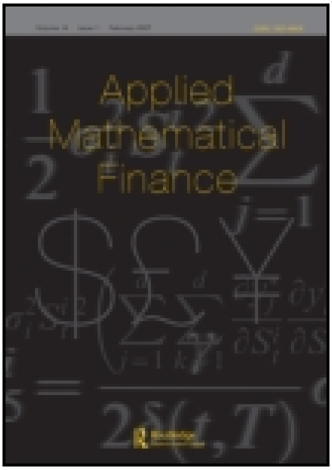

Applied Mathematical Finance
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/ramf20

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Published online: 13 May 2010.

Applied Mathematical Finance **10**, 245–266 (2003)

Routledge
Taylor & Francis Group

Multi-asset barrier options and occupation time derivatives

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Received 14 March 2002 and in revised form 5 March 2003

A general framework is formulated to price various forms of European style multi-asset barrier options and occupation time derivatives with one state variable having the barrier feature. Based on the lognormal assumption of asset price processes, the splitting direction technique is developed for deriving the joint density functions of multi-variate terminal asset prices with provision for single or double barriers on one of the state variables. A systematic procedure is illustrated whereby multi-asset option price formulas can be deduced in a systematic manner as extensions from those of their one-asset counterparts. The formulation has been applied successfully to derive the analytic price formulas of multi-asset options with external two-sided barriers and sequential barriers, multi-asset step options and delayed barrier options. The successful numerical implementation of these price formulas is demonstrated.

Keywords: multi-asset barrier options, occupation time derivatives, splitting direction technique

1. Introduction

Barrier options have become so popular in financial markets that they are no longer considered as exotic options. The inclusion of a barrier provision in the option contract allows the investor to eliminate those unlikely scenarios as viewed by herself, thus achieving option premium reduction. The analytical valuation of the down-and-out call option first appeared in the seminal paper by Merton (1973). Since then there have been numerous articles that consider the pricing of different forms of barrier options (Rubinstein and Reiner, 1991; Rich, 1994). The barrier provision may take more exotic forms, such as two-sided barriers (Kunitomo and Ikeda, 1992; Kolkiewicz, 2002), sequential barriers (Sidenius, 1998) and external barrier (Heynen and Kat, 1994).

The barrier feature is well known to have the undesirable 'circuit breaker' effect. When evaluated at the barrier, the barrier option's delta is discontinuous and the option's gamma tends to an infinite value, thus causing serious hedging difficulties for option writers (Linetsky, 1999). Gradual knock-out options are introduced to modify the abrupt one-touch knock-out feature in traditional barrier options. The terminal payoffs of these options are parameterized by the occupation time, which is the

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Applied Mathematical Finance ISSN 1350-486X print/ISSN 1466–4313 online
http://www.tandf.co.uk/journals
DOI: 10.1080/1350486032000107352

total time spent by the asset price process staying in the knock-out region. Hence, these gradual knock-out options are also called occupation time derivatives. The improvement, from the risk management perspective, of these occupation time derivatives over the one-touch barrier option is well explored in Linetsky's paper (1999). Using the technique of Laplace transform, Linetsky (1999) and Davydov and Linetsky (2001) obtained analytic formulas for these one-asset gradual knock-out options. The price formulas of other one-asset occupation time derivatives were also obtained by Douady (1998) and Hugonnier (1999).

Option models that are multi-variate in nature are quite common in financial markets. For multi-state options, the option value is determined by the stochastic behaviour of several underlying asset price and/or stochastic variables (like interest rates) and the correlation coefficients between these stochastic quantities. Under the Black–Scholes assumption of lognormality of the asset price processes, the option value is governed by a multi-dimensional parabolic diffusion type equation. Unlike the usual diffusion type equations, the multi-dimensional Black–Scholes option equation contains second-order cross-derivative terms due to the non-vanishing of the correlation coefficients among the stochastic state variables.

Analytical valuation of the option price function amounts to the determination of the transition density function of the terminal asset prices conditional on the values of the current asset prices. For most one-asset barrier option models, the transition density functions can be found quite easily using the reflection principle or the method of images. For multi-state models, the transition density of the terminal asset prices for the unrestricted processes can be obtained without great difficulty. However, the integration of the expectation integrals can be quite tedious. By following an ingenious method of choosing a set of appropriate similarity variables, Johnson (1987) managed to obtain the price formulas for various European multi-asset vanilla option models. The price formulas of multi-asset options with one-sided external barrier have also been obtained by Heynen and Kat (1994), Rich and Leipus (1997) and Kwok *et al.* (1998).

It is almost analytically intractable to price the multi-state occupation time derivatives by extending the techniques used by Linetsky (1999), Douady (1998) and Hugonnier (1999) in their pricing frameworks for one-asset option models. In this paper, we formulate the splitting direction technique to derive the transition density functions of the restricted asset price processes associated with the presence of barriers. With this robust formalism, one can deduce in a straightforward manner the price formulas for the multi-asset version of occupation time derivatives from those of their one-asset counterparts.

The paper is organized as follows. In the next section, we state various propositions which formulate the splitting direction technique. We show how to express the density of the joint multi-asset price processes as a product of two simpler known factors: the univariate density of the external barrier variable, and the joint density of the remaining state variables. In Section 3, we apply the splitting direction technique to derive the price formulas of multi-asset options with two-sided barriers and sequential barriers. These price formulas are represented in terms of multi-dimensional cumulative normal distribution functions. The intracacies in the implementation of numerical valuation of these price formulas are discussed. In Section 4, we develop the general pricing methodology for occupation time derivatives with separable barrier variable and payoff variables. The pricing behaviours of the three-asset proportional step option and the three-asset delayed barrier option are examined. The paper ends with concluding remarks.

2. Formulation of the splitting direction technique

The presence of the drift terms in the governing equation of a multi-asset option model is one source of complication in the derivation procedure of finding the fundamental solution of the differential equation. In this section, we present several propositions which show how to decompose the governing equation into simpler structures. First, we deduce a relation that connects the density functions of joint Brownian processes with and without drifts. Next, the splitting direction technique is summarized in Proposition 1. By adopting an appropriate transformation of the independent variables, we can split the density function as a product of two density functions, one is the univariate density of the external barrier variable and the other is the joint density of the remaining variables. We derive several mathematical identities that are essential in the analytic procedures of deriving various price formulas in later sections.

Relation of density functions of joint Brownian processes with and without drifts

If the density function ϕ_n satisfies the following Fokker–Planck equation governing the density function of n-variate unrestricted joint Brownian processes with drifts

$$\frac{\partial \phi_n}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \phi_n}{\partial x_i \partial x_j} - \sum_{j=1}^{n}\mu_j\frac{\partial \phi_n}{\partial x_j}, \qquad t>0, -\infty < x_j < \infty, j=1,2,\cdots,n \tag{2.1}$$

where μ_j is the drift rate of process j and ρ_{ij} is the correlation coefficient between price processes i and j, then ϕ_n can be decomposed into the product of Q and ψ_n

$$\phi_n = Q\psi_n \tag{2.2}$$

where

$$\begin{aligned} Q &= \exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu}t)^T R^{-1}(\mathbf{x}-\boldsymbol{\mu}t) - \mathbf{x}^T R^{-1}\mathbf{x}}{2t}\right) \\ &= e^{\mu R^{-1}\xi}\exp\left(-\frac{(\mathbf{x}-\xi-\boldsymbol{\mu}t)^T R^{-1}(\mathbf{x}-\xi-\boldsymbol{\mu}t) - (\mathbf{x}-\xi)^T R^{-1}(\mathbf{x}-\xi)}{2t}\right) \end{aligned} \tag{2.3}$$

and ψ_n satisfies the following Fokker–Planck equation without the drift terms

$$\frac{\partial \psi_n}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2 \psi_n}{\partial x_i \partial x_j}, \qquad t>0, -\infty < x_j < \infty, \qquad j=1,2,\cdots,n \tag{2.4}$$

Here, $\boldsymbol{x} = (x_1 \cdots x_n)^T$, $\boldsymbol{\mu} = (\mu_1 \cdots \mu_n)^T$, R is the correlation coefficient matrix whose (i, j)th entry is $\rho_{ij}, i, j = 1, 2, \cdots, n$. Theoretically, the vector ξ in Equation 2.3 can be chosen arbitrarily, but an ingenious choice of ξ can be shown to simplify the derivation procedure in later pricing calculations.

The proof of the decomposition (2.2) is argued as follows. In order to eliminate the drift terms in Equation 2.1, the usual procedure is to seek a transformation of the form

$$\phi_n(\mathbf{x}, t) = e^{p_1x_1+\cdots+p_nx_n+qt}\psi_n(\mathbf{x}, t) \tag{2.5}$$

The fundamental solution of Equation 2.4 is

$$\psi_n(\mathbf{x},\, t)=\frac{1}{(2\pi t)^{n/2}}\frac{1}{\sqrt{\det R}}\exp\left(-\frac{\mathbf{x}^T R^{-1}\mathbf{x}}{2t}\right)$$

while that of Equation 2.1 is

$$\phi_n(\mathbf{x},\, t)=\frac{1}{(2\pi t)^{n/2}}\frac{1}{\sqrt{\det R}}\exp\left(-\frac{(\mathbf{x}-\boldsymbol{\mu}t)^T R^{-1}(\mathbf{x}-\boldsymbol{\mu}t)}{2t}\right)$$

These two fundamental solutions must observe the relation (2.5), and taking their ratio leads to the representation shown in Equation 2.3.

