Analytic Approximations for Multi-Asset Option Pricing

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First Version: March 2008
This Version: December 2009

ICMA Centre Discussion Papers in Finance DP2009-05

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The ICMA Centre is supported by the International Capital Market Association

ABSTRACT

We derive general analytic approximations for pricing European basket and rainbow options on N assets. The key idea is to express the option's price as a sum of prices of various compound exchange options, each with different pairs of sub-ordinate multi- or single-asset options. For some multi-asset options a strong condition holds, whereby each compound exchange option is equivalent to a standard single-asset option under a modified measure, and in such cases an almost exact analytic price exists for the multi-asset option. The underlying asset prices are assumed to follow lognormal processes, although the strong condition can be extended to certain other price processes for the underlying. More generally, approximate analytic prices for multi-asset options are derived using a weak lognormality condition, where the approximation stems from making constant volatility assumptions on the price processes that drive the prices of the sub-ordinate basket options. The analytic formulae for multi-asset option prices, and their Greeks, are defined in a recursive framework. For instance, the option delta is defined in terms of the delta relative to sub-ordinate multi-asset options, and the deltas of these sub-ordinate options with respect to the underlying assets. Simulations test the accuracy of our approximations, given some assumed values for the asset volatilities and correlations, and we demonstrate how to calibrate these parameters to market data so that multi-asset option prices are consistent with the implied volatility and correlation skews of the assets.

JEL Code: C02, C29, G12

Keywords: Basket options, Rainbow options, Best-of and Worst-of options, compound exchange options, analytic approximation.

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1. INTRODUCTION

This paper presents a recursive procedure for pricing European basket and rainbow options on N assets. The payoff to these options takes the form $\omega[f(\mathbf{S}_t, K)]^+$ where $\mathbf{S}_t$ is a vector of N asset prices, K is the option strike, f is a function and ω is +1 for a call and -1 for a put. For example, when $N = 3$ then $\mathbf{S}_t = (S_1, S_2, S_3)$ and $f(\mathbf{S}_t, K) = (\theta_1 S_1 + \theta_2 S_2 + \theta_3 S_3) - K$ for the basket option with weights $(\theta_1, \theta_2, \theta_3)$ and $f(\mathbf{S}_t, K) = \max\{S_1, S_2, S_3\} - K$ for a rainbow option. Zero-strike rainbow options are commonly termed best-of-N-assets options, with payoff $\max\{S_{1T}, S_{2T}, ..., S_{NT}\}$ or worst-of-N-assets options, with payoff $\min\{S_{1T}, S_{2T}, ..., S_{NT}\}$.[1]

The most commonly traded two-asset options are exchange options (rainbow options with zero strike) and spread options (basket options with weights 1 and -1). Margrabe [1978] derived an exact solution for the price of an exchange option, under the assumption that the two asset prices follow correlated lognormal processes (in other words, that the asset prices are driven by correlated geometric Brownian motions). However, a straightforward generalization of Margrabe's formula to spread options, or to any two-asset option with non-zero strike, is not possible. An exact analytic solution is elusive because a linear combination of lognormal processes is no longer lognormal, and it is only when the strike of a two-asset option is zero that one may circumvent this problem by reducing the dimension of the correlated lognormal processes to one. For this reason most academic research has focussed on deriving good analytic approximations for pricing two-asset options with non-zero strike, as well as more general multi-asset options.

We now provide a brief overview of the recent research on pricing European basket options: Levy [1992] approximates the basket price distribution with that of a single lognormal variable, matching the first and second moments; Gentle [1993] derives the price by approximating the arithmetic average by a geometric average; Milevsky and Posner [1998a] use the reciprocal gamma distribution and Milevsky and Posner [1998b] use the Johnson [1949] family of distributions to approximate the distribution of the basket price; extending the Asian option pricing approach of Rogers and Shi [1995], Beißer [1999] expresses a basket option price as a weighted sum of single-asset Black-Scholes prices, with adjusted forward price and adjusted strike for every constituent asset; and Ju [2002] uses Taylor expansion to approximate the ratio of the characteristic function of the average of correlated lognormal variables, which is approximately lognormal for short maturities. Krekel et al. [2004] compares the performance of these models concluding that, for the cases they consider, Ju's and Beisser's approximations are most accurate, although they tend to slightly over- and under-price respectively.

Many of these methods have limited validity or scope. They may require a basket value that is always positive, or they may not identify the marginal effect of each individual volatility or pairwise correlation on the multi-asset option price, so calibration of the model price to market prices

[1] Since $\min\{S_{1T}, S_{2T}, ..., S_{NT}\} = -\max\{-S_{1T}, -S_{2T}, ..., -S_{NT}\}$, a pricing model for best-of options also serves for worst-of options.

of liquid options on the same assets may be problematic.

Analytic approximations for pricing rainbow options includes the intuitive inductive formula derived by Johnson [1987], who extends the two-asset rainbow option pricing formula of Stulz [1982] to the general case of N assets. Topper [2001] uses a finite element scheme to solve the associated non-linear parabolic price PDEs for options on two assets with different payoff profiles. Ouwehand and West [2006] verify the results of Johnson [1987] and explain how to prove then using a multivariate normal density approximation derived by West [2005]. Then they explain how to price N asset rainbow options using this approach, providing an explicit approximation for the case $N = 4$.

Our approach is quite novel, in that it is based a simple, recursive decomposition of the basket or rainbow payoff function into a sum of exchange option payoff functions, where the 'assets' in these exchange options are standard call or put options on the original assets. So that there is an exact price for each call or option in the exchange option, we assume that each asset price follows a standard geometric Brownian motion (GBM) process. However, this assumption may be relaxed to allow for more general drift and local volatility components.

The recursive procedure is derived by re-writing the multi-asset option's payoff as a sum of payoffs to various compound exchange options, each with different pairs of sub-ordinate basket options. As a result, the option price can be expressed exactly, as a sum of prices of compound exchange options on various sub-baskets of assets. Hence, the approximate pricing of a compound exchange option (CEO) is central to our work.

Of course, the price of a standard European call or put option follows an Itô's process, but not a lognormal process. However, our approach allows a considerable amount of freedom to choose the strikes of the options in the CEO. In particular, we may choose the strikes so that they are deep in-the-money (ITM); and when an option is deep ITM, its price process can be approximated by a lognormal process. Hence, by using an appropriate choice of strikes for the standard options in each CEO in the recursion, the CEO prices are approximated by the Margrabe [1978] price for exchange options on assets that follow lognormal processes. Then a recursive formula is applied to these CEO prices, to obtain our analytic approximation to the multi-asset option price.

In the remainder of this paper, Section 2 derives a recursive payoff decomposition for multi-asset options, and hence derives the prices of European basket and rainbow options as linear combinations of the prices of standard single-asset options and CEOs. Section 3 explains our approximate price of a CEO, deriving two conditions under which the price processes for the options in the CEO are approximately lognormal. These are: a strong condition, under which an almost exact CEO price can be obtained; and a weaker condition, under which the CEO price is more approximate. Section 4 presents our approximations for basket option prices based on both strong and weak lognormality conditions. Section 5 presents simulations to test the accuracy of our approximations, Section 6 illustrates our recommended approach to model calibration and Section 7 concludes.

2. Pricing Framework

In this section we derive a payoff decomposition for N-asset basket and rainbow options, which is based on the Lemma below. Then we use this decomposition to express the prices of European basket and rainbow options in terms of the prices of standard single-asset options. Price decompositions are considered for a three-asset basket option, four-asset basket and rainbow options, and a seven-asset rainbow option.

Our first result is a simple Lemma that will be applied to derive a recursive formula for multi-asset option payoff decompositions:

Lemma 1. *Let f and g be two real-valued functions. Then,*

$$\begin{aligned} [f+g]^+ &= [f^+ - g^-]^+ + [g^+ - f^-]^+ \\ [f-g]^+ &= [f^+ - g^+]^+ + [g^+ - f^+]^+, \end{aligned} \qquad (1)$$

where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Proof. First consider $[f-g]^+$. We may write

$$\begin{aligned} V = [f-g]^+ &= [f^+ - f^- - (g^+ - g^-)]^+ \\ &= [(f^+ - g^+) + (g^- - f^-)]^+. \end{aligned} \qquad (2)$$

Now:

- If $f > g > 0$, then $(f^+ - g^+) > 0$ and $(g^- - f^-) = 0$;
- If $g > f > 0$, then $(f^+ - g^+) < 0$ and $(g^- - f^-) = 0$;
- If $f < g < 0$, then $(f^+ - g^+) = 0$ and $(g^- - f^-) < 0$;
- If $g < f < 0$, then $(f^+ - g^+) = 0$ and $(g^- - f^-) > 0$;
- If $f > 0, g < 0$, then $(f^+ - g^+) = f^+ > 0$ and $(g^- - f^-) = g^- > 0$;
- If $f < 0, g > 0$, then $(f^+ - g^+) = 0$ and $(g^- - f^-) = -f^- < 0$.

Hence, we may split equation (2) into two separate components and rewrite V as

$$V = [f - g]^{+} = [f^{+} - g^{+}]^{+} + [g^{-} - f^{-}]^{+}. \tag{3}$$

The equality for $[f+g]^{+}$ can be proved similarly, on noting that $[f+g]^{+} = [f-(-g)]^{+}$. □

Now consider a basket option on N assets with prices $\mathbf{S}_t = (S_{1t}, S_{2t}, \ldots, S_{Nt})'$ at time t with weights $\mathbf{\Theta}_N = (\theta_1, \theta_2, \ldots, \theta_N)$, where θ_i are real constants. Denote this basket by $\mathbf{b}_N = (\theta_1 S_{1t}, \theta_2 S_{2t}, \ldots, \theta_N S_{Nt})'$. Then the price of the basket at time t is $B_t = \sum_{i=1}^{N} \theta_i S_{it}$. Denote by V_{Nt} the price of an option on basket $\mathbf{b}_N$ with strike price K and maturity T, at any time t with $0 \le t \le T$. Note that K can be any real number: positive, negative or zero. The option's payoff (i.e. the price at the expiry time, T) is given by:

$$\begin{aligned} V_{NT} = \left[\omega \left(B_T - K\right)\right]^{+} &= \left[\omega \left(\mathbf{\Theta}\, \mathbf{S}_T - K\right)\right]^{+} \\ &= \left[\omega \mathbf{\Theta} \left(\mathbf{S}_T - \mathbf{K}\right)\right]^{+} \end{aligned} \tag{4}$$

where $\omega = 1$ and -1 for calls and puts respectively and $\mathbf{K} = (K_1, K_2, \ldots, K_N)'$ is a column vector of strikes such that $\mathbf{\Theta K} = K$.

Let $\mathbf{b}_m = (\theta_1 S_{1t}, \theta_2 S_{2t}, \ldots, \theta_m S_{mt})'$ and $\mathbf{b}_n = (\theta_{m+1} S_{(m+1)t}, \theta_{m+2} S_{(m+2)t}, \ldots, \theta_N S_{Nt})'$ denote sub-baskets of $\mathbf{b}$ of sizes m and n respectively, with $m + n = N$ and denote the weights vectors of the corresponding sub-baskets by $\mathbf{\Theta}_m$ and $\mathbf{\Theta}_n$, such that $\mathbf{\Theta}_N = (\mathbf{\Theta}_m, \mathbf{\Theta}_n)$. Similarly, let $\mathbf{S}'_t = (\mathbf{S}'_{mt}, \mathbf{S}'_{nt})$ and $\mathbf{K}' = (\mathbf{K}'_m, \mathbf{K}'_n)$. Define the call and put sub-basket option prices as:

$$\begin{aligned} C_{mT} &= \mathbf{\Theta}_m \left[\mathbf{S}_{mT} - \mathbf{K}_m\right]^{+}, \\ P_{mT} &= -\mathbf{\Theta}_m \left[\mathbf{S}_{mT} - \mathbf{K}_m\right]^{+}, \end{aligned}$$

and similarly for n. Now, with this notation and following Lemma 1, equation (4) may be rewritten as

$$\begin{aligned} V_{NT} &= \left[\omega \left[\mathbf{\Theta}_m \left(\mathbf{S}_{mT} - \mathbf{K}_m\right) + \mathbf{\Theta}_n \left(\mathbf{S}_{nT} - \mathbf{K}_n\right)\right]\right]^{+} \\ &= \left[\omega \Big(\mathbf{\Theta}_m \left[\mathbf{S}_{mT} - \mathbf{K}_m\right]^{+} - \mathbf{\Theta}_n \left[(\mathbf{K}_n - \mathbf{S}_{nT})\right]^{+} \Big)\right]^{+} \\ &\qquad + \left[\omega \Big(\mathbf{\Theta}_n \left[(\mathbf{S}_{nT} - \mathbf{K}_n)\right]^{+} - \mathbf{\Theta}_m \left[\mathbf{K}_m - \mathbf{S}_{mT}\right]^{+} \Big)\right]^{+} \\ &= \left[C_{mT} - P_{nT}\right]^{+} + \left[C_{nT} - P_{mT}\right]^{+}. \end{aligned} \tag{5}$$

Alternatively, we may choose $\mathbf{\Theta}_m$ and $\mathbf{\Theta}_n$ such that $\mathbf{\Theta}_N = (\mathbf{\Theta}_m, -\mathbf{\Theta}_n)$. Then the N-asset basket option price may be written as

$$V_{NT} = \left[C_{mT} - C_{nT}\right]^{+} + \left[P_{nT} - P_{mT}\right]^{+}, \tag{6}$$

The European basket option price at any time t before expiry may now be computed as the discounted sum of the risk-neutral expectations of the two replicating CEO payoffs, E_{1T}, E_{2T}, which appear on the right hand side of (5) or (6). That is, in case (5), E_{1T} and E_{2T} are payoffs to exchange options on a basket call and a basket put; and in case (6) they are payoffs to exchange options on two basket calls and two basket puts with a different number of assets in each basket. Hence,

$$V_{Nt} = e^{-r(T-t)} \left(\mathbb{E}_{\mathbb{Q}} \left\{ E_{1T} \middle| \mathcal{F}_t \right\} + \mathbb{E}_{\mathbb{Q}} \left\{ E_{2T} \middle| \mathcal{F}_t \right\} \right). \tag{7}$$

In the recursive framework that we use for pricing, decompositions of the form (5) or (6) are applied to each of C_{mT}, P_{mT}, C_{nT} and P_{nT} in turn, choosing suitable partitions for m and n which determine the number of assets in the sub-ordinate basket calls and puts. By applying the payoff decomposition repeatedly, each time decreasing the number of assets in the sub-ordinate options, one eventually expresses the payoff to an N-asset basket option as a sum of payoffs to compound exchange options in which the sub-ordinate options are standard single-asset calls and puts, and ordinary exchange options. This way, the price of the original N-asset basket options is computed as the sum of the prices of compound exchange options and standard exchange options.

To illustrate our recursive pricing framework we consider a four-asset basket option. Figure 1 depicts a tree that illustrates how the basket option is recursively priced. Due to lack of space we have only shown one leg of the tree, but the other leg can be priced in a similar manner. Considering the entire tree, we have to compute the prices of:

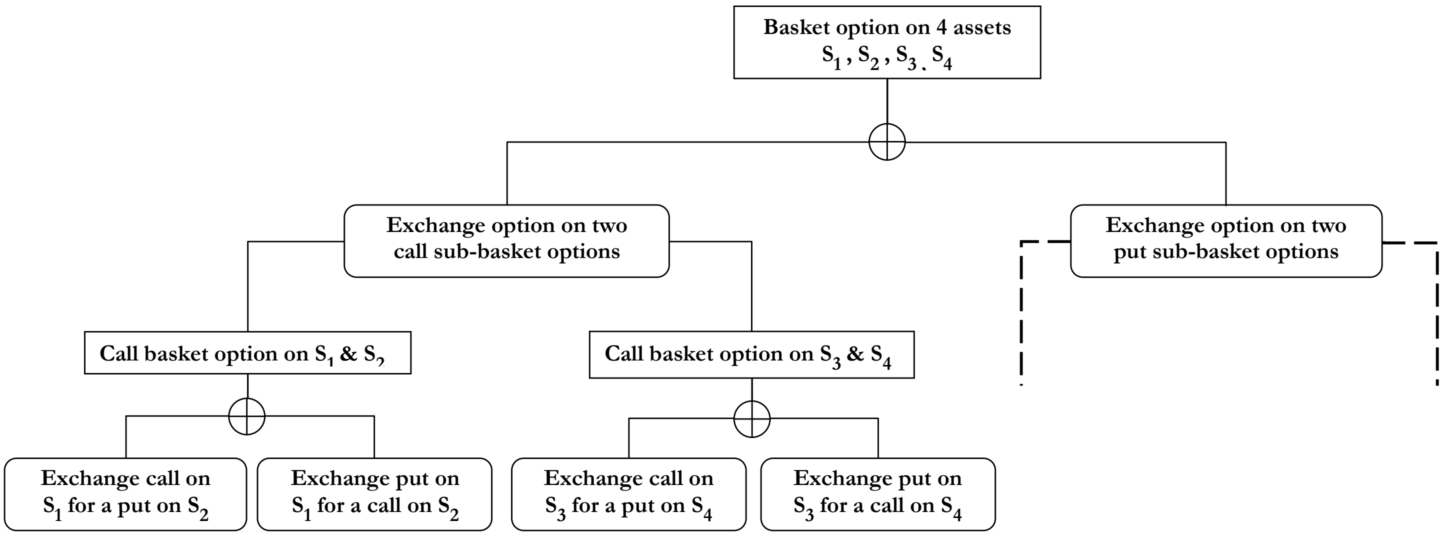
- Two CEOs: one to exchange two call two-asset sub-basket options, and one to exchange two call two-asset sub-basket options;
- Four two-asset basket options (two basket calls and two basket puts); and
- Eight standard CEOs (each is to exchange a call for a put);
- Eight standard options (i.e. calls and puts on each of the four assets).

