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Analytical valuation of autocallable notes

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Abstract

In this paper, a general form of autocallable note is analytically valued, which includes the following features: regular coupons, reverse convertible provision and possible participation in the growth of the underlying equity asset. Simpler notes can be designed and analytically priced on the basis of this general structure. The equity asset follows a jump-diffusion process, while interest rates are driven by a two-factor model. Equity and interest rate sources of randomness are correlated. The numerical implementation is easy and very efficient compared to alternative valuation techniques. The formula provided in this paper can thus be expected to be a valuable tool for both buyers and issuers in terms of pricing and risk management.

Keywords: Autocallable; valuation; jump-diffusion; equity-rate correlation; multivariate normal distribution.

JEL Classifications: G13; C63

1. Introduction

Autocallables, also known as auto-trigger structures or kick-out plans, are very popular in the world of structured products. They have captured a large part of the market share in recent years. Product providers use them to offer higher payoffs than those on structured products that automatically run to a full term. In its standard form, an autocallable is a note that is linked to an underlying risky asset (usually a single stock, a basket of stocks or an equity index) and that has no fixed maturity. What is referred to as the maturity of the autocallable is actually the

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maximum duration this product can stay alive. Several observation dates within the product's life are prespecified in the contract. At each observation date, if the value of the underlying is at or above a prespecified level, usually called the autocall trigger level or autocall barrier, then the principal amount is paid back by the issuer to the holder of the note, along with a coupon rate. It is said then that the note autocalls. The prespecified autocall trigger level is often defined as the level of the underlying asset at the contract's inception, but it does not have to be. It may also vary in time. If there is no early redemption, the note proceeds to the next observation date, where there is again the possibility of early redemption. A lot of plans kick out in year one or two, leaving investors with the choice to reinvest in rollover substitutes from the same provider, switch to another offer or opt back into the markets. Another level can be prespecified for each observation date, below the autocall trigger level, such that if the note does not autocall but the underlying is above that lower level, usually called the coupon barrier, then the note pays a coupon rate.

Several variants of the standard structure are traded on the markets. One way to allow for yield enhancement is to include a reverse convertible mechanism: If the note has not been autocalled and if the underlying asset has fallen below a prespecified conversion level at maturity, which is a kind of safety threshold from the point of view of the investor, then the principal amount is not redeemed and shares (or their equivalent cash amount) are paid back instead, which entails a capital loss for the investor. This amounts to the sale of an out-of-the-money put by the investor to the issuer. This is a suitable feature in the current market environment. Indeed, low interest rates leave little room for yield enhancement if investors want full capital protection, as providers must use more of the initial capital to guarantee its complete return. Moreover, in high volatility markets, volatility has to be sold in order to generate a higher income. The answer is then to introduce capital-at-risk structures, typically losing money if the underlying index has fallen 50% or further from its initial level.

Investors may also want to be given the opportunity to participate in potential increases in the value of the underlying instead of receiving fixed coupons. More specifically, some contracts define a prespecified upper level such that, if the underlying is above that level at a given observation date, then the note autocalls and the coupon paid to the holder of the note is a percentage of the spot value of the underlying.

Despite their widespread use in the markets, there are not many academic studies on autocallable products. While De Weert (2008), as well as Bouzoubaa and Osseiran (2010) describe a variety of payoffs and analyze risk management issues, all the other contributions focus on numerical pricing schemes. Fries and Joshi (2011) study a product specific variance reduction scheme for Monte Carlo simulation purposes. Deng *et al.* (2011) discuss finite difference methods in a numerical partial differential equation framework. Kamtchueng (2011) discusses smoothing algorithms for the Monte Carlo simulation of the Greeks. Alm *et al.* (2013) developed a Monte Carlo algorithm that allows stability with respect to differentiation. No research article has yet come up with an analytical solution to the valuation problem raised by the autocallable structured products traded in the markets, i.e., autocallables with discrete observation dates. The only formula published so far is the one by Deng *et al.* (2011) and it prices products with continuous autocall dates, a feature that is hardly ever encountered in the markets. Moreover, the analytical framework they use is the standard Black–Scholes model. That set of assumptions is notoriously flawed for a number of reasons. First, it has long been known that equity prices are not purely continuous and exhibit jumps. The latter are a way to account for the skew observed in the options market, especially the steeper skew for short expirations (Gatheral, 2006). Second, the assumption that interest rates are constant is particularly spurious for the valuation of autocallable notes. Indeed, the latter are a combination of fixed income and equity components and they are usually long-dated. Moreover, the correlation between equity and interest rate sources of randomness has a significant impact. When the stock market goes up, the duration of the autocallable structure goes down. If there is a positive correlation between equity and interest rate, sellers of the note make losses while hedging their interest rate exposure, whether equity increases — as they have to sell longer-dated zero coupon bonds and buy more short term zero coupon bonds under higher interest rates, or decreases — as they need to sell short term bonds and buy more longer-dated bonds under lower interest rates. Conversely, if there is a negative correlation between equity and interest rate, sellers of the notes make a net profit while hedging their interest rate exposure, whether equity goes up and down because of the opposite directions of equity and interest rate. As a consequence, pricing models that do not take the correlation between equity and interest rate into account will underprice the autocallable structure when this correlation is positive and overprice it when that correlation is negative.

The main contribution of this paper is to provide an analytical formula for the fair price of an autocallable note with discrete observation dates that includes the following features: regular coupons, reverse convertible provision and possible early exit through participation in the growth of the underlying equity asset. The modeling framework is a jump-diffusion equity process correlated with a two-factor stochastic interest rate process. To achieve an analytical solution, the structure is priced as a whole in the form of an option valuation formula that is a function of the contract's specifications. From a computational perspective, our approach has two distinctive features: representations of the multivariate normal density function are expanded in dimension up to five, and changes of measure are avoided through preservation of multivariate normality and integral calculus. Dimension reduction rules for multidimensional Gaussian integrals are also provided.

The paper is organized as follows: Section 2 provides a precise description of the payoff under consideration and states the valuation formula, along with a discussion of its numerical implementation. Section 3 provides a proof of the formula given in Sec. 2.

2. Analytical Formula for the Value of a General Autocallable Note Under a Jump-Diffusion Equity Model Correlated with a Two-Factor Stochastic Interest Rate Process

Several observation dates $t_1, t_2, \ldots, t_n = T$ are set within the product life $[t_0 = 0, T]$, where T is the maximum duration of the product (its "maturity"). The underlying equity asset is denoted by S. At each observation date t_m, $1 \leq m < n$, prior to expiry, autocall may occur in two ways:

(i) If the value of S at time t_m, denoted by $S(t_m)$, lies within a range $[D_m, U_m[$, $D_m < U_m$, then the investor's initial capital or notional M is redeemed at time t_m, along with a coupon rate y_m. Both the range $[D_m, U_m[$ and the coupon rate y_m are prespecified in the contract. The autocall trigger level D_m is typically the value of the underlying at inception, i.e., $S(t_0)$.

(ii) If $S(t_m)$ is greater than the level U_m, the note autocalls but, instead of yielding a prespecified coupon rate, it provides the investor with a percentage $(1 + \alpha)$ of the return ratio $S(t_m)/S(t_0)$.

Besides, if $S(t_m)$ lies within a range $[C_m, D_m[$, $C_m < D_m$, then the note does not autocall but a prespecified coupon rate z_m is paid out to the investor.

At expiry or maximum duration $t_n = T$, the notional M is fully redeemed if and only if the asset price $S(t_n)$ stands above a prespecified level H. Tables 1 and 2 show all possible events and their consequences, before and at expiry.

The modeling framework is now introduced.

Let $\{W_1(t), t \geq 0\}$, $\{W_2(t), t \geq 0\}$ and $\{W_3(t), t \geq 0\}$ be three correlated Brownian motions, whose constant pairwise correlation coefficients are denoted by $\rho_{1.2}$, $\rho_{1.3}$ and $\rho_{2.3}$.

The default-free interest rate process $\{r(t), t \geq 0\}$ is driven by:

$$dr(t) = \theta(t)dt + \upsilon_1 dW_1(t) + \upsilon_2 dW_2(t) \tag{1}$$

where υ_1 and υ_2 are two positive constants and $\theta(t)$ is a non-random, piecewise continuous function satisfying a linear growth condition. Equation (1) is a two-factor Ho and Lee model (Ho and Lee, 1986). The reason that it was chosen is

Table 1. Possible payoffs at each observation date $t_m, 1 \leq m \leq n$ prior to expiry.

