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Fast Monte Carlo Simulation for Pricing Equity-Linked Securities

Hanbyeol Jang[1] · Sangkwon Kim[2] · Junhee Han[1] · Seongjin Lee[1] · Jungyup Ban[1] · Hyunsoo Han[1] · Chaeyoung Lee[2] · Darae Jeong[3] · Junseok Kim[2]

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Abstract
In this paper, we present a fast Monte Carlo simulation (MCS) algorithm for pricing equity-linked securities (ELS). The ELS is one of the most popular and complex financial derivatives in South Korea. We consider a step-down ELS with a knock-in barrier. This derivative has several intermediate and final automatic redemptions when the underlying asset satisfies certain conditions. If these conditions are not satisfied until the expiry date, then it will be checked whether the stock path hits the knock-in barrier. The payoff is given depending on whether the path hits the knock-in barrier. In the proposed algorithm, we first generate a stock path for redemption dates only. If the generated stock path does not satisfy the early redemption conditions and is not below the knock-in barrier at the redemption dates, then we regenerate a daily path using Brownian bridge. We present numerical algorithms for one-, two-, and three-asset step-down ELS. The computational results demonstrate the efficiency and accuracy of the proposed fast MCS algorithm. The proposed fast MCS approach is more than 20 times faster than the conventional standard MCS.

Keywords Monte Carlo simulation · Equity-linked securities · Option pricing · Brownian bridge

✉ Junseok Kim
cfdkim@korea.ac.kr

1 Department of Financial Engineering, Korea University, Seoul 02841, Republic of Korea

2 Department of Mathematics, Korea University, Seoul 02841, Republic of Korea

3 Department of Mathematics, Kangwon National University, Chuncheon-si, Gangwon-do 24341, Republic of Korea

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1 Introduction

The equity-linked security (ELS) is the financial derivative whose return on investment is dependent on the performance of the linked underlying equity. It has been one of the most popular financial derivatives since it was introduced to South Korea in 2003. The value of annual issuance for this derivative is over half-trillion US dollars (Jo and Kim 2013). In 2015, ELS was issued with about 0.65 trillion US dollars. However, the collapse of China's stock market took place in June 2015. The global stock market was influenced by the black swan incident occurred two months later on August 24th. On that day, the Dow Jones Industrial Average of the US plummeted by more than 1000 points as the two major composite indices of China plunged 8%. Furthermore, Japanese and European stock indices dropped more than 4% (Tsai 2017). For this reason, ELS has faced critical crisis during the second half of that year. Therefore, there is a need for more detailed studies about the structure of ELS to manage risk related to ELS. In this paper, we consider a step-down ELS with a knock-in barrier and describe the one-, two-, and three-asset step-down ELS.

Figure 1 illustrates the option payoff in which there are two early repayments and maturity repayment. Here, $K_1 \geq K_2 \geq K_3$ and $c_1 < c_2 < c_3$ are respectively repayment criteria percentages of the underlying asset and coupon rates at times $T_1 < T_2 < T_3$. Let $X(t) = 100S(t)/S(0)$, where $S(t)$ is the underlying asset value at time t. At the first early redemption time T_1, if $X(T_1) \geq K_1$, then the contract is closed with a return of $(1 + c_1)F$. Here, F is the face value. Otherwise, the contract will be maintained until the second early redemption time T_2. At time T_2, it repeats the first step. At maturity time T_3, we check whether $X(T_3) \geq K_3$ or not. If it is true, then the contract is closed with $(1 + c_3)F$ return. Otherwise, we check whether the underlying asset has hit the knock-in barrier D during the period $[0, T_3]$. That is, if $\min_{0 \leq t \leq T_3} X(t) \leq D$, then the return is $X(T_3)F/100$. Otherwise, it is $(1+d)F$, where d is a dummy rate. Figure 1 schematically illustrates the above payoff condition.

We now summarize the payoff function by using the characteristic function $\chi_i = \chi_{A_i}$, where $A_i = \{X \geq K_i\}$ $(i = 1, 2, 3)$. We construct the payoff function of one-asset step-down ELS as follows:

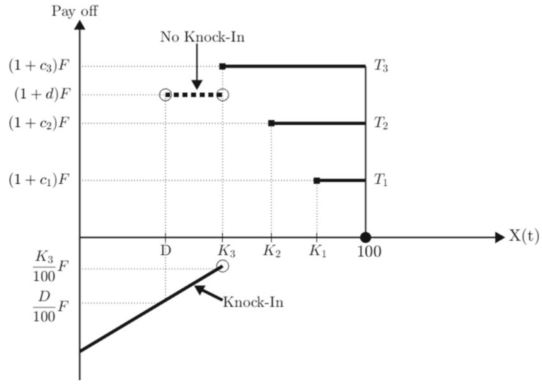


Fig. 1 Payoff structure of the one-asset step-down ELS at early redemptions and maturity

$$\begin{cases} \chi_1 = 1 & \text{Payoff} = (1+c_1)F \\ \chi_1 = 0 & \begin{cases} \chi_2 = 1 & \text{Payoff} = (1+c_2)F \\ \chi_2 = 0 & \begin{cases} \chi_3 = 1 & \text{Payoff} = (1+c_3)F \\ \chi_3 = 0 & \begin{cases} \text{If } \min_{0 \le t \le T_3} X(t) > D, \text{ then} \\ \text{Payoff} = (1+d)F; \\ \text{Otherwise} \\ \text{Payoff} = X(T_3)F/X(0). \end{cases} \end{cases} \end{cases} \end{cases}$$

The closed-form exact solution for these type of options can be found with multiple integrals (Deng et al. 2011). However, most of integrations are improper integrals. We have to numerically solve the integrals and this computation is also challenging. Some authors have proposed solutions, such as the lattice model, finite difference method (Kalantari and Shahmorad 2019; Jeong et al. 2018), and Monte Carlo simulation (MCS) (Ma et al. 2017; Leitao et al. 2017; Ghafarian et al. 2018). Among these, MCS is typically employed for pricing an ELS, because it is a convenient method for calculating complex derivatives and is simple to apply. Furthermore, several studies have presented advanced MCS methods. For example, a control variate method for the pricing basket, spread, and average options has been studied (Shiraya and Takahashi 2017), and a more advanced weighted-least squares Monte Carlo simulation has been investigated for pricing American put options (Fabozzi et al. 2017).

