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ORIGINAL RESEARCH

A numerical method for pricing discrete double barrier option by Chebyshev polynomials

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Abstract
In this article, a fast numerical method based on orthogonal Chebyshev polynomials for pricing discrete double barrier option is illustrated. At first, a recursive formula for computing price of discrete double barrier option is obtained. Then, these recursive formulas are estimated by Chebyshev polynomials and expressed in operational matrix form that reduce CPU time of algorithm. Finally, the effectiveness and validity of the presented method is demonstrated by comparison with the obtained numerical results with some other algorithms.

Keywords Double barrier option · Black–Scholes model · Option pricing · Chebyshev polynomial

Mathematics Subject Classification 65D15 · 35E15 · 46A32

Introduction

Option pricing is investigated as one of the most interesting problems in mathematical finance. Barrier options are one of the most applicable types of exotic derivatives in the financial market that has been studied by researchers in the past two decades. As a description, a knockout (knock-in) double barrier option is one that is deactivated (activated) when the price of underlying asset hits each of two predetermined barriers before the expire date. According to the way of how the underlying asset price is monitoring, there are two types of barrier option: continuous and discrete. Discrete barrier options are those that the price of underlying asset is monitored at the specific dates, for example daily, weekly or monthly, but continuous ones is monitored in all the time before expiry date. In option trading, there are two kinds of option, namely put and call. A put (call) option gives the holder the right to sell (buy) the underlying asset. Based on when we can exercise the option, there are two types of option contracts: European and American. European options could be exercised only in expire date. American ones could be exercised each any time before expiry date. Therefore, European options are worth less than American ones. First of all, Kamrad and Ritchken [1] in 1991 introduced the approximating model for financial contract by using the standard trinomial tree method. Kwok [2] used the binomial and trinomial trees for pricing path-dependent options. Dai and Lyuu [3] presented a new tree model. It was bino-trinomial tree which reduces errors, by both linearity and nonlinearity. The most kinds of derivatives have an accurate answer, besides quick and convergent answers. Ahne et al. in [4] showed that besides of difficulty of solving pricing of barrier options, monitoring of double barrier discreetly by lattice method is so difficult to estimate. However, they recommended mesh methods for pricing discrete barriers. In [5], Andricopoulos et al. proposed quadrature methods for discrete monitoring and remarked that they are more accurate and faster than lattice or finite-difference methods. Fusai in [6] solved the discrete barrier options with inverse z-transform method and for this purpose considered the Wiener–Hopf model. In [7], Fusai et al. used the quadrature

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method that combined with an interpolation procedure for evaluating barrier options. In [8], Milev and Tagliani proved that their method is more efficient and faster than others for knockout options with discrete double barrier monitoring. Golbabai et al. [9] approximated the discrete double barrier options using Euler scheme and disserted spaces. In [10, 11], Farnoosh et al. provided the algorithm for pricing discrete single and double barrier options that are viable even for the case of time-dependent parameters. In [12], a numerical method based on projection methods has been provided for pricing barrier options. Sobhani and Milev in [13] presented a numerical algorithm based on Alpert wavelets for pricing double and single barrier options. In [14] Yoon and Kim applied the Mellin transform method for pricing European vulnerable options and generalized the results to double Mellin transform. In [15] Gzyl et al. were used maximum entropy method for estimating option pricing by recovering the probability density function. Fusai et al. [16] priced options with discretely monitored used levy process.

This article is organized as follows: In "The pricing model" section, a recursive formula for evaluating discrete double barrier option is derived based on Black–Scholes partial differential equations and converting it to the heat equation. The Chebyshev polynomials and their properties are explained in "Chebyshev polynomials" section. Next, a fast numerical algorithm based on these polynomials for pricing of discrete double barrier options is introduced in "Pricing by Chebyshev polynomials" section. Finally, numerical results are obtained and compared with some other methods in "Numerical results" section.

