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CONDITIONAL MONTE CARLO SCHEME FOR STABLE GREEKS OF WORST-OF AUTOCALLABLE NOTES*

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It is well known that the application of Monte Carlo method in pricing of products with early termination feature results in a high Monte Carlo error and unstable greeks; see Fries & Joshi (2011). We develop a Monte Carlo scheme that utilizes a special structure of worst-of autocallable notes and produces stable greeks. This scheme clearly demonstrates the variance reduction in Monte Carlo scheme and can be used in pricing of multi-asset worst-of autocallable notes with any number of underlying assets. We suggest an algorithm and analyze its performance for an autocallable note on four assets. The suggested algorithm allows one to calculate stable greeks (delta, gamma, vega and others) and substantially reduce the computational effort to achieve the desired accuracy in comparison to standard Monte Carlo algorithm.

Keywords: Monte Carlo; autocallable; variance reduction; sensitivities; greeks.

1. Introduction

Let us define the structure of worst-of autocallable notes. Given a set of observation dates $0 < t_1 < \cdots < t_n$, consider a worst-of autocallable note based on the basket of assets $S^{(1)}, \ldots, S^{(d)}$. At each observation date t_j, except the final observation date,

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it provides a payoff

$$\begin{aligned} &Nc, && \text{if } \min_{i=\overline{1,d}} \frac{S^{(i)}(t_k)}{S^{(i)}_{\text{ref}}} < B \quad \forall k < j \quad \text{and} \quad C \le \min_{i=\overline{1,d}} \frac{S^{(i)}(t_j)}{S^{(i)}_{\text{ref}}} < B, \\ &N + Nc, && \text{if } \min_{i=\overline{1,d}} \frac{S^{(i)}(t_k)}{S^{(i)}_{\text{ref}}} < B \quad \forall k < j \quad \text{and} \quad \min_{i=\overline{1,d}} \frac{S^{(i)}(t_j)}{S^{(i)}_{\text{ref}}} \ge B. \end{aligned} \tag{1.1}$$

On the final observation date t_n, the payoff is

$$\begin{aligned} &N + Nc, \quad \text{if } \min_{i=\overline{1,d}} \frac{S^{(i)}(t_k)}{S^{(i)}_a} < B \quad \forall k < n \quad \text{and} \quad C \le \min_{i=\overline{1,d}} \frac{S^{(i)}(t_n)}{S^{(i)}_a}, \\ &N, \quad \text{if } \min_{i=\overline{1,d}} \frac{S^{(i)}(t_k)}{S^{(i)}_a} < B \quad \forall k < n \quad \text{and} \quad L \le \min_{i=\overline{1,d}} \frac{S^{(i)}(t_n)}{S^{(i)}_a} < C, \\ &Nq(S^{(1)}(t_n), \dots, S^{(d)}(t_n)), \quad \text{if } \min_{i=\overline{1,d}} \frac{S^{(i)}(t_k)}{S^{(i)}_a} < B \quad \forall k < n \quad \text{and} \\ &\qquad\qquad\qquad \min_{i=\overline{1,d}} \frac{S^{(i)}(t_n)}{S^{(i)}_a} < L, \end{aligned} \tag{1.2}$$

where N is a notional amount, C, B and L are, respectively, called the coupon, autocall and lower barriers. We always assume that $C < B$, c is the coupon rate for each observation period and is fixed upon the initiation of the note. $S^{(i)}_a := S^{(i)}(t_a)$, $i = 1, \dots, d$ denotes the value of ith asset at initiation moment $t_a < t_0$ of the note and used to define the performance of the each asset. Thus, $S^{(i)}(t_k)/S^{(i)}_a$ defines the performance of ith asset at time t_k. Function $q(S^{(1)}(t_n), \dots, S^{(d)}(t_n))$ defines reduced amount paid to the holder of autocallable note and usually equals $\min_{i=\overline{1,d}} S^{(i)}(t_n)/S^{(i)}_a$ (worst-performing asset).

