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A decomposition formula for option prices in the Heston model and applications to option pricing approximation

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Abstract By means of classical Itô calculus, we decompose option prices as the sum of the classical Black–Scholes formula, with volatility parameter equal to the root-mean-square future average volatility, plus a term due to correlation and a term due to the volatility of the volatility. This decomposition allows us to develop first- and second-order approximation formulas for option prices and implied volatilities in the Heston volatility framework, as well as to study their accuracy for short maturities. Numerical examples are given.

Keywords Stochastic volatility · Heston model · Itô calculus

Mathematics Subject Classification 91B28 · 91B70

JEL Classification G13

1 Introduction

Stochastic volatility models are a natural extension of the classical Black–Scholes model that have been introduced as a way to manage the *skew* and *smiles* observed in real market data (see, for example, Hull and White [14], Scott [16], Stein and Stein [15], Ball and Roma [5] and Heston [13]). The study of these models has introduced new important mathematical and practical challenges, in particular related with the option pricing problem and the calibration of the corresponding parameters. In fact, we do not have closed-form option pricing formulas for the majority of the stochastic volatility models, and even in the case when closed-form pricing solutions can be

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derived (see, for example, Heston [13] or Schöbel and Zhu [17]), they do not allow in general for fast calibration of the parameters.

A recent trend in the literature has been the development of approximate closed-form option pricing formulas. To this end, some authors have presented a perturbation analysis of the corresponding PDE with respect to a specific model parameter, like the volatility (see Hagan et al. [12]), the mean reversion (see Fouque et al. [10] and Fouque et al. [11]) or the correlation (see Antonelli and Scarlatti [4]). In all these techniques, the region of validity of the results is restricted to either short or long maturities.The obtained approximations for option prices allow for fast calibration and give a better understanding of the role of model parameters. More recently, another approach has been proposed by Benhamou et al. [6–8], where the authors focus directly on the law of the log stock price at maturity, given its initial condition. They expand prices with respect to the volatility of the volatility, computing the correction terms using Malliavin calculus. This approach allows the authors to deal with short and long maturities, as well as with time-dependent coefficients. Another point of view has been presented in Alòs [1], where by means of Malliavin calculus the author extends the classical Hull and White formula by decomposing option prices as the sum of the same derivative price if there were no correlation and a correction due to correlation. As an application, the author develops a method to construct first-order option pricing approximation formulas that only require some regularity conditions (in the Malliavin calculus sense) for the volatility process and that can be applied for a very general class of volatility models, including the case of long-memory volatilities.

Even though the conditions required in Alòs [1] are satisfied by the majority of stochastic volatility models, they are not trivial in the case of the Heston model. In Alòs and Ewald [2], the authors studied the Malliavin differentiability of the Heston volatility to adapt the results in Alòs [1] to the Heston case, but unfortunately the accuracy of the approximation could be proved only in the case when the dimension δ of the underlying Bessel process is greater than 6.

This paper is devoted to obtaining a new decomposition formula for option prices, similar to the one presented in Alòs [1], but valid even when the Malliavin regularity conditions needed in that work are not satisfied. Instead of expanding option prices around the Hull and White term by means of anticipating stochastic calculus (Malliavin calculus), we use the classical Itô formula to expand prices around the classical Black–Scholes formula with volatility parameter equal to the root-mean-square future average volatility. This will allow us to describe option prices as the sum of this last term plus a term due to the correlation and a term due to the volatility of the volatility. This method needs only some general integrability conditions that are satisfied by the Heston model, and then it allows us to extend the results in Alòs and Ewald [2] to the case $\delta > 2$ and to prove, in the case $\delta > 3$, a new second-order approximation formula. Even if the paper is focused on the Heston case, the results can be easily extended to other volatility models with good integrability conditions.

The paper is organized as follows. In Sect. 2, we introduce the main notations and hypotheses and prove our decomposition formula for option prices. In Sect. 3, we use the results from Sect. 2 to obtain first- and second-order option pricing approximation formulas. Some numerical examples are presented in Sect. 4. The main conclusions are summarized in Sect. 5.

2 A decomposition formula for option prices

We consider the Heston model for stock prices in a time interval $[0,T]$ under a risk neutral probability P^*,

$$dS_t = rS_t\,dt + \sigma_t S_t\big(\rho\,dW_t^* + \sqrt{1-\rho^2}\,dB_t^*\big),\quad t\in[0,T], \tag{2.1}$$

where

$$d\sigma_t^2 = \kappa\big(\theta-\sigma_t^2\big)\,dt + \nu\sqrt{\sigma_t^2}\,dW_t^*,$$

where r is the instantaneous interest rate (supposed to be constant), W^* and B^* are independent standard Brownian motions defined on a probability space $(\Omega,\mathcal{F},P)$, $\rho\in[-1,1]$ and κ,θ and ν are constants satisfying the condition $2\kappa\theta>\nu^2$. In the following, we denote by $\mathcal{F}^{W^*}, \mathcal{F}^{B^*}$ the filtrations generated by W^* and B^*, respectively. Moreover we define $\mathcal{F}:=\mathcal{F}^{W^*}\vee\mathcal{F}^{B^*}$. It will be convenient in the following sections to make the change of variable $X_t=\log(S_t)$, $t\in[0,T]$. It is well known that the price of a contingent claim of the form $h(X_T)$ at time t is given by

$$V_t = e^{-r(T-t)}E^*\big(h(X_T)\mid\mathcal{F}_t\big),$$

where E^* denotes the expectation with respect to P^*.

We use the following notation:

- $v_t^2=\frac{1}{T-t}\int_t^T E^*(\sigma_s^2\mid\mathcal{F}_t)\,ds$. That is, v_t^2 denotes the average squared future volatility.
- $M_t=\int_0^T E^*(\sigma_s^2\mid\mathcal{F}_t)\,ds$. Notice that $v_t^2=\frac{1}{T-t}(M_t-\int_0^t\sigma_s^2\,ds)$. Moreover, we recall that $dM_t=\nu\sqrt{\sigma_t^2}(\int_t^T e^{-\kappa(s-t)}\,ds)\,dW_t^*$.
- For any $\tau>0$, $p(x,\tau)$ will denote the centered Gaussian kernel with variance τ^2. If $\tau=1$ we write $p(x)$.
- $BS(t,x,\sigma)$ will denote the price of a European call option under the classical Black–Scholes model with constant volatility σ, current log stock price x, time to maturity $T-t$, strike price K and interest rate r. Remember that in this case

  $$BS(t,x,\sigma)=e^xN(d_+)-Ke^{-r(T-t)}N(d_-),$$

  where N denotes the cumulative probability function of the standard normal law and

  $$d_\pm:=\frac{x-x_t^*}{\sigma\sqrt{T-t}}\pm\frac{\sigma}{2}\sqrt{T-t},$$

  with $x_t^*:=\ln K-r(T-t)$.
- $\mathcal{L}_{BS}(\sigma)$ will denote the Black–Scholes differential operator (in the log variable) with volatility σ,

  $$\mathcal{L}_{BS}(\sigma)=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}+\left(r-\frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x}-r\,\cdot\,.$$

  It is well known that $\mathcal{L}_{BS}(\sigma)BS(\cdot,\cdot;\sigma)=0$.
- $G(t,x,\sigma):=(\partial_{xx}^2-\partial_x)BS(t,x,\sigma)$.