Proposition 1. (Splitting direction technique)

If the density function ϕ_n satisfies the forward Fokker–Planck equation with semi-infinite range in the first independent variable x_1 and infinite range in the remaining independent variables

$$\frac{\partial\phi_n}{\partial t}=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2\phi_n}{\partial x_i\partial x_j}-\sum_{j=1}^{n}\mu_j\frac{\partial\phi_n}{\partial x_j}, \tag{2.6}$$

$$t>0,\, b_1<x_1<\infty, -\infty<x_j<\infty, j=2,\cdots,n$$

then the following linear transformation of the independent variables

$$z_j=\begin{cases} x_1 & \text{if } j=1 \\ \frac{x_j-\rho_{1j}x_1}{\sqrt{1-\rho_{1j}^2}} & \text{if } j=2,3,\cdots,n \end{cases} \tag{2.7}$$

leads to the splitting of ϕ_n in the following sense

$$\phi_n=(z_1,\, z_2,\cdots,\, z_n,\, t)=\phi_1(z_1,\, t)\phi_{n-1}(z_2,\cdots,\, z_n,\, t) \tag{2.8}$$

The reduced density functions $\phi_1(z_1,t)$ and $\phi_{n-1}(z_2,\cdots,z_n,t)$ satisfy, respectively, the following equations

$$\frac{\partial\phi_1}{\partial t}=\frac{1}{2}\frac{\partial^2\phi_1}{\partial z_1^2}-\mu_1\frac{\partial\phi_1}{\partial z_1}, \qquad t>0,\, b_1<z_1<\infty \tag{2.9a}$$

$$\frac{\partial\phi_{n-1}}{\partial t}=\frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\tilde{\rho}_{ij}\frac{\partial^2\phi_{n-1}}{\partial z_i\partial z_j}-\sum_{j=2}^{n}\tilde{\mu}_j\frac{\partial\phi_{n-1}}{\partial z_j}, \qquad t>0,-\infty<z_j<\infty, j=2,\cdots,n \tag{2.9b}$$

where

$$\tilde{\rho}_{ij}=\frac{\rho_{ij}-\rho_{1i}\rho_{1j}}{\sqrt{\left(1-\rho_{1i}^2\right)\left(1-\rho_{1j}^2\right)}} \text{ and } \tilde{\mu}_j=\frac{\mu_j-\rho_{1j}\mu_1}{\sqrt{1-\rho_{1j}^2}}, \qquad i,j=2,3,\cdots,n \tag{2.9c}$$

Both $\phi_1(z_1,\, t)$ and $\phi_n(z_1,\cdots,z_n,\, t)$ share the same absorbing boundary condition at $z_1=b_1$.

Remark

The probabilistic argument for the choice of the transformation (2.7) is presented as follows. Consider an n-dimensional Brownian motion $(X_1\ X_2 \cdots X_n)$ with correlation matrix R whose entries are ρ_{ij}, $i, j, = 1, 2, \cdots, n$. Suppose we define

$$Z_j = \begin{cases} X_1 & \text{if } j = 1 \\ \frac{X_j - \rho_{1j} X_1}{\sqrt{1-\rho_{1j}^2}} & \text{if } j = 2, \cdots, n \end{cases} \tag{2.10}$$

then the joint process $(Z_1\ Z_2 \cdots Z_n)$ is Brownian and $\mathrm{COV}(Z_1, Z_j) = 0, j = 2, \cdots, n$. This establishes the independence of Z_1 with $Z_j, j = 2, \cdots, n$.

The proof of Proposition 1 is given in Appendix A. This splitting direction technique is particularly useful to deal with multi-state option models where only one state variable (say x_1) has the barrier feature. Now, the barrier variable z_1 (z_1 is set to be x_1) is uncorrelated with $z_2, \cdots, z_n$, by virtue of the transformation given in Equation 2.7.

Corollary of Proposition 1. Let $\psi_n(\mathbf{x}, t; R)$ denote the fundamental solution of Equation 2.4, that is

$$\psi_n = (\mathbf{x}, t; R) = \frac{1}{(2\pi t)^{n/2}\sqrt{\det R}} \exp\left(-\frac{\mathbf{x}^T R^{-1} \mathbf{x}}{2t}\right) \tag{2.11}$$

Write $\tilde{\mathbf{z}} = (z_2 \cdots z_n)^T$, where $z_2, \cdots, z_n$ are related to $x_1, \cdots, x_n$ as defined in Equation 2.7; and $\tilde{R}$ is the $(n-1) \times (n-1)$ correlation coefficient matrix whose entries are $\tilde{\rho}_{ij}$, $i, j = 2, \cdots, n$, as defined in Equation 2.9*c*. We then have

$$\psi_n(\mathbf{x} - \alpha R \mathbf{e}_1, t; R) = \psi_1(z_1 - \alpha, t)\psi_{n-1}(\tilde{\mathbf{z}}, t; \tilde{R}) \det J \tag{2.12}$$

where α is any scalar, $\mathbf{e}_1 = (1\ 0 \cdots 0)^T$ and

$$J = \frac{\partial(z_1, z_2, \cdots, z_n)}{\partial(x_1, x_2, \cdots, x_n)}$$

For its proof, see Appendix B.

3. Multi-asset options with external two-sided barriers

We consider the class of multi-asset option models with an external barrier variable. The barrier variable does not determine the payoff of the option. Rather, it determines whether the option is knocked out when the value of the barrier variable breaches some predetermined level (one-sided barrier) or stays outside a certain range of values (two-sided barriers).

The valuation of multi-asset options with an external one-sided barrier has been considered by Heynen and Kat (1994), Rich and Leipus (1997) and Kwok *et al.* (1998). In this section, we illustrate how to apply the formalism in Section 2 to derive the price formulas of multi-asset options whose external barrier variable has two-sided barriers.

Let S_1^t denote the value of the barrier variable and S_i^t denote the value of asset i, $i=2,\cdots,n$, at time t. For the multi-asset maximum call option with an external barrier, the terminal payoff is given by $\max\left(\max\left(S_2^T,\cdots,S_n^T\right)-X,0\right)$, where X is the strike price. We adopt the usual Black–Scholes assumptions on the capital market. Under the risk neutral valuation framework, we assume S_i^t, $i=1,2,\cdots,n$ to follow the lognormal diffusion processes

$$\frac{\mathrm{d}S_i^t}{S_i^t}=r\,\mathrm{d}t+\sigma_i\,\mathrm{d}z_i,\ i=1,2,\cdots,n \tag{3.1}$$

where r is the riskless interest rate, σ_i is the volatility of asset i, $\mathrm{d}z_i$ is the Wiener process for asset i, $i=1,2,\cdots,n$. Let ρ_{ij} denote the correlation coefficient between $\mathrm{d}z_i$ and $\mathrm{d}z_j$. We define

$$x_i=\frac{1}{\sigma_i}\ln\frac{S_i^t}{S_i}\text{ and }\mu_i=\frac{r-\frac{\sigma_i^2}{2}}{\sigma_i},\ i=1,2,\cdots,n \tag{3.2}$$

where S_i, $i=1,2,\cdots,n$ are the asset values at the current time (taken to be time zero). Let H and L denote the upstream and downstream barriers of the barrier variable. The call option will be knocked out when $S_1^t>H$ or $S_1^t<L$ at any time t during the life of the option. We define

$$M_1=\frac{1}{\sigma_1}\ln\frac{H}{S_1}\text{ and }m_1=\frac{1}{\sigma_1}\ln\frac{L}{S_1}$$

3.1. *Joint density function with provision of two-sided barrier levels*

Let $\Phi(x_1,x_2,\cdots,x_n,t;R)$ denote the density function of the joint process of the asset prices and barrier variable with the provision of two-sided barrier levels on x_1, $m_1<x_1<M_1$. To find $\Phi(x_1,x_2,\cdots,x_n,t)$, we apply the splitting direction technique presented in Section 2. First, we consider the following one-dimensional diffusion equation:

$$\frac{\partial\phi_1}{\partial t}=\frac{1}{2}\frac{\partial^2\phi_1}{\partial x_1^2},\qquad m_1<x_1<M_1, t>0 \tag{3.3}$$

Its fundamental solution is known to be (Kevorkian, 1990)

$$\phi_1(x_1,t)=\sum_{k=-\infty}^{\infty}\left[\psi_1(x_1-2\alpha_k,t)-\psi_1\left(x_1-2\alpha_k',t\right)\right] \tag{3.4}$$

where

$$\psi_1(x_1,t)=\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x_1^2}{2t}\right)$$

$\alpha_k=k(M_1-m_1)$ and $\alpha_k'=\alpha_k+m_1$. By applying the formalism in Proposition 1 and its Corollary, the fundamental solution to the following n-dimensional Fokker–Planck equation

$$\frac{\partial\phi_n}{\partial t}=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2\phi_n}{\partial x_i\partial x_j},\qquad m_1<x_1<M_1,-\infty<x_j<\infty, j=2,\cdots,n,\ t>0 \tag{3.5}$$

is given by

$$\begin{aligned}\phi_n(\mathbf{x},t) &= \sum_{k=-\infty}^{\infty} \left[\psi_1(z_1-2\alpha_k,\, t)-\psi_1\left(z_1-2\alpha'_k,\, t\right)\right]\psi_{n-1}\left(\tilde{\mathbf{z}},\, t;\tilde{R}\right)\det J \\ &= \sum_{k=-\infty}^{\infty} \left[\psi_n(\mathbf{x}-2\alpha_k\mathbf{e}_1,\, t;R)-\psi_n\left(\mathbf{x}-2\alpha'_k\mathbf{e}_1,\, t;R\right)\right]\end{aligned} \tag{3.6}$$

where $\psi_n(\mathbf{x},\ t;\ R)$ is defined in Equation 2.11. Let $\Phi(\mathbf{x},\ t;\ R)$ denote the fundamental solution to the following n-dimensional Fokker–Planck equation

$$\begin{aligned}&\frac{\partial\Phi}{\partial t} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}\frac{\partial^2\Phi}{\partial x_i\partial x_j} - \sum_{j=1}^{n}\mu_j\frac{\partial\Phi}{\partial x_j}, \\ &\qquad m_1 < x_1 < M_1,\ -\infty < x_j < \infty, j=2,\cdots,n,\ t>0\end{aligned} \tag{3.7}$$