In other words, this contract pays a contingent coupon until a knockout event, defined as the first observation date at which the worst-performing asset exceeds level B. If at one of the observation dates the barrier level B is exceeded, the notional amount N plus coupon Nc is paid and the contract terminates. However, if the contract survives until the final observation date, and at the final observation date the worst performer in the basket is less than L, the reduced amount $Nq(S^{(1)}(t_n), \dots, S^{(d)}(t_n))$ to the holder of the note is paid. If the worst performer in the basket is less than C, but greater than L, the notional N is redeemed without a coupon payment.

For simplicity, let us denote the trigger function by

$$I_j \triangleq \min_{i=\overline{1,d}} \frac{S^{(i)}(t_j)}{S^{(i)}_a}. \tag{1.3}$$

The arbitrage-free value $V(0)$ of specified worst-of autocallable note at time 0 is given by the expectation $\mathbb{E}(\cdot) := \mathbb{E}^{\mathbb{Q}}(\cdot)$ of discounted future cash flows where $\mathbb{Q}$

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denotes risk-neutral measure and $\beta(\cdot)$ is a money-market account.

$$
\begin{aligned}
V(0) = \mathbb{E}\Bigg\{ & \sum_{j=1}^{n-1} (\beta(t_j))^{-1} \left[Nc \prod_{k=1}^{j-1} \mathbb{1}_{I_k<B}\, \mathbb{1}_{C\le I_j<B} + (N+Nc) \prod_{k=1}^{j-1} \mathbb{1}_{I_k<B}\, \mathbb{1}_{I_j\ge B} \right] \\
& + (\beta(t_n))^{-1} \Bigg[(N+Nc) \prod_{k=1}^{n-1} \mathbb{1}_{I_k<B}\, \mathbb{1}_{I_n\ge C} + N \prod_{k=1}^{n-1} \mathbb{1}_{I_k<B}\, \mathbb{1}_{L\le I_n<C} \\
& + Nq(S^{(1)}(t_n),\dots,S^{(d)}(t_n)) \prod_{k=1}^{n-1} \mathbb{1}_{I_k<B}\, \mathbb{1}_{I_n<L} \Bigg] \Bigg\}. \qquad (1.4)
\end{aligned}
$$

For the rigorous treatment of the subject, we refer to authoritative monograph of Andersen & Piterbarg (2010). Next, following the ideas of smoothing out the discontinuities by survival conditioning, we can develop a scheme given by the following proposition.

Proposition 1. *A worst-of autocallable note with payoff is defined by* (1.1)–(1.2) *can be priced as follows:*

$$
\begin{aligned}
V(0) = \widetilde{\mathbb{E}}\Bigg\{ & \sum_{j=1}^{n-1} \beta^{-1}(t_j)(Nc\psi_{j,2} + (N+Nc)\psi_{j,1}) + \beta^{-1}(t_n)[(N+Nc)(\psi_{n,2}+\psi_{n,1}) \\
& + N\psi_{n,3} + Nq(S^{(1)}(t_n),\dots,S^{(d)}(t_n))\psi_{n,4}] \Bigg\}, \qquad (1.5)
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_{j,1} &= \mathbb{Q}\left(\bigcap_{k=1}^{j-1} \{I_k < B\} \cap \{I_j \ge B\} \right), \quad j = \overline{1,n}, \\
\psi_{j,2} &= \mathbb{Q}\left(\bigcap_{k=1}^{j-1} \{I_k < B\} \cap \{C \le I_j < B\} \right), \quad j = \overline{1,n}, \\
\psi_{n,3} &= \mathbb{Q}\left(\bigcap_{k=1}^{n-1} \{I_k < B\} \cap \{L \le I_n < C\} \right), \\
\psi_{n,4} &= \mathbb{Q}\left(\bigcap_{k=1}^{n-1} \{I_k < B\} \cap \{I_n < L\} \right).
\end{aligned} \qquad (1.6)
$$

The outer expectation is taken under the survival measure $\widetilde{\mathbb{Q}}$.

This subject is discussed in the monograph of Andersen & Piterbarg (2010).

