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Generalizing the reflection principle of Brownian motion, and closed-form pricing of barrier options and autocallable investments[☆]

Hangsuck Lee[a], Soohan Ahn[b], Bangwon Ko[c,]*

[a] Dept. of Actuarial Science/Mathematics, Sungkyunkwan University, Seoul 03063, Republic of Korea
[b] Dept. of Statistics, University of Seoul, Seoul 02504, Republic of Korea
[c] Dept. of Statistics and Actuarial Science, Soongsil University, Seoul 06978, Republic of Korea

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ABSTRACT

In this paper, we intend to generalize the well-known reflection principle, one of the most interesting properties of the Brownian motion. The essence of our generalization lies in its ability to stochastically eliminate arbitrary number of partial maximums (or minimums) in the joint events associated with the Brownian motion, thereby allowing us to express the joint probabilities in terms of the multivariate normal distribution functions. Due to the simplicity and versatility, our generalized reflection principle can be used to solve many probabilistic problems pertaining to the Brownian motion. To illustrate, we consider evaluating barrier options and autocallable structured product. Using the basic inclusion-exclusion principle, we obtain integrated pricing formulas for various barrier options under the Black-Scholes model, and derive an explicit pricing formula for the autocallable product, which is not known yet despite its popularity. These formulas are explored through numerical examples. The method of Esscher transform demonstrates its time-honored value during the derivation process.

1. Introduction

In this paper, we are interested in generalizing the reflection principle of the Brownian motion. For a Brownian motion $X(t)$ with drift μ and volatility σ, it is well-known that

$$P(X(t) \leqslant x, \max_{0 \leqslant \tau \leqslant t} X(\tau) > m) = e^{\frac{2\mu m}{\sigma^2}} \Phi\left(\frac{x - 2m - \mu t}{\sigma\sqrt{t}}\right), \quad \text{for } x \leqslant m,\ m \geqslant 0, \tag{1}$$

where Φ is the standard normal distribution function. Basically, if the drift is zero, (1) is easy to see by drawing a reflected sample path at the first time that the Brownian motion hits the barrier m, whence the name reflection principle. However, the case with nonzero drift does not seem trivial and needs some mathematical treatment such as measure change. For the proof, we refer the readers to the discussion paper by Huang and Shiu (2001) or Harrison (1985) among others. The former utilizes the actuarial method of Esscher transform, which will play a crucial role in our simplifying tedious calculations later. Needless to say, the main point of (1)

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* Corresponding author.
E-mail address: bko@ssu.ac.kr (B. Ko).

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is to eliminate the maximum in the given event and express the desired probability in terms of the simple normal distribution function.

As can be seen in Theorem 1 later, our generalization of (1) develops in two perspectives. First, arbitrary number of observations before time t is to be included and secondly, arbitrary number of maximums taken over nonoverlapping partial periods of $[0, t]$. This line of generalization was first motivated by the so-called autocallable structured product, explained in more detail in Section 5, and has been discussed in the recent literature. Lee and Ko (2018) presented an earliest version which is stated as Example 2 of this paper, and Lee, Ko, and Song (2018) considered changing the barrier m in a piecewise constant function of time. Now, in this paper, we propose a very general version of the reflection principle with arbitrary observations and partial maximums. Obviously, the main difficulty of such generalization arises because the general form of the solution is not known in advance. Considering the complexity involved, our solution is quite simple to use and inherits the same attractive feature of (1). Therefore, it can be used to further generalize or solve existing probabilistic problems related to the reflection principle.

As practical applications, this paper focuses on deriving explicit pricing formulas for various barrier options and the autocallable product under the Black-Scholes (BS) model. As shown in Section 4, the generalized reflection principle allows us to obtain an integrated pricing framework for various barrier options, which yields many existing formulas as special cases. In particular, it can naturally incorporate the concept of icicles introduced by Lee and Ko (2018). The icicle is a single vertical branch of barrier attached along with the horizontal barrier of the ordinary barrier option, acting as an additional knock-in or knock-out trigger. Now, due to the generalized reflection principle, we are able to obtain explicit pricing formulas for barrier options with multiple icicles. Furthermore, it enables us to find an explicit pricing formula for the autocallable product with knock-in (KI) feature. Because of the complexity involved in the payoff structure, the formula is not known yet in the literature despite the popularity.

Our derivation is completely probabilistic, so it does not require mathematical backgrounds regarding partial differential equation (PDE). Instead, we exploit the Esscher transform. This method was originally developed to approximate the aggregate claim amount distribution, but now has become more popular in option pricing theory since the work of Gerber and Shiu Gerber and Shiu, 1994. For complete market model, see Tiong (2000), Lee (2003), Gerber, Shiu, and Yang (2012), Lee and Ko (2018) and more recently, Lee et al. (2018) to name a few. The readers are also referred to Bühlmann, Delbaen, Embrechts, and Shiryaev (1996), Chan (1999) and Shoutens (2003) for the application under an incomplete market model.

The rest of the paper is organized as follows. In Section 2, we briefly review the Esscher transform and the factorization theorem. In Section 3, we state our main result as a theorem and provide a heuristic proof. The theorem will be illustrated with several examples, each of which might look interesting. In Section 4, using the basic inclusion-exclusion principle, we derive an integrated pricing formula for barrier options, and explore how it works for multiple icicled barrier options. In Section 5, we derive an explicit pricing formula for the autocallable product with KI feature. These pricing formulas are investigated with numerical examples at the end of each section. And Section 6 concludes the paper, while discussing practical limitations. Finally, a more rigorous proof on Theorem 1 by induction is provided in Appendix A.

2. Preliminaries

First, we denote by $S(t)$ the time-t price of the underlying asset or index and assume that it pays no dividend for simplicity. Thus, under the BS model, we may write

$$S(t) = S(0)e^{X(t)}, \quad t \geqslant 0$$

for a Brownian motion $X(t)$ with drift μ and volatility σ. Denote the partial maximum during the time interval from s to t by

$$M(s, t) = \max_{s \leqslant \tau \leqslant t} X(\tau).$$

As mentioned, the generalized reflection principle allows us to express the probability of the event including these partial maximums in terms of the normal distribution function. When considering n subperiods in a fixed time horizon $[0, T]$, we will need n-dimensional standard multivariate normal distribution $\Phi(x_1, x_2, ..., x_n; \rho_{12}, \rho_{13}, ..., \rho_{n-1,n})$. Here ρ_{ij} represents the correlation between the i-th and the j-th variables.

For completeness, we briefly review the Esscher transform and the factorization formula as in Lee and Ko (2018). Since the stochastic process $\{e^{hX(t)}/E[e^{hX(t)}]\}$ is a positive martingale, it can be used to define the Radon-Nikodym derivative dQ/dP, where P is the original physical measure and Q is the Esscher measure of parameter h. For Y which is a measurable function of $\{X(t): 0 \leqslant t \leqslant T\}$, the expectation of Y under Q is calculated as $E^Q[Y] = E^P\left[Y\frac{e^{hX(T)}}{E^P[e^{hX(T)}]}\right]$. To emphasize the dependence upon the parameter h, we denote the expectation by $E[Y;h]$. Similarly, the probability of event A under the Esscher measure of parameter h is denoted by $P(A;h)$ throughout the paper. Under the Esscher measure of parameter h, the process $X(t)$ is also a Brownian motion with the same volatility σ, but the drift μ changed to $\mu + h\sigma^2$.

Under the BS model, the risk-neutral measure is the Esscher measure of parameter h^*, the solution to equation $\mu + h^*\sigma^2 = r - \sigma^2/2$. Here r is the interest rate. Thus, the drift under the Esscher measure of parameter h^* is $r - \sigma^2/2$. When we calculate expectations under the risk neutral measure, the following version of factorization theorem can be particularly useful: For an event B determined by $\{X(t): 0 \leqslant t \leqslant T\}$, we have

$$
\begin{aligned}
E[e^{X(T)}I(B);h^*] &= E\left[e^{X(T)}I(B)\frac{e^{h^*X(T)}}{E[e^{h^*X(T)}]}\right]\\
&= \frac{E[e^{(h^*+1)X(T)}]}{E[e^{h^*X(T)}]}\frac{E[I(B)e^{(h^*+1)X(T)}]}{E[e^{(h^*+1)X(T)}]}\\
&= E[e^{X(T)};h^*]E[I(B);h^*+1]\\
&= e^{rT}P(B;h^*+1).
\end{aligned}
$$

Here $I(\cdot)$ is an indicator function. We remark that the drift changes to $r+\sigma^2/2$ under the Esscher measure of parameter h^*+1. For more detail, the readers are referred to Gerber and Shiu (1994).

3. Main result with examples

Now we state the generalized reflection principle in Theorem 1. As mentioned earlier, it provides a general formula to determine the probabilities associated with the events that the Brownian motion $X(t)$ satisfies multiple conditions defined in terms of a discrete tenor of dates, $t_0 = 0, t_1, \ldots, t_n = T$. These multiple conditions can be:

- restrictions in terms of the partial maximum values for any subset of time intervals, i.e., several restrictions of type

  $M(t_{i-1}, t_i) > m,$

- restrictions on the values of the process itself at any subset of tenor dates, i.e., several restrictions of type

  $X(t_i) \leqslant x_i.$

Fig. 1 illustrates a sample path of the Brownian motion satisfying $M(t_0, t_1) > m$, $M(t_2, t_3) >$, $X(t_1) \leqslant x_1$, $X(t_2) \leqslant x_2$ and $X(t_3) \leqslant x_3$ for $n = 3$. These are realistic restrictions motivated by existent structured financial products. In this section, we provide a heuristic proof of the theorem based on the Esscher transform, and a more rigorous proof by induction can be found at the appendix.

