Structural Safety 32 (2010) 35–41

Contents lists available at ScienceDirect

Structural Safety

journal homepage: www.elsevier.com/locate/strusafe

Evaluation of multivariate normal integrals for general systems by sequential compounding

Won-Hee Kang, Junho Song *

Department of Civil and Environmental Engineering, University of Illinois, Urbana-Champaign, Urbana, IL 61801, USA

ARTICLE INFO

Article history:
Received 17 February 2009
Received in revised form 17 June 2009
Accepted 21 June 2009
Available online 21 July 2009

Keywords:
Complex system
Cut-set system
Link-set system
Multivariate normal integral
Parallel system
Series system
System reliability

ABSTRACT

System reliability analysis often requires efficient and accurate evaluation of a multivariate normal integral. Despite recent advances in system reliability analysis methods, it is still a challenging task especially when the definition of the system event is complex; the system has a large number of components; and/or the component events have significant statistical dependence. This paper presents a new method developed for evaluating multivariate normal integrals defined for general system events including series, parallel, cut-set and link-set systems. The method compounds two components coupled by union or intersection sequentially until the system becomes a single compound event. Efficient numerical procedures are developed for obtaining the reliability index of the new compound event, and the correlation coefficients between the compound event and the remaining component events, at each step of the sequential compounding. The accuracy and efficiency of the proposed method, and its applicability to various types and sizes of multivariate normal integrals are demonstrated by numerical examples.

1. Introduction

For risk management of an engineering system, it is essential to evaluate the likelihood of its failure event efficiently and accurately. Sometimes, the failure needs to be described by a *system* event, i.e. a logical (or Boolean) function of multiple *component* events, each of which representing the failure of a physical component, the occurrence of a failure mode, or the structural damage at a particular location. The probability of such a system event is often formulated as a multivariate normal integral

$$P(E_{sys}) = \int_{\Omega} \varphi_n(\mathbf{z};\mathbf{R})d\mathbf{z} = \int_{\Omega} \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{R}}} \exp\left(-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z}\right)d\mathbf{z} \quad (1)$$

where E_{sys} is the system event of interest; Ω denotes the domain of a system event defined in the space of n standard normal random variables $\mathbf{Z}$; $\varphi_n(.)$ is the joint probability density function (PDF) of $\mathbf{Z}$; $\mathbf{R}$ is the correlation coefficient matrix of $\mathbf{Z}$; and $\det \mathbf{R}$ denotes the determinant of $\mathbf{R}$. For example, if the system failure event is defined as $E_{sys} = (E_1 \cap E_2) \cup E_3$, the domain Ω is determined as

$$\Omega = \{(z_1, z_2, z_3) | [(z_1 \leqslant -\beta_1) \cap (z_2 \leqslant -\beta_2)] \cup (z_3 \leqslant -\beta_3)\} \quad (2)$$

where β_i is the reliability index of the ith component event, $i = 1, 2, 3$. If the first order reliability method (FORM) is used for component reliability analysis, the correlation coefficient matrix $\mathbf{R}$ can be determined from the inner products of negative normalized gradient vectors evaluated at the design points [1].

There exists no closed form solution for this multi-fold integration. A direct numerical integration is impractical in many cases because, as the number of component events increases, the computational cost increases rapidly and significant numerical errors are accumulated [2]. Furthermore, a complex system failure domain may render a numerical integration challenging. A number of methods have been developed to compute the multivariate normal integral in Eq. (1) mostly for parallel systems (i.e., intersection of component events) and series systems (i.e., union of component events): first order approaches to multivariate normal integration (FOMN) [3,4], sequentially conditioned importance sampling (SCIS) [5], the product of conditional marginal (PCM) method [6–8], and quasi Monte Carlo simulation methods employing transformation and conditional expectations [9–11].

One of the few methods developed for non-series or non-parallel system problems can be found in the software RELSYS [12]. The method estimates the system probability using the concept of "equivalent component" [13], which sequentially replaces sub-series or sub-parallel systems into equivalent normal components until the system is simplified to a single equivalent component. In this method, individual component reliability indexes and the correlation coefficients between components are first evaluated by

* Corresponding author. Tel.: +1 217 244 9307; fax: +1 217 265 8040.
E-mail addresses: wkang3@illinois.edu (W.-H. Kang), junho@illinois.edu (J. Song).

0167-4730/$ - see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.strusafe.2009.06.001

FORM. The reliability indexes of the equivalent components are then estimated by use of a multivariate normal integral evaluation method while the equivalent directional cosines are approximately obtained by a finite difference method. It is known that this method provides accurate results for parallel systems with five or fewer components. However, it may result in significant error for series systems consisting of components with the same reliability indexes [12], and the errors may accumulate as the size of a general system increases.

The authors of this paper recently developed a matrix-based system reliability (MSR) method [14,15]. This method depends on neither sampling methods nor a finite difference based method, and computes the probability of a general system event including series, parallel, cut-set and link-set system and its parameter sensitivities efficiently and accurately using a convenient matrix-based framework. The method identifies the sources of statistical dependence between components as "common source random variables (CSRVs)" by using, e.g. a generalized Dunnett–Sobel class correlation coefficient matrix [15]. If the MSR method is used for evaluating multivariate normal integrals [16], the dimension of the multivariate integral is reduced from the number of components to that of the CSRVs. However, when it is required to have many CSRVs to describe the correlation coefficients between components accurately, this approach may require a time-consuming multi-fold integration, which prevents rapid evaluations.