2. Pricing Model

Assume that the dynamics of asset prices $S^{(1)}(t), \ldots, S^{(d)}(t)$ are given by multi-dimensional stochastic process as

$$dS^{(i)}(t) = \mu^{(i)}(t)dt + \sigma^{(i)}(t)\, dW^{(i)}(t). \quad i = \overline{1, d}, \tag{2.1}$$

where $W(t) = (W^{(1)}(t), \ldots, W^{(d)}(t))$ with

$$d\langle W^{(i)}(t), W^{(j)}(t)\rangle = \rho_{ij} dt.$$

Here, R denotes the correlation matrix

$$R \triangleq \begin{pmatrix} \rho_{11} & \cdots & \rho_{1d} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots} \\ \rho_{d1} & \cdots & \rho_{dd} \end{pmatrix}. \tag{2.2}$$

We allow the coefficients of (2.1) to be $S^{(i)}(t)$-dependent, but for notational simplicity the variable $S^{(i)}$ does not appear as a variable in $\mu^{(i)}$, $\sigma^{(i)}$.

We shall use the ideas of smoothing out the discontinuities by survival conditioning technique of Glasserman & Staum (1999) and importance sampling technique (see Geweke (1991)) to sample only paths that remain in the survival region, where the trigger function I_j doesn't cross the level B for *all* observation dates.

Consider a step from observation date t_j to t_{j+1}. We use Log-Euler scheme to discretize (2.1); as for the volatility $\sigma^{(i)}(t)$ that is close to proportional in $S^{(i)}(t)$ (e.g. lognormal specification), Log-Euler scheme produces lower bias than analogous Euler scheme. In addition, Log-Euler discretization keeps asset prices positive, whereas Euler discretization will generally not. See Andersen & Piterbarg (2010) for in-depth discussion.

Log-Euler discretization for (2.1) is represented by

$$S^{(i)}(t_{j+1}) = S^{(i)}(t_j) \exp\left\{ \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2 \right) \right.$$

$$\left. \Delta t_j + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)} \sqrt{\Delta} t_j Z^{(i)} \right\}, \quad i = \overline{1, d}, \tag{2.3}$$

where $Z = (Z^{(1)}, \ldots, Z^{(d)})$ is a d-dimensional Gaussian vector with the correlation matrix R.

First, we apply a Cholesky decomposition on the correlation matrix R and transform vector Z into d-dimensional Gaussian vector Y with independent coordinates, $Y \sim N(0, I)$ by

$$Z = AY, \tag{2.4}$$

where $AA^\top = R$.

Next, we apply additional orthogonal transformation to Y by setting

$$Y = QX. \tag{2.5}$$

As illustrated by formula (4.2), the survival region in Z variables is the exterior of shifted positive orthant. The transformation (2.4) translates it to the exterior of

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shifted 'orthant' with non-orthogonal faces. Finally, the rotation step (2.5) rotates this shifted 'orthant' in such a way, that its projection on first $d-1$ coordinates is $\mathbb{R}^{d-1}$.

Thus, combining (2.3), (2.4) and (2.5), the updated discretization for (2.1) is

$$S^{(i)}(t_{j+1}) = S^{(i)}(t_j)\exp\left\{\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\right.$$
$$\left.\times\,\Delta t_j + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}(AQX)^{(i)}\right\}, \quad i=\overline{1,d}, \tag{2.6}$$

where $X \sim N(0,I)$ is a d-dimensional Gaussian vector with independent coordinates.

3. Numerical Scheme

Assume the note has not been terminated by time t_j. As we move from step t_j to t_{j+1}, we see that it terminates if

$$I_{j+1} = \min_{i=\overline{1,d}} \frac{S^{(i)}(t_{j+1})}{S_a^{(i)}} \geq B,$$

i.e.

$$\frac{S^{(i)}(t_j)}{S_a^{(i)}}\exp\left\{\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j\right.$$
$$\left. + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}(AQX)^{(i)}\right\} \geq B, \quad \forall\, i=\overline{1,d}.$$

$$\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}(AQX)^{(i)} \geq \ln\left(\frac{BS_a^{(i)}}{S^{(i)}(t_j)}\right)$$
$$-\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j, \quad \forall\, i=\overline{1,d}.$$

$$(AQX)^{(i)} \geq \frac{\ln\left(\frac{BS_a^{(i)}}{S^{(i)}(t_j)}\right) - \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j}{\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}}, \quad \forall\, i=\overline{1,d}. \tag{3.1}$$

Let us denote

$$M = AQ \tag{3.2}$$

and $X' = (X_1,\ldots,X_{d-1},0)$. Here M_i denotes ith row of matrix M and m_i denotes the last element of vector M_i (we suppress the dependence of M and m_i on index j).