In this study, we propose a fast and efficient Monte Carlo simulation method using Brownian bridge (Boyle et al. 1997; Boyle 1977) to evaluate the price and Greeks of ELS. Brownian bridge is used to calculate the price of financial instruments (Shreve 2004; Baldi et al. 1999; Ruf and Scherer 2011).

This paper is organized as follows. In Sect. 2, we describe the proposed algorithm in detail. In Sect. 3, we present numerical experiments to validate the fastness and efficiency of the proposed MCS algorithm. In Sect. 4, conclusions are presented.

2 Fast Monte Carlo Simulation Algorithm

With the assumption of geometric Brownian motion of the underlying asset and risk-neutrality, we generate the sample asset paths at times $t = t_i = i\Delta t$ using the following formula:

$$S(t_{i+1}) = S(t_i)e^{(r-0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_i}, \tag{1}$$

where $S(t_i)$ is the underlying asset price at time $t = t_i$, r is the risk-free interest rate, σ is the constant volatility, Δt is the time-step size, and Z_i is a normally distributed sample with mean zero and variance one (Higham 2004). From Eq. (1), we have

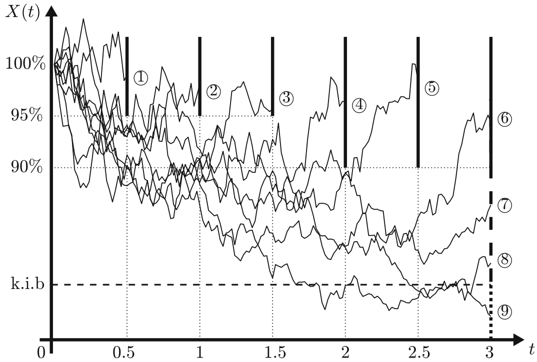
$$X(t_{i+1}) = X(t_i)e^{(r-0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_i}. \tag{2}$$

Now, let us examine nine possible cases for stock paths arising in the step-down ELS. Table 1 lists the early redemption dates (T), strike percentages (K), and coupon

Table 1 Early redemption dates, strike percentages, and coupon rates for the step-down ELS

Redemption date	$T_1 = 0.5$	$T_2 = 1$	$T_3 = 1.5$	$T_4 = 2$	$T_5 = 2.5$	$T_6 = 3$
Strike percentage	$K_1 = 95$	$K_2 = 95$	$K_3 = 95$	$K_4 = 90$	$K_5 = 90$	$K_6 = 90$
Coupon rate	$c_1 = 0.025$	$c_2 = 0.05$	$c_3 = 0.075$	$c_4 = 0.1$	$c_5 = 0.125$	$c_6 = 0.15$

Fig. 2 Nine possible random path cases for the step-down ELS



rates (c) for the step-down ELS. The other parameters used are the face value $F = 100$, knock-in barrier $D = 65$, dummy rate $d = 0.15$, the risk-free interest rate $r = 0.0166$, volatility $\sigma = 0.196$, and time-step size $\Delta t = 1/360$.

Then, we can obtain the nine possible paths corresponding to each case of the step-down ELS using Eq. (2). We mark each case as a circled number in Fig. 2. ①–⑤ are early redemption cases at $t = T_1, \ldots, T_5$, respectively. Case ⑥ represents the occurrence of obligatory redemption at maturity. Case ⑦ illustrates the situation in which the dummy is paid because the stock path has not hit the knock-in barrier (k.i.b) until maturity. If the stock path hits the knock-in barrier at least once without redemption, then the principal will be lost, see the cases of ⑧ and ⑨. We call this method by the standard MCS in which we generate the random path on a daily basis.

Next, we describe how to evaluate the step-down ELS price using the Brownian bridge construction. First, we generate discrete stock prices at early redemption and maturity dates only. That is,

$$X(T_{i+1}) = X(T_i)e^{(r-0.5\sigma^2)(T_{i+1}-T_i)+\sigma\sqrt{T_{i+1}-T_i}Z_i}, \quad i = 0, \ldots, 5, \tag{3}$$

where $X(T_0) = 100$ and $T_0 = 0$.

Then, we check the early redemptions and the maturity condition. If all these conditions are not satisfied and $\min\{X(T_1), X(T_2), \ldots, X(T_6)\} \leq D$, then the payoff is $X(T_6)F/100$. If $\min\{X(T_1), X(T_2), \ldots, X(T_6)\} > D$, then we regenerate a full path passing the generated prices at the checked days using the Brownian bridge approach, see Fig. 3. Using the regenerated full path, if $\min_{1\leq i\leq T_6/\Delta t} X(t_i) \leq D$, then the return is $X(T_6)F/100$. Otherwise, it is $(1 + d)F$, where d is a dummy rate. To obtain the option price, we take a weighted average of the return values from the sample paths.