The next result is similar to Lemma 4.1 in Alòs et al. [3]:

Lemma 2.1 *Let* $0 \le t \le s \le T$. *Then for every* $n \ge 0$, *there exists* $C = C(n, \rho)$ *such that*

$$\left|\partial_x^n G(s, X_s, v_s)\right| \le C\left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s\right) d\theta\right)^{-\frac{1}{2}(n+1)}.$$

Proof A simple calculation gives us that

$$G(s, X_s, v_s) = e^{X_s} p\left(X_s - \eta, v_s\sqrt{T-s}\right) = Ke^{-r(T-s)} p\left(X_s - \mu, v_s\sqrt{T-s}\right),$$

where $\eta = \ln K - (r + v_s^2/2)(T-s)$ and $\mu = \ln K - (r - v_s^2/2)(T-s)$. This allows us to write

$$\partial_x^n G(s, X_s, v_s) = (-1)^n Ke^{-r(T-s)} \partial_\mu^n p\left(X_s - \mu, v_s\sqrt{T-s}\right).$$

A simple calculation and the fact that for all positive constants c, d, the function $x^c e^{-dx}$ is bounded give us that

$$\left|\partial_\mu^n p\left(X_s - \mu, v_s\sqrt{T-s}\right)\right| \le C\left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s\right) d\theta\right)^{-\frac{1}{2}(n+1)},$$

as we wanted to prove. □

Now we are in a position to prove the main result of this section.

Theorem 2.2 (Decomposition formula) *Assume the model* (2.1), *where the volatility process* $\sigma = \{\sigma_s, s \in [0, T]\}$ *satisfies the condition* $2\kappa\theta > \nu^2$. *Then for all* $t \in [0, T]$,

$$\begin{aligned} V_t = BS(t, X_t; v_t) &+ \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} H(s, X_s, v_s)\,\sigma_s\, d\langle M, W^*\rangle_s \,\Big|\, \mathcal{F}_t\right) \\ &+ \frac{1}{8} E^*\left(\int_t^T e^{-r(s-t)} K(s, X_s, v_s)\, d\langle M, M\rangle_s \,\Big|\, \mathcal{F}_t\right), \end{aligned} \tag{2.2}$$

where

$$H(s, X_s, v_s) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s)$$

and

$$K(s, X_s, v_s) := \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s).$$

Proof Notice that $BS(T, X_T; v_T) = V_T$. As $(e^{-rt} V_t)$ is a P^*-martingale, we can write

$$e^{-rt} V_t = E^*\left(e^{-rT} V_T \mid \mathcal{F}_t\right) = E^*\left(e^{-rT} BS(T, X_T; v_T) \mid \mathcal{F}_t\right).$$

Now our idea is to apply Itô's formula to the process $(e^{-rt}BS(t, X_t; v_t))$. As the derivatives of $BS(t, x; y)$ are not bounded, we use an approximating argument. Take $\varepsilon > 0$ and consider the process $(e^{-rt}BS(t, X_t; v_t^\varepsilon))$, where

$$v_t^\varepsilon := \sqrt{\frac{1}{T-t}\left(\varepsilon + \int_t^T E^*\left(\sigma_s^2 \mid \mathcal{F}_t\right) ds\right)} = \sqrt{\frac{1}{T-t}\left(\varepsilon + M_t - \int_0^t \sigma_s^2\, ds\right)}.$$

Applying the classical Itô formula and the relationship between the Gamma, the Vega and the Delta

$$\frac{\partial BS}{\partial \sigma}\left(s, X_s, v_s^\varepsilon\right) \frac{1}{v_s^\varepsilon\,(T-s)} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, v_s^\varepsilon\right),$$

we deduce that

$$\begin{aligned}
&e^{-rT} BS\left(T, X_T; v_T^\varepsilon\right) \\
&\quad = e^{-rt} BS\left(t, X_t; v_t^\varepsilon\right) \\
&\qquad + \int_t^T e^{-rs}\left(\mathcal{L}_{BS}\left(v_s^\varepsilon\right) + \frac{1}{2}\left(\sigma_s^2 - \left(v_s^\varepsilon\right)^2\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\right) BS\left(s, X_s, v_s^\varepsilon\right) ds \\
&\qquad + \int_t^T e^{-rs}\left(\frac{\partial BS}{\partial x}\right)\left(s, X_s, v_s^\varepsilon\right)\sigma_s\left(\rho\, dW_s^* + \sqrt{1-\rho^2}\, dB_s^*\right) \\
&\qquad + \frac{1}{2}\int_t^T e^{-rs}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, v_s^\varepsilon\right) dM_s \\
&\qquad + \frac{\rho}{2}\int_t^T e^{-rs}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s^\varepsilon\right)\sigma_s\, d\langle M, W^*\rangle_s \\
&\qquad + \frac{1}{8}\int_t^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s^\varepsilon\right) d\,\langle M, M\rangle_s \\
&\qquad - \frac{1}{2}\int_t^T e^{-rs}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, v_s^\varepsilon\right)\left(\sigma_s^2 - \left(v_s^\varepsilon\right)^2\right) ds,
\end{aligned}$$

that is,

$$\begin{aligned}
&e^{-rT} BS\left(T, X_T; v_T^\varepsilon\right) \\
&\quad = e^{-rt} BS\left(t, X_t; v_t^\varepsilon\right) \\
&\qquad + \int_t^T e^{-rs}\left(\frac{\partial BS}{\partial x}\right)\left(s, X_s, v_s^\varepsilon\right)\sigma_s\left(\rho\, dW_s^* + \sqrt{1-\rho^2}\, dB_s^*\right) \\
&\qquad + \frac{1}{2}\int_t^T e^{-rs}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) BS\left(s, X_s, v_s^\varepsilon\right) dM_s \\
&\qquad + \frac{\rho}{2}\int_t^T e^{-rs}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s^\varepsilon\right)\sigma_s\, d\left\langle M, W^*\right\rangle_s \\
&\qquad + \frac{1}{8}\int_t^T e^{-rs}\left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) BS\left(s, X_s, v_s^\varepsilon\right) d\langle M, M\rangle_s.
\end{aligned}$$

Taking conditional expectations and multiplying by e^{rt}, we obtain that

$$
\begin{aligned}
&e^{-r(T-t)} E^*\big(BS\left(T, X_T; v_T^{\varepsilon}\right) \mid \mathcal{F}_t\big) \\
&\quad = BS(t, X_t; v_t^{\varepsilon}) \\
&\qquad + \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} H\left(s, X_s, v_s^{\varepsilon}\right) \sigma_s \, d\langle M, W^* \rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\qquad + \frac{1}{8} E^*\left(\int_t^T e^{-r(s-t)} K\left(s, X_s, v_s^{\varepsilon}\right) d\langle M, M \rangle_s \,\Big|\, \mathcal{F}_t \right).
\end{aligned}
$$

Letting now $\varepsilon \searrow 0$, using the facts that $d\langle M, W^*\rangle_s = \nu\sigma_s \int_s^T e^{-\kappa(r-s)}\,dr\,ds$, $d\langle M, M\rangle_s = \nu^2\sigma_s^2(\int_s^T e^{-\kappa(r-s)}\,dr)^2\,ds$, Lemma 2.1 and the dominated convergence theorem, the result follows. □

Remark 2.3 The proof above uses only some integrability and regularity conditions of the volatility process and can therefore be extended to other volatility models, even non-Markovian or non-continuous volatilities.

Remark 2.4 Formula (2.2) gives us a tool to describe the impact on option prices of the correlation and the volatility of the volatility. Notice that the second term on the right-hand side of (2.2) becomes zero in the uncorrelated case $\rho = 0$.

3 Approximate option pricing formulas

This section presents a first- and a second-order approximation for option prices in the Heston volatility framework and their accuracy for short maturities.