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We note that the projection of the survival region in first $d-1$ coordinates is $\mathbb{R}^{d-1}$, thus the survival condition in (3.1) depends on its ultimate coordinate X_d. Therefore, the *trigger* condition (3.1) is equivalent to

$$m_i X_d \geq \frac{\ln\left(\dfrac{BS_a^{(i)}}{S^{(i)}(t_j)}\right) - \left(\dfrac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \dfrac{1}{2}\left(\dfrac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j}{\dfrac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}} - M_i X', \quad \forall\, i = \overline{1,d},$$

$$X_d \geq \frac{1}{m_i}\left\{\frac{\ln\left(\dfrac{BS_a^{(i)}}{S^{(i)}(t_j)}\right) - \left(\dfrac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \dfrac{1}{2}\left(\dfrac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j}{\dfrac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}} - M_i X'\right\}, \quad \forall\, i = \overline{1,d},$$

Therefore, the trigger condition reduces to

$$X_d \geq \max_{i=\overline{1,d}} \frac{1}{m_i}\left\{\frac{\ln\left(\dfrac{BS_a^{(i)}}{S^{(i)}(t_j)}\right) - \left(\dfrac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \dfrac{1}{2}\left(\dfrac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j}{\dfrac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}} - M_i X'\right\}. \tag{3.3}$$

4. Choice of Matrix $\boldsymbol{Q}$

According to the specification (1.1) and (1.2), if autocallable note has not been terminated before t_{j+1}, it survives at date t_{j+1} if

$$\min_{i=\overline{1,d}} \frac{S^{(i)}(t_j)}{S_a^{(i)}} \exp\left\{\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j} Z^{(i)}\right\} < B. \tag{4.1}$$

Inverting the inequality, we see that the termination condition

$$\min_{i=\overline{1,d}} \frac{S^{(i)}(t_j)}{S_a^{(i)}} \exp\left\{\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j} Z^{(i)}\right\} \geq B$$

is equivalent to

$$\frac{S^{(i)}(t_j)}{S_a^{(i)}} \exp\left\{\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j} Z^{(i)}\right\} \geq B, \quad \forall\, i = \overline{1,d}.$$

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$$\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta t_j}Z^{(i)} \geq \ln\left(\frac{BS_a^{(i)}}{S^{(i)}(t_j)}\right) - \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - \frac{1}{2}\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta t_j, \quad \forall\, i = \overline{1,d}. \tag{4.2}$$

So, the complement of region defined by (4.2) represents a survival region at step from t_j to t_{j+1}. Reducing both sides of inequality by $\sigma^{(i)}(t_j)/S^{(i)}(t_j)\sqrt{\Delta t_j} \neq 0$, we see that the survival region is the exterior of shifted positive orthant. In Z–variables, the survival region's boundaries are orthogonal hyperplanes, parallel to

$$Z_i = 0, \quad i = \overline{1,d}. \tag{4.3}$$

Such region obviously does not have a Lipschitz-continuous boundary parametrization. Change of variables $Z = AY$ induced by Cholesky decomposition of correlation matrix R, translates the survival region into a shifted 'orthant' for which the boundaries hyperplanes are no longer orthogonal. The boundaries in Y-variables are hyperplanes, parallel to

$$a_{i1}Y_1 + \cdots + a_{id}Y_d = 0, \quad i = \overline{1,d}, \tag{4.4}$$

where $\{a_{ij}\}_{1\leq i\leq j\leq d}$ are elements of matrix A.

Let us denote by $\mathbf{n}^{(i)} \triangleq (a_{i1},\ldots,a_{id}) \in \mathbb{R}^d$ the normal vector of hyperplane defined by (4.4) and $\mathbf{n}$ denote the vector that forms equal angles with every normal vectors $\mathbf{n}^{(i)}$, $1 \leq i \leq d$. Assuming that $\mathbf{n}$, $\mathbf{n}^{(i)}$ have unit length, $\mathbf{n}$ can be found from the relation

$$(\mathbf{n}^{(1)},\mathbf{n}) = \cdots = (\mathbf{n}^{(d)},\mathbf{n}), \tag{4.5}$$

This system of equations defines a set of linear equations and determines the solution $\mathbf{n}$ up to its length and direction.

