to the identity argument as in Theorem 1 and then use the Functional Itô formula to

$$F\left(t,S(t),\sigma_t^2\right)=e^{-rt}\mathrm{BS}\left(t,S,f\left(t,\sigma_t^2\right)\right). \tag{37}$$

We deduce that

$$\begin{aligned}&F\left(T,S(T),\sigma_T^2\right)-F\left(t,S(t),\sigma_t^2\right)\\&\quad=\int_t^T\mathscr{D}_uF\left(u,S(u),\sigma_u^2\right)du\\&\qquad+\int_t^T\nabla_SF\left(u,S(u),\sigma_u^2\right)dS(u)\\&\qquad+\int_t^T\nabla_{\sigma^2}F\left(u,S(u),\sigma_u^2\right)du\\&\qquad+\frac{1}{2}\int_t^T\nabla_S^2F\left(t,S(t),\sigma_t^2\right)d[S,S](u)\\&\qquad+\frac{1}{2}\int_t^T\nabla_{\sigma^2}^2F\left(u,S(u),\sigma_u^2\right)d\left[\sigma_u^2,\sigma_u^2\right]\\&\qquad+\frac{1}{2}\int_t^T\mathscr{D}_u^2F\left(u,S(u),\sigma_u^2\right)du\\&\qquad+\int_t^T\nabla_{S,\sigma^2}^2F\left(u,S(u),\sigma_u^2\right)d\left[S(u),\sigma_u^2\right]\\&\qquad+\int_t^T\partial_f\left(\nabla_SF\left(u,S(u),\sigma_u^2\right)\right)d\left[S(u),f\left(u,\sigma_u^2\right)\right]\\&\qquad+\int_t^T\partial_f\left(\nabla_{\sigma^2}F\left(u,S(u),\sigma_u^2\right)\right)d\left[\sigma^2,f\left(u,\sigma_u^2\right)\right].\end{aligned} \tag{38}$$

Note that

(i) as $S(t)$ is not path-dependent, we have that $\nabla_S(\cdot) = \partial_S(\cdot)$;

(ii) as $u > t$ and f is a nonanticipative functional, then $\nabla_{\sigma^2(u)} f(t, \sigma_t^2) = 0$.

So, we have

$$\begin{aligned} F\left(T, S(T), \sigma_T^2\right) - F\left(t, S(t), \sigma_t^2\right) &= \int_t^T \mathscr{D}_u F\left(u, S(u), \sigma_u^2\right) du \\ &+ \int_t^T \partial_S F\left(u, S(u), \sigma_u^2\right) dS(u) \\ &+ \frac{1}{2}\int_t^T \partial_S^2 F\left(u, S(u), \sigma_u^2\right) d[S,S](u) \\ &+ \frac{1}{2}\int_t^T \mathscr{D}_u^2 F\left(u, S(u), \sigma_u^2\right) du \\ &+ \int_t^T \partial_{f,S}^2 F\left(u, S(u), \sigma_u^2\right) d\left[S(u), f\left(u, \sigma_u^2\right)\right]. \end{aligned} \tag{39}$$