The four-asset basket option has price equal to the sum of the two CEO prices on the penultimate level of the tree. However, in order to compute the price of these CEOs, we have first to compute the prices of the four two-asset basket options on the level below. To price these, we must first price eight CEOs, and to price the CEOs we need to know the prices of the standard calls and puts on the four assets.

However, we may use put-call parity to obtain

$$\mathbb{E}_{\mathbb{Q}} \left\{ \left[B_T - K \right]^+ \right\} - \mathbb{E}_{\mathbb{Q}} \left\{ \left[K - B_T \right]^+ \right\} = \mathbb{E}_{\mathbb{Q}} \left\{ B_T - K \right\},$$

FIGURE 1: *Pricing Tree for Four-Asset Basket Option*



$\oplus$ denotes sum of prices of the two daughter nodes.

and thus

$$C_{Nt} - P_{Nt} = \sum_{i=1}^{N} \theta_i S_{it} e^{-q_i(T-t)} - K e^{-r(T-t)}, \tag{8}$$

where q_i is the dividend yield of asset i. Therefore, we only need to compute the call option prices, because we can deduce the corresponding put prices using (8). Alternatively, we can compute the put prices, and derive the call prices using (8). Hence, in the general case of N assets, our approach requires the evaluation of $2(N-1)$ CEO prices and N standard option prices.

The options that appear in the terminal nodes of a pricing tree, such as that illustrated in Figure 1, will be either vanilla or exchange options. We only require the weighted sum of their strikes to be equal to the strike of the basket option. For instance, in a three-asset basket option with payoff $[S_{1T} - S_{2T} + S_{3T} - K]^+$, we may write the payoff as

$$\left[[S_{1T} - S_{2T}]^+ - [K - S_{3T}]^+\right]^+ + \left[[S_{3T} - K]^+ - [S_{2T} - S_{1T}]^+\right]^+$$

and hence choose a pair of exchange options on assets 1 and 2 and a pair of vanilla options on asset 3 in the terminal nodes of the tree.

In general, the choice of strikes depends on the moneyness of the basket option and the current price levels of the individual assets. For instance, in the three-asset basket option above, suppose that $S_{10} = 50, S_{20} = 140, S_{30} = 100$ and $K = 10$. Then the decomposition suggested above is inappropriate, since a vanilla call on S_3 with strike 10 will be too deep ITM. There is no general rule.

Alternatively, the terminal nodes might always be regarded as vanilla options because, if the individual strikes are are zero, the penultimate nodes would be exchange options anyway. For example,

$$[S_{1T} - S_{2T} - K]^+ = [[S_{1T} - K_1]^+ - [S_{2T} - K_2]^+]^+ + [[K_2 - S_{2T}]^+ - [K_1 - S_{1T}]^+]^+ ,$$

so if K_1 and K_2 are zero, the first term is simply an exchange option and the second term disappears, since $[0 - S_{1T}]^+ = [0 - S_{2T}]^+ = 0$. So, effectively the terminal nodes are exchange options.

We now provide an example which illustrates how the price of a rainbow option on N assets may be expressed in terms of an N-asset basket option price and exchange option prices.[2] Consider a general rainbow option with payoff $\max\{S_{1T} - K, S_{2T} - K, ..., S_{NT} - K\}$. Let $(n_1, n_2, ..., n_N)$ be a permutation of $(1, 2, ..., N)$ and choose k to be some integer between 1 and N. By splitting the basket of N assets into two sub-baskets, where k determine the size of the sub-baskets and the permutation $(n_1, n_2, ..., n_N)$ determines the assets in these sub-baskets, we have

$$\begin{aligned}\max\{S_{1T} - K, S_{2T} - K, ..., S_{NT} - K\} &= \max\left\{S_{n_{k+1}T}, S_{n_{k+2}T}, ..., S_{n_N T}\right\} - K \\ &+ \left[\max\left\{S_{n_1 T}, S_{n_2 T}, ..., S_{n_k T}\right\} - \max\left\{S_{n_{k+1}T}, S_{n_{k+2}T}, ..., S_{n_N T}\right\}\right]^+ . \end{aligned} \qquad (9)$$

This shows that the rainbow option's payoff may be written as sum of two payoffs, one to a best-of option on a sub-basket and the other to a compound option.

For every permutation $(n_1, n_2, ..., n_N)$ and index k we obtain a different payoff decomposition for the rainbow option. An illustration of two alternative decompositions, for the case that $N = 4$, are given below. Obviously, the value of the payoff will be the same in each case, so the model should be calibrated in such a way that the option price is invariant to the choice of $(n_1, n_2, ..., n_N)$ and k.

The best-of option payoff terms on the right hand side of (9) may themselves be represented as the sum of two such payoffs, until all sub-baskets are on one or two assets. Once the sub-basket size is eventually reduced to two, we use the identity

$$\max\left\{S_{iT}, S_{jT}\right\} \quad = \quad S_{jT} + [S_{iT} - S_{jT}]^+ .$$

For illustration, consider a rainbow option on four assets. Here it is convenient to use the notation $X_{n_i t}$ for the price of an option to receive asset n_i in exchange for selling asset n_{i+1}. Choosing

[2] Note that rainbow options on N assets have been priced under the GBM assumption in Ouwehand and West [2006].

$(n_1, n_2, n_3, n_4) = (1, 2, 3, 4)$ and $k = 2$, the rainbow option's payoff P_{4T} may be written:

$$\begin{aligned}
P_{4T} &= \max\{S_{1T} - K, S_{2T} - K, S_{3T} - K, S_{4T} - K\} \\
&= \max\{S_{3T}, S_{4T}\} - K + [\max\{S_{1T}, S_{2T}\} - \max\{S_{3T}, S_{4T}\}]^+ \\
&= S_{4T} + [S_{3T} - S_{4T}]^+ + \left[S_{2T} + [S_{1T} - S_{2T}]^+ - S_{4T} - [S_{3T} - S_{4T}]^+\right]^+ - K \\
&= S_{4T} + X_{3T} + [S_{2T} + X_{1T} - S_{4T} - X_{3T}]^+ - K \\
&= S_{4T} + X_{3T} + [B_T]^+ - K. \qquad (10)
\end{aligned}$$

Thus, the price of the rainbow option is

$$P_{4t} = S_{4t} + X_{3t} + V_t - Ke^{-r(T-t)},$$

where $V_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\{[B_T]^+\}$ is the price of a zero-strike basket option with four assets whose prices are $\{X_{1t}, S_{2t}, X_{3t}, S_{4t}\}$ and with weights $\{1, 1, -1, -1\}$. Recall that, under the correlated log-normal assumption, an analytic solution exists for X_{it}. Hence P_{4t} may be evaluated because we have already derived the price V_t of the basket option; it may be expressed in terms of CEO prices, as in (7).

We have chosen $k = 2$ above because this choice leads to the simplest form of payoff decomposition for a four-asset rainbow option. In fact, for any permutation (n_1, n_2, n_3, n_4), a similar argument to that above may be applied to show that the payoff decomposition is:

$$P_{4T} = S_{n_4T} + X_{n_3T} + [S_{n_2T} - X_{n_1T} - S_{n_4T} + X_{n_3T}]^+ - K.$$

Hence, a general expression for the price of a four-asset rainbow option is

$$P_{4t} = S_{n_4t} + X_{n_3t} + V_{4t} - Ke^{-r(T-t)},$$

where $V_{4t} = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\{[B_T]^+\}$ denotes the price of a zero-strike basket option with four assets whose prices are $\{X_{n_1t}, S_{n_2t}, X_{n_3t}, S_{n_4t}\}$ and with weights $\{1, 1, -1, -1\}$.

Finally, we provide an example that illustrates how to extend this argument to rainbow options on more than four assets. Following the lines of the four-asset rainbow option, the payoff to a rainbow option on three assets, having prices S_5, S_6 and S_7, may be written as:

$$\begin{aligned}
P_{3T} &= \max\{S_{5T}, S_{6T}, S_{7T}\} \\
&= S_{7T} + [S_{6T} - S_{7T}]^+ + \left[S_{5T} - S_{7T} - [S_{6T} - S_{7T}]^+\right]^+ \\
&= S_{7T} + X_{6T} + [S_{5T} - S_{7T} + X_{6T}]^+,
\end{aligned}$$

and the price of a seven-asset rainbow option is given by

$$P_{7t} = P_{3t} + \mathbb{E}_{\mathbb{Q}} \left\{ [P_{4T} - P_{3T}]^{+} \right\}. \tag{11}$$

3. Pricing Compound Exchange Options

A compound exchange option (CEO) is an option to exchange one option for another. The two underlying options may be on different assets, or may themselves be compound options. In general, the underlying options may also have different maturities. However, we shall always assume that both of the underlying options have the same maturity as the CEO, because this will always be the case for the framework described in the previous section.

The payoff decompositions illustrated above have employed options on a basket which may contain the assets themselves, options written on these assets, and options to exchange these assets. For instance, the basket on the right-hand side of equation (10) is defined on two assets and two standard exchange options.

Having applied a suitable payoff decomposition, we use equation (7) to express the price of the basket options in the decomposition as a sum of CEO prices. Thus, pricing CEOs is central to our framework. In this section we derive an analytic approximation for the price of CEO, first under the assumption that the underlying assets of these options follow correlated lognormal processes, and then under more general assumptions for the asset price processes. Our key idea is to express the price of such a CEO as the price of a single-asset option, which can be easily derived.

Let $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q} \right)$ be a filtered probability space, where Ω is the set of all possible events such that $S_{1t}, S_{2t} \in (0, \infty)$, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration produced by the sigma algebra of the price pair $(S_{1t}, S_{2t})_{t\geq 0}$ and $\mathbb{Q}$ is a bivariate risk-neutral probability measure. Assume that the risk-neutral price dynamics are governed by :

$$\begin{aligned} dS_{it} &= rS_{it}dt + \sigma_i S_{it} dW_{it} \\ \langle dW_{1t}, dW_{2t} \rangle &= \rho dt \qquad i = 1,\ 2, \end{aligned}$$

where, W_1 and W_2 are Wiener processes under the risk-neutral measure $\mathbb{Q}$, σ_i is the volatility of asset i (assumed constant) and ρ is the correlation between dW_1 and dW_2 (assumed constant).

We now consider compound options to exchange a standard European call or put on asset 1 with a standard European call or put on asset 2, where all options have the same maturity, but the standard options may have different strikes. Let U_{1t} and V_{1t} denote the prices of a pair of standard European call and put options with common maturity T and common strike K_1. Thus, U_{1T} is the payoff to a call on asset 1 and V_{1T} is the payoff to a put on asset 1. Similarly, let U_{2t} and V_{2t}

denote the prices of a pair of standard European call and put options on asset 2 with strike K_2 and maturity T.

If the CEO is on two calls the payoff is $[\omega(U_{1T}-U_{2T})]^+$, if the CEO is on two puts the payoff is $[\omega(V_{1T}-V_{2T})]^+$, and the payoff is either $[\omega(U_{1T}-V_{2T})]^+$ or $[\omega(V_{1T}-U_{2T})]^+$ if the CEO is on a call and a put, where $\omega=1$ for a call CEO and -1 for a put CEO.

The price of a CEO can be obtained as a risk-neutral expectation of the terminal payoff. For instance, if the CEO is on two calls then its price is:

$$f_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\Big\{ [\omega(U_{1T}-U_{2T})]^+ \,|\mathcal{F}_t\Big\}. \tag{12}$$

But the application of risk-neutral valuation requires a description of the price processes U_{it} and V_{it}, for $i=1,2$.

To this end we apply Itô's Lemma and the Black-Scholes differential equation to (12), which yields:

$$\begin{aligned} dU_{it} &= \frac{\partial U_{it}}{\partial t}dt + \frac{\partial U_{it}}{\partial S_{it}}dS_{it} + \frac{1}{2}\frac{\partial^2 U_{it}}{\partial S_{it}^2}dS_{it}^2 \\ &= \left(\frac{\partial U_{it}}{\partial t} + rS_{it}\frac{\partial U_{it}}{\partial S_{it}} + \frac{1}{2}\sigma_i^2 S_{it}^2\frac{\partial^2 U_{it}}{\partial S_{it}^2}\right)dt + \sigma_i S_{it}\frac{\partial U_{it}}{\partial S_{it}}dW_{it} \\ &= rU_{it}dt + \xi_{it}U_{it}dW_{it}, \end{aligned} \tag{13}$$

where, $\xi_{it} = \sigma_i \dfrac{S_{it}}{U_{it}}\dfrac{\partial U_{it}}{\partial S_{it}}$.

Applying a similar argument to the standard put option, its price process may be written:

$$dV_{it} = rV_{it}dt + \eta_{it}V_{it}dW_{it}, \tag{14}$$

where, $\eta_{it} = \sigma_i \dfrac{S_{it}}{V_{it}}\left|\dfrac{\partial V_{it}}{\partial S_{it}}\right|$.[3]

For brevity we shall restrict our subsequent analysis to pricing a CEO on two calls, because the derivations in the remainder of this section are very similar when one or both of the standard options are puts. We first solve for the prices U_{it} of the underlying options and their volatilities ξ_{it}, for $i=1,2$. Then we derive 'weak' and 'strong' lognormality conditions, under which the CEO price is approximately equal to a single-asset option price under an equivalent measure. Hence we derive an approximate closed-form solution to the approximate CEO price.

Lemma 2. *When the asset price S_{it} follows a GBM with Wiener process W_{it}, a standard call option on asset*

[3]Since the delta of a put option is negative, we take the absolute value here. However, due to the symmetry of Wiener process - i.e., upward and downward movements have equal probabilities - the sign of volatility does not affect the distribution of the diffusion term $\eta_i dW_{it}$. The use of the absolute value here merely changes the sign of the correlation between the Wiener process driving the put option and any other process.

i has price process (13) with volatility ξ_{it} following the process:

$$d\xi_{it} = \xi_{it}\left(\sigma_i - \xi_{it} + \sigma_i S_{it}\frac{\Gamma_{it}}{\Delta_{it}}\right)\left[-\xi_{it}dt + dW_{it}\right], \tag{15}$$

where $\Delta_{it} = \dfrac{\partial U_{it}}{\partial S_{it}}$ and $\Gamma_{it} = \dfrac{\partial \Delta_{it}}{\partial S_{it}}$ are the delta and gamma of the call option.

