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Computation of the quadrivariate and pentavariate normal cumulative distribution functions

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ABSTRACT
This article provides explicit integration rules for the quadrivariate and the pentavariate normal distribution. By analytically reducing the dimension of the problem and simplifying the functions to be integrated, these rules form the basis for a numerical evaluation scheme yielding an observed maximum error in the order of 10^{-7} and a computational time of less than 10^{-6} s. The implementation is very straightforward as it is based on a classical Gauss–Legendre quadrature. Order statistics are also dealt with.

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1. Introduction

The need for a fast and accurate numerical evaluation of the multivariate normal cumulative distribution function arises in many statistical applications; numerous examples can be found, e.g., in Genz & Bretz (2009), such as multiple comparison procedures, integrated log-likelihood problems, applications of the multivariate probit model, and Bayesian statistics. The computation of multivariate normal integrals is also of central importance in many other scientific subjects, from computational physics to mathematical finance. Yet, the famous "curse of dimensionality" makes this problem uneasy. The rise of modern computer technology has fostered the development of computationally intensive algorithms, most of them based on Monte Carlo methods. For a survey of this vast area of research, the reader is referred to Kotz, Balakrishnan & Johnson (2000) and Genz & Bretz (2009). The main advantage of Monte Carlo integration is that it can cope with high dimension. The well-known problems associated with this approach are its relatively poor accuracy and its slow rate of convergence, so that, in low or moderate dimension, it may not be competitive compared to quadrature-based methods. The latter are all the more powerful as they draw on preliminary analytical efforts to reduce dimension or simplify function evaluations, instead of relying solely on brute force computing power. In this respect, specific algorithms have been devised to tackle the bivariate and trivariate normal integrals, based on analytics rather than pure numerics. In the bivariate case, Drezner and Wesolowsky (1990) developed a semi-analytical scheme that rapidly became the standard reference, and that was slightly improved on by Genz (2004). Based on seminal work by Plackett (1954), Drezner (1994) handled the trivariate case. The numerical properties of Plackett's method are analyzed in

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detail by Gassmann (2003), in comparison with alternative numerical techniques described by Gassmann et al. (2002). Genz (2004) developed an alternative analytical approach for the trivariate normal integral, based on a reduction formula by Owen (1956). No analogous exact rules of integration, based on analytical dimension reduction, have yet been published in the quadrivariate and pentavariate cases. The purpose of this article is to provide these rules, as well as a closed form formula for the order statistics associated with a set of up to five correlated normal random variables. So far, the analytical results known about the evaluation of the quadrivariate normal integral have had limited scope. They deal with orthant probabilities, i.e., probabilities that all four correlated normal random variables have the same sign. A few exact results about quadrivariate normal orthant probabilities have been found, not in general but for special correlation matrices (David and Mallows, 1961; Sondhi, 1961; Cheng, 1969; Poznyakov, 1971); other contributions have focused on analytical approximations of orthant quadrivariate normal probabilities for general correlation structure (McFadden, 1960; Abrahamson, 1964; Gerhlein, 1979; Drezner, 1990). More recently, Sinn and Keller (2011) expressed the quadrivariate normal orthant probability as the sum of four one-dimensional integrals. As for the evaluation of the pentavariate normal integral, no exact result can yet be cited, to the best of our knowledge; at most, one can mention David (1953), pointing out how to derive the pentavariate normal orthant probability from the quadrivariate normal one, assuming the latter is already known. Hence, the need for an exact, general formula, that is not contingent on a particular correlation structure or limited to the special case of the orthant probability, and that admits a simple, fast and accurate numerical implementation.

This article is organized as follows. In Section 2, the main results (Propositions 1–3) are stated; Section 3 provides numerical results; Section 4 deals with the proof of the main results.