The following result is similar to Lemma A.1 in Bossy and Diop [9].

Lemma 3.1 *Let* $\delta := \frac{4\kappa\theta}{\nu^2} \geq 4$ *and* $n \in [2, \delta - 2]$. *Then, for all* $(s,t) \in [0,T]$ *with* $s > t$,

$$
E\left(\frac{1}{\sigma_s^n} \,\Big|\, \mathcal{F}_t \right) \leq C_n(T, \sigma_t),
$$

where $C_n(T, \sigma_t)$ *is a positive constant which is nondecreasing as a function of* T.

Proof For the sake of simplicity, we take $t = 0$. From the proof of Lemma A.1 in Bossy and Diop [9], we can write

$$
\begin{aligned}
E\left(\frac{1}{\sigma_s^n}\right) &\leq \frac{1}{2^{n/2-1}\Gamma(n/2)\,L(s)^{n/2}} \int_0^1 u^{\frac{n}{2}-1}(1-u)^{2\kappa\theta/\nu^2-n/2-1} \\
&\qquad \times \exp\left(-\frac{\sigma_0 e^{-\kappa s} u}{2L(s)}\right) du \\
&= \frac{1}{2^{n/2-1}\Gamma(n/2)\,L(s)} \int_0^1 u^{\frac{n}{2}-1}(1-u)^{2\kappa\theta/\nu^2-n/2-1} \left(\frac{e^{\kappa s}}{\sigma_0 u}\right)^{n/2-1} \\
&\qquad \times \left(\frac{\sigma_0 e^{-\kappa s} u}{L(s)}\right)^{n/2-1} \exp\left(-\frac{\sigma_0 e^{-\kappa s} u}{4L(s)}\right) \exp\left(-\frac{\sigma_0 e^{-\kappa s} u}{4L(s)}\right) du,
\end{aligned}
$$

where $L(s) := \frac{\nu^2}{4\kappa}(1-e^{-\kappa s})$. Now, using the fact that $y^{n/2-1}e^{-y/4} \leq C(n)$ for some positive constant $C(n)$ and any $y>0$, it follows that

$$\begin{aligned}
E\left(\frac{1}{\sigma_s^n}\right) &\leq \frac{4^{n/2-1}C(n)}{2^{n/2-1}\Gamma(n/2)L(s)}\left(\frac{e^{\kappa s}}{\sigma_0}\right)^{n/2-1} \\
&\quad \times \int_0^1 (1-u)^{2\kappa\theta/\nu^2-n/2-1}\exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{4L(s)}\right)du \\
&\leq \frac{C(n)}{L(s)}\left(\frac{e^{\kappa s}}{\sigma_0}\right)^{n/2-1}\int_0^1 \exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{4L(s)}\right)du \\
&\leq C(n)\left(\frac{e^{\kappa s}}{\sigma_0}\right)^{n/2},
\end{aligned}$$

and this allows us to complete the proof. □

We need a similar result in the case $\delta < 4$.

Lemma 3.2 *Assume the condition* $2\kappa\theta > \nu^2$. *Assume* $\delta := \frac{4\kappa\theta}{\nu^2} < 4$. *Then, for all* $(s,t) \in [0,T]$ *with* $s > t$ *and for all* $p < \frac{2}{4-\delta}$,

$$E\left(\frac{1}{\sigma_s^2}\,\middle|\,\mathcal{F}_t\right) \leq \frac{C(T,\sigma_t)}{[(s-t)^2\nu^2[p(\delta/2-2)+1]]^{\frac{1}{p}}},$$

where $C(T,\sigma_t)$ *is a positive constant which is nondecreasing as a function of* T.

Proof For the sake of simplicity, we can take $t=0$. From the proof of Lemma A.1 in Bossy and Diop [9], we know that

$$E\left(\frac{1}{\sigma_s^2}\right) \leq \frac{C}{L(s)}\int_0^1 (1-u)^{2\kappa\theta/\nu^2-2}\exp\left(-\frac{\sigma_0 e^{-\kappa s}u}{2L(s)}\right)du,$$

where $L(s) := \frac{\nu^2}{4\kappa}(1-e^{-\kappa s})$. Now, by Hölder's inequality, we know that we have for all $\frac{1}{p}+\frac{1}{q}=1$ such that $\frac{1}{p} > 2-2\kappa\theta/\nu^2 = 2-\delta/2$,

$$\begin{aligned}
E\left(\frac{1}{\sigma_s^2}\right) &\leq \frac{C}{L(s)}\left(\int_0^1 (1-u)^{p(\delta/2-2)}du\right)^{\frac{1}{p}}\left(\int_0^1 \exp\left(-\frac{q\sigma_0 e^{-\kappa s}u}{L(s)}\right)du\right)^{\frac{1}{q}} \\
&\leq \frac{C}{L(s)[p(\delta/2-2)+1]^{\frac{1}{p}}}\left(\frac{q\sigma_0 e^{-\kappa s}}{L(s)}\right)^{-\frac{1}{q}} \\
&\leq \frac{C(T,\sigma_0)}{L(s)^{1-1/q}[p(\delta/2-2)+1]^{\frac{1}{p}}} \\
&\leq \frac{C(T,\sigma_0)}{\nu^{2(1-1/q)}(1-e^{-\kappa s})^{1-1/q}[p(\delta/2-2)+1]^{\frac{1}{p}}}.
\end{aligned}$$

Now, using that $1 - e^{-\kappa s} \geq ske^{-\kappa s}$, it follows that

$$E\left(\frac{1}{\sigma_s^2}\right) \leq \frac{C(T, \sigma_0)}{s^{1-1/q} \nu^{2(1-1/q)} [p(\delta/2 - 2) + 1]^{\frac{1}{p}}},$$

and now the proof is complete. □

Now we are in a position to prove our first approximation result, valid in the short-time-to-maturity regime.

Theorem 3.3 (First-order approximation formula) *Assume the model* (2.1), *where the volatility process* $\sigma = \{\sigma_s, s \in [0, T]\}$ *satisfies the condition* $2\kappa\theta > \nu^2$. *Then, if* $\delta \geq 4$, *we have for all* $t \in [0, T]$ *that*

$$\begin{aligned} &\left| V_t - BS(t, X_t; v_t) - \frac{\rho}{2} H(t, X_t, v_t) E^*\left(\int_t^T \sigma_s \, d\langle M, W^* \rangle_s \,\middle|\, \mathcal{F}_t \right) \right| \\ &\quad \leq C(T, \sigma_t) \nu^2 (T - t)^{\frac{3}{2}}. \end{aligned} \tag{3.1}$$

Moreover, if $\delta < 4$, *then*

$$\begin{aligned} &\left| V_t - BS(t, X_t; v_t) - \frac{\rho}{2} H(t, X_t, v_t) E^*\left(\int_t^T \sigma_s \, d\langle M, W^* \rangle_s \,\middle|\, \mathcal{F}_t \right) \right| \\ &\quad \leq C(T, \sigma_t) \nu^{2 - 2\sqrt{2-\delta/2}} \left(\frac{1}{1 - \sqrt{2 - \delta/2}} \right)^{1+\sqrt{2-\delta/2}} (T - t)^{\frac{1}{2}(3 - \sqrt{2-\delta/2})}, \end{aligned} \tag{3.2}$$

where $C(T, \sigma_t)$ *is a positive constant which is nondecreasing as a function of* T.