Theorem 1. *Let*$[0, T]$*be a time interval with n non-overlapping subperiods*$[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$, $t_0 = 0$*and*$t_n = T$. *Let J denote a subset of*$\{1, 2, \ldots, n\}$*and*$h = \mu/\sigma^2$. *Denote by*$U(t_i)$*and*$R(t_i)$*the number of subperiods with at least one upcrossing (or the number of restrictions on partial maxima) up to time*t_i*and after time*t_i*, respectively. Then, for*$m \geqslant 0$*and*$x_i \leqslant m$*, we have*

$$
P\left(\bigcap_{i=1}^{n}(X(t_i) \leqslant x_i), \bigcap_{j\in J}(M(t_{j-1}, t_j) > m)\right) = e^{2hm_n}P\left(\bigcap_{i=1}^{n}(a_iX(t_i) + 2m_i \leqslant x_i)\right), \tag{2}
$$

where

$$
m_i = \begin{cases} m, & \text{if } U(t_i) \text{ is odd,}\\ 0, & \text{otherwise,}\end{cases} \qquad a_i = \begin{cases} -1, & \text{if } R(t_i) \text{ is odd,}\\ 1, & \text{otherwise.}\end{cases}
$$

Proof. Let us first consider where the drift is zero. To emphasize, we add a subscript 0 to the Brownian motion and the partial maximum as in $X_0(t)$ and $M_0(s, t)$, respectively. We can then obtain Eq. (2) by reflecting the sample path of interest at the times of the first upcrossing in each subperiod. See Fig. 2 to get some idea of how the reflection works. In this figure, we consider the same sample

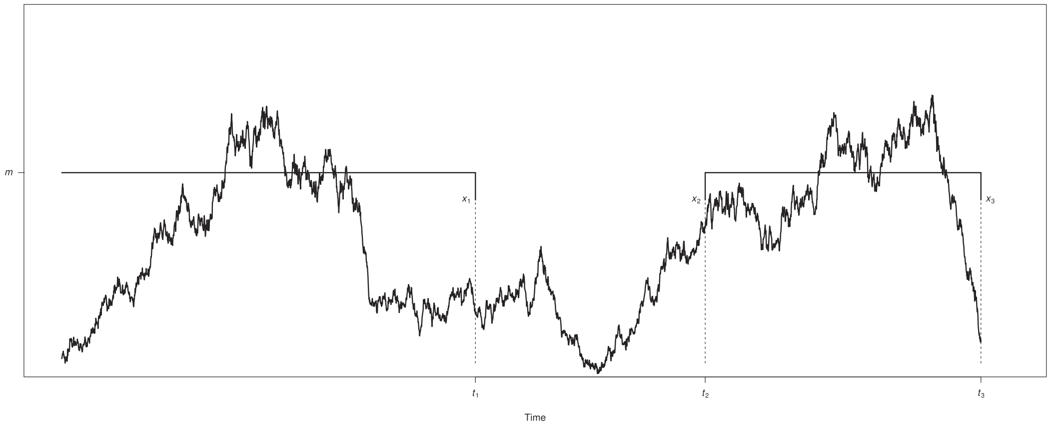


Fig. 1. A sample path of Brownian motion satisfying multiple conditions such that $M(t_0, t_1) > m$, $M(t_2, t_3) > m$, $X(t_1) \leqslant x_1$, $X(t_2) \leqslant x_2$ and $X(t_3) \leqslant x_3$ for $n = 3$.

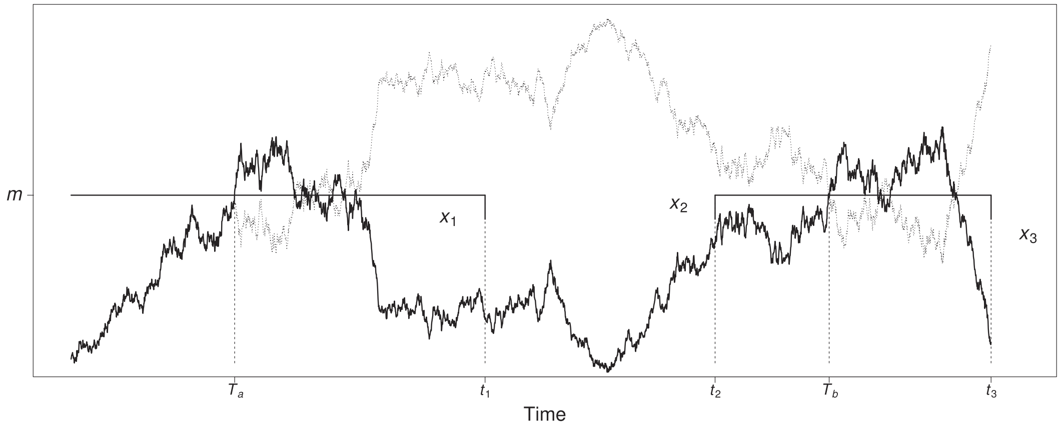


Fig. 2. Brownian motion with zero drift (solid) and its reflection (dotted) for $n = 3$.

path with Fig. 1 and its reflection at time T_a and T_b. Here T_a represents the time when the Brownian motion upcrosses the barrier m for the first time in the first subperiod, and T_b in the third subperiod. Then, in order to get rid of the partial maximums, we need to reflect the sample path once at T_a, and twice at T_b. Obviously, the part which is reflected twice after T_b equals the original, and hence we get

$$\begin{aligned}
&P(X_0(t_1) \leqslant x_1, X_0(t_2) \leqslant x_2, X_0(t_3) \leqslant x_3, M_0(t_0, t_1) > m, M_0(t_2, t_3) > m)\\
&= P(X_0(t_1) \geqslant 2m - x_1, X_0(t_2) \geqslant 2m - x_2, X_0(t_3) \leqslant x_3)\\
&= P(-X_0(t_1) + 2m \leqslant x_1, -X_0(t_2) + 2m \leqslant x_2, X_0(t_3) \leqslant x_3).
\end{aligned}$$

Now, by definition, we should have $U(t_1) = 1, U(t_2) = 1, U(t_3) = 2, R(t_1) = 1, R(t_2) = 1$ and $R(t_3) = 0$, and thus $m_1 = m, m_2 = m, m_3 = 0, a_1 = -1, a_2 = -1$ and $a_3 = 1$. Therefore, Eq. (2) holds for this particular case. By the same argument, one may expect that Eq. (2) with zero drift would hold in general.

Next, to establish the case with nonzero drift, we employ the method of Esscher transform, following the discussion paper by Wang Wang, 2016:

$$\begin{aligned}
&P\left(\bigcap_{i=1}^{n}(X(t_i) \leqslant x_i), \bigcap_{j\in J}(M(t_{j-1}, t_j) > m)\right)\\
&= E\left[e^{hX(T)-hX(T)} I\left(\bigcap_{i=1}^{n}(X(t_i) \leqslant x_i), \bigcap_{j\in J}(M(t_{j-1}, t_j) > m)\right)\right]\\
&= E[e^{-hX(T)}]E\left[e^{hX(T)} I\left(\bigcap_{i=1}^{n}(X(t_i) \leqslant x_i), \bigcap_{j\in J}(M(t_{j-1}, t_j) > m)\right); -h\right]\\
&= E[e^{-hX(T)}]E\left[e^{hX_0(T)} I\left(\bigcap_{i=1}^{n}(X_0(t_i) \leqslant x_i), \bigcap_{j\in J}(M_0(t_{j-1}, t_j) > m)\right)\right]\\
&= E[e^{-hX(T)}]E\left[e^{h(X_0(T)+2m_n)} I\left(\bigcap_{i=1}^{n}(a_i X_0(t_i) + 2m_i \leqslant x_i)\right)\right]\\
&= e^{2hm_n} E[e^{-hX(T)}]E[e^{hX_0(T)}]E\left[I\left(\bigcap_{i=1}^{n}(a_i X_0(t_i) + 2m_i \leqslant x_i)\right); h\right]\\
&= e^{2hm_n} P\left(\bigcap_{i=1}^{n}(a_i X(t_i) + 2m_i \leqslant x_i)\right).
\end{aligned}$$

Here, the second and the fifth identities follow from the definition of the Esscher transform. The third identity is based on the fact that the drift becomes zero under the Esscher measure of parameter $(-h)$, and the fourth identity uses our previous observation for the case with zero drift. Finally, the last identity can be seen from a simple algebra

$$1 = E[e^{-hX(T)}e^{hX(T)}] = E[e^{-hX(T)}]E[e^{hX(T)}; -h] = E[e^{-hX(T)}]E[e^{hX_0(T)}]. \quad \square$$

The usefulness of the theorem lies in its ability to eliminate arbitrary partial maximums in a very simple way. Note that the total number of subperiods with at least one upcrossing, m_n, determines the multiplying factor. The rest of this section illustrates the theorem in lower dimensions.