In order to overcome these challenges and limitations, this paper presents a new systematic procedure developed for evaluating the multivariate normal integral of any general system efficiently and accurately. The proposed method sequentially compounds two components coupled by union or intersection operation in a general system event into a single equivalent component while the reliability index of the compound event and the correlation coefficients between the compound event and the remaining components are obtained through efficient numerical procedures. The accuracy and efficiency of the proposed method is demonstrated by various numerical examples.

2. Proposed method

The proposed method compounds two components coupled by a logical operation (e.g. union or intersection) sequentially until the system event of interest is simplified into a single compound event. Since it deals with the logical operation of only two components each time, the compounding process would not be encumbered by the complexity of the logical description of a given general system. This section introduces the compounding procedures developed for two components coupled by intersection or union, and explains the overall sequential compounding scheme toward the final solution.

2.1. Compounding two components coupled by intersection

Let us consider two components E_1 and E_2 coupled by intersection. The goal is to compound these into a single equivalent event $E_{1\text{and}2}$. For example, they can appear in a parallel system or a cut-set system, and can be compounded as follows.

$$\begin{aligned} &P(E_1 \cap E_2 \cap \cdots \cap E_{10}) = P(E_{1\text{and}2} \cap E_3 \cap \cdots \cap E_{10}) \\ &P(E_1E_2E_3 \cup E_4E_5E_6 \cup E_7E_8E_9) = P(E_{1\text{and}2}E_3 \cup E_4E_5E_6 \cup E_7E_8E_9) \end{aligned} \tag{3}$$

In order to proceed to the next compounding, we need to find the reliability index of the compound event $E_{1\text{and}2} = E_1 \cap E_2$ (denoted by $\beta_{1\text{and}2}$), and the correlation coefficient between the new compound event $E_{1\text{and}2}$ and each of the remaining component events in the system.

First, the reliability index of the compound event, $\beta_{1\text{and}2}$ is obtained by use of the single-fold numerical integration of $P(E_1 \cap E_2)$ as follows.

$$\beta_{1\text{and}2} = -\Phi^{-1}[P(E_1 \cap E_2)] = -\Phi^{-1}[\Phi_2(-\beta_1, -\beta_2; \rho_{1,2})] \tag{4}$$

where $\Phi(\cdot)$ is the marginal cumulative distribution function (CDF) of the standard normal distribution; $\Phi_2(\cdot)$ denotes the joint CDF of the bi-variate standard normal distribution; β_1 and β_2 are the reliability indexes of E_1 and E_2 respectively; and $\rho_{1,2}$ is the correlation coefficient between the standard normal random variables Z_1 and Z_2 which respectively represent E_1 and E_2 by $Z_1 \leqslant -\beta_1$ and $Z_2 \leqslant -\beta_2$. For simplicity, the correlation coefficients between standard normal random variables will be hereinafter termed as the correlation coefficient between component *events*. The bi-variate CDF $\Phi_2(-\beta_1, -\beta_2; \rho_{1,2})$ in Eq. (4) can be computed by a single-fold numerical integral [17]

$$\Phi_2(-\beta_1, -\beta_2; \rho_{1,2}) = \Phi(-\beta_1)\Phi(-\beta_2) + \int_0^{\rho_{1,2}} \varphi_2(-\beta_1, -\beta_2; \rho) d\rho \tag{5}$$

where $\varphi_2(\cdot)$ is the bi-variate joint PDF of standard normal distribution. The efficiency of the evaluation of the bi-variate joint CDF can be further improved by using special algorithms. In all the numerical examples in this paper, we use the algorithm that was developed in [18] and later modified in [19].

Next, we obtain the correlation coefficients between the new compound event $E_{1\text{and}2}$ and each of the other remaining component events in the system. Consider a triple of E_1, E_2, and E_k, $k = 3, \ldots, n$ where n is the current total number of components in the system during a sequential compounding process. We define the equivalent correlation coefficient $\rho_{(1\text{and}2),k}$, $k = 3, \ldots, n$ as the one that provides the same estimate on the probability of the event $\Omega_p = [(Z_1 \leqslant -\beta_1) \cap (Z_2 \leqslant -\beta_2)] \cap (Z_k \leqslant -\beta_k)$ even after compounding, i.e.

$$\Phi_3(-\beta_1, -\beta_2, -\beta_k; \rho_{1,2}, \rho_{1,k}, \rho_{2,k}) = \Phi_2(-\beta_{1\text{and}2}, -\beta_k; \rho_{(1\text{and}2),k}) \tag{6}$$

where $\Phi_3(\cdot)$ denotes the joint CDF of the tri-variate standard normal distribution. Since all the terms in Eq. (6) except $\rho_{(1\text{and}2),k}$ are already given, one can solve the equation for $\rho_{(1\text{and}2),k}$ numerically by finding $\rho_{(1\text{and}2),k}$ that minimizes the difference between the two CDFs by use of a nonlinear optimization algorithm with the constraint $-1 \leqslant \rho_{(1\text{and}2),k} \leqslant 1$. However, this is a time-consuming task because the numerical analysis would require repeated evaluations of the multi-fold integrals.