Event	Consequence
$S(t_m) < C_m$	$\Rightarrow$ the note proceeds to the next observation date
$D_m > S(t_m) > C_m$	$\Rightarrow$ the note pays out a prespecified coupon $M \times z_m$ and the note proceeds to the next observation date
$U_m > S(t_m) > D_m$	$\Rightarrow$ early exit with coupon: the investor's initial capital M is fully redeemed and the note pays out a final prespecified coupon $M \times y_m$
$S(t_m) > U_m$	$\Rightarrow$ early exit with participation in S: the investor's initial capital M is fully redeemed and the note pays out a final coupon equal to a prespecified percentage $1 + \alpha$ of the ratio $S(t_m)/S(t_0)$

Notes: This table shows the possible payoffs provided by the autocallable structure under consideration at an observation date prior to expiry.

Table 2. Possible payoffs at expiry or maximum duration $t_n = T$.

Event	Consequence
$S(t_n) < H$	$\Rightarrow$ the reverse convertible mechanism is triggered: a fraction $S(t_n)/S(t_0)$ of the investor's initial capital M is redeemed
$H > S(t_n) > D_n$	$\Rightarrow$ the investor's initial capital M is fully redeemed
$U_n > S(t_n) > D_n$	$\Rightarrow$ the investor's initial capital M is fully redeemed and the note pays out a prespecified coupon $M \times y_n$
$S(t_n) \geq U_n$	$\Rightarrow$ the investor's initial capital M is fully redeemed and the note pays out a final coupon equal to a prespecified percentage $1 + \alpha$ of the ratio $S(t_n)/S(t_0)$

Notes: This table shows the possible payoffs provided by the autocallable structure under consideration at expiry or maximum duration.

two-fold: (i) it allows calibration to the observed data by fitting a suitable $\theta(t)$; (ii) the finite-dimensional distributions of $\{r(t), t \geq 0\}$ are multivariate normal, which enables to preserve the analytical tractability of the full model, as will be seen in Sec. 2. The latter point should be emphasized as there are other classical interest rate models that are univariate normal and that can be appropriately calibrated but whose finite-dimensional distributions are not multivariate normal, such as the two-factor Hull and White model (Hull and White, 1994) and its generalization known as the G2++ model. The reader wishing details about interest rate models and how they can be fitted to market data may refer to Brigo and Mercurio (2006).

The underlying equity asset process $\{S(t), t \geq 0\}$ is driven by

$$\frac{dS(t)}{S(t_-)} = (r(t) - \lambda\kappa)dt + \sigma(t)dW_3(t) + I(t)dN(t) \tag{2}$$

where

(i) $\sigma(t)$ is a positive, piecewise continuous, non-random function such that $\int_0^t \sigma^2(s)ds < \infty$, $\forall t \geq 0$.
(ii) $\{N(t), t \geq 0\}$ is a Poisson process of constant intensity $\lambda \geq 0$.
(iii) $I(t) = \sum_n U_n \mathbb{I}_{[\tau_{n-1},\, \tau_n[}(t)$, where $\tau_n = \inf\{t \geq 0, N(t) = n \in \mathbb{N}\}$ and U_n is a sequence of independent, identically distributed random variables taking values in $]-1,\ +\infty]$.

Let $J_n \triangleq \ln(1 + U_n)$ be normally distributed with mean ξ and variance ε^2; then, the parameter κ is defined by: $\kappa = \exp(\xi + \varepsilon^2/2) - 1$.

It is assumed that all random processes are defined on a suitable probability space $(\Omega, \mathbb{F}, \mathcal{F}_t, \mathbb{P})$, in which $\{\mathcal{F}_t, t \geq 0\}$ is the smallest σ-algebra generated by the random variables $W_i(s)$, $N(s)$ and $U_n\mathbb{I}\{n \leq N(t)\}$, $\forall s : 0 \leq s \leq t$, $\forall n \in N$, $\forall i \in \{1, 2, 3\}$. It is recalled that a Brownian motion and a Poisson process relative to the same filtration must be independent (Shreve, 2004).

Thus, the stochastic differential equation for the equity asset price is a jump-diffusion process extending the seminal Merton model (Merton, 1976) by integrating a stochastic interest rate process driven by Eq. (1). The reason for that extension is that it is essential to factor in the correlation between sources of randomness in the equity world and in the fixed income world, as explained in Sec. 1. The main result of this paper can now be stated. It consists in a valuation formula for a contingent claim with payoff defined by Tables 1 and 2, under the assumptions above defined, in the case of $n = 4$ observation dates.

Formula:
The fair value, V, at time $t_0 = 0$, of the autocallable contingent claim above defined, with four observation dates $t_1 \leq t_2 \leq t_3 \leq t_4$, is given by

$$V = M \times \exp(-\lambda t_4) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \frac{\lambda^{n_1+n_2+n_3+n_4} t_1^{n_1} (t_2 - t_1)^{n_2} (t_3 - t_2)^{n_3} (t_4 - t_3)^{n_4}}{n_1! n_2! n_3! n_4!}$$
$$\times \left\{ \begin{aligned} &\bar{\beta}(t_1)((1+y_1)(\mathbb{P}_1 - \mathbb{P}_2) + z_1(\mathbb{P}_4 - \mathbb{P}_5)) + \tilde{\beta}(t_1)(1+\alpha)\mathbb{P}_3 \\ &+ \bar{\beta}(t_2)((1+y_2)(\mathbb{P}_6 - \mathbb{P}_7) + z_2(\mathbb{P}_9 - \mathbb{P}_{10})) + \tilde{\beta}(t_2)(1+\alpha)\mathbb{P}_8 \\ &+ \bar{\beta}(t_3)((1+y_3)(\mathbb{P}_{11} - \mathbb{P}_{12}) + z_3(\mathbb{P}_{14} - \mathbb{P}_{15})) + \tilde{\beta}(t_3)(1+\alpha)\mathbb{P}_{13} \\ &+ \bar{\beta}(t_4)((1+y_4)(\mathbb{P}_{16} - \mathbb{P}_{17}) + \mathbb{P}_{19} - \mathbb{P}_{20}) + \tilde{\beta}(t_4)((1+\alpha)\mathbb{P}_{18} + \mathbb{P}_{21}) \end{aligned} \right\} \tag{3}$$

$$\mathbb{P}_1 = \Phi\left(\frac{\ln(U_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1, t_1)\psi(t_1)\bar{\nu}(t_1)}{\psi(t_1)}\right)$$

$$\mathbb{P}_2 = \Phi\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_1)\psi(t_1)\bar{\nu}(t_1)}{\psi(t_1)}\right)$$

$$\mathbb{P}_3 = \Phi\left(\frac{\ln(S_0/U_1) + \mu(t_1) + \tilde{\varphi}(t_1,t_1)\psi(t_1)\tilde{\nu}(t_1)}{\psi(t_1)}\right)$$

$$\mathbb{P}_4 = \Phi\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_1)\psi(t_1)\bar{\nu}(t_1)}{\psi(t_1)}\right)$$

$$\mathbb{P}_5 = \Phi\left(\frac{\ln(C_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_1)\psi(t_1)\bar{\nu}(t_1)}{\psi(t_1)}\right)$$

$$\mathbb{P}_6 = \Phi_2\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_2)\psi(t_1)\bar{\nu}(t_2)}{\psi(t_1)}, \frac{\ln(U_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_2)\psi(t_2)\bar{\nu}(t_2)}{\psi(t_2)}; \frac{\psi(t_1)}{\psi(t_2)}\right)$$

$$\mathbb{P}_7 = \Phi_2\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_2)\psi(t_1)\bar{\nu}(t_2)}{\psi(t_1)}, \frac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_2)\psi(t_2)\bar{\nu}(t_2)}{\psi(t_2)}; \frac{\psi(t_1)}{\psi(t_2)}\right)$$

$$\mathbb{P}_8 = \Phi_2\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \tilde{\varphi}(t_1,t_2)\psi(t_1)\tilde{\nu}(t_2)}{\psi(t_1)}, \frac{\ln(S_0/U_2) + \mu(t_2) + \tilde{\varphi}(t_2,t_2)\psi(t_2)\tilde{\nu}(t_2)}{\psi(t_2)}; -\frac{\psi(t_1)}{\psi(t_2)}\right)$$

$$\mathbb{P}_9 = \Phi_2\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_2)\psi(t_1)\bar{\nu}(t_2)}{\psi(t_1)}, \frac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_2)\psi(t_2)\bar{\nu}(t_2)}{\psi(t_2)}; \frac{\psi(t_1)}{\psi(t_2)}\right)$$