Proof. Let $\theta_{it} = \dfrac{\partial U_{it}}{\partial t}$ and $X_{it} = \dfrac{\Delta_{it}}{U_{it}}$. Dropping the subscripts, we have, by Itô's Lemma:

$$\begin{aligned}
d\Delta &= \frac{\partial \Delta}{\partial t}dt + \frac{\partial \Delta}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \Delta}{\partial S^2}dS^2 \\
&= \frac{\partial \theta}{\partial S}dt + \frac{\partial \Delta}{\partial S}\left(rSdt + \sigma SdW\right) + \frac{1}{2}\frac{\partial^2 \Delta}{\partial S^2}\sigma^2 S^2 dt \\
&= \frac{\partial}{\partial S}\left(\theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma\right)dt - \left(r\Delta + \Gamma\sigma^2 S\right)dt + \sigma S\Gamma dW \\
&= \frac{\partial}{\partial S}\left(rU\right)dt - \left(r\Delta + \Gamma\sigma^2 S\right)dt + \sigma S\Gamma dW \\
&= -\Gamma\sigma^2 S dt + \sigma S\Gamma dW,
\end{aligned}$$

and

$$\begin{aligned}
dX &= \frac{1}{U}d\Delta - \frac{\Delta}{U^2}dU + \frac{\Delta}{U^3}dU^2 - \frac{1}{U^2}d\Delta dU \\
&= \frac{1}{U}\left(-\Gamma\sigma^2 Sdt + \sigma S\Gamma dW - \Delta\left(rdt + \xi dW\right) + \Delta\xi^2 dt - \sigma S\Gamma\xi dt\right) \\
&= \frac{1}{U}\left(\Delta\xi^2 - \sigma S\Gamma\xi - r\Delta - \Gamma\sigma^2 S\right)dt + \frac{1}{U}\left(\sigma S\Gamma - \xi\Delta\right)dW.
\end{aligned}$$

Since $\xi = \sigma S X$,

$$\begin{aligned}
d\xi &= \sigma\left(XdS + SdX + dSdX\right) \\
&= \sigma\left(XS\left(rdt + \sigma dW\right) + \frac{S}{U}\left(\Delta\xi^2 - \sigma S\Gamma\xi - r\Delta - \Gamma\sigma^2 S\right)dt\right. \\
&\qquad \left. + \frac{S}{U}\left(\sigma S\Gamma - \xi\Delta\right)dW + \frac{\sigma S}{U}\left(\sigma S\Gamma - \xi\Delta\right)dt\right) \\
&= \left(\sigma\xi - \xi^2 + \frac{\sigma^2 S^2}{U}\Gamma\right)\left[-\xi dt + dW\right],
\end{aligned}$$

which can be rewritten as (15). □

Now we define

$$\tilde{\sigma}_i = \sigma_i + c_i, \text{ with } c_i = \sigma_i S_{it}\Gamma_{it}/\Delta_{it}. \tag{16}$$

That is, we set $\sigma_i S_{it}\Gamma_{it}/\Delta_{it}$ to be constant in equation (15). This is the only approximation we make for pricing CEOs.[4] For in-the-money (ITM) options, $\sigma_i S_{it}\Gamma_{it}/\Delta_{it} << (\xi_{it} - \sigma_i)$ and for out-of-the-money (OTM) options, $\sigma_i S_{it}\Gamma_{it}/\Delta_{it} \approx (\xi_{it} - \sigma_i)$, which implies $(\xi_{it} - \sigma_i + \sigma_i S_{it}\Gamma_{it}/\Delta_{it}) \approx 2(\xi_{it} - \sigma_i)$. For example, in the Black-Scholes model, when $S = 100$, $T = 30$ days, $\sigma = 0.2$, $r = 5\%$, then the values of $(\sigma_i S_{it}\Gamma_{it}/\Delta_{it}, (\xi_{it} - \sigma_i))$ are: $(4.13, 2.5)$, $(1.66, 0.2)$, $(0.78, 4 \times 10^{-04})$, $(0.46, 2.9 \times 10^{-09})$, $(0.29, 3.3 \times 10^{-18})$ for call options with strikes 100, 90, 80, 70 and 60 respectively.

Approximating $\sigma_i\left(1 + S_{it}\Gamma_{it}/\Delta_{it}\right)$ with a constant allows one to derive an approximate solution for ξ_{it} irrespective of the moneyness of the option. The approximation error will be highest when the moneyness of the option swings between ITM and OTM (i.e. when the option is close to at-the-money (ATM), because then asset price crosses the strike frequently). But it is important to note that, since the pricing framework described in section 2 requires only the constraint that the weighted sum of strikes of the single-asset options on all assets be equal to K, we can choose the strikes of these options to minimize the approximation error, subject to this constraint. Thus, we should aim to choose either very low or very high strike values. In other words, we should choose the strikes K_1 and K_2 so that the call (or put) options are deep ITM or OTM.

Under the approximation (16), the option volatility processes are given by

$$\begin{aligned} d\xi_{it} &= \xi_{it}(\xi_{it} - \tilde{\sigma}_i)\left(\xi_{it}dt - dW_{it}\right) \\ &= \alpha(\xi_{it})dt + \beta(\xi_{it})dW_{it}. \end{aligned} \tag{17}$$

The next result solves the SDEs (17) and hence provides the volatilities ξ_{it} that are used to price standard European call options at the base of the pricing tree directly, by approximating their future values under the process (13). The volatility processes η_{it} for put option prices, which follow the process (14), are derived analogously. Thus, the following lemma considers the call option volatilities only, for brevity.

Theorem 3. *Let* $k_i = \left(\dfrac{\tilde{\sigma}_i}{\xi_{i0}} - 1\right)$ *and* $W^*_{it} = -\frac{1}{2}\tilde{\sigma}t + W_{it}$. *Then (17) has solution*

$$\xi_{it} = \tilde{\sigma}_i\left(1 + k_i e^{-\tilde{\sigma}_i W^*_{it}}\right)^{-1}. \tag{18}$$

When $\xi_{i0} > \tilde{\sigma}_i$, k_i *is negative and the option volatility process explodes in finite time. Moreover, the boundary at* ∞ *is an exit boundary.*

Proof. The proof is the same for $i = 1$ and 2, so we can drop the subscript i for convenience. If $\xi_0 = \tilde{\sigma}$ then $\xi_t = \tilde{\sigma}$ for all $t > 0$. So in the following we consider two separate cases, according as $\xi_0 < \tilde{\sigma}$ and $\xi_0 > \tilde{\sigma}$.

[4] It is similar to assuming a linear relationship between the price and yield volatility of a bond, where a linear relationship is only exact when the bond price P is related to its yield y as $P = e^{-yT}$, and in that case $\frac{dP}{P} = -Tdy$ so the bond's price and yield volatilities are proportional.

It follows from (17) that $d\xi_t \to 0$ whenever $\xi_t \to 0$ or $\tilde{\sigma}$. So if the process is started with a value $\xi_0 < \tilde{\sigma}$ then ξ_t will remain bounded between 0 and $\tilde{\sigma}$ ($\tilde{\sigma} \geq \xi_t \geq 0$) for all $t \in [0, T]$. On the other hand, when $\xi_0 > \tilde{\sigma}$, ξ_t is bounded below by $\tilde{\sigma}$ but not bounded above, and $\xi_t - \tilde{\sigma} \geq 0$.

Set $x_t = \dfrac{1}{\tilde{\sigma}} \ln \left| \dfrac{\xi_t - \tilde{\sigma}}{\xi_t} \right|$. Then

$$
\begin{aligned}
dx_t &= \frac{1}{\tilde{\sigma}} \left(\frac{1}{\xi_t - \tilde{\sigma}} - \frac{1}{\xi_t} \right) d\xi_t + \frac{1}{2\tilde{\sigma}} \left(\frac{-1}{(\xi_t - \tilde{\sigma})^2} + \frac{1}{\xi_t^2} \right) d\xi_t^2 \\
&= \frac{d\xi_t}{\xi_t(\xi_t - \tilde{\sigma})} + \frac{1}{2}(\tilde{\sigma} - 2\xi_t) \left(\frac{d\xi_t}{\xi_t(\xi_t - \tilde{\sigma})} \right)^2 \\
&= \xi_t dt - dW + \frac{1}{2}(\tilde{\sigma} - 2\xi_t) dt \\
&= \frac{1}{2}\tilde{\sigma} dt - dW.
\end{aligned}
$$

Thus

$$
x_t = x_0 + \frac{1}{2}\tilde{\sigma} t - W_t.
$$

Substituting x_t into the above equation yields

$$
\begin{aligned}
\frac{1}{\tilde{\sigma}} \ln \left| \frac{\xi_t - \tilde{\sigma}}{\xi_t} \right| &= \frac{1}{\tilde{\sigma}} \ln \left| \frac{\xi_0 - \tilde{\sigma}}{\xi_0} \right| + \frac{1}{2}\tilde{\sigma} t - W_t \\
\frac{\tilde{\sigma}}{\xi_t} - 1 &= k e^{\frac{1}{2}\tilde{\sigma}^2 t - \tilde{\sigma} W_t} \\
\xi_t &= \tilde{\sigma} \left(1 + k e^{-\tilde{\sigma} W_t^*} \right)^{-1}.
\end{aligned}
$$

Now we show that the option volatility process explodes when $\xi_{i0} > \tilde{\sigma}_i$ and the boundary at ∞ is an exit boundary. That is, ξ_{it} reaches ∞ in finite time and once it reaches ∞, it stays there. From the above equation, we see that $\xi_t \to \infty$ when $W_t^* \to \tilde{\sigma}^{-1} \ln |k|$. So the volatility process could reach ∞ in finite time. However, when $W_t^* < \tilde{\sigma}^{-1} \ln |k|$, the above equation implies that ξ_t could become negative, which cannot be true. So to prove that ξ_t remains strictly positive, we need to know more about the boundary at ∞. If the boundary is an 'exit' boundary, then ξ_t cannot return back once it enters the region.[5] That is, if $\xi_\tau = \infty$ for some stopping time $0 \leq \tau \leq T$, then $\xi_s = \infty$ for all $s > \tau$.

In fact, we can indeed classify ∞ as an exit boundary, and to show this we perform the test described in Lewis [2000], Durrett [1996] and Karlin and Taylor [1981]. To this end, let $s(y)$, $m(y)$ be functions such that, for $0 < y < \infty$,

$$
\begin{aligned}
s(y) &= \exp \left\{ - \int^y \frac{2\alpha(x)}{\beta(x)^2} dx \right\} \\
m(y) &= \beta(y)^2 s(y).
\end{aligned}
$$

[5]See Lewis [...] and [...] for further details on volatility explosion and types of boundaries.

Define $S(c,d)$, $M(c,d)$ and $N(d)$ as:

$$\begin{aligned} S(c,d) &= \int_c^d s(y)dy \\ M(c,d) &= \int_c^d \frac{1}{m(y)}dy \\ N(d) &= \lim_{c\downarrow 0} \int_c^d \frac{S(c,y)}{m(y)}dy \end{aligned}$$

Then, the Feller test states that the boundary at ∞ is an exit boundary of the process iff $M(0,d) = \infty$ and $N(0) < \infty$.

In our case we have $\alpha(x) = x^2(x - \tilde{\sigma})$ and $\beta(x) = -x(x - \tilde{\sigma})$, so

$$\begin{aligned} s(y) &= \exp\left\{-\int^y \frac{2}{x-\tilde{\sigma}}dx\right\} = \frac{\tilde{\sigma}^2}{(y-\tilde{\sigma})^2} \\ m(y) &= \tilde{\sigma}^2 y^2 \\ S(c,d) &= \tilde{\sigma}^2\left[\frac{1}{\tilde{\sigma}-d} - \frac{1}{\tilde{\sigma}-c}\right] \\ M(c,d) &= \frac{1}{\tilde{\sigma}^2}\left[\frac{1}{c} - \frac{1}{d}\right] \\ N(d) &= \int_0^d S(0,x)m(x)dx = -\frac{d}{\tilde{\sigma}} - \ln|d-\tilde{\sigma}| + \ln\tilde{\sigma}. \end{aligned}$$

This shows that $M(0,d) = \infty$ and $N(0) < \infty$, hence ξ_{it} explodes and the boundary at ∞ is an exit boundary. □

Now that we have characterized the option price volatility, it is straightforward to find the option price under our lognormal approximation, as follows:

Corollary 4. *When the option volatilities are given by (18) the call option price at time t is:*

$$U_{it} = \frac{U_{i0}\, e^{rt}}{1+k_i}\left(e^{\tilde{\sigma}_i W^*_{it}} + k_i\right) \tag{19}$$

where $k_i = \left(\frac{\tilde{\sigma}_i}{\xi_{i0}} - 1\right)$ and $W^*_{it} = -\frac{1}{2}\tilde{\sigma}t + W_{it}$. Moreover, when $\xi_{i0} > \tilde{\sigma}_i$, $U_{it} \to 0$ as $\xi_{it} \to \infty$.

Proof. Given the option price SDE (13), dropping the subscript i, we have:

$$U_t = U_0 \exp\left(rt - \frac{1}{2}\int_0^t \xi_t^2 dt + \int_0^t \xi_t dW_t\right) \tag{20}$$

Also note that

$$\begin{aligned} d(\ln|\tilde{\sigma}-\xi_t|) &= \frac{d\xi_t}{\xi_t-\tilde{\sigma}} - \frac{1}{2}\frac{d\xi_t^2}{(\xi_t-\tilde{\sigma})^2} \\ &= \frac{1}{2}\xi_t^2 dt - \xi_t dW_t. \end{aligned}$$

Substituting this in (20) gives

$$\begin{aligned} U_t &= U_0 \exp\left(rt - \int_0^t d(\ln|\tilde{\sigma}-\xi_t|)\right) \\ &= U_0 \exp\left(rt - \ln|\tilde{\sigma}-\xi_t| + \ln|\tilde{\sigma}-\xi_0|\right) \\ &= U_0 e^{rt}\left(\frac{\tilde{\sigma}-\xi_0}{\tilde{\sigma}-\xi_t}\right). \end{aligned} \tag{21}$$

Now, substituting equation (18) in (21), we may rewrite:

$$\begin{aligned} U_t &= \frac{U_0\xi_0}{\tilde{\sigma}}e^{rt}\left(\exp\left(-\frac{1}{2}\tilde{\sigma}^2 t + \tilde{\sigma}W_t\right) + \left(\frac{\tilde{\sigma}}{\xi_0} - 1\right)\right) \\ &= \frac{U_0 e^{rt}}{1+k}\left(e^{\tilde{\sigma}W_t^*} + k\right). \end{aligned}$$

From Theorem 4, since ξ_t is bounded between 0 and $\tilde{\sigma}$ when $\xi_0 < \tilde{\sigma}$ and bounded below by $\tilde{\sigma}$ when $\xi_0 > \tilde{\sigma}$, we can conclude that U_t will remain strictly positive for all time $t \in [0, T]$. However, for $\xi_0 > \tilde{\sigma}$, if the volatility explodes ($\xi_t \to \infty$), equation (21) shows that $U_t \to 0$. Moreover, once U_t reaches zero, it will stay there. □

The option price process (19) will follow an approximately lognormal process if the initial option volatility satisfies

$$\xi_{i0} \approx \tilde{\sigma}_i, \tag{22}$$

and we shall call (22) the *weak* lognormality condition. But when is $\xi_{i0} \approx \tilde{\sigma}_i$? By definition, we have

$$\xi_{it} = \sigma_i \frac{S_{it}}{U_{it}}\frac{\partial U_{it}}{\partial S_{it}},$$

and so $\xi_{it} \to \sigma_i$ if $\frac{\partial U_{it}}{\partial S_{it}} \to 1$ and $\frac{S_{it}}{U_{it}} \to 1$ as $t \to T$. Then, $\sigma_i \approx \tilde{\sigma}_i$ and therefore $\xi_{i0} \approx \tilde{\sigma}_i$. Moreover, from equation (18), $\xi_{it} \approx \tilde{\sigma}_i$, for all $t \in [0, T]$.

In the following we shall apply the weak lognormality condition to price multi-asset options by choosing the strikes of the exchange or vanilla options in the terminal nodes of the pricing tree so that they are as far as possible from ATM. A justification for this is that the price of a deep ITM exchange option is an almost linear function of the relative price of the two underlying assets and, under the GBM assumption, the relative price distribution is lognormal. The price of a deep OTM exchange option is approximately zero. Similarly, the price of a deep ITM vanilla option is an almost linear function of the price of the underlying asset and the price of a deep OTM vanilla option is approximately zero.

Under the weak lognormality condition, the option price volatilities ξ_{it} and η_{it} are directly approximated as constants in equations (13) and (14), respectively. This allows us to approximate the option price processes as lognormal processes. Hence, we can change the numeraire to be one of the option prices, so that the price of a CEO may be expressed as the price of a single-asset option, and we can price the CEO using the formula given by Margrabe [1978].

It is important to note that, although it is necessary for the underlying price volatility σ_i to be a constant, the weak lognormality condition does not depend on the choice of drift of the stochastic process of the underlying asset price. This is because, on deriving the option price SDEs (13) and (14), the price process drift $\mu_{it}(.)$, which need not be constant or equal to the risk-free rate r in general, is absorbed into the drift term of the Black-Scholes PDE. And since options are non-dividend paying traded instruments, their drifts must be equal to r. Hence, when we apply the weak lognormality condition, many choices of drift are possible. For instance, for non-traded assets such as commodity spots, we may choose $\mu_{it} = \kappa(\theta(t) - \ln S_{it})$ as in Schwartz [1997] and Pilipovic [2007].

The weak lognormality condition will be particularly useful for deriving analytic approximations to basket option prices, in section 4. However, for some multi-asset options it is possible to obtain an almost exact price, which is more accurate than the prices obtained by assuming only the weak lognormality condition. The following Theorem provides a stronger condition, which we therefore call the *strong* lognormality condition, under which the relative option price follows a lognormal process almost exactly. Our next theorem shows that the price of a CEO may again be expressed as the price of a single-asset option, but now we can obtain an almost exact solution for its price.

Theorem 5. *Let $k_i = \tilde{\sigma}_i/\xi_0 - 1$, for $i = 1, 2$. If the following condition holds*

$$U_{10}\frac{k_1}{1+k_1} - U_{20}\frac{k_2}{1+k_2} = 0, \tag{23}$$

the CEO on calls has the same price as a standard single-asset option under a modified yet equivalent measure.