By applying relation (2.2) and choosing ξ to be $2\alpha_k R\mathbf{e}_1$ and $2\alpha'_k R\mathbf{e}_1$ successively in the second form of Q in Equation 2.3, we obtain

$$\begin{aligned}\Phi(\boldsymbol{x},\, t;R) = \sum_{k=-\infty}^{\infty} &\left\{e^{2\mu_1\alpha_k}\psi_n(\mathbf{x}-2\alpha_k R\mathbf{e}_1-\boldsymbol{\mu}t,\, t;R)\right. \\ &\left. -e^{2\mu_1\alpha'_k}\psi_n\left(\mathbf{x}-2\alpha'_k R\mathbf{e}_1-\boldsymbol{\mu}t,\, t;R\right)\right\}\end{aligned} \tag{3.8}$$

3.2. *Option value of maximum call with two-sided external barriers*

Following the discounted risk neutral expectation approach, the value of the multi-asset maximum call option with two-sided external barrier levels H and L is given by

$$\begin{aligned}V(S_1,\, S_2,\cdots,\, S_n,\, T) = e^{-rT}\int_{m_1}^{M_1}\int_{D_{n-1}} &\Phi(x_1,\, x_2,\cdots,\, x_n,\, T;R) \\ &\max(\max(S_2e^{\sigma_2x_2},\cdots,\, S_ne^{\sigma_nx_n})-X,\, 0)\mathrm{d}x_n\cdots\,\mathrm{d}x_2\,\mathrm{d}x_1\end{aligned} \tag{3.9}$$

where T is the expiry time and D_{n-1} is the domain in the $(n-1)$-dimensional $(x_2,\cdots,\, x_n)$-plane inside which $\max(S_2e^{\sigma_2x_2},\cdots,S_ne^{\sigma_nx_n}) > X$ is satisfied.

Theorem 1. The value of the multi-asset maximum call option with two-sided external barriers is found to be

$$V(S_1,\, S_2,\cdots,\, S_n,\, T) = \sum_{k=-\infty}^{\infty} f_k \tag{3.10a}$$

where

$$
\begin{aligned}
f_k = \sum_{\ell=2}^{n} S_\ell \Big\{ & e^{2\mu_1\alpha_k - 2\beta_k^\ell}\left[N_n\left(\mathbf{d}_2^\ell - \mathbf{b}_k^\ell; R^\ell\right) - N_n\left(\mathbf{d}_4^\ell - \mathbf{b}_k^\ell; R^\ell\right)\right] \\
& - e^{2\mu_1\alpha'_k - 2\beta_{k'}^\ell}\left[N_n\left(\mathbf{d}_2^\ell - \mathbf{b}_{k'}^\ell; R^\ell\right) - N_n\left(\mathbf{d}_4^\ell - \mathbf{b}_{k'}^\ell; R^\ell\right)\right]\Big\} \\
& - Xe^{-rT}\Big\{e^{2\mu_1\alpha_k}\left[N_n\left(\mathbf{d}_1^\ell - \mathbf{b}_k^\ell; R^\ell\right) - N_n\left(\mathbf{d}_3^\ell - \mathbf{b}_k^\ell; R^\ell\right)\right] \\
& - e^{2\mu_1\alpha'_k}\left[N_n\left(\mathbf{d}_1^\ell - \mathbf{b}_{k'}^\ell; R^\ell\right) - N_n\left(\mathbf{d}_3^\ell - \mathbf{b}_{k'}^\ell; R^\ell\right)\right]\Big\}
\end{aligned}
\tag{3.10b}
$$

Here, $N_n(\mathbf{x}; R)$ is the n-dimensional cumulative normal distribution function and the jth component of d_1^ℓ is

$$
\mathrm{d}_{1,j}^\ell = \begin{cases} M_1/\sqrt{T} & \text{if } j=1 \\ \frac{1}{\sigma_\ell\sqrt{T}}\ln\frac{S_\ell}{X} & \text{if } j=\ell \\ \frac{1}{\sigma_{\ell j}\sqrt{T}}\ln\frac{S_\ell}{S_j} & \text{otherwise} \end{cases}
\tag{3.11a}
$$

$$
\mathbf{d}_2^\ell = \mathbf{d}_1^\ell + \sigma_\ell\sqrt{T}R^\ell\mathbf{e}_\ell,\ \mathbf{d}_3^\ell = \mathbf{d}_1^\ell - \frac{M_1 - m_1}{\sqrt{T}}\mathbf{e}_1,\ \mathbf{d}_4^\ell = \mathbf{d}_2^\ell - \frac{M_1 - m_1}{\sqrt{T}}\mathbf{e}_1
\tag{3.11b}
$$

$$
\begin{aligned}
&\mathbf{b}_k^\ell = \frac{2\alpha_k R^\ell \mathbf{e}_1 + A^\ell \mu T}{\sqrt{T}} \text{ and } \mathbf{b}_{k'}^\ell = \frac{2\alpha'_k R^\ell \mathbf{e}_1 + A^\ell \mu T}{\sqrt{T}}, \\
&\ell = 2, \cdots, n,\ k \text{ is any integer}
\end{aligned}
\tag{3.11c}
$$

$$
\left[r_{ij}^\ell\right] = R^\ell = A^\ell R A^{\ell^T},\ \beta_k^\ell = r_{1\ell}^\ell \sigma_\ell \alpha_k \text{ and } \beta_{k'}^\ell = r_{1\ell}^\ell \sigma_\ell \alpha'_k
\tag{3.11d}
$$

The proof of Theorem 1 is shown in Appendix C.

3.3. *Extension to maximum call with sequential external barriers*

Unlike the two-sided barrier provision where the option is knocked out when the barrier variable hits either H or L, the sequential barrier provision requires the breaching of the two barrier levels at a predetermined sequential order, say, up then down. For the one-asset case, given the asset price S_1 at the zeroth time, the density function of the asset price S_1^t at time t conditional on non-breaching of the sequential barrier provision (first upstream barrier H then downstream L) is given by (Sidenius, 1998; Li, 1999)

$$
\begin{aligned}
\phi_1(x_1, t) = & \left[e^{2\mu_1\alpha_0}\psi_1(x_1 - \alpha_0 - \mu_1 t, t) - e^{2\mu_1\alpha'_0}\psi_1\left(x_1 - \alpha'_0 - \mu_1 t, t\right)\right] \\
& - \left[e^{2\mu_1\alpha_{-1}}\psi_1(x_1 - \alpha_{-1} - \mu_1 t, t) - e^{2\mu_1\alpha'_{-1}}\psi_1\left(x_1 - \alpha'_{-1} - \mu_1 t, t\right)\right]
\end{aligned}
\tag{3.12}
$$

We consider the multi-asset maximum call option with an external barrier S_1, the terminal payoff of which is given by $\max\left(\max\left(S_2^T, \cdots, S_n^T\right) - X, 0\right)$. This barrier call option is knocked out only if S_1 hits the up-barrier H first then the down-barrier L afterwards, By following a similar derivation procedure to that for the two-sided barrier counterpart, the value of the multi-asset maximum call option with sequential up-then-down barriers is given by

$$V_{up-down}(S_1, S_2, \cdots, S_n, T) = f_0 - f_{-1} \tag{3.13}$$

where f_0 and f_{-1} are given by Equation 3.10*b*.

3.4. *Numerical calculations*

We performed the numerical valuation of the price formula (3.10*a*, *b*) for the three-asset maximum call option with two-sided barriers L and U, and terminal payoff $\max\left(\max\left(S_2^T, S_3^T\right) - X, 0\right)$. We obtained option values with varying volatility σ_1 of the barrier variable and interval width $[L, U]$. The other parameter values of the option model used in the calculations are: $\sigma_2 = \sigma_3 = 20\%$, $r = 0.05$, $T = 0.5$, $X = 100$, $S_1 = S_2 = S_3 = 100$, $\rho_{12} = 0.2$, $\rho_{23} = 0.3$ and $\rho_{13} = 0.3$. Table 1 lists the option values for the double-barrier three-asset call option obtained using different number of summation terms in the price formula (3.10*a*,*b*). In most cases, we observe that nine terms in the summation are sufficient to achieve four significant figure accuracy. The option values are seen to decrease with increasing volatility of the barrier variable and narrowing of the interval $[L, U]$. These results are consistent with the financial intuition that the option is worth less if the chance of knock-out is higher.

In the numerical procedure of computing the values of the three-dimensional cumulative normal distribution functions in the price formula, we adopted the effective numerical algorithm proposed by Genz (1992). The Genz algorithm consists of three transformations that are used to transform the original integral into an integral over a unit hypercube, then the efficient valuation of the transformed integral can be achieved using either Monte Carlo or subregion adaptive numerical integration algorithms. In computing option prices for multi-asset options, the availability of the effective

Table 1. The table lists the option values for the double-barrier three-asset maximum call option that were obtained from the numerical valuation of the price formula (3.10*a*,*b*) using N terms in the summation. The other parameter values used in the calculations are: $\sigma_2 = \sigma_3 = 20\%$, $r = 0.05$, $T = 0.5$, $X = 100$, $S_1 = S_2 = S_3 = 100$, $\rho_{12} = 0.2$, $\rho_{23} = 0.3$ and $\rho_{13} = 0.3$. The option values are seen to decrease with increasing volatility of the barrier variable and narrowing of the interval $[L, U]$.