$$\mathbb{P}_{10} = \Phi_2\left(\frac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_2)\psi(t_1)\bar{\nu}(t_2)}{\psi(t_1)}, \frac{\ln(C_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_2)\psi(t_2)\bar{\nu}(t_2)}{\psi(t_2)}; \frac{\psi(t_1)}{\psi(t_2)}\right)$$

$$\mathbb{P}_{11} = \Phi_3\left(\begin{array}{l} \dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_3)\psi(t_1)\bar{\nu}(t_3)}{\psi(t_1)}, \\ \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_3)\psi(t_2)\bar{\nu}(t_3)}{\psi(t_2)}, \\ \dfrac{\ln(U_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_3)\psi(t_3)\bar{\nu}(t_3)}{\psi(t_3)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)} \end{array}\right)$$

$$\mathbb{P}_{12} = \Phi_3\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_3)\psi(t_1)\bar{\nu}(t_3)}{\psi(t_1)},\\[2ex] \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_3)\psi(t_2)\bar{\nu}(t_3)}{\psi(t_2)},\\[2ex] \dfrac{\ln(D_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_3)\psi(t_3)\bar{\nu}(t_3)}{\psi(t_3)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}\end{array}\right)$$

$$\mathbb{P}_{13} = \Phi_3\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \tilde{\varphi}(t_1,t_3)\psi(t_1)\tilde{\nu}(t_3)}{\psi(t_1)},\\[2ex] \dfrac{\ln(D_2/S_0) - \mu(t_2) - \tilde{\varphi}(t_2,t_3)\psi(t_2)\tilde{\nu}(t_3)}{\psi(t_2)},\\[2ex] \dfrac{\ln(S_0/U_3) + \mu(t_3) + \tilde{\varphi}(t_3,t_3)\psi(t_3)\tilde{\nu}(t_3)}{\psi(t_3)}; \dfrac{\psi(t_1)}{\psi(t_2)}, -\dfrac{\psi(t_2)}{\psi(t_3)}\end{array}\right)$$

$$\mathbb{P}_{14} = \Phi_3\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_3)\psi(t_1)\bar{\nu}(t_3)}{\psi(t_1)},\\[2ex] \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_3)\psi(t_2)\bar{\nu}(t_3)}{\psi(t_2)},\\[2ex] \dfrac{\ln(D_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_3)\psi(t_3)\bar{\nu}(t_3)}{\psi(t_3)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}\end{array}\right)$$

$$\mathbb{P}_{15} = \Phi_3\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_3)\psi(t_1)\bar{\nu}(t_3)}{\psi(t_1)},\\[2ex] \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_3)\psi(t_2)\bar{\nu}(t_3)}{\psi(t_2)},\\[2ex] \dfrac{\ln(C_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_3)\psi(t_3)\bar{\nu}(t_3)}{\psi(t_3)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}\end{array}\right)$$

$$\mathbb{P}_{16} = \Phi_4\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_4)\psi(t_1)\bar{\nu}(t_4)}{\psi(t_1)},\\[2ex] \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_4)\psi(t_2)\bar{\nu}(t_4)}{\psi(t_2)},\\[2ex] \dfrac{\ln(D_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_4)\psi(t_3)\bar{\nu}(t_4)}{\psi(t_3)},\\[2ex] \dfrac{\ln(U_4/S_0) - \mu(t_4) - \bar{\varphi}(t_4,t_4)\psi(t_4)\bar{\nu}(t_4)}{\psi(t_4)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}, \dfrac{\psi(t_3)}{\psi(t_4)}\end{array}\right)$$

$$\mathbb{P}_{17} = \Phi_4\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_4)\psi(t_1)\bar{\nu}(t_4)}{\psi(t_1)},\\ \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_4)\psi(t_2)\bar{\nu}(t_4)}{\psi(t_2)},\\ \dfrac{\ln(D_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_4)\psi(t_3)\bar{\nu}(t_4)}{\psi(t_3)},\\ \dfrac{\ln(D_4/S_0) - \mu(t_4) - \bar{\varphi}(t_4,t_4)\psi(t_4)\bar{\nu}(t_4)}{\psi(t_4)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}, \dfrac{\psi(t_3)}{\psi(t_4)}\end{array}\right)$$

$$\mathbb{P}_{18} = \Phi_4\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \tilde{\varphi}(t_1,t_4)\psi(t_1)\tilde{\nu}(t_4)}{\psi(t_1)},\\ \dfrac{\ln(D_2/S_0) - \mu(t_2) - \tilde{\varphi}(t_2,t_4)\psi(t_2)\tilde{\nu}(t_4)}{\psi(t_2)},\\ \dfrac{\ln(D_3/S_0) - \mu(t_3) - \tilde{\varphi}(t_3,t_4)\psi(t_3)\tilde{\nu}(t_4)}{\psi(t_3)},\\ \dfrac{\ln(S_0/U_4) + \mu(t_4) + \tilde{\varphi}(t_4,t_4)\psi(t_4)\tilde{\nu}(t_4)}{\psi(t_4)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}, -\dfrac{\psi(t_3)}{\psi(t_4)}\end{array}\right)$$

$$\mathbb{P}_{19} = \Phi_4\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_4)\psi(t_1)\bar{\nu}(t_4)}{\psi(t_1)},\\ \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_4)\psi(t_2)\bar{\nu}(t_4)}{\psi(t_2)},\\ \dfrac{\ln(D_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_4)\psi(t_3)\bar{\nu}(t_4)}{\psi(t_3)},\\ \dfrac{\ln(D_4/S_0) - \mu(t_4) - \bar{\varphi}(t_4,t_4)\psi(t_4)\bar{\nu}(t_4)}{\psi(t_4)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}, \dfrac{\psi(t_3)}{\psi(t_4)}\end{array}\right)$$

$$\mathbb{P}_{20} = \Phi_4\left(\begin{array}{l}\dfrac{\ln(D_1/S_0) - \mu(t_1) - \bar{\varphi}(t_1,t_4)\psi(t_1)\bar{\nu}(t_4)}{\psi(t_1)},\\ \dfrac{\ln(D_2/S_0) - \mu(t_2) - \bar{\varphi}(t_2,t_4)\psi(t_2)\bar{\nu}(t_4)}{\psi(t_2)},\\ \dfrac{\ln(D_3/S_0) - \mu(t_3) - \bar{\varphi}(t_3,t_4)\psi(t_3)\bar{\nu}(t_4)}{\psi(t_3)},\\ \dfrac{\ln(H/S_0) - \mu(t_4) - \bar{\varphi}(t_4,t_4)\psi(t_4)\bar{\nu}(t_4)}{\psi(t_4)}; \dfrac{\psi(t_1)}{\psi(t_2)}, \dfrac{\psi(t_2)}{\psi(t_3)}, \dfrac{\psi(t_3)}{\psi(t_4)}\end{array}\right)$$

$$\mathbb{P}_{21} = \Phi_4\left(\begin{array}{l}\dfrac{\ln(D_1/S_0)-\mu(t_1)-\tilde{\varphi}(t_1,t_4)\psi(t_1)\tilde{\nu}(t_4)}{\psi(t_1)},\\[2ex] \dfrac{\ln(D_2/S_0)-\mu(t_2)-\tilde{\varphi}(t_2,t_4)\psi(t_2)\tilde{\nu}(t_4)}{\psi(t_2)},\\[2ex] \dfrac{\ln(D_3/S_0)-\mu(t_3)-\tilde{\varphi}(t_3,t_4)\psi(t_3)\tilde{\nu}(t_4)}{\psi(t_3)},\\[2ex] \dfrac{\ln(H/S_0)-\mu(t_4)-\tilde{\varphi}(t_4,t_4)\psi(t_4)\tilde{\nu}(t_4)}{\psi(t_4)};\dfrac{\psi(t_1)}{\psi(t_2)},\dfrac{\psi(t_2)}{\psi(t_3)},\dfrac{\psi(t_3)}{\psi(t_4)}\end{array}\right)$$

where the following definitions hold:

$$\bar{\beta}(t_i) = \exp\left(-r(0)t_i - \int_0^{t_i}\int_0^u \theta(s)ds du + \frac{t_i^3((\upsilon_1+\upsilon_2\rho_{1.2})^2+\upsilon_2^2\phi_{2|1}^2)}{6}\right) \tag{4}$$

