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A generalization of the Hull and White formula with applications to option pricing approximation

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Abstract By means of Malliavin calculus we see that the classical Hull and White formula for option pricing can be extended to the case where the volatility and the noise driving the stock prices are correlated. This extension will allow us to describe the effect of correlation on option prices and to derive approximate option pricing formulas.

Keywords Continuous-time option pricing model · Stochastic volatility · Malliavin calculus

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1 Introduction

One of the natural extensions of the Black-Scholes model that has been proposed in order to capture the modern market phenomena is to modify the

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specification of volatility to make it a stochastic process. The classical literature devoted to continuous-time stochastic volatility models has been extended in different directions. Some works (like [13,15]) proposed numerical methods for option pricing as Monte Carlo simulations. Others have developed quasi-analytic solutions by series expansions (see [8]). On the other hand, some researchers have set up models with closed form solutions, as in Stein and Stein [14], Heston [7] or more recently in Schöbel and Zhu [12].

The simplest stochastic volatility models assume that the volatility and the noise driving stock prices are uncorrelated (see [2,8,13,14]). In this framework, an important result is given by the so-called *Hull and White formula,* which establishes that the price of a European option is the expectation (under the risk-neutral probability) of the Black–Scholes option pricing formula where the constant volatility is replaced by its future quadratic average. This representation is of practical use for option pricing because it reduces the dimensionality of the problem. Moreover, it allows to prove some important features of these models, as the *smile effect* (see for example [10]).

Nevertheless, it is often found from financial data that there exists a negative instantaneous correlation between the volatility and the price processes, and there are economic arguments that justify this negative correlation. The Hull and White formula can be generalized to the correlated case (see for example [11,16]) in the sense that option prices can be written again as conditional expectations over Black–Scholes prices by appropriately adjusting the arguments to the Black–Scholes formula. The obtained formulas are of practical use for Monte Carlo simulation because the dimensionality of the problem is again reduced, but their mathematical expressions are more complex and, unlike the uncorrelated case, they do not directly reveal any information about the implied volatility curve.

The main goal of this paper is to obtain a new generalization of the classical Hull and White formula to the correlated case. The basic idea is to expand option prices around the classical Hull and White expression by means of Malliavin calculus. The anticipating stochastic calculus (or Malliavin calculus) is a powerful extension of the classical Itô calculus that allows us to work with non-adapted processes (we refer to [9] for a complete introduction to this subject). For this reason it becomes a natural tool for our problem, where the average future volatility is not adapted. This approach does not reduce the dimensionality of the problem, but it decomposes option prices as the sum of the same derivative price if there were no correlation and a correction due to correlation. This decomposition allows us to study the effect of correlation on the option prices. As an application, we present a method to construct option pricing approximation formulas that can be interpreted as a generalization of some recent results presented in Fouque et al. [5] for the case of fast mean-reverting volatilities.

The paper is organized as follows. In Sect. 2 we present a brief introduction to Malliavin calculus. Section 3 is devoted to proving our extension of the Hull and White formula for option prices. Finally, in Sect. 4 we use this decomposition in order to derive approximate option pricing formulas.

2 Preliminaries on Malliavin calculus

We recall here the basic facts of Malliavin calculus required in the paper. For a complete exposition we refer to Nualart [9].

Let us consider a standard Brownian motion $W = \{W_t, t \in [0,T]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Set $H = L^2([0,T])$, and denote by $W(h)$ the Wiener integral of a function $h \in H$. Let $\mathcal{S}$ be the set of random variables of the form $F = f(W(h_1), \dots, W(h_n))$, where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (i.e., f and all its derivatives are bounded), and $h_1, \dots, h_n \in H$. Given a random variable F of this form, we define its derivative as the stochastic process $\left\{D_t^W F, t \in [0,T]\right\}$ given by

$$D_t^W F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n)) h_i(x), \quad t \in [0,T]. \tag{1}$$

The operator D^W and the iterated operators $D^{W,n}$ are closable and unbounded from $L^2(\Omega)$ into $L^2\left([0,T]^n \times \Omega\right)$, for all $n \geq 1$. We denote by $\mathbb{D}_W^{n,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{n,2}^2 := \|F\|_{L^2(\Omega)}^2 + \sum_{k=1}^n \left\|D^{W,k} F\right\|_{L^2\left([0,T]^k \times \Omega\right)}^2.$$