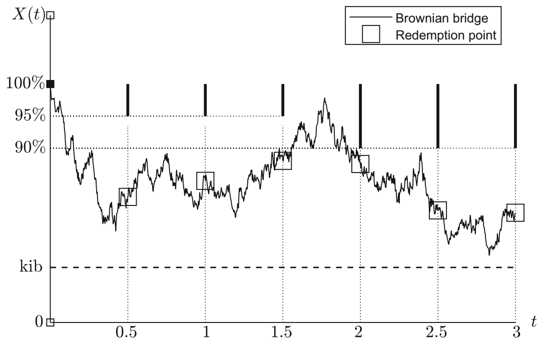


Fig. 3 Generation of the Brownian bridge path

Let us consider the Brownian bridge procedure which forms the core of our proposed algorithm (Shreve 2004; Pemantle and Mathew 1992). When we want more information between the two points, we can apply the Brownian bridge approach to generate a path connecting the specific two points. We define the standard Brownian bridge from 0 to 0 on $[0, T]$ to be the process

$$X(t) = W(t) - \frac{t}{T} W(T), \quad 0 \le t \le T, \tag{4}$$

where $W(t)$ is the Brownian motion and $W(0) = 0$. More generally, we define the Brownian bridge from a to b ($a, b \in \mathbb{R}$) on $[T_i, T_{i+1}]$ as the process

$$\begin{aligned} X^{a \to b}(t) &= a + \frac{(b-a)(t-T_i)}{T_{i+1} - T_i} + W(t - T_i) - \frac{t - T_i}{T_{i+1} - T_i} W(T_{i+1} - T_i), \\ &T_i \le t \le T_{i+1}. \end{aligned} \tag{5}$$

Let $X(T_i)$ and $X(T_{i+1})$ be the two given stock index values, then we generate a path starting from $Y(T_i) = X(T_i)$ with the time step Δt.

$$Y(t_{j+1}) = Y(t_j) e^{w_j}, \quad j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1, \tag{6}$$

where $w_j = (r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z_j$ and $t_j = T_i + j\Delta t$. Let $W_j = \sum_{i=0}^{j} w_i$, then $Y(t_{j+1}) = Y(T_i)e^{W_j}$, $j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1$. In general, $Y(T_{i+1}) \neq X(T_{i+1})$. To construct a path connecting $X(T_i)$ and $X(T_{i+1})$, we apply the Brownian bridge technique to W_j. Let

$$B_j = W_j + \frac{t_j - T_i}{T_{i+1} - T_i} \log \frac{X(T_{i+1})}{Y(T_{i+1})}, \quad j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1. \tag{7}$$

Then, we obtain a full path connecting $X(T_i)$ and $X(T_{i+1})$ as

$$X(t_{j+1}) = X(T_i) e^{B_j}, \quad j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1. \tag{8}$$

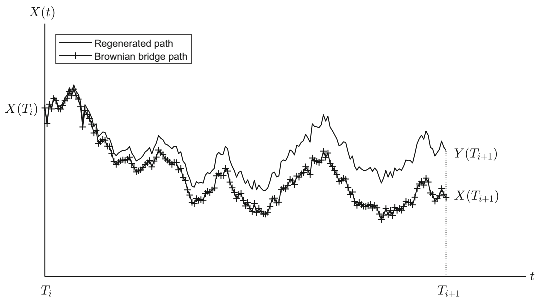


Fig. 4 Regenerated stock index path and its Brownian bridge process passing the two given points

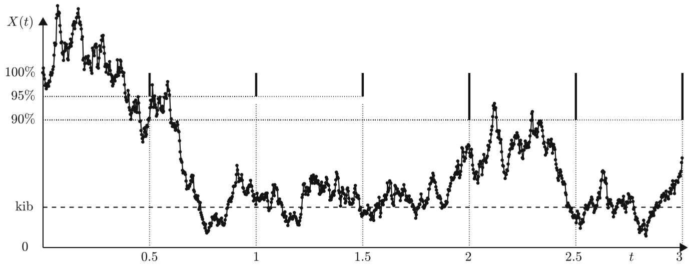


Fig. 5 Daily stock path for 3 years

Figure 4 shows the regenerated stock index path and its Brownian bridge process passing two given points. A sample path process with one time step ($\Delta t = T$) in range $[0, T]$ and a sample path process with a small time step ($\Delta t < T$) using Brownian bridge connecting two end points in range $[0, T]$ has the same mean and variance. For example, the sample path process with one time step has a mean $\frac{W(T)-W(0)}{T}t + W(0)$ at time t because this path is linear ($0 \leq t \leq T$). The mean of the sample path process with a small time step using Brownian bridge is obtained using the conditioning formula (Glasserman 2013):

$$E[W(s)|W(u) = x, W(t) = y] = \frac{(t-s)x + (s-u)y}{t-u}, \tag{9}$$

$$Var[W(s)|W(u) = x, W(t) = y] = \frac{(s-u)(t-s)}{t-u}. \tag{10}$$

Therefore, the sample path process with a small time step using Brownian bridge has mean $\frac{W(T)-W(0)}{T}t + W(0)$. According to the formula, the variance is constant regardless of the two end points of the connection. For this reason we may use the proposed algorithm.

Further details concerning the fast Brownian bridge MCS algorithm are presented in the following **Algorithm 1** in pseudo code.

In summary, we highlight the fundamental difference between the proposed and the conventional standard algorithms. In standard Monte Carlo simulation for ELS pricing, we generate a full daily path for the 3 years as shown in Fig. 5 and check early repayments and knock-in barrier.