Example 1. When $n = 1$ (or $t_1 = T$), it is straightforward to recover Eq. 1 from Theorem 1 since we have $U(t_1) = 1$ and $R(t_1) = 0$, and thus $m_1 = m$ and $a_1 = 1$.

Example 2. When $n = 2$, Theorem 1 can also immediately generate the result of Lee and Ko Lee and Ko, 2018 as follows:

(i) $P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2) = e^{2h\cdot 0}P(1\cdot X(t_1) + 2\cdot 0 \leqslant x_1, 1\cdot X(t_2) + 2\cdot 0 \leqslant x_2) = \Phi_2\left(\frac{x_1 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - \mu t_2}{\sigma\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right)$.

(ii) $P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(0, t_1) > m) = e^{2hm}P(1\cdot X(t_1) + 2m \leqslant x_1, 1\cdot X(t_2) + 2m \leqslant x_2) = e^{2hm}\Phi_2\left(\frac{x_1 - 2m - \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - 2m - \mu t_2}{\sigma\sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}}\right)$.

(iii) $P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(t_1, t_2) > m) = e^{2hm}P(-1\cdot X(t_1) + 2\cdot 0 \leqslant x_1, 1\cdot X(t_2) + 2m \leqslant x_2) = e^{2hm}\Phi_2\left(\frac{x_1 + \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - 2m - \mu t_2}{\sigma\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right)$.

(iv) $P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(0, t_1) > m, M(t_1, t_2) > m,) = e^{2h\cdot 0}P(-1\cdot X(t_1) + 2m \leqslant x_1, 1\cdot X(t_2) + 2\cdot 0 \leqslant x_2) =$
$\Phi_2\left(\frac{x_1 - 2m + \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - \mu t_2}{\sigma\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right)$.

These joint probabilities can be used as a building block for finding other joint probabilities. For instance, the joint distribution of $X(t_1)$, $X(t_2)$ and $M(0, t_2)$ can be obtained as follows.

$$\begin{aligned}
&P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(0, t_2) \leqslant m)\\
&= P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(0, t_1) \leqslant m, M(t_1, t_2) \leqslant m)\\
&= P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2) - P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(0, t_1) > m)\\
&- P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(t_1, t_2) > m)\\
&+ P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, M(0, t_1) > m, M(t_1, t_2) > m).
\end{aligned}$$

□

Example 3. When $n = 3$, one may consider eight possibilities according to the inclusion of partial maximums in each subperiod. For illustration, we exhibit the following two cases:

(i) $P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, X(t_3) \leqslant x_3, M(t_1, t_2) > m) = e^{2hm}P(-1\cdot X(t_1) + 2\cdot 0 \leqslant x_1, 1\cdot X(t_2) + 2m \leqslant x_2, 1\cdot X(t_3) + 2m \leqslant x_3) =$
$e^{2hm}\Phi_3\left(\frac{x_1 + \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - 2m - \mu t_2}{\sigma\sqrt{t_2}}, \frac{x_3 - 2m - \mu t_3}{\sigma\sqrt{t_3}}; -\sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_1}{t_3}}, \sqrt{\frac{t_2}{t_3}}\right)$.

(ii) $P(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, X(t_3) \leqslant x_3, M(t_1, t_2) > m, M(t_2, t_3) > m) =$
$e^{2h\cdot 0}P(1\cdot X(t_1) + 2\cdot 0 \leqslant x_1, -1\cdot X(t_2) + 2m \leqslant x_2, 1\cdot X(t_3) + 2\cdot 0 \leqslant x_3) = \Phi_3\left(\frac{x_1 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - 2m + \mu t_2}{\sigma\sqrt{t_2}}, \frac{x_3 - \mu t_3}{\sigma\sqrt{t_3}}; -\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_1}{t_3}}, -\sqrt{\frac{t_2}{t_3}}\right)$.

4. An integrated framework for pricing barrier options

As a direct application of Theorem 1, this section develops an integrated pricing framework for various barrier options under the Black-Scholes model. It includes partial barrier options whose monitoring period is shorter than the option's lifetime, and icicled barrier options which have been introduced by Lee and Ko (2018). In Lee and Ko (2018), an early version with single icicle was proposed to enhance the yield from equity-linked investment or equity-indexed annuity. Now, due to the generalized reflection principle, explicit pricing formulas for multiple icicled barrier options can be integrated within the same unifying framework. See Fig. 3 for an illustration of icicled barriers, where 3 icicles (L_1, L_2 and L_3) are attached along with an ordinary and a partial barrier.

In a more generalized setting, one may think of eight types of barrier options according to the combinations of (up, down), (in, out) and (call, put) as usual. For instance, an up-and-out put option with icicled barrier could have its payoff at maturity T,

$$(K - S(T))_+ I(\bigcap_{i\in J^*\setminus\{n\}} (S(t_i) \leqslant L_i), \bigcap_{j\in J} (\max_{t_{j-1}\leqslant\tau\leqslant t_j} S(\tau) \leqslant B)), \tag{3}$$

where $(x)_+ = \max\{x, 0\}$, $t_0(=0) < t_1 < \cdots < t_n(=T)$, and J, $J^*(J \subseteq J^*)$ are the subsets of $\{1, 2, ..., n\}$. In the triggering event of (3), we are able to choose some subperiods to attach icicles, and to observe partial maximums. Here, subtracting n from J^* is to make provision for the strike price later. Obviously, we can always remove icicles by setting $L_i = B$ at time t_i and generate various types of barrier. For instance, the icicled barrier in Fig. 3b can be obtained from $n = 4$, $J^* = \{1, 2, 3, 4\}$ and $J = \{2, 4\}$.

For simplicity, in what follows, we denote the triggering event of (3) by A_u, i.e.,

$$A_u = \{\bigcap_{i\in J^*\setminus\{n\}} (S(t_i) \leqslant L_i), \bigcap_{j\in J} (\max_{t_{j-1}\leqslant\tau\leqslant t_j} S(\tau) \leqslant B)\},$$

and similarly, we let

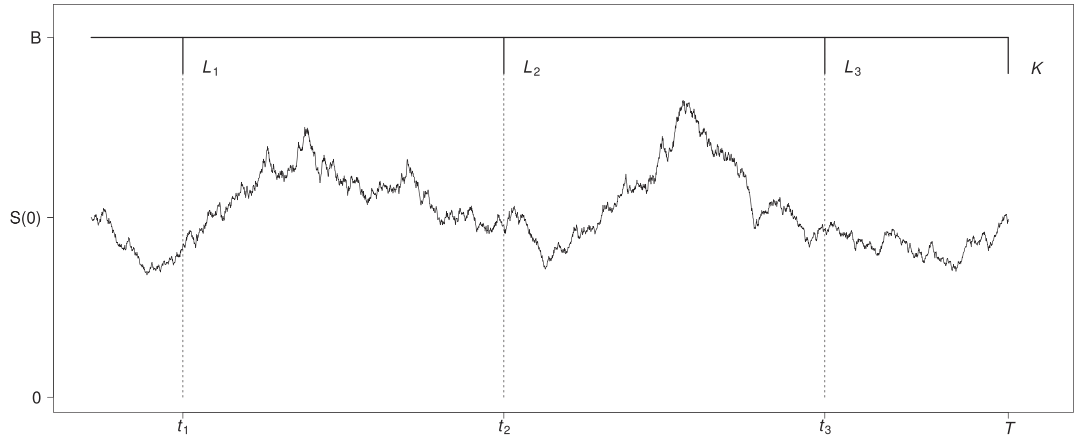
$$A_d = \{\bigcap_{i\in J^*\setminus\{n\}} (S(t_i) \geqslant L_i), \bigcap_{j\in J} (\min_{t_{j-1}\leqslant\tau\leqslant t_j} S(\tau) \geqslant B)\}.$$

Based on these events, the payoffs from the barrier options can be formulated as in Table 1.

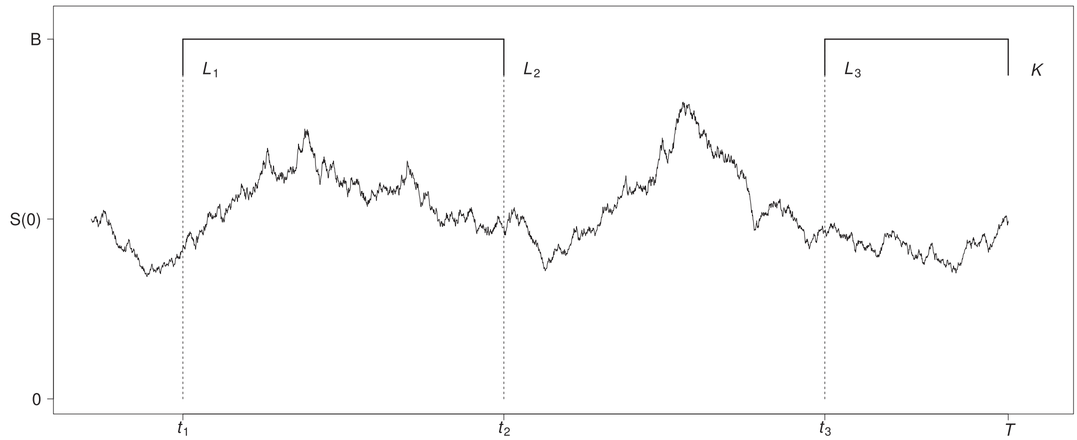
Now, using the fundamental theorem of asset pricing, we can calculate the time-0 price of the UOP option in (3) as

$$E[e^{-rT}(K - S(T))_+ I(A_u); h^*] = Ke^{-rT}P(A_u \cap (S(T) \leqslant K); h^*) - S(0)P(A_u \cap (S(T) \leqslant K); h^* + 1). \tag{4}$$

By letting



(a) Ordinary barrier with 3 icicles: $A_u$with $n=4, J^*=\{1,2,3,4\}$ and $J=\{1,2,3,4\}$



(b) Partial barrier with 3 icicles: A_u with $n=4, J^*=\{1,2,3,4\}$ and $J=\{2,4\}$

Fig. 3. Multiple icicled barriers.