$$\tilde{\beta}(t_i) = \exp\left(-\lambda\kappa t_i - \frac{1}{2}\int_0^{t_i}\sigma^2(s)ds + \xi\sum_{k=1}^{i} n_k + \frac{1}{2}\left(\int_0^{t_i}\sigma^2(s)ds + \varepsilon^2\sum_{k=1}^{i} n_k\right)\right) \tag{5}$$

$$\mu(t_i) = r(0)t_i + \int_0^{t_i}\int_0^u \theta(s)dsdu - \lambda\kappa t_i - \frac{1}{2}\int_0^{t_i}\sigma^2(s)ds + \xi\sum_{k=1}^{i} n_k \tag{6}$$

$$\psi(t_i) = \left(\begin{array}{l}\varepsilon^2\displaystyle\sum_{k=1}^{i} n_k + \left(\frac{t_i^{3/2}(\upsilon_1+\upsilon_2\rho_{1.2})}{\sqrt{3}} + \rho_{1.3}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\right)^2\\ \quad + \left(\dfrac{t_i^{3/2}\upsilon_2\phi_{2|1}}{\sqrt{3}} + \phi_{2.3|1}\sqrt{\displaystyle\int_0^{t_i}\sigma^2(s)ds}\right)^2 + \phi_{3|1.2}^2\displaystyle\int_0^{t_i}\sigma^2(s)ds\end{array}\right)^{1/2} \tag{7}$$

$$\bar{\nu}(t_i) = \left(\frac{t_i^3((\upsilon_1+\upsilon_2\rho_{1.2})^2+\upsilon_2^2\phi_{2|1}^2)}{3}\right)^{1/2} \tag{8}$$

$$\tilde{\nu}(t_i) = \left(\int_0^{t_i}\sigma^2(s)ds + \varepsilon^2\sum_{k=1}^{i} n_k\right)^{1/2} \tag{9}$$

$$\bar{\varphi}(t_i,t_j) = \left(\left(-\frac{t_i^{3/2}(\upsilon_1+\upsilon_2\rho_{1.2})}{\sqrt{3}}\right)\left(\frac{t_i^{3/2}(\upsilon_1+\upsilon_2\rho_{1.2})}{\sqrt{3}}+\rho_{1.3}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\right)\right.$$
$$\left.+\left(\frac{t_i^{3/2}\upsilon_2\phi_{2|1}}{\sqrt{3}}+\phi_{2.3|1}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\right)\left(-\frac{t_i^{3/2}\upsilon_2\phi_{2|1}}{\sqrt{3}}\right)\right)\Big/(\psi(t_i)\bar{\nu}(t_j)) \tag{10}$$

$$\tilde{\varphi}(t_i,t_j) = \left(\rho_{1.3}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\left(\frac{t_i^{3/2}(\upsilon_1+\upsilon_2\rho_{1.2})}{\sqrt{3}}+\rho_{1.3}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\right)\right.$$
$$+\phi_{2.3|1}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\left(\frac{t_i^{3/2}\upsilon_2\phi_{2|1}}{\sqrt{3}}+\phi_{2.3|1}\sqrt{\int_0^{t_i}\sigma^2(s)ds}\right)$$
$$\left.+\phi_{3|1.2}^2\int_0^{t_i}\sigma^2(s)ds+\varepsilon^2\sum_{k=1}^{i}n_k\right)\Big/(\psi(t_i)\tilde{\nu}(t_j)) \tag{11}$$

$$\phi_{2|1}=\sqrt{1-\rho_{1.2}^2},\quad \phi_{2.3|1}=\frac{\rho_{2.3}-\rho_{1.2}\rho_{1.3}}{\phi_{2|1}},\quad \phi_{3|1.2}=\sqrt{1-\rho_{1.3}^2-\phi_{2.3|1}^2}. \tag{12}$$

The functions $\Phi[b_1]$ and $\Phi_2[b_1,b_2;\rho]$ are equal to the univariate and bivariate standard normal cumulative distribution functions, respectively. For $n>2$, the function $\Phi_n[b_1,\ldots,b_{n-1},b_n;\rho_1,\ldots,\rho_{n-2},\rho_{n-1}]$ is defined by the following integral of dimension n:

$$\Phi_n[b_1,\ldots,b_{n-1},b_n;\rho_1,\ldots,\rho_{n-2},\rho_{n-1}]$$
$$=\int_{x_1=-\infty}^{b_1}\cdots\int_{x_{n-1}=-\infty}^{b_{n-1}}\int_{x_n=-\infty}^{b_n}\frac{1}{(2\pi)^{n/2}\prod_{i=1}^{n-1}\sqrt{1-\rho_i^2}}$$
$$\times\exp\left(-\frac{x_1^2}{2}-\frac{1}{2}\sum_{i=1}^{n-1}\left(\frac{x_{i+1}-\rho_i x_i}{\sqrt{1-\rho_i^2}}\right)^2\right)dx_n dx_{n-1},\ldots,dx_1. \tag{13}$$

The numerical implementation of this formula is easy. First, the quadruple infinite series can be truncated in a simple manner by setting a convergence threshold such that no further terms are added once the difference between two successive finite sums of the quadruple series becomes smaller than that pre-specified level. The speed of convergence is inversely related to the intensity of the Poisson process. Numerical experiments were carried out that showed that, when λ was equal to 0.1, only three or four terms had to be computed in each summation operator for the percentage variation in two successive approximations of the quadruple infinite series to decrease below 0.0001%; when λ was equal to 1, seven terms were required to achieve the same level of convergence. It should be pointed out that the rate of decay of the denominator in (3) is "fast", so that numerical errors may arise for "large" values of n_1, n_2, n_3 and n_4. Fortunately, finance applications do not require the level of accuracy that might raise such concerns, i.e., a small number of terms in the infinite series will suffice to achieve adequate convergence for all practical purposes.

Next, the stated formula raises the question of the numerical computation of the function Φ_n when $n > 2$. After a little algebra, one can obtain two formulae that provide an efficient integration rule for the cases $n = 3$ and $n = 4$ required by Proposition 1:

$$\Phi_3[b_1, b_2, b_3; \rho_1, \rho_2] = \int_{x_2=-\infty}^{b_2} \frac{\exp(-x_2^2/2)}{\sqrt{2\pi}} \Phi\left[\frac{b_1 - \rho_1 x_2}{\sqrt{1-\rho_1^2}}\right] \Phi\left[\frac{b_3 - \rho_2 x_2}{\sqrt{1-\rho_2^2}}\right] dx_2 \tag{14}$$

$$\Phi_4[b_1, b_2, b_3, b_4; \rho_1, \rho_2, \rho_3] = \int_{x_2=-\infty}^{b_2} \int_{x_3=-\infty}^{\frac{b_3-\rho_2 x_2}{\sqrt{1-\rho_2^2}}} \frac{\exp\left(-\frac{(x_2^2+x_3^2)}{2}\right)}{2\pi} \Phi\left[\frac{b_1 - \rho_1 x_2}{\sqrt{1-\rho_1^2}}\right] \times \Phi\left[\frac{b_4 - \rho_3\sqrt{1-\rho_2^2}x_3 - \rho_3\rho_2 x_2}{\sqrt{1-\rho_3^2}}\right] dx_2 dx_3. \tag{15}$$

The functions to be integrated in (14) and (15) are so smooth that a mere 16-point Gauss–Legendre quadrature with a lower bound cut off at -5 is enough to achieve a high level of accuracy for pricing purposes. An adaptive Gauss–Legendre quadrature will allow users to determine exactly the approximation error resulting from numerical integration. Combining an adaptive Gauss–Legendre scheme with a Kronrod rule will reduce the number of required iterations (Davis and Rabinowitz, 2007). When $|\rho_2|$ is close to 1, the upper bound of the inner integral in (15) becomes very "large". As the parameters n_1, n_2, n_3 and n_4 of the quadruple infinite series increase, so too will the magnitude of the b_i arguments in the multivariate cumulative distribution functions to be computed. To avoid numerical errors, it is safe to prespecify a maximum value for all possible multiple integral endpoints. Given the standard normal nature of the integrands, bounding at an absolute value of 5 entails a loss of accuracy in the order of 10^{-9}. The fact that the arguments in the $\Phi[x]$ functions in (14) and (15) may rise sharply, however, is not a cause for concern since these univariate integrals can be numerically computed with almost arbitrary precision. Again, it is the smoothness of the Gaussian integrands that makes the numerical integration process very accurate and efficient, even for simple quadrature schemes. The total computational time required by the formula given in (3), on a mainstream commercial PC, ranges between 0.8 and 27 s, depending on the number of terms to be computed in each summation operator (from 4 to 8) and on the quality of the numerical integration scheme used, assuming that a 10^{-5} convergence threshold is set. Such an efficiency is in sharp contrast with the slowness of a Monte Carlo simulation approximation. For only a moderate level of accuracy to be achieved (10^{-3} divergence from the analytical formula), the latter requires between one and three hours of computational time, depending on the level of λ (from 0.1 to 1), and assuming that the maximum expiry of the autocallable structured product under consideration is $t_4 =$ four years. It must be emphasized that, compared with a standard geometric Brownian motion model with constant parameters, the introduction of additional stochastic factors such as jumps and a multi-factor term structure of interest rates heavily deteriorates the quality of Monte Carlo simulation approximations, as the payoff under consideration is path-dependent and requires time discretization. Hence the usefulness of accurate, quickly computed analytical formulae.