In order to obtain the equivalent correlation coefficients efficiently, we first decompose the CDFs in Eq. (6) using the conditional probabilities, i.e.

$$\begin{aligned} &P(Z_1 \leqslant -\beta_1 \cap Z_2 \leqslant -\beta_2 | Z_k \leqslant -\beta_k) \cdot \Phi(-\beta_k) \\ &\quad = P(Z_{1\text{and}2} \leqslant -\beta_{1\text{and}2} | Z_k \leqslant -\beta_k) \cdot \Phi(-\beta_k) \end{aligned} \tag{7}$$

where $Z_{1\text{and}2}$ is the standard normal random variable representing the compound event. For exact evaluations, these conditional probabilities would be obtained from non-normal distributions [6], but the proposed method approximates them by the CDFs of normal random variables. By dividing both terms by $\Phi(-\beta_k)$, Eq. (7) is approximated to

$$\Phi_2(-\beta_{1|k}, -\beta_{2|k}; \rho_{1,2|k}) = \Phi(-\beta_{(1\text{and}2)|k}) \tag{8a}$$

The conditional reliability indexes and conditional correlation coefficients given $Z_k \leqslant -\beta_k$ in this equation are determined exactly by [20]

$$\begin{aligned}
\beta_{1|k} &= (\beta_1 - \rho_{1,k}A)\Big/\sqrt{1-\rho_{1,k}^2 B}\\
\beta_{2|k} &= (\beta_2 - \rho_{2,k}A)\Big/\sqrt{1-\rho_{2,k}^2 B}\\
\rho_{1,2|k} &= (\rho_{1,2} - \rho_{1,k}\rho_{2,k}B)\Big/\left(\sqrt{1-\rho_{1,k}^2 B}\sqrt{1-\rho_{2,k}^2 B}\right)\\
\beta_{(1\text{and}2)|k} &= (\beta_{1\text{and}2} - \rho_{(1\text{and}2),k}A)\Big/\sqrt{1-\rho_{(1\text{and}2),k}^2 B}
\end{aligned} \tag{8b}$$

where $A = \varphi(-\beta_k)/\Phi(-\beta_k)$; and $B = A(-\beta_k + A)$ in which $\varphi(\cdot)$ denotes the PDF of the standard normal distribution. Note that the bi-variate CDF in Eq. (8a) can be computed by a single-fold integral [17], i.e.

$$\Phi_2(-\beta_{1|k}, -\beta_{2|k}; \rho_{1,2|k}) = \Phi(-\beta_{1|k})\Phi(-\beta_{2|k}) + \int_0^{\rho_{1,2|k}} \varphi_2(-\beta_{1|k}, -\beta_{2|k}, \rho)d\rho \tag{9}$$

or by use of an efficient algorithm [18,19].

In summary, Eq. (8) has only one unknown, i.e. $\rho_{(1\text{and}2),k}$ because β_1, β_2, β_k, $\rho_{1,2}$, $\rho_{1,k}$ and $\rho_{2,k}$ are available from the original component analysis or previous compounding processes. The efficiency of numerical analysis for obtaining $\rho_{(1\text{and}2),k}$ is significantly increased by solving Eq. (8) instead of Eq. (6) because it does not require performing any multi-fold numerical integration. At each compounding, this numerical analysis is performed $(n-2)$ times to obtain $\rho_{(1\text{and}2),k}$, $k = 3, \ldots, n$. In all the numerical examples in the paper, we used a sequential quadratic programming method [21] to obtain the correlation coefficients. In order to check the accuracy of the approximate equation in Eq. (8), Fig. 1 compares the equivalent correlation coefficients obtained by Eqs. (6) and (8) when the three components are equi-correlated, i.e. $\rho_{1,2} = \rho_{1,k} = \rho_{2,k} = \rho$, and have equal reliability indexes, i.e. $\beta_1 = \beta_2 = \beta_k = \beta$. Close matches are observed at all levels of correlation and reliability indexes considered.

2.2. Compounding two components coupled by union

Let us consider compounding two components E_1 and E_2 coupled by union into a single equivalent event $E_{1\text{or}2}$. For example, this compounding can appear in a series system or a link-set system, and can be compounded as follows.

$$\begin{aligned}
&P(E_1 \cup E_2 \cup \cdots \cup E_{10}) = P(E_{1\text{or}2} \cup E_3 \cup \cdots \cup E_{10})\\
&P((E_1 \cup E_2 \cup E_3) \cap (E_4 \cup E_5 \cup E_6)) = P((E_{1\text{or}2} \cup E_3) \cap (E_4 \cup E_5 \cup E_6))
\end{aligned} \tag{10}$$

First, using De Morgan's rule and the symmetry of the standard normal distribution, the reliability index of the compound event $E_{1\text{or}2}$ is obtained by

$$\begin{aligned}
\beta_{1\text{or}2} &= -\Phi^{-1}[P(E_1 \cup E_2)] = -\Phi^{-1}[1 - P(\bar{E}_1 \cap \bar{E}_2)]\\
&= \Phi^{-1}[P(\bar{E}_1 \cap \bar{E}_2)] = \Phi^{-1}[\Phi_2(\beta_1, \beta_2; \rho_{1,2})]
\end{aligned} \tag{11}$$

Next, we obtain the correlation coefficients between the new compound event $E_{1\text{or}2}$ and each of the other remaining component events in the system. Consider a triple of E_1, E_2, and E_k, $k = 3, \ldots, n$ where n is the current total number of components in the system during a sequential compounding process. We define the equivalent correlation coefficient $\rho_{(1\text{or}2),k}$ as the one that provides the same estimate on the probability of the event $\Omega_u = [(Z_1 \leqslant -\beta_1) \cup (Z_2 \leqslant -\beta_2)] \cap (Z_k \leqslant -\beta_k)$ after compounding, i.e.

$$\int_{\Omega_u} \varphi_3(z_1, z_2, z_3; \rho_{1,2}, \rho_{1,k}, \rho_{2,k})d\mathbf{z} = \Phi_2(-\beta_{1\text{or}2}, -\beta_k; \rho_{(1\text{or}2),k}) \tag{12}$$

