# Making the best of best-of

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**Abstract**    This paper extends the analytical valuation of options on the maximum or the minimum of several risky assets in several directions. The first extension consists in including more assets in the payoff and making the latter more flexible by adding knock-in and knock-out provisions. The second extension consists in pricing these contracts in a multivariate jump-diffusion framework allowing for a stochastic two-factor term structure of interest rates. In both cases, explicit formulae are provided which yield prices quasi instantaneously and with utmost precision. Hedge ratios can be easily and accurately derived from these formulae.

## 1 Introduction

Multiasset options, or rainbow options, have been traded for a long time in the markets. Among this class of contracts, options on the best or the worst of several risky assets are popular. In its standard form, a European-style call on the maximum (the "best") of $n$ assets provides the investor, at the option expiry, with the difference, if positive, between the highest of the $n$ asset prices and a fixed strike price. Similarly, a put on the minimum (the "worst") provides the difference, if positive, between the strike price and the lowest of the underlying asset prices. It is not hard to see why these products appeal to investors. Indeed, they provide them with powerful tools of diversification. As such, they allow both to reduce their risk exposure and to expand their

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investments' opportunities. If you are long a call option on the maximum of several assets that behave differently when market conditions change, it is quite unlikely that you will end up out of the money. You need not have views about the market's direction as long as there is volatility. Furthermore, compared with a basket option, there is no averaging effect to make a dent in your gains, as you are automatically endowed with the single best performance. Another intensively traded rainbow option is the spread option, which yields the difference between two underlying asset prices at expiry. An interesting combination of spread-like and best-of/worst-of-like features consists in defining a payoff yielding the difference, at expiry, between the best performer and the worst performer of a basket of several risky assets ('best-of spread option'). The payoff to the option holder may or may not turn out to be higher than that of a standard best-of call option written on the same assets depending on the specification of the fixed strike price in the latter contract. Unless all assets are positively correlated, it will usually be more rewarding, except if the standard best-of call option is very in-the-money, because then the worst of the several risky assets would have to dip beneath a low fixed strike price for the payoff of the best-of spread option to be greater than that of the best-of option. Finally, the most attractive payoff to the investor is probably that of the 'lookback best-of spread option', which yields the difference, at expiry, between the highest value attained by any of the several risky assets and the lowest point hit by any of them at any time during the option life. Needless to say, the cost of such a product can be quite substantial.

These contracts are also widely used for various hedging purposes. For example, companies which choose to settle their expenses in different foreign currencies can buy best-of calls, while those that choose to receive earnings in various foreign currencies can buy worst-of puts. This is less costly than buying separate calls and puts, owing to the correlation effects inherent in best/worst-of options. Also, portfolio managers can use these contracts if they know that at some particular time in the future they want to buy the better of two stocks for their portfolio but do not yet know exactly which of both will turn out to be the best.

These options are also embedded in a variety of financial securities issued by firms or in their capital budgeting decision problems (Detemple et al. 2003).

Unfortunately, formulae for this kind of options are very scarce. To the best of our knowledge, one and only one fully explicit formula has been published, the one given in Stulz's seminal paper (Stulz 1982). Another commonly cited reference is Jonhson (1987), but the formulae given in his paper are not completely explicit and their numerical implementation is not discussed. Other contributions have been devoted to numerical approximation algorithms (Boyle 1989; Boyle et al. 1989; Boyle and Tse 1990). Boyle (1989) approximates the value of a worst-of call on $n$ assets with zero strike by drawing on Clark's algorithm to approximate the first four moments of the maximum (or the minimum) of $n$ jointly normal random variables (Clark 1961). His assumptions are quite restrictive since all asset returns must have the same variance and the correlations between each pair of asset returns must be equal. Boyle and Tse (1990) relax these assumptions and design a numerical approximation algorithm for general asset return and correlation values, the accuracy of which is found to be satisfactory for best-of/worst-of options written on three assets. More recently, the issue of parameter estimation was discussed (Fengler and Schwender 2004) and

American-style contracts were analysed by Detemple et al. (2003), who provided lower and upper bounds for an American call option on the minimum of two assets.

This work focuses on the analytical valuation of European-style options on the maximum or the minimum of several risky assets. There are many reasons why one would want to improve on the current state of the published research in this area. First, Stulz's formula is restricted to two assets. This seriously limits the scope of diversification and does not allow to exploit all the potential of best-of contracts. A fundamental reason for this restriction is technical, that is the alleged laboriousness of mathematical computations involving the multivariate standard normal distribution and, more seriously, the lack of knowledge of analytical expressions for multivariate normal densities as soon as dimension goes beyond 3, let alone the issue of the numerical evaluation of the multivariate normal cumulative distribution. The problem of dimension alone is sufficiently involved to lead practitioners to resort to Monte Carlo simulation as soon as the number of assets is greater than two.

A second limitation in Stulz's formula is that it does not accommodate for flexibility provisions that are highly attractive to investors, such as knock-in or knock-out triggering barriers. Yet, these are all the more welcome as an obstacle to the commercial success of best-of calls and worst-of puts is the cost of these products, which can be substantially higher than that of vanilla calls and puts. One way to reduce this cost is for investors not to buy insurance against the scenarios that they consider too unlikely to happen or, conversely, to buy insurance conditional on the manifestation of an event that they regard as almost certain. Again, the main reason why these features are not dealt with in the existing formulae is the supposed absence of analytical tractability of such path-dependent features in a multiasset setting.

A third limitation in Stulz's formula is that, like most closed form option pricing formulae, it relies on the Black-Scholes assumptions (Black and Scholes 1973), which are notoriously at odds with some salient features of real market data, such as the fact that asset prices can "jump" or the fact that interest rates are stochastic. Yet, the latter properties can be incorporated into a pricing model leading to a closed form solution.

This paper is an attempt to adress the above deficiencies. More specifically, Sect. 2 provides an explicit formula for a knock-in/out option on the minimum/maximum of four risky assets, as well as for a regular option on the minimum/maximum of four risky assets. As a consequence of the complexity of the payoff, which involves both dimension and path dependency issues, the only way to preserve analytical tractability is to remain in a Black-Scholes framework. This also enables to keep the number of model parameters relatively under control. Option values are obtained through an extensive use of the change of numeraire approach (Geman et al. 1995), each risky asset successively completing the market. The formulae given in Sect. 2 are fully explicit and immediately computable using an elementary quadrature. This degree of analytical tractability is made possible by expressing the quadrivariate normal density function as a product of univariate normal conditional density functions. This approach stands in contrast with the usual definition of the multivariate normal density function in terms of the inverse of the determinant of the covariance matrix as stated, for example, in Tong (1990). Not only does it make more sense intuitively from a probabilistic perspective, but it results in the explicit statement of the joint density of four correlated standard normal random variables, contrary to the linear algebra

representation. Furthermore, this new approach allows to obtain several expressions of the quadrivariate normal cumulative distribution as a function of the trivariate, the bivariate or the univariate normal distributions, thus leading to highly accurate and efficient evaluations of the integral. Generalisation to higher dimension is analytically straightforward.

Section 3 deals with the valuation of an option on the minimum/maximum of two risky assets when both assets follow jump-diffusion processes, where the jump component consists of two correlated compound Poisson processes and the diffusion component consists of two correlated geometric Brownian motions with a time-dependent two-factor term structure of interest rates. This model is a two-dimensional combination of the classical Merton (1976) and Hull and White (1990) models. It is designed to obtain prices in a more 'realistic' framework, relaxing two stringent assumptions of the Black-Scholes model used in Sect. 2, whilst being sufficiently tractable to attain a closed form formula.

## 2 Knock-in/out call or put option on the best or the worst of four assets

The class of option contract under consideration in this section is written on four asset prices denoted by $S_t^{(i)}, i \in \{1, 2, 3, 4\}, t \geq 0$. Generalisation to a higher number of underlying assets is analytically feasible but results in cumbersome formulae and, more importantly, may raise numerical computation issues, as will be discussed later.

The option contract assigns a positive weight, $\omega_i$, to each asset $S^{(i)}$, so there is no need to assume that initial asset prices (i.e. prices at time $t_0 = 0$) are the same. For the option to have positive value at expiry $T$, the maximum/minimum (best-of/worst-of contract) of the four terminal weighted asset values $\omega_1 S_T^{(1)}, \omega_2 S_T^{(2)}, \omega_3 S_T^{(3)}, \omega_4 S_T^{(4)}$ must be greater/smaller (call/put contract) than a pre-specified strike price, provided that a pre-specified knock-in/out condition has been met prior to the option expiry.

Let $\underline{S}_T^{(i)}$ refer to the worst performer of the four assets at the option expiry $T$. Then, the payoff of an up-and-in worst-of put option can be written in the following manner:

$$\left( K_i - \omega_i \underline{S}_T^{(i)} \right) \mathbb{I} \left\{ \sup_{0 \leq t \leq T} \omega_i \underline{S}_t^{(i)} > H_i \right\} \tag{1}$$

where $\mathbb{I}\{.\}$ will denote, from now on, the indicator function; $K_i$ and $H_i$ are the strike price and the barrier, respectively, associated with the worst performer at expiry in the option contract. Likewise, let $\bar{S}_T^{(i)}$ refer to the best performer of the four assets at the option expiry $T$. Then, the payoff of down-and-out best-of call option is:

$$\left( \omega_i \bar{S}_T^{(i)} - K_i \right) \mathbb{I} \left\{ \inf_{0 \leq t \leq T} \omega_i \bar{S}_t^{(i)} > H_i \right\} \tag{2}$$

All other possible payoffs can obviously be written in a similar manner.

It should be emphasized that, if the knock-in or knock-out condition of the best-performer or the worst-performer at expiry has not been met, then the option expires worthless. In other words, the payoff is defined with regard to the absolute

maximum or minimum, i.e., it does not allow to take the next best/worst asset at expiry that would have fulfilled the knock-in/out condition. If one is concerned this absolute maximum/minimum condition will be deemed as overly stringent by investors, then one can "soften" the payoff rule under consideration by specifying a rebate provision in case of knocking-out or not knocking-in. Of course, such a compensation will make the option more expensive.

Also, it must be pointed out that it is not more difficult to value a second-best/worst or a third-best/worst contract using the techniques developed in this paper, but the resulting formula are bulkier.

To price 4-asset knock-in/out options on the maximum/minimum, a multidimensional Black-Scholes framework is assumed in this Section, which implies that asset prices $S_t^{(i)}$ are modeled by geometric Brownian motions with constant drifts $\alpha_i$ (under the historical probability measure), volatilities $\sigma_i$ and continuous payout rates $\delta_i$, according to the standard stochastic differential equation:

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (\alpha_i - \delta_i)\,dt + \sigma_i dW_t^{(i)} \tag{3}$$

It is assumed that all Brownian motions are defined on an adequate probability product space. The smallest $\sigma$-algebra generated by the set of random variables $W_s^{(i)}$, $0 \le s \le t$, is denoted by $\mathcal{F}_t$.

The assumption that asset price paths are continuous will be relaxed in Sect. 3.

The constant correlation coefficient between $W_t^{(i)}$ and $W_t^{(j)}$ is denoted by $\rho_{i,j}$. The constant riskless interest rate is denoted by $r$; in Sect. 3, a stochastic yield curve will be incorporated.

The volatilities $\sigma_i$ are supposed to be extracted from quoted vanilla option prices. If the set of strikes and maturities available in the market is inadequate, then one may fit a deterministic volatility function, as explained in Dumas et al. (1998). Adopting this simple method in a Black-Scholes framework yields good results in terms of pricing errors, as documented by Christoffersen and Jacobs (2004).

With regard to the correlation parameters $\rho_{i,j}$, it is assumed that there exists a set of quoted 2-asset option prices (e.g. exchange option prices) from which implied correlations can be extracted and used as inputs to the forthcoming Proposition 1. The estimation procedure should ensure that implied volatilities and implied correlations are consistent in the sense that the volatilities extracted from quoted vanilla option prices should be the same as those can be inferred from the quoted 2-asset option prices. In practice, estimation of the $\rho_{i,j}$ parameters will not be as easy as estimation of the $\sigma_i$ ones, since rainbow options are traded over-the-counter. Correlation between the constituents of liquid equity indices, however, is publicly observable. Index option prices can thus be combined with prices of individual options on all index components to infer the implied average correlation between the index components over the option's life (Driessen et al. 2005). Further investigation into this important issue is beyond the scope of this paper.

Under the above assumptions, the no-arbitrage price of the class of options under consideration is given by the following formula.

**Proposition 1** *Under the above assumptions, the value, at current time $t_0 = 0$, of an up-and-in put or a down-and-in call on the minimum or the maximum of four positively weighted assets $\omega_1 S_T^{(1)}, \omega_2 S_T^{(2)}, \omega_3 S_T^{(3)}, \omega_4 S_T^{(4)}$ with strike prices $K_1, K_2, K_3, K_4$, knock-in barriers $H_1, H_2, H_3, H_4$ and maturity $T > t_0$ is given by:*

$$
V\left(\omega_1, \omega_2, \omega_3, \omega_4; S_0^{(1)}, S_0^{(2)}, S_0^{(3)}, S_0^{(4)}; K_1, K_2, K_3, K_4; H_1, H_2, H_3, H_4; T\right)
$$
$$
= \Upsilon \exp(-rT)\left[K_1 Q_1 + K_2 Q_2 + K_3 Q_3 + K_4 Q_4\right]
$$
$$
- \Upsilon\left[\exp(-\delta_1 T)\,\omega_1 S_0^{(1)} Q_{F1} + \exp(-\delta_2 T)\,\omega_2 S_0^{(2)} Q_{F2}\right.
$$
$$
\left. + \exp(-\delta_3 T)\,\omega_3 S_0^{(3)} Q_{F3} + \exp(-\delta_4 T)\,\omega_4 S_0^{(4)} Q_{F4}\right]
$$

*where*

$$
\Upsilon = \begin{cases} 1 & \text{if the option is an up—and—in put} \\ -1 & \text{if the option is a down—and—in call} \end{cases}
$$

$$
Q_i = P_i\left(\mu_1 = \mu_1^{(Q)}, \mu_2 = \mu_2^{(Q)}, \mu_3 = \mu_3^{(Q)}, \mu_4 = \mu_4^{(Q)}\right), \quad \forall i \in \{1, 2, 3, 4\}
$$

$$
\mu_i^{(Q)} = r - \delta_i - \frac{\sigma_i^2}{2}
$$

$$
Q_{F1} = P_1\left( \begin{array}{l} \mu_1 = r - \delta_1 + \frac{\sigma_1^2}{2},\ \mu_2 = r - \delta_2 + \sigma_1\sigma_2\rho_{1.2} - \frac{\sigma_2^2}{2} \\[2mm] \mu_3 = r - \delta_3 + \sigma_1\sigma_3\rho_{1.3} - \frac{\sigma_3^2}{2},\ \mu_4 = r - \delta_4 + \sigma_1\sigma_4\rho_{1.4} - \frac{\sigma_4^2}{2} \end{array} \right)
$$

$$
Q_{F2} = P_2\left( \begin{array}{l} \mu_1 = r - \delta_1 + \sigma_1\sigma_2\rho_{1.2} - \frac{\sigma_1^2}{2},\ \mu_2 = r - \delta_2 + \frac{\sigma_2^2}{2} \\[2mm] \mu_3 = r - \delta_3 + \sigma_2\sigma_3\rho_{2.3} - \frac{\sigma_3^2}{2},\ \mu_4 = r - \delta_4 + \sigma_2\sigma_4\rho_{2.4} - \frac{\sigma_4^2}{2} \end{array} \right)
$$

$$
Q_{F3} = P_3\left( \begin{array}{l} \mu_1 = r - \delta_1 + \sigma_1\sigma_3\rho_{1.3} - \frac{\sigma_1^2}{2},\ \mu_2 = r - \delta_2 + \sigma_2\sigma_3\rho_{2.3} - \frac{\sigma_2^2}{2} \\[2mm] \mu_3 = r - \delta_3 + \frac{\sigma_3^2}{2},\ \mu_4 = r - \delta_4 + \sigma_3\sigma_4\rho_{3.4} - \frac{\sigma_4^2}{2} \end{array} \right)
$$

$$
Q_{F4} = P_3\left( \begin{array}{l} \mu_1 = r - \delta_1 + \sigma_1\sigma_4\rho_{1.4} - \frac{\sigma_1^2}{2},\ \mu_2 = r - \delta_2 + \sigma_2\sigma_4\rho_{2.4} - \frac{\sigma_2^2}{2} \\[2mm] \mu_3 = r - \delta_3 + \sigma_3\sigma_4\rho_{3.4} - \frac{\sigma_3^2}{2},\ \mu_4 = r - \delta_4 + \frac{\sigma_4^2}{2} \end{array} \right)
$$