Proof Consider the process $(e^{-rt} H(t, X_t; v_t) U_t))$, where

$$U_t := \frac{\rho}{2} E^*\left(\int_t^T \sigma_s d\langle M, W^* \rangle_s \,\middle|\, \mathcal{F}_t \right).$$

Since $U_T = 0$, the same arguments as in the proof of Theorem 2.2 allow us to write

$$\begin{aligned} 0 &= H(t, X_t; v_t) U_t \\ &\quad - \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} H(s, X_s, v_s) \sigma_s \, d\langle M, W^* \rangle_s \,\middle|\, \mathcal{F}_t \right) \\ &\quad + \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, X_s, v_s) U_s \sigma_s \, d\langle M, W^* \rangle_s \,\middle|\, \mathcal{F}_t \right) \\ &\quad + \frac{1}{8} E^*\left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H(s, X_s, v_s) U_s \, d\langle M, M \rangle_s \,\middle|\, \mathcal{F}_t \right). \end{aligned}$$

This, together with (2.2), gives us that

$$
\begin{aligned}
V_t &= BS(t, X_t; v_t) + H(t, X_t; v_t) U_t \\
&\quad + \frac{\rho}{2} E^* \left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, X_s, v_s) U_s \sigma_s \, d\langle M, W^* \rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{1}{8} E^* \left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H(s, X_s, v_s) U_s \, d\langle M, M \rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{1}{8} E^* \left(\int_t^T e^{-r(s-t)} K(s, X_s, v_s) \, d\langle M, M \rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&= BS(t, X_t; v_t) + H(t, X_t; v_t) U_t + T_1 + T_2 + T_3. \qquad (3.3)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
|U_s| &\le \frac{\nu\rho}{2} E^* \left(\int_s^T \sigma_r^2 \int_r^T e^{-\kappa(u-r)} \, du \, dr \,\Big|\, \mathcal{F}_s \right) \\
&= \frac{\nu\rho}{2} \int_s^T E^* \left(\sigma_r^2 \mid \mathcal{F}_s \right) \int_r^T e^{-\kappa(u-r)} \, du \, dr.
\end{aligned}
$$

Then Lemma 2.1 gives us that

$$
\begin{aligned}
T_1 &\le C \frac{\nu^2\rho^2}{4} E^* \left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s \right) d\theta \right)^{-\frac{5}{2}} \right. \\
&\quad \times \left. \left(\int_s^T E^*\left(\sigma_r^2 \mid \mathcal{F}_s \right) \int_r^T e^{-\kappa(u-r)} \, du \, dr \right) \sigma_s^2 \int_s^T e^{-\kappa(u-s)} \, du \, ds \,\Big|\, \mathcal{F}_t \right) \\
&\le C \frac{\nu^2\rho^2}{4} E^* \left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s \right) d\theta \right)^{-\frac{3}{2}} \right. \\
&\quad \times \left. \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} \, du \right)^2 ds \,\Big|\, \mathcal{F}_t \right).
\end{aligned}
$$

Taking into account that $\int_s^T E(\sigma_\theta^2 \mid \mathcal{F}_s) \, d\theta \ge \sigma_s^2 \int_s^T e^{-\kappa(r-s)} \, dr$, it follows that

$$
\begin{aligned}
T_1 &\le C \frac{\nu^2\rho^2}{4} E^* \left(\int_t^T e^{-r(s-t)} \left(\sigma_s^2 \int_s^T e^{-\kappa(r-s)} \, dr \right)^{-\frac{3}{2}} \right. \\
&\quad \times \left. \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)} \, du \right)^2 ds \,\Big|\, \mathcal{F}_t \right) \\
&\le C \frac{\nu^2\rho^2}{4} \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-1} \mid \mathcal{F}_t \right) \left(\int_s^T e^{-\kappa(u-s)} \, du \right)^{\frac{1}{2}} ds \\
&\le C \frac{\nu^2\rho^2}{4} \int_t^T e^{-r(s-t)} \sqrt{E^*\left(\sigma_s^{-2} \mid \mathcal{F}_t \right)} \left(\int_s^T e^{-\kappa(u-s)} \, du \right)^{\frac{1}{2}} ds.
\end{aligned}
$$

Now Lemma 3.1 gives us that if $\delta \ge 4$,

$$
\begin{aligned}
T_1 &\le C(T, \sigma_t) \nu^2 \rho^2 \int_t^T e^{-r(s-t)} \left(\int_s^T e^{-\kappa(u-s)} \, du \right)^{\frac{1}{2}} ds \\
&\le C(T, \sigma_t) \nu^2 \rho^2 (T-t)^{\frac{3}{2}}.
\end{aligned}
$$

And if $\delta < 4$, Lemma 3.2 yields

$$
\begin{aligned}
T_1 &\leq \frac{C(T,\sigma_t)\nu^2\rho^2}{\nu^{1/p}[p(\delta/2-2)+1]^{\frac{1}{2p}}} \int_t^T \frac{e^{-r(s-t)}}{(s-t)^{1/2p}} \left(\int_s^T e^{-\kappa(u-s)}\,du \right)^{\frac{1}{2}} ds \\
&\leq \frac{C(T,\sigma_t)\nu^{2-1/p}\rho^2}{[p(\delta/2-2)+1]^{\frac{1}{2p}}} (T-t)^{\frac{3}{2}-1/2p}.
\end{aligned}
$$

Then, taking $p = \frac{1}{\sqrt{2-\delta/2}}$, we obtain

$$
T_1 \leq \frac{C(T,\sigma_t)\rho^2\nu^{2-\sqrt{2-\delta/2}}}{[\sqrt{2-\delta/2}+1]^{\frac{\sqrt{2-\delta/2}}{2}}} (T-t)^{\frac{1}{2}(3-\sqrt{2-\delta/2})}.
$$

On the other hand, the same arguments give us that

$$
\begin{aligned}
T_2 \leq{}& C\frac{\nu^2\rho^2}{4} E^*\left(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s\right) d\theta \right)^{-3} \right. \\
&\times \left. \left(\int_s^T E^*\left(\sigma_r^2 \mid \mathcal{F}_s\right) \int_r^T e^{-\kappa(u-r)}\,du\,dr \right) \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)}\,du \right)^2 ds \,\middle|\, \mathcal{F}_t \right) \\
\leq{}& C\frac{\nu^2\rho^2}{4} E^*\left(\int_t^T e^{-r(s-t)} \left(\sigma_s^2 \int_s^T e^{-\kappa(r-s)}\,dr \right)^{-2} \right. \\
&\times \left. \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)}\,du \right)^3 ds \,\middle|\, \mathcal{F}_t \right) \\
\leq{}& C\frac{\nu^2\rho^2}{4} \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-2} \mid \mathcal{F}_t\right) \int_s^T e^{-\kappa(r-s)}\,dr\,ds.
\end{aligned}
$$

Then, if $\delta \geq 4$,

$$
T_2 \leq C(T,\sigma_t)\nu^2\rho^2(T-t)^2 \leq C(T,\sigma_t)\nu^2\rho^2(T-t)^{\frac{3}{2}}.
$$