Using the same decomposition and approximation used for the intersection case, Eq. (12) is approximated as

$$1 - \Phi_2(\beta_{1|k}, \beta_{2|k}; \rho_{1,2|k}) = \Phi(-\beta_{(1\text{or}2)|k}) \tag{13a}$$

where

$$\beta_{(1\text{or}2)|k} = (\beta_{1\text{or}2} - \rho_{(1\text{or}2),k}A)/\sqrt{1-\rho_{(1\text{or}2),k}^2 B} \tag{13b}$$

The bi-variate CDF in Eq. (13a) can be computed by performing the single-fold numerical integration in Eq. (9) with $-\beta_{1|k}$ and $-\beta_{2|k}$ replaced by $\beta_{1|k}$ and $\beta_{2|k}$, respectively or by use of a special algorithm [18,19]. At each compounding, Eq. (13) is solved numerically for $\rho_{(1\text{or}2),k}, k = 3, \ldots, n$ with the constraint $-1 \leqslant \rho_{(1\text{or}2),k} \leqslant 1$. Fig. 2 compares the equivalent correlation coefficients obtained by Eqs. (12) and (13) when the three components are equi-correlated, i.e. $\rho_{1,2} = \rho_{1,k} = \rho_{2,k} = \rho$, and have equal reliability indexes, i.e. $\beta_1 = \beta_2 = \beta_k = \beta$. Close matches are once again observed at all levels of correlation and reliability indexes considered.

2.3. Sequential compounding processes

For a parallel or series system, the procedures introduced in Section 2.1 or Section 2.2 can be applied to adjacent components sequentially until it becomes a single compound event. For a cut-set or link-set system, one can first compound components in each cut-set or link-set until the system becomes an equivalent series or parallel system, respectively. Then, the compound components in

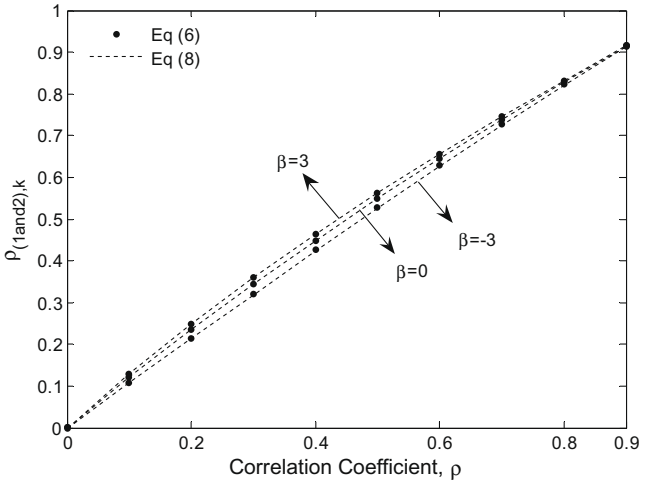

Fig. 1. Comparison of equivalent correlation coefficients $\rho_{(1\text{and}2),k}$ obtained by exact (Eq. (6)) and approximate (Eq. (8)) formulations.

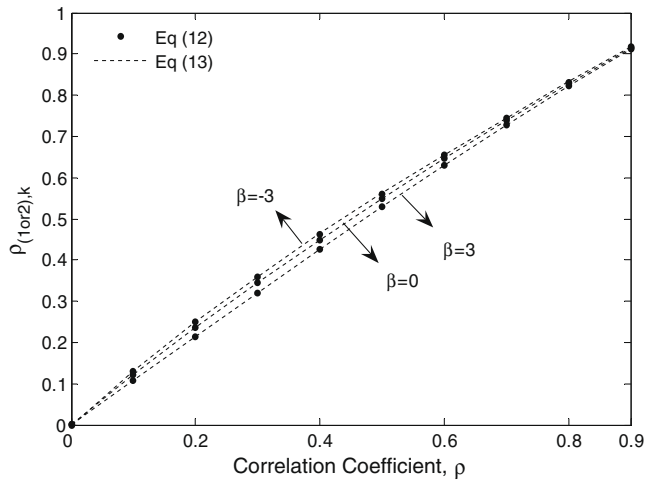

Fig. 2. Comparison of equivalent correlation coefficients $\rho_{(1\text{or}2),k}$ obtained by exact (Eq. (12)) and approximate (Eq. (13)) formulations.

the series or parallel system are compounded sequentially again. One can follow alternative orders of compounding as long as it is compatible with event operation rules (e.g. associative rule and commutative rule). Although each compounding process requires solving the nonlinear equation in Eq. (8) or Eq. (13) numerically for all remaining components, the proposed procedure is efficient because it does not involve sampling or expensive multi-fold numerical integrations. It is also noteworthy that the proposed approach can be used to quantify the statistical dependence between sub-systems, e.g., cut-sets or link-sets by the equivalent correlation coefficients between the compound events.

3. Numerical examples

3.1. Illustrative example: a link-set system with five components

Consider the following link-set system event consisting of five equally reliable components with, $\beta_i = -1,\ i = 1,\ldots,5$:

$$E_{sys} = E_1 \cap E_2 \cap (E_3 \cup E_4) \cap E_5 \tag{14}$$

The correlation coefficient matrix of the standard normal random variables representing the five components is given as

$$\mathbf{R} = \begin{bmatrix} 1 & 0.8 & 0.6 & 0.4 & 0.2 \\ & 1 & 0.8 & 0.6 & 0.4 \\ & & 1 & 0.8 & 0.6 \\ & sym. & & 1 & 0.8 \\ & & & & 1 \end{bmatrix} \tag{15}$$

In this example, each step of sequential compounding is shown along with the updated reliability index and correlation coefficients to demonstrate the proposed method. See Fig. 3 for an illustration of the procedure.