Table 1
Various barrier options and their payoffs.

Option type	Payoff	Option type	Payoff
Up-and-Out Put (UOP)	$(K-S(T))_+ I(A_u)$	Up-and-In Put (UIP)	$(K-S(T))_+ I(A_u^c)$
Up-and-Out Call (UOC)	$(S(T)-K)_+ I(A_u)$	Up-and-In Call (UIC)	$(S(T)-K)_+ I(A_u^c)$
Down-and-Out Put (DOP)	$(K-S(T))_+ I(A_d)$	Down-and-In Put (DIP)	$(K-S(T))_+ I(A_d^c)$
Down-and-Out Call (DOC)	$(S(T)-K)_+ I(A_d)$	Down-and-In Call (DIC)	$(S(T)-K)_+ I(A_d^c)$

$$x_i = \ln(L_i/S(0)), \quad \text{for} \quad i=1,\,2,\,\ldots,n-1,$$

$$x_n = \ln(K/S(0)) \quad \text{and} \quad b=\ln(B/S(0)),$$

we may rewrite the first probability as

$$P(A_u \cap (S(T) \leqslant K);h^*) = P(\bigcap_{i\in J^*}(X(t_i) \leqslant x_i),\ \bigcap_{j\in J}(M(t_{j-1},\,t_j) \leqslant b);h^*),$$

which, in turn, gives rise to

$$P(\bigcap_{i\in J^*}(X(t_i)\leqslant x_i);h^*)+\sum_{k=1}^{n(J)}(-1)^k\sum_{\substack{i_1<\cdots<i_k\\ i_1,\dots,i_k\in J}}P(\bigcap_{i\in J^*}(X(t_i)\leqslant x_i),\bigcap_{j\in\{i_1,\dots,i_k\}}(M(t_{j-1},t_j)\leqslant b);h^*). \tag{5}$$

Here we used the basic inclusion-exclusion principle and $n(J)$ counts the number of elements in set J. Then Theorem 1 comes into play for each term of (5), allowing us to obtain the price in terms of the multivariate normal distribution functions. The second probability of (4) is immediately available by changing the drift from $r-\sigma^2/2$ to $r+\sigma^2/2$ in calculating (5). Once the formula is ready for the UOP option, it is trivial to get the UIP prices because the sum of the two payoffs from the UOP and the UIP equals to the standard European put payoff. Also, the UOC and the UIC options can be handled almost identically to our previous derivation.

The down barrier options can also be dealt with similarly, but need some further treatments. As before, let us first consider the DOP option. At time 0, the DOP option can be evaluated as

$$E[e^{-rT}(K-S(T))_+I(A_d);h^*]=Ke^{-rT}P(A_d\cap(S(T)\leqslant K);h^*)-S(0)P(A_d\cap(S(T)\leqslant K);h^*+1). \tag{6}$$

By letting $Y(t)=-X(t)$, $y_i=-x_i$ for $i=1,2,\dots,n$ and $b_y=-b$, the first probability of (6) can be written as

$$\begin{aligned}P(A_d\cap(S(T)\leqslant K);h^*)&=P(\bigcap_{i\in J^*\setminus\{n\}}(X(t_i)\geqslant x_i),X(t_n)\leqslant x_n,\bigcap_{j\in J}(\min_{t_{j-1}\leqslant\tau\leqslant t_j}X(\tau)\geqslant b);h^*)\\&=P(\bigcap_{i\in J^*\setminus\{n\}}(Y(t_i)\leqslant y_i),Y(t_n)\geqslant y_n,\bigcap_{j\in J}(\max_{t_{j-1}\leqslant\tau\leqslant t_j}Y(\tau)\leqslant b_y);h^*)\\&=P(\bigcap_{i\in J^*\setminus\{n\}}(Y(t_i)\leqslant y_i),\bigcap_{j\in J}(\max_{t_{j-1}\leqslant\tau\leqslant t_j}Y(\tau)\leqslant b_y);h^*)-P(\bigcap_{i\in J^*}(Y(t_i)\leqslant y_i),\bigcap_{j\in J}(\max_{t_{j-1}\leqslant\tau\leqslant t_j}Y(\tau)\leqslant b_y);h^*).\end{aligned} \tag{7}$$

This shows that the two probabilities in the right side of (7) can be calculated as in (5). Similarly, the other down barrier option prices follow from the DOP.

Now we illustrate our pricing formulas with the icicled barrier in Fig. 3a. Obviously, there are many combinations of parameter values, so we have to focus on some simple scenarios: at-the-money ($S(0)=K=100$) up barrier options; the maturity of four months ($T=4/12=t_4$); and three icicles at the end of every month before the maturity ($t_1=1/12$, $t_2=2/12$, $t_3=3/12$). For four different levels of the icicled barriers (L_1, L_2, L_3), Table 2 exhibits the option prices to give some idea of how they behave with respect to the barrier levels. First of all, the sums of the two columns in (a) and (b) (or, (c) and (d)) can serve as a simple check of the calculation, because they should be the standard BS option prices. On the other hand, to verify our formulas, we have performed Monte Carlo simulation of 10,000 iterations, and obtained approximate option prices as parenthesized in Table 2. If $(B, L_1, L_2, L_3)=(130, 130, 130, 130)$, the option prices correspond to the ordinary ones. As can be expected, they should be more valuable than the knock-out barrier options with icicles, but the opposite is true for the knock-in case. According to the option types, these icicles may have a considerable impact on their prices.

5. Pricing autocallable structured product

In this section, we derive an explicit pricing formula for the autocallable structured product under the Black-Scholes model. For those unfamiliar with the autocallable product, we begin by explaining its most popular form in the Korean equity-linked security (ELS) market, usually quoted as 90-90-85-85-80-80/45 KI. In this specification, each number represents 7 (=6 + 1) chances that the investor could receive the initial investment credited at a higher rate of return. To be more concrete, let us assume that the contract is

Table 2
Icicled barrier option prices with $r=3\%$, $S(0)=K=100$, $t_1=1/12$, $t_2=2/12$, $t_3=3/12$, $T=4/12$

σ	(B, L_1, L_2, L_3)	(a) UOP	(b) UIP	(c) UOC	(d) UIC
20%	(130, 130, 130, 130)	4.1008	0.0000	4.3572	0.7386
		(4.0450)	(0.0000)	(4.3688)	(0.7427)
	(130, 110, 110, 110)	4.0488	0.0520	1.9053	3.1905
		(3.9988)	(0.0462)	(1.9099)	(3.2017)
	(130, 120, 110, 100)	3.8018	0.2990	0.3168	4.7789
		(3.7662)	(0.2788)	(0.3352)	(4.7764)
	(130, 100, 110, 120)	3.0498	1.0509	1.0977	3.9980
		(2.9854)	(1.0596)	(1.0890)	(4.0226)
30%	(130, 130, 130, 130)	6.3751	0.0057	3.6256	3.7503
		(6.2913)	(0.0053)	(3.7347)	(3.6614)
	(130, 110, 110, 110)	6.0660	0.3148	1.4762	5.8996
		(5.9913)	(0.3053)	(1.5330)	(5.8632)
	(130, 120, 110, 100)	5.8717	0.5091	0.4450	6.9308
		(5.8055)	(0.4911)	(0.4696)	(6.9266)
	(130, 100, 110, 120)	4.7406	1.6403	1.1184	6.2574
		(4.6515)	(1.6451)	(1.1074)	(6.2888)

Notes: The exact values based on our pricing formulas are compared with the approximate ones in parenthesis using Monte Carlo simulation with 10,000 iterations.