If $\delta < 4$,

$$
\begin{aligned}
T_2 &\leq C(T,\sigma_t)\nu^2\rho^2 \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-2} \mid \mathcal{F}_t\right) \int_s^T e^{-\kappa(r-s)}\,dr\,ds \\
&\leq \frac{C(T,\sigma_t)\nu^2\rho^2}{\nu^{2/p}[p(\delta/2-2)+1]^{\frac{1}{p}}} \int_t^T \frac{e^{-r(s-t)}}{(s-t)^{1/p}} \int_s^T e^{-\kappa(u-s)}\,du\,ds \\
&\leq \frac{C(T,\sigma_t)p\nu^2\rho^2}{(p-1)\nu^{2/p}[p(\delta/2-2)+1]^{\frac{1}{p}}} (T-t)^{2-1/p} \\
&= C(T,\sigma_t)\nu^2\rho^2(T-t)^2 \frac{p}{p-1} \left(\frac{1}{\nu^2(T-t)[p(\delta/2-2)+1]} \right)^{\frac{1}{p}},
\end{aligned}
$$

and then, taking $p = \frac{1}{\sqrt{2-\delta/2}}$, it follows that

$$
T_2 \leq C(T,\sigma_t)\rho^2\left[\nu(T-t)\right]^{2-2\sqrt{2-\delta/2}} \left(\frac{1}{1-\sqrt{2-\delta/2}} \right)^{1+\sqrt{2-\delta/2}}.
$$

Finally,

$$
\begin{aligned}
T_3 &\le C\frac{\nu^2}{8}E^*\bigg(\int_t^T e^{-r(s-t)}\bigg(\int_s^T E\big(\sigma_\theta^2 \mid \mathcal{F}_s\big)\,d\theta\bigg)^{-\frac{3}{2}} \\
&\quad \times \sigma_s^2\bigg(\int_s^T e^{-\kappa(u-s)}\,du\bigg)^2 ds\,\bigg|\,\mathcal{F}_t\bigg) \\
&\le C\nu^2 E^*\bigg(\int_t^T e^{-r(s-t)}\bigg(\sigma_s^2\int_s^T e^{-\kappa(r-s)}\,dr\bigg)^{-\frac{3}{2}} \\
&\quad \times \sigma_s^2\bigg(\int_s^T e^{-\kappa(u-s)}\,du\bigg)^2 ds\,\bigg|\,\mathcal{F}_t\bigg) \\
&\le C\nu^2\int_t^T E^*\big(\sigma_s^{-1} \mid \mathcal{F}_t\big)\bigg(\int_s^T e^{-\kappa(r-s)}\,dr\bigg)^{\frac{1}{2}} ds.
\end{aligned}
$$

Then, using the same arguments as for T_1, it follows that if $\delta \ge 4$,

$$
T_3 \le C(T,\sigma_t)\nu^2(T-t)^{\frac{3}{2}},
$$

and if $\delta < 4$,

$$
T_3 \le C(T,\sigma_t)\frac{\nu^{2-\sqrt{2-\delta/2}}}{[\sqrt{2-\delta/2}+1]^{\frac{\sqrt{2-\delta/2}}{2}}}(T-t)^{\frac{1}{2}(3-\sqrt{2-\delta/2})},
$$

and this allows us to complete the proof. □

Remark 3.4 Notice that when $\delta = 4$, formula (3.1) gives the same order of approximation as (3.2). On the other hand, the accuracy of the approximation given in (3.2) becomes worse as δ tends to 2.

Remark 3.5 Formulas (3.1) and (3.2) show us that for fixed δ, the accuracy of this first-order approximation increases when the volatility of the volatility or the time to maturity decreases.

The decomposition formula (2.2) suggests that we can obtain a second-order approximation formula by approximating its last term. To this end, we need the following lemma.

Lemma 3.6 *Assume $\delta := \frac{4\kappa\theta}{\nu^2} \in (3,5)$. Then, for all $(s,t) \in [0,T]$ with $s > t$ and for all $p < \frac{2}{5-\delta}$,*

$$
E\bigg(\frac{1}{\sigma_s^3}\,\bigg|\,\mathcal{F}_t\bigg) \le \frac{C(T,\sigma_t)}{\nu^{2(1-1/q)}s^{1-1/q}\left[\frac{p}{2}(\delta-5)+1\right]^{\frac{1}{p}}},
$$

where $C(T,\sigma_t)$ is a positive constant which is nondecreasing as a function of T.

Proof For the sake of simplicity, we can take $t = 0$. From the proof of Lemma A.1 of Bossy and Diop [9], we know that

$$E\left(\frac{1}{\sigma_s^3}\right) \leq \frac{C}{L(s)} \int_0^1 (1-u)^{2\kappa\theta/\nu^2 - \frac{5}{2}} \exp\left(-\frac{\sigma_0 e^{-\kappa s} u}{2L(s)}\right) du,$$

where $L(s) := \frac{\nu^2}{4\kappa}(1 - e^{-\kappa s})$. Now by Hölder's inequality, we know that for all $\frac{1}{p} + \frac{1}{q} = 1$ such that $p < \frac{2}{5-\delta}$,

$$\begin{aligned}
E\left(\frac{1}{\sigma_s^3}\right) &\leq \frac{C}{L(s)} \left(\int_0^1 (1-u)^{\frac{p}{2}(\delta-5)} du\right)^{\frac{1}{2}} \left(\int_0^1 \exp\left(-\frac{\sigma_0 e^{-\kappa s} u}{L(s)}\right) du\right)^{\frac{1}{2}} \\
&\leq \frac{C}{L(s)[\frac{p}{2}(\delta-5)+1]^{\frac{1}{p}}} \left(\frac{q\sigma_0 e^{-\kappa s}}{L(s)}\right)^{-\frac{1}{q}} \\
&\leq \frac{C(T,\sigma_0)}{L(s)^{1-1/q}[\frac{p}{2}(\delta-5)+1]^{\frac{1}{p}}} \\
&\leq \frac{C(T,\sigma_0)}{\nu^{2(1-1/q)}(1-e^{-\kappa s})^{1-1/q}[\frac{p}{2}(\delta-5)+1]^{\frac{1}{p}}}.
\end{aligned}$$

Now, using that $1 - e^{-\kappa s} \geq ske^{-\kappa s}$, it follows that

$$E\left(\frac{1}{\sigma_s^3}\right) \leq \frac{C(T,\sigma_0)}{\nu^{2(1-1/q)} s^{1-1/q}[\frac{p}{2}(\delta-5)+1]^{\frac{1}{p}}},$$

as we wanted to prove. □

Theorem 3.7 (Second-order approximation formula) *Assume the model* (2.1), *where the volatility process* $\sigma = \{\sigma_s, s \in [0,T]\}$ *satisfies the condition* $2\kappa\theta > \nu^2$. *Then, if* $\delta \geq 5$, *we have for all* $t \in [0,T]$ *that*

$$\begin{aligned}
&\Bigg| V_t - BS(t, X_t; v_t) - \frac{\rho}{2} H(t, X_t, v_t) E^*\left(\int_t^T \sigma_s \, d\langle M, W^*\rangle_s \,\middle|\, \mathcal{F}_t\right) \\
&\quad - \frac{1}{8} K(t, X_t, v_t) E^*\left(\int_t^T d\langle M, M\rangle_s \,\middle|\, \mathcal{F}_t\right)\Bigg| \\
&\quad \leq C(T,\sigma_t)\left(\nu^2\rho^2(T-t)^{\frac{3}{2}} + \nu^3\rho(T-t)^2 + \nu^4(T-t)^{5/2}\right).
\end{aligned}$$

Moreover, if $\delta \in [4,5)$,

$$\begin{aligned}
&\Bigg| V_t - BS(t, X_t; v_t) - \frac{\rho}{2} H(t, X_t, v_t) E^*\left(\int_t^T \sigma_s \, d\langle M, W^*\rangle_s \,\middle|\, \mathcal{F}_t\right) \\
&\quad - \frac{1}{8} K(t, X_t, v_t) E^*\left(\int_t^T d\langle M, M\rangle_s \,\middle|\, \mathcal{F}_t\right)\Bigg| \\
&\leq C(T,\sigma_t)\Bigg\{\nu^2\rho^2(T-t)^{\frac{3}{2}} + \nu^3\rho(T-t)^2 \\
&\quad + \nu^{4-2\sqrt{5/2-\delta/2}}(T-t)^{5/2-2\sqrt{5/2-\delta/2}}\left(\frac{1}{1-\sqrt{5/2-\delta/2}}\right)^{1+\sqrt{5/2-\delta/2}}\Bigg\}.
\end{aligned}$$