The pricing model

In this article, we consider the parameters of pricing model as follows: r is risk-free rate, σ is volatility, E is exercise price, L is lower barrier and U is upper barrier. In addition, we assume that the underlying asset price process X_t be a geometric Brownian motion, i.e., it satisfies the following stochastic differential equations:

$$dX_t = rX_t + \sigma X_t dB_t,$$

where B_t is the standard Brownian motion. If we have a discrete barrier call option, the payment at the maturity time T is equal to $\max(X_T - E, 0)$, provided that the underlying asset does not touch the barriers at the predetermined monitoring date $\tau_m, m = 0, 1, 2, \dots, N$ and the price of a discrete barrier call option is defined as discounted expectation of payment at the predetermined maturity time. If we denote the price of a discrete barrier call option at time $t \in (\tau_{m-1}, \tau_m)$ as $V(X_t, t, m-1)$, according to the well-known Black–Scholes model, V must satisfy the following partial differential equations:

$$-\frac{\partial V}{\partial t} + r X\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 X^2\frac{\partial^2 V}{\partial X^2} - rV = 0, \tag{1}$$

with the following initial conditions:

$$\begin{aligned} V\big(X, t_0, 0\big) &= (X-E)\mathbf{1}_{(\max(\mathbf{E},\mathbf{L})\le \mathbf{X}\le \mathbf{U})}, \\ V\big(X, \tau_m, 0\big) &= V\big(X, \tau_m, m-1\big)\mathbf{1}_{(\mathbf{L}\le\mathbf{X}\le\mathbf{U})}; \\ m &= 1, 2, \dots, M-1\ , \end{aligned}$$

where $V\big(X, \tau_m, m-1\big) := \lim_{t\to\tau_m} V(X, t, m-1)$.

By change of variable $z = \ln(\frac{X}{L})$, we have:

$$\frac{\partial V}{\partial X} = \frac{\partial V}{\partial Z}\frac{\partial Z}{\partial X} = \frac{1}{x}\frac{\partial V}{\partial Z} \tag{2}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial V}{\partial Z}\right) = -\frac{1}{x^2}\frac{\partial V}{\partial Z} + \frac{1}{x^2}\left(\frac{\partial^2 V}{\partial Z^2}\right) \tag{3}$$

So Eq. 1 and its initial conditions are reduced to the following form:

$$\begin{aligned} &-W_t + \mu W_z + \frac{\sigma^2}{2}W_{zz} = rW, \\ &W\big(z, t_0, 0\big) = L\big(e^z - e^{E^*}\big)\mathbf{1}_{(\boldsymbol{\delta}\le\mathbf{z}\le\boldsymbol{\theta})}, \\ &W\big(z, \tau_m, m\big) \\ &\quad = W\big(z, \tau_m, m-1\big)\mathbf{1}_{(\mathbf{0}\le\mathbf{z}\le\boldsymbol{\theta})}\ ;, m = 1, 2, \dots, M-1. \end{aligned} \tag{4}$$

which $E^* = \ln\left(\frac{E}{L}\right); \mu = r - \frac{\sigma^2}{2}; \theta = \ln\left(\frac{U}{L}\right)$; and $\delta = \max\{E^*, 0\}$. Next by considering $W\big(z, \tau_m, m\big) = e^{\alpha z + \beta t}u(z, t, m)$ and defining

$$\alpha = -\frac{\mu}{\sigma^2};\ c^2 = \frac{\sigma^2}{2};\ \beta = \alpha\mu + \alpha^2\frac{\sigma^2}{2} - r,$$

we have:

$$W_t = \beta e^{\alpha z + \beta t}\,u(z, t, m) + e^{\alpha z + \beta t}u_t \tag{5}$$

$$W_z = \alpha e^{\alpha z + \beta t}\,u(z, t, m) + e^{\alpha z + \beta t}u_z \tag{6}$$

$$W_{zz} = \alpha^2 e^{\alpha z + \beta t}\,u(z, t, m) + 2\alpha e^{\alpha z + \beta t}u_z + e^{\alpha z + \beta t}u_{zz} \tag{7}$$