Parameter values	$N=1$	$N=3$	$N=5$	$N=7$	$N=9$
$\sigma_1 = 20\%$, $L = 90$, $U = 110$	3.1016	2.1881	2.1872	2.1872	2.1872
$\sigma_1 = 20\%$, $L = 92$, $U = 108$	1.9034	1.1947	1.1856	1.1856	1.1856
$\sigma_1 = 20\%$, $L = 92$, $U = 105$	1.3311	0.8381	0.7969	0.7968	0.7968
$\sigma_1 = 30\%$, $L = 92$, $U = 108$	0.7931	0.5123	0.4130	0.4114	0.4114
$\sigma_1 = 30\%$, $L = 92$, $U = 105$	0.5411	0.4464	0.2999	0.2885	0.2884
$\sigma_1 = 40\%$, $L = 92$, $U = 108$	0.4013	0.3618	0.2088	0.1878	0.1871

algorithms for evaluating $N_n(\mathbf{x}; R)$ partially relieves the curse of dimensionality. This would give the numerical valuation of analytic price formula significant computational advantage over other common numerical methods, like Monte Carlo simulation or finite difference methods.

4. Multi-asset occupation time derivatives

We consider the pricing of multi-asset occupation time derivative where the terminal payoff depends on the terminal asset prices and the occupation time associated with a barrier variable. We start with a review of some of the results about the one-asset occupation time derivatives and examine how the pricing formulation of the multi-asset models of occupation time derivatives can be inferred from their one-asset counterparts.

4.1. *Review of the results for one-asset occupation time derivatives*

Let S_1^t be a stochastic variable with the barrier level B. We assume that S_1^t follows the lognormal diffusion process as defined in Equation 3.1. The occupation times of the stochastic variable S_1 staying below and above the barrier level B from the zeroth time to time t are random variables defined by

$$\tau_B^- = \int_0^t H\left(B - S_1^u\right) \mathrm{d}u \text{ and } \tau_B^+ = \int_0^t H\left(S_1^u - B\right) \mathrm{d}u \tag{4.1}$$

respectively, where $H(x)$ is the Heaviside function. The occupation time of the stochastic variable above (below) the barrier level B is the total amount of time that the value of the stochastic variable stays higher (lower) than B. The differentials of τ_B^- and τ_B^+ are given by

$$\mathrm{d}\tau_B^- = H\left(B - S_1^u\right) \mathrm{d}t \text{ and } \mathrm{d}\tau_B^+ = H\left(S_1^u - B\right) \mathrm{d}t \tag{4.2}$$

Consider an occupation time derivative whose terminal payoff function takes the form $F\left(S_1, \tau_B^-\right)$. Let $V\left(S_1, \tau_B^-, t\right)$ denote the derivative value at time t. Taking the usual Black–Scholes assumptions, the governing equation for $V\left(S_1, \tau_B^-, t\right)$ is given by (Linetsky, 1999)

$$\begin{gathered} \frac{\partial V}{\partial t} + \frac{\sigma_1^2}{2} S_1^2 \frac{\partial^2 V}{\partial S_1^2} + r S_1 \frac{\partial V}{\partial S_1} + H(B - S_1) \frac{\partial V}{\partial \tau_B^-} - rV = 0, \\ t > 0,\ 0 < S_1 < \infty,\ \tau_B^- > 0 \end{gathered} \tag{4.3}$$

Note that an additional term

$$H(B - S_1) \frac{\partial V}{\partial \tau_B^-}$$

is added in the usual Black–Scholes equation to reflect the dependence of the derivative value on the occupation time τ_B^-. We write

$$x_1 = \frac{1}{\sigma_1} \ln \frac{S_1^t}{B} \text{ and } \xi_1 = \frac{1}{\sigma_1} \ln \frac{S_1}{B}$$

where S_1 and S_1^t are the respective asset prices at the current time (taken to be zero) and future time t. The transition density function $\psi(x_1, \tau_B^-, t; \xi_1)$ satisfies the following forward Fokker–Planck equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\frac{\partial^2 \psi}{\partial x_1^2} - \mu_1 \frac{\partial \psi}{\partial x_1} - H(-x_1)\frac{\partial \psi}{\partial \tau_B^-}, \qquad t > 0, -\infty < x_1 < \infty, \tau_B^- > 0 \tag{4.4}$$

The associated initial conditions at $t=0$ and $\tau_B^-=0$ are

$$\psi(x_1, \tau_B^-, 0; \xi_1) = \delta(x_1 - \xi_1)\delta(\tau_B^-) \text{ and } \psi(x_1, 0, t; \xi_1) = \psi_B(x_1, t; \xi_1) \tag{4.5}$$

respectively. Here, $\psi_B(x_1, t; \xi_1)$ is the transition density function of the corresponding restricted asset price process without crossing the down barrier B. The condition $\tau_B^-=0$ is equivalent to the situation where the asset price never breaches the down barrier B. Hence, the value of $\psi(x_1, \tau_B^-, t; \xi_1)$ at $\tau_B^-=0$ is equal to $\psi_B(x_1, t; \xi_1)$. Also, the initial condition $\psi(x_1, \tau_B^-, 0; \xi_1)$ is derived from the observation that S_1^t and τ_B^- start at $t=0$ with certainty at S_1 and zero value, respectively.

Linetsky (1999) obtained the solution to $\psi(x_1, \tau_B^-, t; \xi_1)$ corresponding to the zero drift case [that is, setting $\mu_1=0$ in Equation 4.4]. The solution takes different forms in different domains Ω_i, $i=1,\ldots, 4$, namely,

$\Omega_1 = \{(x_1, \xi_1): \xi_1 \geq 0,\ x_1 \geq 0 \text{ and } \xi_1 + x_1 > 0\}$

$$\psi = u_1(x_1, \tau_B^-, t; \xi_1) = \int_0^{t-\tau_B^-} \frac{x_1 + \xi_1}{2\pi(t-u)^{3/2} u^{3/2}} \exp\left(-\frac{(x_1+\xi_1)^2}{2u}\right) \mathrm{d}u \tag{4.6a}$$

$\Omega_2 = \{(x_1, \xi_1): \xi_1 \leq 0 \text{ and } x_1 > 0\}$

$$\begin{aligned}\psi &= u_2(x_1, \tau_B^-, t; \xi_1)\\ &= \int_0^{t-\tau_B^-} \frac{x_1\left(1-\frac{\xi_1^2}{t-u}\right) + \xi_1\left(1-\frac{x_1^2}{u}\right)}{2\pi(t-u)^{3/2}u^{3/2}} \exp\left(-\frac{x_1^2}{2u} - \frac{\xi_1^2}{2(t-u)}\right)\mathrm{d}u\end{aligned} \tag{4.6b}$$

$\Omega_3 = \{(x_1, \xi_1): \xi_1 \geq 0,\ x_1 < 0\}$

$$\psi = u_3(x_1, \tau_B^-, t; \xi_1) = u_2(-x_1, t-\tau_B^-, t; -\xi_1) \tag{4.6c}$$

$\Omega_4 = \{(x_1, \xi_1): \xi_1 \leq 0,\ x_1 \geq 0 \text{ and } \xi_1 + x_1 < 0\}$

$$\psi = u_4(x_1, \tau_B^-, t; \xi_1) = u_1(-x_1, t-\tau_B^-, t; -\xi_1) \tag{4.6d}$$

When $\tau_B^-=0$ or $\tau_B^-=t$, we have

$$\psi = \psi_B(x_1, t; \xi_1) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(x_1-\xi_1)^2}{2t}\right) - \exp\left(-\frac{(x_1+\xi_1)^2}{2t}\right)\right] \tag{4.6e}$$

Remark

1. For the non-zero drift case, we can apply the decomposition rule (2.2) to obtain the corresponding solution for ψ. For example, with $\mu_1 \neq 0$, u_1 becomes

$$u_1\left(x_1, \tau_B^-, t;\ \xi_1, \mu_1\right) = u_1\left(x_1, \tau_B^-, t;\ \xi_1\right) \exp\left(-\frac{(x_1-\mu_1 t)^2 - x_1^2}{2t}\right) \tag{4.7}$$

2. Suppose we count the occupation time starting at an earlier time t_s with $t_s < 0$, that is, before the current time. The accumulation of occupation time from t_s to the current time is a known quantity since it can be evaluated from the already known realization of the asset price path. The terminal payoff of any occupation time derivative can be modified so that the payoff depends on the occupation time counting from the current time to maturity. Without loss of generality, it suffices to consider pricing models where the counting of the occupation time starts at the current time.
3. Successive Laplace transforms on τ_B^- and t are applied to obtain the above solutions for $\psi\left(x_1, \tau_B^-, t;\ \xi_1\right)$. The imposition of the initial condition: $\psi(x_1, 0, t; \xi_1) = \psi_B(x_1, t; \xi_1)$ seems to lead some complexity in the derivation procedure. Fortunately, $\psi_B(x_1, t; \xi_1)$ does not enter into the equation for the Laplace transform function since the factor $H(\text{-}x_1)$ becomes zero when $\tau_B^- = 0$. This is because S_1^t is guaranteed to stay above B when the occupation time τ_B^- is zero.

4.2. *Multi-asset occupation time derivatives*

We would like to examine how the splitting direction technique can be applied to obtain the price formulas for multi-asset occupation time derivatives. Let S_1^t denote the value of the barrier variable at time t and B be the constant down-barrier level associated with S_1^t. We consider the pricing of multi-asset options whose terminal payoff depends on the terminal asset values $S_2^T, \cdots, S_n^T$, and the terminal value of the occupation time variable. The occupation time variable τ_B^- associated with the barrier variable S_1^t is defined by Equation 4.1*a*.