First, we compound the subsystem $E_1 \cap E_2$ into an equivalent component E_A. The reliability index of this compound component, β_A is computed as -0.773 by Eq. (4), and the system definition and correlation coefficient matrix are then updated to

$$E_{sys} = E_A \cap (E_3 \cup E_4) \cap E_5 \tag{16}$$

and

$$\mathbf{R} = \begin{bmatrix} 1 & \mathbf{0.725} & \mathbf{0.522} & \mathbf{0.319} \\ & 1 & 0.8 & 0.6 \\ & sym. & 1 & 0.8 \\ & & & 1 \end{bmatrix} \tag{17}$$

where the correlation coefficients between the new compound component and the remaining components (in bold) are obtained by solving Eq. (8) numerically.

Next, we merge the subsystem $(E_3 \cup E_4)$ in Eq. (16) into a compound component E_B. Using Eq. (11) in Section 2.3, the reliability index of the compound component β_B is computed as -1.295, and the system definition and correlation coefficients are respectively updated to

$$E_{sys} = E_A \cap E_B \cap E_5 \tag{18}$$

and

$$\mathbf{R} = \begin{bmatrix} 1 & \mathbf{0.651} & 0.319 \\ & 1 & \mathbf{0.729} \\ sym. & & 1 \end{bmatrix} \tag{19}$$

where the correlation coefficients between the compound event E_B and the other events (in bold) are computed by solving Eq. (13) numerically. The original system event in Eq. (14) is now transformed to a parallel system with three components E_A, E_B and E_5. By applying the procedure in Section 2.1 two more times, the system in Eq. (18) becomes a single compound event (E_D in Fig. 3). As a result, the system failure probability $P(E_{sys})$ is computed as 0.671 which is fairly close to 0.673 by a Monte Carlo simulation with 10^8 samples.

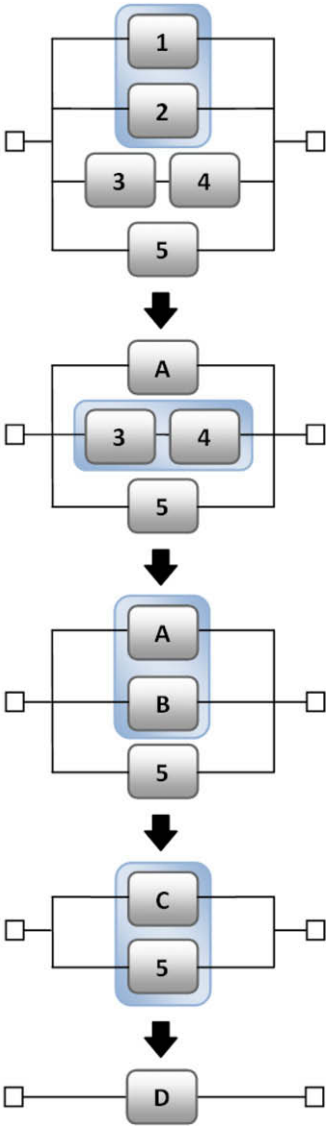

Fig. 3. Sequential compounding procedure for a general system with five components.

3.2. Parallel system consisting of 10 components with equal reliability indexes and equal correlation coefficients

The proposed approach is applied to a parallel system consisting of 10 components with the equal reliability indexes, i.e. $\beta_i = \beta, i = 1,2,\ldots,10$ and equal correlation coefficients $\rho_{i,j} = 0.5,\ i \neq j$. The probabilities computed by the proposed method are compared to those by the PCM method [6] which is known as an efficient and reasonably accurate method for evaluating multivariate normal integrals for series and parallel systems. Due to the equal correlation coefficients, the exact solution can be obtained by a single-fold numerical integration employing the Dunnett–Sobel class correlation model [22], i.e.

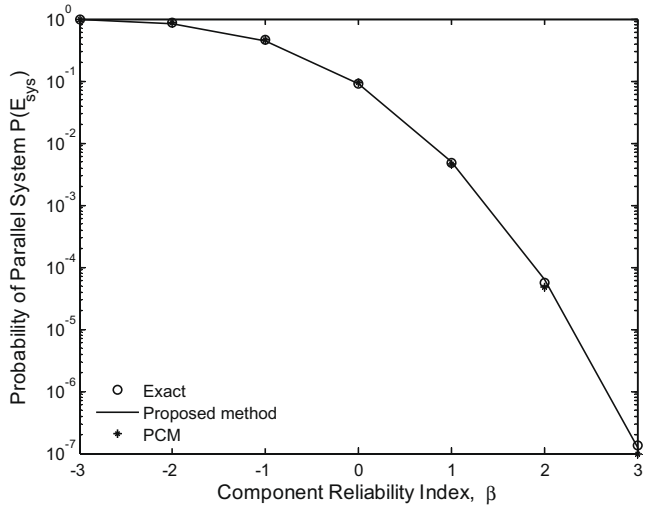

Fig. 4. Comparison of the proposed method and the PCM for parallel systems consisting of 10 components with equal reliability indexes and equal correlation coefficients.