Finally, we choose orthogonal transformation Q in such a way that $Q\mathbf{e}_d = \mathbf{n}$, where $\mathbf{e}_d = (0,0,\cdots,0,1)$ and $\mathbf{n}$ points outward the survival region (it is sufficient to take $\mathbf{n}$'s first coordinate equal to 1 and normalize the solution of (4.5)). This ensures that the survival region in X-variables has Lipschitz-continuous boundary parametrization.

Let us provide some details on the choice of matrix Q. The condition $Q\mathbf{e} = \mathbf{n}$ automatically implies that the last column of Q is vector-column $\mathbf{n}$. As columns of matrix Q must be orthogonal, first $(d-1)$ columns can be found, for instance, by applying Gram-Schmidt orthogonalization process to system $\{\mathbf{n},\mathbf{e}_2,\ldots,\mathbf{e}_d\}$, followed by the removal of the first vector. We note that $\{\mathbf{n},\mathbf{e}_2,\ldots,\mathbf{e}_d\}$ forms a basis in $\mathbb{R}^d$, as $n_1 = 1$, so that $\mathbf{n}$ is not orthogonal to $\mathbf{e}_1$.

5. Algorithm

For the sake of completeness, we provide the full algorithm below assuming that interest rates are deterministic and flat.

(1) Define d as the number of assets, n as the number of observation dates, R as the positive definite correlation matrix, $\{S_a^{(1)}, \ldots, S_a^{(d)}\}$ as the assets' reference values, $\{\sigma^{(1)}(t), \ldots, \sigma^{(d)}(t)\}$ as the deterministic volatility function, $\{\mu^{(1)}(t), \ldots, \mu^{(d)}(t)\}$ as the drift function (as noted above, $\mu^{(i)}$ and $\sigma^{(i)}$ may depend on $S^{(i)}$, but we suppress it for convenience).

(2) Find $A : AA^\top = R$, e.g. use Cholesky decomposition.

(3) Find column vector $\mathbf{n} \triangleq (n_1, \ldots, n_d)^\top$ that satisfies

$$\begin{cases} n_1 = 1, \\ (\mathbf{n}, \mathbf{n}^{(1)}) = \cdots = (\mathbf{n}, \mathbf{n}^{(d)}), \end{cases} \tag{5.1}$$

where $\mathbf{n}^{(i)} \triangleq (a_{i1}, \ldots, a_{id})^\top, i = 1, \ldots, d$. Normalize $\mathbf{n}$ by setting $\mathbf{n} = \mathbf{n}/\|\mathbf{n}\|$.

(4) Apply Gram-Schmidt othogonalization process: $\{\mathbf{n}, \mathbf{e}_2, \ldots, \mathbf{e}_d\} \xrightarrow{\text{Gram-Schmidt}} \{\mathbf{n}, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$. Set $Q \triangleq (\mathbf{v}_d \ldots \mathbf{v}_2\ \mathbf{n})$, i.e. column-vectors $\mathbf{v}_d, \ldots, \mathbf{v}_2, \mathbf{n}$ become columns of matrix Q.

(5) Set $w = 1$, the probability that note has not terminated, and set $V = 0$. Further, for $j = 1, \ldots, n$:

5.1 Let M denote matrix 3.2, denote by M_i the ith row of M and by m_i the last element vector of M_i.

5.2 Set $\Delta_t \triangleq t_j - t_{j-1}$ and for every $i = 1, \ldots, d$ define

$$\begin{aligned} B_i &= \left(\ln\left(\frac{BS_a^{(i)}}{S^{(i)}(t_j)} \right) - \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - 0.5 \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} \right)^2 \right) \Delta_t \right) \cdot \\ &\quad \times \left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)} \sqrt{\Delta_t} \right)^{-1}, \\ C_i &= \left(\ln\left(\frac{CS_a^{(i)}}{S^{(i)}(t_j)} \right) - \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - 0.5 \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} \right)^2 \right) \Delta_t \right) \cdot \\ &\quad \times \left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)} \sqrt{\Delta_t} \right)^{-1}, \\ L_i &= \left(\ln\left(\frac{LS_a^{(i)}}{S^{(i)}(t_j)} \right) - \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - 0.5 \left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} \right)^2 \right) \Delta_t \right) \cdot \\ &\quad \times \left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)} \sqrt{\Delta_t} \right)^{-1}. \end{aligned} \tag{5.2}$$