We assume that the asset price processes S_i^t, $i = 1, 2, \cdots, n$, follow the joint lognormal diffusion processes as defined in Equation 3.1. Similarly, we define

$$x_j = \frac{1}{\sigma_j} \ln \frac{S_j^t}{S_j}, j = 2, \cdots, n$$

where S_j^t and S_j are the value of asset j at the future time t and the current time (taken to be zero), respectively. The valuation of a multi-asset occupation time derivative requires the determination of the transition density function $\psi\left(\mathbf{x}, \tau_B^-, t;\ \xi_I\right)$ of the joint processes of the asset values and the occupation time, where $\mathbf{x}$ denotes the vector $(x_1\ x_2 \cdots x_n)^T$. The forward Fokker–Planck equation that governs $\psi\left(\mathbf{x}, \tau_B^-, t;\ \xi_I\right)$ takes the form

$$\begin{aligned} &\frac{\partial \psi}{\partial t} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \sum_{j=1}^{n} \mu_j \frac{\partial \psi}{\partial x_j} - H(-x_1) \frac{\partial \psi}{\partial \tau_B^-}, \\ &-\infty < x_j < \infty, j = 1, 2, \cdots, n,\ t > 0,\ \tau_B^- > 0 \end{aligned} \tag{4.8}$$

Here, ρ_{ij} is the correlation coefficient between dz_i and dz_j, and

$$\mu_j = \frac{r - \frac{\sigma_j^2}{2}}{\sigma_j}, j = 1, 2, \cdots, n$$

The initial conditions at $t=0$ and $\tau_B^- = 0$ are given by

$$\psi\left(\mathbf{x}, \tau_B^-, 0; \xi_1\right) = \delta(x_1 - \xi_1)\delta(x_2)\cdots\delta(x_n)\delta\left(\tau_B^-\right) \text{ and } \psi(\mathbf{x}, 0, t; \xi_1) = \psi_B(\mathbf{x}, t; \xi_1) \tag{4.9}$$

where $\psi_B(\mathbf{x}, t; \xi_1)$ is the transition density function of the joint processes of $S_1^t, S_2^t, \cdots, S_n^t$ with S_1^t staying above B at all times.

The splitting direction technique stated in Proposition 1 can be applied to solve for $\psi\left(\boldsymbol{x}, \tau_B^-, t; \xi_1\right)$. We employ the linear transformation of the independent variables as given in Equation 2.7, and this leads to the splitting

$$\psi\left(\mathbf{z}, \tau_B^-, t; \xi_1\right) = \psi_1\left(z_1, \tau_B^-, t; \xi_1\right)\psi_{n-1}(z_2, \cdots, z_n, t; \xi_1) \tag{4.10a}$$

The procedure is motivated by observing that the function in the boundary condition in Equation 4.9 can be split by the same linear transformation, i.e.

$$\psi_B(\mathbf{x}, t; \xi_1) = \psi_B(z_1, t; \xi_1)\psi_{n-1}(\hat{\mathbf{z}}, t) \tag{4.10b}$$

The governing equations and auxiliary conditions for $\psi_1\left(z_1, \tau_B^-, t; \xi_1\right)$ and $\psi_{n-1}(z_2, \cdots, z_n,; t; \xi_1)$ are given by

$$\begin{aligned} &\frac{\partial \psi_1}{\partial t} = \frac{1}{2}\frac{\partial^2 \psi_1}{\partial z_1^2} - \mu_1 \frac{\partial \psi_1}{\partial z_1} - H(-z_1)\frac{\partial \psi_1}{\partial \tau_B^-}, \qquad t > 0, \tau_B^- > 0, -\infty < z_1 < \infty, \\ &\psi_1\left(x_1, \tau_B^-, 0; \xi_1\right) = \delta(x_1 - \xi_1)\delta\left(\tau_B^-\right), \psi_1(x_1, 0, t; \xi_1) = \psi_B(x_1, t; \xi_1) \end{aligned} \tag{4.11a}$$

$$\begin{aligned} &\frac{\partial \psi_{n-1}}{\partial t} = \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\tilde{\rho}_{ij}\frac{\partial^2 \psi_{n-1}}{\partial z_i \partial z_j} - \sum_{j=2}^{n}\tilde{\mu}_j \frac{\partial \psi_{n-1}}{\partial z_j}, \qquad t > 0, -\infty < z_j < \infty, j = 2, \cdots, n, \\ &\psi_{n-1}\left(\hat{\mathbf{z}}, \tau_B^-, 0; \xi_1\right) = \delta(\hat{\mathbf{z}}) \end{aligned} \tag{4.11b}$$

respectively. The correlation coefficients $\tilde{\rho}_{ij}$ and the drift parameters $\tilde{\mu}_j$ are defined in Equation 2.9*c*.

We let R^{n-1} denote $\{(x_2, \ldots, x_n) : -\infty < x_j < \infty, j = 2, \ldots, n\}$ and define the domains Ω_i^n, $i = 1, \ldots, 4$, to be $\Omega_i^n = \Omega_i \times R^{n-1}$, where Ω_i are defined in Equation 4.6*a*–4.6*d*. The solution to the transition density function $\psi\left(\mathbf{x}, \tau_B^-, t; \xi_1\right)$ in different domains Ω_i^n, $i = 1, \ldots, 4$, are summarized in Theorem 2.

Theorem 2. Let ψ_k^n be the solution to the joint transition density function $\psi\left(\mathbf{x}, \tau_B^-, t; \xi_1\right)$ in the domain Ω_i^n, $i = 1, \ldots, 4$. We have

$$\psi\left(\mathbf{x}, \tau_B^-, t; \xi_1\right) = 1_{\{\Omega_1^n\}}\psi_1^n + 1_{\{\Omega_2^n\}}\psi_2^n + 1_{\{\Omega_3^n\}}\psi_3^n + 1_{\{\Omega_4^n\}}\psi_4^n + \left[\delta\left(\tau_B^-\right) + \delta\left(\tau_B^- - t\right)\right]\psi_5^n \tag{4.12}$$

where

$$
\begin{aligned}
\psi_1^n &= \sqrt{\frac{t}{2\pi}}\psi_n(\mathbf{x}-\xi_1\mathbf{e}_1-\boldsymbol{\mu}t,\ t;R)\int_0^{t-\tau_B^-}\frac{x_1+\xi_1}{(t-u)^{3/2}u^{3/2}}\exp\left(\frac{(x_1-\xi_1)^2}{2t}-\frac{(x_1+\xi_1)^2}{2u}\right)\mathrm{d}u \\
\psi_2^n &= \sqrt{\frac{t}{2\pi}}\psi_n(\mathbf{x}-\xi_1\mathbf{e}_1-\boldsymbol{\mu}t,\ t;R) \\
&\quad \int_0^{t-\tau_B^-}\frac{\left[x_1\left(1-\frac{\xi_1^2}{t-u}\right)+\xi_1\left(1+\frac{x_1^2}{u}\right)\right]}{(t-u)^{3/2}u^{3/2}}\exp\left(\frac{(x_1-\xi_1)^2}{2t}-\frac{x_1^2}{2u}-\frac{\xi_1^2}{2(t-u)}\right)\mathrm{d}u \\
\psi_3^n&\left(x_1,x_2,\cdots,x_n,\tau_B^-,t;\xi_1\right)=\psi_2^n\left(-x_1,x_2,\cdots,x_n,t-\tau_B^-,t;-\xi_1\right) \\
\psi_4^n&\left(x_1,x_2,\cdots,x_n,\tau_B^-,t;\xi_1\right)=\psi_1^n\left(-x_1,x_2,\cdots,x_n,t-\tau_B^-,t;-\xi_1\right) \\
\psi_5^n&=\psi_n(\mathbf{x}-\xi_1\mathbf{e}_1-\boldsymbol{\mu}t,\ t;R)-e^{2\mu_1\xi_1}\psi_n(\mathbf{x}-\xi_1\mathbf{e}_1-\boldsymbol{\mu}t-2\xi_1R\mathbf{e}_1,\ t;R)
\end{aligned}
\tag{4.13}
$$

and $\psi_n(\mathbf{x},\ t;\ R)$ is the fundamental solution defined in Equation 2.11.

Let $V(S_1,\ldots,S_n,T)$ denote the value of the n-asset occupation time derivative with the down barrier B on S_1^t and terminal payoff function $V_T\left(\mathbf{x},\tau_B^-\right)$. The price function $V(S_1,\ldots,S_n,T)$ is given by

$S_1\geq B$

$$
\begin{aligned}
&V(S_1,\ldots,S_n,T) \\
&=e^{-rT}\left\{\int_{R^{n-1}}\int_0^\infty V_T(\mathbf{x},0)\psi_5^n\,\mathrm{d}\mathbf{x}+\int_0^T\int_{R^{n-1}}\int_0^\infty V_T\left(\mathbf{x},\tau_B^-\right)\psi_1^n\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau_B^-\right. \\
&\quad\left.+\int_0^T\int_{R^{n-1}}\int_{-\infty}^0 V_T\left(\mathbf{x},\tau_B^-\right)\psi_3^n\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau_B^-\right\}
\end{aligned}
\tag{4.14a}
$$

$S_1\leq B$

$$
\begin{aligned}
&V(S_1,\ldots,S_n,T) \\
&=e^{-rT}\left\{\int_{R^{n-1}}\int_{-\infty}^0 V_T(\mathbf{x},T)\psi_5^n\,\mathrm{d}\mathbf{x}+\int_0^T\int_{R^{n-1}}\int_0^\infty V_T\left(\mathbf{x},\tau_B^-\right)\psi_2^n\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau_B^-\right. \\
&\quad\left.+\int_0^T\int_{R^{n-1}}\int_{-\infty}^0 V_T\left(\mathbf{x},\tau_B^-\right)\psi_4^n\,\mathrm{d}\mathbf{x}\,\mathrm{d}\tau_B^-\right\}
\end{aligned}
\tag{4.14b}
$$

In particular, we can apply the above formula to compute the price of the proportional step options, simple step options and delayed barrier options (or called the cumulative Parisian options). They are occupation time derivatives with terminal payoff of the separable form: $f\left(\tau_B^-\right)G(\mathbf{x})$.

1. Proportional step option:

$$f\left(\tau_B^-\right)=e^{-s\tau_B^-}$$

where s is called the killing rate.

2. Simple step option:

$$f(\tau_B^-) = \max(1 - s\tau_B^-, 0)$$

3. Delayed barrier option:

$$f(\tau_B^-) = 1_{\{\tau_B^- < \alpha T\}}$$

where α is a parameter satisfying $0 < \alpha < 1$.