2. Main results

Proposition 1 (pentavariate standard normal integral in rectangular coordinates). *Let $[X_1, X_2, X_3, X_4, X_5]$ be a multivariate standard normal random vector, with each pairwise correlation denoted by $\rho_{i.j}$, $(i, j) \in \{1, \ldots, 4\} \times \{2, \ldots, 5\}$, $i < j$. Let b_1, b_2, b_3, b_4 and b_5 be five real numbers. Let $P(.)$ denote the probability operator. Then, the five-dimensional multivariate (or pentavariate) standard normal integral is given by:*

$$
\begin{aligned}
&P\left(X_1 \leq b_1, X_2 \leq b_2, X_3 \leq b_3, X_4 \leq b_4, X_5 \leq b_5\right) \\
&\quad \triangleq N_5\left[b_1, b_2, b_3, b_4, b_5; \rho_{1.2}, \rho_{1.3}, \rho_{1.4}, \rho_{1.5}, \rho_{2.3}, \rho_{2.4}, \rho_{2.5}, \rho_{3.4}, \rho_{3.5}, \rho_{4.5}\right] \\
&\quad = \int_{x_1=-\infty}^{b_1} \int_{x_2=-\infty}^{f_1(x_1)} \int_{x_3=-\infty}^{f_2(x_1,x_2)} \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{\left(x_1^2+x_2^2+x_3^2\right)}{2}\right) \\
&\qquad \times N_2\left[f_3\left(x_1, x_2, x_3\right), f_4\left(x_1, x_2, x_3\right); \frac{\rho_{4.5|1.2.3}}{\sigma_{5|1.2.3}}\right] dx_3 dx_2 dx_1 \qquad (1) \\
&\quad = \int_{x_1=-\infty}^{b_1} \int_{x_2=-\infty}^{f_1(x_1)} \int_{x_3=-\infty}^{f_2(x_1,x_2)} \int_{x_4=-\infty}^{f_3(x_1,x_2,x_3)} \frac{1}{4\pi^2} \exp\left(-\frac{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}{2}\right) \\
&\qquad \times N\left[f_5\left(x_1, x_2, x_3, x_4\right)\right] dx_4 dx_3 dx_2 dx_1 \qquad (2)
\end{aligned}
$$

where:

$N_2[\alpha, \beta; \theta]$ is the bivariate standard normal integral with upper bounds α and β and correlation coefficient θ.

$N[\alpha]$ is the univariate standard normal integral with upper bound α.

The functions f_1, f_2, f_3, f_4, and f_5 in (2.1) and (2.2) are defined by:

$$f_1(x_1) = \frac{b_2 - \rho_{1.2}x_1}{\sigma_{2|1}},\ \ f_2(x_1, x_2) = \left(\frac{b_3 - \rho_{1.3}x_1}{\sigma_{3|1}} - \frac{\rho_{2.3|1}}{\sigma_{3|1}}x_2\right) \Big/ \sqrt{1 - \rho^2_{2.3|1}/\sigma^2_{3|1}} \tag{3}$$

$$f_3(x_1, x_2, x_3) = \left(\frac{b_4 - \rho_{1.4}x_1}{\sigma_{4|1.2}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}x_2 - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}x_3\right) \Big/ \sqrt{1 - \rho^2_{3.4|1.2}/\sigma^2_{4|1.2}} \tag{4}$$

$$f_4(x_1, x_2, x_3) = (b_5 - \rho_{1.5}x_1 - \rho_{2.5|1.}x_2 - \rho_{3.5|1.2.}x_3)/\sigma_{5|1.2.3} \tag{5}$$

$$f_5(x_1, x_2, x_3, x_4) = (b_5 - \rho_{1.5}x_1 - \rho_{2.5|1.}x_2 - \rho_{3.5|1.2.}x_3 - \rho_{4.5|1.2.3}x_4)/\sigma_{5|1.2.3.4} \tag{6}$$

The constants $\sigma_{j|i}$, $\rho_{j.k|i}$, $\sigma_{k|i.j}$, $\rho_{k.l|i.j}$, $\sigma_{l|i.j.k}$, $\rho_{l.m|i.j.k}$ and $\sigma_{m|i.j.k.l}$ in (2.3)–(2.6) are given by

$$\sigma_{j|i} = \sqrt{1 - \rho^2_{i.j}},\ \rho_{j.k|i} = (\rho_{j.k} - \rho_{i.j}\rho_{i.k})/\sigma_{j|i},\ \sigma_{k|i.j} = \sqrt{1 - \rho^2_{i.k} - \rho^2_{j.k|i}} \tag{7}$$

$$\rho_{k.l|i.j} = (\rho_{k.l} - \rho_{i.k}\rho_{i.l} - \rho_{j.k|i}\rho_{j.l|i})/\sigma_{k|i.j},\ \sigma_{l|i.j.k} = \sqrt{1 - \rho^2_{i.l} - \rho^2_{j.l|i} - \rho^2_{k.l|i.j}} \tag{8}$$

$$\rho_{l.m|i.j.k} = (\rho_{l.m} - \rho_{i.l}\rho_{i.m} - \rho_{j.m|i}\rho_{j.l|i} - \rho_{k.m|i.j}\rho_{k.l|i.j})/\sigma_{l|i.j.k} \tag{9}$$

$$\sigma_{m|i.j.k.l} = \sqrt{1 - \rho^2_{i.m} - \rho^2_{j.m|i} - \rho^2_{k.m|i.j} - \rho^2_{l.m|i.j.k}} \tag{10}$$