We denote by δ^W the adjoint of the derivative operator D^W. Notice that δ^W is an extension of the Itô integral in the sense that the set $L_a^2([0,T] \times \Omega)$ of square integrable and adapted processes is included in $\operatorname{Dom}\delta$ and the operator δ restricted to $L_a^2([0,T] \times \Omega)$ coincides with the Itô stochastic integral. We use the notation $\delta(u) = \int_0^T u_t \mathrm{d}W_t$. We recall that $\mathbb{L}_W^{n,2} := L^2\left([0,T]; \mathbb{D}_W^{n,2}\right)$ is contained in the domain of δ for all $n \geq 1$.

The proof of the following version of Itô's formula for anticipating processes is similar to the proof of Theorem 3 in Alòs and Nualart [1].

Theorem 1 *Let us consider a process of the form $X_t = X_0 + \int_0^t u_s \mathrm{d}W_s + \int_0^t v_s \mathrm{d}s$, where X_0 is an $\mathcal{F}_0$-measurable random variable and $u, v \in L_a^2([0,T] \times \Omega)$. Consider also a process $Y_t = \int_t^T \theta_s \mathrm{d}s$, for some $\theta \in \mathbb{L}_W^{1,2}$. Let $F: \mathbb{R}^3 \to \mathbb{R}$ be a twice continuously differentiable function such that there exists a positive constant C such that, for all $t \in [0,T]$, F and its derivatives evaluated in (t, X_t, Y_t) are bounded by C. Then it follows that*

$$\begin{aligned} F(t, X_t, Y_t) &= F(0, X_0, Y_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s, Y_s)\,\mathrm{d}s \\ &\quad + \int_0^t \frac{\partial F}{\partial x}(s, X_s, Y_s)\,\mathrm{d}X_s + \int_0^t \frac{\partial F}{\partial y}(s, X_s, Y_s)\,\mathrm{d}Y_s \\ &\quad + \int_0^t \frac{\partial^2 F}{\partial x \partial y}(s, X_s, Y_s)\left(\int_s^T D_s^W \theta_r dr\right) u_s \mathrm{d}s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s, Y_s)\, u_s^2 \mathrm{d}s. \end{aligned} \tag{2}$$

3 An extension of the Hull and White formula

3.1 Main notations and hypotheses

We consider the following model for stock prices on a time interval $[0, T]$ under a risk-neutral probability:

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma_t S_t\left(\rho dW_t^* + \sqrt{1-\rho^2}dZ_t^*\right), \quad t \in [0, T], \tag{3}$$

where r is the instantaneous interest rate (supposed to be constant), W^* and Z^* are independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P^*)$, $\rho \in [0, 1]$, and (σ_t) is a square integrable process adapted to the filtration generated by the Brownian motion W^*. It will be convenient in the following sections to make the change of variable $X_t = \log S_t,\ t \in [0, T]$. It is well known that if we price a European option [defined by its nonnegative payoff $h(X_T)$] by the formula

$$V_t = \mathrm{e}^{-r(T-t)}E^*\left[h(X_T)|\,\mathcal{F}_t\right], \tag{4}$$

where E^* denotes expectation with respect to P^* and $\mathcal{F}_t := \sigma\left\{W_s^*, Z_s^*, s \leq t\right\}$, there is no arbitrage opportunity. Thus V_t is a possible price for this derivative. Notice that any allowable choice of P^* leads to an equivalent martingale measure and to a different no-arbitrage price. The approach that we follow here is the same used in Fouque et al. [5], where it is assumed that the market selects a unique equivalent martingale measure under which derivative contracts are priced.

We make use of the following notations:

- $v_t^2 = \frac{1}{T-t}\int_t^T \sigma_s^2 \mathrm{d}s$. That is, v_t^2 is the future average volatility.
- $\mathrm{BS}(t, x; \sigma)$ denotes the price of a European option with payoff $h(X_T)$ for a model with constant volatility equal to σ, current log stock price x, time to maturity $T - t$ and interest rate r. That is,

$$\mathrm{BS}(t,x;\sigma)=\frac{1}{\sigma\sqrt{2\pi\,(T-t)}}\int_{\mathbb{R}} h(y)\exp\left(-\frac{\left(y-x-\left(r-\frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)\mathrm{d}y.$$

- $\mathcal{L}_{BS}(\sigma)$ denotes the Black–Scholes differential operator (in the log variable) with volatility σ, i.e.,

$$\mathcal{L}_{\mathrm{BS}}(\sigma)f = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial f^2}{\partial x^2} + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial f}{\partial x} - rf.$$

It is well known that $\mathcal{L}_{\mathrm{BS}}(\sigma)\,\mathrm{BS}(\cdot,\cdot;\sigma) = 0$.