Consider the maximum call option with S_1 as the external barrier variable and the associated occupation time τ_B^-. Suppose the terminal payoff takes the separable form: $f(\tau_B^-)\ \max(\max(S_2, \cdots, S_n) - K, 0)$. We define $F(T) = \int_0^T f(u)\,\mathrm{d}u$. The corresponding price formula of this n-asset occupation time derivative with the maximum call payoff is found to be

$$\begin{aligned}
&S_1 \geq B \\
&\quad V(S_1, \ldots, S_n, T) \\
&= f(0) C^n_{down}(S_1, \ldots, S_n, T) \\
&\quad + \int_0^T \int_0^\infty F(T-u) c^{n-1}_{\max}(S_2, \cdots, S_n, T; x_1) u_1(x_1, t; T, \xi_1, \mu_1) \mathrm{d}x_1\, \mathrm{d}u \\
&\quad + \int_0^T \int_{-\infty}^0 [F(T) - F(u)] c^{n-1}_{\max}(S_2, \cdots, S_n, T; x_1) u_3(x_1, t; T, \xi_1, \mu_1) \mathrm{d}x_1\, \mathrm{d}u
\end{aligned} \tag{4.15a}$$

$$\begin{aligned}
&S_1 \leq B \\
&\quad V(S_1, \ldots, S_n, T) \\
&= \int_0^T \int_0^\infty F(T-u) c^{n-1}_{\max}(S_2, \cdots, S_n, T; x_1) u_2(x_1, t; T, \xi_1, \mu_1) \mathrm{d}x_1\, \mathrm{d}u \\
&\quad + \int_0^T \int_{-\infty}^0 [F(T) - F(u)] c^{n-1}_{\max}(S_2, \cdots, S_n, T; x_1) u_4(x_1, t; T, \xi_1, \mu_1) \mathrm{d}x_1\, \mathrm{d}u
\end{aligned} \tag{4.15b}$$

where $c^{n-1}_{\max}(S_2, \cdots, S_n, T)$ denotes the price function of a $(n-1)$-asset European maximum call option and $C^n_{down}(S_1, \ldots, S_n, T)$ denotes the price function of the corresponding down-and-out maximum call option. The functions u_j, $j = 1, \ldots, 4$ are given in Equation 4.6*a*–4.6*d* modified according to Equation 4.7 under non-zero drift μ_1.

Remark. For different types of occupation time derivatives with separable terminal payoff function, the function $F(T - \tau_B^-)$ takes different forms.

1. Proportional step option

$$F(T - \tau_B^-) = \int_0^{T - \tau_B^-} e^{-su}\,\mathrm{d}u = \frac{1 - e^{-s(T - \tau_B^-)}}{s} \tag{4.16a}$$

2. Delayed barrier option

$$F\left(T-\tau_B^-\right)=\int_0^{T-\tau_B^-} 1_{u<\alpha T}\,\mathrm{d}u$$
$$=\begin{cases}\alpha T, & 0\le\tau_B^-\le(1-\alpha)T\\ T-\tau_B^-, & (1-\alpha)T<\tau_B^-\le T\end{cases} \tag{4.16b}$$

3. Simple step option

$$F\left(T-\tau_B^-\right)=\int_0^{T-\tau_B^-}\max(1-su,0)\mathrm{d}u$$
$$=\begin{cases}\frac{1}{2s}, & 0\le\tau_B^-\le T-\frac{1}{s}\\ \left(T-\tau_B^-\right)\left[1-\frac{s}{2}\left(T-\tau_B^-\right)\right], & T-\frac{1}{s}<\tau_B^-\le T\end{cases} \tag{4.16c}$$

4.3. *Numerical calculations*

We performed the numerical valuation of the three-asset proportional step call option, delayed barrier call option and down-and-out call option with one-sided down barrier B. The terminal payoff takes the form: $f\left(\tau_B^-\right)\max(\max(S_2, S_3)-X, 0)$, where $\tau_B^-=\int_0^T H\left(B-S_1^u\right)\mathrm{d}u$. We have $f\left(\tau_B^-\right)=e^{-s\tau_B^-}$ for the proportional step option, $f\left(\tau_B^-\right)=1_{\{\tau_B^-<\alpha T\}}$ for the delayed barrier option and $f\left(\tau_B^-\right)=1$ for the down-and-out option. In Fig. 1, we plot the option prices of these occupation time derivatives against the barrier variable S_1. The parameter values of the option models used in the calculations are: $\sigma_1=\sigma_2=\sigma_3=0.2$, $r=0.05$, $T=0.5$, $\rho_{12}=0.2$, $\rho_{13}=0.3$, $\rho_{23}=0.3$, $B=80$, $X=90$, $S_2=S_3=100$, $s=25$, $\alpha=0.5$. The price of the European two-asset maximum call option is also included for comparison. The price functions are seen to be increasing functions of S_1. The prices of the proportional step option and delayed barrier option are included between those of the non-barrier option and the one-touch barrier option counterparts. Such results are obvious from financial intuition. For the given choices of s and α, the proportional step option is always worth less than its delayed barrier option counterpart.

In Fig. 2, we show plots of the delta of the option price against S_1 of the proportional step call options with varying yearly killing rate s. The same set of parameter values as those used in Fig. 1 are chosen in the calculations. Both the top dot-dashed delta curve (corresponding to $s=20$) and the middle dashed delta curve (corresponding to $s=40$) have their peak value at the barrier level $B=80$ and decrease in value when S_1 moves downstream from B. The option delta of the one-touch down-and-out call option (shown as solid curve) is also plotted for comparison. The delta value of the down-and-out call has a discontinuous jump at $B=80$, a manifestation of the undesirable curcuit breaker effect.

5. Conclusions

Since option prices are given by the discounted expectation of the terminal payoff under the risk neutral valuation framework, the derivation of the analytic price formulas of exotic option models

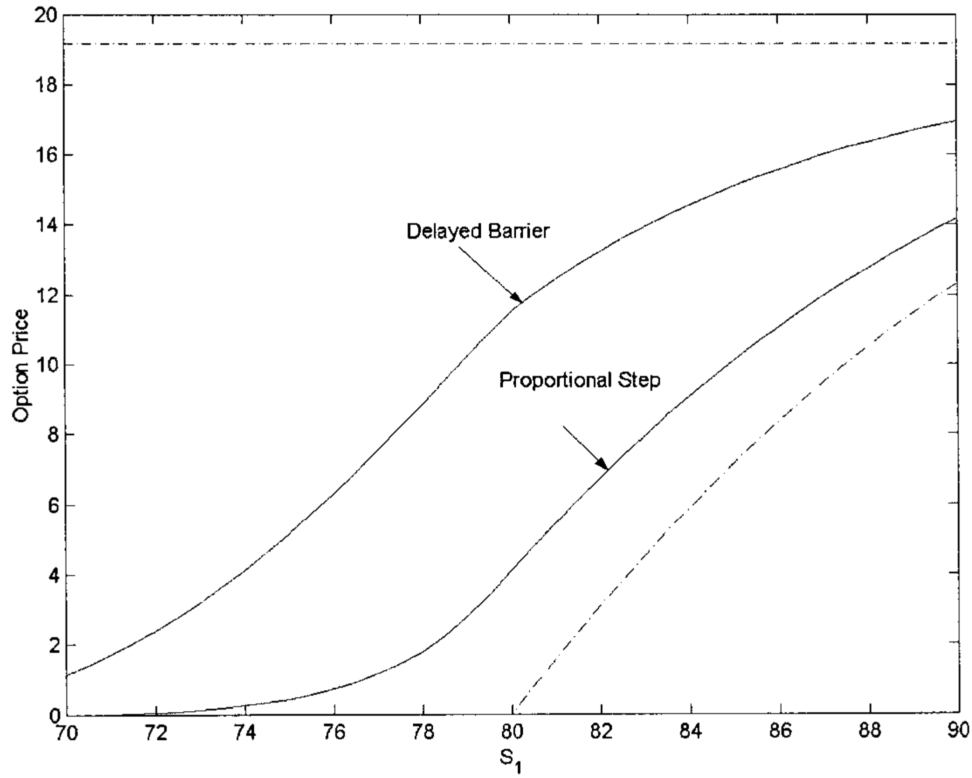

Fig. 1. Plots of the price functions of the proportional step call option (lower solid curve), delayed barrier call option (upper solid curve) and down-and-out call option (lowest dot-dashed curve) against the barrier variable S_1. The plot of the price of the European two-asset maximum call option is included for comparison (shown as the top horizontal dot-dashed line). The parameter values of the option models are: $\sigma_1=\sigma_2=\sigma_3=0.2$, $r=0.05$, $T=0.5$, $\rho_{12}=0.2$, $\rho_{13}=0.3$, $\rho_{23}=0.3$, $B=80$, $X=90$, $S_2=S_3=100$, $s=25$, $\alpha=0.5$.

amounts to the analytical evaluation of expectation integrals. A typical form of the integrand in an expectation integral is given by the product of the transition density function and the terminal payoff function. For multi-asset barrier type options and occupation time derivatives, derivation of the associated density function is known to be mathematically challenging, due primarily to the presence of the correlation terms in the Fokker–Planck equation. In this paper, we develop the splitting direction technique which leads to a systematic derivation approach to find the density functions of multi-asset option models from an extension of their one-asset counterparts. Analytic price formulas of European style multi-asset options with external two-sided barriers and sequential barriers, multi-asset step options and delayed barrier options are obtained in their most succinct forms. Though the analytic price formulas are expressed in terms of multi-dimensional cumulative normal distribution functions (may even involve infinite summation), their numerical valuation has been shown to be viable by virtue