We deduce that

$$\begin{aligned} F\left(T, S(T), \sigma_T^2\right) - F\left(t, S(t), \sigma_t^2\right) &= \int_t^T \mathscr{L}_{f(u,\sigma_u^2)} \mathrm{BS} du \\ &+ \int_t^T e^{-ru} \partial_f \mathrm{BS}\left(u, S(u), f\left(u, \sigma_u^2\right)\right) \mathscr{D}_u f\left(u, \sigma_u^2\right) du \\ &+ \frac{1}{2}\int_t^T e^{-ru} \partial_S^2 \mathrm{BS}\left(u, S(u), f\left(u, \sigma_u^2\right)\right) \\ &\cdot \left(\theta^2\left(u, S(u), \sigma(u)\right) - S^2 f^2\left(u, \sigma_u^2\right)\right) du \\ &+ \int_t^T \partial_S \mathrm{BS}\left(u, S(u), f\left(u, \sigma_u^2\right)\right) \theta\left(u, S(u), \sigma(u)\right) \\ &\cdot \left(\rho dW(u) + \sqrt{1-\rho^2} dB(u)\right) + \frac{1}{2} \\ &\cdot \int_t^T e^{-ru} \partial_f^2 \mathrm{BS}\left(u, S(u), f\left(u, \sigma_u^2\right)\right) d\left[f\left(u, \sigma_u^2\right), f\left(u, \sigma_u^2\right)\right] \\ &+ \rho \int_t^T e^{-ru} \partial_{f,S}^2 \mathrm{BS}\left(u, S(u), f\left(u, \sigma_u^2\right)\right) \\ &\cdot \theta\left(u, S(u), \sigma(u)\right) d\left[W(u), f\left(u, \sigma_u^2\right)\right]. \end{aligned} \tag{40}$$

Taking now conditional expectations, using (15), and multiplying by e^{rt}, we obtain that

$$\begin{aligned} e^{-r(T-t)} \mathbb{E}_t\left[F\left(T, S(T), \sigma_T^2\right)\right] &= \mathrm{BS}\left(t, S(t), f\left(u, \sigma_t^2\right)\right) \\ &+ \mathbb{E}_t\left[\int_t^T e^{-ru} f\left(u, \sigma_u^2\right)(T-t)\right. \\ &\left.\cdot S^2 \partial_S^2 \mathrm{BS}\left(u, S(u), f\left(u, \sigma_u^2\right)\right) \mathscr{D}_u f\left(u, \sigma_u^2\right)\right] du + \frac{1}{2} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-ru} G\left(u, S(u), f\left(u, \sigma_u^2\right)\right)\right. \\ &\left.\cdot \left(L^2\left(u, S(u), \sigma(u)\right) - f^2\left(u, \sigma_u^2\right)\right) du\right] + \frac{1}{2} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-ru} f\left(u, \sigma_u^2\right)^2\right. \\ &\left.\cdot \tau^2 K\left(u, S(u), f\left(u, \sigma_u^2\right)\right) d\left[f\left(u, \sigma_u^2\right), f\left(u, \sigma_u^2\right)\right]\right] \\ &+ \rho \mathbb{E}_t\left[\int_t^T e^{-ru} H\left(u, S(u), f\left(u, \sigma_u^2\right)\right) f\left(u, \sigma_u^2\right)\right. \\ &\left.\cdot \tau\theta\left(u, S(u), \sigma(u)\right) d\left[W(u), f\left(u, \sigma_u^2\right)\right]\right]. \end{aligned} \tag{41}$$

□

Remark 10. Note that Functional Itô formula proved in [11] holds for semimartingales but in [12] is also proved for Dirichlet process. In both cases, the hypothesis hold by definition of f and differentiability of the derivatives of Black-Scholes function when $\tau, S, \sigma > 0$. Therefore, this technique can be applied to these models.

Remark 11. Note that Theorem 9 coincides with Theorem 1 when we choose the volatility function as $f(t, \sigma_t^2) = v(t)$. We found an equivalence of the ideas developed by [6, 10–13] in the decomposition problem. Both formulas come from different points of view; the ideas under [10–13] are based on an extension to functionals of the work [14], while the main idea of [6] is to change a process by its expectation. Realize that standard Itô Calculus also can be applied to Dirichlet processes (for more information see [14]).

Remark 12. Realize that Theorem 9 holds for any nonanticipative $f(t, \sigma_t^2)$. It is not trivial to find a different nonanticipative process $f(t, \sigma_t^2)$ different from the one chosen in [6].

6. Basic Elements of Malliavin Calculus

In the next section, we present a brief introduction to the basic facts of Malliavin calculus. For more information, see [15].