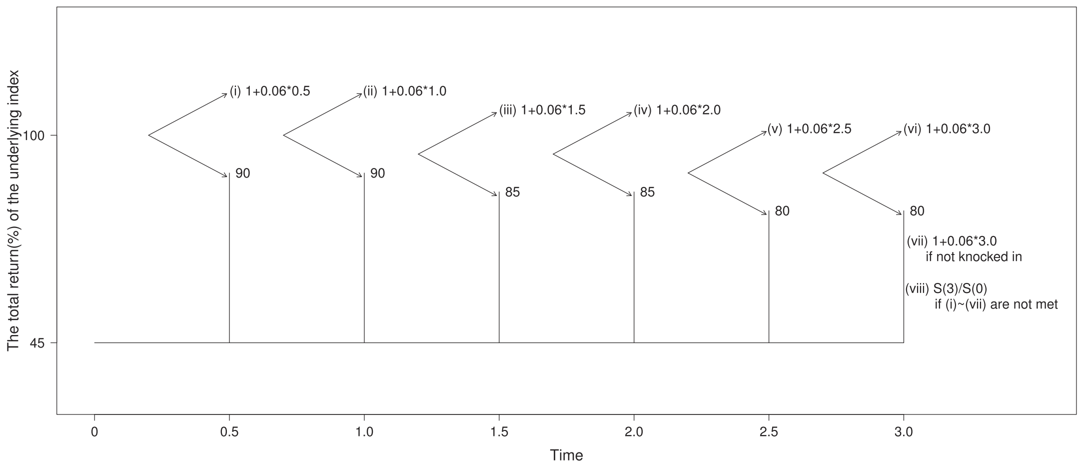


Fig. 4. A popular autocallable payoff structure with 90-90-85-85-80-80/45 KI,3 years of maturity, autocallabe at every 6 months and annual coupon rate of 6%.

based on a certain equity index, autocallable at every 6 months with maturity of 3 years, and offers an annualized coupon rate of 6%. The underlying index would be observed at the end of the first 6 months after the issuance to see whether it is above or below the first autocall threshold, i.e., 90% of the initial level. If above, the principal plus the interest earned at 3% (=6% × 6months) would be redeemed and the contract closed out. Otherwise, it proceeds to the next 6 months and the analogous redemption process would be repeated until the maturity. Finally, even if the underlying indices at all 6 observation dates are below the autocall thresholds, there would be one more chance given by the last knock-in (KI) barrier of 45. This works as follows. If the lowest index value during the entire period of 3 years does not fall below 45% of its initial level, the principal plus the interest earned at 18% (=6% × 3years) would be redeemed at the maturity. If all these are not met, the autocallable would redeem at the maturity the initial investment at the rate of return actually experienced by the underlying index during the period. Thus, the investors could be faced with a loss ranging from −20% to −100% according to the index value at the maturity. See Fig. 4 for a graphical representation of the payoff structure and Kim and Lim (2019) for more details.

There are several reasons behind the popularity. Above all, they provide an attractive coupon rate under the low-interest-rate environment. As shall be seen later, the theoretical breakeven rates can be much higher than the risk-free rate. Also, the investors' market prospect or risk aversion can be easily reflected in the product design. For instance, during the past decade, the Korean equity market has exhibited a box pattern, i.e., an index movement seemingly trapped within a limited range. Capturing their needs and worries, the product could have been quite successful so far. On the other hand, the maturity of the autocallable represents a maximum duration it can stay alive. The actual duration could be much shorter than the maturity, so the investors may opt to reinvest once they get redeemed early.

However, despite the popularity, one may find only a few academic studies on this topic. One reason might be that there exist a variety of payoff structures as much as the popularity. And, above all, the payoff structure with the knock-in feature is very complicated. As a result, the explicit pricing formula is not completely known in the literature, and numerical methods based on simulation or PDE are being discussed in most cases. We refer the readers to Guillaume and France (2015) and Lee and Ko (2018) for some early partial solutions to this problem. The readers are also referred to Bouzoubaa and Osseiran (2010), Deng, Mallett, and McCann (2011), Alm, Harrach, Harrach, and Keller (2013), Albuquerque, Gaspar, and Michel (2015), Kim and Lim (2019) and papers cited therein.

In this section, using the generalized reflection principle, we find the explicit pricing formula for the autocallable product with its payoff defined as (8).

$$
\begin{array}{lll}
1 + c_1 & \text{at time } t_1 & \text{if } S(t_1) \geqslant L_1, \\
1 + c_2 & \text{at time } t_2 & \text{if } S(t_1) < L_1,\, S(t_2) \geqslant L_2, \\
1 + c_3 & \text{at time } t_3 & \text{if } \bigcap_{i=1}^{2} (S(t_i) < L_i),\, S(t_3) \geqslant L_3, \\
 & \cdots & \\
1 + c_6 & \text{at time } T & \text{if } \bigcap_{i=1}^{5} (S(t_i) < L_i),\, S(t_6) \geqslant L_6, \\
1 + c_7 & \text{at time } T & \text{if } \bigcap_{i=1}^{6} (S(t_i) < L_i),\, \min_{0\leqslant\tau\leqslant T S(\tau)\geqslant B}, \\
e^{X(T)} & \text{at time } T & \text{if } \bigcap_{i=1}^{6} (S(t_i) < L_i),\, \min_{0\leqslant\tau\leqslant T S(\tau)< B}.
\end{array}
\tag{8}
$$

The payoff above assumes a unit investment at time 0, a KI barrier of B and 6 autocall dates $(t_1, \ldots, t_6 = T)$. It allows for the possibility that the coupon rates, $c_1, \cdots, c_7$, may not be proportional to the time until redemption. The seventh and the eighth payoffs are related to the KI event, where a loss might occur by crediting the realized rate of return during the period. Similarly as before, we let $x_i = \ln(L_i/S(0))$, $i = 1, 2, \ldots, 6$, and $b = \ln(B/S(0))$. Then, by the fundamental theorem of asset pricing, the time-0 price of the autocallable product can be evaluated as

$$e^{-rt_1}(1+c_1)P(X(t_1)\geqslant x_1;h^*) + \sum_{i=2}^{6} e^{-rt_i}(1+c_i)P\left(\bigcap_{j=1}^{i-1}(X(t_j)<x_j), X(t_i)\geqslant x_i;h^*\right) + e^{-rT}(1+c_7)P \left(\bigcap_{i=1}^{6}(X(t_i)<x_i), \min_{0\leqslant\tau\leqslant T} X(\tau)\geqslant b;h^*\right) + e^{-rT}E\left[e^{X(T)}I\left(\bigcap_{i=1}^{6}(X(t_i)<x_i), \min_{0\leqslant\tau\leqslant T} X(\tau)<b\right);h^*\right]. \tag{9}$$

The first six terms of (9) do not include the minimum, so they can be easily written in terms of the multivariate normal distributions. In order to deal with the seventh and eighth terms, we let $Y(t) = -X(t)$, $y_i = -x_i$ for $i = 1, 2, \ldots, 6$ and $b_y = -b$, and rewrite the probability of the seventh term of (9) as

$$\begin{aligned} P\left(\bigcap_{i=1}^{6}(X(t_i)<x_i), \min_{0\leqslant\tau\leqslant T} X(\tau)\geqslant b;h^*\right) &= P\left(\bigcap_{i=1}^{6}(Y(t_i)>y_i), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*\right)\\ &= P\left(\left(\bigcup_{i=1}^{6}(Y(t_i)<y_i)\right)^c, \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*\right)\\ &= P(\max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*) - P\left(\bigcup_{i=1}^{6}(Y(t_i)<y_i), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*\right). \end{aligned} \tag{10}$$

Using the reflection principle (1), we may find the marginal distribution of $\max_{0\leqslant\tau\leqslant T}Y(\tau)$ as

$$P(\max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*) = \Phi\left(\frac{b_y + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right) - e^{-(2r/\sigma^2-1)b_y}\Phi\left(\frac{-b_y + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Also, the second term of the right side of (10) can be obtained as

$$\begin{aligned} &P\left(\bigcup_{i=1}^{6}(Y(t_i)<y_i), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*\right)\\ &\quad= \sum_{i=1}^{6} P((Y(t_i)<y_i), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*) - \sum_{j_1<j_2} P(\bigcap_{j\in\{j_1,j_2\}}(Y(t_j)<y_j), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*) +\\ &\quad\sum_{j_1<j_2<j_3} P(\bigcap_{j\in\{j_1,j_2,j_3\}}(Y(t_j)<y_j), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*)\cdots - P\left(\bigcap_{j=1}^{6}(Y(t_j)<y_j), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*\right), \end{aligned} \tag{11}$$

and each summand of (11) can be handled similarly as before. For instance, for any given subset J^* of $\{1, 2, \ldots, 6\}$, we have

$$\begin{aligned} P(\bigcap_{j\in J^*}(Y(t_j)<y_j), \max_{0\leqslant\tau\leqslant T} Y(\tau)\leqslant b_y;h^*) &= P(\bigcap_{j\in J^*}(Y(t_j)<y_j);h^*) - P\left(\bigcap_{j\in J^*}(Y(t_j)<y_j), \bigcup_{i=1}^{6}(\max_{t_{i-1}\leqslant\tau\leqslant t_i} Y(\tau)>b_y);h^*\right)\\ &= P(\bigcap_{j\in J^*}(Y(t_j)<y_j);h^*) - \sum_{i=1}^{6} P(\bigcap_{j\in J^*}(Y(t_j)<y_j), \max_{t_{i-1}\leqslant\tau\leqslant t_i} Y(\tau)>b_y;h^*) + \cdots\\ &\quad+P\left(\bigcap_{j\in J^*}(Y(t_j)<y_j), \bigcap_{i=1}^{6}(\max_{t_{i-1}\leqslant\tau\leqslant t_i} Y(\tau)>b_y);h^*\right). \end{aligned}$$

Now we can apply Theorem 1 in order to express the above in terms of the multivariate normal distributions. If the coupon rates, $c_1, \ldots, c_7$, are proportional to the time until redemption, then we can solve for the breakeven coupon rate by equating (9) with the unit investment at inception.