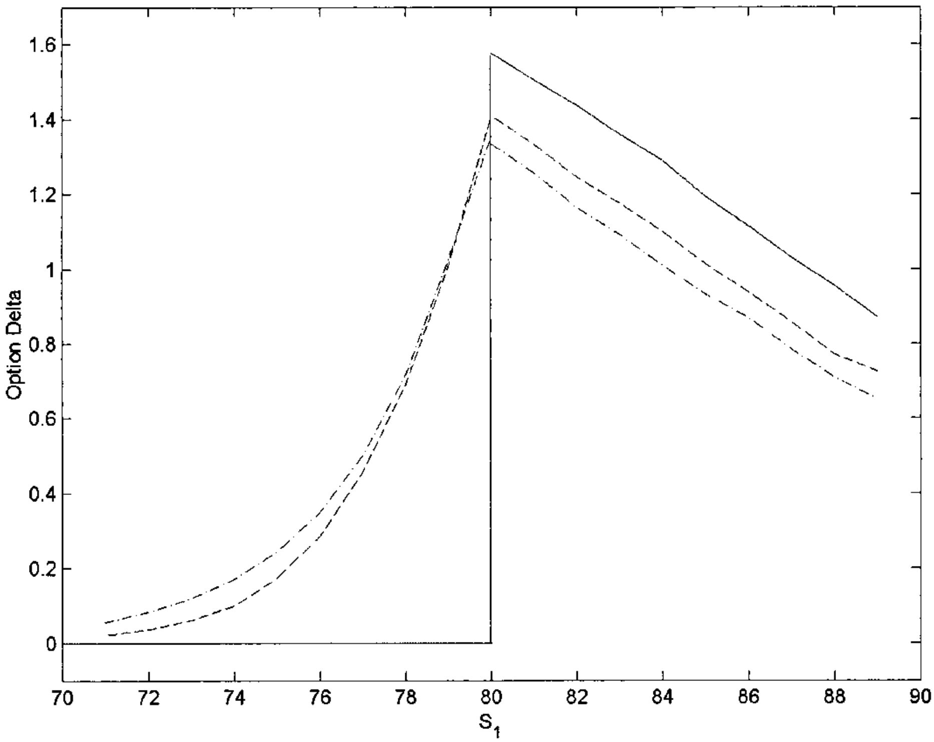

Fig. 2. Plots of the delta of the option price of the proportional step call options with varying yearly killing rate s against the barrier variable S_1. The top dot-dashed delta curve and the middle dashed delta curve correspond to $s=20$ and $s=40$, respectively. The solid curve corresponds to the option delta of the down-and-out barrier option, which exhibits a jump in value at the barrier level, $B=80$.

of the effective numerical algorithm proposed by Genz. Compared to common numerical schemes like finite difference methods, the curse of dimensionality becomes less severe when dealing with numerical valuation of analytic price formulas.

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Appendix A: Proof of Proposition 1

Let $\nabla_{\mathbf{x}} = (\partial_{x_1} \cdots \partial_{x_n})^T$ and let J^T denote the matrix representing the linear transformation between $\boldsymbol{x} = (x_1 \cdots x_n)^T$ and $z = (z_1 \cdots z_n)^T$ as defined in Eq. (2.7), that is, $\mathbf{z} = J^T\mathbf{x}$. We have

$$J = \frac{\partial(z_1, \cdots, z_n)}{\partial(x_1, \cdots, x_n)}$$

so that $\nabla_{\mathbf{x}} = J\nabla_{\mathrm{z}}$. Also, we write $\boldsymbol{\mu} = (\mu_1 \cdots \mu_n)^T$. Now, Equation 2.6 can be expressed as

$$\begin{aligned}\frac{\partial\phi_n}{\partial t} &= \frac{1}{2}\nabla_{\mathbf{x}}^T R \nabla_{\mathbf{x}}\phi_n - \boldsymbol{\mu}^T\nabla_{\mathbf{x}}\phi_n \\ &= \frac{1}{2}\nabla_{\mathbf{z}}^T\left(J^T RJ\right)\nabla_{\mathbf{z}}\phi_n - \left(J^T\boldsymbol{\mu}\right)^T\nabla_{\mathbf{z}}\phi_n\end{aligned} \tag{A.1}$$

Let $\hat{\rho}_{ij}$ denote the (i, j)th entry of $J^T RJ$ and $\hat{\mu}_j$ denote the jth entry of $J^T\boldsymbol{\mu}$. We observe that

$$\begin{aligned}\hat{\rho}_{ij} &= \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{if } i = 1, j \neq 1 \text{ or } i \neq 1, j = 1 \\ \tilde{\rho}_{ij} & \text{if } i \neq 1, j \neq 1 \end{cases} \\ \hat{\mu}_j &= \begin{cases} \mu_1 & j = 1 \\ \tilde{\mu}_j & j \neq 1 \end{cases}\end{aligned} \tag{A.2}$$

where $\tilde{\rho}_{ij}$ and $\tilde{\mu}_j$ are defined in Equation 2.9*c*. In terms of $z_1, \cdots, z_n$, Equation 2.6 can be expressed as

$$\frac{\partial \phi_n}{\partial t} = \left[\frac{1}{2}\frac{\partial^2 \phi_n}{\partial z_1^2} - \mu_1 \frac{\partial \phi_n}{\partial z_1}\right] + \left[\frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\tilde{\rho}_{ij}\frac{\partial^2 \phi_n}{\partial z_i \partial z_j} - \sum_{j=2}^{n}\tilde{\mu}_j \frac{\partial \phi_n}{\partial z_j}\right], \tag{A.3}$$

$$t > 0,\, b_1 < z_1 < \infty,\; -\infty < z_j < \infty, j = 2, \cdots, n$$

The auxiliary conditions of $\phi_n(z_1, z_2, \cdots, z_n, t)$ are given by

$$\phi_n(b_1, z_2, \cdots, z_n, t) = 0 \text{ (absorbing boundary condition at } z_1 = b_1)$$

$$\phi_n(z_1, z_2, \cdots, z_n, 0) = \delta(z_1)\delta(z_2)\cdots\delta(z_n)$$

where $\delta(z_j)$ is the Dirac function, $j = 1, 2, \cdots, n$. We explore the feasibility of the decomposition of ϕ_n into the form as specified in Equation 2.8. Suppose ϕ_1 and ϕ_{n-1} satisfy Equation 2.9*a* and Equation 2.9*b*, respectively; in addition, ϕ_1 (z_1, t) satisfies the auxiliary conditions (i) $\phi_1(b, t) = 0$ and (ii) $\phi_1(z_1, 0) = \delta(z_1)$, while $\phi_{n-1}(z_2, \cdots, z_n, t)$ satisfies the initial condition: $\phi_2(z_2, \cdots, z_n, 0) = \delta(z_2)\cdots\delta(z_n)$, then the product $\phi_1\phi_{n-1}$ satisfies the same set of governing equation and auxiliary conditions of ϕ_n. By uniqueness of solution, ϕ_n must be equal to the product $\phi_1\phi_{n-1}$. This is the decomposition of ϕ_n as specified by Equation 2.8. Now, ϕ_1 is the univariate density function for the Brownian motion in the semi-infinite range $b_1 < z_1 < \infty$ subject to the absorbing boundary condition $\phi_1(b_1, t) = 0$. Also, ϕ_{n-1} is the free space density function for the remaining variables $z_2, \cdots, z_n$.

Remark. The above proof is based on semi-infinite range and absorbing boundary condition for the barrier variable. By following a similar proof, decomposition of the density function remains plausible for cases with barrier variable having a finite range and a different set of boundary conditions.

Appendix B: Proof of Equation 2.12

Let $\psi_1(z_1, t)$ and $\psi_{n-1}(\tilde{\mathbf{z}}, t; \tilde{R})$ denote the respective fundamental solutions to the following Fokker–Planck equations

$$\frac{\partial \psi_1}{\partial t} = \frac{1}{2}\frac{\partial^2 \psi_1}{\partial z_1^2}, \qquad t > 0, -\infty < z_1 < \infty, \tag{B.1}$$

$$\frac{\partial \psi_{n-1}}{\partial t} = \frac{1}{2}\sum_{i=2}^{n}\sum_{j=2}^{n}\tilde{\rho}_{ij}\frac{\partial^2 \psi_{n-1}}{\partial z_i \partial z_j},\; t > 0, -\infty < z_j < \infty, j = 2, \cdots, n \tag{B.2}$$

As deduced from relation (2.8), the fundamental solutions $\psi_n(\mathbf{x}, t; R)$, $\psi_1(z_1, t)$ and $\psi_{n-1}(\tilde{\mathbf{z}}, t; \tilde{R})$ are related by

$$\psi_n(\mathbf{x}, t; R) = \psi_1(z_1, t)\psi_{n-1}(\tilde{\mathbf{z}}, t; \tilde{R}) \det J \tag{B.3}$$

Note that the Jacobian

$$\det J = \left|\frac{\partial(z_1, \cdots, z_n)}{\partial(x_1, \cdots, x_n)}\right|$$

is included due to the change of the vector of independent variables from $\mathbf{x}$ to $\mathbf{z}$.

Suppose we apply a shifting transformation on $\mathbf{x}$: $\mathbf{x}_{new}=\mathbf{x}_{old}-\alpha R\mathbf{e}_1$, where α is any scalar, then $\mathbf{z}_{new}=J^T\mathbf{x}_{new}=J^T(\mathbf{x}_{old}-\alpha R\mathbf{e}_1)=\mathbf{z}_{old}-\alpha\mathbf{e}_1$. We observe that $\mathbf{z}_{new}$ is obtained from $\mathbf{z}_{old}$ by changing only the first component z_1 to $z_1-\alpha$ while keeping all the other $n-1$ components in $\mathbf{z}_{old}$ unchanged. Accordingly, the relation for the fundamental solutions as stated in Equation B.3 is modified to

$$\psi_n(\mathbf{x}-\alpha R\mathbf{e}_1,\, t; R)=\psi_1(z_1-\alpha,\, t)\psi_{n-1}\left(\tilde{z},\, t; \tilde{R}\right)\det J \tag{B.4}$$

where $\tilde{z}$ and the Jacobian remain unchanged under this shifting transformation on $\mathbf{x}$.