3. Proof of Formula

The proof relies on three lemmas called Lemma 1, Lemma 2 and Lemma 3. Unless stated otherwise, the notations used in this section have been defined in Sec. 2.

Lemma 1. *Let* $\{r(t), t \geq 0\}$ *and* $\{S(t), t \geq 0\}$ *be driven by Eqs.* (1) *and* (2), *respectively. Let* b *be a positive number. Then, given* $N(t)$, *we have*:

$$\mathbb{P}(S(t) \leq b) = \Phi\left(\frac{\ln(b/S_0) - \mu(t)}{\psi(t)}\right). \tag{16}$$

Proof of Lemma 1. Given $N(t)$, let $P = \mathbb{P}(S(t) \leq b) = \mathbb{P}(\ln(S(t)/S_0) \leq \ln(b/S_0))$.

For $t \geq 0$, the solution of Eq. (2) yields, conditional on $N(t)$:

$$\ln\left(\frac{S(t)}{S(0)}\right)\bigg|N(t) = \int_0^t r(s)ds - \lambda\kappa t - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_3(s) + \sum_{n=1}^{N(t)} J_n \tag{17}$$

where each J_n is a normal random variable with expectation ξ and variance ε^2, which will be denoted as follows:

$$J_n \sim \mathcal{N}(\xi; \varepsilon^2).$$

Integrating Eq. (1) yields:

$$r(t) = r(0) + \int_0^t \theta(s)ds + \upsilon_1 W_1(t) + \upsilon_2 W_2(t). \tag{18}$$

Let us denote by L_2 the Hilbert space of random variables with finite variance on (Ω, F, P). Then, given $W_1(t)$, it is well known that $W_2(t)$ admits the following orthogonal decomposition in L_2 (see e.g., Shreve, 2004):

$$W_2(t) = \rho_{1.2} W_1(t) + \phi_{2|1} B_2(t) \tag{19}$$

where B_2 is a standard Brownian motion defined on the same probability space as W_1 and W_2 and independent of W_1. Furthermore, for any given Brownian motion $\{W(s), s \geq 0\}$ defined on a suitable probability space and any fixed $t \geq 0$, we have

$$\int_0^t W(s)ds = \int_0^t (t-s)dW(s) \sim \mathcal{N}\left(0; \frac{t^3}{3}\right) \tag{20}$$

which implies that, for any given $x \in \mathbb{R}$:

$$\mathbb{P}\left(\int_0^t W(s)ds \leq x\right) = \mathbb{P}\left(\frac{t}{\sqrt{3}} W(t) \leq x\right) \tag{21}$$

Thus, as far as the computation of P is concerned, the integral $\int_0^t r(s)ds$ can be expanded as follows:

$$r(0)t + \int_0^t \int_0^u \theta(s)dsdu + \frac{t(\upsilon_1 + \upsilon_2 \rho_{1.2})}{\sqrt{3}} W_1(t) + \frac{t\upsilon_2 \phi_{2|1}}{\sqrt{3}} B_2(t). \tag{22}$$

Next, given $W_1(t)$ and $W_2(t)$, the random variable $W_3(t)$ admits an orthogonal decomposition in L_2 as follows:

$$W_3(t) = \rho_{1.3} W_1(t) + a_2 B_2(t) + a_3 B_3(t) \tag{23}$$

where a_2 and a_3 are real-valued scalars and B_3 is a standard Brownian motion defined on the same probability space as the processes W_1, W_2 and W_3 and independent of the processes W_1, W_2 and B_2. Note that a_3 must be positive since, by definition of the multivariate normal random vector $[W_1(t), W_2(t), W_3(t)]$, $a_3\sqrt{t}$ is the standard deviation of $W_3(t)$ conditional on $W_1(t)$ and $W_2(t)$ From the definition of linear correlation and the bilinearity of covariance, we obtain:

$$\begin{aligned}
\rho_{2.3} &= \frac{\mathrm{cov}[W_2(t), W_3(t)]}{t} \\
&= \frac{1}{t}(\mathrm{cov}[\rho_{1.2} W_1(t), \rho_{1.3} W_1(t)] + \mathrm{cov}[\phi_{2|1} B_2(t), a_2 B_2(t)]) \\
\leftrightarrow a_2 &= \frac{\rho_{2.3} - \rho_{1.2}\rho_{1.3}}{\phi_{2|1}} \triangleq \phi_{2.3|1}.
\end{aligned} \tag{24}$$

The real number $\phi_{2.3|1}$ is the partial correlation between $W_2(t)$ and $W_3(t)$ conditional on $W_1(t)$. Next, from the standard deviation of $W_3(t)$ and the stability under addition of a set of uncorrelated normal random variables, we obtain

$$\sqrt{t(\rho_{1.3}^2+\phi_{2.3|1}^2+a_3^2)}=\sqrt{t} \leftrightarrow a_3=\sqrt{1-\rho_{1.3}^2-\phi_{2.3|1}^2} \triangleq \phi_{3|1.2} \tag{25}$$

Now, as the function $\sigma(t)$ is non-random, the isometry property of the Ito integral implies that the random variable $\int_0^t \sigma(s)dW_3(s)$ is normally distributed with mean zero and variance $\int_0^t \sigma^2(s)ds$, $\forall\, t \geq 0$.

Hence, with regard to the computation of P, the integral $\int_0^t \sigma(s)dW_3(s)$ can be expanded as

$$\sqrt{\frac{1}{t}\int_0^t \sigma^2(s)ds}(W_1(t)\rho_{1.3}+B_2(t)\phi_{2.3|1}+B_3(t)\phi_{3|1.2}). \tag{26}$$

Since the random variables J_n are pairwise independent and are also independent of $W_1(t)$, $W_2(t)$ and $W_3(t)$, the following equality thus holds in distribution for any given $t \geq 0$:

$$\begin{aligned}\ln\left(\frac{S(t)}{S(0)}\right)\Bigg|N(t) &= r(0)t+\int_0^t\int_0^u \theta(s)dsdu-\lambda\kappa t-\frac{1}{2}\int_0^t \sigma^2(s)ds+N(t)\xi\\ &\quad+Z_1\left(\frac{t^{3/2}(\upsilon_1+\upsilon_2\rho_{1.2})}{\sqrt{3}}+\rho_{1.3}\sqrt{\int_0^t \sigma^2(s)ds}\right)\\ &\quad+Z_2\left(\frac{t^{3/2}\upsilon_2\phi_{2|1}}{\sqrt{3}}+\phi_{2.3|1}\sqrt{\int_0^t \sigma^2(s)ds}\right)\\ &\quad+Z_3\phi_{3|1.2}\sqrt{\int_0^t \sigma^2(s)ds}+\varepsilon\sqrt{N(t)}Z_4\end{aligned} \tag{27}$$

where Z_1, Z_2, Z_3 and Z_4 are four uncorrelated standard normal random variables.