Finally, we consider the following hypotheses:

(H1) The payoff function $h : \mathbb{R} \to \mathbb{R}_+$ is continuous and piecewise $\mathcal{C}_b^1$.
(H2) There exists a positive real constant a such that $a \leq \sigma_t^2$ for all $t \in [0, T]$.
(H3) $\sigma^2 \in \mathbb{L}_{W^*}^{1,2}([0, T])$.
(H4) For all $t \in [0, T]$, there exists a positive constant C such that for all $s \in [t, T]$,

$$\left| E^* \left(\left(\int_s^T D_s^{W^*} \sigma_r^2 \mathrm{d}r \right) \sigma_s \middle| \mathcal{F}_t \right) \right| \leq C.$$

The proof of the next lemma is similar to the proof of Lemma 5 in Fouque et al. [6].

Lemma 2 *Fix* $t \in [0, T]$. *Consider the model* (3) *and assume that hypotheses* (H1), (H2) *and* (H3) *hold. Then there exists a positive constant C such that for all* $s \in [t, T[$,

$$E^* \left(\left| \frac{\partial^n \mathrm{BS}}{\partial x^n} (s, X_s; v_s) \right| \middle| W_u^*, u \leq s \right) \leq C (T - s)^{1 - n/2}.$$

Now we are in a position to prove the main result of this paper.

3.2 An extension of the Hull and White formula

Theorem 3 *Consider the model* (3)*, and assume that hypotheses* (H1) *to* (H4) *hold. Then, for all* $t \in [0, T]$,

$$V_t = E^* \left(\mathrm{BS} (t, X_t; v_t) \middle| \mathcal{F}_t \right) + \frac{\rho}{2} E^* \left(\int_t^T \mathrm{e}^{-r(s-t)} H (s, X_s; v_s) \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right), \qquad (5)$$

where $H (s, X_s; v_s) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) \mathrm{BS} (s, X_s; v_s)$ *and* $\Lambda_s := \left(\int_s^T D_s^{W^*} \sigma_r^2 \mathrm{d}r \right) \sigma_s$.

Proof Notice that $\mathrm{BS} (T, X_T; v_T) = V_T$. As $(\mathrm{e}^{-rt} V_t)$ is a P^*-martingale we can then write

$$\mathrm{e}^{-rt} V_t = E^* \left(\mathrm{e}^{-rT} V_T \middle| \mathcal{F}_t \right) = E^* \left(\mathrm{e}^{-rT} \mathrm{BS} (T, X_T; v_T) \middle| \mathcal{F}_t \right). \qquad (6)$$

Our idea is to apply Itô's formula (2) to the process $\mathrm{e}^{-rt} \mathrm{BS} (t, X_t; v_t)$. As the derivatives of $\mathrm{BS} (t, x; y)$ are not bounded we make use of an approximation argument. Take $\delta > 0$ and consider the process

$$\mathrm{e}^{-rt} \mathrm{BS} \left(t, X_t; v_t^\delta \right),$$

where $v_t^\delta := \sqrt{\frac{1}{T-t}\left(\delta + \int_t^T \sigma_s^2 \mathrm{d}s\right)}$. Applying now Itô's formula (2) with $Y_t = \int_t^T \sigma_\theta^2 \mathrm{d}\theta$ (notice that $v_t^\delta := \sqrt{\frac{1}{T-t}(\delta + Y_t)}$), we deduce that