Appendix C: Proof of Theorem 1

Let D^{ℓ}_{n-1} denote the domain inside which $S_\ell e^{\sigma_\ell x_\ell}$ is maximum among the $n-1$ quantities: $S_2e^{\sigma_2x_2}, \cdots, S_ne^{\sigma_nx_n}$. The representation of D^{ℓ}_{n-1} is given by

$$D^{\ell}_{n-1}=\left\{(x_2,\cdots,x_n): x_\ell\geq\frac{1}{\sigma_\ell}\ln\frac{X}{S_\ell},\, x_\ell\geq\frac{\sigma_j}{\sigma_\ell}x_j-\frac{1}{\sigma_\ell}\ln\frac{S_\ell}{S_j}, j=2,\cdots,n \text{ but } j\neq\ell\right\},\ \ell=2,\cdots,n \tag{C.1}$$

The terminal payoff becomes $S_\ell e^{\sigma_\ell x_\ell}-X$ inside the domain D^{ℓ}_{n-1}. The integral in Equation 3.9 can be decomposed into the sum of $n-1$ integrals. The integration domain of a typical term I_ℓ is $[m_1, M_1]\times D^{\ell}_{n-1}$. Now, I_ℓ is formally represented by

$$\begin{aligned} I_\ell=e^{-rT}&\sum_{k=-\infty}^{\infty}\int_{m_1}^{M_1}\int_{D^{\ell}_{n-1}}\left(S_\ell e^{\sigma_\ell x_\ell}-X\right)\\ &\left[e^{2\mu_1\alpha_k}\psi_n(\mathbf{x}-2\alpha_kR\mathbf{e}_1-\boldsymbol{\mu}T,\, T; R)-e^{2\mu_1\alpha'_k}\psi_n\left(\mathbf{x}-2\alpha'_kR\mathbf{e}_1-\boldsymbol{\mu}T,\, T; R\right)\right]\mathrm{d}x_n\cdots\mathrm{d}x_2\,\mathrm{d}x_1 \end{aligned} \tag{C.2}$$

To facilitate the integration in I_ℓ, we apply the following linear transformation of the independent variables: $\mathbf{y}^\ell=A^\ell\mathbf{x}$, where

$$y^\ell_j=\begin{cases} x_1 & \text{if } j=1\\ -x_\ell & \text{if } j=\ell\\ \frac{\sigma_j}{\sigma_{j\ell}}\left(x_j-\frac{\sigma_\ell}{\sigma_j}x_\ell\right) & \text{otherwise}\end{cases} \tag{C.3}$$

with $\sigma^2_{j\ell}=\sigma^2_j-2\rho_{j\ell}\sigma_j\sigma_\ell+\sigma^2_\ell$. The integration domain for I_ℓ becomes

$$\begin{aligned}(m_1, M_1)\times D^{\ell}_{n-1}=\Big\{\left(y^\ell_1, y^\ell_2,\cdots,y^\ell_n\right): m_1<y^\ell_1<M_1,\, y^\ell_\ell\leq\frac{1}{\sigma_\ell}\ln\frac{S_\ell}{X},\\ y^\ell_j\leq\frac{1}{\sigma_{j\ell}}\ln\frac{S_\ell}{S_j},\quad j=2,\,\cdots,n \text{ and } j\neq\ell\Big\}\end{aligned} \tag{C.4}$$

Also, we have

$$\begin{aligned}&(\mathbf{x}-2\alpha_kR\mathbf{e}_1-\boldsymbol{\mu}T)^TR^{-1}(\mathbf{x}-2\alpha_kR\mathbf{e}_1-\boldsymbol{\mu}T)\\ &=\left(y^\ell-2\alpha_kR^\ell\mathbf{e}_1-A^\ell\boldsymbol{\mu}T\right)^T\left(R^\ell\right)^{-1}\left(y^\ell-2\alpha_kR^\ell\mathbf{e}_1-A^\ell\boldsymbol{\mu}T\right)\end{aligned} \tag{C.5}$$

where $R^{\ell} = A^{\ell} R A^{\ell T}$. Consider the typical term in I_{ℓ} [see Equation C.2]

$$
\begin{aligned}
I_{\ell}^{k} &= \int_{m_1}^{M_1} \int_{D_{n-1}^{\ell}} (S_{\ell} e^{\sigma_{\ell} x_{\ell}} - X) \psi_n(\mathbf{x} - 2\alpha_k R \mathbf{e}_1 - \boldsymbol{\mu} T,\ T; R) dx_n \cdots \mathrm{d}x_2\, \mathrm{d}x_1 \\
&= \int_{m_1}^{M_1} \int_{-\infty}^{\frac{1}{\sigma_{2\ell}} \ln \frac{S_{\ell}}{S_2}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{\ell}} \ln \frac{S_{\ell}}{X}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{n\ell}} \ln \frac{S_{\ell}}{S_n}} \left(S_{\ell} e^{-\sigma_{\ell} y_{\ell}^{\ell}} - X \right) \psi_n\left(\mathbf{w}^{\ell}, T; R^{\ell} \right) \mathrm{d}y_n^{\ell} \cdots \mathrm{d}y_2^{\ell}\, \mathrm{d}y_1^{\ell},
\end{aligned}
\tag{C.6}
$$

where $\mathbf{w}^{\ell} = \mathbf{y}^{\ell} - 2\alpha_k R^{\ell}\ \mathbf{e}_1 - A^{\ell}\ \boldsymbol{\mu} T$. For the term involving X, it is straightforward to observe that

$$
\begin{aligned}
&-X \int_{m_1}^{M_1} \int_{-\infty}^{\frac{1}{\sigma_{2\ell}} \ln \frac{S_{\ell}}{S_2}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{\ell}} \ln \frac{S_{\ell}}{X}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{n\ell}} \ln \frac{S_{\ell}}{S_n}} \psi_n\left(\mathbf{w}^{\ell},\ T; R^{\ell} \right) \mathrm{d}y_n^{\ell} \cdots \mathrm{d}y_2^{\ell}\, \mathrm{d}y_1^{\ell} \\
&= -XN\left(\mathbf{d}_1^{\ell} - \mathbf{b}_k^{\ell}; R^{\ell} \right)
\end{aligned}
\tag{C.7}
$$

where d_1^{ℓ} and b_k^{ℓ} are defined in Equation 3.1*a*, *b*, *c*. To deal with the integration with respect to the integrand $S_{\ell} e^{-\sigma_{\ell} y_{\ell}^{\ell}} \psi_n\left(\mathbf{w}^{\ell}, T; R^{\ell} \right)$, we consider the following identity

$$
\begin{aligned}
&\left(\mathbf{w}^{\ell} + \sigma_{\ell} T R^{\ell} \mathbf{e}_{\ell} \right)^T \left(R^{\ell} \right)^{-1} \left(\mathbf{w}^{\ell} + \sigma_{\ell} T R^{\ell} \mathbf{e}_{\ell} \right) \\
&= \mathbf{w}^{\ell T} \left(R^{\ell} \right)^{-1} \mathbf{w}^{\ell} + 2\sigma_{\ell} T \left(\boldsymbol{y}^{\ell} - 2\alpha_k R^{\ell} \mathbf{e}_1 - A^{\ell} \boldsymbol{\mu} T \right)^T \mathbf{e}_{\ell} + \sigma_{\ell}^2 T^2 \left(R^{\ell} \mathbf{e}_{\ell} \right)^T \mathbf{e}_{\ell} \\
&= \mathbf{w}^{\ell T} \left(R^{\ell} \right)^{-1} \mathbf{w}^{\ell} + 2\sigma_{\ell} y_{\ell}^{\ell} T - 4 r_{1\ell}^{\ell} \sigma_{\ell} \alpha_k T + 2rT^2
\end{aligned}
\tag{C.8}
$$

so that

$$
\begin{aligned}
&\int_{m_1}^{M_1} \int_{-\infty}^{\frac{1}{\sigma_{2\ell}} \ln \frac{S_{\ell}}{S_2}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{\ell}} \ln \frac{S_{\ell}}{X}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{n\ell}} \ln \frac{S_{\ell}}{S_n}} S_{\ell} e^{-\sigma_{\ell} y_{\ell}^{\ell}} \psi_n\left(\boldsymbol{w}^{\ell}, T; R^{\ell} \right) \mathrm{d}y_n^{\ell} \cdots \mathrm{d}y_1^{\ell} \\
&= S_{\ell} \int_{m_1}^{M_1} \int_{-\infty}^{\frac{1}{\sigma_{2\ell}} \ln \frac{S_{\ell}}{S_2}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{\ell}} \ln \frac{S_{\ell}}{X}} \cdots \int_{-\infty}^{\frac{1}{\sigma_{n\ell}} \ln \frac{S_{\ell}}{S_n}} \\
&\quad \psi_n\left(\mathbf{w}^{\ell} + \sigma_{\ell} T R^{\ell} \mathbf{e}_{\ell}, T; R^{\ell} \right) e^{-2 r_{1\ell}^{\ell} \sigma_{\ell} \alpha_k} e^{rT}\, \mathrm{d}y_n^{\ell} \cdots \mathrm{d}y_1^{\ell} \\
&= S_{\ell} e^{rT} e^{-2\beta_k^{\ell}} N_n\left(\mathbf{d}_2^{\ell} - \mathbf{b}_k^{\ell}; R^{\ell} \right)
\end{aligned}
\tag{C.9}
$$

where $\mathbf{d}_2^{\ell}$ and β_k^{ℓ} are defined in Equation 3.1*b*, *d* Combining all the results, we obtain the price formula (3.10*a*, *b*).