Proof. We shall prove the result for a compound call exchange option on two single-asset call options. Results for a compound put exchange option on two calls can be derived in an exactly parallel fashion.[6] The call CEO on two calls has price given by:

$$\begin{aligned}
f_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{[\omega(U_{1T}-U_{2T})]^+\right\} \\
&= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left\{\left[\frac{U_{10}e^{rT}}{1+k_1}\left(e^{\tilde{\sigma}_1 W^*_{1T}}-k_1\right)-\frac{U_{20}e^{rT}}{1+k_2}\left(e^{\tilde{\sigma}_2 W^*_{2T}}-k_2\right)\right]^+\right\} \\
&= \mathbb{E}_{\mathbb{Q}}\left\{\left[\frac{U_{10}}{1+k_1}e^{\tilde{\sigma}_1 W^*_{1T}}-\frac{U_{20}}{1+k_2}e^{\tilde{\sigma}_2 W^*_{2T}}-\left(U_{10}\frac{k_1}{1+k_1}-U_{20}\frac{k_2}{1+k_2}\right)\right]^+\right\} \\
&= \mathbb{E}_{\mathbb{Q}}\left\{\left[\frac{U_{10}}{1+k_1}\exp\left(\int_0^T -\frac{1}{2}\tilde{\sigma}_1^2 ds+\tilde{\sigma}_1 dW_{1s}\right)-\frac{U_{20}}{1+k_2}\exp\left(\int_0^T -\frac{1}{2}\tilde{\sigma}_2^2 ds+\tilde{\sigma}_2 dW_{2s}\right)\right]^+\right\}.
\end{aligned}$$

Let $dW_{1t} = \rho dW_{2t} + \rho' dZ_{1t}$, where W_2 and Z_1 are independent Wiener processes, $\rho' = \sqrt{1-\rho^2}$ and $\mathbb{P}$ is a probability measure whose Radon-Nikodym derivative with respect to $\mathbb{Q}$ is given by:

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\tilde{\sigma}_2^2 t + \tilde{\sigma}_2 W_{2t}\right).$$

Let $Y_t = \frac{U_{1t}(1+k_2)}{U_{2t}(1+k_1)}$ and $Z_{2t} = W_{2t} - \tilde{\sigma}_2 t$. Then Z_1 and Z_2 are independent Brownian motions under the measure $\mathbb{P}$ and the dynamics of Y can be described by

$$\begin{aligned}
d(\ln Y_t) &= \left(-\frac{1}{2}\tilde{\sigma}_1^2+\frac{1}{2}\tilde{\sigma}_2^2\right)dt + (\rho\tilde{\sigma}_1-\tilde{\sigma}_2)dW_{2t} - \rho'\tilde{\sigma}_1 dZ_{1t} \\
&= \left(-\frac{1}{2}\tilde{\sigma}_1^2-\frac{1}{2}\tilde{\sigma}_2^2+\rho\tilde{\sigma}_1\tilde{\sigma}_2\right)dt + (\rho\tilde{\sigma}_1-\tilde{\sigma}_2)(dW_{2t}-\tilde{\sigma}_2 dt) - \rho'\tilde{\sigma}_1 dZ_{1t} \\
&= -\frac{1}{2}\tilde{\sigma}^2 dt + (\rho\tilde{\sigma}_1-\tilde{\sigma}_2)dZ_{2t} - \rho'\tilde{\sigma}_1 dZ_{1t} \\
&= -\frac{1}{2}\tilde{\sigma}^2 dt + \tilde{\sigma}_t dW_t
\end{aligned}$$

where W is a Brownian motion under $\mathbb{P}$ and

$$\tilde{\sigma}^2 = (\rho\tilde{\sigma}_1-\tilde{\sigma}_2)^2 + (\rho'\tilde{\sigma}_1)^2 = \tilde{\sigma}_1^2+\tilde{\sigma}_2^2-2\rho\tilde{\sigma}_1\tilde{\sigma}_2.$$

Now the price of the CEO can be written as the price of a single-asset option written on Y, as

$$\begin{aligned}
f_0 &= \frac{U_{20}}{1+k_2}\mathbb{E}_{\mathbb{P}}\left\{\left[Y_t\exp\left(-\frac{1}{2}\int_0^T \tilde{\sigma}_s^2 ds + \int_0^T \tilde{\sigma}_s dW_s\right)-1\right]^+\right\} \\
&= \frac{U_{20}}{1+k_2}\mathbb{E}_{\mathbb{P}}\left\{[Y_T-1]^+\right\}.
\end{aligned}$$

When $\tilde{\sigma}_i > \xi_{i0}$, for $i = 1,2$, the above expectation simply yields the Black-Scholes price of a European option on Y_t with strike equal to one. But when $\tilde{\sigma}_i < \xi_{i0}$, $\xi_{i\tau}$ could reach ∞ for some stopping time $\tau \leq T$. However, when the volatility explodes $U_{is} = 0$ for $s \geq \tau$, since the boundary at ∞ is an exit boundary, and the expectation in equation (24) need only be computed over the

[6] As may the analogous results for exchange options on two puts, or exchange options on puts and calls.

paths for which the individual option volatilities remain finite.

In Theorem 3 we saw that, for ξ_{it} to remain finite, we must have $W_{it} > \frac{1}{2}\tilde{\sigma}_i t + \tilde{\sigma}_i^{-1} \ln |k_i|$ for all $t \in [0, T]$. Now,

$$\begin{aligned} Z_{1t} &> \frac{1}{\rho'} \left(\frac{1}{2}(\tilde{\sigma}_1 - \tilde{\sigma}_2)t + \tilde{\sigma}_1^{-1} \ln |k_1| - \tilde{\sigma}_2^{-1} \ln |k_2| \right) = \mu_1; \\ Z_{2t} &> -\frac{1}{2}\tilde{\sigma}_2 t + \tilde{\sigma}_2^{-1} \ln |k_2| = \mu_2. \end{aligned}$$

Hence, setting $m_i = \min \{Z_{is} : 0 \leq s \leq T\}$, the price of the CEO may be written:

$$f_0 = \frac{U_{20}}{1+k_2} \mathbb{E}_{\mathbb{P}} \left\{ \mathbf{1}_{m_1 > \mu_1; m_2 > \mu_2} \left[Y_T - 1 \right]^+ \right\} + \frac{U_{10}}{1+k_1} \mathbb{E}_{\mathbb{Q}} \left\{ \mathbf{1}_{m_1 > \mu_1; m_2 < \mu_2} \left[e^{\tilde{\sigma}_1 W_{1T}^*} - k_1 \right]^+ \right\} \tag{24}$$

where $\mathbf{1}$ is the indicator function. The first term on the right hand side gives the expected value of the payoff when neither of the volatilities explode. This is equal to the price of a down-and-out barrier exchange option, where the option expires if either of the asset prices cross the barrier. The second term is equivalent to a single-asset external barrier option when only ξ_2 explodes.[7] In that case, U_2 becomes zero and the CEO payoff reduces to U_{1T}.

Single-asset barrier options can be priced by an application of the reflection principle (see Karatzas and Shreve [1991]) and the case of a two-asset barrier exchange option is an extension of that. Carr [1995], Banerjee [2003] and Kwok et al. [1998] discuss the pricing of two-asset and multi-asset external barrier options that knock out if an external process crosses the barrier. Lindset and Persson [2006] discuss the pricing of two-asset barrier exchange options, where the option knocks out if the price of one asset equals the other.

Using the reflection principle, the first term in the r.h.s. of equation (24) may be written:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left\{ \mathbf{1}_{Y_T > 1;\, m_1 > \mu_1;\, m_2 > \mu_2} \right\} &= \mathbb{E}_{\mathbb{P}} \left\{ \mathbf{1}_{Y_T > 1;\, m_1 > \mu_1} \right\} - \mathbb{E}_{\mathbb{P}} \left\{ \mathbf{1}_{Y_T > 1;\, m_2 < \mu_2;\, m_1 > \mu_1} \right\} \\ &= \mathbb{E}_{\mathbb{P}} \left\{ \mathbf{1}_{Y_T > 1;\, m_1 > \mu_1} \right\} - \mathbb{E}_{\mathbb{P}} \left\{ \mathbf{1}_{\ln Y_T < \, 2\mu_2 \,;\, m_1 > \mu_1} \right\} \end{aligned}$$

The two expectation terms in the above equation are equivalent to the ITM probabilities of a call option ($Y_T > 1$) and a put option ($Y_T < \exp(2\mu_2)$) with external barriers. These may be computed using the results of Carr [1995] and Kwok et al. [1998]. Similarly, by using reflection principle, the second term on the r.h.s. of equation (24) may also be evaluated as a combination of external barrier options. □

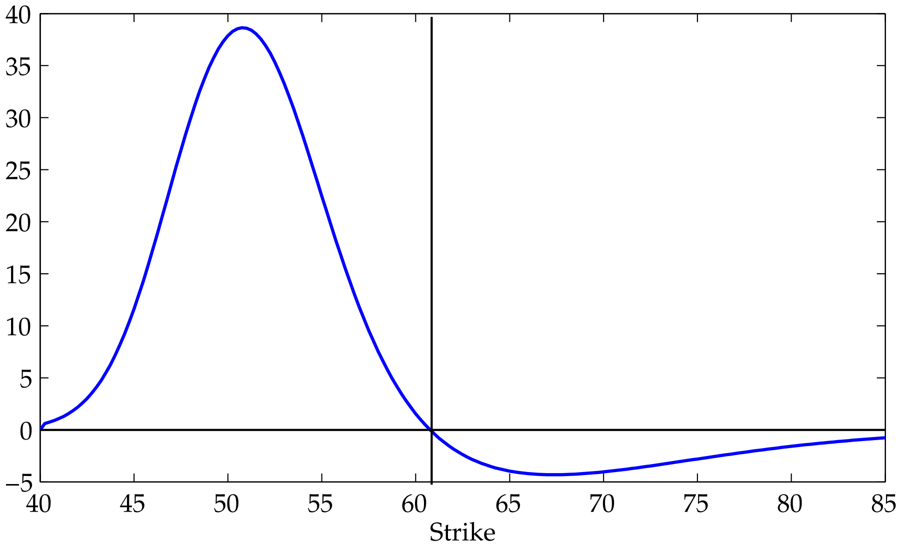
When applying Theorem 5 to pricing basket options, we must calibrate the strikes K_i of the under-

[7] An external barrier option is one that expires before maturity if an external process other than the underlying asset crosses the barrier.

lying vanilla options of the CEOs that replicate the basket option so that $\sum \theta_i K_i = K$. Furthermore, we aim to find particular strikes K_i for which the relative price of these vanilla options is lognormal. When the underlying option prices satisfy condition (23) almost exactly, Theorem 5 allows one to price CEOs almost exactly. As discussed earlier, the approximation error can be extremely small for certain strikes of the vanilla options and, since are calibrating these strikes, our approximation can be justified. The strikes K_1 and $K_2 = (K_1 - K)$ for which condition (23) holds can be found by using a simple one-dimensional solver. Figure 2 plots the behaviour of condition (23) for two sample vanilla options. It shows that the condition holds when the strikes of the options are equal to 60.8.

FIGURE 2: *Plot of* $U_{1t}k_1/(1+k_1) - U_{2t}k_2/(1+k_2)$ *against strike*

($S_1 = 75$, $S_2 = 65$, $\sigma_1 = 0.25$, $\sigma_2 = 0.25$, $r = 4\%$, $T = 3$ months)



In fact, we can price a CEO by dimension reduction under both the weak and the strong lognormality conditions. In both cases a CEO becomes equivalent to a simple lognormal exchange option, and the price of such an option can be found by change of numeraire, as in Margrabe [1978]. Under the weak lognormality condition (22) the vanilla option prices follow approximate lognormal processes, and under the strong lognormality condition (23) their *relative* prices follow lognormal processes. In general, the option prices follow displaced lognormal processes, but the displacement terms cancel out under the condition (23) and hence the relative option price process is exactly lognormal.

Finally, we remark that Theorem 5 may be extended to cases where the underlying asset prices follow certain non-lognormal processes. To see this, suppose the risk-neutral price dynamics are governed by a general two-factor model:

$$\begin{aligned} dS_{it} &= \mu_i(S_{it},t)dt + \sigma_i(S_{it},t)dW_{it} \\ \langle dW_{1t}, dW_{2t} \rangle &= \rho dt \qquad i = 1,\ 2, \end{aligned}$$

where ρ is assumed constant. It is easy to show that when $\mu_i(S_{it},t)$ and $\sigma_i(S_{it},t)$ satisfy

$$\left(\frac{\partial \sigma_{it}}{\partial t} + \mu_{it} \frac{\partial \sigma_{it}}{\partial S_{it}} + \frac{1}{2} \sigma_{it}^2 \frac{\partial^2 \sigma_{it}}{\partial S_{it}^2} \right) = \sigma_{it} \frac{\partial \mu_{it}}{\partial S_{it}}, \tag{25}$$

the option price processes will still be given by (19). For example, this holds for affine functions $\mu_i = a_i + b_i S_i$ and $\sigma_i = \alpha_i + \beta_i S_i$ with $a_i/\alpha_i = b_i/\beta_i$; where a_i, b_i, α_i and β_i are real constants.

4. Pricing Basket Options

Here we derive analytic approximations to basket option prices using the recursive framework described in section 2. To price the CEOs, we apply Theorem 5 and hence apply the lognormal exchange option pricing formula of Margrabe [1978] recursively. Specific examples of basket options with two or three assets are shown to have almost exact prices. However, when the number of assets is four or more the lognormality condition in Theorem 5 is too strong. Thus we employ a more general, but weaker condition under which we derive approximate analytic prices for basket options on N assets.

4.1. Pricing under the Strong Lognormality Condition

Recall that when the single-asset option price processes are described by equations (13) or (14), then a CEO on them can be priced by dimension reduction under (23). Now consider a CEO written on two lognormal exchange options, both having a common asset. Margrabe [1978] has shown that the exchange options can be priced as vanilla options on a single asset, by choosing the price of the common asset as numeraire. Then, under the strong lognormality condition introduced in section 3, the exchange option price processes may be described by equations (13) or (14), and the CEO can be priced by applying Theorem 5.[8]

To illustrate this we consider a two-asset basket options with non-zero strike and a three-asset basket options with zero strike.

[8]Carr [1988] introduced an alternative change of numeraire approach to price sequential exchange options where an exchange may lead to further exchanges. His approach is based on Geske [1977] and Margrabe [1978] who discuss the pricing of compound vanilla options and lognormal exchange options respectively. However, he only discusses the pricing of CEOs on a lognormal exchange option and a lognormal asset that is the same as the asset delivered in the exchange option. His approach is difficult to extend to higher dimensions and also does not apply to our case where both the underlying assets of the CEO are options.

(a) The payoff to a two-asset basket option can be written as a sum of payoffs of two CEOs on single-asset call and put options, as in section 3. Since the CEOs are written on vanilla options it is straightforward to compute their prices using Theorem 5.

(b) Consider a three-asset basket option with zero strike, when the signs of the asset weights $\mathbf{\Theta}$ are a permutation of $(1, 1, -1)$ or $(-1, -1, 1)$. This is just an extension of the two-asset case, where we have an additional asset instead of the strike. The three-asset basket option can be priced as a CEO either to exchange a two-asset exchange option for the third asset or to exchange two 2-asset exchange options with a common asset. For example, a 3 : 2 : 1 spread option, which is commonly traded in energy markets, has payoff decomposition:[9]

$$\begin{aligned} P_T &= [3S_{1T} - 2S_{2T} - S_{3T}]^+ \\ &= \left[3\,[S_{1T} - S_{2T}]^+ - [S_{3T} - S_{2T}]^+\right]^+ + \left[[S_{2T} - S_{3T}]^+ - 3\,[S_{2T} - S_{1T}]^+\right]^+ . \end{aligned}$$

Now the CEOs above can be priced by dimension reduction, choosing S_2 as the numeraire.

In the general case of basket options on N underlying assets, except for the ones discussed above, the two replicating CEOs are no longer written on plain vanilla or lognormal exchange options, but on sub-basket options. Since the prices of these sub-basket options are computed as a sum of prices of CEOs, it is not straightforward to express their processes in a form that would lead to an exact solution to the basket option price. For instance, consider a four-asset basket option with zero strike. Here we can write the payoff as:

$$\begin{aligned} P_T &= [S_{1T} - S_{2T} - S_{3T} + S_{4T}]^+ \\ &= \left[[S_{1T} - S_{2T}]^+ - [S_{3T} - S_{4T}]^+\right]^+ + \left[[S_{4T} - S_{3T}]^+ - [S_{2T} - S_{1T}]^+\right]^+ . \end{aligned}$$

Since the two replicating CEOs are written on lognormal exchange options with no common asset, the CEO prices cannot be priced using Theorem 5. Instead, we can adjust the volatilities of the CEOs using the weak lognormality condition, so that the sub-basket option price processes are approximately lognormal processes. Then the relative sub-basket option prices also follow approximate lognormal processes and the two replicating CEO prices can be computed by using the formula of Margrabe [1978].