$$
\begin{aligned}
&\mathrm{e}^{-rT}\mathrm{BS}\left(T, X_T; v_T^\delta\right) \\
&= \mathrm{e}^{-rt}\mathrm{BS}\left(t, X_t; v_t^\delta\right) \\
&\quad + \int_t^T \mathrm{e}^{-rs}\left(\mathcal{L}_{\mathrm{BS}}\left(v_s^\delta\right) + \frac{1}{2}\left(\sigma_s^2 - \left(v_s^\delta\right)^2\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\right)\mathrm{BS}\left(s, X_s; v_s^\delta\right)\mathrm{d}s \\
&\quad + \int_t^T \mathrm{e}^{-rs}\frac{\partial \mathrm{BS}}{\partial x}\left(s, X_s; v_s^\delta\right)\sigma_s\left(\rho \mathrm{d}W_s^* + \sqrt{1-\rho^2}\mathrm{d}Z_s^*\right) \\
&\quad + \frac{\rho}{2}\int_t^T \mathrm{e}^{-rs}\frac{\partial^2 \mathrm{BS}}{\partial x \partial \sigma}\left(s, X_s; v_s^\delta\right)\frac{1}{v_s^\delta (T-s)}\left(\int_s^T D_s^{W^*}\sigma_r^2 \mathrm{d}r\right)\sigma_s \mathrm{d}s \\
&\quad - \frac{1}{2}\int_t^T \mathrm{e}^{-rs}\frac{\partial \mathrm{BS}}{\partial \sigma}\left(s, X_s; v_s^\delta\right)\frac{\left(\sigma_s^2 - \left(v_s^\delta\right)^2\right)}{v_s^\delta (T-s)}\mathrm{d}s.
\end{aligned}
$$

Then, from the classical relationship between the Gamma, the Vega and the Delta,

$$
\frac{\partial \mathrm{BS}}{\partial \sigma}(s, x; \sigma)\frac{1}{\sigma(T-s)} = \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right)\mathrm{BS}(s, x; \sigma), \tag{7}
$$

we can write

$$
\begin{aligned}
\mathrm{e}^{-rT}\mathrm{BS}\left(T, X_T; v_T^\delta\right) = \mathrm{e}^{-rt}\mathrm{BS}\left(t, X_t; v_t^\delta\right) &+ \int_t^T \mathrm{e}^{-rs}\frac{\partial \mathrm{BS}}{\partial x}\left(s, X_s; v_s^\delta\right)\sigma_s\left(\rho \mathrm{d}W_s^* + \sqrt{1-\rho^2}\mathrm{d}Z_s^*\right) \\
&+ \frac{\rho}{2}\int_t^T \mathrm{e}^{-rs} H\left(s, X_s; v_s^\delta\right)\left(\int_s^T D_s^{W^*}\sigma_r^2 \mathrm{d}r\right)\sigma_s \mathrm{d}s.
\end{aligned}
$$

Taking now conditional expectations and multiplying by e^{rt} we obtain that

$$
\begin{aligned}
&E^*\left[\mathrm{BS}\left(T, X_T; v_T^\delta\right)\middle| \mathcal{F}_t\right] = \mathrm{BS}\left(t, X_t; v_t^\delta\right) \\
&\quad + E^*\left(\int_t^T \mathrm{e}^{-r(s-t)} H\left(s, X_s; v_s^\delta\right)\left(\int_s^T D_s^{W^*}\sigma_r^2 \mathrm{d}r\right)\sigma_s \mathrm{d}s\middle| \mathcal{F}_t\right).
\end{aligned}
$$

Notice that the second term on the right-hand side of this equality can be written as

$$E^*\Big(\int_t^T \mathrm{e}^{-r(s-t)} E^*\left(H\left(s, X_s; v_s^\delta\right) \middle| W_u^*, u \le s\right)\Big(\int_s^T D_s^{W^*}\sigma_r^2 \mathrm{d}r\Big)\sigma_s \mathrm{d}s \Big| \mathcal{F}_t\Big).$$

Letting now $\delta \to 0$, using Lemma 2 and dominated convergence arguments, the result follows. □

Remark 4 The proof of the above theorem does not require the volatility to be Markovian.

Remark 5 This proof only needs some integrability and regularity conditions (in the Malliavin calculus sense) on the volatility process. Then hypothesis (H2) can be replaced by appropriate integrability conditions.