Let us consider a Brownian motion $W = \{W(t),\ t \in [0,T]\}$ defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Set $H = L^2([0,T])$, and denote by $W(h)$ the Wiener integral of a function $h \in H$. Let $\mathcal{S}$ be the set of random variables of the form $F = f(W(h_1), \ldots, W(h_n))$, where $n \geq 1$, $f \in \mathscr{C}_b^\infty$, and $h_1, \ldots, h_n \in H$. Given a random variable F of this form, we define its derivative as the stochastic process $\{D_t^W F,\ t \in [0,T]\}$ given by

$$D_t^W F = \sum_{i=1}^n \partial_{x_i} f\left(W(h_1), \ldots, W(h_n)\right) h_i(x), \quad t \in [0,T]. \tag{42}$$

The operator D^W and the iterated operators $D^{W,n}$ are closable and unbounded from $L^2(\Omega)$ into $L^2([0,T]^n \times \Omega)$, for all $n \geq 1$. We denote the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{n,2}^2 := \|F\|_{L^2(\Omega)}^2 + \sum_{k=1}^{n} \left\|D^{W,k}F\right\|_{L^2([0,T]^k\times\Omega)}^2 . \quad (43)$$

We denote by δ^W the adjoint of the derivative operator D^W. Note that δ^W is an extension of the Itô integral in the sense that the set $L_a^2([0,T] \times \Omega)$ of square integrable and adapted processes is included in $\operatorname{Dom}\delta$ and the operator δ restricted to $L_a^2([0,T]\times\Omega)$ coincides with the Itô stochastic integral. We use the notation $\delta(u) = \int_0^T u(t)dW(t)$. We recall that $\mathbb{L}_W^{n,2} := L^2([0,T];\mathbb{D}_W^{n,2})$ is contained in the domain of δ for all $n \geq 1$.

We will use the next Itô formula for anticipative processes.

Proposition 13. *Let us consider the processes $X(t) = x(0) + \int_0^t u(s)dW(s) + \int_0^t v(s)ds$, where $u, v \in L_a^2([0,T]\times\Omega)$. Furthermore, consider also a process $Y(t) = \int_t^T \theta(s)ds$, for some $\theta \in \mathbb{L}^{1,2}$. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated in $(t, X(t), Y(t))$ are bounded by C. Then it follows that*

$$\begin{aligned} F(t, X(t), Y(t)) &= F(0, X(0), Y(0)) \\ &+ \int_0^t \partial_s F(s, X(s), Y(s))\, ds \\ &+ \int_0^t \partial_x F(s, X(s), Y(s))\, dX(s) \\ &+ \int_0^t \partial_y F(s, X(s), Y(s))\, dY(s) \\ &+ \int_0^t \partial_{x,y}^2 F(s, X(s), Y(s)) \left(D^- Y\right)(s)\, u(s)\, ds + \frac{1}{2} \\ &\cdot \int_0^t \partial_x^2 F(s, X(s), Y(s))\, u^2(s)\, ds, \end{aligned} \quad (44)$$

where $(D^-Y)(s) := \int_s^T D_s^W Y(r)dr$.

Proof. See [4]. □

The next proposition is useful when we want to calculate the Malliavin derivative.

Proposition 14. *Let σ and b be continuously differential functions on $\mathbb{R}$ with bounded derivatives. Consider the solution $X = \{X_t,\ t \in [0,T]\}$ of the stochastic differential equation:*

$$X(t) = x(0) + \int_0^t \sigma(X(s))\, dW(s) + \int_0^t b(X(s))\, ds. \quad (45)$$

Then, we have

$$\begin{aligned} D_s X(t) &= \sigma(X(s)) \\ &\cdot \exp\left(\int_s^t \sigma'(X(s))\, dW(s) + \int_s^t \lambda(s)\, ds\right) \mathbb{1}_{[0,t]}(s), \end{aligned} \quad (46)$$

where $\lambda(s) = [b' - (1/2)(\sigma')^2](X(s))$.

Proof. See [15, Section 2.2]. □

7. Decomposition Formula Using Malliavin Calculus

In this section, we use the Malliavin calculus to extend the call option price decomposition in an anticipative framework. This time, the decomposition formula has one term less than in the Itô formula's setup.