In Tables 3 and 4, we summarize our numerical results based on the explicit pricing formula. In these tables, we assume an autocallable product with an initial investment of 100, 3 years of maturity, autocallable at every 6 months and 90-90-85-85-80-80/45

Table 3
The time-0 prices of the autocallable product with 90-90-85-85-80-80/45 KI at different levels of interest rate, volatility and coupon rate.

r	3%			4%			5%		
$\tilde{c}\backslash\sigma$	20%	25%	30%	20%	25%	30%	20%	25%	30%
5.0%	**99.365**	97.324	95.076	**99.024**	96.953	94.684	98.691	96.595	94.310
6.5%	**100.591**	98.526	96.208	**100.262**	98.156	95.813	**99.936**	97.798	95.436
8.0%	101.818	99.727	97.341	101.498	99.358	96.943	**101.181**	99.000	96.561

Table 4
The breakeven coupon rates of the autocallable product with 90-90-85-85-80-80/45 KI at different levels of interest rate and volatility

$r \backslash \sigma$	20%	25%	30%
3%	5.8%	8.3%	11.5%
4% 6.2%	8.8%	12.1%	
5%	6.6%	9.2%	12.6%

KI. Also, the coupon rates are set proportional to the time until redemption, i.e., $c_1 = \widetilde{c}/2,\ c_2 = \widetilde{c},\ \cdots, c_6 = 3\widetilde{c}$ and $c_7 = 3\widetilde{c}$ for some coupon rate $\widetilde{c}$. Table 3 contains the time-0 prices of the autocallable product evaluated at different levels of interest rate r, volatility σ and coupon rate c. As can be expected, the prices are decreasing in r and σ, but increasing in $\widetilde{c}$. In the table, bold-faced numbers represent the endpoints of the intervals to which the breakeven coupon rates should belong. For instance, if $(r, \sigma, \widetilde{c}) = (3\%, 20\%, 5\%)$, the price is 99.3649, so the investor is paying more than its theoretical value. Likewise, for $(r, \sigma, \widetilde{c}) = (3\%, 20\%, 6.5\%)$, we have the price greater than 100, and thus the breakeven rate for $(r, \sigma) = (3\%, 20\%)$ would lie between 5% and 6.5%. In Table 4, we present the breakeven rates for the autocallable product. Expectedly, the breakeven rates are increasing in both r and σ. At $(r, \sigma) = (3\%, 20\%)$, the breakeven rate turns out to be 5.8%, almost twice of the interest rate. In practice, it is usual that the minimum of multiple assets or indices are considered as the underlying. If so, the breakeven rate would be higher than our current level.

6. Conclusion

In this paper, we generalized the well-known reflection principle of the Brownian motion. For practical applications, we exhibited how it could be applied to computational finance. More specifically, using the basic inclusion-exclusion principle, we could find an integrated pricing framework for various barrier options and an explicit pricing formula for the autocallable product with KI feature. These formulas are new and interesting in themselves, but we would like to put more emphasis on the generalized reflection principle. As shown in the pricing examples, it eliminates the annoying partial maximums (or minimums) very efficiently, and thus the associated joint events could be handled by the multivariate normal distributions only. Following the method presented in the paper, we expect that many probabilistic problems related to the original reflection principle be similarly generalized.

Needless to say, explicit option pricing formulas have definite advantages over numerical solutions. However, our pricing formulas might be devaluated due to the complexity or the fast-growing number of terms. Moreover, they inherit the shortcomings from the BS model, and other limitations exist in incorporating risk factors such as multiple assets or currency. But this requires extending the reflection principle (1) in the multidimensional Brownian motions, which has not been successful to the best of our knowledge. Nevertheless, our formulas could be used as a starting point to find the fair prices or manage the relevant risks. In particular, the explicit expression for the breakeven coupon rates of the autocallable would reduce the computational load substantially.

Appendix A. Proof of Theorem 1

The appendix uses the usual vector notation as follows. Bold letter may represent vector or matrix, and the prime notation on its superscript the transpose operator. For a given vector $\mathbf{v}$, the subvectors with the first k elements and the k-th element are denoted by $[\mathbf{v}]_{1:k}$ and $[\mathbf{v}]_k$, respectively. Similarly, for a given matrix $\mathbf{v}$, $[\mathbf{v}]_{ij}$ represents its (i, j)-th element. Also, we denote by $\mathbf{e}_k$ the k-th unit vector of dimension k, that is, k-dimensional column vector with the k-th element being 1 and the others 0. To write the joint event of Theorem 1 in a simpler form, we introduce a set $\mathscr{I}^n = \{(i_1, i_2, \ldots, i_n): i_k = 0 \text{or} 1 \text{for} k = 1, 2, \ldots, n\}$. And, for $k = 1, 2, \ldots$, we let

$$B_k^{(0)}(x) = (X(t_k) \leqslant x) \text{and} B_k^{(1)}(x) = (X(t_k) \leqslant x,\ M(t_{k-1}, t_k) > m).$$

Here the superscript (0) or (1) specifies whether the upcrossings in the particular subinterval are taken into account or not. For instance, if $n = 2$, the four events of Example 2 can be written as $B_1^{(0)}(x_1) \cap B_2^{(0)}(x_2)$, $B_1^{(1)}(x_1) \cap B_2^{(0)}(x_2)$, $B_1^{(0)}(x_1) \cap B_2^{(1)}(x_2)$ and $B_1^{(1)}(x_1) \cap B_2^{(1)}(x_2)$, respectively. Moreover, by using $\mathbf{i}_n \in \mathscr{I}^n$ and $\mathbf{x}_n = (x_1, x_2, \ldots, x_n)$, these sets can be further simplified to $B^{\mathbf{i}_n}(\mathbf{x}_n) = \cap_{k=1}^n B_k^{(i_k)}(x_k)$.

For a given $\mathbf{i}_n \in \mathscr{I}^n$, we define the cumulative sum of i_k's as $\upsilon_k^n = \sum_{j=1}^k i_j$. Since υ_k^n counts the number of subintervals with at least one upcrossing up to time t_k, it should correspond to $U(t_k)$ if n is fixed. However, the appendix uses the induction method on n, and hence the notation becomes more complicated and different from that of the main body. We define $\boldsymbol{\delta}_n = (\delta_1^n, \ldots, \delta_n^n)$, where

$$\delta_k^n = \mathrm{mod}(\upsilon_k^n, 2) \text{for} k = 1, 2, \ldots, n.$$

We then have $m_k = m\delta_k^n$. Using $\boldsymbol{\delta}_n$, we also define the mean vector and the variance-covariance matrix as follows:

(i) $\boldsymbol{\mu}^{\mathbf{i}_n} = (\mu_k^{\mathbf{i}_n}; k = 1, 2, \ldots, n) \text{where} \mu_k^{\mathbf{i}_n} = 2m\delta_k^n + (-1)^{\mathrm{mod}(\upsilon_n^n - \upsilon_k^n, 2)}\mu t_k$,
(ii) $\boldsymbol{\Sigma}^{\mathbf{i}_n} = (\Sigma_{j,k}^{\mathbf{i}_n}; j, k = 1, 2, \ldots, n) \text{where} \Sigma_{j,k}^{\mathbf{i}_n} = (-1)^{\mathrm{mod}(\upsilon_{j \vee k}^n - \upsilon_{j \wedge k}^n, 2)}(t_j \wedge t_k)\sigma^2$,

where $a \vee b$ and $a \wedge b$ represent the maximum and the minimum between a and b, respectively. In what follows, the elementwise superscripts in $\mu_k^{\mathbf{i}_n}$ and $\Sigma_{j,k}^{\mathbf{i}_n}$ would be dropped out if there is no confusion. For instance, when n is fixed, $\mu_n^{\mathbf{i}_n} = 2m\delta_n^n + \mu t_n := \mu_n$ and

$\Sigma^{\mathbf{i}_n}_{n,n} = \sigma^2 t_n := \Sigma_{n,n}$. In the exponent of definition (i), $\upsilon^n_n - \upsilon^n_k$ counts the number of subintervals with upcrossings after time t_k, thereby, matching with $R(t_k)$ of Theorem 1. Similarly, $a_k = (-1)^{\mathrm{mod}(\upsilon^n_n - \upsilon^n_k, 2)}$ for $k = 1, 2, \ldots, n$. Now we are ready to rephrase Theorem 1 using these symbols.

Theorem 2. *(identical to Theorem 1) For $n \geqslant 1$ and $\mathbf{i}_n \in \mathscr{I}^n$, it holds that*

$$P(B^{\mathbf{i}_n}(\mathbf{x}_n)) = e^{2hm\delta^n_n} F_n(\mathbf{x}_n; \boldsymbol{\mu}^{\mathbf{i}_n}, \boldsymbol{\Sigma}^{\mathbf{i}_n}), \tag{12}$$

where $h = \mu/\sigma^2$, and $F_n(\mathbf{x};\boldsymbol{\mu}, \boldsymbol{\Sigma})$ denotes the n-dimensional normal distribution function with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$.