Finally, if $\delta \in [3,4)$,

$$\begin{aligned}
&\Bigg| V_t - BS(t, X_t; v_t) - \frac{\rho}{2} H(t, X_t, v_t)\, E^*\left(\int_t^T \sigma_s \, d\langle M, W^* \rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\qquad - \frac{1}{8} K(t, X_t, v_t)\, E^*\left(\int_t^T d\langle M, M \rangle_s \,\Big|\, \mathcal{F}_t \right) \Bigg| \\
&\quad \le C(T, \sigma_t) \Bigg\{ \nu^{2-2\sqrt{2-\delta/2}} \rho^2 \left(\frac{1}{1-\sqrt{2-\delta/2}} \right)^{1+\sqrt{2-\delta/2}} \\
&\qquad \times \left[\rho^2 (T-t)^{\frac{1}{2}(3-\sqrt{2-\delta/2})} + \nu\rho (T-t)^{2(1-\sqrt{2-\delta/2})} \right] \\
&\qquad + \nu^{4-2\sqrt{5/2-\delta/2}} (T-t)^{5/2-2\sqrt{5/2-\delta/2}} \left(\frac{1}{1-\sqrt{5/2-\delta/2}} \right)^{1+\sqrt{5/2-\delta/2}} \Bigg\},
\end{aligned}$$

where $C(T, \sigma_t)$ *is a positive constant which is nondecreasing as a function of* T.

Proof For the process $(e^{-rt} K(t, X_t; v_t) R_t)$ with $R_t := \frac{1}{8} E^*(\int_t^T d\langle M, M\rangle_s \mid \mathcal{F}_t)$, we have $R_t = 0$ and so the same arguments as in the proof of Theorem 3.3 give us that

$$\begin{aligned}
0 &= K(t, X_t; v_t)\, R_t \\
&\quad - \frac{1}{8} E^*\left(\int_t^T e^{-r(s-t)} K(s, X_s, v_s)\, d\langle M, M\rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) K(s, X_s, v_s)\, R_s \sigma_s \, d\langle M, W^*\rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{1}{8} E^*\left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) K(s, X_s, v_s)\, R_s \, d\langle M, M\rangle_s \,\Big|\, \mathcal{F}_t \right).
\end{aligned}$$

This, together with (2.2) and (3.3), allows us to write

$$\begin{aligned}
V_t &= BS(t, X_t; v_t) + H(t, X_t; v_t)\, U_t + K(t, X_t; v_t)\, R_t \\
&\quad + \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H(s, X_s, v_s)\, U_s \sigma_s \, d\langle M, W^*\rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{1}{8} E^*\left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H(s, X_s, v_s)\, U_s \, d\langle M, M\rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{\rho}{2} E^*\left(\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) K(s, X_s, v_s)\, R_s \sigma_s \, d\langle M, W^*\rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&\quad + \frac{1}{8} E^*\left(\int_t^T e^{-rs} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) K(s, X_s, v_s)\, R_s \, d\langle M, M\rangle_s \,\Big|\, \mathcal{F}_t \right) \\
&= BS(t, X_t; v_t) + H(t, X_t; v_t)\, U_t + K(t, X_t; v_t)\, R_t + T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

Then, by the proof of Theorem 3.3, we know that if $\delta \ge 4$,

$$T_1 + T_2 \le C(T, \sigma_t) \nu^2 \rho^2 (T-t)^{\frac{3}{2}},$$

and if $\delta < 4$,

$$T_1 + T_2 \le C(T, \sigma_t) \rho^2 \nu^{2-\sqrt{2-\delta/2}} (T-t)^{\frac{1}{2}(3-\sqrt{2-\delta/2})}.$$

On the other hand,

$$|R_s| \le \frac{\nu^2}{2} E^* \left(\int_s^T \sigma_r^2 \left(\int_r^T e^{-\kappa(u-r)}\, du \right)^2 dr \,\Big|\, \mathcal{F}_t \right)$$
$$= \frac{\nu^2}{2} \int_s^T E^*\left(\sigma_r^2 \mid \mathcal{F}_s\right) \left(\int_r^T e^{-\kappa(u-r)}\, du \right)^2 dr.$$

Then, using Lemma 2.1, we obtain as in the proof of Theorem 2.2

$$\begin{aligned}
T_3 &\le \frac{\nu^3 \rho}{2} E^* \bigg(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s\right) d\theta \right)^{-3} \\
&\quad \times \left(\int_s^T E^*\left(\sigma_r^2 \mid \mathcal{F}_s\right) \left(\int_r^T e^{-\kappa(u-r)}\, du \right)^2 dr \right) \\
&\quad \times \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)}\, du \right) ds \,\Big|\, \mathcal{F}_t \bigg) \\
&\le C(T, \sigma_t) \nu^3 \rho \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-2} \mid \mathcal{F}_t\right) \int_s^T e^{-\kappa(r-s)}\, dr\, ds.
\end{aligned}$$

Now, by Lemma 3.1, if $\delta \ge 4$,

$$T_3 \le C(T, \sigma_t) \nu^3 \rho (T-t)^2$$

and if $\delta < 4$, by Lemma 3.2,

$$T_3 \le C(T, \sigma_t) \rho \nu \left[\nu(T-t)\right]^{2-2\sqrt{2-\delta/2}} \left(\frac{1}{1-\sqrt{2-\delta/2}} \right)^{1+\sqrt{2-\delta/2}}.$$

Finally, by Lemma 2.1 and using the same arguments as before, we can write

$$\begin{aligned}
T_4 &\le \frac{\nu^4}{8} E^* \bigg(\int_t^T e^{-r(s-t)} \left(\int_s^T E\left(\sigma_\theta^2 \mid \mathcal{F}_s\right) d\theta \right)^{-\frac{7}{2}} \\
&\quad \times \left(\int_s^T E^*\left(\sigma_r^2 \mid \mathcal{F}_s\right) \left(\int_r^T e^{-\kappa(u-r)}\, du \right)^2 dr \right) \\
&\quad \times \sigma_s^2 \left(\int_s^T e^{-\kappa(u-s)}\, du \right)^2 ds \,\Big|\, \mathcal{F}_t \bigg) \\
&\le \frac{\nu^4}{8} E^* \left(\int_t^T e^{-r(s-t)} \sigma_s^{-3} \left(\int_s^T e^{-\kappa(r-s)}\, dr \right)^{\frac{3}{2}} ds \,\Big|\, \mathcal{F}_t \right) \\
&\le \frac{\nu^4}{8} \int_t^T e^{-r(s-t)} E^*\left(\sigma_s^{-3} \mid \mathcal{F}_t\right) \left(\int_s^T e^{-\kappa(r-s)}\, dr \right)^{\frac{3}{2}} ds.
\end{aligned}$$