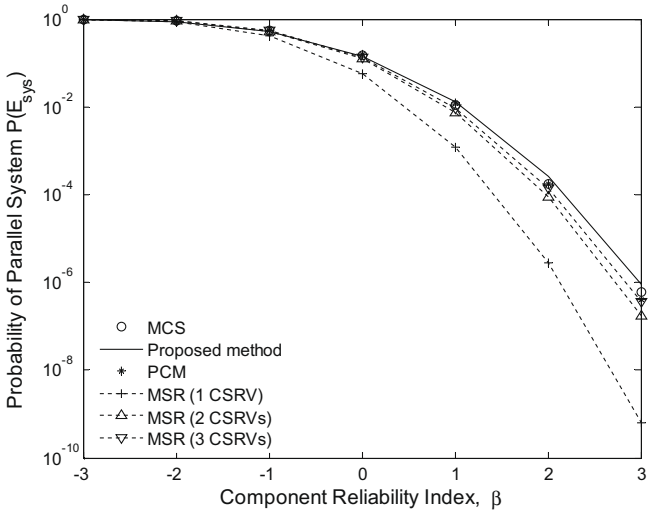

Fig. 5. Comparison of the proposed method, the PCM and MSR method for parallel systems consisting of 10 components with equal reliability indexes but unequal correlation coefficients.

$$P(E_{sys}) = \int_{-\infty}^{\infty} \left[\Phi\left(\frac{-\beta - \sqrt{0.5} \cdot s}{\sqrt{1-0.5}} \right) \right]^N \varphi(s) ds \tag{20}$$

where $N = 10$. Fig. 4 and Table 1 show the comparison as the equal component reliability index β varies from -3 to 3. Both methods provide quite accurate results while the proposed method shows slightly better accuracy overall.

3.3. Parallel system consisting of 10 components with equal reliability indexes but unequal correlation coefficients

Consider a parallel system with 10 components having equal reliability indexes but their unequal correlation coefficients are determined by

$$\rho_{i,j} = 1 - \frac{|i-j|}{10-1}, i,j = 1,\ldots,10, i \neq j \tag{21}$$

which satisfies the positive definite condition. The system probabilities are computed by the proposed method, the PCM method, and the MSR method using one, two and three CSRVs. The MSR method requires numerical integration whose dimension is the same as the number of CSRVs while the accuracy of the MSR method gradually enhances with the number of CSRVs increased. Since there is no closed form solution available for this example, we perform Monte Carlo simulation (MCS) with 10^8 samples for each case. The results in Fig. 5 show that the proposed method has a good agreement with the MCS results for the entire range of the reliability index considered.

3.4. Cut-set system consisting of 10 components with equal reliability indexes but unequal correlation coefficients

Consider the following cut-set system consisting of the components used in the previous example:

$$E_{sys} = E_1E_2 \cup E_3E_4 \cup \ldots \cup E_9E_{10} \tag{22}$$

Fig. 6 compares the results by the proposed method with those by the MSR method and MCS (10^8 samples). It is noted that most existing methods for multivariate normal integral including the PCM method are developed only for parallel or series systems, thus can not be used for this cut-set system. The proposed method provides the results that are close to those by MCS and its accuracy is even better than that of the MSR using three CSRVs.

3.5. Parallel system consisting of 10 components with unequal reliability indexes and unequal correlation coefficients

Consider a parallel system with 10 components having the equal correlation coefficients 0.5 except $\rho_{1,5} = \rho_{2,4} = 0.4$ and $\rho_{1,7} = \rho_{2,6} = \rho_{3,5} = 0.9$. The unequal reliability indexes of the 10 components are given by $\beta_i = (4 \cdot i - 22)/9$, $i = 1,\ldots,10$. For this example, the MSR and PCM method show significant errors, as shown in Table 2. By contrast, the proposed method shows a good agreement with the MCS result (10^8 samples).

3.6. Parallel system consisting of 5–50 components with equal reliability indexes and equal correlation coefficients

In order to examine the potential effect of the system size on the performance of the proposed method, we consider parallel systems with 5–50 components having the equal reliability indexes ($\beta = 1, 2$ or 3) and equal correlation coefficients 0.5. The exact solution can be obtained by Eq. (20) as N is varied from 5 to 50. As shown by the results in Fig. 7, in contrast to the PCM, the proposed method does not show significant error accumulation as the system size increases. Although the proposed method is not as efficient as the PCM method, it requires affordable computational cost. For example, the computations for 5, 10, …, 50-component

Table 1
Comparison of the proposed method with the PCM for parallel systems consisting of 10 components with equal reliability indexes and equal correlation coefficients.

β	−3	−2	−1	0	1	2	3
Exact	9.890×10^{-1}	8.669×10^{-1}	4.606×10^{-1}	9.091×10^{-2}	4.791×10^{-3}	5.658×10^{-5}	1.361×10^{-7}
Proposed method	9.884×10^{-1}	8.658×10^{-1}	4.621×10^{-1}	9.238×10^{-2}	5.013×10^{-3}	6.158×10^{-5}	1.549×10^{-7}
PCM	9.878×10^{-1}	8.628×10^{-1}	4.697×10^{-1}	9.332×10^{-2}	4.552×10^{-3}	4.735×10^{-5}	9.982×10^{-8}

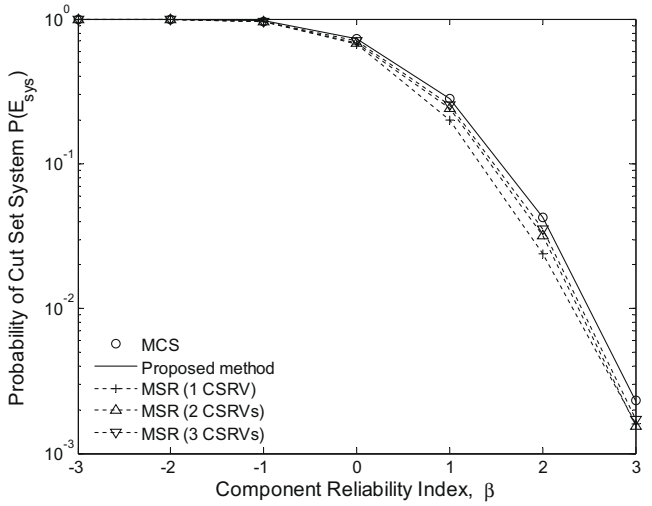

Fig. 6. Comparison of the proposed method and MSR method for cut-set systems consisting of 10 components with equal reliability indexes but unequal correlation coefficients.