$$
P_1(\mu_1, \mu_2, \mu_3, \mu_4)
$$
$$
= \exp\left(\frac{2\mu_1}{\sigma_1^2} h_1\right) \int_{-\infty}^{\Upsilon \frac{k_1 - 2h_1 - \mu_1 T}{\sigma_1\sqrt{T}}} \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\, N_3\left[ \frac{\Theta(\psi(1,2) - \Upsilon x)}{\sigma_{2|1}/(\sigma_1/\sigma_2 - \rho_{1.2})}, \frac{\Theta(\psi(1,3) - \Upsilon x)}{\sigma_{3|1}/(\sigma_1/\sigma_3 - \rho_{1.3})}, \right.
$$
$$
\left. \frac{\Theta(\psi(1,4) - \Upsilon x)}{\sigma_{4|1.2}/(\sigma_1/\sigma_4 - \rho_{1.4})}; \Theta\Upsilon\, \frac{\rho_{2.3|1}}{\sigma_{3|1}}, \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}, \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}} \right] dx
$$

$$P_2\left(\mu_1, \mu_2, \mu_3, \mu_4\right)$$

$$= \exp\left(\frac{2\mu_2}{\sigma_2^2} h_2\right) \int_{-\infty}^{\Upsilon\frac{k_2 - 2h_2 - \mu_2 T}{\sigma_2\sqrt{T}}} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} N_3\left[\frac{\Theta(\psi(2,1) - \Upsilon x)}{\sigma_{1|2}/(\sigma_2/\sigma_1 - \rho_{1.2})}, \frac{\Theta(\psi(2,3) - \Upsilon x)}{\sigma_{3|2}/(\sigma_2/\sigma_3 - \rho_{2.3})}, \right.$$

$$\left. \frac{\Theta(\psi(2,4) - \Upsilon x)}{\sigma_{4|2.1}/(\sigma_2/\sigma_4 - \rho_{2.4})}; \Theta\Upsilon\frac{\rho_{1.3|2}}{\sigma_{3|2}}, \frac{\rho_{1.4|2}}{\sigma_{4|2.1}}, \frac{\rho_{3.4|2.1}}{\sigma_{4|2.1}}\right] dx$$

$$P_3\left(\mu_1, \mu_2, \mu_3, \mu_4\right)$$

$$= \exp\left(\frac{2\mu_3}{\sigma_3^2} h_3\right) \int_{-\infty}^{\Upsilon\frac{k_3 - 2h_3 - \mu_3 T}{\sigma_3\sqrt{T}}} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} N_3\left[\frac{\Theta(\psi(3,1) - \Upsilon x)}{\sigma_{1|3}/(\sigma_3/\sigma_1 - \rho_{1.3})}, \frac{\Theta(\psi(3,2) - \Upsilon x)}{\sigma_{2|3}/(\sigma_3/\sigma_2 - \rho_{2.3})}, \right.$$

$$\left. \frac{\Theta(\psi(3,4) - \Upsilon x)}{\sigma_{4|3.1}/(\sigma_3/\sigma_4 - \rho_{3.4})}; \Theta\Upsilon\frac{\rho_{1.2|3}}{\sigma_{2|3}}, \frac{\rho_{1.4|3}}{\sigma_{4|3.1}}, \frac{\rho_{2.4|3.1}}{\sigma_{4|3.1}}\right] dx$$

$$P_4\left(\mu_1, \mu_2, \mu_3, \mu_4\right)$$

$$= \exp\left(\frac{2\mu_4}{\sigma_4^2} h_4\right) \int_{-\infty}^{\Upsilon\frac{k_4 - 2h_4 - \mu_4 T}{\sigma_4\sqrt{T}}} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} N_3\left[\frac{\Theta(\psi(4,1) - \Upsilon x)}{\sigma_{1|4}/(\sigma_4/\sigma_1 - \rho_{1.4})}, \frac{\Theta(\psi(4,2) - \Upsilon x)}{\sigma_{2|4}/(\sigma_4/\sigma_2 - \rho_{2.4})}, \right.$$

$$\left. \frac{\Theta(\psi(4,3) - \Upsilon x)}{\sigma_{3|4.1}/(\sigma_4/\sigma_3 - \rho_{3.4})}; \Theta\Upsilon\frac{\rho_{1.2|4}}{\sigma_{2|4}}, \frac{\rho_{1.3|4}}{\sigma_{3|4.1}}, \frac{\rho_{2.3|4.1}}{\sigma_{3|4.1}}\right] dx$$

*where,* $\forall a, b, c, d, i \in \{1, 2, 3, 4\}$,

$$k_i = \ln\left(\frac{K_i}{\omega_i S_0^{(i)}}\right), \quad h_i = \ln\left(\frac{H_i}{\omega_i S_0^{(i)}}\right)$$

$$\Theta = \begin{cases} 1 & \text{if the option is on the minimum} \\ -1 & \text{if the option is on the maximum} \end{cases}$$

$$\rho_{b\cdot c|a} = \frac{\rho_{b\cdot c} - \rho_{a\cdot b}\rho_{a\cdot c}}{\sigma_{b|a}}, \quad \sigma_{b|a} = \sqrt{1 - \rho_{b\cdot a}^2}$$

$$\rho_{c\cdot d|a\cdot b} = \frac{\rho_{c\cdot d} - \rho_{a\cdot c}\rho_{a\cdot d} - \rho_{b\cdot c|a}\rho_{b\cdot d|a}}{\sigma_{c|a\cdot b}}, \quad \sigma_{c|a\cdot b} = \sqrt{1 - \rho_{a\cdot c}^2 - \rho_{b\cdot c|a}^2}$$

$$\mu(a, b) = (\mu_b - \mu_a)\sqrt{T}, \quad \theta(a, b) = \sigma_a - \rho_{a\cdot b}\sigma_b, \quad \omega(a, b) = \ln\left(\frac{\omega_a S_0^{(a)}}{\omega_b S_0^{(b)}}\right)$$

$$\psi(a, b) = \frac{\mu(a, b)}{\theta(a, b)} + \frac{(2h_a + \omega(a, b))}{\sqrt{T}}\left(\frac{\rho_{a\cdot b}}{\sigma_a(\sigma_a/\sigma_b - \rho_{a\cdot b})} - \frac{1}{\theta(a, b)}\right)$$

$N_3\left(., ., .; \rho_1, \rho_2, \rho_3\right)$, *where the first three arguments are real numbers and the last three arguments are ordered correlation coefficients, is the trivariate standard normal cumulative distribution function, given by the upcoming Proposition 3.*

Proof of Proposition 1 is given in Appendix 1. The next formula provides the Black-Scholes value of a 'regular' (that is, without knock-in or knock-out conditions) call or put on the minimum or the maximum of four assets. This payoff is tackled not only because it is useful in itself but also because, combined with Proposition 1, it yields values of up-and-out puts and down-and-out calls on the minimum or the maximum of four assets, by virtue of the simple parity relation: knock-in + knock-out = regular.

**Proposition 2** *The Black-Scholes value, at time $t_0 = 0$, of an option on the minimum or the maximum of four positively weighted assets $\omega_1 S_T^{(1)}$, $\omega_2 S_T^{(2)}$, $\omega_3 S_T^{(3)}$, $\omega_4 S_T^{(4)}$ with strike prices $K_1$, $K_2$, $K_3$, $K_4$, and maturity $T > t_0$ is given by:*

$$
V\left(\omega_1, \omega_2, \omega_3, \omega_4; S_0^{(1)}, S_0^{(2)}, S_0^{(3)}, S_0^{(4)}; K_1, K_2, K_3, K_4; T\right)
$$
$$
= \Upsilon \exp\left(-rT\right)\left[K_1 Q_1 + K_2 Q_2 + K_3 Q_3 + K_4 Q_4\right]
$$
$$
- \Upsilon \left[\exp\left(-\delta_1 T\right)\omega_1 S_0^{(1)} Q_{F1} + \exp\left(-\delta_2 T\right)\omega_2 S_0^{(2)} Q_{F2}\right.
$$
$$
\left. + \exp\left(-\delta_3 T\right)\omega_3 S_0^{(3)} Q_{F3} + \exp\left(-\delta_4 T\right)\omega_4 S_0^{(4)} Q_{F4}\right]
$$

*where $Q_1$, $Q_2$, $Q_3$, $Q_4$, $Q_{F1}$, $Q_{F2}$, $Q_{F3}$, $Q_{F4}$ are as in Proposition 1, with:*

$$
P_1 = N_4 \begin{bmatrix} \Upsilon \frac{k_1 - \mu_1 T}{\sigma_1 \sqrt{T}}, \Theta\left(\omega(2,1) + \frac{\mu(1,2)}{\sigma(1,2)}\right), \Theta\left(\omega(3,1) + \frac{\mu(1,3)}{\sigma(1,3)}\right), \Theta\left(w(4,1) + \frac{\mu(1,4)}{\sigma(1,4)}\right); \\ \Upsilon\Theta\frac{\theta(1,2)}{\sigma(1,2)}, \Upsilon\Theta\frac{\theta(1,3)}{\sigma(1,3)}, \Upsilon\Theta\frac{\theta(1,4)}{\sigma(1,4)}, \frac{\sigma(1,2,3)}{\sigma(1,2)\sigma(1,3)}, \frac{\sigma(1,2,4)}{\sigma(1,2)\sigma(1,4)}, \frac{\sigma(1,3,4)}{\sigma(1,3)\sigma(1,4)} \end{bmatrix}
$$

$$
P_2 = N_4 \begin{bmatrix} \Upsilon \frac{k_2 - \mu_2 T}{\sigma_2 \sqrt{T}}, \Theta\left(\omega(1,2) + \frac{\mu(2,1)}{\sigma(2,1)}\right), \Theta\left(\omega(3,2) + \frac{\mu(2,3)}{\sigma(2,3)}\right), \Theta\left(\omega(4,2) + \frac{\mu(2,4)}{\sigma(2,4)}\right); \\ \Upsilon\Theta\frac{\theta(2,1)}{\sigma(2,1)}, \Upsilon\Theta\frac{\theta(2,3)}{\sigma(2,3)}, \Upsilon\Theta\frac{\theta(2,4)}{\sigma(2,4)}, \frac{\sigma(2,1,3)}{\sigma(2,1)\sigma(2,3)}, \frac{\sigma(2,1,4)}{\sigma(2,1)\sigma(2,4)}, \frac{\sigma(2,3,4)}{\sigma(2,3)\sigma(2,4)} \end{bmatrix}
$$

$$
P_3 = N_4 \begin{bmatrix} \Upsilon \frac{k_3 - \mu_3 T}{\sigma_3 \sqrt{T}}, \Theta\left(\omega(1,3) + \frac{\mu(3,1)}{\sigma(3,1)}\right), \Theta\left(\omega(2,3) + \frac{\mu(3,2)}{\sigma(3,2)}\right), \Theta\left(\omega(4,3) + \frac{\mu(3,4)}{\sigma(3,4)}\right); \\ \Upsilon\Theta\frac{\theta(3,1)}{\sigma(3,1)}, \Upsilon\Theta\frac{\theta(3,2)}{\sigma(3,2)}, \Upsilon\Theta\frac{\theta(3,4)}{\sigma(3,4)}, \frac{\sigma(3,1,2)}{\sigma(3,1)\sigma(3,2)}, \frac{\sigma(3,1,4)}{\sigma(3,1)\sigma(3,4)}, \frac{\sigma(3,2,4)}{\sigma(3,2)\sigma(3,4)} \end{bmatrix}
$$

$$
P_4 = N_4 \begin{bmatrix} \Upsilon \frac{k_4 - \mu_4 T}{\sigma_4 \sqrt{T}}, \Theta\left(\omega(1,4) + \frac{\mu(4,1)}{\sigma(4,1)}\right), \Theta\left(\omega(2,4) + \frac{\mu(4,2)}{\sigma(4,2)}\right), \Theta\left(\omega(3,4) + \frac{\mu(4,3)}{\sigma(4,3)}\right); \\ \Upsilon\Theta\frac{\theta(4,1)}{\sigma(4,1)}, \Upsilon\Theta\frac{\theta(4,2)}{\sigma(4,2)}, \Upsilon\Theta\frac{\theta(4,3)}{\sigma(4,3)}, \frac{\sigma(4,1,2)}{\sigma(4,1)\sigma(4,2)}, \frac{\sigma(4,1,3)}{\sigma(4,1)\sigma(4,3)}, \frac{\sigma(4,2,3)}{\sigma(4,2)\sigma(4,3)} \end{bmatrix}
$$

$$
\Upsilon = \begin{cases} 1 & \text{if the option is a put} \\ -1 & \text{if the option is a call} \end{cases}
$$

$$
\Theta = \begin{cases} 1 & \text{if the option is on the minimum} \\ -1 & \text{if the option is on the maximum} \end{cases}
$$

$$
\forall i \in \{1, 2, 3, 4\}, k_i = \ln\left(\frac{K_i}{\omega_i S_0^{(i)}}\right)
$$
$$
\mu(a,b) = (\mu_b - \mu_a)\sqrt{T}, \quad \theta(a,b) = \sigma_a - \rho_{a \cdot b}\sigma_b,
$$
$$
\omega(a,b) = \ln\left(\frac{\omega_a S_0^{(a)}}{\omega_b S_0^{(b)}}\right),
$$
$$
\sigma(x,y) = \sqrt{\sigma_x^2 - 2\sigma_x\sigma_y\rho_{x \cdot y} + \sigma_y^2},
$$
$$
\sigma(x,y,z) = \sigma_x^2 - \sigma_x\sigma_y\rho_{x \cdot y} - \sigma_x\sigma_z\rho_{x \cdot z} + \sigma_y\sigma_z\rho_{y \cdot z}
$$

*$N_4\left(., ., ., .; \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\right)$, where the first four arguments are real numbers and the last six arguments are ordered correlation coefficients, is the quadrivariate standard normal cumulative distribution function, as defined by the upcoming Proposition 3.*

Proof of Proposition 2 is outlined in Appendix 4, which also shows how the cases of an up-and-in/out call and a down-and-in/out put can be easily dealt with by means of Proposition 1 and Proposition 2.

From a practical point of view, Proposition 1 and Proposition 2 need to be efficiently and accurately computed. This requirement can be met by applying the following integration rule:

**Proposition 3** *Let: $X_a$, $X_b$, $X_c$, $X_d$ be four standard normal random variables; $u_a$, $u_b$, $u_c$, $u_d$ be four real numbers; and $\rho_{i.j}$ denote the correlation coefficient between $X_i$ and $X_j$, with $i$ and $j$ taking values in the set $\{a, b, c, d\}$. Then,*

(i) *The joint cumulative distribution function of $X_a$, $X_b$ and $X_c$ is given by:*

$$
\begin{aligned}
&P\left(X_a \le u_a, X_b \le u_b, X_c \le u_c\right) \\
&\triangleq N_3\left(u_a, u_b, u_c; \rho_{a \cdot b}, \rho_{a \cdot c}, \rho_{b \cdot c}\right) \\
&= \int_{-\infty}^{u_a} \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} \, N_2\left(\frac{u_b - \rho_{a \cdot b} x}{\sigma_{b|a}}, \frac{u_c - \rho_{a \cdot c} x}{\sigma_{c|a}}; \frac{\rho_{b \cdot c|a}}{\sigma_{c|a}}\right) dx
\end{aligned}
\tag{4}
$$

$$
= \int_{x=-\infty}^{u_a} \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} \left( \int_{y=-\infty}^{\frac{u_b - \rho_{a \cdot b} x}{\sigma_{b|a}}} \frac{\exp\left(-y^2/2\right)}{\sqrt{2\pi}} \right.
$$

$$
\left. N\left(\frac{u_c - \rho_{a \cdot c} x - \rho_{b \cdot c|a} \, y}{\sigma_{c|a \cdot b}}\right) dy \right) dx
\tag{5}
$$