Algorithm 1 Fast MCS algorithm for one-asset ELS

Require: Set initial price S_0, maturity T, the number of checking days N_c, the number of sample paths N_m, the number of total time steps N_T, time-step size $\Delta t = T/N_T$, face value F, volatility σ of underlying asset, risk-neutral interest rate r, early redemption dates T_i, coupon rates c_i for early and final redemptions, strike percentages K_i, dummy d, and knock-in barrier D. Set $M_i = 0$ and $X(t) = 100S(t)/S_0$. Here, $1 \le i \le N_c$ and $T_0 = 0$.

for $k = 1$ to N_m **do**
  ▷ Generate stock path for T_i only as
  for $i = 0$ to $N_c - 1$ **do**
    $X(T_{i+1}) = X(T_i)\exp((r - 0.5\sigma^2)(T_{i+1} - T_i) + \sigma\sqrt{T_{i+1} - T_i}Z_i), \quad Z_i \sim N(0, 1)$
  end for
  ▷ Check the value of the stock path at checking days
  if $X(T_1) \ge K_1$ **then** $M_1 = M_1 + (1 + c_1)F$
  else if $X(T_2) \ge K_2$ **then** $M_2 = M_2 + (1 + c_2)F$
  $\vdots$
  else if $X(T_{N_c}) \ge K_{N_c}$ **then** $M_{N_c} = M_{N_c} + (1 + c_{N_c})F$
  else if $\min_{1\le i\le N_c}\{X(T_i)\} \le D$ **then** $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
  else
    ▷ Generate a daily stock path passing through X as
    for $i = 0$ to $N_c - 1$ **do**
      Set $Y(T_i) = X(T_i)$
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $Y(t_{j+1}) = Y(t_j)\exp(w_j)$,
        where $w_j = (r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_j, \; Z_j \sim N(0, 1)$, and $t_j = j\Delta t$
      end for
      ▷ Apply the Brownian bridge
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $Y(t_{j+1}) = Y(T_i)\exp(W_j), \quad W_j = \sum_{p=T_i/\Delta t}^{j} w(t_p)$
      end for
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $X(t_j) = X(T_i)\exp(B_j), \quad B_j = W_j + \frac{t_j - T_i}{T_{i+1} - T_i}\log\frac{X(T_{i+1})}{Y(T_{i+1})}$
      end for
    end for
    if $\min_{1\le j\le N_c/\Delta t}\{X(t_j)\} \le D$ **then** $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
    else
      $M_{N_c} = M_{N_c} + (1 + d)F$
    end if
  end if
end for
▷ Take average and discount to present value.
$V^0 = \sum_{i=1}^{N_c} e^{-rT_i} M_i/N_m$

To speed up the computation, we use Brownian bridge. First, we generate a semiannual stock path for 3 years, see Fig. 6 and check early repayments and knock-in barrier. Second, only in the case of all the conditions are not satisfied, we generate a daily path passing the semiannual stock path using Brownian bridge, see Fig. 7. This process allows us to calculate the ELS price much faster than standard Monte Carlo simulation.

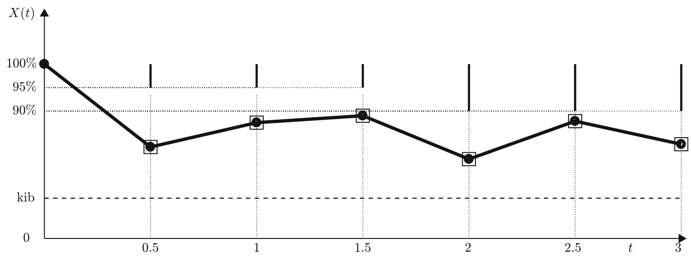


Fig. 6 Semiannual stock path for 3 years

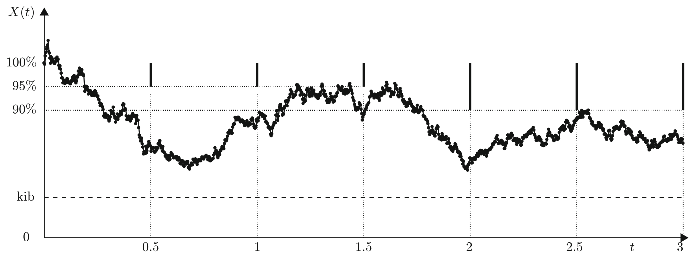


Fig. 7 Daily path passing the semiannual stock path using Brownian bridge

3 Numerical Experiment

In this section, we present numerical tests such as the convergence test, computation of the Greeks, and a comparison of the CPU time between the standard MCS and Brownian bridge MCS. These tests demonstrate that the proposed algorithm is faster than the standard MCS with an equivalent accuracy. We use the same parameter values as in Sect. 2. All computations are run in MATLAB version R2017a on a quad 3.60 GHz Intel PC with 8 GB RAM.

3.1 Convergence Test

Figure 8 illustrates the convergence of the price of the ELS with respect to the number of samples. Open circles and plus marks represent the distribution of the ELS price obtained using the Brownian bridge MCS and standard MCS, respectively. We plot 100 simulation results using the different number of samples. We can observe that the two methods converge to the same value as the number of samples increases.

Table 2 lists the mean and variance of the ELS price with the standard and Brownian Bridge MCS approaches. Each simulation is performed with 10^5 samples. The mean and variance are obtained with 500 simulations. The results demonstrate the equivalence between Brownian bridge MCS and standard MCS in terms of accuracy.

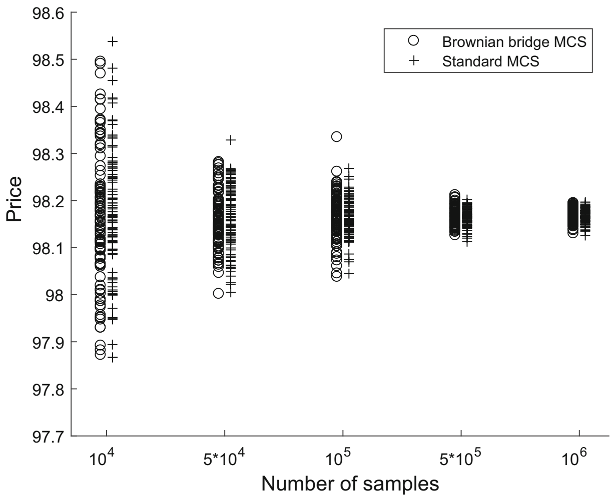


Fig. 8 ELS price versus the number of samples. Here, we plot 100 simulation results for each case

Table 2 Mean and variance of the ELS prices with two different approaches

Case	Mean	Variance
Standard MCS	98.1675	0.0048
Brownian bridge MCS	98.1662	0.0053