We recall the definition of the future average volatility as

$$\overline{\sigma}(t) := \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(s)\, ds}. \quad (47)$$

Theorem 15 (decomposition formula). *For all $t \in [0,T)$, we have*

$$\begin{aligned} V(t) &= \mathbb{E}_t\left[BS(t, S(t), \overline{\sigma}(t))\right] + \frac{1}{2} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-ru} G(u, S(u), \overline{\sigma}(u)) \right. \\ &\cdot \left.\left(L^2(u, S(u), \sigma(u)) - \sigma^2(u)\right) du\right] + \frac{\rho}{2} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} L(u, S(u), \sigma(u))\, H\left(u, S(u), \overline{\sigma}_u\right)\right. \\ &\cdot \left.\left(\int_u^T D_u^W \sigma^2(r)\, dr\right) du\right], \end{aligned} \quad (48)$$

where

$$\begin{aligned} G(t, S(t), \sigma(t)) &:= S^2(t)\, \partial_S^2 BS(t, S(t), \sigma(t)), \\ H(t, S(t), \sigma(t)) &:= S(t)\, \partial_S G(t, S(t), \sigma(t)), \\ L(t, S(t), \sigma(t)) &:= \frac{\theta(t, S(t), \sigma(t))}{S(t)}. \end{aligned} \quad (49)$$

Proof. Notice that $e^{-rT}\mathrm{BS}(T, S(T), \overline{\sigma}(T)) = e^{-rT}V_T$. As $e^{-rt}V(t)$ is a martingale, we can write

$$\begin{aligned} e^{-rt}V(t) &= \mathbb{E}_t\left(e^{-rT}V(T)\right) \\ &= \mathbb{E}_t\left(e^{-rT}\mathrm{BS}(T, S(T), \overline{\sigma}(T))\right). \end{aligned} \quad (50)$$

So, using the approximation to the identity argument and applying the Itô formula presented in Proposition 13 to

$$e^{-rt}\mathrm{BS}(t, S(t), \overline{\sigma}(t)), \quad (51)$$

we deduce by using (15) and (7) that

$$\begin{aligned}
&e^{-rT}\mathrm{BS}(T,S(T),\overline{\sigma}(T))-e^{-rt}\mathrm{BS}(t,S(t),\overline{\sigma}(t))\\
&=\int_t^T e^{-ru}\mathscr{L}_{\overline{\sigma}S}\mathrm{BS}(u,S(u),\overline{\sigma}(u))\,du+\frac{1}{2}\\
&\cdot\int_t^T e^{-ru}S^2(u)\,\partial_S^2\mathrm{BS}(u,S(u),\overline{\sigma}(u))\\
&\cdot\left(\frac{\theta(u,S(u),\sigma(u))}{S(u)}\right)^2du-\frac{1}{2}\int_t^T e^{-ru}S^2(u)\\
&\cdot\partial_S^2\mathrm{BS}(u,S(u),\overline{\sigma}(u))\,\sigma^2(u)\,du\\
&+\int_t^T e^{-ru}\partial_S\mathrm{BS}(u,S(u),\overline{\sigma}_u)\,\theta(u,S(u),\sigma(u))\\
&\cdot\left(\rho dW(u)+\sqrt{1-\rho^2}dB(u)\right)+\frac{\rho}{2}\\
&\cdot\int_t^T e^{-ru}\theta(u,S(u),\sigma(u))\\
&\cdot\partial_S\left(S^2(u)\,\partial_S^2\mathrm{BS}(u,S(u),\overline{\sigma}(u))\right)\\
&\cdot\left(\int_u^T D_u^W\sigma^2(r)\,dr\right)du.
\end{aligned}\tag{52}$$