Table 2
Comparison of the existing methods and the proposed method for a parallel system with unequal reliability indexes and unequal correlation coefficients.

	Proposed method	MSR (3 CSRVs)	PCM	MCS
Probability ($\times 10^{-3}$)	4.585	3.613	2.373	4.480

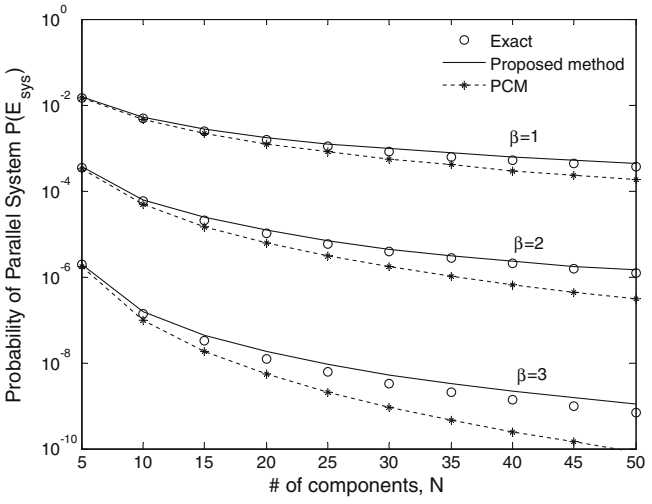

Fig. 7. Comparison of the proposed method and the PCM for parallel systems consisting of 5–50 components with equal reliability indexes and equal correlation coefficients.

systems took 0.16, 0.75, 1.64, 2.59, 3.69, 5.01, 6.38, 8.31, 9.82, and 11.90 seconds, respectively (Matlab® on a personal computer with AMD dual core 2.0 GHz.) The accuracy and efficiency of the proposed method can be further improved by using a numerical solver with higher-performance in computing equivalent correlation coefficients.

3.7. Series system consisting of 10–100 components with equal reliability indexes and equal correlation coefficients

The size effect is examined for series systems as well. Consider series systems having 10–100 components having the equal reliability indexes ($\beta = 1$, 2 or 3) and equal correlation coefficients 0.5. The exact solution of this problem can be obtained by

$$P(E_{sys}) = \int_{-\infty}^{\infty} \left\{ 1 - \left[1 - \Phi\left(\frac{-\beta - \sqrt{0.5} \cdot s}{\sqrt{1-0.5}} \right) \right]^N \right\} \varphi(s) ds \qquad (23)$$

where N is varied from 10 to 100. Fig. 8 confirms that the size effect of the proposed method is negligible in contrast to the PCM method. This example uses an improved version of the PCM method (IPCM) [8] that was developed for more accurate evaluation of multivariate normal integrals of series systems.

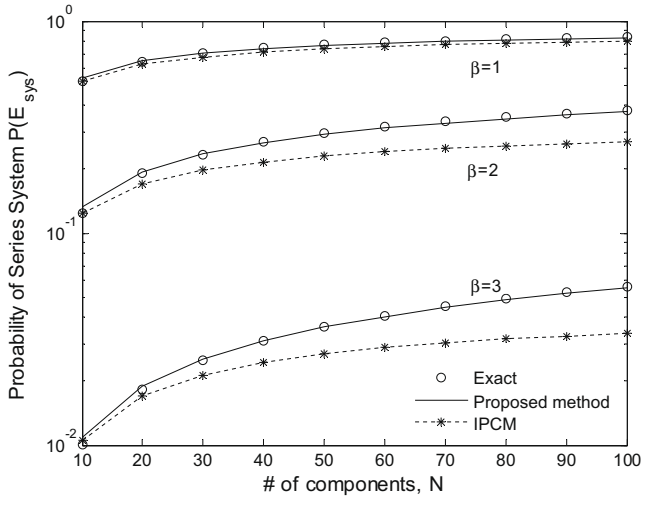

Fig. 8. Performance of the proposed method for series systems consisting of 10–100 components with equal reliability indexes and equal correlation coefficients.

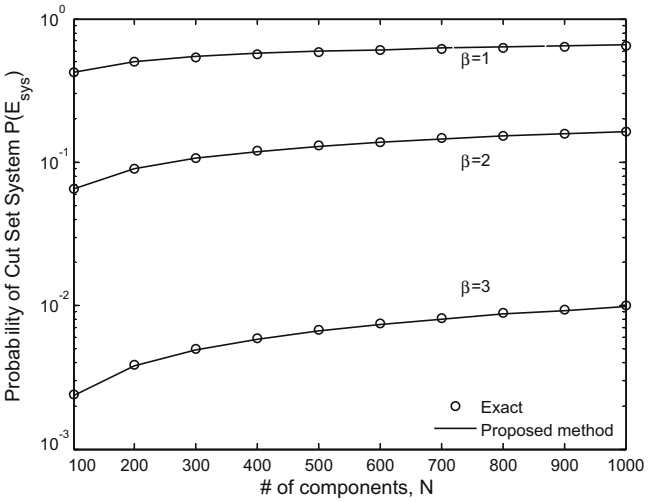

Fig. 9. Performance of the proposed method for cut-set systems consisting of 100–1000 components with equal reliability indexes and equal correlation coefficients.