Remark 6 It is known that the Hull and White formula can also be extended in the following way (see for example Willard [16] and Romano and Touzi [11]):

$$V_t = E^*\left(\mathrm{BS}\left(t, x\xi_t; \sqrt{\frac{1}{T-t}\int_t^T \left(1-\rho^2\right)\sigma_s^2 \mathrm{d}s}\right) \middle| W_s^*, s \le t\right), \tag{8}$$

where $\xi_t := \exp\Big(\rho\int_t^T \sigma_s \mathrm{d}Z_s^* - \frac{1}{2}\rho^2\int_t^T \sigma_s^2 \mathrm{d}s\Big)$. Expression (8) is of practical use for Monte Carlo simulation, because only one Brownian path needs to be generated. Although this result is presented usually in a Markovian context, it can be generalized to the case of non-Markovian volatilities (see for example Comte et al. [3]). Notice that formula (5) does not allow us to reduce the dimensionality of the problem, but its main contribution is to identify the impact of correlation on the option prices as the term $\frac{\rho}{2}E^*\left(\int_t^T \mathrm{e}^{-r(s-t)} H\left(s, X_s; v_s\right)\Lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right)$.

4 Approximate option pricing formulas

In this section we use the decomposition formula proved in the last section in order to construct approximate option pricing formulas. In order to prove our approximation results, we need to introduce the following hypotheses:

(H5) $\sigma^2 \in \mathbb{L}_{W^*}^{2,2}([0,T])$.

(H6) For all $t \in [0,T]$, there exist two constants $C > 0$ and $\gamma > 1$ such that, for all $s \in [t,T]$,

$$\left|E^*\Big(\Big(\int_s^T \Lambda_r \mathrm{d}r\Big)\Lambda_s \Big| \mathcal{F}_t\Big)\right| \le C(T-s)^\gamma$$

and

$$\left| E^*\left(\left(\int_s^T D_s^W \Lambda_r \mathrm{d}r \right) \sigma_s \middle| \mathcal{F}_t \right) \right| \leq C.$$

Proposition 7 *Fix* $t \in [0,T]$. *Consider the model* (3) *and assume that hypotheses* (H1) *to* (H6) *hold. Then there exists a positive constant C such that*

$$\left| V_t - E^*\left(BS\,(t, X_t; v_t) + \frac{\rho}{2} H\,(t, X_t; v_t) \left(\int_t^T \Lambda_s \mathrm{d}s \right) \middle| \mathcal{F}_t \right) \right|$$

$$\leq C\rho^2 E^* \left(\int_t^T (T-s)^{-2} \left(\int_s^T \Lambda_r \mathrm{d}r \right) \Lambda_s \mathrm{d}s \right.$$

$$\left. + \int_t^T (T-s)^{-\frac{1}{2}} \left(\int_s^T D_s^W \Lambda_r \mathrm{d}r \right) \sigma_s \mathrm{d}s \middle| \mathcal{F}_t \right).$$

Proof By Theorem 1 we know that

$$V_t = E^*\left(\mathrm{BS}\,(t, X_t; v_t) \middle| \mathcal{F}_t \right) + \frac{\rho}{2} E^*\left(\int_t^T \mathrm{e}^{-r(s-t)} H\,(s, X_s; v_s)\, \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right). \tag{9}$$

Consider the process $\left(\mathrm{e}^{-rt} H\,(t, X_t; v_t) \left(\int_t^T \Lambda_u \mathrm{d}u \right) \right)$. It is easy to check that it vanishes at $t = T$. Then, using Itô's formula as in the proof of Theorem 1, it follows that

$$0 = E^* \left(H\,(t, X_t; v_t) \left(\int_t^T \Lambda_u \mathrm{d}u \right) - \int_t^T \mathrm{e}^{-r(s-t)} H\,(s, X_s; v_s)\, \Lambda_s \mathrm{d}s \right.$$

$$+ \frac{\rho}{4} \int_t^T \mathrm{e}^{-r(s-t)} G\,(s, X_s; v_s) \left(\int_s^T \Lambda_r \mathrm{d}r \right) \Lambda_s \mathrm{d}s$$

$$\left. + \frac{\rho}{2} \int_t^T \mathrm{e}^{-r(s-t)} H\,(s, X_s; v_s) \left(\int_s^T D_s^W \Lambda_r \mathrm{d}r \right) \sigma_s \mathrm{d}s \middle| \mathcal{F}_t \right),$$

where $G\,(s, X_s; v_s) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H\,(s, X_s; v_s)$. Lemma 2 gives us that

$$\left| E^*\left(H\,(t, X_t; v_t) \left(\int_t^T \Lambda_s \mathrm{d}s \right) - \int_t^T \mathrm{e}^{-r(s-t)} H\,(s, X_s; v_s)\, \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right) \right|$$