Proposition 2 (quadrivariate standard normal integral in rectangular coordinates). *Using the definitions of Proposition 1, the four-dimensional multivariate (or quadrivariate) standard normal integral is given by*

$$\begin{aligned} &P(X_1 \le b_1, X_2 \le b_2, X_3 \le b_3, X_4 \le b_4) \\ &\triangleq N_4[b_1, b_2, b_3, b_4; \rho_{1.2}, \rho_{1.3}, \rho_{1.4}, \rho_{2.3}, \rho_{2.4}, \rho_{3.4}] \\ &= \int_{-\infty}^{b_1}\int_{-\infty}^{f_1(x_1)} \frac{1}{2\pi}\exp\left(-\frac{(x_1^2 + x_2^2)}{2}\right) N_2\left[f_2(x_1, x_2), f_6(x_1, x_2); \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}\right] dx_2 dx_1 \end{aligned} \tag{11}$$

$$\begin{aligned} &= \int_{x_1=-\infty}^{b_1}\int_{x_2=-\infty}^{f_1(x_1)}\int_{x_3=-\infty}^{f_2(x_1,x_2)} \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{(x_1^2 + x_2^2 + x_3^2)}{2}\right) \\ &\times N\left[f_3(x_1, x_2, x_3)\right] dx_3 dx_2 dx_1 \end{aligned} \tag{12}$$

where:

$$f_6(x_1, x_2) = \frac{b_4 - \rho_{1.4}x_1}{\sigma_{4|1.2}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}x_2 \tag{13}$$

Corollary of Proposition 2 (reduction formulae for special cases of the correlation structure).*:*

(i) If $\rho_{1.4} = 0$, i.e. if X_1 and X_4 are uncorrelated, then the quadrivariate standard normal integral can be evaluated by the following single quadrature:

$$\begin{aligned} &N_4[b_1, b_2, b_3, b_4; \rho_{1.2}, \rho_{1.3}, 0, \rho_{2.3}, \rho_{2.4}, \rho_{3.4}] \\ &= \int_{-\infty}^{b_1} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x_1^2}{2}\right) N_3\left[\frac{b_2 - \rho_{1.2}x_1}{\sigma_{2|1}}, \frac{b_3 - \rho_{1.3}x_1}{\sigma_{3|1}}, \frac{b_4}{\sigma_{4|1.2}}; \frac{\rho_{2.3|1}}{\sigma_{3|1}}, \rho_{2.4|1}, \frac{\rho_{3.4}}{\sigma_{3|1}}\right] dx_1 \end{aligned} \tag{14}$$

(ii) If $\rho_{1.3} = \rho_{1.4} = \rho_{2.4} = 0$, then we have:

$$N_4\left[b_1, b_2, b_3, b_4; \rho_{1.2}, 0, 0, \rho_{2.3}, 0, \rho_{3.4}\right] = \int_{x_2=-\infty}^{b_2} \int_{x_3=-\infty}^{\frac{b_3-\rho_{2.3}x_2}{\sigma_{3|2}}} \frac{\exp\left(-\frac{\left(x_2^2+x_3^2\right)}{2}\right)}{2\pi} N\left[\frac{b_1-\rho_{1.2}x_2}{\sigma_{2|1}}\right] N\left[\frac{b_4-\rho_{3.4}\sigma_{3|2}x_3-\rho_{3.4}\rho_{2.3}x_2}{\sigma_{4|3}}\right] dx_3 dx_2 \tag{15}$$