(ii) *The joint cumulative distribution function of $X_a$, $X_b$, $X_c$ and $X_d$ is given by:*

$$
\begin{aligned}
&P\left(X_a \le u_a, X_b \le u_b, X_c \le u_c, X_d \le u_d\right) \\
&\triangleq N_4\left(u_a, u_b, u_c, u_d; \rho_{a \cdot b}, \rho_{a \cdot c}, \rho_{a \cdot d}, \rho_{b \cdot c}, \rho_{b \cdot d}, \rho_{c \cdot d}\right) \\
&= \int_{-\infty}^{u_a} \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} \\
&\quad N_3\left(\frac{u_b - \rho_{a \cdot b} x}{\sigma_{b|a}}, \frac{u_c - \rho_{a \cdot c} x}{\sigma_{c|a}}, \frac{u_d - \rho_{a \cdot d} x}{\sigma_{d|a \cdot b}}; \frac{\rho_{b \cdot c|a}}{\sigma_{c|a}}, \frac{\rho_{b \cdot d|a}}{\sigma_{d|a \cdot b}}, \frac{\rho_{c \cdot d|a \cdot b}}{\sigma_{d|a \cdot b}}\right) dx
\end{aligned}
\tag{6}
$$

$$
= \int_{x=-\infty}^{u_a} \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} \left( \int_{y=-\infty}^{\frac{u_b - \rho_{a \cdot b} x}{\sigma_{b|a}}} \frac{\exp\left(-y^2/2\right)}{\sqrt{2\pi}} \, N_2\left( \frac{\frac{u_c - \rho_{a \cdot c} x}{\sigma_{c|a}} - \frac{\rho_{b \cdot c|a}}{\sigma_{c|a}} y}{\left(1 - \frac{\rho_{b \cdot c|a}^2}{\sigma_{c|a}^2}\right)^{1/2}}, \right. \right.
$$

$$\left. \left. \frac{\frac{u_d-\rho_{a\cdot d}x}{\sigma_{d|a\cdot b}}-\frac{\rho_{b\cdot d|a}}{\sigma_{d|a\cdot b}}y}{\left(1-\frac{\rho_{b\cdot d|a}^2}{\sigma_{d|a\cdot b}^2}\right)^{1/2}};\ \frac{\frac{\rho_{c\cdot d|a\cdot b}}{\sigma_{d|a\cdot b}}-\frac{\rho_{b\cdot c|a}}{\sigma_{c|a}}\frac{\rho_{b\cdot d|a}}{\sigma_{d|a\cdot b}}}{\left(1-\frac{\rho_{b\cdot c|a}^2}{\sigma_{c|a}^2}\right)^{1/2}\left(1-\frac{\rho_{b\cdot d|a}^2}{\sigma_{d|a\cdot b}^2}\right)^{1/2}}\right)dy\right]dx \tag{7}$$

$$=\int_{x=-\infty}^{u_a}\frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}}\left(\int_{y=-\infty}^{\frac{u_b-\rho_{a\cdot b}x}{\sigma_{b|a}}}\frac{\exp\left(-y^2/2\right)}{\sqrt{2\pi}}\left(\int_{z=-\infty}^{\frac{\frac{u_c-\rho_{a\cdot c}x}{\sigma_{c|a}}-\frac{\rho_{b\cdot c|a}}{\sigma_{c|a}}y}{\left(1-\frac{\rho_{b\cdot c|a}^2}{\sigma_{c|a}^2}\right)^{1/2}}}\frac{\exp\left(-z^2/2\right)}{\sqrt{2\pi}}\right.\right.$$

$$N\left[\left(\frac{\frac{\frac{u_d-\rho_{a\cdot d}x}{\sigma_{d|a\cdot b}}-\frac{\rho_{b\cdot d|a}}{\sigma_{d|a\cdot b}}y}{\left(1-\frac{\rho_{b\cdot d|a}^2}{\sigma_{d|a\cdot b}^2}\right)^{1/2}}-\frac{\frac{\rho_{c\cdot d|a\cdot b}}{\sigma_{d|a\cdot b}}-\frac{\rho_{b\cdot c|a}}{\sigma_{c|a}}\frac{\rho_{b\cdot d|a}}{\sigma_{d|a\cdot b}}}{\left(1-\frac{\rho_{b\cdot c|a}^2}{\sigma_{c|a}^2}\right)^{1/2}\left(1-\frac{\rho_{b\cdot d|a}^2}{\sigma_{d|a\cdot b}^2}\right)^{1/2}}z}{\left(1-\left(\frac{\frac{\rho_{c\cdot d|a\cdot b}}{\sigma_{d|a\cdot b}}-\frac{\rho_{b\cdot c|a}}{\sigma_{c|a}}\frac{\rho_{b\cdot d|a}}{\sigma_{d|a\cdot b}}}{\left(1-\frac{\rho_{b\cdot c|a}^2}{\sigma_{c|a}^2}\right)^{1/2}\left(1-\frac{\rho_{b\cdot d|a}^2}{\sigma_{d|a\cdot b}^2}\right)^{1/2}}\right)^2\right)^{1/2}}\right)\right]dz\right)dy\right)dx \tag{8}$$

Proof of Proposition 3 is outlined in Appendix 4, which also discusses its numerical implementation.

The above formulae give the no-arbitrage prices of knock-in/out call or put options on the best or the worst of four assets, under the modeling assumptions described at the beginning of this Section. As the latter account for a complete market, dynamical hedging seems the natural course of action. It suffices to differentiate the valuation formula with respect to the relevant parameters, to obtain the amount of each underlying asset that the replicating portfolio should be invested in. Likewise, the availability of a valuation formula allows to monitor positions easily by analytically extracting the sensitivities of the option price with respect to the various risk factors. Of course, the usual issues associated with dynamical hedging will have to be taken into consideration, such as transaction costs. It must be stressed that the presence of knock-in/out barriers compounds the hedging problem, as it is well known that deltas can get out of control near the barrier. Practical hedges, such as trading call spreads in the region of the barrier, can help mitigate the problem in these instances. More material on this important topic can be found in Carr et al. (1998).

Let us now illustrate how option prices are affected by changes in volatility and correlation parameters. The market is made up of four risky assets denoted by: $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, $S^{(4)}$, paying out no dividend rate, and a riskless bond. The initial price, i.e. at time $t_0$, of all risky assets is 100 (monetary units), and there is a unique weight on all stocks equal to 1. Call options are specified as either OTM (out of the money) when struck at $K = 105$, or as ATM (at the money), or as ITM (in the money) when struck

at $K = 95$. Expiry is 1 year. Option values are computed in four different settings: (i) low volatility and low correlation; (ii) low volatility and high correlation; (iii) high volatility and low correlation; (iv) high volatility and high correlation. In each setting, the prices of six different products are compared:

(i) 2-asset call option on maximum, whose payoff reads $\left[ \max \left( S_T^{(1)}, S_T^{(2)} \right) - K \right]^+$;

(ii) 4-asset call option on maximum, whose payoff reads $\left[ \max \left( S_T^{(1)}, S_T^{(2)}, S_T^{(3)}, \right. \right.$
$\left. \left. S_T^{(4)} \right) - K \right]^+$;

(iii) 4-asset down-and-in call option on maximum, with a unique barrier $H = 95$, whose payoff reads $\left[ \bar{S}_T - K \right]^+ \mathbb{I} \left\{ \inf_{0 \leq t \leq T} \bar{S}_t < H \right\}$, where $\bar{S}_T$ refer to the best performer of the four assets at the option expiry;

(iv) 4-asset down-and-out call option on maximum, with a unique barrier $H = 90$, whose payoff reads $\left[ \bar{S}_T - K \right]^+ \mathbb{I} \left\{ \inf_{0 \leq t \leq T} \bar{S}_t > H \right\}$;

(v) 4-asset equally-weighted basket call option, whose payoff reads

$$\left[ \frac{1}{4} \left( S_T^{(1)} + S_T^{(2)} + S_T^{(3)} + S_T^{(4)} \right) - K \right]^+;$$

(vi) 4-asset zero-strike spread option, whose payoff reads

$$\max \left( S_T^{(1)}, S_T^{(2)}, S_T^{(3)}, S_T^{(4)} \right) - \min \left( S_T^{(1)}, S_T^{(2)}, S_T^{(3)}, S_T^{(4)} \right)$$

For the comparison between the 2-asset call and the other 4-asset call options not to be distorted, the volatilities of $S^{(3)}$ and $S^{(4)}$ are taken to be identical to the volatilities of $S^{(1)}$ and $S^{(2)}$, respectively. In the "low volatility" environment, the volatilities of $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$ are, respectively: $\sigma_1 = 0.16$, $\sigma_2 = 0.15$, $\sigma_3 = 0.16$ and $\sigma_4 = 0.15$; in the "high volatility" environment, we have: $\sigma_1 = 0.42$, $\sigma_2 = 0.48$, $\sigma_3 = 0.42$ and $\sigma_4 = 0.48$.

In order to define a consistent correlation matrix in a simple manner, we refer to a stylized stock market comprising "defensive" and "cyclical" stocks: $S^{(1)}$ and $S^{(4)}$ are defensive, $S^{(2)}$ and $S^{(3)}$ are cyclical; thus, there is negative correlation between $S^{(1)}$ and $S^{(2)}$, $S^{(1)}$ and $S^{(3)}$, $S^{(3)}$ and $S^{(4)}$; there is positive correlation between $S^{(1)}$ and $S^{(4)}$, $S^{(2)}$ and $S^{(3)}$. In the "low correlation" setting, the correlation matrix is given by Table 1.

In the "high correlation" setting, the coefficients of the previous correlation matrix are simply multiplied by 2 so as not to change the structure of the correlation between the asset returns and thus allow meaningful comparison. This yields the values reported in Table 2.

The numerical values obtained for the option prices are reported in Tables 3,4,5 and 6.

In general, best-of call options will be attractive when at least one of the underlying assets has performed very well at expiry. No wonder that adding more assets increases

**Table 1** Correlation parameters used in the computations reported in Tables 3 and 5 ("low correlation")

|         | Asset 1 | Asset 2 | Asset 3 | Asset 4 |
|---------|---------|---------|---------|---------|
| Asset 1 | 1       | −0.18   | −0.2    | 0.15    |
| Asset 2 | −0.18   | 1       | 0.1     | −0.22   |
| Asset 3 | −0.2    | 0.1     | 1       | −0.24   |
| Asset 4 | 0.15    | −0.22   | −0.24   | 1       |

**Table 2** Correlation parameters used in the computations reported in Tables 4 and 6 ("high correlation")

|         | Asset 1 | Asset 2 | Asset 3 | Asset 4 |
|---------|---------|---------|---------|---------|
| Asset 1 | 1       | −0.36   | −0.4    | 0.3     |
| Asset 2 | −0.36   | 1       | 0.2     | −0.44   |
| Asset 3 | −0.4    | 0.2     | 1       | −0.48   |
| Asset 4 | 0.3     | −0.44   | −0.48   | 1       |

**Table 3** Prices of various multiasset options with low volatility and low correlation[a]

|                                              | OTM    | ATM    | ITM    |
|----------------------------------------------|--------|--------|--------|
| 2-asset call option on maximum               | 11.195 | 15.048 | 19.357 |
| 4-asset call option on maximum               | 17.644 | 22.269 | 27.006 |
| 4-asset down-and-in call option on maximum   | 4.835  | 6.494  | 8.230  |
| 4-asset down-and-out call option on maximum  | 16.738 | 20.954 | 25.246 |
| 4-asset basket call option                   | 2.676  | 5.704  | 9.787  |
| 4-asset spread option                        | 33.181 | 33.181 | 33.181 |

[a] This table presents several option values, whose parameters and contract specifications are defined in Sect. 2 after the end of Proposition 3. All best-of option prices were obtained using the formulae provided in propositions 1, 2 and 3. Basket and spread option prices were obtained using 5,000,000 Monte Carlo simulations

**Table 4** Prices of various multiasset options with low volatility and high correlation[a]

|                                              | OTM    | ATM    | ITM    |
|----------------------------------------------|--------|--------|--------|
| 2-asset call option on maximum               | 11.535 | 15.527 | 19.948 |
| 4-asset call option on maximum               | 18.089 | 22.803 | 27.573 |
| 4-asset down-and-in call option on maximum   | 5.068  | 6.772  | 8.483  |
| 4-asset down-and-out call option on maximum  | 17.112 | 21.407 | 25.756 |
| 4-asset basket call option                   | 2.064  | 5.276  | 9.668  |
| 4-asset spread option                        | 33.984 | 33.984 | 33.984 |

[a] See footnote of Table 3

the value of these instruments, as manifest when comparing 2-asset call options on maximum and 4-asset call options on maximum, particularly when there is some negative correlation between the underlying assets' returns, because then bad surprises will go along with good ones. To mitigate the deterrent effect of higher prices on investors' demand resulting from the introduction of a larger number of assets in the

**Table 5** Prices of various multiasset options with high volatility and low correlation[a]

|  | OTM | ATM | ITM |
|---|---|---|---|
| 2-asset call option on maximum | 32.974 | 36.335 | 39.929 |
| 4-asset call option on maximum | 54.629 | 59.036 | 63.561 |
| 4-asset down-and-in call option on maximum | 37.793 | 41.237 | 44.797 |
| 4-asset down-and-out call option on maximum | 29.733 | 31.606 | 33.491 |
| 4-asset basket call option | 8.065 | 10.462 | 13.365 |
| 4-asset spread option | 95.231 | 95.231 | 95.231 |

[a] See footnote of Table 3

**Table 6** Prices of various multiasset options with high volatility and high correlation[a]

|  | OTM | ATM | ITM |
|---|---|---|---|
| 2-asset call option on maximum | 33.847 | 37.344 | 41.079 |
| 4-asset call option on maximum | 55.802 | 60.350 | 64.996 |
| 4-asset down-and-in call option on maximum | 38.226 | 41.865 | 45.589 |
| 4-asset down-and-out call option on maximum | 30.525 | 32.383 | 34.256 |
| 4-asset basket call option | 6.497 | 8.938 | 11.969 |
| 4-asset spread option | 97.342 | 97.342 | 97.342 |

[a] See footnote of Table 3

payoff, knock-in/out barriers are welcome. Compared with a regular 4-asset best-of call, the price cut caused by a knock-in barrier provision is, in percentage, 70.98% in Table 3, and 70.51% in Table 4, averaging across all strikes. Of course, investors must be aware that the knock-in provision makes their claim riskier. One way to assess this increased risk is to compare the probability of reaching the breakeven point and thus having positive return on investment at expiry when holding the regular contract and when holding the knock-in contract. The breakeven point is the premium of the option accrued at the riskless rate (in the risk-neutral world) until expiry. In the low volatility/high correlation environment, for instance, the probability of reaching the breakeven point for the holder of a regular at-the-money 4-asset best-of call, that is, the probability that the maximum of the four assets at expiry is greater than 123.972, is 43.58% in the risk-neutral world[1]; while the probability of reaching the breakeven point for the holder of a down-and-in at-the-money 4-asset best-of call, that is, the probability that the maximum of the four assets at expiry is greater than 107.119 and that it has dipped below 95 prior to expiry, is equal to 31.958%. This is only a moderate increase in risk, whereas the reduction in the cost of the option is dramatic, which will make the knock-in contract look very attractive to a lot of investors.

If we now turn to the high volatility environment, the average price cut caused by a knock-out barrier provision is 46.45% in Table 5 and 46.31% in Table 6, when compared with a regular 4-asset best-of call. In the high volatility/high correlation

[1] This probability would be computed by an investor in the "real" world, not in the risk-neutral world; this would imply to introduce a positive equity premium reflecting risk aversion.

environment, the probability of reaching the breakeven point for the holder of a regular at-the-money 4-asset best-of call, that is, the probability that the maximum of the four assets at expiry is greater than 163.444, is 39.51% in the risk-neutral world; while the probability of reaching the breakeven point for the holder of a down-and-out at-the-money 4-asset best-of call, that is, the probability that the maximum of the four assets at expiry is greater than 134.043 and that it has not dipped below 90 prior to expiry, is equal to 33.92%. Thus, the price cut is not as dramatic as in the previous example, but it is still big and, considering the modest increase in risk entailed, there is little doubt that the introduction of the knock-out provision will be viewed as a good opportunity by many investors.

With regard to volatility, the sensitivity of knock-in/out best-of options is always positive. It is very high for knock-in options, as greater volatility increases not only the likelihood of ending up in-the-money but also the chances of being activated prior to expiry. Tables 5 and 6 show that, despite a sharp increase in value with regard to the low volatility environment, the price reduction with respect to regular 4-asset best-of options remains significant. In the risk-neutral world, the probability that the down-and-in call option on maximum will expire worthless due to non-activation is 20.828% (Table 5) or 21.407% (Table 6). The risk born by the investor is thus relatively weak, which makes these options quite attractive, in view of their relatively cheap price, compared with regular 4-asset best-of contracts. The effect of increased volatility on knock-out best-of call options is more ambiguous. The sensitivity of the latter is positive but substantially lower and less linear than that of knock-in contracts. This is because increased volatility raises the odds of knocking-out before expiry. But when volatility is low and the risk of knocking-out thus weak, there is not much difference between knock-out best-of prices and regular best-of prices (Tables 3 and 4); to attain substantial price reduction, the option contract should then locate the barrier nearer to the spot.

Knock-in/out best-of options depend positively on correlation, but only moderately. Particularly when volatility is low, the change in price resulting from an increase in correlation is quite small. The introduction of barriers does not alter this apparently weak functional dependence of best-of options on the overall level of correlation, since the latter phenomenon can be observed as well with regular best-of contracts. The current modest numerical experiment thus suggests that mispricings caused by errors in terms of the magnitude of correlation inputs might not be overstated, or at least that they will not weigh as heavily as errors in the magnitude of volatility inputs.