$$= \left| E^* \left(\frac{\rho}{4} \int_t^T \mathrm{e}^{-r(s-t)} E^* \left(G\,(s, X_s; v_s) \middle| W_u, u \leq t \right) \left(\int_s^T \Lambda_r \mathrm{d}r \right) \Lambda_s \mathrm{d}s \right. \right.$$

$$+\frac{\rho}{2}\int_t^T e^{-r(s-t)}E^*\left(H\left(s,X_s;v_s\right)\middle| W_u,u\leq t\right)\left(\int_s^T D_s^W\Lambda_r\mathrm{d}r\right)\sigma_s\mathrm{d}s\middle|\mathcal{F}_t\right)\Bigg|$$

$$\leq C\rho E^*\left(\int_t^T (T-s)^{-2}\left(\int_s^T\Lambda_r\mathrm{d}r\right)\Lambda_s\mathrm{d}s\right.$$

$$\left.+\int_t^T (T-s)^{-\frac{1}{2}}\left(\int_s^T D_s^W\Lambda_r\mathrm{d}r\right)\sigma_s\mathrm{d}s\middle|\mathcal{F}_t\right)$$

for some positive constant C, and now the proof is complete. □

Remark 8 If Λ_r and its Malliavin derivative are small enough, this result suggests that the quantity $E^*\left(\mathrm{BS}\left(t,X_t;v_t\right)+\frac{\rho}{2}H\left(t,X_t;v_t\right)\int_t^T\Lambda_s\mathrm{d}s\middle|\mathcal{F}_t\right)$ can be interpreted as an approximation to option prices. Moreover, it is reasonable to assume that, if the variability of the volatility is small, this expression can be approximated by

$$V_{\text{approx}}:=\mathrm{BS}\left(t,X_t;v_t^*\right)+\frac{\rho}{2}H\left(t,X_t;v_t^*\right)E^*\left(\int_t^T\Lambda_s\mathrm{d}s\middle|\mathcal{F}_t\right),\tag{10}$$

where $v_t^*:=\sqrt{\frac{1}{T-t}\int_t^T E^*\left(\sigma_s^2\middle|\mathcal{F}_t\right)\mathrm{d}s}$. In the next section we study the goodness of this approximation.

4.1 Approximation results

For the sake of simplicity we assume that the volatility process can be written as $\sigma_r=f\left(Y_r\right)$, where $\left(Y_r\right)$ is a mean-reverting OU process of the form[1]

$$dY_r=\alpha\left(m-Y_r\right)dr+\lambda\sqrt{\alpha}dW_r^*\tag{11}$$

and $f\in\mathcal{C}^2$. We assume that for every fixed $t<T$ and $p\geq 1$, there exists a constant C (maybe depending on the initial value Y_t of the OU process) such that, for all $r\in[t,T]$,

$$E^*\left(|f(Y_r)|^p+\left|f'(Y_r)\right|^p+\left|f''(Y_r)\right|^p\middle|\mathcal{F}_t\right)\leq C.\tag{12}$$

From the definition of the Malliavin derivative operator we can deduce that, for all $t<s<r$, $D_s^W\sigma_r^2=2\lambda\sqrt{\alpha}f\left(Y_r\right)f'\left(Y_r\right)e^{-\alpha(r-s)}$. Then

[1] The equation for the volatility process depends on the market price of volatility risk λ which is determined by the risk-neutral probability P^*. Nevertheless, the risk premium is usually chosen in a way that the form of the risk-neutralized process does not change (see for example [8,12,14]).

$$V_{\text{approx}} = \mathrm{BS}\left(t, X_t; v_t^*\right) \tag{13}$$
$$+\lambda\sqrt{\alpha}\rho H\left(t, X_t; v_t^*\right) E^*\left(\int_t^T \left(\int_s^T f\left(Y_r\right) f'\left(Y_r\right) \mathrm{e}^{-\alpha(r-s)} \mathrm{d}r\right) f(Y_s) \mathrm{d}s \middle| \mathcal{F}_t\right).$$

Usually, f is taken to be a simple function (for example $f(x) = e^x$ in Scott [13], or $f(x) = x$ in Stein and Stein [14] and in Schöbel and Zhu [12]), which implies that the expression (13) can be easily evaluated. Notice that the second term on the right-hand side of this equation is bounded by $C\lambda\rho/\sqrt{\alpha}$, for some positive constant C.