3.2 Greeks

In this section, we calculate the *delta* ($\Delta = \partial V^0/\partial S$) and *gamma* ($\Gamma = \partial^2 V^0/\partial S^2$) of the ELS. To compute these Greeks, we apply the central finite difference approximation, i.e., $\Delta \approx [V^0(S+\Delta S) - V^0(S-\Delta S)]/(2\Delta S)$ and $\Gamma \approx [V^0(S-\Delta S) - 2V^0(S) + V^0(S+\Delta S)]/\Delta S^2$, where V^0 is the ELS price, S is the underlying asset, and $\Delta S = 3$. We compare the results from the standard MCS and Brownian bridge MCS. Figure 9a–c show the option price, *deltas*, and *gammas* of the ELS. The rows from top to bottom are the results with M=10^4, 10^5, and M=10^6, respectively. As the number of samples increases, we can see that the *delta* and *gamma* obtained from the standard and Brownian bridge MCS approaches converge to the same values.

Figure 10a–c show the option price, *deltas*, and *gammas* of the ELS. The rows from top to bottom are the results with M=10^4, 10^5, and M=10^6, respectively. In this test, the same sample paths for the different underlying stock index values are used. We can observe the stable Greek values.

3.3 Comparison of CPU Time

We consider the elapsed time required to calculate the ELS price using the Brownian bridge MCS and standard MCS approaches with the number of samples set to : $10^4, 5\times 10^4, 10^5, 5\times 10^5$ and 10^6. We compare the elapsed times for the Brownian bridge and

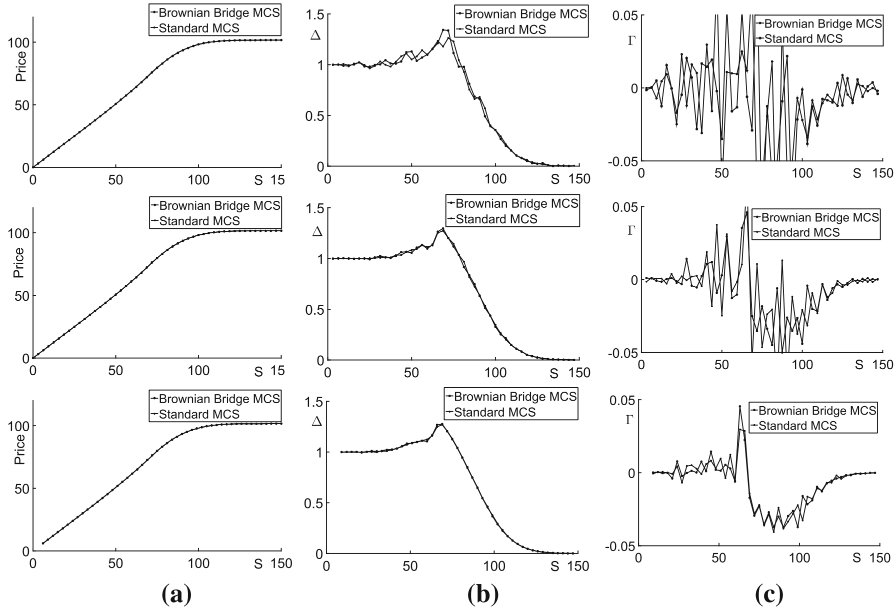


Fig. 9 **a**, **b**, and **c** show the option price, *deltas*, and *gammas* of the ELS. The rows from top to bottom are the results with M=10^4, 10^5, and M=10^6, respectively

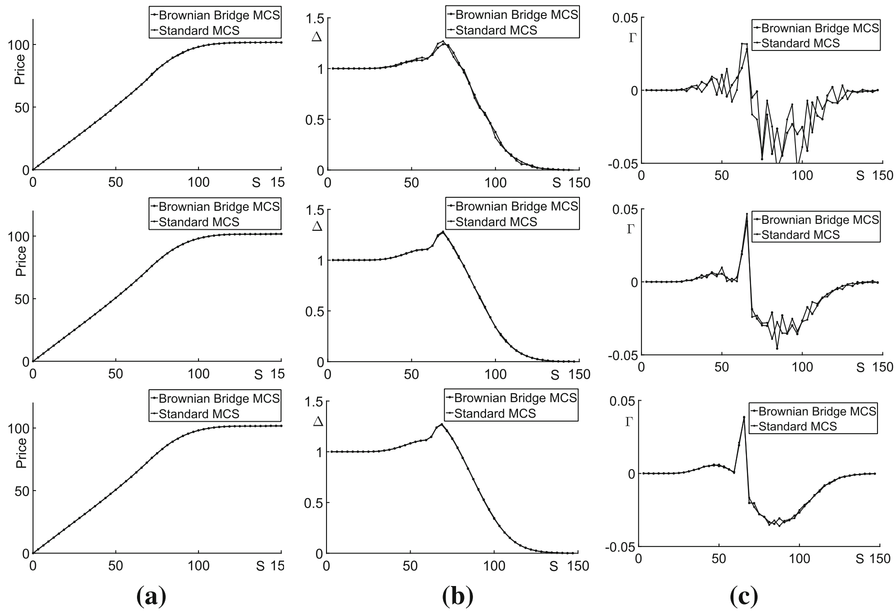


Fig. 10 **a**, **b**, and **c** show the option price, *deltas*, and *gammas* of the ELS. The rows from top to bottom are the results with M=10^4, 10^5, and M=10^6, respectively. Here, we used the same sample paths for the different underlying stock index values

Table 3 Comparison of the elapsed time (in seconds) for the two different approaches

M	10^4	5×10^4	10^5	5×10^5	10^6
Brownian bridge MCS	0.0243	0.1210	0.2411	1.2054	2.4101
Standard MCS	0.6882	3.4435	6.8884	34.4489	68.8472
Ratio	28.320	28.461	28.570	28.578	28.566