Taking conditional expectation and multiplying by e^{rt}, we have

$$\begin{aligned}
&\mathbb{E}_t\left[e^{-r(T-t)}\mathrm{BS}(T,S(T),\overline{\sigma}(T))\right]\\
&=\mathbb{E}_t\left[\mathrm{BS}(t,S(t),\overline{\sigma}(t))\right]+\frac{1}{2}\\
&\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}G(u,S(u),\overline{\sigma}(u))\right.\\
&\left.\cdot\left(L^2(u,S(u),\sigma(u))-\sigma^2(u)\right)du\right]+\frac{\rho}{2}\\
&\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}L(u,S(u),\sigma(u))\right.\\
&\left.\cdot H(u,S(u),\overline{\sigma}(u))\left(\int_u^T D_u^W\sigma^2(r)\,dr\right)du\right].
\end{aligned}\tag{53}$$

□

Remark 16. As it is expected, a new term emerges when it is considered (1) like it happens in Theorem 1.

Remark 17. In particular, when $\theta(t,S(T),\sigma(t))=\sigma(t)S(t)$,

$$\begin{aligned}
V(t)&=\mathrm{BS}(t,S_t,\overline{\sigma}(t))+\frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\sigma(u)\right.\\
&\left.\cdot H(u,S(u),\overline{\sigma}_u)\left(\int_u^T D_u^W\sigma^2(r)\,dr\right)du\right].
\end{aligned}\tag{54}$$

Additionally, the gamma effect is cancelled as we have seen in the Itô formula section.

Remark 18. Note that when $v(t)$ is a deterministic function, we have that all decomposition formulas are equal.

Remark 19. When $\rho=0$, we have

$$\begin{aligned}
&\mathbb{E}_t\left[\mathrm{BS}(t,S(t),\overline{\sigma}(t))-\mathrm{BS}(t,S_t,v(t))\right]=\frac{1}{2}\\
&\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(G(u,S(u),\overline{\sigma}(u))\right.\\
&\left.-G(u,S(u),v(u)))\,L^2(u,S(u),\sigma(u))\,du\right]-\frac{1}{2}\\
&\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}(G(u,S(u),\overline{\sigma}(u))\right.\\
&\left.-G(u,S(u),v(u)))\,\sigma^2(u)\,du\right]-\frac{1}{8}\\
&\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}K(u,S(u),v(u))\,d[M,M](u)\right].
\end{aligned}\tag{55}$$

In particular, when $\theta(t,S(t),\sigma(t))=\sigma(t)S(t)$,

$$\begin{aligned}
&\mathbb{E}_t\left[\mathrm{BS}(t,S(t),\overline{\sigma}(t))-\mathrm{BS}(t,S_t,v(t))\right]=-\frac{1}{8}\\
&\cdot\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}K(u,S(u),v(u))\,d[M,M](u)\right].
\end{aligned}\tag{56}$$

The difference between the two approaches is given by the vol-vol of the option.

8. An Expression for the Derivative of the Implied Volatility

In this section, we give a general expression for the derivative of the implied volatility under the framework of Itô Calculus and Malliavin calculus. A previous calculation of this derivative in the case of exponential models by using Malliavin calculus is given in [5].

Let $I(S(t))$ denote the implied volatility process, which satisfies by definition $V(t)=\mathrm{BS}(t,S(t),I(S(t)))$. We calculate the derivative of the implied volatility in the standard Itô case.

Proposition 20. *Under (1), for every fixed $t\in[0,T)$ and assuming that $(v(t))^{-1}<\infty$ a.s., we have that*

$$\begin{aligned}
\partial_S I(S^*(t))&=\frac{\mathbb{E}_t\left[\int_t^T\partial_S F_2(u,S^*(u),v(u))\,du\right]}{\partial_\sigma BS(t,S^*(t),I(S^*(t)))}\\
&-\frac{\mathbb{E}_t\left[\int_t^T(F_1(u,S^*(u),v(u))+\partial_S F_3(u,S^*(u),v(u)))\,du\right]}{2S\partial_\sigma BS(t,S^*(t),I(S^*(t)))},
\end{aligned}\tag{57}$$