[9] Note that there is a closed form formula for the price of this option of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\ln\frac{3}{2}+x} \int_{\infty}^{\ln(3e^x - 2e^y)} \left(3e^x - 2e^y - e^x\right) f(x, y, z)\, dz dy dx$$

where f is the trivariate normal density function and x, y, z are the log stock price processes. However, the triple integral is not easy to compute

4.2. Pricing under the Weak Lognormality Condition

Exact pricing under the strong lognormality condition is only possible in special cases. In the general case we must use the weak lognormality condition to find an approximate price, as described in this sub-section.

Let $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q} \right)$ be a filtered probability space, where Ω is the set of all possible events such that $(S_{1t}, S_{2t}, \ldots, S_{Nt}) \in (0, \infty)^N$, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration produced by the sigma algebra of the N-tuplet $(S_{1t}, S_{2t}, \ldots, S_{Nt})_{t\geq 0}$ of asset prices and $\mathbb{Q}$ is a multi-variate risk-neutral probability measure. We assume that the underlying asset prices processes S_i are described by:

$$\begin{aligned} dS_{it} &= \mu_i(S_{it}, t)S_{it}dt + \sigma_i S_{it}dW_{it} \\ \langle dW_{it}, dW_{jt} \rangle &= \rho_{ij}dt, \;\; 1 \leq i, j \leq N, \end{aligned} \tag{26}$$

where W_i are Wiener processes under the risk-neutral measure $\mathbb{Q}$, σ_i is the volatility of i^{th} asset (assumed constant), $\mu_i(.)$ is a well-defined function of S_{it} and t, and ρ_{ij} is the correlation between i^{th} and j^{th} assets (assumed constant).

We now describe the price process V_{Nt} of the basket option on N assets. Using a recursive argument, we begin by assuming that the prices of the call and put sub-basket options on m and n assets follow lognormal processes. Then we show that, when the basket option volatility is approximated as a constant, the basket option price process V_{Nt} can be expressed as a lognormal process. Since C_{mt}, C_{nt}, P_{mt} and P_{nt} are prices of basket options themselves, we may also express their processes as lognormal process, assuming their sub-basket option prices follow lognormal processes. In the end, these assumptions that sub-basket call and put option prices follow lognormal processes yields an approximate lognormal process for the price of a basket option on N assets.

As before, let $\mathbf{\Theta}_N = (\mathbf{\Theta}_m, -\mathbf{\Theta}_n)$. Then the basket option price may be computed as a sum of two CEO prices, with one CEO on two call sub-basket options and the other CEO on two put sub-basket options. Consider the CEO on calls. For $i = m$ or n, we assume

$$dC_{it} = rC_{it}dt + \sigma_{C_i}C_{it}d\tilde{W}_{it}, \tag{27}$$

where σ_{C_m}, σ_{C_n} are the volatilities and $\tilde{W}_m$, $\tilde{W}_n$ are the Wiener processes driving the two call basket options. Then, by Itô's Lemma:

$$
\begin{aligned}
dE_{1t} &= \frac{\partial E_{1t}}{\partial t}dt + \sum_{i=m,n}\left(\frac{\partial E_{1t}}{\partial C_{it}}dC_{it} + \sum_{j=m,n}\frac{1}{2}\frac{\partial^2 E_{1t}}{\partial C_{it}\partial C_{jt}}dC_{it}dC_{jt}\right) \\
&= \left(\frac{\partial E_{1t}}{\partial t} + \sum_{i=m,n}\left(rC_{it}\frac{\partial E_{1t}}{\partial C_{it}} + \sum_{j=m,n}\gamma_{ij}\sigma_{Ci}\sigma_{Cj}C_{it}C_{jt}\frac{\partial^2 E_{1t}}{\partial C_{it}C_{jt}}\right)\right)dt + \sum_{i=m,n}\sigma_{Ci}C_{it}\frac{\partial E_{1t}}{\partial C_{it}}d\tilde{W}_{it} \\
&= rE_{1t}dt + \sum_{i=m,n}\sigma_{Ci}C_{it}\Delta_{C_{it}}d\tilde{W}_{it}. \qquad (28)
\end{aligned}
$$

The price of the CEO on puts E_{2t} will follow a similar process to the one described by equation (28) with C_i replaced by P_i.[10]

Now, by (7), $V_{Nt} = E_{1t} + E_{2t}$ and so we have

$$
\begin{aligned}
dV_{Nt} &= dE_{1t} + dE_{2t} \\
&= r\left(E_{1t} + E_{2t}\right)dt + \sum_{i=m,n}\sigma_{Ci}C_{it}\Delta_{C_{it}}d\tilde{W}_{it} - \sum_{i=m,n}\sigma_{Pi}P_{it}\Delta_{P_{it}}d\tilde{W}_{it} \\
&= rV_{Nt}dt + V_{Nt}\left(\left(\sigma_{Cm}\tfrac{C_{mt}}{V_{Nt}}\tfrac{\partial E_{1t}}{\partial C_{mt}} - \sigma_{Pm}\tfrac{P_{mt}}{V_{Nt}}\tfrac{\partial E_{2t}}{\partial P_{mt}}\right)d\tilde{W}_{mt} - \left(\sigma_{Cn}\tfrac{C_{nt}}{V_{Nt}}\tfrac{\partial E_{1t}}{\partial C_{nt}} - \sigma_{Pn}\tfrac{P_{nt}}{V_{Nt}}\tfrac{\partial E_{nt}}{\partial P_{nt}}\right)d\tilde{W}_{nt}\right) \\
&= rV_{Nt}dt + V_{Nt}\Big(\left(\xi_{mt} - \eta_{mt}\right)d\tilde{W}_{mt} - \left(\xi_{nt} - \eta_{nt}\right)d\tilde{W}_{nt}\Big).
\end{aligned}
$$

Write

$$
d\tilde{W}_{nt} = \gamma_{mn}\, d\tilde{W}_{mt} + \sqrt{1-\gamma_{mn}^2}\, d\tilde{W}_t
$$

where $\tilde{W}$ is a Wiener process, independent of $\tilde{W}_m$, and γ_{mn} is the correlation between the basket options written on $\mathbf{b}_m$ and $\mathbf{b}_n$. Define, $\tilde{\sigma}_{mt} = (\xi_{mt} - \eta_{mt})$ and $\tilde{\sigma}_{nt} = (\xi_{nt} - \eta_{nt})$, with

$$
\xi_{it} = \sigma_{Ci}\frac{C_{it}}{V_{Nt}}\frac{\partial E_{1t}}{\partial C_{it}} \quad \text{and} \quad \eta_{it} = \sigma_{Pi}\frac{P_{it}}{V_{Nt}}\frac{\partial E_{2t}}{\partial P_{it}}\,. \qquad (29)
$$

Then V_{Nt} may be written

$$
dV_{Nt} = rV_{Nt}dt + \tilde{\sigma}_t V_{Nt} d\tilde{W}_t, \qquad (30)
$$

where the basket option volatility $\tilde{\sigma}_t$ is given by

$$
\tilde{\sigma}_t \;=\; \sqrt{\tilde{\sigma}_{mt}^2 + \tilde{\sigma}_{nt}^2 - 2\gamma_{mn}\tilde{\sigma}_{mt}\tilde{\sigma}_{nt}}, \qquad (31)
$$

[10]When two CEOs are written on call and put sub-basket options their price processes will be similar to (28) but now each has a call and put option component. For instance,

$$
dE_{1t} = rE_{1t}dt + \sigma_{Cm}C_{mt}\Delta_{C_{mt}}d\tilde{W}_{mt} + \sigma_{Pn}P_{nt}\Delta_{P_{nt}}d\tilde{W}_{nt}.
$$

the exchange option volatilities σ_{Eit} is given by

$$\begin{aligned}\tilde{\sigma}_{E1t} &= \sqrt{\tilde{\sigma}^2_{Cmt} + \tilde{\sigma}^2_{Cnt} - 2\gamma_{mn}\tilde{\sigma}_{Cmt}\tilde{\sigma}_{Cnt}}\\ \tilde{\sigma}_{E2t} &= \sqrt{\tilde{\sigma}^2_{Pmt} + \tilde{\sigma}^2_{Pnt} - 2\gamma_{mn}\tilde{\sigma}_{Pmt}\tilde{\sigma}_{Pnt}},\end{aligned}$$

and the covariance between the sub-basket options written on baskets $\mathbf{b}_m$ and $\mathbf{b}_n$ is given by

$$\begin{aligned}\gamma_{mn}dt &= \mathscr{C}\left(\mathbf{b}_m, \mathbf{b}_n\right)\\ &= \mathscr{C}\left(\mathbf{b}_{m1}, \mathbf{b}_{n1}\right) - \mathscr{C}\left(\mathbf{b}_{m1}, \mathbf{b}_{n2}\right) - \mathscr{C}\left(\mathbf{b}_{m2}, \mathbf{b}_{n1}\right) + \mathscr{C}\left(\mathbf{b}_{m2}, \mathbf{b}_{n2}\right), \qquad (32)\end{aligned}$$

for $i = m, n$, $j = 1, 2$, $\mathscr{C}\left(a, b\right)$ is the covariance between the Wiener processes driving the assets in baskets a and b, and the $\mathbf{b}_{ij}$s are the sub-baskets of $\mathbf{b}_i$ (i.e. $\mathbf{b}_i = \{\mathbf{b}_{i1}, \mathbf{b}_{i2}\}$).

As before, the weak lognormality condition is most likely to hold when we choose the two CEOs to be deep in-the-money (ITM). That is, we shall choose the strikes of the sub-basket call and put options so that the call option on m assets and the put option on n assets are both deep ITM. The reason for such a choice is that, as $t \to T$,

$$\frac{\partial E_{1t}}{\partial C_{mt}} \to 1,\ \frac{\partial E_{2t}}{\partial P_{nt}} \to 1,\ \frac{\partial E_{1t}}{\partial C_{nt}} \to 0 \text{ and } \frac{\partial E_{2t}}{\partial P_{mt}} \to 0.$$

Thus, by choosing the strikes of the sub-basket options so that the basket option price is obtained from deep ITM call and put options, we may approximate ξ_{it} and η_{it} as constants. Thus, $\xi_m \approx \sigma_{Cm}$, $\eta_n \approx \sigma_{Pn}$ and $\xi_n, \eta_m \approx 0$ and the basket option volatility is approximately constant, so we write $\tilde{\sigma}_t = \tilde{\sigma}$. The basket option price will then follow an approximate lognormal process described by

$$dV_{Nt} = rV_{Nt}dt + \tilde{\sigma}V_{Nt}d\tilde{W}_t.$$

Equation (27), describing the two call sub-basket option prices, can be derived in a similar fashion, starting with their sub-basket option price processes. Ultimately, when the sub-basket size reduces to one, we will have a plain vanilla option whose price process will be described by (13) if it is a call, or (14) if it is a put.

Theorem 6. *The price of the basket option on* $\mathbf{B}$ *at time t is given by the recursive formula:*

$$\begin{aligned}V_{Nt}(\mathbf{\Theta}, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \mathbb{E}_{\mathbb{Q}}\left\{V_{NT}\,\middle|\,\mathcal{F}_t\right\}\\ &= E_{1t}(\mathbf{\Theta}, \mathbf{S}_t, \mathbf{K}, T, \omega) + E_{2t}(\mathbf{\Theta}, \mathbf{S}_t, \mathbf{K}, T, \omega), \qquad (33)\end{aligned}$$

where

$$\begin{aligned} E_{1t}(\mathbf{\Theta}, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \omega\left(V_{mt}(\mathbf{\Theta}_m, \mathbf{S}_{mT}, \mathbf{K}_m, +1)\Phi(\omega d_{11}) - V_{nt}(\mathbf{\Theta}_n, \mathbf{S}_{nT}, \mathbf{K}_n, -\chi)\Phi(\omega d_{12})\right) \\ &= \omega\left(V^1_{mt}\Phi(\omega d_{11}) - V^1_{nt}\Phi(\omega d_{12})\right), \ say \end{aligned}$$

$$\begin{aligned} E_{2t}(\mathbf{\Theta}, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \omega\left(V_{nt}(\mathbf{\Theta}_n, \mathbf{S}_{nT}, \mathbf{K}_n, \chi)\Phi(\omega d_{21}) - V_{mt}(\mathbf{\Theta}_m, \mathbf{S}_{mT}, \mathbf{K}_m, -1)\Phi(\omega d_{22})\right) \\ &= \omega\left(V^2_{nt}\Phi(\omega e_1) - V^2_{mt}\Phi(\omega e_2)\right), \ say \end{aligned} \quad (34)$$

and

$$d_{i1} = \frac{ln\left(V^i_{mt}/V^i_{nt}\right) + \frac{1}{2}\sigma^2_{Ei}(T-t)}{\sigma_{Ei}\sqrt{T-t}}; \qquad d_{i2} = d_{i1} - \sigma_{Ei}\sqrt{T-t};$$

where σ_{E1} and σ_{E2} are the volatilities of the two CEO prices E_1 and E_2 respectively.

Proof. Recall that C_m, P_m and C_n, P_n are themselves prices of options on baskets of sizes m and n respectively. Therefore these prices can be computed by applying equation (33) recursively. Their volatilities σ_{Cm}, σ_{Cn}, σ_{Pm}, and σ_{Pn} will be given by equation (31). This procedure is followed until the size of a sub-basket reaches one.

When the size of the basket reduces to one, the basket option price is merely the price of a vanilla option under the chosen model. Then for $\mathbf{S}_t = (S_{it})$, $\mathbf{K}_1 = (K_i)$ and $\mathbf{\Theta}_1 = \theta_i$, for some $1 \leq i \leq N$, the single-asset option price is given by

$$\begin{aligned} E_1(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= \omega e^{-r(T-t)}\theta_i\left(F_{it,T}\Phi(\omega d_1) - K_i\Phi(\omega d_2)\right), \\ E_2(\Theta, \mathbf{S}_t, \mathbf{K}, T, \omega) &= 0, \end{aligned} \quad (35)$$

where $F_{it,T}$ is the i^{th} asset futures price and

$$d_1 = \frac{ln\left(\frac{F_{itT}}{K_i}\right) + \left(r + \frac{1}{2}\Sigma^2_i\right)(T-t)}{\Sigma_i\sqrt{T-t}}; \qquad d_2 = d_1 - \Sigma_i\sqrt{T-t}.$$

For instance, when $\mu_i = (r - q_i)$ in equation (26), $F_{it,T} = S_{it}e^{(r-q_i)(T-t)}$ and $\Sigma_i = \sigma_i$. Or more generally, when $\mu_i = \kappa(\theta(t) - \ln S_{it})$, as in Pilipovic [2007]:

$$\begin{aligned} F_{it,T} &= \exp\left(e^{-\kappa(T-t)}\ln S_{it} + \int_t^T e^{-\kappa(T-s)}\theta(s)ds + \frac{\sigma_i^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)\right), \\ \Sigma_i &= \sigma_i\sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}} \end{aligned}$$

□

One of the main advantages of our approximation is that we can derive analytic formulae for the multi-asset option Greeks which, unlike most other approximations, capture the effects that individual asset's volatilities and correlations have on the hedge ratios.

Below we present the deltas, gammas and vegas of a basket option; the corresponding formulae for a rainbow option may then be obtained from it's specific basket-option representation, which we have shown how to derive, using some illustrative examples in Section 2. Differentiating the basket option price given in Theorem 6, using the chain rule, yields the following:

Proposition 7. *The basket option deltas, gammas and vegas of our basket option price f are given by:*

$$\begin{aligned}
\Delta^f_{S_i} &= \Delta^f_{C_j}\Delta^{C_j}_{S_i} + \Delta^f_{P_j}\Delta^{P_j}_{S_i} \\
\Gamma^f_{S_i} &= \Gamma^f_{C_j}\left(\Delta^{C_j}_{S_i}\right)^2 + \Gamma^{C_j}_{S_i}\Delta^f_{C_j} + \Gamma^f_{P_j}\left(\Delta^{P_j}_{S_i}\right)^2 + \Gamma^{P_j}_{S_i}\Delta^f_{P_j} \\
\mathcal{V}^f_{\sigma_i} &= \mathcal{V}^f_{\sigma_{E_1}}\frac{\partial\sigma_{E_1}}{\partial\sigma_i} + \mathcal{V}^f_{\sigma_{E_2}}\frac{\partial\sigma_{E_2}}{\partial\sigma_i} + \mathcal{V}^{C_j}_{\sigma_i}\Delta^f_{C_j} + \mathcal{V}^{P_j}_{\sigma_i}\Delta^f_{P_j}
\end{aligned} \tag{36}$$

where j is equal to m when $1 \leq i \leq m$ and equal to n when $m+1 \leq i \leq N$. Here Δ^z_x, Γ^z_x and $\mathcal{V}^z_x$ denote the delta, gamma and vega of z with respect to x respectively.

5. Simulation Results

This section presents the results of several simulation experiments that are designed to test the accuracy of our lognormal approximations, for several multi-asset options of increasingly complexity. For each option we compute its price using simulation in two different ways: (1) by simulating the underlying asset prices themselves, using correlated geometric Brownian motions; and (2) by simulating the option prices directly using our lognormal approximations. In each case we employed 2 million simulations, including antithetic sampling to reduce the sampling variance. The standard deviation of the GBM price over all 2 millions simulations is shown next to the GBM price in each table.