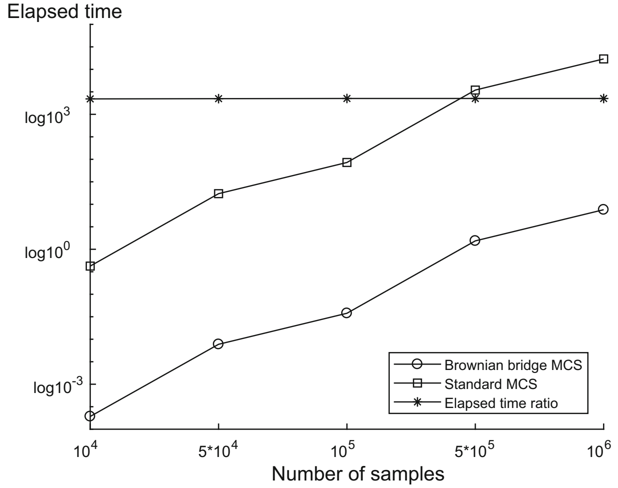


Fig. 11 Comparison of the elapsed time (in seconds) on a log scale

standard MCS approaches. Table 3 shows the elapsed times for the Brownian bridge MCS and standard MCS with respect to the number of samples. Also it demonstrates the ratio of the elapsed times for both the methods. Figure 11 shows the log scale plot of the results in Table 3. This shows that the Brownian bridge MCS is approximately more than 20 times faster than the standard MCS.

3.4 ELS with Two or Three Underlying Assets

In this section, we calculate the prices of ELS products with two and three underlying assets using **Algorithms 2** and **3**, respectively.

Most ELS products traded in South Korea financial market are products with two or three underlying assets. To calculate a derivative price with two or three assets, we need to generate a pair or triple of random numbers with correlations. We generate correlated random numbers Z_1^*, Z_2^* from a standard bivariate normal distribution using Cholesky factorization (Glasserman 2013):

$$Z_1^* = Z_1 + \rho Z_2, \ Z_2^* = \sqrt{1-\rho^2} Z_2,$$

Algorithm 2 Fast MCS algorithm for two-asset ELS

Require: Set maturity T, the number of checking days N_c, the number of sample paths N_m, the number of total time steps N_T, time-step size $\Delta t = T/N_T$, face value F, volatilities σ_1, σ_2, correlation coefficient ρ, correlated random numbers Z_1^*, Z_2^*, risk-neutral interest rate r, early redemption dates T_i, coupon rates c_i for early and final redemptions, strike percentages K_i, dummy d, and knock-in barrier D. Set $M_i = 0$, $X_1(0) = X_2(0) = 100$, and worst performer $WP(t) = \min(X_1(t), X_2(t))$. Here, $1 \le i \le N_c$ and $T_0 = 0$.

for $k = 1$ to N_m **do**
  ▹ Generate stock path for T_i only as
  for $i = 0$ to $N_c - 1$ **do**
    $X_l(T_{i+1}) = X_l(T_i)\exp((r - 0.5\sigma_l)(T_{i+1} - T_i) + \sigma_l\sqrt{T_{i+1} - T_i}Z_l^*)$, $Z_l^* \sim N(0, 1)$
    for $l = 1, 2$
    $WP(T_i) = \min(X_1(T_{i+1}), X_2(T_{i+1}))$
  end for
  ▹ Check the value of the stock path at checking days
  if $WP(T_1) \ge K_1$ **then** $M_1 = M_1 + (1 + c_1)F$
  else if $WP(T_2) \ge K_2$ **then** $M_2 = M_2 + (1 + c_2)F$
  $\vdots$
  else if $WP(T_{N_c}) \ge K_{N_c}$ **then** $M_{N_c} = M_{N_c} + (1 + c_{N_c})F$
  else if $\min_{1 \le i \le N_c}\{WP(T_i)\} \le D$ **then** $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
  else
    ▹ Generate a daily stock path passing through WP as
    for $i = 0$ to $N_c - 1$ **do**
      Set $Y_1(T_i) = X_1(T_i)$ and $Y_2(T_i) = X_2(T_i)$
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $Y_l(t_{j+1}) = Y_l(t_j)\exp(w_l^j)$,
        $w_l^j = (r - 0.5\sigma_l^2)\Delta t + \sigma_l\sqrt{\Delta t}Z_l^*$, $Z_l^* \sim N(0, 1)$ for $l = 1, 2$
      end for
      ▹ Apply the Brownian bridge
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $Y_l(t_{j+1}) = Y_l(T_i)\exp(W_l^j)$, $W_l^j = \sum_{p=T_i/\Delta t}^{j} w_l(t_p)$ for $l = 1, 2$
      end for
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $X_l(t_j) = X_l(T_j)\exp(B_l^j)$, $B_l^j = W_l^j + \frac{t_j - T_i}{T_{i+1} - T_i}\log\frac{X_l(T_{i+1})}{Y_l(T_{i+1})}$ for $l = 1, 2$
        $WP(t_j) = \min(X_1(t_j), X_2(t_j))$
      end for
    end for

    if $\min_{1 \le j \le N_c/\Delta t}\{WP(t_j)\} \le D$ **then** $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
    else
      $M_{N_c} = M_{N_c} + (1 + d)F$
    end if
  end if
end for
▹ Take average and discount to present value.
$V^0 = \sum_{i=1}^{N_c} e^{-rT_i} M_i/N_m$

Table 4 Comparison of the elapsed time (in seconds) for 2 asset with strike prices $K_1 = 90,\ K_2 = 90,\ K_3 = 90,\ K_4 = 85,\ K_5 = 85,\ K_6 = 80$, knock-in barrier $D = 65$, volatilities $\sigma_1 = 0.24,\ \sigma_2 = 0.22$, the correlation coefficient $\rho = 0.5$, and the risk-free interest free $r = 0.02$