So u must satisfy the following partial differential equation:

$$\begin{aligned} &-u_t + c^2 u_{zz} = 0, \\ &u\big(z, t_0, 0\big) = Le^{-\alpha z}\big(e^z - e^{E^*}\big)\mathbf{1}_{(\boldsymbol{\delta}\le\mathbf{z}\le\boldsymbol{\theta})};\ m = 0, \\ &u\big(z, \tau_m, m\big) = u\big(z, \tau_m, m-1\big)\mathbf{1}_{(\mathbf{0}\le\mathbf{z}\le\boldsymbol{\theta})};\ m = 1, \dots, M-1. \end{aligned} \tag{8}$$

The partial differential equation (8) is the well-known heat equation that admits a closed-form solution as follows:

$$u(z,t,m) = \begin{cases} L\int_{\delta}^{\theta} \hat{k}(z-\xi,t)e^{-\alpha\xi}\left(e^{\xi}-e^{E^*}\right)\mathrm{d}\xi\,;\ m=0 \\ \int_0^{\theta} \hat{k}\left(z-\xi,t-\tau_m\right)u\left(\xi,\tau_m,m-1\right)\mathrm{d}\xi\,;\ m=1,2,\dots,M-1 \end{cases}$$

where the kernel $\hat{k}$ is:

$$\hat{k}(z,t) = \frac{1}{\sqrt{4\pi c^2 t}}e^{-\frac{z^2}{4c^2 t}}. \tag{9}$$

In the rest of paper, we have additionally assumed that the monitoring dates are equally spaced, i.e., $\tau_m = m\tau$ which $\tau = T/M$. Now, by denoting $\hat{g}_m(z) := u(z,\tau_m,m-1)$, we have:

$$\hat{g}_1(z) = \int_0^{\theta} \hat{k}(z-\xi,\tau)\hat{g}_0(\xi)\mathrm{d}\xi, \tag{10}$$

$$\hat{g}_m(z) = \int_0^{\theta} \hat{k}(z-\xi,\tau)\hat{g}_{m-1}(\xi)\mathrm{d}\xi; m=2,3,\dots,M\,; \tag{11}$$

where

$$\hat{g}_0(z) = Le^{-\alpha z}\left(e^{z}-e^{E^*}\right)\mathbf{1}_{(\boldsymbol{\delta}\le\mathbf{z}\le\boldsymbol{\theta})}.$$

Because we want to work on the interval $[-1,1]$ instead of the interval $[0,\theta]$, we denote $g_m(z) = \hat{g}(\frac{\theta}{2}(z+1))$, and

$$k(z,t) := \frac{\theta}{2}\hat{k}\left(\frac{\theta}{2}(z),t\right),$$

thus, relations (10) and (11) are converted to the below ones:

$$g_1(z) = \int_{-1}^{1} k(z-\xi,\tau)g_0(\xi)\mathrm{d}\xi, \tag{12}$$

$$g_m(z) = \int_{-1}^{1} k(z-\xi,\tau)g_{m-1}(\xi)\mathrm{d}\xi; m=2,3,\dots,M; \tag{13}$$

where

$$g_0(z) = Le^{-\alpha\frac{\theta}{2}(z+1)}\left(e^{\frac{\theta}{2}(z+1)}-e^{E^*}\right)\mathbf{1}_{\left(\frac{2\boldsymbol{\delta}-\boldsymbol{\theta}}{\boldsymbol{\theta}}\le\mathbf{z}\le\mathbf{1}\right)}.$$

Chebyshev polynomials

The Chebyshev polynomials of degree n are defined as:

$$\tilde{T}_n(x) = \cos(n\cos^{-1}(x)), \quad -1 \le x \le 1.$$

So we have

$$\tilde{T}_0(x) = 1,\ \tilde{T}_1(x) = x,\ \tilde{T}_2(x) = 2x^2-1,\dots \tag{14}$$

they also satisfy the following three-term recurrence formula:

$$\tilde{T}_{n+1}(x) = 2x\,\tilde{T}_n(x) - \tilde{T}_{n-1}(x)$$

Chebyshev polynomials are orthogonal polynomials with respect to the weight function $w(x) = 1/\sqrt{(1-x^2)}$ on $[-1,1]$, i.e.,