Proposition 3 (Order statistics). *Let $[X_1, X_2, X_3, X_4, X_5]$ be a multivariate normal random vector. For $i \in \{1, .., 5\}$, each $E[X_i]$ is denoted by μ_i, and each $\operatorname{var}[X_i]$ is denoted by σ_i^2. Each pairwise correlation is denoted by $\rho_{i.j}$, $(i, j) \in \{1, \ldots, 4\} \times \{2, \ldots, 5\}$, $i < j$. Then, the probability, denoted as $p_{[n,a]}(i, j, k, l, m)$, that the variable $X(i)$ will be the $n-$th order statistic, $n \in \{1, 2, 3, 4, 5\}$, among the set of correlated variables $\{X(i), X(j), X(k), X(l), X(m)\}$, and that it will be less than $a \in \mathbb{R}$, is given by:*

$$p_{[n,a]}\left(i, j, k, l, m\right) = N_5\begin{bmatrix} \frac{a-\mu_i}{\sigma_i}, \lambda_1\Phi\left(i, j\right), \lambda_2\Phi\left(i, k\right), \lambda_3\Phi\left(i, l\right), \lambda_4\Phi\left(i, m\right); \lambda_5\Psi\left(i, j\right), \\ \lambda_6\Psi\left(i, k\right), \lambda_7\Psi\left(i, l\right), \lambda_8\Psi\left(i, m\right), \\ \lambda_9\Upsilon\left(i, j, k\right), \lambda_{10}\Upsilon\left(i, j, l\right), \lambda_{11}\Upsilon\left(i, j, m\right), \lambda_{12}\Upsilon\left(i, k, l\right), \\ \lambda_{13}\Upsilon\left(i, k, m\right), \lambda_{14}\Upsilon\left(i, l, m\right) \end{bmatrix} \tag{16}$$

where the following notations hold:

$$\Phi\left(r, s\right) = \frac{\mu_s - \mu_r}{\varepsilon\left(r, s\right)} \tag{17}$$

$$\Psi\left(r, s\right) = \frac{\sigma_r - \sigma_s\rho_{r.s}}{\varepsilon\left(r, s\right)} \tag{18}$$

$$\Upsilon\left(r, s, t\right) = \frac{\sigma_r^2 - \sigma_r\sigma_s\rho_{r.s} - \sigma_r\sigma_t\rho_{r.t} + \sigma_s\sigma_t\rho_{s.t}}{\varepsilon\left(r, s\right)\varepsilon\left(r, t\right)} \tag{19}$$

$$\varepsilon\left(r, s\right) = \sqrt{\sigma_r^2 - 2\sigma_r\sigma_s\rho_{r.s} + \sigma_s^2} \tag{20}$$

and the λ_i's are given by:

$$\begin{aligned}
\lambda_1 &= (-1)\left(\mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} \\
\lambda_2 &= (-1)\left(\mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} \\
\lambda_3 &= (-1)\left(\mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} \\
\lambda_4 &= (-1)\left(\mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} \\
\lambda_5 &= (-1)\left(\mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} \\
\lambda_6 &= (-1)\left(\mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} \\
\lambda_7 &= (-1)\left(\mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} \\
\lambda_8 &= (-1)\left(\mathbb{I}_{\{n=1\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} \\
\lambda_9 &= (-1)\left(\mathbb{I}_{\{n=4\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}} \\
\lambda_{10} &= (-1)\left(\mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}} \\
\lambda_{11} &= (-1)\left(\mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=1\}} \\
\lambda_{12} &= (-1)\left(\mathbb{I}_{\{n=3\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=2\}} + \mathbb{I}_{\{n=1\}} \\
\lambda_{13} &= (-1)\left(\mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=2\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=1\}} \\
\lambda_{14} &= (-1)\left(\mathbb{I}_{\{n=2\}}\right) + \mathbb{I}_{\{n=5\}} + \mathbb{I}_{\{n=4\}} + \mathbb{I}_{\{n=3\}} + \mathbb{I}_{\{n=1\}}
\end{aligned}$$

Corollary of Proposition 3:

(i) Let R_n denote the n-th order statistic, $n \in \{1, 2, 3, 4, 5\}$, among the set $\{X(i), X(j), X(k), X(l), X(m)\}$ and $a \in \mathbb{R}$; the cumulative distribution function of R_n is given by:

$$P(R_n \leq a) = p_{[n,a]}(1, 2, 3, 4, 5) + p_{[n,a]}(2, 1, 3, 4, 5) + p_{[n,a]}(3, 1, 2, 4, 5) + p_{[n,a]}(4, 1, 2, 3, 5) + p_{[n,a]}(5, 1, 2, 3, 4) \quad (21)$$

(ii) The probability that the variable $X(i)$ will be the $n-$th order statistic, $n \in \{1, 2, 3, 4, 5\}$, among the set of correlated variables $\{X(i), X(j), X(k), X(l), X(m)\}$, and that it will be greater than $a \in \mathbb{R}$, is given by:

$$\lim_{a\to\infty} p_{[n,a]}(i, j, k, l, m) - p_{[n,a]}(i, j, k, l, m)$$

3. Numerical results

The numerical implementation of (1), (2), (11) and (12) is now discussed.