It can also be observed that spread options are very expensive, which makes their marketing uneasy, while basket call options are cheap, due notably to the smoothing effect of averaging at expiry. For the return of the basket call option to become attractive, it is not required that one of the underlying assets perform very well, but that all underlying assets perform reasonably well (as far as an equally weighted basket is concerned). Thus, this kind of option will be well suited to an upward market with moderate volatility, even if the trend is not pronounced. A necessary condition to attain maximum payoff is that assets be positively correlated. On another hand, negative correlation, although precluding maximum payoff at expiry, will provide protection in adverse market conditions. This explains the slightly negative sensitivity of basket call options to an increase in correlation in Tables 4 and 6.

## 3 The value of an option on the best or the worst of two assets with correlated jump diffusions and a stochastic term structure of interest rates

A rather stringent assumption made in Sect. 2 is that asset price paths are continuous. In effect, it is well-known that asset price paths exhibit jumps, which results in higher prices than those generated by a Black-Scholes model, especially for short-lived, out-of-the-money options. The assumption of continuous asset price paths is necessary to achieve analytical tractability while introducing knock-in/out features in a multi-asset setting. When dealing with standard best-of/worst-of options, that assumption can be relaxed without having to resort to slow and inaccurate numerical approximations for pricing and hedging purposes.

Furthermore, the use of a constant riskless interest rate in Sect. 2, may lead to non-negligible pricing errors as the option maturity increases, making a case for the introduction of a stochastic yield curve.

The goal of this section is to show how to price regular best-of/worst-of contracts in closed form with a model designed to fit market data better than standard Black-Scholes. In the sequel, $W_t^{(i)}$ stands for a standard Brownian motion and the constant correlation coefficient between $W_t^{(i)}$ and $W_t^{(j)}$ is denoted by $\rho_{i,j}$, as in Sect. 2. The riskless interest rate is now driven by the following two-factor time-dependent Vasicek-type stochastic differential equation:

$$dr_t = a\left(b\left(t\right) - r_t\right)dt + \sigma_r dW_t^{(1)} + \sigma_r dW_t^{(2)} \tag{9}$$

where $(a, \sigma_r) \in \mathbb{R}^2$ and $b(t)$ is a deterministic function of $t$ satisfying a linear growth condition.

The choice of this model for the riskless interest rate is motivated by several reasons. First, principal component analysis shows that at least two factors are needed to capture the main changes in the yield curve (Martellini et al. 2003). Note that it would be straightforward, in our setting, to incorporate three or even four factors, at the cost of making formulae more cumbersome. Second, making the drift of the riskless rate time-dependent enables to make the model consistent with the currently observed yield curve by choosing an appropriate fitting function $b(t)$. Third, the mean-reverting feature is confirmed by statistical data. Finally, the model is tractable, making it possible to derive explicit pricing formulae.

Let $Q$ be the equivalent martingale measure under which the numeraire is the money market account $\beta_t$, defined by $\beta_t = \exp\left(\int_0^t r_s ds\right)$. Under $Q$, the dynamics of the two underlying asset prices, $S_t^{(1)}$ and $S_t^{(2)}$, are driven by:

$$\frac{dS_t^{(1)}}{S_{t_-}^{(1)}} = (r_t - \delta_1 - \lambda_1\kappa_1)\,dt + \sigma_{S1}dW_t^{(3)} + I_t^{(1)}dN_t^{(1)} \tag{10}$$

$$\frac{dS_t^{(2)}}{S_{t_-}^{(2)}} = (r_t - \delta_2 - \lambda_2\kappa_2)\,dt + \sigma_{S2}dW_t^{(4)} + I_t^{(2)}dN_t^{(2)} \tag{11}$$

Justification for (10) and (11), in terms of the pricing measure, is provided in Appendix 4. Meanwhile, let us define the notations used: $\sigma_{S1}$ is the constant diffusive volatility of $S^{(1)}$ and $\sigma_{S2}$ is the constant diffusive volatility of $S^{(2)}$; $\delta_1$ and $\delta_2$ are constant payout rates associated with assets $S^{(1)}$ and $S^{(2)}$, respectively; $N_t^{(1)}$ and $N_t^{(2)}$ are two Poisson processes with intensities $\lambda_1$ and $\lambda_2$ which admit the following decompositions:

$$N_t^{(1)} = Z_t^{(1)} + Z_t^{(12)}, \quad N_t^{(2)} = Z_t^{(2)} + Z_t^{(12)} \tag{12}$$

where $Z_t^{(1)}$, $Z_t^{(2)}$ and $Z_t^{(12)}$ are independent Poisson processes with intensities $\lambda_1'$, $\lambda_2'$ and $\lambda_{12}'$ respectively. Thus, $N_t(1)$ and $N_t^{(2)}$ have positive correlation given by:

$$\rho_{1.2}^{(N)} = \frac{\lambda_{12}'}{\sqrt{\left(\lambda_1' + \lambda_{12}'\right)\left(\lambda_2' + \lambda_{12}'\right)}} \tag{13}$$

Let $n \in \mathbb{N}$, $\tau_n^{(i)} = \inf\left\{t \geq 0, N_t^{(i)} = n\right\}$ and $U_n^{(i)}$ be a sequence of independent, identically distributed random variables taking values in $]-1, +\infty]$. Then, $I_t^{(i)} = \sum_n U_n^{(i)} \mathbb{I}_{\left[\tau_{n-1}^{(i)}, \tau_n^{(i)}\right[}(t)$ is a right-continuous process providing the magnitudes of the jumps of asset $S_t^{(i)}$. Set: $J_n^{(i)} = \ln\left(1 + U_n^{(i)}\right)$. Assume that $J_n^{(i)}$ is normally-distributed with mean $\xi_i$ and variance $\varepsilon_i^2$. Then,

$$E\left[U_n^{(i)}\right] \triangleq \kappa_i = \exp\left(\xi_i + \varepsilon_i^2/2\right) - 1 \tag{14}$$

It is assumed that all Brownian and compound Poisson processes implied by the model are defined on an adequate probability product space in which the smallest $\sigma$-algebra generated by the random variables $W_s^{(i)}$, $N_s^{(i)}$, for $s \leq t$, and $U_n^{(i)}\mathbb{I}\left\{n \leq N_t^{(i)}\right\}$, for $n \geq 1$ is denoted by $\mathcal{F}_t$.

Thus, the model used in this section is a two-dimensional combination of Merton (1976) and Hull and White (1990). The classical jump-diffusion framework by Merton (1976) is extended to a bivariate setting allowing for correlation between the jumps of the stocks. As shown in Appendix 4, three major assumptions make it possible to end up with a closed form formula: (i) the jump sizes are lognormally distributed, (ii) the jump processes are of finite activity, (iii) interest rates are Gaussian. The most stringent hypothesis of the model, however, and consequently its main weakness, is probably that the diffusive volatilities of the stocks remain constant. To try and overcome this limitation, the option position will have to be monitored with regularly updated implied volatility inputs.

It must be stressed that, due to the introduction of jumps, the market is now incomplete. This means that perfect hedging is not possible. The classical argument by Merton (1976) is to assume that jump risk is diversifiable and therefore not rewardable with excess return. However, since industry wide and country wide shocks do

exist, there are clearly times when this assumption is flawed. The literature on mean-variance hedging (Schweizer 1992) and quantile hedging (Föllmer and Leukert 1999) can be consulted for alternative approaches.

Proposition 4 can now be stated.

**Proposition 4** *Under the above assumptions, the value, at current time $t_0 = 0$, of a call or a put option on the maximum or the minimum of two positively weighted assets $\omega_1 S_T^{(1)}$ and $\omega_2 S_T^{(2)}$ with strike prices $K_1$, $K_2$ and maturity $T > t_0$ is given by:*

$$V\left(\omega_1, \omega_2; S_0^{(1)}, S_0^{(2)}; K_1, K_2; T\right)$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_{12}=0}^{\infty} e^{-\lambda_1' T - \lambda_2' T - \lambda_{12}' T} \frac{\left(\lambda_1' T\right)^{n_1} \left(\lambda_2' T\right)^{n_2} \left(\lambda_{12}' T\right)^{n_{12}}}{n_1! n_2! n_{12}!}$$

$$\Upsilon \, \omega_1 S_0^{(1)} \exp\left(v_1 + \frac{\gamma_1^2}{2}\right) N_2 \left[\Upsilon \frac{\ln\left(\frac{S_0^{(1)}}{\omega_1 K_1}\right) + \omega(1,2) + v_2 + \Psi_{12}}{\gamma_2}, \Theta \frac{v_3 + \Psi_{13}}{\gamma_3}; \Upsilon \Theta \frac{\Psi_{23}}{\gamma_2 \gamma_3}\right]$$

$$+ \Upsilon \, \omega_2 S_0^{(2)} \exp\left(v_4 + \frac{\gamma_4^2}{2}\right) N_2 \left[\Upsilon \frac{\ln\left(\frac{S_0^{(2)}}{\omega_2 K_2}\right) + \omega(2,1) + v_5 + \Phi_{12}}{\gamma_5}, \Theta \frac{-v_3 + \Phi_{13}}{\gamma_3}; \Upsilon \Theta \frac{\Phi_{23}}{\gamma_5 \gamma_3}\right]$$

$$- \Upsilon \, B(0,T) \left( K_1 N_2 \left[\Upsilon \frac{\ln\left(\frac{S_0^{(1)}}{\omega_1 K_1}\right) + \omega(1,2) + v_1^{(B)}}{\gamma_2}, \Theta \frac{v_2^{(B)}}{\gamma_3}; \Upsilon \Theta \frac{\Psi_{23}}{\gamma_2 \gamma_3}\right]\right.$$

$$\left. + K_2 N_2 \left[\Upsilon \frac{\ln\left(\frac{S_0^{(2)}}{\omega_2 K_2}\right) + \omega(2,1) + v_3^{(B)}}{\gamma_5}, \Theta \frac{-v_2^{(B)}}{\gamma_3}; \Upsilon \Theta \frac{\Phi_{23}}{\gamma_5 \gamma_3}\right]\right)$$

*where*

$$\Upsilon = \begin{cases} 1 & \text{if the option is a call} \\ -1 & \text{if the option is a put} \end{cases}$$

$$\Theta = \begin{cases} 1 & \text{if the option is on the maximum} \\ -1 & \text{if the option is on the minimum} \end{cases}$$

$$v_1 = (n_1 + n_{12}) \xi_1 - \left(\lambda_1 \kappa_1 + \delta_1 + \sigma_{S1}^2/2\right) T, \quad v_2 = \bar{\mu}_r + v_1$$

$$v_3 = \left(\lambda_2 \kappa_2 - \lambda_1 \kappa_1 + (\delta_2 - \delta_1) + \left(\sigma_{S2}^2 - \sigma_{S1}^2\right)/2\right) T + (n_1 + n_{12}) \xi_1$$
$$\quad - (n_2 + n_{12}) \xi_2$$

$$v_4 = (n_2 + n_{12}) \xi_2 - \left(\lambda_2 \kappa_2 + \delta_2 + \sigma_{S2}^2/2\right) T, \quad v_5 = \bar{\mu}_r + v_4$$

$$\gamma_1 = \left(\sigma_{S1}^2 T + (n_1 + n_{12}) \varepsilon_1^2\right)^{1/2}$$

$$\gamma_2 = \left(\left(2\bar{\sigma}_r^2 (1 + \rho_{1.2}) + 2\bar{\sigma}_r \sigma_{S1} (\rho_{1.3} + \rho_{2.3}) + \sigma_{S1}^2\right) T + (n_1 + n_{12}) \varepsilon_1^2\right)^{1/2}$$

$$\gamma_3 = \left(\left(\sigma_{S1}^2 - 2\sigma_{S1} \sigma_{S2} \rho_{3.4} + \sigma_{S2}^2\right) T + (n_1 + n_{12}) \varepsilon_1^2 + (n_2 + n_{12}) \varepsilon_2^2\right)^{1/2}$$

$$\gamma_4 = \left(\sigma_{S2}^2 T + (n_2 + n_{12})\,\varepsilon_2^2\right)^{1/2}$$

$$\gamma_5 = \left(\left(2\bar{\sigma}_r^2\,(1+\rho_{1.2}) + 2\bar{\sigma}_r\sigma_{S2}\,(\rho_{1.4}+\rho_{2.4}) + \sigma_{S2}^2\right)T + (n_2+n_{12})\,\varepsilon_2^2\right)^{1/2}$$

$$\Psi_{12} = \sigma_{S1}^2 T + \bar{\sigma}_r\sigma_{S1}\,(\rho_{1.3}+\rho_{2.3})\,T + (n_1+n_{12})\,\varepsilon_1^2, \quad \Psi_{13} = \left(\sigma_{S1}^2 - \sigma_{S1}\sigma_{S2}\rho_{3.4}\right)T$$
$$+ (n_1+n_{12})\,\varepsilon_1^2$$

$$\Psi_{23} = \left(\bar{\sigma}_r\,(1+\rho_{1.2})\,(\sigma_{S1}\rho_{1.3} - \sigma_{S2}\rho_{1.4}) + \bar{\sigma}_r\sigma_{2|1}\,(\sigma_{S1}\rho_{2.3|1} - \sigma_{S2}\rho_{2.4|1}) + \sigma_{S1}^2 - \sigma_{S1}\sigma_{S2}\rho_{3.4}\right)T$$
$$+ (n_1+n_{12})\,\varepsilon_1^2$$

$$\Phi_{12} = \sigma_{S2}^2 T + \bar{\sigma}_r\sigma_{S2}\,(\rho_{1.4}+\rho_{2.4})\,T + (n_2+n_{12})\,\varepsilon_2^2, \quad \Phi_{13} = \left(\sigma_{S2}^2 - \sigma_{S1}\sigma_{S2}\rho_{3.4}\right)T$$
$$+ (n_2+n_{12})\,\varepsilon_2^2$$

$$\Phi_{23} = \left(\bar{\sigma}_r\,(1+\rho_{1.2})\,(\sigma_{S2}\rho_{1.4} - \sigma_{S1}\rho_{1.3}) + \bar{\sigma}_r\sigma_{2|1}\,(\sigma_{S2}\rho_{2.4|1} - \sigma_{S1}\rho_{2.3|1}) + \sigma_{S2}^2 - \sigma_{S1}\sigma_{S2}\rho_{3.4}\right)T$$
$$+ (n_2+n_{12})\,\varepsilon_2^2$$

$$v_1^{(B)} = \bar{\mu}_r + \frac{1}{a}\left(\sigma_{S1}\rho_{1.3}\sigma_r\,(1+\rho_{1.2}) + \sigma_{S1}\rho_{2.3|1}\,\sigma_r\sigma_{2|1}\right)\left(\left(1-e^{-aT}\right)\Big/a - T\right)$$
$$- \left(\lambda_1\kappa_1 + \delta_1 + \sigma_{S1}^2/2\right)T + (n_1+n_{12})\,\xi_1$$

$$v_2^{(B)} = \frac{1}{a}\left(\sigma_r\,(1+\rho_{1.2})\,(\sigma_{S1}\rho_{1.3} - \sigma_{S2}\rho_{1.4}) + \sigma_r\sigma_{2|1}(\sigma_{S1}\rho_{2.3|1} - \sigma_{S2}\rho_{2.4|1})\right)\left(\left(1-e^{-aT}\right)\Big/a - T\right)$$
$$+ \left(\lambda_2\kappa_2 - \lambda_1\kappa_1 + (\delta_2-\delta_1) + \left(\sigma_{S2}^2 - \sigma_{S1}^2\right)\Big/2\right)T + (n_1+n_{12})\,\xi_1 - (n_2+n_{12})\,\xi_2$$

$$v_3^{(B)} = \bar{\mu}_r + \frac{1}{a}\left(\sigma_{S2}\rho_{1.4}\sigma_r\,(1+\rho_{1.2}) + \sigma_{S2}\rho_{2.4|1}\,\sigma_r\sigma_{2|1}\right)\left(\left(1-e^{-aT}\right)\Big/a - T\right)$$
$$- \left(\lambda_2\kappa_2 + \delta_2 + \sigma_{S2}^2/2\right)T + (n_2+n_{12})\,\xi_2$$

$$\bar{\mu}_r = \frac{r_0}{a}\left(1 - e^{-aT}\right) + a\int_0^T\left(\int_0^t e^{-a(t-u)}b\,(u)\,du\right)dt$$

$$\bar{\sigma}_r = \frac{\sigma_r}{a\sqrt{T}}\,(T - (2/a)\,(1 - \exp(-aT)) + (1/2a)\,(1 - \exp(1 - 2aT)))^{1/2}$$

$$A\,(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

$$C\,(t,T) = -a\int_t^T\left(\int_t^u e^{-a(u-s)}b\,(s)\,ds\right)du + \frac{\sigma_r^2}{a^2}\,(1+\rho_{1.2})$$
$$\left(T + \frac{1}{2a}\left(1 - e^{-2aT}\right) + \frac{2}{a}\left(e^{-aT} - 1\right)\right)$$

$B\,(t,T) = \exp\,(-A\,(t,T)\,r_t + C\,(t,T))$, $0 \le t \le T$, *is the price, at time t, of a zero-coupon bond maturing at time T All other symbols that have not been defined in Sect. 3 are the same as in Sect. 2.*

Proof of Proposition 4 is provided in Appendix 4.