$$\int_{-1}^{1} w(x)\tilde{T}_n(x)\tilde{T}_m(x)\mathrm{d}x = l_i\delta_{i,j}$$

where $l_0 = \pi$ and $l_i = \frac{\pi}{2}$. Let $L^2_w[-1,1]$ be the Hilbert space of all square-weighted integrable functions with the following inner product and norm:

$$\begin{aligned} <f,g>_w &= \int_{-1}^{1} w(x)f(x)g(x)\mathrm{d}x \\ \|g\|_w &= <g,g>_w^{\frac{1}{2}}\,. \end{aligned} \tag{15}$$

Now we define normalized Chebyshev polynomials as $T_i(x) = \sqrt{l_i^{-1}}\tilde{T}_i(x)$. These polynomials set an orthonormal basis function for this space. So we could write any $f \in L^2_w[-1,1]$ as follows:

$$f(x) = \sum_{i=0}^{\infty} a_i T_i(x)$$

where

$$a_i = \int_{-1}^{1} w(x)f(x)T_i(x)\mathrm{d}x.$$

Pricing by Chebyshev polynomials

By defining the linear integral operator $\mathcal{K} : L^2_w[-1,1] \to L^2_w[-1,1]$ as follows:

$$\mathcal{K}(g)(z) := \int_{-1}^{1} \kappa(z-\xi,\tau)g(\xi)\mathrm{d}\xi, \tag{16}$$

we could rewrite Eqs. (12) and (13) in the following operational form:

$$g_1 = \mathcal{K}g_0, \tag{17}$$

$$g_m = \mathcal{K}g_{m-1},\ \ m=2,3,\dots,M. \tag{18}$$

Now, consider $\Pi_n = span\left\{T_i(x)\right\}_{i=0}^{n}$ be the space of all polynomials with degrees up to n and $P_n : L^2_w[-1,1] \to \Pi_n$ be the orthogonal projection operator, which is defined as below:

$$\forall g \in L^2_w[-1,1] \quad P_n(g) = \sum_{i=0}^{n} \langle g, T_i(x)\rangle_w T_i(x), \tag{19}$$

where $\langle\, .\, ,\, .\, \rangle_w$ indicates the inner product that is defined in (15). We approximate operator $\mathcal{K}$ by $P_n\mathcal{K}$ and denote $\tilde{g}_{m,n}, m \geq 1$ as follows:

$$\tilde{g}_{1,n} = P_n\mathcal{K}(g_0), \tag{20}$$

$$\tilde{g}_{m,n} = P_n\mathcal{K}(\tilde{g}_{m-1,n}) = (P_n\mathcal{K})^m(g_0)\ ,\ m \geq 2. \tag{21}$$

We note that P_n is a pointwise convergent operator to identity operator I, i.e.,

$$\forall g \in L^2_w[-1,1] \quad \lim_{n\to\infty} \|P_n(g) - g\|_w = 0,$$

so because $\mathcal{K}$ is a compact operator, the operator $P_n\mathcal{K}$ is also a compact operator, and we have a norm convergency of $P_n\mathcal{K}$ to $\mathcal{K}$:

$$\lim_{n\to\infty} \|P_n\mathcal{K} - \mathcal{K}\|_w = 0. \tag{22}$$

Now let $\lim_{n\to\infty} \left\|(P_n\mathcal{K})^{m-1} - \mathcal{K}^{m-1}\right\|_w = 0$ (induction hypothesis). On the other hand:

$$\begin{aligned}&\left\|(P_n\mathcal{K})^m - \mathcal{K}^m\right\|_w \\ &\quad = \left\|(P_n\mathcal{K})^m - (P_n\mathcal{K})\mathcal{K}^{m-1} + (P_n\mathcal{K})\mathcal{K}^{m-1} - \mathcal{K}^m\right\|_w\end{aligned} \tag{23}$$

$$= \left\|(P_n\mathcal{K})((P_n\mathcal{K})^{m-1} - \mathcal{K}^{m-1}) + (P_n\mathcal{K} - \mathcal{K})\mathcal{K}^{m-1}\right\|_w \tag{24}$$

$$\begin{aligned}&\leq \left\|(P_n\mathcal{K})\right\|_w \left\|(P_n\mathcal{K})^{m-1} - \mathcal{K}^{m-1}\right\|_w \\ &\quad + \left\|P_n\mathcal{K} - \mathcal{K}\right\|_w \|\mathcal{K}\|_w^{m-1}.\end{aligned} \tag{25}$$