where

$$\begin{aligned}
&\mathbb{E}_t\left[\int_t^T F_1(u,S(u),v(u))\,du\right] = \frac{1}{2} \\
&\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u,S(u),v(u)) \right.\\
&\left.\cdot \left(L^2(u,S(u),\sigma(u)) - \sigma^2(u)\right) du\right] + \frac{1}{8} \\
&\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u,S(u),v(u))\, d[M,M](u)\right] \\
&+ \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} \right.\\
&\cdot \frac{\theta(u,S(u),\sigma(u))}{S(u)} H(u,S(u),v(u))\, d[W,M] \\
&\left.\cdot (u)\right], \\
&\mathbb{E}_t\left[\int_t^T F_2(u,S(u),v(u))\,du\right] = \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)}\right. \\
&\cdot \frac{\theta(u,S(u),\sigma(u))}{S(u)} H(u,S(u),v(u))\, d[W,M] \\
&\left.\cdot (u)\right], \\
&\mathbb{E}_t\left[\int_t^T F_3(u,S(u),v(u))\,du\right] = \frac{1}{2} \\
&\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u,S(u),v(u)) \right.\\
&\left.\cdot \left(L^2(u,S(u),\sigma(u)) - \sigma^2(u)\right) du\right] + \frac{1}{8} \\
&\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u,S(u),v(u))\, d[M,M](u)\right].
\end{aligned} \tag{58}$$

Proof. Taking partial derivatives with respect to $S(t)$ on the expression $V(t) = \mathrm{BS}(t,S(t),I(S(T)))$, we obtain

$$\begin{aligned}
\partial_S V(t) &= \partial_S \mathrm{BS}(t,S(t),I(S(T))) \\
&\quad + \partial_\sigma \mathrm{BS}(t,S(t),I(S(T)))\,\partial_S I(S(t)).
\end{aligned} \tag{59}$$

On the other hand, from Theorem 1, we deduce that

$$\begin{aligned}
V(t) &= \mathrm{BS}(t,S(t),v(t)) \\
&\quad + \mathbb{E}_t\left[\int_t^T F_1(u,S(u),v(u))\,du\right],
\end{aligned} \tag{60}$$

which implies that

$$\begin{aligned}
\partial_S V(t) &= \partial_S \mathrm{BS}(t,S(t),v(t)) \\
&\quad + \mathbb{E}_t\left[\int_t^T \partial_S F_1(u,S(u),v(u))\,du\right].
\end{aligned} \tag{61}$$

Using the fact that $(v(t))^{-1} < \infty$, we can check that $\partial_S V(t)$ is well defined and finite a.s. Thus, using the fact that $S^*(t) = K\exp(r(T-t))$, (59), and (61), we obtain

$$\begin{aligned}
&\partial_S I(S^*(t)) \\
&= \frac{\partial_S \mathrm{BS}(t,S^*(t),v(t)) - \partial_S \mathrm{BS}(t,S^*(t),I(S(t)))}{\partial_\sigma \mathrm{BS}(t,S^*(t),I(S(t)))} \\
&\quad + \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_1(u,S^*(u),v(u))\,du\right]}{\partial_\sigma \mathrm{BS}(t,S^*(t),I(S(t)))}.
\end{aligned} \tag{62}$$

From [16], we know that $\partial_S I^0(t) = 0$, where $I^0(t)$ is the implied volatility in the case $\rho = 0$, so

$$\begin{aligned}
&\partial_S \mathrm{BS}(t,S^*(t),v(t)) \\
&= \partial_S \mathrm{BS}\left(t,S^*(t),I^0(S(t))\right) \\
&\quad - \mathbb{E}_t\left[\int_t^T \partial_S F_3(u,S^*(u),v(u))\,du\right].
\end{aligned} \tag{63}$$