The presence of bivariate normal cumulative distribution functions in the integrands of (1) and (11) should not be a cause for concern, as these functions can be evaluated with the accuracy and efficiency required for all practical purposes by means of the algorithm by Genz (2004), which slightly improves on the well-known algorithm of Drezner and Wesolowsky (1990). Since the integrands are identically smooth in (1) and in (2) and the computational cost of evaluating the $N_2(.,.;.)$ function is only negligibly greater than that of evaluating the $N(.)$ function, better results will be achieved by implementing the triple quadrature in (1) than by implementing the quadruple quadrature in (2). For the same reasons, the double quadrature in (11) is superior to the triple quadrature in (12).

The simplest implementation of Propositions 1 and 2 consists in selecting an appropriate cutoff value for the negative infinity lower bounds and then applying a fixed-degree quadrature rule. Given the smoothness of the rapidly decaying exponential functions in the integrands, even a low-degree rule can be expected to perform well. The nature of the integrands makes them good candidates for a Gauss–Legendre rule. A modified Gauss–Hermite rule can also be applied after an elementary transformation of the integrals described by Drezner (1992), but it proved to be slightly less accurate in our testing so it was discarded. Following Genz (2004), cutoff values of −5.5 and −8.5 were selected for respective single and double precision targets.

Another, more sophisticated, form of implementation of Proposition 1 and Proposition 2 consists in replacing the fixed-degree quadrature rule by a subregion adaptive algorithm, as explained by Bernsten, Espelid & Genz (1991). This second approach adapts the number of integrand evaluations in each subregion according to the rate of change of the integrand, thus concentrating the computational effort where it is most needed. The subdivision of the integration domain stops when the sum of the local error deterministic estimates becomes smaller than some prespecified requested accuracy. Adaptive integration is more accurate than fixed-degree rules but it can also be more time-consuming.

In the forthcoming Tables 1 and 2 reporting numerical results, three different implementations of (1) in Proposition 1 and (11) in Proposition 2 have been carried out:

- a fixed-degree 8-point Gauss–Legendre, denoted by Imp1; the total number of function evaluations involved is thus equal to 64 for the double quadrature in (11) and to 512 for the triple quadrature in (1)

Table 1. Numerical tests based on benchmarks available for special covariance matrices.

	Imp1	Imp2	Imp3 10^{-5} requested accuracy	Imp3 10^{-7} requested accuracy
Test 1				
Average absolute error	6×10^{-8}	3×10^{-9}	4×10^{-5}	5×10^{-7}
Maximum absolute error	2×10^{-7}	6×10^{-8}	9×10^{-5}	9×10^{-7}
Test 2				
Average absolute error Proposition 1	4×10^{-8}	2×10^{-9}	3×10^{-5}	5×10^{-7}
Maximum absolute error Proposition 1	4×10^{-6}	3×10^{-8}	9×10^{-5}	9×10^{-7}
Average absolute error Proposition 2	3×10^{-8}	4×10^{-9}	6×10^{-5}	4×10^{-7}
Maximum absolute error Proposition 2	4×10^{-7}	2×10^{-8}	9×10^{-5}	9×10^{-7}
Test 3				
Average absolute error Proposition 1	9×10^{-7}	4×10^{-8}	5×10^{-5}	5×10^{-7}
Maximum absolute error Proposition 1	5×10^{-5}	8×10^{-7}	9×10^{-5}	9×10^{-7}
Average absolute error Proposition 2	2×10^{-8}	7×10^{-9}	5×10^{-5}	4×10^{-7}
Maximum absolute error Proposition 2	4×10^{-6}	8×10^{-8}	9×10^{-5}	9×10^{-7}

- a fixed-degree 16-point Gauss–Legendre, denoted by Imp2; the total number of function evaluations involved is thus equal to 256 for the double quadrature in (11) and to 4096 for the triple quadrature in (1)
- the Cuhre adaptive integration algorithm as implemented by Hahn (2005), based on Bernsten, Espelid & Genz (1991), denoted by Imp3, with two levels of requested accuracy (10^{-5} and 10^{-7}).