The numerical implementation of Proposition 4 is easy. Using a standard algorithm to compute the bivariate normal integrals, such as Drezner and Wesolowsky (1990),

or its improved version by Genz (2004), option prices are obtained in usually less than one second as convergence of the infinite series is reached with very few iterations for realistic parameters. This should be all the more emphasized as, alternatively, a Monte Carlo simulation for the model under consideration is slow and quite involved to implement.

To illustrate the impact of the introduction of jumps and stochastic interest rates, Proposition 4 is now applied to compute option prices and compare them with prices obtained in a Black-Scholes framework. Two risky assets, paying out no dividend rate, each having weight equal to 1, start at an initial price of 100 (monetary units). The diffusive volatilities of asset 1 (parameter $\sigma_{S1}$) and asset 2 (parameter $\sigma_{S2}$) are 25% and 28%, respectively. As 2-asset options are structured in order to take advantage of correlation, our two stocks are supposed to have negative diffusive correlation. Also, in accordance with observed data, it is assumed that the two main factors driving the yield curve are negatively correlated. The current value of the instantaneous riskless rate (parameter $r_0$) is 3%, its long-term "equilibrium" value (parameter $b(t)$) is a constant also equal to 3%, and the mean-reversion speed (parameter $a$) is 60%. The constant riskless rate used in the Black-Scholes model is the current rate of 3%.

At-the-money and out-of-the-money best-of call option values, with both short and long maturities (3-month and 2-year, respectively), are computed in four different settings:

  (i)   low jump intensity and constant short-term interest rate
  (ii)  low jump intensity and stochastic short-term interest rate
  (iii) high jump intensity and constant short-term interest rate
  (iv)  high jump intensity and stochastic short-term interest rate

Besides, two kinds of jumps can be considered: idiosyncratic and systemic ones. The former refer to company-specific events, not affecting the market nor the industry as a whole. This kind of jumps is diversifiable. Within our model, purely idiosyncratic risk can be specified by setting the parameter $\lambda'_{12}$ equal to zero. In contrast, systemic jumps affect all stocks, such as during stock market crashes. Such a risk can be specified by adjusting the parameter $\lambda'_{12}$.

The magnitude of the parameters $\lambda'_1$ and $\lambda'_2$ priced by the option dealer can be interpreted as a measure of the imperfect diversification of his/her portfolio. If the latter is adequately diversified, those parameters should be close to zero. In this case, the compound Poisson processes driving the discontinuous dynamics of the two stocks tend to have the same intensity $\lambda'_{12}$.

The magnitude of the parameter $\lambda'_{12}$ can be interpreted as a measure of imperfect hedging due to the discontinuities in the underlying assets' paths. As imperfect hedging is almost surely inevitable, $\lambda'_{12}$ should always be positive. Too low a value for $\lambda'_{12}$ should be regarded with suspicion by risk managers.

The numerical values obtained for option prices are reported in Table 7. It is assumed that the option dealer's portfolio is perfectly diversified, so that he/she prices no idiosyncratic risk.

One can notice that the impact of jumps is substantial, especially on out-of-the-money short-maturity options. Not taking the stochastic nature of interest rates into account, the price increase for the latter type of contract, compared to the

**Table 7** Prices of two-asset best-of call options with systemic jumps only[a]

| | 3-month expiry at-the-money 2-asset best-of call option | 3-month expiry out-of-the-money 2-asset best-of call option ($K_1 = K_2 = 110$) | 2-year expiry at-the-money 2-asset best-of call option | 2-year expiry out-of-the-money 2-asset best-of call option ($K_1 = K_2 = 110$) |
|---|---|---|---|---|
| Black-Scholes | 10.684 | 4.266 | 33.439 | 26.280 |
| Low intensity jumps constant short-term interest rate | 11.335 | 5.113 | 36.116 | 29.036 |
| Low intensity jumps stochastic short-term interest rate | 11.312 | 5.084 | 34.065 | 27.062 |
| High intensity jumps constant short-term interest rate | 12.961 | 6.831 | 40.643 | 34.370 |
| High intensity jumps stochastic short-term interest rate | 12.934 | 6.804 | 38.754 | 32.296 |

[a] Prices were computed by means of Proposition 4. Low intensity jumps are specified by setting $\lambda'_1 = 0$, $\lambda'_2 = 0$ and $\lambda'_{12} = 0.35$. High intensity jumps are specified by setting $\lambda'_1 = 0$, $\lambda'_2 = 0$ and $\lambda'_{12} = 1$. The jump parameters are $\xi_1 = \xi_2 = 0$, $\varepsilon_1 = 0.2$ and $\varepsilon_2 = 0.3$. The stochastic short-term interest rate is assumed to have a volatility of 12%. The correlation parameters driving the Brownian motions are the following: $\rho_{1.2} = -0.25$, $\rho_{1.3} = -0.25$, $\rho_{1.4} = 0.26$, $\rho_{2.3} = 0.24$, $\rho_{2.4} = -0.28$, $\rho_{3.4} = -0.4$. All other parameters and contract specifications are given in Sect. 3 after the end of Proposition 4

Black-Scholes value, is equal to 19.85% when jumps have low intensity, and it reaches 60.13% when jumps have high intensity. This simple example hints at the extent to which certain options may be underpriced when assuming that asset paths are purely continuous. At the other end of the options' spectrum, when jumps have low intensity and the contract is at-the-money, the price increases are "only" 6.09% (3-month expiry) and 8% (2-year expiry). It must be stressed that, while the increase in the option premium resulting from the introduction of jumps is a growing function of time-to-expiry, it is nevertheless primarily dependent on the moneyness of the contract. Obviously, these features make jump-diffusion modeling an appropriate candidate to account for the famous "smile" effect observed in the options' markets.

If we now look at the impact of stochastic interest rates, we can see that it is rather negligible for short maturities. However, as maturity increases, it becomes quite significant. Thus, the discrepancy between prices computed with jumps and prices computed with jumps and stochastic interest rates is as high as 6% for 2-year-expiry options, whether at-the-money or out-of-the-money, although the volatility parameter assigned to the short rate is quite low. It is therefore spurious to ignore the fact that interest rates are stochastic, not only when dealing with fixed income products, as is unanimously acknowledged, but also when dealing with equity options, which is less frequently taken into consideration in practice. The fact that option prices decrease due to the introduction of stochastic interest rates in Table 7 is not true in general, but relies on

the specification of parameters $r_0$, $b(t)$ and $a$; if one takes $r_0 < b(t)$, best-of call option prices will increase at a rate proportional to $a$.

## 4 Conclusion

In this paper, fully explicit valuation formulae are obtained for knock-in/out options on the maximum or the minimum of up to four assets, as well as for regular best-of/worst-of options in a framework allowing for correlated jumps and stochastic interest rates. It is shown how analytical tractability can be attained in a context where numerical approximations are often alleged to be the only resort. However, a number of issues are only touched upon. In particular, the formulae in this paper inevitably rely on the quality of the parameter estimation of the underlying model. This is a particularly important matter in the case of multiasset options as correlations between asset prices are notoriously unstable. Cointegration and copulae are two alternative approaches that have been shown to be generally more robust than linear correlation to measure the linkage between several financial prices, but there is currently no known way to obtain analytical prices and hedge ratios using these statistical techniques. One can also try a stochastic correlation model, as explained in Da Fonseca et al. (2007). Eventually, the demand for closed form formulae from practitioners remains strong, at least as reliable benchmarks against which they can test more general models.

## Appendix 1

Proof of Proposition 1 is provided for an up-and-in put on the minimum of four assets. The method involved in the other cases is identical.

Within an arbitrage-free framework, the fair value of an option, in a complete market, is the expectation at present time, under the equivalent martingale measure, of its discounted payoff at expiry (Harrison and Pliska 1981).

Let $(a, b, c, d)$ be a sequence of distinct positive integers taking values in the set $\{1, 2, 3, 4\}$. Define:

$$I\{a, b, c, d\} \triangleq$$
$$\mathbb{I}\left\{ \sup_{t_0 \leq t \leq T} \omega_a S_t^{(a)} \geq H_a, \omega_a S_T^{(a)} \leq K_a, \omega_a S_T^{(a)} \leq \omega_b S_T^{(b)}, \omega_a S_T^{(a)} \leq \omega_c S_T^{(c)}, \right.$$
$$\left. \omega_a S_T^{(a)} \leq \omega_d S_T^{(d)} \right\}$$

Then, the valuation of an up-and-in put on the minimum of four assets requires the computation of the following expectations:

$$\mathrm{E}_Q\left[I\{a, b, c, d\}\right] \tag{15}$$

$$\mathrm{E}_Q\left[S_T^{(a)} I\{a, b, c, d\}\right] \tag{16}$$

for the following sequence of parameters:

$$(a = 1, b = 2, c = 3, d = 4), (a = 2, b = 1, c = 3, d = 4),$$
$$(a = 3, b = 1, c = 2, d = 4), (a = 4, b = 1, c = 2, d = 3)$$

$E_Q[.]$ is the expectation operator under the risk-neutral measure defined by taking the riskless rate as numeraire. Actually, it suffices to calculate (15). It will be shown later in this section how appropriate changes of numeraire yield (16).

Let $P(.)$ stand for the probability operator. Under the historical measure, using elementary conditioning, one can write:

$$P\left(\sup_{t_0 \leq t \leq T} \omega_a S_t^{(a)} \geq H_a, \omega_a S_T^{(a)} \leq K_a, \omega_a S_T^{(a)} \leq \omega_b S_T^{(b)}, \omega_a S_T^{(a)} \leq \omega_c S_T^{(c)}, \omega_a S_T^{(a)} \leq \omega_d S_T^{(d)}\right)$$

$$\triangleq P_a\left(\mu_a^{(P)}, \mu_b^{(P)}, \mu_c^{(P)}, \mu_d^{(P)}\right) \text{ where } \mu_i^{(P)} = \alpha_i - \delta_i - \frac{\sigma_i^2}{2}, \ i \in \{a, b, c, d\}$$

$$= P\left(\sup_{t_0 \leq t \leq T} \omega_a S_t^{(a)} \geq H_a, \omega_a S_T^{(a)} \leq K_a\right)$$

$$\times P\left(\omega_b S_T^{(b)} \geq \omega_a S_T^{(a)} \left| \sup_{t_0 \leq t \leq T} \omega_a S_t^{(a)} \geq H_a, \omega_a S_T^{(a)} \leq K_a\right.\right)$$

$$\times P\left(\omega_c S_T^{(c)} \geq \omega_a S_T^{(a)} \left| \omega_b S_T^{(b)} \geq \omega_a S_T^{(a)}, \sup_{t_0 \leq t \leq T} \omega_a S_t^{(a)} \geq H_a, \omega_a S_T^{(a)} \leq K_a\right.\right)$$

$$\times P\left(\omega_d S_T^{(d)} \geq \omega_a S_T^{(a)} \left| \omega_c S_T^{(c)} \geq \omega_a S_T^{(a)}, \omega_b S_T^{(b)} \geq \omega_a S_T^{(a)}, \sup_{t_0 \leq t \leq T} \omega_a S_t^{(a)} \geq H_a, \omega_a S_T^{(a)} \leq K_a\right.\right)$$

(17)

Let $X_t^{(i)} \triangleq \ln\left(\frac{S_t^{(i)}}{S_0^{(i)}}\right)$. Integrating with respect to all admissible values of $X_T^{(a)}$, $X_T^{(b)}$, $X_T^{(c)}$ and $X_T^{(d)}$, and applying the Markov property of diffusion processes $X_t^{(b)}$, $X_t^{(c)}$ and $X_t^{(d)}$, Eq. 17 turns into:

$$P_a\left(\mu_a^{(P)}, \mu_b^{(P)}, \mu_c^{(P)}, \mu_d^{(P)}\right)$$

$$= \int_{-\infty}^{k_a} \int_{x_a+\omega(a,b)}^{\infty} \int_{x_a+\omega(a,c)}^{\infty} \int_{x_a+\omega(a,d)}^{\infty} P\left(\sup_{t_0 \leq t \leq T} X_t^{(a)} \geq h_a, X_T^{(a)} \in dx_a\right)$$

$$P\left(X_T^{(b)} \in dx_b \left| X_T^{(a)} \in dx_a\right.\right) P\left(X_T^{(c)} \in dx_c \left| X_T^{(b)} \in dx_b, X_T^{(a)} \in dx_a\right.\right)$$

$$P\left(X_T^{(d)} \in dx_d \left| X_T^{(c)} \in dx_c, X_T^{(b)} \in dx_b, X_T^{(a)} \in dx_a\right.\right) dx_d dx_c dx_b dx_a \quad (18)$$

In Eq. 18, $P\left(\sup_{t_0 \leq t \leq T} X_t^{(a)} \geq h_a, X_T^{(a)} \in dx_a\right)$ and $P\left(X_T^{(b)} \in dx_b \left| X_T^{(a)} \in dx_a\right.\right)$ are the derivatives of known cumulative distribution functions (cf., e.g., Karatzas and Shreve 1991). The densities $P\left(X_T^{(c)} \in dx_c \left| X_T^{(b)} \in dx_b, X_T^{(a)} \in dx_a\right.\right)$ and $P(X_T^{(d)} \in$

$dx_d \left| X_T^{(c)} \in dx_c, X_T^{(b)} \in dx_b, X_T^{(a)} \in dx_a \right.$), however, are not standard results. To find the former, let us define $X_a$, $X_b$ and $X_c$ as three correlated standardized normal random variables and write $X_c$ as a linear combination of $X_a$ and two independent standardized normal random variables $\bar{X}_b$ and $\bar{X}_c$ defined on the same probability space as $X_b$ and $X_c$:

$$X_c = \lambda_1 X_a + \lambda_2 \bar{X}_b + \lambda_3 \bar{X}_c \qquad (19)$$

The real coefficients $\lambda_1$, $\lambda_2$ and $\lambda_3$ are to be determined. From the definition of linear correlation, we obtain:

$$\rho_{a \cdot c} = \frac{\text{cov}(X_a, X_c)}{\sigma_a \sigma_c} = \text{cov}(X_a, \lambda_1 X_a) \leftrightarrow \lambda_1 = \rho_{a \cdot c} \qquad (20)$$

$$\rho_{b \cdot c} = \frac{\text{cov}(X_b, X_c)}{\sigma_b \sigma_c} = \text{cov}(\rho_{a \cdot b} X_a, \rho_{a \cdot c} X_a) + \text{cov}\left(\sigma_{b|a} \bar{X}_b, \lambda_2 \bar{X}_b\right)$$
$$\leftrightarrow \lambda_2 = \frac{\rho_{b \cdot c} - \rho_{a \cdot b} \rho_{a \cdot c}}{\sigma_{b|a}} \triangleq \rho_{b \cdot c|a} \qquad (21)$$

The coefficient $\rho_{b \cdot c|a}$ in (21) is the correlation between $X_b$ and $X_c$ conditional on $X_a$.

Since $X_c \sim N(0, 1)$ and $X_a$, $\bar{X}_b$ and $\bar{X}_c$ are mutually independent, we have:

$$\sqrt{\rho_{a \cdot c}^2 + \rho_{b \cdot c|a}^2 + \lambda_3^2} = 1 \leftrightarrow \lambda_3 = \sqrt{1 - \rho_{a \cdot c}^2 - \rho_{b \cdot c|a}^2} \triangleq \sigma_{c|a \cdot b} \qquad (22)$$

where $\sigma_{c|a \cdot b}$ is the standard deviation of $X_c$ conditional on $X_a$ and $X_b$.