So, we have that

$$\begin{aligned}
&\partial_S I(S^*(t)) \\
&= \frac{\partial_S \mathrm{BS}\left(t,S^*(t),I^0(t)\right) - \partial_S \mathrm{BS}(t,S^*(t),I(S^*(t)))}{\partial_\sigma \mathrm{BS}(t,S^*(t),I(S^*(t)))} \\
&\quad + \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_2(u,S^*(u),v(u))\,du\right]}{\partial_\sigma \mathrm{BS}(t,S^*(t),I(S^*(t)))}.
\end{aligned} \tag{64}$$

On the other hand, we have that

$$\begin{aligned}
\partial_S \mathrm{BS}(t,S^*(t),v(t)) &= \phi(d), \\
\mathrm{BS}(t,S^*(t),v(t)) &= S(\phi(d) - \phi(-d)),
\end{aligned} \tag{65}$$

where ϕ is the standard Gaussian density. Then

$$\begin{aligned}
&\partial_S \mathrm{BS}(t,S^*(t),v(t)) = \frac{\mathrm{BS}(t,S^*(t),v(t)) + S}{2S}, \\
&\partial_S \mathrm{BS}\left(t,S^*(t),I^0(t)\right) - \partial_S \mathrm{BS}(t,S^*(t),I(S^*(t))) \\
&= \frac{1}{2S}\left(\mathrm{BS}\left(t,S^*(t),I^0(t)\right) - \mathrm{BS}(t,S^*(t),\right. \\
&\left. I(S^*(t)))\right) = -\frac{1}{2S}\mathbb{E}_t\left[\int_t^T (F_1(u,S^*(u),v(u))\right. \\
&\left. + \partial_S F_3(u,S^*(u),v(u)))\,du\right].
\end{aligned} \tag{66}$$

□

Now, we derive the implied volatility using Malliavin calculus. This is done in [5] for the case $\theta(t,S(T),\sigma(t)) = \sigma(t)S(t)$.

Proposition 21. *Under (1), for every fixed $t \in [0,T)$ and assuming that $(\bar{\sigma}(t))^{-1} < \infty$ a.s., we have that*

$$\partial_S I(S^*(t)) = \frac{\mathbb{E}_t\left[\int_t^T \partial_S F_2(u, S^*(u), \bar{\sigma}(u))\,du\right]}{\partial_\sigma BS(t, S^*(t), I(S^*(t)))} - \frac{\mathbb{E}_t\left[\int_t^T (F_1(u, S^*(u), \bar{\sigma}(u)) + \partial_S F_3(u, S^*(u), \bar{\sigma}(u)))\,du\right]}{2S\partial_\sigma BS(t, S^*(t), I(S^*(t)))}, \tag{67}$$

where

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^T F_1(u, S(u), \bar{\sigma}(u))\,du\right] &= \frac{1}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u, S(u), \bar{\sigma}(u)) \cdot \left(L^2(u, S(u), \sigma(u)) - \sigma^2(u)\right) du\right] + \frac{\rho}{2} \\
&\quad \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} L(u, S(u), \sigma(u)) \cdot H(u, S(u), \bar{\sigma}(u)) \left(\int_u^T D_u^W \sigma^2(r)\,dr\right) d[W, M] \cdot (u)\right], \\
\mathbb{E}_t\left[\int_t^T F_2(u, S(u), \bar{\sigma}(u))\,du\right] &= \frac{\rho}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} L^2(u, S(u), \sigma(u)) \cdot \left(\int_u^T D_u^W \sigma^2(r)\,dr\right) d[W, M](u)\right], \\
\mathbb{E}_t\left[\int_t^T F_3(u, S(u), \bar{\sigma}(u))\,du\right] &= \frac{1}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u, S(u), \bar{\sigma}(u)) \cdot \left(L^2(u, S(u), \sigma(u)) - \sigma^2(u)\right) du\right].
\end{aligned} \tag{68}$$

Proof. See [5] or the previous proof. □

Remark 22. Note that this is generalization of the formula proved in [5]. In that case, $F_1 = F_2$ and $F_3 = 0$.

9. Examples

In this section, we provide some applications of the decomposition formula to well-known models in Finance.