One may see the equivalence between the two theorems by noting the relationship between the constants. First, by letting $Y_k = a_k X(t_k) + 2m_k$ in Theorem 1, we see that $E[Y_k] = a_k E[X(t_k)] + 2m_k = \mu_k$ and $Cov(Y_j, Y_k) = a_j a_k (t_j \wedge t_k)\sigma^2$. Since the product between a_j and a_k becomes 1 or -1 depending on the number of subintervals with upcrossings between the times t_j and t_k, it is equal to $(-1)^{\mathrm{mod}(\upsilon^n_{j\vee k} - \upsilon^n_{j\wedge k}, 2)}$. Before we prove Theorem 2, it would be helpful to consider the following lemma.

Lemma 1. *Assume that $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2)$ is normally distributed with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, where the dimensions of $\mathbf{X}_1$ and $\mathbf{X}_2$ are m and n, respectively. The following is the partitioned form of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in accordance with $\mathbf{X}$:*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \text{and} \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

Then we have the followings.

(a) *$\mathbf{X}_1$ is normally distributed with mean vector $\boldsymbol{\mu}_1$ and variance-covariance matrix $\boldsymbol{\Sigma}_{11}$. Furthermore, the conditional distribution of $\mathbf{X}_2$ given $\mathbf{X}_1$ is normal with mean $\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{X}_1 - \boldsymbol{\mu}_1)$ and variance-covariance matrix $\boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$. That is,*

$$P(\mathbf{X}_1 \leqslant \mathbf{x}_1, \mathbf{X}_2 \leqslant \mathbf{x}_2) = \int_{-\infty}^{\mathbf{x}_1} P(\mathbf{X}_2 \leqslant \mathbf{x}_2 | \mathbf{X}_1 = \mathbf{z}_1) dF_m(\mathbf{z}_1; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$$
$$= \int_{-\infty}^{\mathbf{x}_1} F_n(\mathbf{x}_2; \boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}(\mathbf{z}_1 - \boldsymbol{\mu}_1), \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}) dF_m(\mathbf{z}_1; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}).$$

(b) *For n-dimensional normal density function $f_n(\cdot; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{x}_n = (x_1, \ldots, x_n)'$,*

$$e^{-\frac{2\mu x_n}{\sigma^2}} f_n(\mathbf{x}_n; \boldsymbol{\mu}^{\mathbf{i}_n}, \boldsymbol{\Sigma}^{\mathbf{i}_n}) = e^{-\frac{4\mu m\delta^n_n}{\sigma^2}} f_n(\mathbf{x}_n; \boldsymbol{\mu}^*_n, \boldsymbol{\Sigma}^{\mathbf{i}_n}),$$

where the k-th element of $\boldsymbol{\mu}^*_n$ is defined as

$$[\boldsymbol{\mu}^*_n]_k = 2m\delta^n_k - (-1)^{\mathrm{mod}(\upsilon^n_n - \upsilon^n_k, 2)} \mu t_k \text{ for } k = 1, 2, \ldots, n,$$

which implies that $\boldsymbol{\mu}^*_n = \boldsymbol{\mu}^{\mathbf{i}_n} - \frac{2\mu}{\sigma^2} \boldsymbol{\Sigma}^{\mathbf{i}_n} \mathbf{e}_n$.

Proof.

(a) It is a well-known result on the multivariate normal distributions and the proof can be found in the literature, e.g., Tong (2012).

(b) In accordance with $\mathbf{i}_n = (\mathbf{i}_{n-1}, i_n)$, we partition the mean and the variance-covariance matrix as

$$\boldsymbol{\mu}^{\mathbf{i}_n} = \begin{pmatrix} \boldsymbol{\mu}_{n-1} \\ \mu_n \end{pmatrix}, \text{ and } \boldsymbol{\Sigma}^{\mathbf{i}_n} = \begin{pmatrix} \boldsymbol{\Sigma}_{n-1,n-1} & \boldsymbol{\Sigma}_{n-1,n} \\ \boldsymbol{\Sigma}_{n,n-1} & \Sigma_{n,n} \end{pmatrix}.$$

Observe that, for $k = 1, 2, \ldots, n$,

$$[\boldsymbol{\mu}^*_n]_k = \mu_k - \frac{2\mu}{\sigma^2} [\boldsymbol{\Sigma}^{\mathbf{i}_n}]_{k,n} = \left[\boldsymbol{\mu}^{\mathbf{i}_n} - \frac{2\mu}{\sigma^2} \boldsymbol{\Sigma}^{\mathbf{i}_n} \mathbf{e}_n \right]_k.$$

In particular, since $\Sigma_{n,n} = \sigma^2 t_n$, we have $\mu^*_n := [\boldsymbol{\mu}^*_n]_n = \mu_n - 2\mu t_n = 2m\delta^n_n - \mu t_n$ and $\boldsymbol{\mu}^*_{n-1} := [\boldsymbol{\mu}^*_n]_{1:n-1} = \boldsymbol{\mu}_{n-1} - (2\mu/\sigma^2)\boldsymbol{\Sigma}_{n-1,n}$. Let us temporarily write $\boldsymbol{\Sigma}_{n-1|n} := \boldsymbol{\Sigma}_{n-1,n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}\boldsymbol{\Sigma}_{n,n-1}$. Then it follows from part (a) that, for $\mathbf{x}_n = (\mathbf{x}'_{n-1} x_n)'$,

$$\begin{aligned} e^{-\frac{2\mu x_n}{\sigma^2}} |2\pi\boldsymbol{\Sigma}_{n-1|n}|^{1/2} \sqrt{2\pi\sigma^2 t_n} f_n(\mathbf{x}_n; \boldsymbol{\mu}^{\mathbf{i}_n}, \boldsymbol{\Sigma}^{\mathbf{i}_n}) &= e^{-\frac{1}{2}(\mathbf{x}_{n-1} - \boldsymbol{\mu}_{n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}(x_n - \mu_n))' \boldsymbol{\Sigma}^{-1}_{n-1|n} (\mathbf{x}_{n-1} - \boldsymbol{\mu}_{n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}(x_n - \mu_n))'} e^{-\frac{(x_n - \mu_n)^2}{2\sigma^2 t_n}} e^{-\frac{2\mu x_n}{\sigma^2}} \\ &= e^{-\frac{1}{2}(\mathbf{x}_{n-1} - \boldsymbol{\mu}_{n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}(x_n - \mu_n))' \boldsymbol{\Sigma}^{-1}_{n-1|n} (\mathbf{x}_{n-1} - \boldsymbol{\mu}_{n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}(x_n - \mu_n))'} e^{-\frac{(x_n - \mu^*_n)^2}{2\sigma^2 t_n}} e^{-\frac{4\mu m\delta^n_n}{\sigma^2}} \\ &= e^{-\frac{1}{2}(\mathbf{x}_{n-1} - \boldsymbol{\mu}^*_{n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}(x_n - \mu^*_n))' \boldsymbol{\Sigma}^{-1}_{n-1|n} (\mathbf{x}_{n-1} - \boldsymbol{\mu}^*_{n-1} - \boldsymbol{\Sigma}_{n-1,n}\Sigma^{-1}_{n,n}(x_n - \mu^*_n))'} e^{-\frac{(x_n - \mu^*_n)^2}{2\sigma^2 t_n}} e^{-\frac{4\mu m\delta^n_n}{\sigma^2}} \\ &= |2\pi\boldsymbol{\Sigma}_{n-1|n}|^{1/2} \sqrt{2\pi\sigma^2 t_n} f_n(\mathbf{x}_n; \boldsymbol{\mu}^*_n, \boldsymbol{\Sigma}^{\mathbf{i}_n}) e^{-\frac{4\mu m\delta^n_n}{\sigma^2}}, \end{aligned}$$

which completes the proof of Lemma 1.