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5.3 Generate $X' \sim N(0, I)$, $(d-1)$-dimensional Gaussian vector with independent coordinates, and define

$$
\begin{aligned}
B_0 &= \max_{i=1,\dots,d} \frac{B_i - M_i X'}{m_i}, \\
C_0 &= \max_{i=1,\dots,d} \frac{C_i - M_i X'}{m_i}, \\
L_0 &= \max_{i=1,\dots,d} \frac{L_i - M_i X'}{m_i}, \\
p_B &= \Phi(B_0), \\
p_C &= \Phi(C_0), \\
p_L &= \Phi(L_0).
\end{aligned} \tag{5.3}
$$

5.4 Generate $U \sim \mathrm{Unif}(0,1)$ and choose $X_d = \Phi^{-1}(p_B U)$. Concatenate $(d-1)$–dimensional Gaussian vector X' with variable X_d, i.e. $X = (X', X_d)$ and define vector $Z \triangleq (Z_1, \dots, Z_d)^\top$ by $Z = AQX$.

5.5 Use Z in log-Euler discretization, i.e.

$$
S^{(i)}(t_{j+1}) = S^{(i)}(t_j) \exp\left(\left(\frac{\mu^{(i)}(t_j)}{S^{(i)}(t_j)} - 0.5\left(\frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\right)^2\right)\Delta_t + \frac{\sigma^{(i)}(t_j)}{S^{(i)}(t_j)}\sqrt{\Delta_t} Z_i\right). \tag{5.4}
$$

5.6 Update V_p:

$$
V_p = V_p + w\left[(1 - p_B)(N + Nc) + (p_B - p_C)Nc\right] e^{-rt_j}. \tag{5.5}
$$

5.7 Update the weight: $w = p_B w$.

(6) Update V_p:

$$
V_p = V_p + w\left[(1 - p_C)(N + Nc) + (p_C - p_L)N + p_L N q(S^{(1)}(t_n), \dots, S^{(d)}(t_n))\right] e^{-rt_n}. \tag{5.6}
$$

Repeating steps 2–6 sufficient number of times, say P, we form a sample $V_1, \dots, V_P$ and estimate $V(0)$ by the sample mean

$$
\overline{V}(0) = \frac{V_1 + \cdots + V_P}{P}. \tag{5.7}
$$

6. Numerical Results

We apply our algorithm in pricing and estimating the greeks of worst-of autocallable notes. Results demonstrate that the algorithm presented in 5, produces stable greeks. As we avoid the discontinuities that arise from crossing the barrier

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Table 1. Results of delta calculation with respect to $S^{(1)}(0)$. 'Direct Simulation' denotes standard Monte Carlo scheme, 'Efficient Algorithm' denotes Monte Carlo scheme with survival conditioning presented in Sec. 5.

# simulations	Efficient algorithm		Direct simulation	
	mean	s.e.	mean	s.e.
3000	$2.10 \cdot 10^{-1}$	$6.87 \cdot 10^{-3}$	$2.16 \cdot 10^{-1}$	$3.61 \cdot 10^{-2}$
5000	$2.09 \cdot 10^{-1}$	$8.38 \cdot 10^{-3}$	$2.04 \cdot 10^{-1}$	$5.64 \cdot 10^{-2}$
10000	$2.10 \cdot 10^{-1}$	$3.74 \cdot 10^{-3}$	$2.02 \cdot 10^{-1}$	$4.47 \cdot 10^{-2}$
20000	$2.09 \cdot 10^{-1}$	$4.04 \cdot 10^{-3}$	$2.13 \cdot 10^{-1}$	$2.73 \cdot 10^{-2}$
30000	$2.08 \cdot 10^{-1}$	$3.69 \cdot 10^{-3}$	$2.12 \cdot 10^{-1}$	$3.04 \cdot 10^{-2}$

level B, compared to brute force Monte Carlo method, our algorithm demonstrates significant variance reduction in the computation of risk sensitivities.