With the help of the above inequality and (22), we obtain

$$\lim_{n\to\infty} \left\|(P_n\mathcal{K})^m - \mathcal{K}^m\right\|_w = 0. \tag{26}$$

Therefore, $\tilde{g}_{m,n}$ is convergent to g_m as n tends to infinity:

$$\begin{aligned}\|\tilde{g}_{m,n} - g_m\|_w =& \left\|(P_n\mathcal{K})^m(g_0) - \mathcal{K}^m(g_0)\right\|_w \\ \leq& \left\|(P_n\mathcal{K})^m - \mathcal{K}^m\right\|_w \|g_0\|_w \to 0 \ as\ n \to \infty.\end{aligned} \tag{27}$$

By considering each $\tilde{g}_{m,n} \in \Pi_n$ as follows:

$$\tilde{g}_{m,n} = \sum_{j=0}^{n} a_{mi}T_i(z) = \Phi'_n F_m,$$

where $F_m = [a_{m0}, a_{m1}, \cdots, a_{mn}]'$ and $\Phi_n = [T_0, T_1, \cdots, T_n]'$. From Eq. (21), one can obtain:

$$\tilde{g}_{m,n} = (P_n\mathcal{K})^{m-1}(\tilde{g}_{1,n}), \tag{28}$$

but since Π_n is a finite-dimensional space, we can rewrite operator $P_n\mathcal{K}$ in the following matrix relation form on Π_n:

$$P_n\mathcal{K}(y) = \Phi'_n K\mathbf{a},$$

where $\mathbf{a} = [a_0, a_1, \cdots, a_n]'$,

$$\begin{aligned}y &= \sum_{j=0}^{n} a_j T_j(z), \\ K &= (k_{ij})_{(n+1)\times(n+1)}, \\ k_{ij} &= \int_{-1}^{1}\int_{-1}^{1} w(\eta)T_i(\eta)T_j(\xi)\kappa(\eta-\xi,\tau)\mathrm{d}\xi\mathrm{d}\eta\ .\end{aligned}$$

Therefore, we have also the following matrix relation:

$$\begin{aligned}\tilde{g}_{m,n} &= \Phi'_n K^{m-1} F_1, \\ a_{1i} &= \int_0^1 \int_{\frac{2\delta}{\theta}-1}^{1} w(\eta)T_i(\eta)\kappa(\eta-\xi,\tau)g_0(\xi)\mathrm{d}\xi\mathrm{d}\eta\ ,\ 0 \leq i \leq n.\end{aligned} \tag{29}$$

Thus, finally we could estimate the price of double barrier option with the following relation:

$$\mathcal{P}(X_0, \tau_m, M-1) \simeq e^{\alpha z_0 + \beta T}\tilde{g}_{M,n}\left(\frac{2z_0 - \theta}{\theta}\right), \tag{30}$$

where $z_0 = \log\left(\frac{X_0}{L}\right)$ and $\tilde{g}_{M,n}$ from (29).

Numerical results

In this section, the presented method is applied for pricing knockout call discrete double barrier option and the obtained numerical results are presented and compared with the other methods in the following examples. The numerical results are obtained from relation (30) with n Chebyshev basis functions.

Example 1 In Table 1, we present the numerical results of valuing knockout discrete double barrier call option with the upper barrier $U = 120$ and different lower barriers L. The other parameters of the problem are set as follows: the risk-free rate $r = 0.05$, volatility $\sigma = 0.25$, the maturity time $T = 0.5$, initial price $S_0 = 100$. These results are compared with some different ones in [17] for various numbers of monitoring date. The benchmark result is related to the quadrature method QUAD-K200. These results show the validity of our presented method; in addition, comparison between CPU time of algorithms shows the efficiency of our algorithm.