Proof (*Proof of Theorem 2***).** We prove by induction on n. Eq. (1) implies that (12) holds for $n = 1$. Let us assume that (12) holds for

$n > 1$. To prove that it also holds for $n+1$ and $\mathbf{i}_{n+1} = (i_1, \ldots, i_n, i_{n+1}) = (\mathbf{i}_n, i_{n+1})$, we consider two possible cases depending on the value of i_{n+1}, that is, whether $i_{n+1} = 0$ or 1. By definition, we have $v_k^{n+1} = v_k^n$, and thus $\delta_k^{n+1} = \delta_k^n$ for $k \leqslant n$, irrespective of the value of i_{n+1}.Let us first consider where $i_{n+1} = 0$. In this case, we should have $v_{n+1}^{n+1} = v_n^n$ and $\delta_{n+1}^{n+1} = \delta_n^n$, which yields that

$$\boldsymbol{\mu}^{\mathbf{i}_{n+1}} = \begin{bmatrix} \boldsymbol{\mu}^{\mathbf{i}_n} \\ 2m\delta_n^n + \mu t_{n+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{\mu}_n \\ \mu_{n+1} \end{bmatrix},$$

and

$$\boldsymbol{\Sigma}^{\mathbf{i}_{n+1}} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathbf{i}_n} & \boldsymbol{\Sigma}^{\mathbf{i}_n}\mathbf{e}_n \\ \mathbf{e}_n'\boldsymbol{\Sigma}^{\mathbf{i}_n} & \sigma^2 t_{n+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{\Sigma}_{n,n} & \boldsymbol{\Sigma}_{n,n+1} \\ \boldsymbol{\Sigma}_{n+1,n} & \Sigma_{n+1,n+1} \end{bmatrix}.$$

Since $B^{\mathbf{i}_{n+1}}(\mathbf{x}_{n+1}) = B^{\mathbf{i}_n}(\mathbf{x}_n) \cap (X(t_{n+1}) \leqslant x_{n+1})$, it holds that

$$\begin{aligned} P(B^{\mathbf{i}_{n+1}}(\mathbf{x}_{n+1})) &= \int_{-\infty}^{\mathbf{x}_n} P(X(t_{n+1}) < x_{n+1} | X(t_n) = z_n) dP(B^{\mathbf{i}_n}(\mathbf{z}_n)) \\ &= \int_{-\infty}^{\mathbf{x}_n} F_1(x_{n+1}; z_n + \mu(t_{n+1} - t_n), \sigma^2(t_{n+1} - t_n)) de^{\frac{2\mu m\delta_n^n}{\sigma^2}} F_n(\mathbf{z}_n; \boldsymbol{\mu}^{\mathbf{i}_n}, \boldsymbol{\Sigma}^{\mathbf{i}_n}) \\ &= e^{\frac{2\mu m\delta_{n+1}^{n+1}}{\sigma^2}} \int_{-\infty}^{\mathbf{x}_n} F_1(x_{n+1}; z_n + \mu(t_{n+1} - t_n), \sigma^2(t_{n+1} - t_n)) dF_n(\mathbf{z}_n; \boldsymbol{\mu}^{\mathbf{i}_n}, \boldsymbol{\Sigma}^{\mathbf{i}_n}). \end{aligned}$$

Therefore, by part (a) of Lemma 1, it is enough to show that

$$z_n + \mu(t_{n+1} - t_n) = \mu_{n+1} + \boldsymbol{\Sigma}_{n+1,n}\boldsymbol{\Sigma}_{n,n}^{-1}(\mathbf{z}_n - \boldsymbol{\mu}_n) \tag{13}$$

and

$$\sigma^2(t_{n+1} - t_n) = \Sigma_{n+1,n+1} - \boldsymbol{\Sigma}_{n+1,n}\boldsymbol{\Sigma}_{n,n}^{-1}\boldsymbol{\Sigma}_{n,n+1}. \tag{14}$$

Concerning (13), we have

$$\begin{aligned} \mu_{n+1} + \boldsymbol{\Sigma}_{n+1,n}\boldsymbol{\Sigma}_{n,n}^{-1}(\mathbf{z}_n - \boldsymbol{\mu}_n) &= \mu_{n+1} + \mathbf{e}_n'(\mathbf{z}_n - \boldsymbol{\mu}_n) \\ &= 2m\delta_n^n + \mu t_{n+1} + z_n - (2m\delta_n^n + \mu t_n) \\ &= z_n + \mu(t_{n+1} - t_n). \end{aligned}$$

And, for (14), we see that

$$\begin{aligned} \Sigma_{n+1,n+1} - \boldsymbol{\Sigma}_{n+1,n}\boldsymbol{\Sigma}_{n,n}^{-1}\boldsymbol{\Sigma}_{n,n+1} &= \sigma^2 t_{n+1} - \mathbf{e}_n'\boldsymbol{\Sigma}_{n,n+1} \\ &= \sigma^2 t_{n+1} - \mathbf{e}_n'\boldsymbol{\Sigma}^{\mathbf{i}_n}\mathbf{e}_n = \sigma^2(t_{n+1} - t_n). \end{aligned}$$

Now we consider where $i_{n+1} = 1$. In this case, we have $v_{n+1}^{n+1} = v_n^n + 1$ and $\delta_{n+1}^{n+1} = \mathrm{mod}(\delta_n^n + 1, 2) = 1 - \delta_n^n$. Then, for $k = 1, 2, \ldots, n$,

$$\begin{aligned} \mu_k^{\mathbf{i}_{n+1}} &= 2m\delta_k^{n+1} + (-1)^{\mathrm{mod}(v_{n+1}^{n+1} - v_k^{n+1}, 2)}\mu t_k \\ &= 2m\delta_k^{n+1} - (-1)^{\mathrm{mod}(v_n^n - v_k^n, 2)}\mu t_k \\ &= [\boldsymbol{\mu}^{\mathbf{i}_n}]_k - \tfrac{2\mu}{\sigma^2}[\boldsymbol{\Sigma}^{\mathbf{i}_n}]_{k,n} \end{aligned}$$

and $\mu_{n+1}^{\mathbf{i}_{n+1}} = 2m \cdot \mathrm{mod}(\delta_n^n + 1, 2) + \mu t_{n+1}$. Therefore, we obtain

$$\boldsymbol{\mu}^{\mathbf{i}_{n+1}} = \begin{bmatrix} \boldsymbol{\mu}^{\mathbf{i}_n} - \frac{2\mu}{\sigma^2}\boldsymbol{\Sigma}^{\mathbf{i}_n}\mathbf{e}_n \\ 2m \cdot \mathrm{mod}(\delta_n^n + 1, 2) + \mu t_{n+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{\mu}_n^* \\ \mu_{n+1} \end{bmatrix},$$

and, by definition,

$$\boldsymbol{\Sigma}^{\mathbf{i}_{n+1}} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathbf{i}_n} & -\boldsymbol{\Sigma}^{\mathbf{i}_n}\mathbf{e}_n \\ -\mathbf{e}_n'\boldsymbol{\Sigma}^{\mathbf{i}_n} & \sigma^2 t_{n+1} \end{bmatrix} := \begin{bmatrix} \boldsymbol{\Sigma}_{n,n} & \boldsymbol{\Sigma}_{n,n+1} \\ \boldsymbol{\Sigma}_{n+1,n} & \Sigma_{n+1,n+1} \end{bmatrix}.$$

Since $B^{\mathbf{i}_{n+1}}(\mathbf{x}_{n+1}) = B^{\mathbf{i}_n}(\mathbf{x}_n) \cap (X(t_{n+1}) \leqslant x_{n+1}, M(t_n, t_{n+1}) > m)$, we get

$$\begin{aligned} P(B^{\mathbf{i}_{n+1}}(\mathbf{x}_{n+1})) &= \int_{-\infty}^{\mathbf{x}_n} P(X(t_{n+1}) - X(t_n) \leqslant x_{n+1} - X(t_n), \max_{t_n \leqslant u \leqslant t_{n+1}} \{X(u) - X(t_n)\} > m - X(t_n) | X(t_n) = z_n) \times dP(B^{\mathbf{i}_{n+1}}(\mathbf{z}_n)) \\ &= \int_{-\infty}^{\mathbf{x}_n} e^{\frac{2\mu(m - z_n)}{\sigma^2}} F_1(x_{n+1}; -z_n + 2m + \mu(t_{n+1} - t_n), \sigma^2(t_{n+1} - t_n)) de^{\frac{2\mu m\delta_n^n}{\sigma^2}} F_n(\mathbf{z}_n; \boldsymbol{\mu}^{\mathbf{i}_n}, \boldsymbol{\Sigma}^{\mathbf{i}_n}) \\ &= e^{\frac{2\mu m\delta_{n+1}^{n+1}}{\sigma^2}} \int_{-\infty}^{\mathbf{x}_n} F_1(x_{n+1}; -z_n + 2m + \mu(t_{n+1} - t_n), \sigma^2(t_{n+1} - t_n)) dF_n(\mathbf{z}_n; \boldsymbol{\mu}_n^*, \boldsymbol{\Sigma}^{\mathbf{i}_n}). \end{aligned}$$

Here, in the last identity, we used part (b) of Lemma 1. Similarly as before, we see that

$$\begin{aligned}\mu_{n+1} + \boldsymbol{\Sigma}_{n+1,n}\boldsymbol{\Sigma}_{n,n}^{-1}(\mathbf{z}_n - \boldsymbol{\mu}_n^*) &= \mu_{n+1} - \mathbf{e}_n'(\mathbf{z}_n - \boldsymbol{\mu}_n^*) \\ &= 2m\cdot\mathrm{mod}(\delta_n^n + 1, 2) + \mu t_{n+1} - z_n + (2m\delta_n^n - \mu t_n) \\ &= -z_n + 2m + \mu(t_{n+1} - t_n),\end{aligned}$$

for $\mathrm{mod}(\delta_n^n + 1, 2) + \delta_n^n = 1$ by definition. Moreover,

$$\Sigma_{n+1,n+1} - \boldsymbol{\Sigma}_{n+1,n}\boldsymbol{\Sigma}_{n,n}^{-1}\boldsymbol{\Sigma}_{n,n+1} = \sigma^2 t_{n+1} + \mathbf{e}_n'\boldsymbol{\Sigma}_{n,n+1} = \sigma^2 t_{n+1} - \mathbf{e}_n'\boldsymbol{\Sigma}^{\mathbf{i}_n}\mathbf{e}_n = \sigma^2(t_{n+1} - t_n).$$

Hence we complete the proof. □

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