M	10^4	5×10^4	10^5	5×10^5	10^6
Brownian bridge MCS	0.0246	0.1240	0.2498	1.2592	2.5238
Standard MCS	0.6352	3.1772	6.3552	31.8028	63.6232
Ratio	25.8471	25.6123	25.4406	25.2555	25.2092

where Z_1 and Z_2 are independent standard normal distribution. Here, ρ is the correlation coefficient between the two underlying assets. We generate the two correlated asset paths using the following formula:

$$X_1(t_{i+1}) = X_1(t_i)e^{(r-0.5\sigma^2)\Delta t+\sigma_1\sqrt{\Delta t}Z_{1i}^*},$$
$$X_2(t_{i+1}) = X_2(t_i)e^{(r-0.5\sigma^2)\Delta t+\sigma_2\sqrt{\Delta t}Z_{2i}^*}.$$

Next, we define the worst performer ($WP(t_i)$) of the two asset paths:

$$WP(t_i) = \min(X_1(t_i), X_2(t_i)) \tag{11}$$

Then, using the worst performer, we can calculate the ELS price with two underlying assets in the same way as we calculated the ELS price for the one underlying asset. Table 4 shows the elapsed time of the ELS prices with two underlying assets calculated by the standard MCS and the proposed algorithm. This shows that the Brownian bridge MCS is approximately 25 times faster than the standard MCS. For the interested reader, we provide a MATLAB source code in "Appendix".

Next, the ELS price of the three underlying assets is calculated in a similar way to the ELS price of the two underlying assets. We can generate correlated random numbers Z_1^*, Z_2^*, Z_3^* from a standard multivariate normal distribution using Cholesky factorization (Glasserman 2013):

$$Z_1^* = Z_1 + \rho_{12}Z_2 + \rho_{13}Z_3,\ \ Z_2^* = \sqrt{1-\rho_{12}^2}Z_2 + \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}Z_3,$$

$$Z_3^* = \sqrt{1-\rho_{13}^2 - \frac{(\rho_{23}-\rho_{12}\rho_{13})^2}{1-\rho_{12}^2}}Z_3,$$

where Z_1,Z_2,Z_3 are independent standard normal distribution. Here, ρ_{12}, ρ_{13}, and ρ_{23} are the correlation coefficients between the three underlying assets.

We create the three correlated asset paths using the following formula:

$$X_1(t_{i+1}) = X_1(t_i)e^{(r-0.5\sigma^2)\Delta t+\sigma_1\sqrt{\Delta t}Z_{1i}^*},$$
$$X_2(t_{i+1}) = X_2(t_i)e^{(r-0.5\sigma^2)\Delta t+\sigma_2\sqrt{\Delta t}Z_{2i}^*},$$

Algorithm 3 Fast MCS algorithm for three-asset ELS

Require: Set maturity T, the number of checking days N_c, the number of sample paths N_m, the number of total time steps N_T, time-step size $\Delta t = T/N_T$, face value F, volatilities σ_1, σ_2, σ_3, correlation coefficients ρ_{12}, ρ_{13}, ρ_{23}, correlated random numbers Z_1^*, Z_2^*, Z_3^*, risk-neutral interest rate r, early redemption dates T_i, coupon rates c_i for early and final redemptions, strike percentages K_i, dummy d, and knock-in barrier D. Set $M_i = 0$, $X_1(0) = X_2(0) = X_3(0) = 100$, and worst performer $WP(t) = \min(X_1(t), X_2(t), X_3(t))$. Here, $1 \le i \le N_c$ and $T_0 = 0$.

for $k = 1$ to N_m **do**
  ▷ Generate stock path for T_i only as
  for $i = 0$ to $N_c - 1$ **do**
    $X_l(T_{i+1}) = X_l(T_i)\exp((r - 0.5\sigma_l)(T_{i+1} - T_i) + \sigma_l\sqrt{T_{i+1} - T_i}Z_l^*)$, $Z_l^* \sim N(0, 1)$
    for l = 1, 2, 3
    $WP(T_i) = \min(X_1(T_{i+1}), X_2(T_{i+1}), X_3(T_{i+1}))$
  end for
  ▷ Check the value of the stock path at checking days
  if $WP(T_1) \ge K_1$ **then** $M_1 = M_1 + (1 + c_1)F$
  else if $WP(T_2) \ge K_2$ **then** $M_2 = M_2 + (1 + c_2)F$
  ⋮
  else if $WP(T_{N_c}) \ge K_{N_c}$ **then** $M_{N_c} = M_{N_c} + (1 + c_{N_c})F$
  else if $\min_{1\le i\le N_c}\{WP(T_i)\} \le D$ **then** $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
  else
    ▷ Generate a daily stock path passing through WP as
    for $i = 0$ to $N_c - 1$ **do**
      Set $Y_1(T_i) = X_1(T_i)$, $Y_2(T_i) = X_2(T_i)$ and $Y_3(T_i) = X_3(T_i)$
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $Y_l(t_{j+1}) = Y_l(t_j)\exp(w_l^j)$,
        $w_l^j = (r - 0.5\sigma_l^2)\Delta t + \sigma_l\sqrt{\Delta t}Z_l^*$, $Z_l^* \sim N(0, 1)$ for l = 1, 2, 3
      end for
      ▷ Apply the Brownian bridge
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $Y_l(t_{j+1}) = Y_l(T_i)\exp(W_l^j)$, $W_l^j = \sum_{p=T_i/\Delta t}^{j} w_l(t_p)$ for l = 1, 2, 3
      end for
      for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
        $X_l(t_j) = X_l(T_j)\exp(B_l^j)$, $B_l^j = W_l^j + \frac{t_j - T_i}{T_{i+1} - T_i}\log\frac{X_l(T_{i+1})}{Y_l(T_{i+1})}$ for l = 1, 2, 3
        $WP(t_j) = \min(X_1(t_j), X_2(t_j), X_3(t_j))$
      end for
    end for

    if $\min_{1\le j\le N_c/\Delta t}\{WP(t_j)\} \le D$ **then** $M_{N_c} = M_{N_c} + FX(T_{N_c+1})/100$
    else
      $M_{N_c} = M_{N_c} + (1 + d)F$
    end if
  end if
end for
▷ Take average and discount to present value.
$V^0 = \sum_{i=1}^{N_c} e^{-rT_i} M_i/N_m$