Consequently, given $N(t)$, the random variable $\ln(\frac{S(t)}{S(0)})$ is normally distributed with mean given by $\mu(t)$ and standard deviation given by $\psi(t)$. □

Lemma 2. *Let $\{r(t), t \geq 0\}$ and $\{S(t), t \geq 0\}$ be driven by Eqs.* (1) *and* (2), *respectively. Let t_1,t_2,t_3,t_4 be four non-random times such that*: $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$. *Let b_1, b_2, b_3 and b_4 be four positive numbers. Then,*

$$\begin{aligned}\mathbb{P}(S(t_1)\leq b_1, S(t_2)\leq b_2, S(t_3)\leq b_3, S(t_4)\leq b_4) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\\ &\times\exp(-\lambda t_4)\frac{\lambda^{n_1+n_2+n_3+n_4}t_1^{n_1}(t_2-t_1)^{n_2}(t_3-t_2)^{n_3}(t_4-t_3)^{n_4}}{n_1!n_2!n_3!n_4!}\end{aligned}$$

$$\times \Phi_4 \left(\begin{array}{l} \frac{\ln(b_1/S_0) - \mu(t_1)}{\psi(t_1)}, \frac{\ln(b_2/S_0) - \mu(t_2)}{\psi(t_2)}, \\ \frac{\ln(b_3/S_0) - \mu(t_3)}{\psi(t_3)}, \frac{\ln(b_4/S_0) - \mu(t_4)}{\psi(t_4)}; \frac{\psi(t_1)}{\psi(t_2)}, \frac{\psi(t_2)}{\psi(t_3)}, \frac{\psi(t_3)}{\psi(t_4)} \end{array} \right). \quad (28)$$

Proof of Lemma 2. Given $N(t_4)$, let $P = \mathbb{P}(S(t_1) \leq b_1, S(t_2) \leq b_2, S(t_3) \leq b_3, S(t_4) \leq b_4)$. The first step is to shift to log coordinates as in the Proof of Lemma 1. Each random variable $\ln(S(t_i)/S_0), i \in \{1, 2, 3, 4\}$, is continuous. Hence, one can express P as the following quadruple integral

$$P = \int_{-\infty}^{\ln(b_1/S_0)} \int_{-\infty}^{\ln(b_2/S_0)} \int_{-\infty}^{\ln(b_3/S_0)} \int_{-\infty}^{\ln(b_4/S_0)} dx_4 dx_3 dx_2 dx_1$$
$$\mathbb{P}\left(\ln\left(\frac{S(t_1)}{S_0}\right) \in dx_1, \ln\left(\frac{S(t_2)}{S_0}\right) \in dx_2, \ln\left(\frac{S(t_3)}{S_0}\right) \in dx_3, \ln\left(\frac{S(t_4)}{S_0}\right) \in dx_4 \right). \quad (29)$$

Then, by conditioning and using the weak Markov property of the process $\{S(t), t \geq 0\}$, we have

$$P = \int_{-\infty}^{\ln(b_1/S_0)} \int_{-\infty}^{\ln(b_2/S_0)} \int_{-\infty}^{\ln(b_3/S_0)} \int_{-\infty}^{\ln(b_4/S_0)} dx_4 dx_3 dx_2 dx_1$$
$$\mathbb{P}\left(\ln\left(\frac{S(t_1)}{S_0}\right) \in dx_1 \right) \times \mathbb{P}\left(\ln\left(\frac{S(t_2)}{S_0}\right) \in dx_2 \middle| \ln\left(\frac{S(t_1)}{S_0}\right) \in dx_1 \right)$$
$$\times \mathbb{P}\left(\ln\left(\frac{S(t_3)}{S_0}\right) \in dx_3 \middle| \ln\left(\frac{S(t_2)}{S_0}\right) \in dx_2 \right)$$
$$\times \mathbb{P}\left(\ln\left(\frac{S(t_4)}{S_0}\right) \in dx_4 \middle| \ln\left(\frac{S(t_3)}{S_0}\right) \in dx_3 \right). \quad (30)$$

Using (27), it is straightforward to show that the correlation between $\ln(S(t_i)/S_0)$ and $\ln(S(t_j)/S_0)$, $i < j$, is equal to the standard deviation of $\ln(S(t_i)/S_0)$ divided by the standard deviation of $\ln(S(t_j)/S_0)$. Thus, from the definition of the bivariate normal distribution, we have

$$\mathbb{P}\left(\ln\left(\frac{S(t_j)}{S_0}\right) \in dx_j \middle| \ln\left(\frac{S(t_i)}{S_0}\right) \in dx_i \right) = \frac{1}{\psi(t_j)\sqrt{2\pi(1 - \psi^2(t_i)/\psi^2(t_j))}}$$
$$\times \exp\left(-\frac{1}{2(1 - \psi^2(t_i)/\psi^2(t_j))} \left(\frac{x_j - \mu(t_j)}{\psi(t_j)} - \frac{\psi(t_i)}{\psi(t_j)} \frac{x_i - \mu(t_i)}{\psi(t_i)} \right)^2 \right). \quad (31)$$

Hence, (30) becomes:

$$P = \int_{-\infty}^{\frac{\ln(b_1/S_0)-\mu(t_1)}{\psi(t_1)}} \int_{-\infty}^{\frac{\ln(b_2/S_0)-\mu(t_2)}{\psi(t_2)}} \int_{-\infty}^{\frac{\ln(b_3/S_0)-\mu(t_3)}{\psi(t_3)}} \int_{-\infty}^{\frac{\ln(b_4/S_0)-\mu(t_4)}{\psi(t_4)}} dx_4 dx_3 dx_2 dx_1$$
$$\frac{1}{4\pi^2 \prod_{i=1}^{3} \sqrt{1-\psi^2(t_i)/\psi^2(t_{i+1})}}$$
$$\times \exp\left(-\frac{x_1^2}{2} - \frac{1}{2}\sum_{i=1}^{3}\left(\frac{x_{i+1}-(\psi(t_i)/\psi(t_{i+1}))x_i}{\sqrt{1-\psi^2(t_i)/\psi^2(t_{i+1})}}\right)^2\right). \tag{32}$$

Summing over all possible values of the process $N(t)$ in each time interval $[t_i, t_{i+1}]$ and multiplying by their respective probabilities of occurrence yields Lemma 2.

The smaller-dimensional case of $\mathbb{P}(S(t_1) \le b_1, S(t_2) \le b_2, S(t_3) \le b_3)$ is nested by Lemma 2 in an obvious manner. □

Lemma 3. *Let Y be a normal random variable with mean μ_Y and variance σ_Y^2. Let $[X_1, X_2, X_3, X_4]$ be a quadrivariate normal random vector such that*

$$\begin{aligned}&\mathbb{P}(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3, X_4 \le x_4)\\ &\quad = \Phi_4\left(\frac{x_1-\mu_{X_1}}{\sigma_{X_1}}, \frac{x_2-\mu_{X_2}}{\sigma_{X_2}}, \frac{x_3-\mu_{X_3}}{\sigma_{X_3}}, \frac{x_4-\mu_{X_4}}{\sigma_{X_4}}; \theta_{X_1.X_2}, \theta_{X_2.X_3}, \theta_{X_3.X_4}\right)\end{aligned} \tag{33}$$

where μ_{X_i} and σ_{X_i} are the mean and the standard deviation of X_i, respectively, and $\theta_{X_i.X_j}$ is the correlation coefficient between X_i and X_j, $\forall\,(i,j) \in \mathbb{N}^2$. Then, if the random vector $[Y, X_1, X_2, X_3, X_4]$ is multivariate normal, we have

$$\begin{aligned}&E[\exp(Y)\mathbb{I}_{\{X_1<x_1, X_2<x_2, X_3<x_3, X_4<x_4\}}] = \exp\left(\mu_Y + \frac{\sigma_Y^2}{2}\right)\Phi_4\\ &\quad\times\left(\begin{aligned}&\frac{x_1-\mu_{X_1}-\theta_{X_1.Y}\sigma_{X_1}\sigma_Y}{\sigma_{X_1}}, \frac{x_2-\mu_{X_2}-\theta_{X_2.Y}\sigma_{X_2}\sigma_Y}{\sigma_{X_2}},\\ &\frac{x_3-\mu_{X_3}-\theta_{X_3.Y}\sigma_{X_3}\sigma_Y}{\sigma_{X_3}}, \frac{x_4-\mu_{X_4}-\theta_{X_4.Y}\sigma_{X_4}\sigma_Y}{\sigma_{X_4}};\\ &\theta_{X_1.X_2}, \theta_{X_2.X_3}, \theta_{X_3.X_4}\end{aligned}\right)\end{aligned} \tag{34}$$

where $\theta_{X_i.Y}$ is the correlation coefficient between X_i and Y.

Proof of Lemma 3. Let $f_{\bar{Y},\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4}(y, x_1, x_2, x_3, x_4)$ denote the joint density of the standardized random vector $[\bar{Y}, \bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4]$, where $\bar{Y} = \frac{Y-\mu_Y}{\sigma_Y}$ and $\bar{X}_i = \frac{X_i-\mu_{X_i}}{\sigma_{X_i}}$.