Equation 19 can now be rewritten as:

$$X_c = \rho_{a \cdot c} X_a + \rho_{b \cdot c|a} \left( \frac{X_b - \rho_{a \cdot b} X_a}{\sigma_{b|a}} \right) + \sigma_{c|a \cdot b} \bar{X}_c \qquad (23)$$

Thus, given $X_a$ and $X_b$, $X_c$ is normally distributed with mean $\rho_{a \cdot c} X_a + \rho_{b \cdot c|a} \left( \frac{X_b - \rho_{a \cdot b} X_a}{\sigma_{b|a}} \right)$ and standard deviation $\sigma_{c|a \cdot b}$. Hence, the density of $X_c$ conditional on $X_a$ and $X_b$ writes:

$$f_{X_c | X_a, X_b} (x_c | X_a \in dx_a, X_b \in dx_b)$$
$$= \exp\left( -\frac{1}{2\sigma_{c|a \cdot b}^2} \left( x_c - \rho_{a \cdot c} x_a - \rho_{b \cdot c|a} \left( \frac{x_b - \rho_{a \cdot b} x_a}{\sigma_{b|a}} \right) \right)^2 \right) / \left( \sigma_{c|a \cdot b} \sqrt{2\pi} \right)$$
$$\qquad (24)$$

Considering now four correlated standard normal random variables $X_a$, $X_b$, $X_c$ and $X_d$, it can be shown in a similar manner that:

$$X_d = \rho_{a \cdot d} X_a + \rho_{b \cdot d|a} \left( \frac{X_b - \rho_{a \cdot b} X_a}{\sigma_{b|a}} \right)$$
$$+ \rho_{c \cdot d|a \cdot b} \left( X_c - \rho_{a \cdot c} X_a - \rho_{b \cdot c|a} \frac{X_b - \rho_{a \cdot b} X_a}{\sigma_{b|a}} \right)$$
$$+ \sigma_{d|a \cdot b \cdot c} \bar{X}_d \tag{25}$$

where $\bar{X}_d$ is an independent standard normal random variable defined on the same probability space as $X_d$; the coefficient $\rho_{c \cdot d|a \cdot b} = \frac{\rho_{c \cdot d} - \rho_{a \cdot c} \rho_{a \cdot d} - \rho_{b \cdot c|a} \rho_{b \cdot d|a}}{\sigma_{c|a \cdot b}}$ is the correlation between $X_c$ and $X_d$ conditional on $X_a$ and $X_b$; and the coefficient $\sigma_{d|a \cdot b \cdot c} = \left( 1 - \rho_{a \cdot d}^2 - \rho_{b \cdot d|a}^2 - \rho_{c \cdot d|a \cdot b}^2 \right)^{1/2}$ is the standard deviation of $X_d$ conditional on $X_a$, $X_b$ and $X_c$.

Hence, the density of $X_d$ conditional on $X_a$, $X_b$ and $X_c$ writes:

$$f_{X_d|X_a, X_b, X_c} (x_d | X_a \in dx_a, X_b \in dx_b, X_c \in dx_c)$$
$$= \frac{1}{\sigma_{d|a \cdot b \cdot c} \sqrt{2\pi}} \times \exp\left( -\frac{1}{2\sigma_{d|a \cdot b \cdot c}^2} \left( x_d - \rho_{a \cdot d} x_a - \rho_{b \cdot d|a} \left( \frac{x_b - \rho_{a \cdot b} x_a}{\sigma_{b|a}} \right) \right. \right.$$
$$\left. \left. - \rho_{c \cdot d|a \cdot b} \left( x_c - \rho_{a \cdot c} x_a - \rho_{b \cdot c|a} \left( \frac{x_b - \rho_{a \cdot b} x_a}{\sigma_{b|a}} \right) \right) \right)^2 \right) \tag{26}$$

It is apparent that this method of generating joint densities of several correlated normal random variables easily extends to higher dimensions.

Substituting (24) and (26) into (18) and then performing the necessary calculations, one can obtain:

$$P_a \left( \mu_a^{(P)}, \mu_b^{(P)}, \mu_c^{(P)}, \mu_d^{(P)} \right)$$
$$= \exp\left( \frac{2\mu_a^{(P)}}{\sigma_a^2} h_a \right) \int_{-\infty}^{\frac{k_a - 2h_a - \mu_a^{(P)} T}{\sigma_a \sqrt{T}}} \frac{\exp\left( -x^2/2 \right)}{\sqrt{2\pi}}$$
$$N_3 \left[ \frac{\psi(a, b) - x}{\sigma_{b|a} / (\sigma_a/\sigma_b - \rho_{a \cdot b})}, \frac{\psi(a, c) - x}{\sigma_{c|a} / (\sigma_a/\sigma_c - \rho_{a \cdot c})}, \right.$$
$$\left. \frac{\psi(a, d) - x}{\sigma_{d|a \cdot b} / (\sigma_a/\sigma_d - \rho_{a \cdot d})}; \frac{\rho_{b \cdot c|a}}{\sigma_{c|a}}, \frac{\rho_{b \cdot d|a}}{\sigma_{d|a \cdot b}}, \frac{\rho_{c \cdot d|a \cdot b}}{\sigma_{d|a \cdot b}} \right] dx \tag{27}$$

Under $Q$, a routine change of measure yields: $\mu_i^{(Q)} = r - \delta_i - \frac{\sigma_i^2}{2}$, $i \in \{a, b, c, d\}$, so that:

$$E_Q [I \{a, b, c, d\}] = P_a \left( \mu_a^{(Q)}, \mu_b^{(Q)}, \mu_c^{(Q)}, \mu_d^{(Q)} \right) \tag{28}$$

We now turn to the calculation of $E_Q\left[S_T^{(a)} I\{a, b, c, d\}\right]$. In order to be able to use the multidimensional version of Girsanov's theorem, we apply the previous orthogonalisation results to express the stochastic differential equations for $S_t^{(1)}$, $S_t^{(2)}$, $S_t^{(3)}$, $S_t^{(4)}$ under $Q$ in terms of four mutually independent standard Brownian motions $W_t^{(1)}$, $\bar{W}_t^{(2)}$, $\bar{W}_t^{(3)}$, $\bar{W}_t^{(4)}$, defined on the same probability space as $W_t^{(1)}$, $W_t^{(2)}$, $W_t^{(3)}$, $W_t^{(4)}$:

$$\frac{dS_t^{(1)}}{S_t^{(1)}} = (r - \delta_1)\, dt + \sigma_1 dW_t^{(1)} \tag{29}$$

$$\frac{dS_t^{(2)}}{S_t^{(2)}} = (r - \delta_2)\, dt + \sigma_2 \rho_{1.2} dW_t^{(1)} + \sigma_2 \sigma_{2|1}\, d\bar{W}_t^{(2)} \tag{30}$$

$$\frac{dS_t^{(3)}}{S_t^{(3)}} = (r - \delta_3)\, dt + \sigma_3 \rho_{1.3} dW_t^{(1)} + \sigma_3 \rho_{2.3|1}\, d\bar{W}_t^{(2)} + \sigma_3 \sigma_{3|1.2}\, d\bar{W}_t^{(3)} \tag{31}$$

$$\frac{dS_t^{(4)}}{S_t^{(4)}} = (r - \delta_4)\, dt + \sigma_4 \rho_{1.4} dW_t^{(1)} + \sigma_4 \rho_{2.4|1}\, d\bar{W}_t^{(2)} + \sigma_4 \rho_{3.4|1.2}\, d\bar{W}_t^{(3)}$$
$$+ \sigma_4 \sigma_{4|1.2.3}\, d\bar{W}_t^{(4)} \tag{32}$$

The first expectation to calculate is: $E_Q\left[S_T^{(1)} \mathbb{I}\{a = 1, b = 2, c = 3, d = 4\}\right]$. A standard use of Girsanov's theorem yields:

$E_Q\left[S_T^{(1)} \mathbb{I}\{a = 1, b = 2, c = 3, d = 4\}\right] = E_{Q_{F1}}\left[\mathbb{I}\{a = 1, b = 2, c = 3, d = 4\}\right]$

where $Q_{F1}$ is the risk-neutral measure under which asset $S^{(1)}$ is chosen as numeraire (often called the $S_t^{(1)}$−forward neutral measure), so that $dW_t^{(1)} = d\left(W_t^{(1)F_1} + \sigma_1 t\right)$, where $W_t^{(1)F_1}$ is a standard Brownian motion under $Q_{F1}$.

The second expectation to calculate is: $E_Q\left[S_T^{(2)} \mathbb{I}\{a = 2, b = 1, c = 3, d = 4\}\right]$. After solving Eq. 30, one can define a new measure $Q_{F2}$, equivalent to $Q$, such that: $dW_t^{(1)} = d\left(W_t^{(1)F_2} + \sigma_2 \rho_{1.2} t\right)$ and $d\bar{W}_t^{(2)} = d\left(\bar{W}_t^{(2)F_2} + \sigma_2 \sigma_{2|1}\, t\right)$, where $W_t^{(1)F_2}$ and $\bar{W}_t^{(2)F_2}$ are independent standard Brownian motions under $Q_{F2}$.

Similarly, to calculate the third expectation, that is: $E_Q\left[S_T^{(3)} \mathbb{I}\{a = 3, b = 1, c = 2, d = 4\}\right]$, one can solve Eq. 31 and then identify a new measure $Q_{F3}$, equivalent to $Q$, such that: $dW_t^{(1)} = d\left(W_t^{(1)F_3} + \sigma_3 \rho_{1.3} t\right)$, $d\bar{W}_t^{(2)} = d\left(\bar{W}_t^{(2)F_3} + \sigma_3 \rho_{2.3|1} t\right)$ and $d\bar{W}_t^{(3)} = d\left(\bar{W}_t^{(3)F_3} + \sigma_3 \sigma_{3|1.2} t\right)$, where the processes $W_t^{(1)F_3}$, $\bar{W}_t^{(2)F_3}$, $\bar{W}_t^{(3)F_3}$ are mutually independent standard Brownian motions under $Q_{F3}$.

Eventually, to calculate the last expectation, that is: $E_Q\left[S_T^{(4)} \mathbb{I}\{a = 4, b = 1, c = 2, d = 3\}\right]$, one can solve Eq. 32 and, again, use Girsanov's multidimensional theorem to turn to the $S_t^{(4)}$−forward measure, $Q_{F4}$ by the following transformations:

$$dW_t^{(1)} = d\left(W_t^{(1)F_4} + \sigma_4 \rho_{1.4} t\right), \quad d\bar{W}_t^{(2)} = d\left(\bar{W}_t^{(2)F_4} + \sigma_4 \rho_{2.4|1} t\right),$$

$d\bar{W}_t^{(3)} = d\left(\bar{W}_t^{(3)F_4} + \sigma_4\rho_{3.4|1.2}\,t\right)$ and $d\bar{W}_t^{(4)} = d\left(\bar{W}_t^{(4)F_4} + \sigma_4\sigma_{4|1.2.3}\,t\right)$, where the processes $W_t^{(1)F_4}$, $\bar{W}_t^{(2)F_4}$, $\bar{W}_t^{(3)F_4}$, $\bar{W}_t^{(4)F_4}$ are mutually independent standard Brownian motions under $Q_{F4}$.

Then, by substituting the relevant Brownian motions $W_t^{(1)F_a}$ and $\bar{W}_t^{(a)F_a}$, $a \in \{2, 3, 4\}$, into Eqs. 29–32, one easily checks that:

$$\begin{aligned}
\mathrm{E}_Q\left[S_T^{(a)} I\{a, b, c, d\}\right] &= \mathrm{E}_{Q_{Fa}}\left[I\{a, b, c, d\}\right] \\
&= \mathrm{P}_a\left(\begin{array}{l}
\mu_a = r - \delta_a + \frac{\sigma_a^2}{2}, \mu_b = r - \delta_b + \sigma_a\sigma_b\rho_{a\cdot b} - \frac{\sigma_b^2}{2}, \mu_c = r - \delta_c + \sigma_a\sigma_c\rho_{a\cdot c} - \frac{\sigma_c^2}{2} \\
\mu_d = r - \delta_d + \sigma_a\sigma_d\rho_{a\cdot d} - \frac{\sigma_d^2}{2}
\end{array}\right)
\end{aligned}$$

$$(33)$$

## Appendix 2

First, proof of Proposition 2 is outlined. The case of a put option on the minimum of four assets is dealt with. The three other cases can be handled in the same way. The valuation of a put option on the minimum of four assets requires the computation of the following joint probability:

$$P\left(\begin{array}{l}
\ln\left(\frac{S_T^{(a)}}{S_0^{(a)}}\right) \le \ln\left(\frac{K_a}{\omega_a S_0^{(a)}}\right), \ln\left(\frac{S_T^{(a)} S_0^{(b)}}{S_T^{(b)} S_0^{(a)}}\right) \le \ln\left(\frac{\omega_b S_0^{(b)}}{\omega_a S_0^{(a)}}\right), \\
\ln\left(\frac{S_T^{(a)} S_0^{(c)}}{S_T^{(c)} S_0^{(a)}}\right) \le \ln\left(\frac{\omega_c S_0^{(c)}}{\omega_a S_0^{(a)}}\right), \ln\left(\frac{S_T^{(a)} S_0^{(d)}}{S_T^{(d)} S_0^{(a)}}\right) \le \ln\left(\frac{\omega_d S_0^{(d)}}{\omega_a S_0^{(a)}}\right)
\end{array}\right)$$

$$(34)$$

for the following sequence of parameters: $(a = 1, b = 2, c = 3, d = 4)$, $(a = 2, b = 1, c = 3, d = 4)$, $(a = 3, b = 1, c = 2, d = 4)$, $(a = 4, b = 1, c = 2, d = 3)$. These are cumulative distribution functions of four normally distributed random variables and they are quadrivariate normal. By solving Eqs. 29–32, the expectation and the variance of $\ln\left(\frac{S_T^{(i)} S_0^{(j)}}{S_T^{(j)} S_0^{(i)}}\right)$, $(i, j) \in \{a, b, c, d\}$, $i \ne j$, are easily found to be: $(\mu_i - \mu_j)\,T$ and $\left(\sigma_i^2 - 2\sigma_i\sigma_j\rho_{i.j} + \sigma_j^2\right)T$, respectively. It is then straightforward to deduce the correlation coefficients between $\ln\left(\frac{S_T^{(a)}}{S_{t_0}^{(a)}}\right)$, $\ln\left(\frac{S_T^{(a)} S_0^{(b)}}{S_T^{(b)} S_0^{(a)}}\right)$, $\left(\frac{S_T^{(a)} S_0^{(c)}}{S_T^{(c)} S_0^{(a)}}\right)$ and $\ln\left(\frac{S_T^{(a)} S_0^{(d)}}{S_T^{(d)} S_0^{(a)}}\right)$. Applying the changes of measure defined in the proof of Proposition 1, one can obtain Proposition 2.

It is important to notice that the cases of an up-and-in/out call and a down-and-in/out put can be easily dealt with using Proposition 1 and Proposition 2. This is achieved by performing appropriate decompositions of the relevant payoffs. For example, take an up-and-in call on the maximum of four assets. The following expectation has to be worked out:

$$E_Q\left[\left(\omega_a S_T^{(a)} - K_a\right)\mathbb{I}\left\{\begin{array}{l}\omega_a S_T^{(a)} > K_a,\ \sup_{0\leq t\leq T}\omega_a S_t^{(a)} > H_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \\ \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)} > \omega_d S_T^{(d)}\end{array}\right\}\right]$$
(35)

The indicator function inside the expectation operator in (35) can be expressed as the following difference:

$$\mathbb{I}\left\{\omega_a S_T^{(a)} > K_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)} > \omega_d S_T^{(d)}\right\}$$

$$-\left(\mathbb{I}\left\{\sup_{0\leq t\leq T}\omega_a S_t^{(a)} < H_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)} > \omega_d S_T^{(d)}\right\}\right.$$

$$-\mathbb{I}\left\{\omega_a S_T^{(a)} < K_a,\ \sup_{0\leq t\leq T}\omega_a S_t^{(a)} < H_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)},\right.$$

$$\left.\left.\omega_a S_T^{(a)} > \omega_d S_T^{(d)}\right\}\right) \triangleq I_1 - (I_2 - I_3)$$
(36)

$I_1$ is the indicator function of the set of events defining a call on the maximum of four assets; the expectation $E_Q\left[\left(\omega_a S_T^{(a)} - K_a\right)I_1\right]$ is therefore given by Proposition 2.

$I_3$ is the indicator function of the set of events defining an up-and-out put on the maximum of four assets; the expectation $E_Q\left[\left(\omega_a S_T^{(a)} - K_a\right)I_3\right]$ is thus given by Proposition 2 (to value a put on the maximum of four assets) minus Proposition 1 (to value an up-and-in put on the maximum of four assets).

$I_2$ is identical to:

$$\mathbb{I}\left\{\omega_a S_T^{(a)} < H_a,\ \sup_{0\leq t\leq T}\omega_a S_t^{(a)} < H_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)}\right.$$

$$\left. > \omega_d S_T^{(d)}\right\}$$
(37)

The expectation $E_Q\left[\left(\omega_a S_T^{(a)} - K_a\right)I_2\right]$ is thus given by Proposition 2 (to value a put on the maximum of four assets) minus Proposition 1 (to value an up-and-in put on the maximum of four assets), taking $k_i = h_i$ in $P_i\left(\mu_a, \mu_b, \mu_c, \mu_d\right), i \in \{1, 2, 3, 4\}$.