Tables 1 - 4 report the average terminal option price at time T, i.e. the average payoff, without discounting to time 0. For instance, in Table 1 we report the simulated terminal prices of a vanilla call option using the payoff $\max(S_T - K, 0)$ under GBM and U_T under our approximation. In each case we report results for a range of different parameter values. The option maturities were 1 month or 6 months; the discount rates were 0 or 4%; the asset prices and (when relevant) the vanilla option strikes took values from the set (80, 90, 100, 110); the asset volatilities were 10% and 20%; and asset correlations took values from the set (-0.5, 0, 0.5, 0.8).

Tables 1 - 4 report the simulation results for standard call options, exchange options and compound call exchange options. Table 1 compares vanilla call prices when the underlying asset price is simulated by GBM, with the prices obtained when the option price process is simulated using

TABLE 1: *Vanilla call prices: GBM versus weak lognormal approximation*

$(S = 100)$

K	T	r	σ	**GBM Price**	**Std Dev**	**Approx Price**
60	1	0.00	0.1	40.0000	(0.0001)	40.0000
60	1	0.00	0.2	39.9999	(0.0003)	39.9999
60	1	0.04	0.1	40.3339	(0.0001)	40.3339
60	1	0.04	0.2	40.3338	(0.0003)	40.3338
80	1	0.00	0.1	20.0000	(0.0001)	20.0000
80	1	0.00	0.2	20.0000	(0.0003)	20.0000
80	1	0.04	0.1	20.3339	(0.0001)	20.3339
80	1	0.04	0.2	20.3338	(0.0003)	20.3339
100	1	0.00	0.1	1.1504	(0.0009)	0.9673
100	1	0.00	0.2	2.3006	(0.0019)	2.1185
100	1	0.04	0.1	1.3270	(0.0009)	1.1076
100	1	0.04	0.2	2.4752	(0.0019)	2.3188
120	1	0.00	0.1	0.0000	(0.0000)	0.0000
120	1	0.00	0.2	0.0013	(0.0001)	0.0011
120	1	0.04	0.1	0.0000	(0.0000)	0.0000
120	1	0.04	0.2	0.0017	(0.0001)	0.0013
60	6	0.00	0.1	39.9999	(0.0004)	39.9999
60	6	0.00	0.2	39.9998	(0.0015)	40.0002
60	6	0.04	0.1	42.0200	(0.0004)	42.0200
60	6	0.04	0.2	42.0198	(0.0015)	42.0200
80	6	0.00	0.1	20.0012	(0.0004)	20.0019
80	6	0.00	0.2	20.3088	(0.0025)	20.3077
80	6	0.04	0.1	22.0204	(0.0004)	22.0208
80	6	0.04	0.2	22.2423	(0.0023)	22.2411
100	6	0.00	0.1	2.8175	(0.0024)	2.3563
100	6	0.00	0.2	5.6314	(0.0050)	5.1930
100	6	0.04	0.1	3.9695	(0.0023)	3.4905
100	6	0.04	0.2	6.7550	(0.0051)	6.4651
120	6	0.00	0.1	0.0124	(0.0002)	0.0108
120	6	0.00	0.2	0.7211	(0.0023)	0.6821
120	6	0.04	0.1	0.0294	(0.0003)	0.0257
120	6	0.04	0.2	0.9754	(0.0027)	0.9232

TABLE 2: *Exchange option prices: Correlated GBM versus weak lognormal approximation*

$(S_1 = 100)$

S_2	r	T	σ_1	σ_2	ρ	**GBM Price**	**Std Dev**	**Approx Price**
80	0.00	1	0.1	0.1	0.5	20.0000	(0.0001)	19.9999
80	0.00	1	0.2	0.2	0.5	20.0000	(0.0003)	19.9999
80	0.00	1	0.1	0.1	0.8	20.0000	(0.0001)	20.0009
80	0.00	1	0.2	0.1	0.8	20.0003	(0.0003)	20.0023
80	0.00	6	0.1	0.1	0.5	20.0012	(0.0004)	20.0013
80	0.00	6	0.2	0.2	0.5	20.3073	(0.002)	20.3123
80	0.00	6	0.1	0.1	0.8	20.0003	(0.0003)	20.0052
80	0.04	1	0.1	0.1	0.5	20.0667	(0.0001)	20.0667
80	0.04	1	0.2	0.1	0.5	20.0667	(0.0003)	20.0664
80	0.04	6	0.1	0.1	0.8	20.4043	(0.0003)	20.4093
80	0.04	6	0.2	0.1	0.8	20.4327	(0.0015)	20.4444
90	0.00	1	0.1	0.1	0.5	10.0001	(0.0001)	10.0000
90	0.00	1	0.2	0.2	0.5	10.0727	(0.0005)	10.0744
90	0.00	1	0.2	0.1	0.8	10.0039	(0.0003)	10.0084
90	0.00	6	0.2	0.1	0.8	10.6073	(0.0024)	10.6123
90	0.00	6	0.1	0.1	0.8	10.0133	(0.0004)	10.0225
90	0.04	1	0.1	0.1	0.5	10.0335	(0.0001)	10.0334
90	0.04	1	0.2	0.1	0.5	10.0633	(0.0004)	10.0635
90	0.04	6	0.2	0.1	0.5	11.4786	(0.0034)	11.4833
90	0.04	6	0.2	0.2	0.5	12.0095	(0.0033)	12.0155
90	0.04	6	0.2	0.1	0.8	10.8216	(0.0025)	10.8267
100	0.00	1	0.2	0.1	0.5	1.9934	(0.0016)	1.9924
100	0.00	1	0.1	0.1	0.5	1.1516	(0.0009)	1.1487
100	0.00	6	0.2	0.1	0.5	4.8802	(0.0043)	4.8767
100	0.00	6	0.1	0.1	0.5	2.8203	(0.0022)	2.8131
100	0.04	1	0.2	0.2	0.5	2.3106	(0.0018)	2.3048
100	0.04	6	0.2	0.1	0.5	4.9788	(0.0044)	4.9753
100	0.04	6	0.1	0.1	0.5	2.8773	(0.0022)	2.8699
110	0.00	1	0.1	0.1	0.5	0.0004	(0.0001)	0.0000
110	0.00	1	0.2	0.1	0.5	0.0569	(0.0004)	0.0413
110	0.00	1	0.2	0.2	0.5	0.1242	(0.0006)	0.0903
110	0.00	1	0.2	0.2	0.8	0.0053	(0.0001)	0.0000
110	0.00	1	0.2	0.1	0.8	0.0091	(0.0002)	0.0003
110	0.04	1	0.1	0.1	0.5	0.0004	(0.0001)	0.0000
110	0.04	1	0.2	0.1	0.5	0.0571	(0.0004)	0.0414
110	0.04	1	0.2	0.2	0.5	0.1246	(0.0006)	0.0906
110	0.04	1	0.2	0.2	0.8	0.0054	(0.0001)	0.0000
110	0.04	6	0.2	0.1	0.5	1.6339	(0.0031)	1.5934
110	0.04	6	0.1	0.1	0.5	0.3103	(0.001)	0.2612

TABLE 3: *CEO prices: Correlated GBM versus weak lognormal approximation*

$(S_1 = 100, \rho = 0.5)$

S_2	K_1, K_2	r	T	σ_1	σ_2	**GBM Price**	**Std Dev**	**Approx Price**
80	60	0.00	1	0.1	0.1	20.0000	(0.0001)	19.9999
80	60	0.00	6	0.1	0.1	20.0011	(0.0004)	20.0052
80	60	0.04	1	0.1	0.1	20.0667	(0.0001)	20.0667
80	60	0.04	1	0.2	0.2	20.0667	(0.0003)	20.0670
80	60	0.04	6	0.1	0.1	20.4052	(0.0005)	20.4086
80	80	0.00	1	0.2	0.1	19.0788	(0.0008)	19.0833
80	80	0.00	1	0.1	0.1	19.0788	(0.0008)	19.0860
80	80	0.04	1	0.2	0.1	19.2714	(0.0007)	19.2756
80	80	0.04	1	0.1	0.1	19.2715	(0.0008)	19.2782
80	80	0.04	6	0.2	0.1	19.1265	(0.0027)	19.1327
80	100	0.00	1	0.1	0.1	1.1504	(0.0009)	1.1516
80	100	0.00	6	0.1	0.1	2.8161	(0.0023)	2.8193
80	100	0.04	1	0.1	0.1	1.3270	(0.0009)	1.3282
80	100	0.04	6	0.1	0.1	3.9659	(0.0023)	3.9688
80	120	0.00	1	0.2	0.1	0.0013	(0.0001)	0.0011
80	120	0.00	1	0.2	0.2	0.0013	(0.0001)	0.0011
80	120	0.00	6	0.1	0.1	0.0124	(0.0002)	0.0113
80	120	0.04	6	0.1	0.1	0.0294	(0.0003)	0.0277
90	80	0.00	1	0.1	0.1	10.0001	(0.0001)	10.0009
90	80	0.04	1	0.1	0.1	10.0334	(0.0001)	10.0342
90	100	0.00	1	0.1	0.1	1.1503	(0.0009)	1.1516
90	100	0.04	6	0.2	0.1	6.4379	(0.0049)	6.4384
90	120	0.00	1	0.2	0.1	0.0013	(0.0001)	0.0011
90	120	0.00	6	0.1	0.1	0.0124	(0.0002)	0.0112
100	60	0.00	1	0.1	0.1	1.1516	(0.0009)	1.1484
100	60	0.04	1	0.1	0.1	1.1555	(0.0009)	1.1523
100	80	0.00	1	0.1	0.1	1.1516	(0.0009)	1.1475
100	80	0.00	1	0.2	0.1	1.9934	(0.0016)	1.9859
100	80	0.04	1	0.1	0.1	1.1555	(0.0009)	1.1513
100	80	0.04	1	0.2	0.1	2.0001	(0.0016)	1.9927
100	120	0.00	6	0.1	0.1	0.0116	(0.0002)	0.0094
100	120	0.00	6	0.2	0.1	0.7146	(0.0023)	0.7092
100	120	0.04	6	0.1	0.1	0.0266	(0.0003)	0.0225
100	120	0.04	6	0.2	0.1	0.9595	(0.0026)	0.9525
110	80	0.00	1	0.2	0.2	0.1242	(0.0006)	0.1190
110	80	0.00	6	0.1	0.1	0.3041	(0.001)	0.3105
110	80	0.04	1	0.2	0.2	0.1246	(0.0006)	0.1185
110	80	0.04	6	0.1	0.1	0.3103	(0.001)	0.3078

TABLE 4: *CEO prices: Correlated GBM versus weak lognormal approximation*

$(S_1 = 100, \rho = 0.8)$

S_2	K_1, K_2	r	T	σ_1	σ_2	**GBM Price**	**Std Dev**	**Approx Price**
80	60	0.00	1	0.1	0.1	20.0000	(0.0001)	20.0009
80	60	0.00	6	0.1	0.1	20.0003	(0.0003)	20.0052
80	60	0.04	1	0.1	0.1	20.0668	(0.0001)	20.0676
80	60	0.04	6	0.1	0.1	20.4043	(0.0003)	20.4093
80	80	0.00	1	0.1	0.1	19.0785	(0.0007)	19.0845
80	80	0.00	1	0.2	0.1	19.0788	(0.0007)	19.0827
80	80	0.04	1	0.1	0.1	19.2712	(0.0007)	19.2767
80	80	0.04	1	0.2	0.1	19.2715	(0.0007)	19.2751
80	100	0.00	1	0.1	0.1	1.1523	(0.0009)	1.1526
80	100	0.00	6	0.1	0.1	2.8208	(0.0023)	2.8215
80	100	0.00	6	0.2	0.1	5.6400	(0.005)	5.6406
80	100	0.04	1	0.1	0.1	1.3289	(0.0009)	1.3293
80	120	0.00	6	0.1	0.1	0.0122	(0.0002)	0.0110
80	120	0.00	1	0.2	0.1	0.0013	(0.0001)	0.0011
80	120	0.04	6	0.1	0.1	0.0292	(0.0003)	0.0273
80	120	0.04	6	0.2	0.1	0.9781	(0.0027)	0.9695
80	120	0.04	1	0.2	0.2	0.0016	(0.0001)	0.0014
90	80	0.00	1	0.2	0.1	10.0039	(0.0003)	10.0049
90	80	0.04	1	0.2	0.1	10.0373	(0.0003)	10.0383
90	100	0.00	1	0.1	0.1	1.1522	(0.0009)	1.1525
90	100	0.00	6	0.2	0.1	5.4414	(0.0047)	5.4458
90	100	0.04	1	0.1	0.1	1.3287	(0.0009)	1.3292
90	100	0.04	6	0.2	0.1	6.3936	(0.0046)	6.4011
90	120	0.00	1	0.2	0.1	0.0013	(0.0001)	0.0011
90	120	0.00	6	0.2	0.1	0.7235	(0.0023)	0.7148
90	120	0.00	1	0.2	0.2	0.0013	(0.0001)	0.0011
90	120	0.04	1	0.2	0.1	0.0016	(0.0001)	0.0014
90	120	0.04	6	0.2	0.1	0.9780	(0.0027)	0.9694
90	120	0.04	1	0.2	0.2	0.0016	(0.0001)	0.0014
100	120	0.00	1	0.2	0.1	0.0013	(0.0001)	0.0011
100	120	0.00	6	0.2	0.1	0.7120	(0.0022)	0.7023
100	120	0.00	1	0.2	0.2	0.0010	(0.0001)	0.0004
100	120	0.04	6	0.1	0.1	0.0196	(0.0003)	0.0146
100	120	0.04	1	0.2	0.1	0.0016	(0.0001)	0.0014
110	60	0.00	1	0.2	0.1	0.0091	(0.0002)	0.0036
110	60	0.04	1	0.2	0.1	0.0092	(0.0002)	0.0036
110	100	0.00	6	0.1	0.1	0.0213	(0.0002)	0.0234
110	100	0.04	6	0.1	0.1	0.0240	(0.0002)	0.0295

our approximation (19). As explained in section 3, the approximation error is greatest for ATM options, and this fact is verified by our results. For deep ITM calls ($K = 60$) and ITM calls ($K = 80$) the error is minuscule, for OTM calls ($K = 110$) it is also very small (but the option prices are nearly zero anyway) and the maximum error occurs for calls with strike $K = 100$. It is for this reason that we choose ITM strikes for the vanilla call options in the terminal nodes of the multi-asset option pricing tree.

Table 2 presents simulations for the prices of various exchange options, to receive asset 1 in exchange for asset 2. The payoff is therefore $[S_1 - S_2]^+$. We compare the terminal prices at 1 month and 6 months when the underlying asset prices themselves are simulated to follow two correlated GBMs, with the prices obtained when the exchange option price process is simulated directly, using our lognormal approximation. The current price of asset 1 was fixed at 100 and, to change the moneyness of the exchange option, that of asset 2 took values 80, 90, 100 or 110. The approximation errors increase with the option maturity, the asset correlation, and the current price of asset 2. However, in each case the errors were typically of order 10^{-2} or less.[11]

Now we compare the two simulated prices for a compound call exchange option, i.e. an option to receive a call on asset 1 in exchange for a call on asset 2. The payoff is

$$P_T = [U_{1T} - U_{2T}]^+, \text{ with } U_{1t} = [S_{1t} - K_1]^+ \text{ and } U_{2t} = [S_{2t} - K_2]^+.$$

Under the assumption that the asset prices follow correlated Brownian motions, we simulate U_{1t} and U_{2t} and then compute $[U_{1T} - U_{2T}]^+$. This gives the prices shown in the column headed 'GBM'. The 'Approx.' results are obtained by simulating an approximately lognormal process for the option price itself, i.e.

$$\frac{dP_t}{P_t} = r dt + \xi_1 \frac{\partial P_t}{\partial U_{2t}} \frac{U_{1t}}{P_t} dW_{1t} - \xi_2 \frac{\partial P_t}{\partial U_{2t}} \frac{U_{2t}}{P_t} dW_{2t} \tag{37}$$

where P_t denotes the CEO price at time t. Note that the deltas used here are those of the CEO with respect to the vanilla options, and ξ_1 and ξ_2 are the option volatilities, which have already been approximated using (19). Hence, in this case the approximation is done twice, once for each ξ.

The results of these experiments may be influenced by the moneyness of the vanilla options and the CEO. Provided we can change these, the actual level and relative values of the strikes K_1 and K_2 have no influence on the qualitative nature of our results. So, we change the moneyness of the options by changing S_2, and to limit the number of cases considered we have simply assumed that the two vanilla calls have equal strikes, i.e. $K_1 = K_2$, taking the values shown in the second column. The price of asset 1 was again fixed at 100, with the asset 2 prices taking values 80, 90, 100 or 110. The GBM price column shows the option prices when the simulations are on the correlated lognormal asset prices themselves, and the 'Approx.' column shows the option prices when the

[11] Note that the relative errors were greater for OTM options ($S_2 = 110$) than for ITM options ($S_2 = 80$) since the value of an option to exchange asset 2 for asset 1 decreases with the relative price of asset 2 to asset 1.