Then,

$$E[\exp(Y)\mathbb{I}_{\{X_1<x_1,X_2<x_2,X_3<x_3,X_4<x_4\}}]$$
$$= \int_{y=-\infty}^{\infty}\int_{z_1=-\infty}^{\frac{x_1-\mu_{X_1}}{\sigma_{X_1}}}\int_{z_2=-\infty}^{\frac{x_2-\mu_{X_2}}{\sigma_{X_2}}}\int_{z_3=-\infty}^{\frac{x_3-\mu_{X_3}}{\sigma_{X_3}}}\int_{z_4=-\infty}^{\frac{x_4-\mu_{X_4}}{\sigma_{X_4}}}$$
$$\times \exp(y) f_{\bar{Y},\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4}(y,z_1,z_2,z_3,z_4)dydz_4dz_3dz_2dz_1. \quad (35)$$

Following the same approach as in the Proof of Lemma 1, a little algebra shows that the normal random variables $\bar{X}_i$, $i \in \{1,2,3,4\}$, admit the following orthogonal decompositions:

$$\bar{X}_1 = \theta_{\bar{X}_1.\bar{Y}}\bar{Y} + \phi_{\bar{X}_1|\bar{Y}}Z_1 \quad (36)$$
$$\bar{X}_2 = \theta_{\bar{X}_2.\bar{Y}}\bar{Y} + \theta_{\bar{X}_1.\bar{X}_2|\bar{Y}}Z_1 + \phi_{\bar{X}_2|\bar{Y}.\bar{X}_1}Z_2 \quad (37)$$
$$\bar{X}_3 = \theta_{\bar{X}_3.\bar{Y}}\bar{Y} + \theta_{\bar{X}_1.\bar{X}_3|\bar{Y}}Z_1 + \theta_{\bar{X}_2.\bar{X}_3|\bar{Y}.\bar{X}_1}Z_2 + \phi_{\bar{X}_3|\bar{Y}.\bar{X}_1.\bar{X}_2}Z_3 \quad (38)$$
$$\bar{X}_4 = \theta_{\bar{X}_4.\bar{Y}}\bar{Y} + \theta_{\bar{X}_1.\bar{X}_4|\bar{Y}}Z_1 + \theta_{\bar{X}_2.\bar{X}_4|\bar{Y}.\bar{X}_1}Z_2 + \theta_{\bar{X}_3.\bar{X}_4|\bar{Y}.\bar{X}_1.\bar{X}_2}Z_3$$
$$+ \phi_{\bar{X}_4|\bar{Y}.\bar{X}_1.\bar{X}_2.\bar{X}_3}Z_4 \quad (39)$$

where the Z_i's are pairwise independent standard normal random variables that are all independent of $\bar{Y}$ and where the following definitions hold:

$$\phi_{\bar{X}_a|\bar{Y}} = \sqrt{1-\theta^2_{\bar{X}_a.\bar{Y}}}, \quad \theta_{\bar{X}_a.\bar{X}_b|\bar{Y}} = \frac{\theta_{\bar{X}_a.\bar{X}_b} - \theta_{\bar{X}_a.\bar{Y}}\theta_{\bar{X}_b.\bar{Y}}}{\phi_{\bar{X}_a|\bar{Y}}},$$
$$\phi_{\bar{X}_b|\bar{Y}.\bar{X}_a} = \sqrt{1-\theta^2_{\bar{X}_b.\bar{Y}} - \theta^2_{\bar{X}_a.\bar{X}_b|\bar{Y}}} \quad (40)$$
$$\theta_{\bar{X}_b.\bar{X}_c|\bar{Y}.\bar{X}_a} = \frac{\theta_{\bar{X}_b.\bar{X}_c} - \theta_{\bar{X}_b.\bar{Y}}\theta_{\bar{X}_c.\bar{Y}} - \theta_{\bar{X}_a.\bar{X}_b|\bar{Y}}\theta_{\bar{X}_a.\bar{X}_c|\bar{Y}}}{\phi_{\bar{X}_b|\bar{Y}.\bar{X}_a}} \quad (41)$$
$$\phi_{\bar{X}_c|\bar{Y}.\bar{X}_a.\bar{X}_b} = \sqrt{1-\theta^2_{\bar{X}_b.\bar{Y}} - \theta^2_{\bar{X}_a.\bar{X}_b|\bar{Y}} - \theta^2_{\bar{X}_b.\bar{X}_c|\bar{Y}.\bar{X}_a}} \quad (42)$$
$$\theta_{\bar{X}_c.\bar{X}_d|\bar{Y}.\bar{X}_a.\bar{X}_b} = \frac{1}{\phi_{\bar{X}_c|\bar{Y}.\bar{X}_a.\bar{X}_b}}(\theta_{\bar{X}_c.\bar{X}_d} - \theta_{\bar{X}_c.Y}\theta_{\bar{X}_d.Y} - \theta_{\bar{X}_a.\bar{X}_d|Y}\theta_{\bar{X}_a.\bar{X}_c|Y}$$
$$- \theta_{\bar{X}_b.\bar{X}_d|Y.\bar{X}_a}\theta_{\bar{X}_b.\bar{X}_c|Y.\bar{X}_a}) \quad (43)$$
$$\phi_{\bar{X}_d|\bar{Y}.\bar{X}_a.\bar{X}_b.\bar{X}_c} = \sqrt{1-\theta^2_{\bar{X}_b.\bar{Y}} - \theta^2_{\bar{X}_a.\bar{X}_b|\bar{Y}} - \theta^2_{\bar{X}_b.\bar{X}_c|\bar{Y}.\bar{X}_a} - \theta^2_{\bar{X}_c.\bar{X}_d|\bar{Y}.\bar{X}_a.\bar{X}_b}}. \quad (44)$$

Thus, we have

$$\bar{X}_1|\bar{Y} \sim \mathcal{N}(\theta_{\bar{X}_1.\bar{Y}}\bar{Y}; \phi_{\bar{X}_1|\bar{Y}}) \quad (45)$$
$$\bar{X}_2|\bar{Y},\bar{X}_1 \sim \mathcal{N}\left(\theta_{\bar{X}_2.\bar{Y}}\bar{Y} + \theta_{\bar{X}_2.\bar{X}_1|\bar{Y}}\frac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}}; \phi_{\bar{X}_2|\bar{Y}.\bar{X}_1}\right) \quad (46)$$

$$\bar{X}_3|\bar{Y},\bar{X}_1,\bar{X}_2 \sim \mathcal{N}\left(\begin{array}{l}\theta_{\bar{X}_3.\bar{Y}}\bar{Y} + \theta_{\bar{X}_3.\bar{X}_1|\bar{Y}}\dfrac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}} + \dfrac{\theta_{\bar{X}_3.\bar{X}_2|\bar{Y}.\bar{X}_1}}{\phi_{\bar{X}_2|\bar{Y}.\bar{X}_1}} \\ \left(\bar{X}_2 - \theta_{\bar{X}_2.\bar{Y}}\bar{Y} - \theta_{\bar{X}_2.\bar{X}_1|\bar{Y}}\dfrac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}}\right); \phi_{\bar{X}_3|\bar{Y}.\bar{X}_1.\bar{X}_2}\end{array}\right) \tag{47}$$

$$\begin{aligned}&\bar{X}_4|\bar{Y},\bar{X}_1,\bar{X}_2,\bar{X}_3 \\ &\quad\sim \mathcal{N}\Bigg(\theta_{\bar{X}_4.\bar{Y}}\bar{Y} + \theta_{\bar{X}_4.\bar{X}_1|\bar{Y}}\frac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}} + \frac{\theta_{\bar{X}_4.\bar{X}_2|\bar{Y}.\bar{X}_1}}{\phi_{\bar{X}_2|\bar{Y}.\bar{X}_1}} \\ &\qquad\left(\bar{X}_2 - \theta_{\bar{X}_2.\bar{Y}}\bar{Y} - \theta_{\bar{X}_2.\bar{X}_1|\bar{Y}}\frac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}}\right) + \frac{\theta_{\bar{X}_4.\bar{X}_3|\bar{Y}.\bar{X}_1.\bar{X}_2}}{\phi_{\bar{X}_3|\bar{Y}.\bar{X}_1.\bar{X}_2}} \\ &\qquad\Bigg(\bar{X}_3 - \theta_{\bar{X}_3.\bar{Y}}\bar{Y} - \theta_{\bar{X}_3.\bar{X}_1|\bar{Y}}\frac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}} - \frac{\theta_{\bar{X}_3.\bar{X}_2|\bar{Y}.\bar{X}_1}}{\phi_{\bar{X}_2|\bar{Y}.\bar{X}_1}} \\ &\qquad\left(\bar{X}_2 - \theta_{\bar{X}_2.\bar{Y}}\bar{Y} - \theta_{\bar{X}_2.\bar{X}_1|\bar{Y}}\frac{\bar{X}_1 - \theta_{\bar{X}_1.\bar{Y}}\bar{Y}}{\phi_{\bar{X}_1|\bar{Y}}}\right)\Bigg); \phi_{\bar{X}_4|\bar{Y}.\bar{X}_1.\bar{X}_2.\bar{X}_3}\Bigg).\end{aligned} \tag{48}$$