Table 1 Double barrier option pricing of Example 1: $T = 0.5$, $r = 0.05$, $\sigma = 0.25$, $S_0 = 100$, $E = 100$

M	L	Presented method ($n = 15$)	Milev (200)	Milev (400)	Trinomial	Quad-K20	Quad-K30	AMM-8	Benchmark
5	80	2.4499	–	–	2.4439	2.4499	2.4499	2.4499	2.4499
	90	2.2028	–	–	2.2717	2.2028	2.2028	2.2027	2.2028
	95	1.6831	1.6831	1.6831	1.6926	1.6831	1.6831	1.6830	1.6831
	99	1.0811	1.0811	1.0811	0.3153	1.0811	1.0811	1.0811	1.0811
	99.9	0.9432	0.9432	0.9432	–	0.9432	0.9432	0.9433	0.9432
25	80	1.9420	–	–	1.9490	1.9420	1.9420	1.9419	1.9420
	90	1.5354	–	–	1.5630	1.5354	1.5354	1.5353	1.5354
	95	0.8668	0.8668	0.8668	0.8823	0.8668	0.8668	0.8668	0.8668
	99	0.2931	0.2931	0.2931	0.3153	0.2931	0.2931	0.2932	0.2931
	99.9	0.2023	0.2023	0.2023	–	0.2023	0.2023	0.2024	0.2023
125	80	1.6808	–	–	1.7477	1.6803	16808	1.6807	1.6808
	90	1.2029	–	–	1.2370	1.2026	1.2029	1.2028	1.2029
	95	0.5532	0.5528	0.5531	0.5699	0.5531	0.5532	0.5531	0.5532
	99	0.1042	0.1042	0.1042	0.1201	0.1042	0.1042	0.1043	0.1042
	99.9	0.0513	0.0513	0.0513	–	0.0513	0.0513	0.0513	0.0513
250	80	1.6165	–	–	1.8631	1.8581	1.6164	1.6163	1.6165
	90	1.1237	–	–	1.2334	1.1234	1.1237	1.1236	1.1237
	95	0.4867	–	–	0.5148	0.4864	0.4867	0.4867	0.4867
	99	0.0758	–	–	0.0772	0.0758	0.0758	0.0759	0.0758
	99.9	0.0311	–	–	–	0.0311	0.0311	0.0311	0.0311

Example 2 In the second instance, the upper and lower barriers are fixed as 110 and 95, respectively. The price of option for different values of initial price S_0 is obtained and compared with numerical algorithms that are presented by Milev in [8] and Crank–Nicolson [18] (Table 2).

Table 2 Double barrier option pricing of Example 2: $T = 0.5$, $r = 0.05$, $\sigma = 0.25$,$E = 100$, $U = 110$ and $L = 95$

s_0	Presented method $n = 15$	Crank–Nicolson (1000)	Milev (200)	Milev (1000)
95	0.174498	0.1656	0.174503	0.174498
95.0001	0.174499	$\simeq$ 0.1656	0.174501	0.174499
95.5	0.182428	0.1732	0.182429	0.182428
99.5	0.229349	0.2181	0.229356	0.229349
100	0.232508	0.2212	0.232514	0.232508
100.5	0.234972	0.2236	0.234978	0.234972
109.5	0.174462	0.1658	0.174463	0.174462
109.9999	0.167394	$\simeq$ 0.1591	0.167399	0.167394
110	0.167393	0.1591	0.167398	0.167393
CPU	0.5 s			

Conclusion and remarks

In this article, projection operator based on Chebyshev polynomials is implemented for approximating the price of discrete double barrier options. By applying this operator, we achieved to reduce the solution to a matrix form (29) that makes our algorithm considerably fast. In particular, when the number of monitoring dates increases, the complexity of the presented method does not increase. Actually, the complexity of our algorithm is a function of number basis function that we use in expansion (19).

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