Similarly, the valuation of a down-and-in put on the maximum of four assets requires the calculation of:

$$E_Q\left[\left(K_a - \omega_a S_T^{(a)}\right)\right.$$

$$\mathbb{I}\left\{\omega_a S_T^{(a)} < K_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)} > \omega_d S_T^{(d)}\right\}$$

$$-\left(\mathbb{I}\left\{\inf_{0\leq t\leq T}\omega_a S_t^{(a)} > H_a, \omega_a S_T^{(a)} > H_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)} > \omega_d S_T^{(d)}\right\}\right.$$

$$- \mathbb{I}\left\{\inf_{0\leq t\leq T} \omega_a S_t^{(a)} > H_a, \omega_a S_T^{(a)} > K_a, \omega_a S_T^{(a)} > \omega_b S_T^{(b)}, \omega_a S_T^{(a)} > \omega_c S_T^{(c)}, \omega_a S_T^{(a)} > \omega_d S_T^{(d)}\right\}\right)\right]\right]$$

(38)

and, again, all three expectations are obtained by using Propositions 1 and 2.

## Appendix 3

Proof of Propositions 3 is outlined. By definition of conditional probability, the joint cumulative distribution function of $X_a$, $X_b$, $X_c$ and $X_d$ is given by:

$$N_4\left(u_a, u_b, u_c, u_d; \rho_{a\cdot b}, \rho_{a\cdot c}, \rho_{a\cdot d}, \rho_{b\cdot c}, \rho_{b\cdot d}, \rho_{c\cdot d}\right)$$

$$= \int_{-\infty}^{u_a} \int_{-\infty}^{u_b} \int_{-\infty}^{u_c} \int_{-\infty}^{u_d} \mathrm{P}\left(X_a \in dx_a\right) \mathrm{P}\left(X_b \in dx_b \mid X_a \in dx_a\right) \mathrm{P}(X_c \in dx_c \mid X_b \in dx_b,$$

$$X_a \in dx_a) \mathrm{P}\left(X_d \in dx_d \mid X_c \in dx_c, X_b \in dx_b, X_a \in dx_a\right) dx_d dx_c dx_b dx_a \quad (39)$$

Substituting the relevant conditional densities given by (24) and (26) into (39), and then performing a little algebra, one can obtain Proposition 3.

Note that the representation of the trivariate normal integral in terms of the bivariate normal integral in Eq. 4 was provided by Owen as early as 1956 (Owen 1956).

We focus on the numerical implementation of the quadrivariate normal integral, as the trivariate normal integral has already been studied (Genz 2004). From Proposition 3, it can be observed that there are three different ways of computing the quadrivariate normal integral: by integrating the trivariate or the bivariate or the univariate normal integral. Extensive testing shows that it is surprisingly simple to obtain a very high degree of accuracy. Indeed, it suffices to integrate the univariate normal integral by means of a sixteen-point Gauss-Legendre quadrature. This is due mainly to the smoothness of the integrand. Even though this implementation implies a triple quadrature, the function evaluations are so elementary and the number of points is so moderate that computational time is less than one second on an ordinary personal computer. To assess the accuracy of this implementation, we first tested the rare cases where analytical results are known in terms of the arcsine function. They are orthant probabilities where the correlation matrix takes on a very specific form. For instance, it can be shown (Kotz et al. 2000) that:

$$N_4\left(0, 0, 0, 0; \rho, 0, \rho, -\rho, -\frac{1}{2}, \rho\right) = \frac{1}{24} + \frac{1}{4\pi} \arcsin(\rho) \quad (40)$$

$$N_4\left(0, 0, 0, 0; \rho, 0, 0, 0, 0, \rho\right) = \left(\frac{1}{4} + \frac{1}{2\pi} \arcsin(\rho)\right)^2 \quad (41)$$

We computed (40), (41) and a couple of other similar cases for hundreds of randomly drawn values of $\rho$ and compared with the results obtained with our implementation of Proposition 3. The results always matched to at least $10^{-12}$ accuracy. To test general correlation matrices, we carried out a second series of tests consisting of a comparison with a powerful adaptive integration scheme referred to as CUHRE (Berntsen et al. 1991).[2] The CUHRE algorithm was selected because it has been shown to be extremely reliable in moderate dimensions (Hahn 2005). For hundreds of randomly drawn correlation matrices, the results always matched to at least $10^{-10}$ accuracy.

In view of such a level of efficiency and accuracy, and considering the simplicity of the quadrature rule, it seems to us that the analytical formulae provided by Proposition 1 and Proposition 2 can be rightfully regarded as "closed form" as the existing valuation formulae involving univariate or bivariate normal distributions.

One can notice that the formulae written down in Proposition 1 and Proposition 2 display a structure and repetitive patterns that make an extension to a greater number of assets relatively easy as far as the analytics are concerned. The real question is how to compute mutivariate normal cumulative distributions when the number of assets grows. This is a big issue in numerical integration (Kotz et al. 2000). In Genz (1992), it is argued that an accuracy to two or three decimal digits can be reached in 1 or 2 s for problems with as many as ten variables.

## Appendix 4

Proof of Proposition 4 is provided for a call on the maximum of $\omega_1 S_T^{(1)}$ and $\omega_2 S_T^{(2)}$. One can expand the value of such an option as follows:

$$
\begin{aligned}
&E_Q\left[\exp\left(-\int_0^T r_t dt\right)\omega_1 S_T^{(1)}\mathbb{I}\left\{\ln\left(\frac{S_T^{(1)}}{S_0^{(1)}}\right)\geq\ln\left(\frac{K_1}{\omega_1 S_0^{(1)}}\right),\ln\left(\frac{S_T^{(1)}S_0^{(2)}}{S_T^{(2)}S_0^{(1)}}\right)\geq\ln\left(\frac{\omega_2 S_0^{(2)}}{\omega_1 S_0^{(1)}}\right)\right\}\right]\\
&-K_1 E_Q\left[\exp\left(-\int_0^T r_t dt\right)\mathbb{I}\left\{\ln\left(\frac{S_T^{(1)}}{S_0^{(1)}}\right)\geq\ln\left(\frac{K_1}{\omega_1 S_0^{(1)}}\right),\ln\left(\frac{S_T^{(1)}S_0^{(2)}}{S_T^{(2)}S_0^{(1)}}\right)\geq\ln\left(\frac{\omega_2 S_0^{(2)}}{\omega_1 S_0^{(1)}}\right)\right\}\right]\\
&+E_Q\left[\exp\left(-\int_0^T r_t dt\right)\omega_2 S_T^{(2)}\mathbb{I}\left\{\ln\left(\frac{S_T^{(2)}}{S_0^{(2)}}\right)\geq\ln\left(\frac{K_2}{\omega_2 S_0^{(2)}}\right),\ln\left(\frac{S_T^{(2)}S_0^{(1)}}{S_T^{(1)}S_0^{(2)}}\right)\geq\ln\left(\frac{\omega_1 S_0^{(1)}}{\omega_2 S_0^{(2)}}\right)\right\}\right]\\
&-K_2 E_Q\left[\exp\left(-\int_0^T r_t dt\right)\mathbb{I}\left\{\ln\left(\frac{S_T^{(2)}}{S_0^{(2)}}\right)\geq\ln\left(\frac{K_2}{\omega_2 S_0^{(2)}}\right),\ln\left(\frac{S_T^{(2)}S_0^{(1)}}{S_T^{(1)}S_0^{(2)}}\right)\geq\ln\left(\frac{\omega_1 S_0^{(1)}}{\omega_2 S_0^{(2)}}\right)\right\}\right]\\
&\triangleq E_1 - K_1.E_2 + E_3 - K_2.E_4
\end{aligned}
\tag{42}
$$

To justify the dynamics of $S_t^{(1)}$ and $S_t^{(2)}$ given by Eqs. 10 and 11, under $Q$, the following lemma is introduced:

---

[2] The implementation of the CUHRE algorithm that we relied on is the one featured in the computer algebra system Maple.

**Lemma 1** *Using the assumptions and notations defined at the beginning of Sect. 3, let $\{S_t, t \geq 0\}$ be a process driven by: $\frac{dS_t}{S_{t_-}} = \mu_t dt + \sigma dW_t + I_t dN_t$, with $0 \leq t \leq T < \infty$, where $\mu_t$ is a continuous $\mathcal{F}_t$-adapted process such that the latter stochastic differential equation possess a unique, strong, global solution. Then, $\tilde{S}_T \triangleq \exp\left(-\int_0^T r_t dt\right) S_T$ is a martingale if and only if $\mu_t = r_t - \lambda \kappa$.*

*Proof of Lemma 1*

$$E\left[\tilde{S}_T \mid \mathcal{F}_t\right]$$

$$= E\left[\exp\left(-\int_0^t r_s ds\right) S_t \exp\left(-\int_t^T (r_s - \mu_s)\, ds\right)\right.$$

$$\left. - \frac{\sigma^2}{2}(T-t) + \sigma(W_T - W_t) \prod_{i=N_t+1}^{N_T} (1 + U_i) \mid \mathcal{F}_t\right]$$

$$= \tilde{S}_t\, E\left[\exp\left(-\int_t^T (r_s - \mu_s)\, ds\right) - \frac{\sigma^2}{2}(T-t)\right.$$

$$\left. + \sigma(W_T - W_t) \prod_{i=1}^{N_T - N_t} \left(1 + U_{N_t+i}\right)\right]$$

$$= \tilde{S}_t \exp\left(-\int_t^T (r_s - \mu_s)\, ds\right) E\left[\prod_{i=N_t+1}^{N_T} (1 + U_i)\right]$$

$$= \tilde{S}_t \exp\left(-\int_t^T (r_s - \mu_s)\, ds + \lambda(T-t) E[U_i]\right)$$

$$= \tilde{S}_t \exp\left(-\int_t^T (r_s - \mu_s)\, ds + \lambda(T-t)\kappa\right)$$

It is clear then that $\mu_t = r_t - \lambda \kappa$ is a necessary condition for the discounted asset value to be a martingale. $\square$

Then, we begin by calculating $E_2$. By taking the stochastic differential of $\exp(at) r_t$ and integrating it on $[0, t]$, Eq. 9 easily yields:

$$r_t = r_0 e^{-at} + a \int_0^t e^{-a(t-u)} b(u) \, du + \sigma_r \int_0^t e^{-a(t-u)} dW_u^{(1)}$$

$$+ \sigma_r \int_0^t e^{-a(t-u)} dW_u^{(2)} \tag{43}$$

The time integral of $r_t$ on $[0, T]$ is normally distributed since $r_t$ is a Gaussian process.

Fubini's theorem yields: $E\left[\int_0^T r_t dt\right] = \bar{\mu}_r$, as given by Proposition 4.

The variance of $\int_0^T r_t dt$ can be written as: $var\left[\int_0^T r_t dt\right] = \int_0^T \int_0^t cov\,[r_t, r_u] \, du \, dt$.

Introduce the following lemma:

**Lemma 2** *Let $\left\{W_u^{(1)}, u \geq 0\right\}$ and $\left\{W_u^{(2)}, u \geq 0\right\}$ be two standard Brownian motions with constant correlation coefficient $\rho$. Let $X_u^{(1)}$ and $X_u^{(2)}$ be two left-continuous processes adapted to the natural filtration, $\mathcal{F}_u$, generated by $W_u^{(1)}$ and $W_u^{(2)}$, and such that $E\left[\int_0^T \left(X_u^{(1)}\right)^2 du\right] < \infty$ and $E\left[\int_0^T \left(X_u^{(2)}\right)^2 du\right] < \infty$. Then, for $0 \leq t \leq T$, we have:*

$$cov\left[\int_0^t X_u^{(1)} dW_u^{(1)}, \int_0^T X_u^{(2)} dW_u^{(2)}\right] = \rho \int_0^t E\left[X_u^{(1)} X_u^{(2)}\right] du$$

*Proof of Lemma 2* The processes $\left\{X_u^{(1)}, u \geq 0\right\}$ and $\left\{X_u^{(2)}, u \geq 0\right\}$ are predictable with respect to $\mathcal{F}_u$, the smallest $\sigma$-algebra generated by $\left(W_u^{(1)}, W_u^{(2)}\right)$. Furthermore, the inequalities $E\left[\int_0^T \left(X_u^{(1)}\right)^2 du\right] < \infty$ and $E\left[\int_0^T \left(X_u^{(2)}\right)^2 du\right] < \infty$ hold. Therefore the integrals $\int_0^t X_u^{(1)} dW_u^{(1)}$ and $\int_0^t X_u^{(2)} dW_u^{(2)}, 0 \leq t \leq T$, are defined, and they are martingales with respect to $\mathcal{F}_u$. We have:

$$cov\left[\int_0^t X_u^{(1)} dW_u^{(1)}, \int_0^T X_u^{(2)} dW_u^{(2)}\right]$$

$$= E\left[\int_0^t X_u^{(1)} dW_u^{(1)} \left(\int_0^t X_u^{(2)} dW_u^{(2)} + \int_t^T X_u^{(2)} dW_u^{(2)}\right)\right] \tag{44}$$

First, note that:

$$E\left[\int_0^t X_u^{(1)} dW_u^{(1)} \int_t^T X_u^{(2)} dW_u^{(2)}\right]$$

$$= E\left[E\left[\int_0^t X_u^{(1)} dW_u^{(1)} \int_t^T X_u^{(2)} dW_u^{(2)} | \mathcal{F}_t\right]\right]$$

$$= E\left[\int_0^t X_u^{(1)} dW_u^{(1)} E\left[\int_t^T X_u^{(2)} dW_u^{(2)} | \mathcal{F}_t\right]\right] = 0 \qquad (45)$$

Next, $X_u^{(1)}$ and $X_u^{(2)}$ can be approximated by the following simple left-continuous processes:

$$X_{t_0}^{(1)(n)} = X_{t_0}^{(1)}, \ X_{t_0}^{(2)(n)} = X_{t_0}^{(2)}$$

and, for $u > 0$:

$$X_u^{(1)(n)} = \sum_{i=0}^{n-1} \theta_i^{(1)} \mathbb{I}_{\left(t_i^{(n)}, t_{i+1}^{(n)}\right]}(u), \ X_u^{(2)(n)} = \sum_{i=0}^{n-1} \theta_i^{(2)} \mathbb{I}_{\left(t_i^{(n)}, t_{i+1}^{(n)}\right]}(u) \qquad (46)$$

where:

(i) $\left\{t_i^{(n)}\right\}$ is a sequence of partitions of $[0 = t_0, t = t_n]$ with $\delta_n = \sup_{i \in \{0,...,n-1\}} \left(t_{i+1}^{(n)} - t_i^{(n)}\right) \to 0$ as $n \to \infty$,

(ii) $\theta_i^{(1)}$ and $\theta_i^{(2)}$ are constants if $\left(X_u^{(1)}, X_u^{(2)}\right)$ is a pair of deterministic processes or square integrable $\mathcal{F}_{t_i}$-adapted random variables if $\left(X_u^{(1)}, X_u^{(2)}\right)$ is a pair of stochastic processes.