From (45)–(48), one can derive $f_{\bar{Y},\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4}(y,z_1,z_2,z_3,z_4)$ as the following product of conditional densities:

$$\begin{aligned}&f_{\bar{Y},\bar{X}_1,\bar{X}_2,\bar{X}_3,\bar{X}_4}(y,z_1,z_2,z_3,z_4) \\ &\quad= f_{\bar{Y}}(y)f_{\bar{X}_1|\bar{Y}}(z_1|y)f_{\bar{X}_2|\bar{Y},\bar{X}_1}(z_2|y,z_1)f_{\bar{X}_3|\bar{Y},\bar{X}_1,\bar{X}_2}(z_3|y,z_1,z_2) \\ &\qquad\times f_{\bar{X}_4|\bar{Y},\bar{X}_1,\bar{X}_2,\bar{X}_3}(z_4|y,z_1,z_2,z_3).\end{aligned} \tag{49}$$

Substituting (49) into (35) and performing the necessary calculations then yields Lemma 3. □

The proof of the formula given in (3) can now proceed. According to the no-arbitrage theory of valuation in a complete market (Harrison and Pliska, 1981), the price of the contingent claim under consideration is equal to the expectation of its discounted payoff under the equivalent martingale measure. But the market here is incomplete, due to the introduction of jumps. This raises the question of the choice of a relevant equivalent martingale measure. A necessary condition for the discounted price process $\{\exp(-\int_0^t r(s)ds)S(t), t \geq 0\}$ to be a martingale is to set the drift coefficient in the dynamics of $\{S(t), t \geq 0\}$ equal to $r(t) - \lambda\kappa$. This can be shown in exactly the same way as when the riskless rate process $\{r(t), t \geq 0\}$ is constant (see e.g., Lamberton and Lapeyre, 1997), so the details are omitted.

According to the classical argument by Merton (1976) that jump risk is diversifiable and therefore not rewardable with excess return, this condition should be both necessary and sufficient. If we consider this argument to be true, then the dynamics of $\{S(t), t \geq 0\}$ can only be given by Eq. (2). However, one must bear in mind that this argument is debatable, as empirical evidence suggests that there are industry wide shocks and even country wide shocks that are not easily diversifiable, so that the possibility of perfect hedging remains theoretical. The literature on mean-variance hedging (Schweizer, 1992) and quantile hedging (Föllmer and Leukert, 1999) can be consulted for alternative approaches.

Denoting by $\mathbb{Q}$ the probability measure under which Eq. (2) holds, and assuming that this is the correct pricing measure, then the autocallable payoff defined in Tables 1 and 2 implies that, at each fixing date t_j, $j \in \{1, 2, 3\}$, prior to maximum expiry t_4, the following expectations must be computed:

$$\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\left(-\int_0^{t_j} r(s)ds\right) \times M \times (1 + y_j) \\ \times \mathbb{I}_{\{S(t_1)\leq D_1,\ldots,S(t_{j-1})\leq D_{j-1},D_j\leq S(t_j)\leq U_j\}}|N(t_j)\Bigg] \tag{50}$$

$$\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\left(-\int_0^{t_j} r(s)ds\right) \times M \times (1 + \alpha) \times \frac{S(t_j)}{S(0)} \\ \times \mathbb{I}_{\{S(t_1)\leq D_1,\ldots,S(t_{j-1})\leq D_{j-1},S(t_j)>U_j\}}|N(t_j)\Bigg] \tag{51}$$

$$\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\left(-\int_0^{t_j} r(s)ds\right) \times M \times z_j \\ \times \mathbb{I}_{\{S(t_1)\leq D_1,\ldots,S(t_{j-1})\leq D_{j-1},C_j\leq S(t_j)\leq D_j\}}|N(t_j)\Bigg]. \tag{52}$$

At maximum expiry t_4, the following expectation is also required:

$$\mathbb{E}_{\mathbb{Q}}\Bigg[\exp\left(-\int_0^{t_4} r(s)ds\right) \times M \times \frac{S(t_4)}{S(0)} \\ \times \mathbb{I}_{\{S(t_1)\leq D_1,S(t_2)\leq D_2,S(t_3)\leq D_3,S(t_4)\leq H\}}|N(t_4)\Bigg]. \tag{53}$$

It is clear that the valuation problem comes down to computing two kinds of expectations, denoted by E_1 and E_2, $\forall\,(i,j) \in \{1, 2, 3, 4\}, i \leq j$:

$$E_1 = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_j} r(s)ds\right) \times \mathbb{I}_{\{[S(t_i)]\leq[c_i]\}}|N(t_j)\right] \tag{54}$$

$$E_2 = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^{t_j} r(s)ds\right) \times \frac{S(t_j)}{S(0)} \times \mathbb{I}_{\{[S(t_i)]\leq[c_i]\}}|N(t_j)\right] \tag{55}$$

where the symbol [.] inside the indicator functions is a vector notation and where the components of $[c_i]$, $0 < i \leq j$, are positive constants.

Thus, the value of the contingent claim under consideration can be obtained by applying Lemma 3, provided the right parameters are entered into the formula.

Let $Y \triangleq -\int_0^{t_j} r(s)ds$ and let $X(t_i) = \ln(\frac{S(t_i)}{S(0)})|N(t_i)$. From Eq. (22), the expectation of Y is given by

$$-r(0)t_j - \int_0^{t_j}\int_0^u \theta(s)dsdu \tag{56}$$

while the standard deviation of Y is given by $\bar{\nu}(t_j)$.
From Eq. (27), the expectation and standard deviation of $X(t_i)$ are given by $\mu(t_i)$ and by $\psi(t_i)$, respectively. Hence, the covariance between Y and $X(t_i)$ is given by the numerator in $\bar{\varphi}(t_i, t_j)$.

Now, let $Y \triangleq -\int_0^{t_j} r(s)ds + \ln(\frac{S(t_j)}{S(0)})|N(t_j)$. Then, the expectation and the variance of Y are given, respectively, by

$$E[Y] = -\lambda\kappa t_j - \frac{1}{2}\int_0^{t_j} \sigma^2(s)ds + \xi\sum_{k=1}^{j} n_k \tag{57}$$

$$\text{var}[Y] = \int_0^{t_j} \sigma^2(s)ds + \varepsilon^2\sum_{k=1}^{j} n_k. \tag{58}$$

Hence, the covariance between Y and $X(t_i)$ is given by the numerator in $\tilde{\varphi}(t_i, t_j)$.

Adding all required E_1-type and E_2-type expectations together and then summing over all possible values of the process $N(t)$ in each time interval $[t_i, t_{i+1}]$ weighted by their respective probabilities of occurrence, one can obtain the formula given in (3).

It must be emphasized that the validity of the proof depends on the multivariate normality of the random vector $[Y, X_1, X_2, X_3, X_4]$ in Lemma 3. It is well known that the univariate normality of each marginal does not imply the multivariate normality of the joint density (see e.g., Tong, 1990).

4. Conclusion

In this paper, a general autocallable structure with four possible exit dates is analytically priced, under the assumption that equity prices follow a jump-diffusion model and that interest rates are driven by a two-factor stochastic differential equation. It is quite straightforward to extend these analytical results to a number of observation dates greater than four, using the same approach as in Sec. 3.

However, the resulting formulae become more cumbersome. Moreover, for large values of n, the efficiency of the analytical approach developed in this paper deteriorates, as the dimension of the multiple series in the formula rises. It should be emphasized, though, that the vast majority of the autocallable structured products traded in the markets have a moderate number of exit dates. Typically, expirations range between two and five years, with early exit allowed at the end of each year. Therefore, most traded contracts can be tackled with the valuation approach developed in this paper. Formulae for five and six observation observation dates were not given in this paper but they are available from the author upon request, along with efficient numerical quadrature rules similar to those given in Eqs. (14) and (15).

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