3.8. Cut-set system consisting of 100–1000 components with equal reliability indexes and equal correlation coefficients

Consider the following cut-set system having the equal reliability indexes ($\beta = 1$, 2 or 3) and equal correlation coefficients 0.5:

$$E_{sys} = E_1E_2 \cup E_3E_4 \cup \ldots \cup E_{N-1}E_N \qquad (24)$$

The exact solution of this problem can be obtained by

$$P(E_{sys}) = \int_{-\infty}^{\infty} \left\{ 1 - \left[1 - \Phi\left(\frac{-\beta - \sqrt{0.5} \cdot s}{\sqrt{1-0.5}} \right) \right]^{N/2} \right\} \varphi(s) ds \qquad (25)$$

where N is varied from 100 to 1,000. As shown in Fig. 9, the results by the proposed method show good agreement with the exact solutions.

4. Summary and conclusion

A new method is developed for evaluating multivariate normal integrals defined for general systems efficiently and accurately. It sequentially compounds components in a general system event until the system becomes one compound event. For two components coupled by intersection and union, efficient procedures are developed to obtain the equivalent reliability index of the new compound event and the correlation coefficients between the compound event and the remaining components. In order to obtain the correlation coefficients efficiently, the numerical analysis employs approximate decomposition of bi- and tri-variate normal integrals. An important merit of the proposed method is that one can quantify the statistical dependence between sub-systems by the equivalent correlation coefficients between compound events. The proposed method's accuracy and efficiency for series, parallel, cut-set and link-set systems are demonstrated through comparison with other existing methods such as the PCM method and MSR method in various numerical examples. It is also confirmed that the accuracy of the method is not significantly affected by the large number of components in a system. Due to its wide applicability, accuracy and efficiency, the method is expected to enhance the computational capability in various system reliability analysis applications.

References

[1] Der Kiureghian A. In: Nikolaidis E, Ghiocel DM, Singhal S, editors. First- and second-order reliability methods, engineering design reliability handbook. Boca Raton, FL: CRC Press; 2005 [chapter 14].
[2] Greig GL. An assessment of higher order bounds for structural reliability. Struct Saf 1992;11:213–25.
[3] Hohenbichler M, Rackwitz R. First-order concepts in system reliability. Struct Saf 1983;1:177–88.
[4] Tang LK, Melchers RE. Improved approximation for multinormal integral. Struct Saf 1987;4:81–93.
[5] Ambartzumian R, Der Kiureghian A, Ohanian V, Sukiasian H. Multinormal probability by sequential conditioned importance sampling: theory and application. Probabilist Eng Mech 1998;13(4):299–308.
[6] Pandey MD. An effective approximation to evaluate multinormal integrals. Struct Saf 1998;20:51–67.
[7] Pandey MD, Sarkar A. Comparison of a simple approximation for multinormal integration with an importance sampling-based simulation method. Probabilist Eng Mech 2002;17:215–8.
[8] Yuan X-X, Pandey MD. Analysis of approximations for multinormal integration in system reliability computation. Struct Saf 2006;28:361–77.
[9] Genz A. Numerical computation of multivariate normal probabilities. J Comput Graph Stat 1992;1:141–50.
[10] Genz A. Comparison of methods for the computation of multivariate normal probabilities. Comput Sci Stat 1993;25:400–5.
[11] Joe H. Approximations to multivariate normal rectangle probabilities based on conditional expectations. J Am Stat Assoc 1995;90(431):957–64.
[12] Estes AC, Frangopol DM. RELSYS: a computer program for structural system reliability analysis. Struct Eng Mech 1998;6(8):901–19.
[13] Gollwitzer S, Rackwitz R. Equivalent components in first-order system reliability. Reliab Eng 1983;5:99–115.
[14] Kang W-H, Song J, Gardoni P. Matrix-based system reliability method and applications to bridge networks. Reliab Eng Syst Saf 2008;93:1584–93.
[15] Song J, Kang W-H. System reliability and sensitivity under statistical dependence by matrix-based system reliability method. Struct Saf 2009;31:148–56.
[16] Kang W-H, Song J. Evaluation of multinormal integral and sensitivity by matrix-based system reliability method. In: Proc. 10th AIAA nondeterministic approaches conference 2008, April 7–10, Schaumburg, IL.
[17] Ditlevsen O, Madsen HO. Structural reliability methods. New York: J. Wiley & Sons; 1996.
[18] Drezner Z, Wesolowsky GO. On the computation of the bivariate normal integral. J Stat Comput Simulat 1990;35:101–7.
[19] Genz A. Numerical computation of rectangular bivariate and trivariate normal and t-probabilities. Stat Comput 2004;14:251–60.
[20] Birnbaum ZW. Effect of linear truncation on a multinormal population. Ann Math Stat 1950;21:272–9.
[21] Biggs MC. Constrained minimization using recursive quadratic programming. In: Dixon LCW, Szergo GP, editors. Towards global optimization. North-Holland; 1975. p. 341–9.
[22] Dunnett CW, Sobel M. Approximations to the probability integral and certain percentage points of a multivariate analogue of Student's t-distribution. Biometrika 1955;42:258–60.