In the following we considered autocallable note on the basket of four assets, $d = 4$. We assume that asset prices follow Geometric Brownian Motion and are driven by 4-dimensional Brownian motion, $m = 4$.

To test our implementation, we used stylized contract with parameters presented in Table 1.

Parameters for autocallable note on basket of four assets.

Observation dates	$\{t_1, t_2, \ldots, t_{12}) = \{0.25, 0.5, 0.75, 1, \ldots, 2.75, 3\}$
Reduced amount payoff	$q(S^{(1)}(t_{12}), \ldots, S^{(4)}(t_{12})) = \min(S^{(1)}(t_{12}), \ldots, S^{(4)}(t_{12}))$
Notional amount	$N = 100$
Barriers	$B = 1$, $C = 0.8$, $L = 0.6$
Coupon rate	$c = 1\%$ (i.e. 12% per annum)

For illustrative purposes, we assumed that assets prices follow a Geometric Brownian Motion:

$$dS^{(i)}(t) = S^{(i)}(t)\mu^{(i)}dt + S^{(i)}\sigma^{(i)}(t)\,dW^{(i)}(t). \quad i = \overline{1, d}. \tag{6.1}$$

We assume that risk-free rate is flat at 0%, assets pay no dividends. Other parameters are as follows:

Market data for on basket of four assets.

Reference levels:	$\{S_a^{(1)}, S_a^{(2)}, S_a^{(3)}, S_a^{(4)}\} = \{105, 110, 105, 110\}$
Risk-free interest rate	$r = 0\%$
Volatities	$\sigma_1 = 30\%, \sigma_2 = 35\%, \sigma_3 = 35\%, \sigma_4 = 40\%$
Correlation matrix	$\begin{pmatrix} 1 & 0.7 & 0.5 & 0.3 \\ 0.7 & 1 & 0.6 & 0.4 \\ 0.5 & 0.6 & 1 & 0.5 \\ 0.3 & 0.4 & 0.5 & 1 \end{pmatrix}$

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6.1. *Sensitivities of worst-of autocallable notes*

In what follows, we compare mean and standard deviation of standard Monte Carlo method and the Algorithm in Sec. 5. Each mean and standard error are based on ten independent Monte Carlo runs.

6.1.1. *delta*

We estimate the delta with respect to first underlying's price $\partial V/\partial S^{(1)}(0)$ using the forward difference

$$\frac{\partial V}{\partial S^{(1)}(0)} \approx \frac{V(S^{(1)}(0)+\Delta_S) - V(S^{(1)}(0))}{\Delta_S}, \tag{6.2}$$

with $\Delta_S = 1$.

As seen from Table 1, standard deviation of proposed Monte Carlo scheme is almost ten times smaller, thus we see a substantial variance reduction compared to the standard Monte Carlo scheme.

6.1.2. *vega*

Now, we estimate the vega with respect to first underlying's volatility $\partial V/\partial \sigma_1$ using forward difference

$$\frac{\partial V}{\partial \sigma_1} \approx \frac{V(\sigma_1+\Delta_\sigma) - V(\sigma_1)}{\Delta_\sigma}, \tag{6.3}$$

with $\Delta_\sigma = 1\%$.

Similar to the results presented in Table 2, compared to the standard Monte-Carlo scheme, we see a substantial variance reduction in vega calculation.

6.1.3. *gamma*

Finally, we estimate the gamma with respect to first underlying's price $\partial^2 V/\partial (S^{(1)}(0))^2$ using forward difference

$$\frac{\partial V}{\partial S^{(1)}(0)} \approx \frac{V(S^{(1)}(0)+\Delta_S) - 2V(S^{(1)}(0)) + V(S^{(1)}(0)-\Delta_S)}{(\Delta_S)^2}, \tag{6.4}$$

with $\Delta_S = 1$.

Table 2. Results of vega calculation with respect to σ_1. 'Direct Simulation' denotes standard Monte-Carlo scheme, 'Efficient Algorithm' denotes Monte Carlo scheme with survival conditioning presented in sec. 5.