CEO price process is simulated directly, based on our lognormal approximation above.

Tables 3 and 4 show that the approximation errors for CEOs are of the same order of magnitude as the exchange option approximation errors in Table 2. Again they are small, and they tend to increase with the option maturity, the asset correlation, and the current price of asset 2. But even though the volatility is approximated twice in the CEO case, and only once in the standard exchange option case, the accuracy does not deteriorate significantly. Our approximation remains very accurate indeed, especially for ITM exchange and compound exchange options.

Tables 5 and 6 compare the calibrated model prices with simulated prices for two-asset call spread option and two-asset basket call options. In these and the following tables we first report the values of the fixed parameters (option strike, maturity, discount rate, volatilities and correlations), then the calibrated strikes, and then the current option price based on simulating correlated GBMs for the underlying assets. Again, the average price from 2 million simulations is reported, followed by the standard deviation of the simulated prices. The final column reports the approximation error, relative to the GBM price, from using our weak lognormal approximation.

Recall that a two-asset option payoff can be replicated as a sum of payoffs to two CEOs, each to exchange a call or put on one asset for a call or put on the other asset. Since the correlations and volatilities are fixed, the calibration for Tables 5 and 6 involves finding the strikes of these single-asset options (K_1 and K_2, with $K_1 + K_2 = K$) for which the model price matches the simulated price. These strike values are reported in the centre two columns of the tables. When the basket option is near ATM the calibrated strikes need to be similar, and such that both options are near to ATM. This is because the basket option price is the sum of two CEO prices, and setting a strike so that one of these CEOs is ITM would yield too high a price for the basket option. Nevertheless, since we have one free strike parameter to calibrate, the pricing errors are very small indeed. The absolute difference between the simulated and model prices are shown in the last column of each table. As expected, the approximation errors are greatest for ATM two-asset basket options, but even these are of order 10^{-3} or less.

Tables 7 and 8 compare our model prices with the results provided in Ju [2002] for a five-asset basket call option with payoff $[S_{1T} + S_{2T} + S_{3T} + S_{4T} + S_{5T}]^+$. In this case we calibrate the strikes $K_1, ..., K_5$ with $K_1 + ... + K_5 = K$, for the five pairs of vanilla call and put options at the base of the pricing tree. Since we replicate a five-asset basket option payoff as the sum of payoffs to five CEOs, there is more flexibility to calibrate the strikes than there is in the two-asset case. For instance, for ATM five-asset basket options, we could calibrate some strikes so that the single-asset options in the CEO are deep ITM or OTM, unlike the two-asset case. As a result, our approximation errors actually decrease as the number of assets in the basket increase, because the number of strike parameters available for calibration increases.

Finally we remark that we have assumed flat implied volatility skew for all single asset European

TABLE 5: *Call spread option prices: Correlated GBM versus weak lognormal approximation*

$(S_1 = 50, S_2 = 40)$

K	r	T	σ_1	σ_2	ρ	K_1	K_2	**GBM Price**	**Std Dev**	**Pricing Error**
0	0.00	1	0.1	0.1	-0.50	45.00	45.00	10.0000	(0.0001)	8.992E-06
0	0.00	1	0.1	0.1	0.00	45.00	45.00	10.0000	(0.0001)	1.052E-05
0	0.00	1	0.1	0.1	0.50	45.00	45.00	9.9999	(0.0001)	6.122E-05
0	0.00	6	0.2	0.2	-0.50	54.62	54.62	11.0726	(0.0023)	2.439E-05
0	0.00	6	0.2	0.1	0.00	44.99	44.99	10.2524	(0.0014)	1.221E-03
0	0.00	6	0.2	0.1	0.50	39.03	39.03	10.0722	(0.001)	1.526E-04
0	0.04	1	0.1	0.1	-0.50	44.90	44.90	10.0000	(0.0001)	1.192E-05
0	0.04	1	0.1	0.1	0.00	44.95	44.95	10.0000	(0.0001)	1.273E-05
0	0.04	1	0.1	0.1	0.50	44.98	44.98	9.9999	(0.0001)	5.962E-05
0	0.04	6	0.2	0.1	-0.50	48.86	48.86	10.4749	(0.0018)	1.796E-05
0	0.04	6	0.2	0.1	0.50	39.84	39.84	10.0722	(0.001)	7.247E-06
5	0.00	1	0.1	0.1	-0.50	47.50	42.50	5.0101	(0.0001)	2.816E-03
5	0.00	1	0.1	0.1	0.00	47.50	42.50	5.0017	(0.0001)	3.225E-03
5	0.00	1	0.1	0.1	0.50	47.50	42.50	5.0000	(0.0001)	2.914E-03
5	0.00	6	0.2	0.1	-0.50	54.86	49.86	6.4916	(0.0026)	1.666E-05
5	0.00	6	0.2	0.2	-0.50	62.62	57.62	7.3235	(0.0031)	2.579E-04
5	0.00	6	0.2	0.1	0.00	52.51	47.51	6.0823	(0.0022)	9.247E-06
5	0.00	6	0.2	0.1	0.50	49.42	44.42	5.6273	(0.0017)	5.263E-05
5	0.04	1	0.1	0.1	-0.50	47.45	42.45	5.0265	(0.0001)	3.184E-03
5	0.04	1	0.1	0.1	0.00	47.47	42.47	5.0183	(0.0001)	2.859E-03
5	0.04	1	0.1	0.1	0.50	47.49	42.49	5.0166	(0.0001)	2.646E-03
5	0.04	6	0.2	0.1	-0.50	55.79	50.79	6.5615	(0.0026)	1.315E-05
5	0.04	6	0.2	0.1	0.00	53.38	48.38	6.1553	(0.0022)	8.685E-06
5	0.04	6	0.2	0.1	0.50	50.21	45.21	5.7054	(0.0017)	3.478E-05
10	0.00	6	0.2	0.2	-0.50	90.00	80.00	4.3949	(0.0035)	2.410E-03
10	0.04	1	0.1	0.1	-0.50	28.20	18.20	0.9159	(0.0007)	1.732E-04
10	0.04	1	0.2	0.2	-0.50	22.21	12.21	1.8144	(0.0014)	9.363E-04
10	0.04	6	0.1	0.1	-0.50	18.00	8.00	2.3014	(0.0018)	1.407E-03
10	0.04	6	0.1	0.1	0.00	18.00	8.00	1.9062	(0.0015)	1.856E-03
20	0.00	1	0.2	0.2	-0.50	55.00	35.00	0.0230	(0.0002)	4.417E-03
20	0.00	6	0.2	0.1	0.00	53.70	33.70	0.4353	(0.0013)	7.217E-05
20	0.00	6	0.2	0.2	0.00	50.66	30.66	0.6629	(0.0016)	1.045E-04
20	0.00	6	0.2	0.1	0.50	57.11	37.11	0.2171	(0.0009)	2.789E-04
20	0.04	1	0.2	0.2	-0.50	55.01	35.01	0.0239	(0.0002)	3.695E-03
20	0.04	6	0.2	0.1	-0.50	51.85	31.85	0.7221	(0.0018)	8.393E-06
20	0.04	6	0.2	0.2	0.00	51.15	31.15	0.7178	(0.0017)	1.627E-06
20	0.04	6	0.2	0.1	0.50	57.64	37.64	0.2431	(0.0009)	4.568E-04
20	0.04	6	0.2	0.2	0.50	69.76	49.76	0.2443	(0.0009)	4.093E-06

TABLE 6: *Two-asset basket call prices: Correlated GBM versus weak lognormal approximation*

$(S_1 = 30, S_2 = 20)$

K	r	T	σ_1	σ_2	ρ	K_1	K_2	**GBM Price**	**Std Dev**	**Pricing Error**
40	0.00	1	0.1	0.1	-0.50	25.0078	14.9922	10.0000	(0.0001)	3.00E-05
40	0.00	1	0.2	0.2	-0.50	25.0150	14.9850	9.9999	(0.0001)	1.85E-04
40	0.00	1	0.1	0.1	0.00	25.0098	14.9902	10.0000	(0.0001)	7.54E-06
40	0.00	1	0.2	0.1	0.00	25.0322	14.9678	10.0000	(0.0001)	4.62E-05
40	0.00	1	0.2	0.2	0.00	25.0189	14.9811	10.0000	(0.0001)	2.31E-05
40	0.00	1	0.1	0.1	0.50	25.0138	14.9862	10.0000	(0.0001)	3.20E-06
40	0.00	6	0.1	0.1	-0.50	25.0180	14.9820	9.9998	(0.0002)	7.70E-04
40	0.00	6	0.1	0.1	0.00	25.0227	14.9773	10.0000	(0.0002)	4.11E-04
40	0.00	6	0.1	0.1	0.50	25.0323	14.9677	10.0001	(0.0002)	2.67E-04
40	0.00	6	0.2	0.1	0.50	25.1091	14.8909	10.0197	(0.0006)	3.50E-03
40	0.04	1	0.1	0.1	-0.50	25.0241	14.9759	10.1331	(0.0001)	3.00E-05
40	0.04	1	0.1	0.1	0.00	25.0360	14.9640	10.1331	(0.0001)	7.54E-06
40	0.04	1	0.2	0.2	0.00	25.0317	14.9683	10.1331	(0.0001)	1.44E-05
40	0.04	1	0.1	0.1	0.50	25.0587	14.9413	10.1331	(0.0001)	3.20E-06
40	0.04	6	0.2	0.1	-0.50	25.0957	14.9043	10.7918	(0.0005)	2.86E-02
40	0.04	6	0.2	0.1	0.00	25.0859	14.9141	10.7946	(0.0005)	1.96E-02
40	0.04	6	0.2	0.2	0.00	25.0684	14.9316	10.8050	(0.0006)	1.29E-02
50	0.00	1	0.1	0.1	-0.50	29.9999	20.0001	0.3045	(0.0003)	6.15E-02
50	0.00	1	0.2	0.1	-0.50	29.9997	20.0003	0.6090	(0.0005)	9.57E-02
50	0.00	1	0.1	0.1	0.00	30.0000	20.0000	0.4151	(0.0004)	4.84E-02
50	0.00	1	0.2	0.1	0.00	30.0003	19.9997	0.7280	(0.0006)	7.99E-02
50	0.00	1	0.2	0.2	0.00	30.0002	19.9998	0.8304	(0.0007)	9.69E-02
50	0.00	6	0.2	0.2	0.00	46.0000	4.0000	2.0366	(0.0018)	1.23E-02
50	0.04	1	0.1	0.1	0.00	30.0002	19.9998	0.5029	(0.0004)	4.73E-02
50	0.04	1	0.1	0.1	0.50	45.9487	4.0513	0.5888	(0.0004)	9.26E-02
50	0.04	6	0.1	0.1	0.00	30.0008	19.9992	1.5783	(0.0008)	5.28E-02
50	0.04	6	0.2	0.1	0.00	37.1800	12.8200	2.2974	(0.0016)	2.79E-05
50	0.04	6	0.2	0.2	0.00	45.9092	4.0908	2.5492	(0.0018)	3.32E-02
60	0.00	6	0.2	0.1	-0.50	38.1322	21.8678	0.0186	(0.0002)	3.68E-02
60	0.00	6	0.2	0.1	0.00	34.9876	25.0124	0.0468	(0.0004)	3.02E-02
60	0.00	6	0.2	0.2	0.00	34.9916	25.0084	0.0853	(0.0005)	1.34E-02
60	0.00	6	0.2	0.1	0.50	34.9835	25.0165	0.0892	(0.0005)	2.93E-02
60	0.04	6	0.2	0.1	-0.50	37.9676	22.0324	0.0348	(0.0003)	5.28E-02
60	0.04	6	0.2	0.1	0.00	34.9805	25.0195	0.0789	(0.0005)	2.81E-02
60	0.04	6	0.2	0.2	0.00	34.9824	25.0176	0.1349	(0.0006)	4.07E-03

TABLE 7: *Five-asset basket call prices: Ju [2002] prices versus weak lognormal approximation*

$(S_1 = S_2 = S_3 = S_4 = S_5 = 100, \theta_1 = 35, \theta_2 = 25, \theta_3 = 20, \theta_4 = 15, \theta_5 = 5, T = 1 \text{ year})$

K	r	σ	ρ	K_1	K_2	K_3	K_4	K_5	**GBM Price**	**Std Dev**	**Pricing Error**
90	0.05	0.2	0.0	21.19	16.19	21.15	18.24	13.24	14.6254	(0.0011)	4.07E-07
90	0.05	0.2	0.5	20.36	15.36	21.70	18.79	13.79	10.3070	(0.0011)	6.79E-06
90	0.05	0.5	0.0	16.78	12.04	21.68	21.93	17.57	8.4260	(0.005)	4.19E-06
90	0.05	0.5	0.5	12.65	9.54	24.35	23.97	19.49	21.2996	(0.0065)	9.23E-06
90	0.10	0.2	0.0	15.38	12.16	21.68	22.88	17.90	8.8929	(0.0004)	9.57E-06
90	0.10	0.2	0.5	15.16	11.97	22.31	22.77	17.79	6.5267	(0.0003)	6.34E-06
90	0.10	0.5	0.0	12.38	9.55	22.62	24.80	20.65	22.8694	(0.0029)	3.26E-07
90	0.10	0.5	0.5	9.64	7.05	26.02	25.87	21.41	20.2037	(0.0028)	2.20E-05
100	0.05	0.2	0.0	40.45	44.53	5.00	6.95	3.07	2.2074	(0.0007)	8.52E-06
100	0.05	0.2	0.5	24.28	19.02	26.72	17.56	12.42	18.6285	(0.0012)	1.64E-07
100	0.05	0.5	0.0	19.97	15.08	24.66	22.54	17.75	12.6438	(0.0054)	8.58E-06
100	0.05	0.5	0.5	16.00	11.13	26.69	25.42	20.77	10.5148	(0.0052)	1.89E-06
100	0.10	0.2	0.0	20.80	15.80	24.59	21.91	16.90	15.6479	(0.0005)	2.36E-07
100	0.10	0.2	0.5	20.10	15.10	25.11	22.35	17.34	11.9199	(0.0005)	2.48E-07
100	0.10	0.5	0.0	16.64	11.88	25.15	25.39	20.93	13.8766	(0.0028)	2.42E-07
100	0.10	0.5	0.5	13.15	8.68	27.96	27.39	22.82	25.3757	(0.0031)	5.89E-07
110	0.05	0.2	0.0	44.73	39.71	12.46	9.03	4.07	6.8143	(0.0009)	1.59E-06
110	0.05	0.2	0.5	42.25	37.48	6.91	14.12	9.23	4.2398	(0.0007)	1.06E-06
110	0.05	0.5	0.0	22.63	17.73	26.89	23.92	18.83	18.3388	(0.0062)	5.05E-08
110	0.05	0.5	0.5	18.19	13.23	29.37	27.18	22.04	15.2241	(0.0058)	2.78E-07
110	0.10	0.2	0.0	24.76	19.93	29.48	20.50	15.33	4.3969	(0.0004)	4.23E-07
110	0.10	0.2	0.5	23.47	18.46	27.04	23.03	18.00	19.2149	(0.0006)	3.74E-06
110	0.10	0.5	0.0	19.62	14.73	28.09	26.17	21.39	17.8991	(0.0028)	2.56E-06
110	0.10	0.5	0.5	15.68	10.80	30.48	28.88	24.16	15.9268	(0.0028)	7.48E-08

options in these simulations. However, in practice, one of the main advantages of our approach is that it allows us to feed in the implied volatility skew information to our calibrated model price of a multi-asset option. In fact, we may calibrate the single-asset implied volatilities and two-asset implied correlations to be consistent with any liquid market prices that are available.