9.1. Heston Model. We consider that the stock price follows the Heston Model (1). Using Theorem 1 or Theorem 9, we have

$$\begin{aligned}
V(t) &= BS(t, X(t), v(t)) + \frac{\rho}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} H(u, X(u), v(u)) \cdot \left(\int_u^T e^{-k(r-s)}\,dr\right) \sigma^2(u)\,\nu\,du\right] + \frac{1}{8} \\
&\quad \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u, X(u), v(u)) \cdot \left(\int_u^T e^{-k(r-s)}\,dr\right)^2 \nu^2 \sigma^2(u)\,du\right].
\end{aligned} \tag{69}$$

Using Theorem 15, we have that

$$V(t) = BS(t, S_t, \bar{\sigma}(t)) + \frac{\rho}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} H(u, S(u), \bar{\sigma}(u)) \cdot \left(\int_u^T D_u^W \sigma^2(r)\,dr\right) \sigma(u)\,du\right], \tag{70}$$

where $D_u^W \sigma^2(r) = \nu\sigma(u)\exp((\nu/2)\int_u^r (1/\sigma(s))dW(s) + \int_u^r [-k - \nu^2/8\sigma^2(s)]ds)$.

9.2. SABR Model. We consider that the stock price follows the SABR model (3). Using Theorem 1 or Theorem 9, we have

$$\begin{aligned}
V(t) &= BS(t, S(t), v(t)) + \frac{1}{2} \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u, S(u), v(u)) \sigma^2(u) \cdot \left(S^{2(\beta-1)}(u) - 1\right) du\right] + \frac{1}{8} \\
&\quad \cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} K(u, S(u), v(u))\,d[M, M]\right](u) \\
&\quad + \frac{\rho}{2}\mathbb{E}_t\left[\int_t^T e^{-r(u-t)} H(u, S(u), v(u)) \sigma(u)\,d[W, M] \cdot (u)\right],
\end{aligned} \tag{71}$$

where

$$\begin{aligned}
d[M, M] &= 4\alpha^2\sigma^4(t)\left(\int_t^T e^{\alpha^2(s-t)}\,ds\right)^2 dt, \\
d[M, W] &= 2\alpha\sigma^2(t)\left(\int_t^T e^{\alpha^2(s-t)}\,ds\right) dt.
\end{aligned} \tag{72}$$

Using Theorem 15, we have that

$$\begin{aligned} V(t) &= \mathbb{E}_t\left[\mathrm{BS}(t, S(t), \overline{\sigma}(t))\right] + \frac{1}{2} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} G(u, S(u), \overline{\sigma}(u))\, \sigma^2(u) \right. \\ &\left. \cdot \left(S^{2(\beta-1)}(u) - 1\right) du\right] + \frac{\rho}{2} \\ &\cdot \mathbb{E}_t\left[\int_t^T e^{-r(u-t)} H(u, S(u), \overline{\sigma}(u)) \right. \\ &\left. \cdot \left(\int_u^T D_u^W \sigma^2(r)\, dr\right) \sigma(u)\, du\right], \end{aligned} \tag{73}$$

where $D_u^W \sigma^2(r) = 2\alpha\sigma^2(u) \mathbb{1}_{[0,r]}(u)$.

10. Conclusion

In this paper, we notice that the idea used in [6] can be used for a generic Stochastic Differential Equation (SDE). There is no need to specify the volatility process. Only existence and uniqueness of the solution of the SDE are needed, allowing much more flexibility in the decomposition formula. We observe what is the effect of assuming that the stock price follows an exponential process and how a new term arises in a general framework. Additionally, we have computed the decomposition using three different methods: Itô formula, Functional Itô Calculus, and Malliavin calculus. In the case of call options, the idea used in [6] is equivalent to the use of the Functional Itô formula developed in [10–13] but without the need of the theory behind the Functional Itô Calculus. Both formulas can be applied to Dirichlet process, in particular, to the fractional Brownian motion with Hurst parameter equal or bigger than 1/2. Furthermore, we realize that the Feynman-Kac formula has a key role in the decomposition process.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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