Then Lemma 3.1 gives us that, if $\delta \ge 5$,

$$T_4 \le C(T, \sigma_t) \nu^4 (T-t)^{5/2},$$

and applying Lemma 3.6 we obtain that, if $\delta \in (3, 5)$,

$$
\begin{aligned}
T_4 &\le C(T,\sigma_t)\nu^4 \int_t^T e^{-r(s-t)} E^*\big(\sigma_s^{-3} \mid \mathcal{F}_t\big)\left(\int_s^T e^{-\kappa(r-s)}\,dr\right)^{\frac{3}{2}} ds \\
&\le C(T,\sigma_t)\frac{\nu^4}{\nu^{2/p}[p(\delta/2-5/2)+1]^{\frac{1}{p}}} \int_t^T \frac{e^{-r(s-t)}}{(s-t)^{1/p}}\left(\int_s^T e^{-\kappa(u-s)}\,du\right)^{\frac{3}{2}} ds \\
&\le C(T,\sigma_t)\frac{p\nu^4}{(p-1)\nu^{2/p}[p(\delta/2-5/2)+1]^{\frac{1}{p}}}(T-t)^{5/2-1/p} \\
&= C(T,\sigma_t)\nu^4(T-t)^{5/2}\frac{p}{p-1}\left(\frac{1}{\nu^2(T-t)[p(\delta/2-5/2)+1]}\right)^{\frac{1}{p}}.
\end{aligned}
$$

Now, taking $p=\sqrt{\frac{2}{5-\delta}}$, it follows that

$$
\begin{aligned}
T_4 \le{}& C(T,\sigma_t)\nu^{4-2\sqrt{5/2-\delta/2}}(T-t)^{5/2-2\sqrt{5/2-\delta/2}} \\
&\times\left(\frac{1}{1-\sqrt{5/2-\delta/2}}\right)^{1+\sqrt{5/2-\delta/2}},
\end{aligned}
$$

and this allows us to complete the proof. □

Remark 3.8 As in Theorem 3.3, the accuracy of this second-order approximation increases when the volatility of the volatility or the time to maturity decreases. On the other hand, we can observe that when $\rho=\pm 1$, the accuracy of Theorem 3.7 is of the same order $O(\nu^2)$ as in Theorem 3.3, while when the correlation decreases, this accuracy becomes significantly better and is of the order $O(\nu^4)$ when $\rho=0$.

Remark 3.9 For a European call option, it is easy to check that

$$
H(t,x,\sigma):=\frac{e^x}{\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{d_+^2}{2}\right)\left(1-\frac{d_+}{\sigma\sqrt{T-t}}\right)
$$

and

$$
\begin{aligned}
K(t,x,\sigma) ={}& \frac{e^x}{\sigma\sqrt{2\pi(T-t)}} \\
&\times\exp\left(-\frac{d_+^2}{2}\right)\left[\left(-\frac{d_+}{\sigma\sqrt{T-t}}+\frac{d_+^2}{\sigma^2(T-t)}\right)-\frac{1}{\sigma^2(T-t)}\right].
\end{aligned}
$$

Moreover, in the case of the Heston volatility, we can easily see that

$$
\begin{aligned}
E^*\left(\int_t^T \sigma_s^2\,ds \,\middle|\, \mathcal{F}_t\right) &= \theta(T-t)+\frac{\sigma_t^2-\theta}{\kappa}\big(1-e^{-\kappa(T-t)}\big), \\
E^*\left(\int_t^T \sigma_s\,d\langle M,W^*\rangle_s \,\middle|\, \mathcal{F}_t\right) &= \frac{\nu\rho}{\kappa^2}\Big(\theta\kappa(T-t)-2\theta+\sigma_t^2+e^{-\kappa(T-t)}\big(2\theta-\sigma_t^2\big) \\
&\quad -\kappa(T-t)e^{-\kappa(T-t)}\times\big(\sigma_t^2-\theta\big)\Big),
\end{aligned}
$$

and

$$E^*\left(\int_t^T d\langle M,M\rangle_s \,\middle|\, \mathcal{F}_t\right)=\frac{\nu^2}{\kappa^2}\left\{\theta(T-t)+\frac{\sigma_t^2-\theta}{\kappa}\left(1-e^{-k(T-t)}\right)\right.$$
$$-\frac{2\theta}{\kappa}\left(1-e^{-k(T-t)}\right)-2\left(\sigma_t^2-\theta\right)(T-t)\,e^{-k(T-t)}$$
$$+\frac{\theta}{2\kappa}\left(1-e^{-2k(T-t)}\right)+\frac{\sigma_t^2-\theta}{\kappa}$$
$$\left.\times\left(e^{-k(T-t)}-e^{-2k(T-t)}\right)\right\}.$$

Then we can easily obtain explicit first- and second-order approximation formulas by substituting the above quantities in the expressions in Theorems 3.3 and 3.7.

Remark 3.10 (Approximations for the implied volatility) It is easy to deduce from the expressions in Theorems 3.3 and 3.7, by using Taylor expansions as in Fouque et al. [10], the first- and second-order approximations for the implied volatility

$$\hat{I}_1 := v_t+\frac{\rho}{2v_t(T-t)}\left(1-\frac{d_+}{v_t\sqrt{T-t}}\right)E^*\left(\int_t^T \sigma_s\, d\langle M,W^*\rangle_s \,\middle|\, \mathcal{F}_t\right),$$
$$\hat{I}_2 := v_t+\frac{\rho}{2v_t(T-t)}\left(1-\frac{d_+}{v_t\sqrt{T-t}}\right)E^*\left(\int_t^T \sigma_s\, d\langle M,W^*\rangle_s \,\middle|\, \mathcal{F}_t\right)$$
$$+\frac{1}{8v_t(T-t)}\left[\left(-\frac{d_+}{v_t\sqrt{(T-t)}}+\frac{d_+^2}{v_t^2(T-t)}\right)-\frac{1}{v_t^2(T-t)}\right]$$
$$\times E^*\left(\int_t^T d\langle M,M\rangle_s \,\middle|\, \mathcal{F}_t\right).$$

Notice that due to

$$d_+=\frac{x-x_t^*}{v_t\sqrt{T-t}}+\frac{v_t}{2}\sqrt{T-t},$$

the first expression is linear in the initial log stock price x, and the second is quadratic in x. Hence we deduce that the first-order approximation formula will help us to describe the skew effect, while the second one will be necessary if we try to describe a smile. Notice that these approximation formulas do not have divergent terms because

$$E^*\left(\int_t^T \sigma_s\, d\langle M,W^*\rangle_s \,\middle|\, \mathcal{F}_t\right)=\nu\int_t^T E^*\left(\sigma_s^2 \mid \mathcal{F}_t\right)\int_s^T e^{-\kappa(r-s)}\,dr\,ds$$
$$\le C\,(T-t)^2$$

and

$$E^*\left(\int_t^T d\langle M,M\rangle_s \,\middle|\, \mathcal{F}_t\right)=\nu^2\int_t^T E^*\left(\sigma_s^2 \mid \mathcal{F}_t\right)\left(\int_s^T e^{-\kappa(r-s)}\,dr\right)^2 ds$$
$$\le C\,(T-t)^3\,.$$

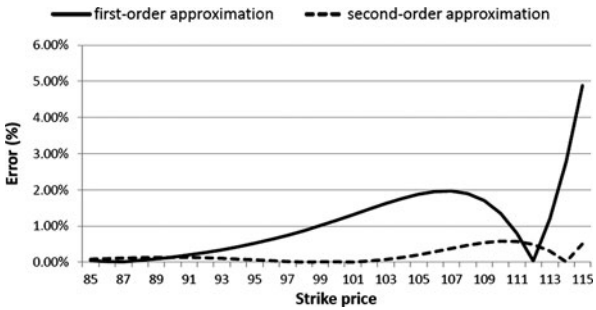

Fig. 1 Error of approximation as a function of the strike price when $T = 0.25$, $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = -0.5$

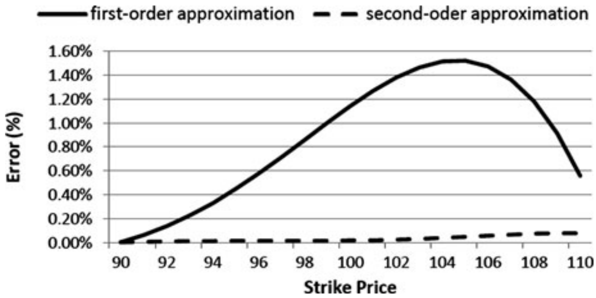

Fig. 2 Error of approximation as a function of the strike price when $T = 0.25$, $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = 0$

4 Numerical examples

This section is devoted to exemplifying the results in the previous section. For the sake of simplicity, we take $t = 0$.