Proposition 9 *Fix $t < T$. Assume the model* (3)*, where the volatility process can be written as $\sigma_t = f(Y_t)$, where $f \in \mathcal{C}^2$ and (Y_t) is a mean-reverting Ornstein–Uhlenbeck process of the form* (11) *such that* (H1), (H2) *and inequality* (12) *hold. Then the accuracy of the approximation of option prices is given by* $\left|V_t - V_{\text{approx}}\right| \leq \frac{C\lambda^2}{\alpha}\left(1 + |\ln \alpha|\right)$, *for some positive constant C.*

Proof The proof will be decomposed into several steps. In the sequel, C denotes a positive constant that can change from line to line and that can depend on the initial value Y_t of the Ornstein–Uhlenbeck process.

Step 1 Let us prove that

$$\left|V_t - E^*\left(\mathrm{BS}\left(t, X_t; v_t\right) + \frac{\rho}{2} H\left(t, X_t; v_t\right) \int_t^T \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right)\right| \leq \frac{C\lambda^2}{\alpha}\left(1 + |\ln \alpha|\right). \tag{14}$$

It is easy to check from the definition of the Malliavin derivative operator and inequality (12) that hypotheses (H3) and (H5) hold and that

$$D_s^{W^*} \sigma_r^2 = 2\lambda\sqrt{\alpha} f\left(Y_r\right) f'\left(Y_r\right) \mathrm{e}^{-\alpha(r-s)}, \tag{15}$$

$$D_u^{W^*} D_s^{W^*} \sigma_r^2 = 2\lambda^2 \alpha \left(f'\left(Y_r\right) f'\left(Y_r\right) + f\left(Y_r\right) f''\left(Y_r\right)\right) \mathrm{e}^{-\alpha(r-u)} \mathrm{e}^{-\alpha(r-s)}. \tag{16}$$

Then

$$\left|E^*\left(\left(\int_s^T D_s^{W^*} \sigma_r^2 \mathrm{d}r\right) \sigma_s \middle| \mathcal{F}_t\right)\right| = 2\lambda\sqrt{\alpha}\left|E^*\left(\left(\int_s^T f\left(Y_r\right) f'\left(Y_r\right) \mathrm{e}^{-\alpha(r-s)} \mathrm{d}r\right) f\left(Y_s\right) \middle| \mathcal{F}_t\right)\right|$$

and, by Hölder's inequality and (12), hypothesis (H4) follows. Using similar arguments we can see that hypothesis (H6) also holds.

Now, Proposition 7 and condition (12) allow us to write

$$
\begin{aligned}
&\left| V_t - E^*\left(\mathrm{BS}\,(t, X_t; v_t) + \frac{\rho}{2} H\,(t, X_t; v_t) \int_t^T \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right) \right| \\
&\le C\lambda^2 \alpha \rho^2 \int_t^T (T-s)^{-1} \left(\int_s^T \mathrm{e}^{-\alpha(\theta - s)} \mathrm{d}\theta \right)^2 \mathrm{d}s \\
&\le C\lambda^2 \alpha \rho^2 \int_t^T (T-s)^{-1} \left(\frac{1}{\alpha} \wedge (T-s) \right)^2 \mathrm{d}s \\
&\le \frac{C\lambda^2 \rho^2}{\alpha} (1 + |\ln \alpha|),
\end{aligned}
$$

which proves (14).

Step 2 Let us prove that

$$
\left| E^*\left(\mathrm{BS}\,(t, X_t; v_t) + \frac{\rho}{2} H\,(t, X_t; v_t) \int_t^T \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right) - V_{\text{approx}} \right| \le \frac{C\lambda^2}{\alpha}.
$$

In fact, using a second order Taylor development it is easy to see that

$$
E^*\left(\mathrm{BS}\,(t, X_t; v_t) - \mathrm{BS}\left(t, X_t; v_t^*\right) \middle| \mathcal{F}_t \right) \le \mathrm{CE}^*\left(\left(v_t^2 - \left(v_t^*\right)^2 \right)^2 \middle| \mathcal{F}_t \right) \le \frac{C\lambda^2}{\alpha},
$$

and using similar arguments we can prove that

$$
E^*\left(H\left(t, X_t; v_t^*\right) \int_t^T \Lambda_s \mathrm{d}s - H\,(t, X_t; v_t) \int_t^T \Lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right) \le C\frac{\lambda^2}{\alpha},
$$

which allows us to complete the proof. □

Remark 10 The above result indicates that V_{approx} is a good approximation for the option price if λ^2 is small or if α is big enough. That is, if the variance of the volatility is small or if the mean reversion is fast.