The Cuhre adaptive integration algorithm was preferred to the classical Schervish's MULNOR (Schervish, 1984) because the latter presents inconsistencies and efficiency issues documented by Genz (2004). The Imp1 and Imp2 routines have been implemented in VBA (the code is available from the author upon request), while the Cuhre algorithm uses a C++ interface.

Two kinds of tests have been conducted to assess the accuracy and the efficiency of Proposition 1 and Proposition 2. The first one is based on special covariance matrices for which exact analytical benchmarks are available that can be numerically evaluated by means of elementary functions at best, or by means of lower-dimensional integrals at least.

"Test 1" in Table 1 refers to a test in which the analytical benchmarks are specific orthant probabilities that admit simple formulae in terms of inverse sine functions (cf. Kotz et al., Vol.1, p.150). "Test 1" applies specifically to Proposition 2.

"Test 2" in Table 1 refers to a test in which the benchmarks are derived from correlation matrices that allow dimension reduction, so that the multivariate normal integrals to be computed come down to products of univariate, bivariate, and trivariate normal integrals at most. "Test 2" in Table 1 applies to both Proposition 1 and Proposition 2. More specifically, the following identities have been used:

$$\begin{aligned} & N_5\left[b_1, b_2, b_3, b_4, b_5; \rho_{1.2}, \rho_{1.3}, 0, 0, \rho_{2.3}, 0, 0, 0, 0, \rho_{4.5}\right] \\ & \quad = N_3\left[b_1, b_2, b_3; \rho_{1.2}, \rho_{1.3}, \rho_{2.3}\right] N_2\left[b_4, b_5; \rho_{4.5}\right] \end{aligned} \tag{22}$$

Table 2. Convergence of Monte Carlo approximations to Proposition 1 and Proposition 2 using the Imp3 10^{-7} method of implementation for general covariance matrices.

	10^{-3} convergence	10^{-4} convergence	10^{-5} convergence	$> 10^{-5}$ convergence
1,000,000 simulations	18.4%	72.7%	8.7%	0.2%
10,000,000 simulations	3.8%	26.5%	53.4%	16.3%
100,000,000 simulations	0%	0.7%	3.2%	96.1%

$$N_4\left[b_1, b_2, b_3, b_4; \rho_{1.2}, 0, 0, 0, 0, \rho_{3.4}\right] = N_2\left[b_1, b_2; \rho_{1.2}\right] N_2\left[b_3, b_4; \rho_{3.4}\right] \quad (23)$$

$$N_4\left[b_1, b_2, b_3, b_4; 0, \rho_{1.3}, \rho_{1.4}, 0, 0, \rho_{3.4}\right] = N_3\left[b_1, b_3, b_4; \rho_{1.3}, \rho_{1.4}, \rho_{3.4}\right] N\left[b_2\right] \quad (24)$$

The $N_3[., ., .; ., ., .]$ functions are evaluated using the algorithm by Genz (2004).

"Test 3" in Table 1 refers to a test based on the condition that all correlations are equal and positive. Denoting by ρ the constant correlation coefficient and by b_i's, $i \in \{1, \ldots, d\}$, the upper bounds of the d-dimensional standard normal integral, Tong (1990) has shown the following result, which serves as a benchmark in "Test 3":

$$N_d\left[b_1, .., b_d; \rho\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-x^2/2\right) \prod_{i=1}^{d} N\left[\left(b_i - x\sqrt{\rho}\right)/\sqrt{1-\rho}\right] dx \quad (25)$$

"Test 3" in Table 1 applies to both Proposition 1 and Proposition 2.

Next, in Table 2, general covariance matrices are randomly drawn, along with a check for non-singularity. A comparison is made between numerical values obtained applying Proposition 1 and Proposition 2, and Monte Carlo approximations using pseudo-random numbers drawn from the Mersenne twister generator. The X_i standard normal random variables, $i \in \{1, .., 5\}$, are simulated using Eqs. (30)–(33) in Section 4. The numbers reported in Table 2 are the proportions of Monte carlo approximations that achieve a given level of convergence to the values obtained using the Imp3 method of implementation of Proposition 1 and Proposition 2, with a requested accuracy of 10^{-7}.