6. MODEL CALIBRATION

This section illustrates how to calibrate the model in practice, by choosing the parameters to match a benchmark. This benchmark would be a market price, if a sufficiently liquid market exists. We shall consider the pricing of a specific four-asset basket option with payoff $[S_1 - S_2 - S_3 + S_4]^+$ and assume the benchmark is obtained by simulating correlated lognormal processes using the following volatilities and correlations for the assets:

$$\mathbf{\Sigma} = \begin{pmatrix} 0.10 \\ 0.15 \\ 0.18 \\ 0.20 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0.8 & 0.6 & 0.2 \\ 0.8 & 1 & 0.55 & 0.65 \\ 0.6 & 0.55 & 1 & 0.57 \\ 0.2 & 0.65 & 0.57 & 1 \end{pmatrix}. \tag{38}$$

TABLE 8: *Five-asset basket call prices: Ju [2002] prices versus weak lognormal approximation*

($S_1 = S_2 = S_3 = S_4 = S_5 = 100$, $\theta_1 = 35$, $\theta_2 = 25$, $\theta_3 = 20$, $\theta_4 = 15$, $\theta_5 = 5$, $T = 3$ years)

K	r	σ	ρ	K_1	K_2	K_3	K_4	K_5	**GBM Price**	**Std Dev**	**Pricing Error**
90	0.05	0.2	0.0	21.91	16.92	19.97	18.10	13.10	23.0121	(0.0017)	1.501E-07
90	0.05	0.2	0.5	21.05	16.05	20.51	18.69	13.71	26.1671	(0.0017)	1.173E-05
90	0.05	0.5	0.0	18.85	14.05	21.61	19.86	15.64	21.0184	(0.0098)	5.882E-05
90	0.05	0.5	0.5	14.38	9.80	23.32	22.95	19.55	37.2287	(0.0107)	5.375E-07
90	0.10	0.2	0.0	17.07	12.08	20.91	22.43	17.51	18.5798	(0.0007)	1.105E-07
90	0.10	0.2	0.5	16.99	11.99	21.79	22.07	17.15	21.7596	(0.0008)	5.973E-08
90	0.10	0.5	0.0	15.19	10.49	23.99	22.14	18.18	36.828	(0.0057)	4.073E-05
90	0.10	0.5	0.5	11.37	7.16	24.59	25.03	21.84	38.5906	(0.0057)	2.829E-06
100	0.05	0.2	0.0	22.06	22.65	14.01	28.31	12.98	9.8016	(0.0013)	1.222E-06
100	0.05	0.2	0.5	25.40	20.22	22.37	18.57	13.44	33.3711	(0.0018)	1.105E-07
100	0.05	0.5	0.0	23.09	18.27	21.97	20.84	15.83	25.161	(0.01)	8.528E-05
100	0.05	0.5	0.5	17.44	12.57	25.61	24.47	19.92	27.6233	(0.0101)	4.210E-05
100	0.10	0.2	0.0	22.71	17.71	22.88	20.85	15.84	24.8104	(0.0008)	1.310E-06
100	0.10	0.2	0.5	22.00	17.00	23.41	21.30	16.30	27.5462	(0.0008)	6.849E-07
100	0.10	0.5	0.0	19.57	14.77	24.63	22.68	18.34	29.1034	(0.0059)	2.239E-08
100	0.10	0.5	0.5	15.04	10.30	27.20	25.76	21.70	42.7673	(0.0058)	3.766E-06
110	0.05	0.2	0.0	43.18	37.97	14.18	9.83	4.85	15.678	(0.0016)	8.567E-08
110	0.05	0.2	0.5	42.55	38.15	7.63	13.16	8.51	19.4368	(0.0016)	1.348E-05
110	0.05	0.5	0.0	27.51	22.89	21.22	22.01	16.36	29.9998	(0.0103)	1.500E-05
110	0.05	0.5	0.5	20.06	15.13	25.51	27.97	21.32	32.1145	(0.0104)	8.878E-05
110	0.10	0.2	0.0	27.93	23.08	24.78	19.67	14.55	13.4902	(0.0006)	3.717E-05
110	0.10	0.2	0.5	26.70	21.53	25.64	20.63	15.50	34.0088	(0.0008)	1.468E-07
110	0.10	0.5	0.0	23.86	18.97	25.48	23.36	18.33	32.7054	(0.0058)	2.576E-10
110	0.10	0.5	0.5	18.28	13.12	29.16	27.58	21.84	34.8364	(0.0057)	9.057E-06

The assets pay no dividends and their current prices are $S_1 = 100$, $S_2 = 90$, $S_3 = 85$ and $S_4 = 75$, so the basket price is currently at zero. The discount rate is 4%.

Following the four-asset basket decomposition described in section 2, we may write the option's payoff as a payoff to a CEO written on two simple exchange options, each on a pair of assets. In a general pricing tree, the model parameters would be the strikes for the single-asset call and put options in the terminal step of the recursive procedure, and the volatilities and correlations of the assets. However, since the strike of the basket option in this case is zero, there are no strikes for vanilla options to calibrate. Nevertheless, for an N-asset basket option, there are N asset price volatilities and $N(N-1)/2$ correlation parameters to calibrate, and there are infinitely many possible combinations of these values for which the model price of the basket option would be equal to its benchmark price. But not all values would yield implied volatility and implied correlation skew consistent prices and hedge ratios.

By choosing the size of the sub-baskets to be equal to 2 (i.e. setting $k = 2$), the payoff decomposition can be done in four distinct ways.[12] That is, the two pairs of assets can be permuted in four different ways. However, due to symmetry, a call option to exchange asset 1 for 2 is equivalent to

[12]While there are 4! permutations possible, most of them lead to redundant representations while some others, for instance, $(1, 4, 2, 3)$, lead to a CEO on two 2-asset options written on the sum of asset prices. This is equivalent to an exchange option with negative strike.

a put option to exchange asset 2 for 1. Hence the permutations $(1,2,3,4)$ and $(1,3,2,4)$ are equivalent to $(4,3,2,1)$ and $(4,2,3,1)$ respectively and yield the same prices. We therefore consider only $p_1 = (1,2,3,4)$ and $p_2 = (1,3,2,4)$.

Our approximation provides a natural convention for choosing the underlying asset volatilities; in general, they should be set equal to the implied volatilities of the vanilla options on individual assets with the calibrated strikes. But the strike of the basket option is zero in this example, so the base of the pricing tree contains exchange options rather than vanilla options, and we must calibrate implied volatilities to exchange option prices. That is, for the exchange option on assets 1 and 2, we set σ_1 equal to the implied volatility of an option on S_1 with strike S_2, and σ_2 equal to the implied volatility of an option on S_2 with strike S_1. The volatilities σ_3 and σ_4 are chosen in a similar manner. This yields prices and hedge ratios that would, in practice, be volatility skew consistent.

In general, certain asset correlations could be set equal to any implied correlations that are backed-out from the market prices of any liquid spread options, or two-asset basket options, using the volatilities that are calibrated as above. Then we calibrate the remaining correlations so that the basket option model prices match each other under different permutations, to eliminate any bias from using a certain permutation.

The approximation error creeps when we approximate the basket option volatility as in equation (31), whose sensitivity to different correlations depends on the chosen permutation. For a given permutation, correlations between assets that appear in different legs of the pricing tree affect the basket option price only through the correlation γ between the sub-basket options. However, if the assets belong to the same leg of the tree, then the correlations between these assets affect the basket option price through the sub-basket prices and the sub-basket volatilities. For instance, the correlations ρ_{12} and ρ_{34} affect the basket option price through the exchange option prices and volatilities under p_1, while under p_2 they affect the price only through the correlation γ between the sub-baskets.[13]

To illustrate this, in Figure 3 we plot the squared difference between the prices from permutations p_1 and p_2 against these correlations. For a given correlation ρ_{ij} shown on the horizontal axis, we set values for the other correlations using (38) and we then vary ρ_{ij} between -1 and 1, computing the squared price differences for each value of ρ_{ij}. To keep the sub-basket correlation (32) within -1 and 1, the values of ρ_{ij} may need to be constrained a little. Nevertheless, the prices from both permutations p_1 and p_2 can still be matched by an appropriate choice for ρ_{ij}, as shown in Figure 3. For instance, the two basket option prices will be equal when $\rho_{24} = 0.73$.[14]

[13] Similarly, the correlations ρ_{13} and ρ_{24} affect the basket option price through the exchange option prices and volatilities under p_2, while they affect the price only through the correlation γ between the sub-baskets under p_1.

[14] The convexity of the ρ_{34} curve in the region of 0 is very low, but there is actually only one value for which the two prices are exactly equal.

FIGURE 3: *Squared difference between our model prices from permutations* $(1,2,3,4)$ *and* $(1,3,2,4)$

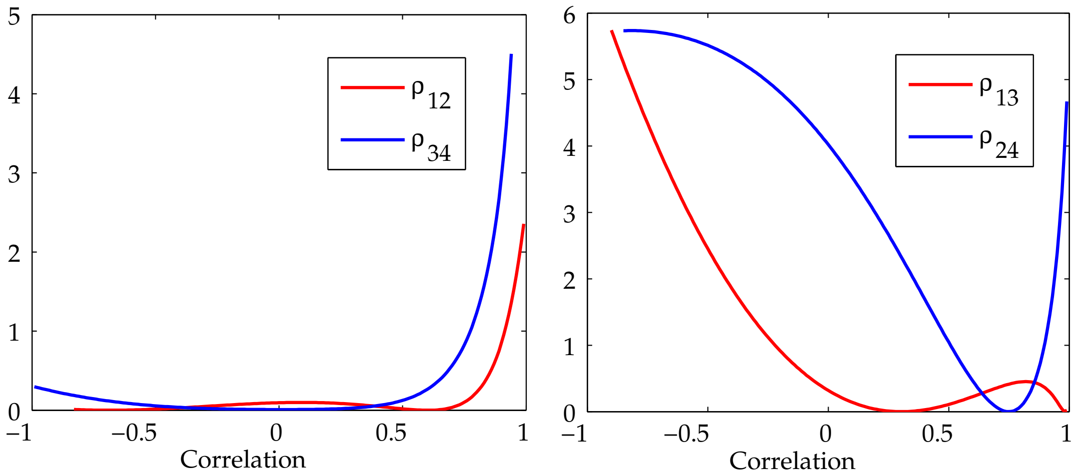


FIGURE 4: *Basket option price variation with respect to correlation between assets 1 and 4, and assets 2 and 3*

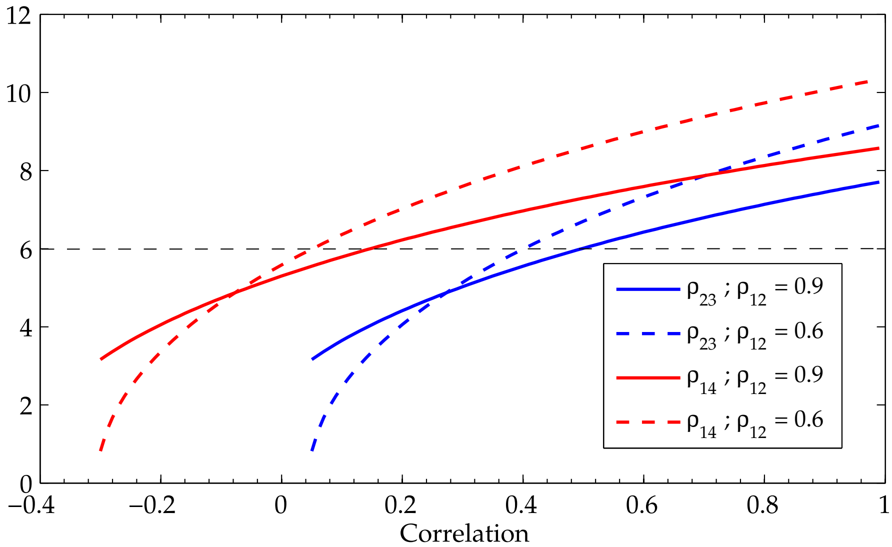
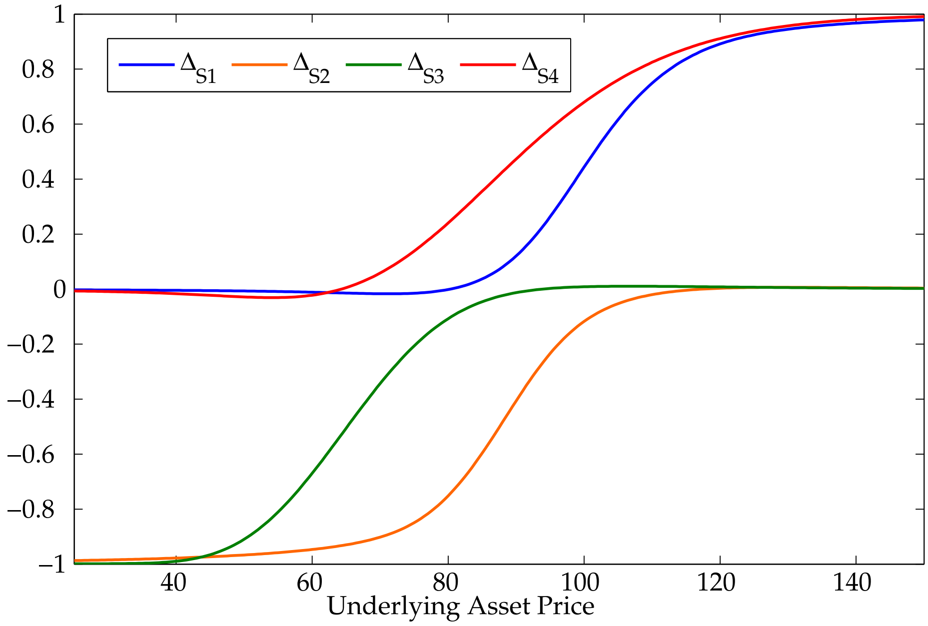


FIGURE 5: *Deltas with respect to all 4 underlying asset prices*



Irrespective of the permutation, some correlations only affect the sub-basket option correlation γ and not the sub-basket option prices or volatilities. Therefore, these would not have been calibrated in the previous step, but they can be calibrated now by matching the model price to the benchmark price of the basket option. In this example, the correlations ρ_{23} and ρ_{14}, unlike the rest, affect the basket option price only through the sub-basket correlation γ. Hence, having chosen ρ_{12}, ρ_{34}, ρ_{13} and ρ_{24} to match the prices obtained under the two possible permutations p_1 and p_2, we now may choose ρ_{23} and ρ_{14} so that the model price is equal to the benchmark price. Figure 4 plots the behaviour of the basket option price with respect to these correlations for two possible values of ρ_{12}, i.e. 0.6 and 0.9. For example, if the benchmark basket option price is 6, as indicated by the dotted horizontal line in the figure, then we calibrate $\rho_{23} = 0.4$ and $\rho_{14} = 0.04$ if $\rho_{12} = 0.6$, or $\rho_{23} = 0.5$ and $\rho_{14} = 0.14$ if $\rho_{12} = 0.9$.

Finally Figure 5 plots the deltas of the basket option with respect to the four underlying assets. These were computed using equation (36). The basket has positive weights on assets 1 and 4, so the deltas with respect to S_1 and S_4 resemble the deltas of vanilla call options, whereas it has negative weights on assets 2 and 3, so the deltas with respect to S_2 and S_3 resemble the deltas of vanilla put options. Due to differences in the assets volatilities and correlations, the current deltas differ. For instance, at $S_1 = 100$, $\Delta_1 \approx 0.5$ while at $S_2 = 90$, $\Delta_2 \approx -0.4$. This property is not captured by any other existing approaches to analytic approximations for multi-asset options, because they ignore the effects of asset price volatilities and correlations on the basket option deltas.

7. Conclusion

Most of the existing approaches to pricing basket options are based on approximating the distribution of the basket price, or they are limited to pricing average price basket options, or they apply only to options on a small number of assets. This paper develops a recursive framework for pricing and hedging European basket options which has no such constraints, and which may also be extended to rainbow options. Our key idea it is write the option payoff as a sum of payoffs to compound exchange options on sub-basket options. By writing the payoffs to these sub-basket options in terms of payoffs to compound exchange options on smaller sub-baskets, and repeating, we can draw a pricing tree that applies to any given basket option. This yields an approximate pricing formula for a general, N-asset basket option, which expresses the basket option price in terms of the prices of $2(N-1)$ compound exchange options and N standard European single-asset option prices.

The error stems from our approximation of various option price process as a lognormal processes with constant volatility, which is possible under a 'weak' lognormality condition. The approximation error can be minimised by a judicious choice of the strikes of the options in the base of the pricing tree, so that the weak lognormality condition holds. Moreover, our recursive procedure provides an almost exact price for certain options on baskets containing no more than three assets, because they satisfy what we call the 'strong' lognormality condition where exact lognormal option price processes may be applied, under a change of measure.

Simulations test the accuracy of our 'weak' lognormal approximations for pricing various vanilla options, exchange options, compound exchange options, spread options, two-asset basket options and the five-asset basket options considered in Ju [2002]. The results show that the approximation errors are very small and, in the five-asset case, the freedom to calibrate more strike parameters as the number of assets in the basket increases allows very accurate approximations indeed.

Our recursive approach is quite novel, and has several advantages over those already developed in the literature. Firstly, the underlying asset prices may follow heterogeneous lognormal processes. For instance, some asset prices could follow mean-reverting processes whilst others follow standard lognormal processes. Secondly, our framework provides a convention for selecting the implied volatilities of vanilla options on the individual underlying assets that are used to price the basket option. This yields volatility skew consistent prices. Moreover, our prices may also be consistent with implied correlations from any two-asset options used in the calibration set, although there is no convention for setting these, as we have for the implied volatilities. Thirdly, we can derive analytic approximations for multi-asset option Greeks, and unlike other approaches, these Greeks will be influenced by the individual asset price volatilities and correlations. Hence hedge ratios are consistent with the individual asset implied volatility and implied correlation skews.

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