Example 4.1 In Fig. 1, we can see the corresponding error of approximation (%) relative to the option price evaluated analytically, for the parameters $T = 0.25$, $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = -0.5$. We can observe the error in the second approximation is smaller than in the first one.

Example 4.2 In Fig. 2, we can see the percentage errors when changing the above parameters to $\rho = 0$. Then the last term in (2.2) becomes more significant and we can observe a bigger difference in the corresponding percentage errors, as expected from Remark 3.8.

Example 4.3 In Fig. 3, we can see the percentage errors corresponding to the above parameters but now taking $\rho = -1$. As expected again from Remark 3.8, the difference in the corresponding percentage errors is clearly larger than in the uncorrelated case $\rho = 0$.

Example 4.4 In this example, we have taken $T = 0.25$, $X_0 = \ln 100$, $K = 90$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, $\rho = -0.5$, and $\kappa \in (0.8, 5)$, in such a way that $\delta = 4\kappa\theta/\nu^2 \in (2.13, 13.33)$. Figure 4 shows the percentage errors as a function of δ. As expected from Theorems 3.3 and 3.7, the accuracy of the approximation depends strongly on δ. Moreover, the error in the second approximation formula becomes

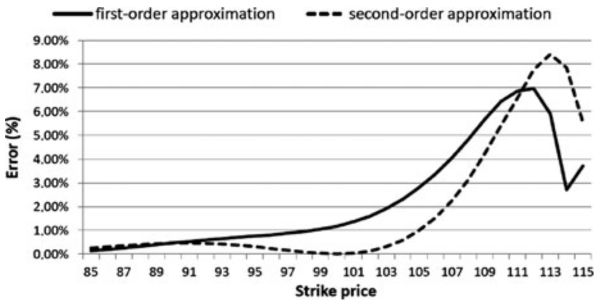

Fig. 3 Error of approximation as a function of the strike price when $T = 0.25$, $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = -1$

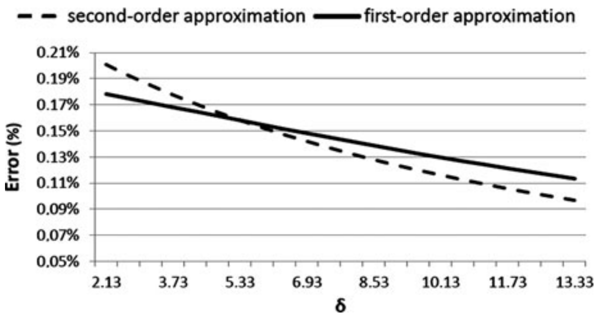

Fig. 4 Error of approximation as a function of δ when $T = 0.25$, $X_0 = \ln 100$, $K = 90$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, $\rho = -0.5$, and $\delta \in (2.13, 12.8)$

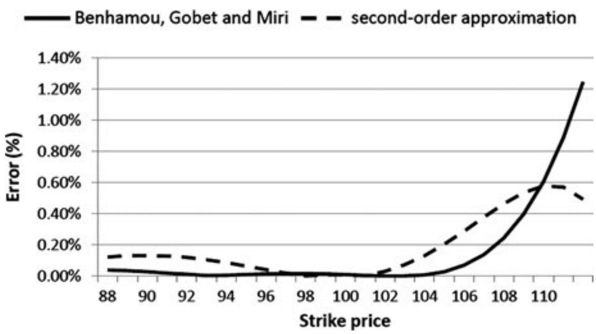

Fig. 5 Error of approximation as a function of the strike price when $T = 0.25$, $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = -0.5$

smaller than in the first one only for δ big enough. The study of the validity of our formulas for small δ is an interesting problem that is left for future research.

Example 4.5 In Benhamou et al. [8], the authors have proposed an approximate option pricing formula deduced from the expansion of option prices with respect to the volatility of the volatility. This approximation formula, where the correction terms are calculated by Malliavin calculus, consists of the sum of $BS(t, X_t; v_t)$ plus four correcting terms. It is easy to see that this approximation coincides with our second-order approximation when $\rho = 0$. In the negatively correlated case, the approximation in Benhamou et al. [8] seems to be more accurate for near-the-money options, as we can see in the following graphs (Figs. 5 and 6), even when both approximations become very precise.

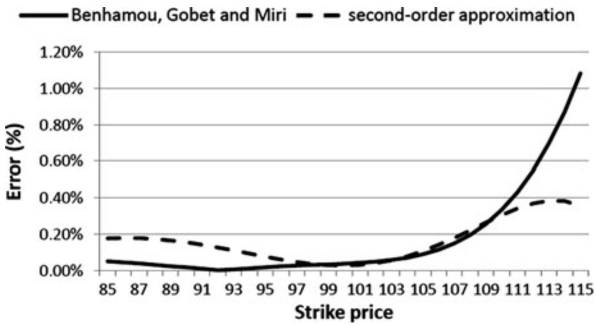

Fig. 6 Error of approximation as a function of the strike price when $T = 0.5$, $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = -0.5$

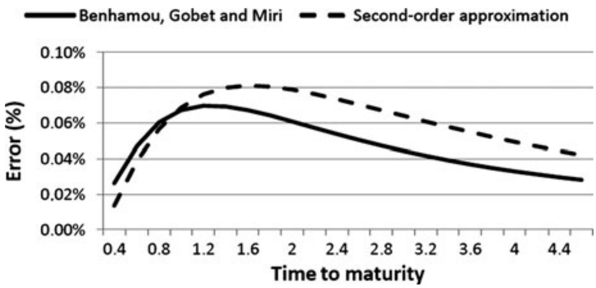

Fig. 7 Error of approximation as a function of time to maturity when $X_0 = \ln 100$, $\kappa = 3$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, and $\rho = -0.5$

Example 4.6 Figure 7 shows the percentage errors for the second-order approximation formula as a function of time to maturity, with parameters $X_0 = \ln 100$, $K = 100$, $\theta = 0.06$, $\sigma_0 = 0.2$, $\nu = 0.3$, $\rho = -0.5$, and $\kappa = 3$. As forecasted, the errors decrease when time to maturity tends to zero. We can also see that, as in the approximations presented in Benhamou et al. [8], the error seems to flatten out for long maturities.

5 Conclusions

By means of classical Itô calculus, we have decomposed option prices in the Heston volatility framework as the sum of the classical Black–Scholes formula, with volatility parameter equal to the root-mean-square future average volatility, plus a term due to correlation and a term due to the volatility of the volatility. This decomposition formula allows us to construct first- and second-order option pricing approximation formulas that are extremely easy to compute, as well as to study their accuracy for short maturities. Moreover, we have seen that the corresponding approximations for the implied volatility are linear (first-order approximation) and quadratic (second-order approximation) in the log stock price variable x. The presented methods need only some general integrability conditions and extend some recent results in Alòs and Ewald [2].

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