4.2 Examples

Example 11 Assume, as in Stein and Stein [14] and in Schöbel and Zhu [12], that the volatility process is given by $\sigma_t = Y_t$, where (Y_t) is a mean-reverting Ornstein–Uhlenbeck process of the form (11). In the following table we compare the values of the approximate call option prices obtained from the approximation formula (13) with the corresponding exact prices obtained by analytical computations in Schöbel and Zhu [12]. We have chosen $T-t = 0.5$, $X_t = \ln 100$,

$\alpha = 4, m = 0.2$, $\lambda = 0.05$, $r = 0.0953, \sigma_t = 0.2$, $\rho = -0.5$, $\rho = 0.5$ and varying values for the strike price K. The last column gives the pricing errors expressed as percentages of the corresponding option prices. Notice that for the chosen parameters the maximum observed deviation of the approximation from the exact option price is by 0.865% of this price (Table 1).

Remark 12 The simplicity of formula (13) makes it easy to apply in new different frameworks. For example, we can use it to obtain approximate option pricing formulas for long-memory volatility models (see for example Comte and Renault [4] and Comte et al. [3]), as we can see in the following example.

Example 13 Denote $\tilde{\sigma}_t = f(Y_t)$, where (Y_t) is a mean-reverting Ornstein–Uhlenbeck process of the form (11) and $f \in \mathcal{C}^2$ is a function such that inequality (12) holds. Assume the model (3), where the volatility process is given by $\sigma_t^2 = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \tilde{\sigma}_s^2 \mathrm{d}s$, for some $\beta \in \left(0, \frac{1}{2}\right)$. Using the same arguments as in Comte et al. [3] we can see that the process (σ_t^2) has long-memory properties, in the sense that the function $r(h) := \operatorname{cov}\left(\sigma_t^2, \sigma_{t+h}^2\right)$ is of order $O\left(|h|^{2\beta-1}\right)$ *for* $h \to \infty$. On the other hand, it is easy to see that $\int_s^T \sigma_r^2 \mathrm{d}r = \frac{1}{\Gamma(\beta+1)} \int_s^T (T-r)^\beta \tilde{\sigma}_r^2 \mathrm{d}r + \int_s^T \left(\int_0^s \frac{(r-u)^{\beta-1}}{\Gamma(\beta)} \tilde{\sigma}_u^2 \mathrm{d}u\right) \mathrm{d}r$, from which we deduce that the term due to correlation is now given by

$$\frac{\lambda\sqrt{\alpha}\rho}{\beta} H\left(t, X_t; v_t^*\right) E^*\left(\int_t^T \left(\int_s^T f(Y_r) f'(Y_r)(T-r)^\beta \mathrm{e}^{-\alpha(r-s)} \mathrm{d}r\right) f(Y_s) \Big| \mathcal{F}_t\right) \quad (17)$$

which is bounded again by $\frac{C\lambda\rho}{\sqrt{\alpha}}$. It is easy to check that the accuracy of this approximation is also given by $\left|V_t - V_{\text{approx}}\right| \leq \frac{C\lambda^2}{\alpha}(1 + |\ln\alpha|)$, for some positive constant C.

Table 1 Accuracy of the approximation

ρ	K	Exact price	Approximation	Pricing error (%)
−0.5	90	15.292	15.298	0.039
−0.5	95	11.503	11.521	0.156
−0.5	100	8.243	8.270	0.328
−0.5	105	5.595	5.627	0.572
−0.5	110	3.582	3.613	0.865
0.5	90	15.003	15.006	0.020
0.5	95	11.243	11.261	0.160
0.5	100	8.106	8.135	0.358
0.5	105	5.640	5.672	0.567
0.5	110	3.803	3.832	0.763

5 Conclusions

By means of Malliavin calculus we have proved that, in the stochastic volatility framework, option prices can be written as the sum of the classical Hull and White term and a correction due to correlation. This result allows us to describe the impact of correlation on derivative prices and to construct option pricing approximation formulas for a very general case of volatility models, including the case of long-memory volatilities.

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