Table 5 Comparison of the elapsed time (in seconds) for 3 asset with strike prices $K_1 = 90,\ K_2 = 90,\ K_3 = 90,\ K_4 = 85,\ K_5 = 85,\ K_6 = 80$, knock-in barrier $D = 65$, volatilities $\sigma_1 = 0.25,\ \sigma_2 = 0.24,\ \sigma_3 = 0.23$,the correlation coefficient $\rho_{12} = 0.5$, $\rho_{13} = 0.5$, $\rho_{23} = 0.5$, the risk-free interest free $r = 0.02$

M	10^4	5×10^4	10^5	5×10^5	10^6
Brownian bridge MCS	0.0449	0.2226	0.4477	2.2468	4.5165
Standard MCS	0.9783	4.8933	9.7851	48.9536	97.9710
Ratio	21.7882	21.9798	21.8574	21.7877	21.6917

$$X_3(t_{i+1}) = X_3(t_i)e^{(r-0.5\sigma^2)\Delta t+\sigma_3\sqrt{\Delta t}Z^*_{3i}}.$$

Then, we define the worst performer($WP(t_i)$) between three asset paths:

$$WP(t_i) = \min(X_1(t_i), X_2(t_i), X_3(t_i)). \tag{12}$$

Then, using the worst performer, we can calculate the ELS price with three underlying assets. Table 5 shows comparison results of elapsed times. This demonstrates that the Brownian bridge MCS is approximately 21 times faster than the standard MCS.

4 Conclusion

In this article, we presented a fast Brownian bridge MCS algorithm for pricing a step-down ELS with a knock-in barrier. In the proposed algorithm, we first generate a stock path for redemption dates only. If the generated stock path does not satisfy the conditions for early redemption and the path has not been below the knock-in barrier at the redemption dates, then we recreate a daily path using Brownian bridge that passes values at the redemption dates. We have provided the detailed numerical algorithms for a one-asset step-down ELS. The computational results demonstrated the efficiency and the accuracy of proposed fast MCS algorithm. The new algorithm is more than 20 times faster than a conventional one with an equivalent accuracy. In future work, we will implement the proposed method on mobile devices.

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Appendix

In this appendix, we provide a MATLAB source code for two asset ELS pricing.

```
% Tow asset ELS pricing by using Brownian Bridge

S0=100; % Initial price
T=3;    % Maturity
ddt=0.5; Nc=T/ddt; % The number of checking days
oneyear=360; NT=oneyear*T; % The number of total time steps
dt=1/oneyear; % time-step size
Nm=1.0e6; % The number of sample paths
F=100; % Facevalue
sigma1=0.25; sigma2=0.24; % Volatility of underlying assets
r=0.02; % risk-neutral interest rate
Ti=ceil(3*oneyear*cumsum(ones(1,Nc))/Nc); % Early redemption dates
% Coupon rates for early and final redemptions
ci=[0.05 0.10 0.15 0.20 0.25 0.30];
Ki=[0.90 0.90 0.90 0.85 0.85 0.80]*S0; % Strike percentages
d=ci(end); % Dummy
D=0.65*S0; % Knock-in barrier
rho=0.5;c=chol([1 rho;rho 1]);
payment=(1+ci)*F;
day=[1 Ti+1];
tot_payoff=zeros(1,Nc);
coef11=(r-0.5*sigma1^2)*ddt;coef21=sigma1*sqrt(ddt);
coef12=(r-0.5*sigma2^2)*ddt;coef22=sigma2*sqrt(ddt);
coef13=(r-0.5*sigma1^2)*dt;coef23=sigma1*sqrt(dt);
coef14=(r-0.5*sigma2^2)*dt;coef24=sigma2*sqrt(dt);
X1=zeros(1,Nc+1);X1(1)=S0;X2=X1;WP=zeros(1,Nc);
n=180;t=0:n;tn=t/n;
Z=randn(2,Nc,Nm);Z=reshape(Z,[2,Nc*Nm]);
w=c*Z;w=reshape(w,[2,Nc,Nm]);
```

```
for m=1:Nm
re_array=1;
M=zeros(1,Nc);
for i=1:Nc
X1(i+1)=X1(i)*exp(coef11+coef21*w(1,i,m));
X2(i+1)=X2(i)*exp(coef12+coef22*w(2,i,m));
WP(i)=min(X1(i+1),X2(i+1));
if WP(i)>=Ki(i)
M(i)=payment(i);
re_array=0;
break
end
end
if re_array==1
if min(WP)<=D
M(end)=WP(end);
else
for k=1:Nc
re_w=c*randn(2,n);
win1=[0 cumsum(coef13+coef23*re_w(1,:))];
win2=[0 cumsum(coef14+coef24*re_w(2,:))];
wb1=log(X1(k+1)/X1(k)); wb2=log(X2(k+1)/X2(k));
win1=((wb1-win1(n+1))*tn)+win1;
win2=((wb2-win2(n+1))*tn)+win2;
Re_X1=X1(k)*exp(win1);
Re_X2=X2(k)*exp(win2);
Re_WP=min(Re_X1,Re_X2);
if min(Re_WP)<=D
M(end)=F*WP(end)/S0;
break
elseif k==Nc
M(end)=F*(1+d);
end
end
end
end
tot_payoff=tot_payoff+M;
end
BB2_price=sum(tot_payoff/Nm.*exp(-r*Ti/oneyear))
```

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