If $X_u^{(1)}$ and $X_u^{(2)}$ are square integrable, then it is a classical result from the theory of continuous-time processes that $\int_0^t X_u^{(1)} dW_u^{(1)}$ and $\int_0^t X_u^{(2)} dW_u^{(2)}$ can be approximated by $\lim_{n\to\infty} \sum_{i=0}^{n-1} \theta_i^{(1)} \left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)$ and $\lim_{n\to\infty} \sum_{i=0}^{n-1} \theta_i^{(2)} \left(W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)}\right)$ respectively, where the approximating sums converge in mean square (Lipster and Shiryayev 1989) . Hence, as $\delta_n \to 0$,

$$E\left[\int_0^t X_u^{(1)} dW_u^{(1)} \int_0^t X_u^{(2)} dW_u^{(2)}\right]$$

$$= E\left[\sum_{i=0}^{n-1} \theta_i^{(1)} \left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right) \sum_{j=0}^{n-1} \theta_j^{(2)} \left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$= \sum_{i=0}^{n-1} E\left[\theta_i^{(1)}\theta_i^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)}\right)\right]$$

$$+ \sum_{i=0}^{n-1}\sum_{\substack{j=0 \\ j \neq i}}^{n-1} E\left[\theta_i^{(1)}\theta_j^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$\triangleq S_1 + S_2 \tag{47}$$

$$S_2 = \sum_{i=0}^{n-1}\sum_{\substack{j=0 \\ j \neq i}}^{n-1} E\left[E\left[\theta_i^{(1)}\theta_j^{(2)}\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\Big| \mathcal{F}_{t_i \wedge t_j}\right]\right]$$

$$= \sum_{i=0}^{n-1}\sum_{\substack{j=0 \\ j \neq i}}^{n-1} E\left[\theta_i^{(1)} E\left[\theta_j^{(2)} |\mathcal{F}_{t_i}\right] E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]\right]\mathbb{I}_{\{t_i < t_j\}}$$

$$+ \sum_{i=0}^{n-1}\sum_{\substack{j=0 \\ j \neq i}}^{n-1} E\left[\theta_j^{(2)} E\left[\theta_i^{(1)} |\mathcal{F}_{t_j}\right] E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]\right]\mathbb{I}_{\{t_j < t_i\}}$$

$$\tag{48}$$

Denoting by $\left\{\bar{W}_t^{(2)}, t \geq 0\right\}$ a standard Brownian motion independent of $\{W_t^1, t \geq 0\}$ defined on the same filtered probability space as $\left\{W_t^{(2)}, t \geq 0\right\}$, one can write:

$$E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$= E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(\rho W_{t_{j+1}}^{(1)} + \sqrt{1-\rho^2}\bar{W}_{t_{j+1}}^{(2)} - \rho W_{t_j}^{(1)} - \sqrt{1-\rho^2}\bar{W}_{t_j}^{(2)}\right)\right]$$

$$= \rho\text{cov}\left[W_{t_{i+1}}^{(1)}, W_{t_{j+1}}^{(1)}\right] - \rho\text{cov}\left[W_{t_{i+1}}^{(1)}, W_{t_j}^{(1)}\right] - \rho\text{cov}\left[W_{t_i}^{(1)}, W_{t_{j+1}}^{(1)}\right]$$

$$+ \rho\text{cov}\left[W_{t_i}^{(1)}, W_{t_j}^{(1)}\right] \tag{49}$$

If $i$ and $j$ are two natural integers such that $i < j$, then $\sup_i ((i+1) - j) = 0$; similarly, if $j < i$, then $\sup_j ((j+1) - i) = 0$. Thus,

$$E\left[\left(W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)}\right)\left(W_{t_{j+1}}^{(2)} - W_{t_j}^{(2)}\right)\right]$$

$$= \rho\left(t_{i+1} - t_{i+1} - t_i + t_i\right)\mathbb{I}_{\{i<j\}} + \rho\left(t_{j+1} - t_j - t_{j+1} + t_j\right)\mathbb{I}_{\{i>j\}} = 0 \tag{50}$$

Hence, $S_2 = 0$. Eventually,

$$
\begin{aligned}
S_1 &= \sum_{i=0}^{n-1} E\left[ E\left[ \theta_i^{(1)} \theta_i^{(2)} \left( W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)} \right) \left( W_{t_{i+1}}^{(2)} - W_{t_i}^{(2)} \right) \big| \mathcal{F}_{t_i} \right] \right] \\
&= \sum_{i=0}^{n-1} E\left[ \theta_i^{(1)} \theta_i^{(2)} E\left[ \left( W_{t_{i+1}}^{(1)} - W_{t_i}^{(1)} \right) \right. \right. \\
&\qquad \left. \left. \left( \rho W_{t_{i+1}}^{(1)} + \sqrt{1-\rho^2} \bar{W}_{t_{i+1}}^{(2)} - \rho W_{t_i}^{(1)} - \sqrt{1-\rho^2} \bar{W}_{t_i}^{(2)} \right) \right] \right] \\
&= \sum_{i=0}^{n-1} E\left[ \theta_i^{(1)} \theta_i^{(2)} \rho \left( t_{i+1} - t_i \right) \right] = \rho \sum_{i=0}^{n-1} E\left[ \theta_i^{(1)} \theta_i^{(2)} \right] (t_{i+1} - t_i) \qquad (51)
\end{aligned}
$$

where the last sum converges to $\rho \int_0^t E\left[ X_u^{(1)} X_u^{(2)} \right] du$. $\qquad\square$

Using Lemma 2:

$$
\mathrm{cov}\left[ r_t, r_u \right] = \frac{\sigma_r^2}{a} \left( e^{a(u-t)} - e^{-a(u+t)} \right) (1 + \rho_{1.2}) \qquad (52)
$$

so that:

$$
\mathrm{var}\left[ \int_0^T r_t \, dt \right] = \frac{2\sigma_r^2}{a^2} (1 + \rho_{1.2}) \left( T + \frac{1}{2a} \left( 1 - e^{-2aT} \right) + \frac{2}{a} \left( e^{-aT} - 1 \right) \right) \qquad (53)
$$

Then, from the formula for the moment generating function of a normal random variable, we get $B(t, T) = \exp\left( -A(t, T) r_t + C(t, T) \right)$ as given by Proposition 4.

Applying Ito's lemma to $\ln(B(t, T))$ and then integrating on $[0, t]$, one can obtain, under $Q$:

$$
B(t, T) = B(0, T) \beta_t L(t, T) \qquad (54)
$$

with:

$$
L(t, T) = \exp\left(
\begin{array}{l}
-\sigma_r (1 + \rho_{1.2}) \int_0^t A(s, T) \, dW_s^{(1)} - \frac{\sigma_r^2 (1+\rho_{1.2})^2}{2} \int_0^t A^2(s, T) \, ds \\[2mm]
-\sigma_r \sigma_{2|1} \int_0^t A(s, T) \, d\bar{W}_s^{(2)} - \frac{\sigma_r^2 \sigma_{2|1}^2}{2} \int_0^t A^2(s, T) \, ds
\end{array}
\right)
$$

where $\bar{W}_t^{(2)}$ denotes a standard Brownian motion independent of $W_t^{(1)}$, as in Sect. 2. Thus, using a change of numeraire:

$$
\begin{aligned}
E_2 &= E_Q\left[\frac{B\,(t,T)}{\beta_t}\mathbb{I}\left\{\ln\left(\frac{S_T^{(1)}}{S_0^{(1)}}\right)\geq\ln\left(\frac{K_1}{\omega_1 S_0^{(1)}}\right),\ln\left(\frac{S_T^{(1)}S_0^{(2)}}{S_T^{(2)}S_0^{(1)}}\right)\geq\ln\left(\frac{\omega_2 S_0^{(2)}}{\omega_1 S_0^{(1)}}\right)\right\}\right]\\
&= B\,(0,T)\,\mathrm{P}_{B_T}\left(\ln\left(\frac{S_T^{(1)}}{S_0^{(1)}}\right)\geq\ln\left(\frac{K_1}{\omega_1 S_0^{(1)}}\right),\ln\left(\frac{S_T^{(1)}S_0^{(2)}}{S_T^{(2)}S_0^{(1)}}\right)\geq\ln\left(\frac{\omega_2 S_0^{(2)}}{\omega_1 S_0^{(1)}}\right)\right)
\end{aligned}
$$

$$(55)$$

where $\mathrm{P}_{B_T}$ is the risk-neutral measure under which the numeraire is the zero-coupon bond, whose Radon-Nikodym derivative is given by:

$$
\frac{d\mathrm{P}_{B_T}}{dQ}\,|\mathcal{F}_t = L\,(t,T)
$$

By Girsanov's theorem, we have:

$$
\begin{aligned}
dW_t^{(1)} &= dW_t^{(1)(B)} - \sigma_r\,(1+\rho_{1.2})\,A\,(t,T)\,dt\\
d\bar{W}_t^{(2)} &= d\bar{W}_t^{(2)(B)} - \sigma_r\sigma_{2|1}\,A\,(t,T)\,dt
\end{aligned}
$$

$$(56)$$

where $W_t^{(1)(B)}$ and $\bar{W}_t^{(2)(B)}$ are independent standard Brownian motions under $\mathrm{P}_{B_T}$.

Using an extended Fubini's theorem to interchange the order of a stochastic and a classical integral:

$$
\begin{aligned}
\int_0^T r_t\,dt &= \bar{\mu}_r + \sigma_r\int_0^T\left(\left(1-e^{-a(T-t)}\right)\Big/a\right)dW_t^{(1)}\\
&\quad +\sigma_r\int_0^T\left(\left(1-e^{-a(T-t)}\right)\Big/a\right)dW_t^{(2)}\\
&\stackrel{law}{=} \bar{\mu}_r + \frac{\sigma_r}{a\sqrt{T}}\sqrt{T-\frac{2}{a}\left(1-e^{-aT}\right)+\frac{1}{2a}\left(1-e^{-2aT}\right)}\\
&\quad \left(W_T^{(1)}\,(1+\rho_{1.2})+\sigma_{2|1}\,\bar{W}_T^{(2)}\right)
\end{aligned}
$$

$$(57)$$

Given $N_T^{(1)}$ and $N_T^{(2)}$, one can now solve Eqs. 10 and 11 under $\mathrm{P}_{B_T}$. Equation 11, for example, yields:

$$S_T^{(2)} = S_0^{(2)} \exp\left\{ \bar{\mu}_r + (1/a)\left(\sigma_{S2}\sigma_r\left(\rho_{1.4}(1+\rho_{1.2}) + \sigma_{2|1}\rho_{2.4|1}\right)\right)\left(1 - e^{-aT} - T\right)\right.$$

$$- \left(\lambda_2\kappa_2 + \sigma_{S2}^2/2\right)T + N_T^{(2)}\xi_2 + W_T^{(1)(B)}\left(\bar{\sigma}_r(1+\rho_{1.2}) + \sigma_{S2}\rho_{1.4}\right)$$

$$+ \bar{W}_T^{(2)(B)}\left(\bar{\sigma}_r\sigma_{2|1} + \sigma_{S2}\rho_{2.4|1}\right) + \bar{W}_T^{(3)(B)}\sigma_{S2}\rho_{3.4|1.2} + \bar{W}_T^{(4)(B)}\sigma_{S2}\sigma_{4|1.2.3}$$

$$+ \left.\sqrt{N_T^{(2)}\varepsilon_2^2}\phi\right\} \tag{58}$$

where the processes $\left\{W_t^{(1)(B)}, t \geq 0\right\}$, $\left\{\bar{W}_t^{(2)(B)}, t \geq 0\right\}$, $\left\{\bar{W}_t^{(3)(B)}, t \geq 0\right\}$ and $\left\{\bar{W}_t^{(4)(B)}, t \geq 0\right\}$ are mutually independent standard Brownian motions under $P_{B_T}$; the variable $\bar{\sigma}_r$ is equal to: $\bar{\sigma}_r = \frac{\sigma_r}{a\sqrt{T}}\sqrt{T - \frac{2}{a}\left(1 - e^{-aT}\right) + \frac{1}{2a}\left(1 - e^{-2aT}\right)}$; and $\phi$ is an independent standard normal random variable.

Then, it only remains to compute the covariance between $\ln\left(S_T^{(1)}/S_0^{(1)}\right)$ and $\ln\left(\frac{S_T^{(1)}S_0^{(2)}}{S_T^{(2)}S_0^{(1)}}\right)$ to obtain $E_2$. The calculation of $E_4$ is similar.

The next step in the proof of Proposition 4 is to calculate $E_1$ and $E_3$ in Eq. 42. This could be achieved by a new change of measure but the application of Girsanov's theorem is more involved here because of the jump components before the indicator functions inside $E_1$ and $E_3$. Instead, one can draw on the joint normality of all the random variables featured inside $E_1$ and $E_3$ to easily terminate the calculation by means of the following lemma:

**Lemma 3** Let $X \sim N\left(\mu_X, \sigma_X^2\right)$, $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$ and $Z \sim N\left(\mu_Z, \sigma_Z^2\right)$ be three normal random variables with constant correlation coefficients denoted by $\rho_{X\cdot Y}$, $\rho_{X\cdot Z}$, $\rho_{Y\cdot Z}$. Let $a$ and $b$ be two real numbers. Then,

$$E\left[\exp(X)\,\mathbb{I}\{Y \leq a, Z \leq b\}\right]$$

$$= \exp\left(\mu_X + \frac{\sigma_X^2}{2}\right)N_2\left[\frac{a - \mu_Y - \rho_{X\cdot Y}\sigma_X\sigma_Y}{\sigma_Y}, \frac{b - \mu_Z - \rho_{X\cdot Z}\sigma_X\sigma_Z}{\sigma_Z}; \rho_{Y\cdot Z}\right]$$

*Proof of Lemma 3* Let $f_{X,Y,Z}$ denote the joint density of $X$, $Y$ and $Z$. Then,

$$E\left[\exp(X)\,\mathbb{I}\{Y \leq a, Z \leq b\}\right]$$

$$= \int\limits_{x=-\infty}^{\infty}\int\limits_{y=-\infty}^{a}\int\limits_{z=-\infty}^{b} e^x f_{X,Y,Z}(x, y, z)\,dz\,dy\,dx \tag{59}$$

Using Eq. 24:

$E\left[\exp\left(X\right)\mathbb{I}\{Y \le a, Z \le b\}\right]$

$$
= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{a} \int_{z=-\infty}^{b} \left\{ \frac{\exp\left(x - \frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - \frac{1}{2\sigma_{Y|X}^2}\left(\frac{y-\mu_Y}{\sigma_Y} - \rho_{X\cdot Y}\frac{x-\mu_X}{\sigma_X}\right)^2\right)}{2\pi\sigma_X\sigma_Y\sigma_{Y|X}} \right.
$$

$$
\left. \frac{\exp\left(-\frac{1}{2\sigma_{Z|X\cdot Y}^2}\left(\frac{z-\mu_Z}{\sigma_Z} - \rho_{X\cdot Z}\frac{x-\mu_X}{\sigma_X} - \rho_{Y\cdot Z|X}\left(\frac{1}{\sigma_{Y|X}}\left(\frac{y-\mu_Y}{\sigma_Y} - \rho_{X\cdot Y}\frac{x-\mu_X}{\sigma_X}\right)\right)\right)^2\right)}{\sigma_Z\sigma_{Z|X\cdot Y}\sqrt{2\pi}} \right\} dz\,dy\,dx \tag{60}
$$

Apply the following chain of changes of variables:

$$
\tilde{x} = \frac{x - \mu_X}{\sigma_X}, \tilde{y} = \frac{y - \mu_Y}{\sigma_Y}, \hat{x} = \tilde{x} - \sigma_X, \hat{y} = \tilde{y} - \rho_{X\cdot Y}\sigma_X, \tilde{z} = \frac{z - \mu_Z}{\sigma_Z} - \rho_{X\cdot Z}\sigma_X
$$

and then use the identity:

$$
\int_{-\infty}^{\infty} f_{X,Y,Z}\left(x, y, z\right) dx = f_{Y,Z}\left(y, z\right)
$$

to obtain Lemma 3. □

To calculate $E_1$, it suffices to apply Lemma 3 under the $Q$−measure with:

$$
X = \left(-\int_0^T r_t dt\right) + \ln\left(\frac{S_T^{(1)}}{S_0^{(1)}}\right), \quad Y = \ln\left(\frac{S_T^{(1)}}{S_0^{(1)}}\right), \quad Z = \ln\left(\frac{S_T^{(1)} S_0^{(2)}}{S_T^{(2)} S_0^{(1)}}\right) \tag{61}
$$

Indeed, the integral $\int_0^T r_t dt$ is normally distributed and its expectation and variance have already been computed. The sum $\sum_{n=1}^{N_T^{(1)}} J_n^{(1)}$ is normal too, with expectation equal to $N_T^{(1)}\xi_1$ and variance equal to $N_T^{(1)}\varepsilon_1^2$. Hence, the three random variables $X, Y, Z$ are jointly normal and, by rewriting $W_T^{(1)}, W_T^{(2)}, W_T^{(3)}$ as linear combinations of independent standard Brownian motions $W_T^{(1)}, \bar{W}_T^{(2)}, \bar{W}_T^{(3)}$, as shown in Sect. 2, one can find strong solutions to stochastic differential Eqs. 10 and 11 under $Q$, from which the expectation and variance of $X, Y, Z$ are obtained. Then, it only remains to compute $\text{cov}\left(X, Y\right), \text{cov}\left(X, Z\right)$ and $\text{cov}\left(Y, Z\right)$ to be supplied with all the inputs required by Lemma 3.

Likewise, to calculate $E_3$, one can apply Lemma 3 under the $Q$—measure with:

$$
X = \left(-\int_0^T r_t dt\right) + \ln\left(\frac{S_T^{(2)}}{S_0^{(2)}}\right), \quad Y = \ln\left(\frac{S_T^{(2)}}{S_0^{(2)}}\right), \quad Z = \ln\left(\frac{S_T^{(2)} S_0^{(1)}}{S_T^{(1)} S_0^{(2)}}\right) \tag{62}
$$

Eventually, Proposition 4 is obtained by summing $E_1 - K \cdot E_2 + E_3 - K \cdot E_4$ over the joint law of $\left( N_T^{(1)}, N_T^{(2)} \right)$, which is the product of the laws of $Z_T^{(1)}$, $Z_T^{(2)}$ and $Z_T^{(12)}$.

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