# simulations	Efficient algorithm		Direct simulation	
	mean	s.e.	mean	s.e.
3000	$-2.60 \cdot 10^1$	$6.06 \cdot 10^{-1}$	$-3.02 \cdot 10^1$	$8.02 \cdot 10^0$
5000	$-2.63 \cdot 10^1$	$9.36 \cdot 10^{-1}$	$-2.69 \cdot 10^1$	$6.09 \cdot 10^0$
10000	$-2.60 \cdot 10^1$	$4.65 \cdot 10^{-1}$	$-2.64 \cdot 10^1$	$3.38 \cdot 10^0$
20000	$-2.61 \cdot 10^1$	$4.35 \cdot 10^{-1}$	$-2.61 \cdot 10^1$	$4.46 \cdot 10^0$
30000	$-2.61 \cdot 10^1$	$3.72 \cdot 10^{-1}$	$-2.55 \cdot 10^1$	$3.04 \cdot 10^0$

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Table 3. Results of gamma calculation with respect to $S^{(1)}(0)$. 'Direct Simulation' denotes standard Monte Carlo scheme, 'Efficient Algorithm' denotes Monte Carlo scheme with survival conditioning presented in Sec 5.

# simulations	Efficient algorithm		Direct simulation	
	mean	s.e.	mean	s.e.
3000	$-1.32 \cdot 10^{-2}$	$1.20 \cdot 10^{-3}$	$-1.79 \cdot 10^{-2}$	$1.68 \cdot 10^{-2}$
5000	$-1.33 \cdot 10^{-2}$	$1.09 \cdot 10^{-3}$	$-7.99 \cdot 10^{-3}$	$1.84 \cdot 10^{-2}$
10000	$-1.39 \cdot 10^{-2}$	$4.39 \cdot 10^{-4}$	$-1.94 \cdot 10^{-2}$	$1.08 \cdot 10^{-2}$
20000	$-1.38 \cdot 10^{-2}$	$3.38 \cdot 10^{-4}$	$-1.29 \cdot 10^{-2}$	$7.20 \cdot 10^{-3}$
30000	$-1.37 \cdot 10^{-2}$	$4.93 \cdot 10^{-4}$	$-1.41 \cdot 10^{-2}$	$4.05 \cdot 10^{-3}$

As seen from Table 3, standard deviation of proposed Monte Carlo scheme is almost 10 times smaller, thus we see a substantial variance reduction compared to the standard Monte Carlo scheme.

7. Conclusions

We have presented the description of worst-of autocallable notes on a basket of assets and constructed a Monte Carlo scheme suitable for their pricing. It is well known that the digital feature in their payoff reduces stability of Monte Carlo estimates for risk sensitivities. Reformulating the pricing by switching to survival measure, we have been able to handle the product's digital feature. It should be noted that when autocallable's 'trigger function' is a sufficiently smooth function, there exist excellent generic methods to handle the discontinuity in payoff; see Fries & Joshi (2011). We used the ideas of smoothing out the discontinuities by the survival conditioning technique of Glasserman & Staum (1999) and sampled only paths that remain in the survival region using the special structure of worst-of autocallable notes.

As seen from the results, the computation of risk sensitivities using finite differences under this scheme shows substantial variance reduction compared to standard Monte Carlo method.

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References

L. B. G. Andersen & V. V. Piterbarg (2010) *Interest Rate Modeling*. Atlantic Financial Press.

C. P. Fries & M. S. Joshi (2011) Perturbation stable conditional analytic Monte-Carlo pricing scheme for auto-callable products, *International Journal of Theoretical and Applied Finance* **14** (2), 197–219.

J. Geweke (1991) Efficient simulation from the multivariate normal and Student-t-distributions subject to linear constraints and the evaluation of constraint probabilities, *Computing science and statistics: Proceedings of the 23rd Symposium on the Interface*, ed. E. M. Keramidas, 571–578. Fairfax Station, VA: Interface Foundation of North America, Inc.

P. Glasserman & J. Staum (1999) Conditioning on one-step survival for barrier option sumulation *Operations Research* **49** (6), 923–927.

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