In both tables (Tables 1 and 2), the reported numerical results have been obtained out of a sample of 5,000 randomly drawn sets of parameters.

Table 3 reports the computational times required by the various numerical methods on an i7-6700 HQ personal computer equipped with 16 Go RAM. The values reported for the Imp3 method are ranges with observed minimum and maximum, as the number of iterations necessary to achieve a requested level of accuracy depends on the integration problem under consideration.

Looking at Table 1 first, it can be observed that Test 1, Test 2, and Test 3 yield approximately the same results. In all three cases, the magnitude of the maximum error for a plain fixed-degree 16-point Gauss–Legendre quadrature is never greater than 10^{-7}, for both Proposition 1 and Proposition 2. Considering the speed of execution and the simplicity of implementation of this method, it can be recommended as a default choice for most practical purposes. Using the Imp3 10^{-7} implementation is theoretically more reliable thanks to the error estimate, but

Table 3. Computational time required by the various numerical methods.

	Computational time (s)
Imp 1 Proposition 1	$< 10^{-6}$
Imp 2 Proposition 1	0.02425
Imp 1 Proposition 2	$< 10^{-6}$
Imp 2 Proposition 2	$< 10^{-6}$
Imp 3×10^{-5} requested accuracy Proposition 1	$< 10^{-6}$
Imp 3×10^{-7} requested accuracy Proposition 1	[1.4; 3.6]
Imp 3×10^{-5} requested accuracy Proposition 2	$< 10^{-6}$
Imp 3×10^{-7} requested accuracy Proposition 2	[0.008; 0.2]
MC 1,000,000 Proposition 1	12.5
MC 10,000,000 Proposition 1	124.3
MC 100,000,000 Proposition 1	1341.7
MC 1,000,000 Proposition 2	9.4
MC 10,000,000 Proposition 2	87.3
MC 100,000,000 Proposition 2	992.4

it should be pointed out that the Imp2 method has the lowest maximum absolute error in all tests, while being far superior to Imp3 10^{-7} in terms of efficiency as shown by Table 3. Likewise, there does not seem to be any gain in using Imp3 10^{-5} instead of Imp1 since it is much slower but not more precise.

Table 2 provides a quick and crude check of (1) and (11) when no analytical benchmark is available. Table 2 shows a clear pattern of convergence of Monte Carlo approximations to the values obtained using Proposition 1 and Proposition 2, as more and more simulations are performed. The computational burden, however, is huge and the accuracy is poor. Clearly, more efficient Monte Carlo techniques could be implemented, leading to shorter computational times, but this is not the subject of this article. Very similar orders of convergence as those of Table 2 can be observed when using Monte Carlo simulation to approximate the benchmarks of Table 1, which suggests that the numerical performance of Proposition 1 and Proposition 2 is not contingent on the choice of a special correlation structure.

The Plackett's method of numerical integration was also tested (Plackett, 1954). As explained by Gassmann (2003), this method involves adding an easily computable reference probability and a probability correction term which consists of a sum of one-dimensional integrals. Since powerful algorithms are available for the computation of the trivariate normal integral as mentioned in the introduction of this article, the following reference probability can be chosen when it comes to the numerical evaluation of the quadrivariate normal integral:

$$N[b_1]\,N_3[b_2, b_3, b_4; \rho_{2.3}, \rho_{2.4}, \rho_{3.4}] \tag{26}$$

Following Drezner's technique of implementation of the trivariate normal integral (Drezner, 1993), the probability correction term should then be equal to:

$$\begin{aligned}
&\rho_{1.2}\int_0^1 \frac{\exp\left(-\frac{b_1^2-2x\rho_{1.2}b_1b_2+b_2^2}{2(1-x^2\rho_{1.2}^2)}\right)}{2\pi\sqrt{1-x^2\rho_{1.2}^2}} N_2\left[\begin{array}{l}\frac{b_3(1-x^2\rho_{1.2}^2)-(\rho_{1.3}-\rho_{2.3}\rho_{1.2})xb_1-(\rho_{2.3}-x^2\rho_{1.3}\rho_{1.2})b_2}{((1-x^2\rho_{1.2}^2)f(x))^{1/2}},\\ \frac{b_4(1-x^2\rho_{1.2}^2)-(\rho_{1.4}-\rho_{2.4}\rho_{1.2})xb_1-(\rho_{2.4}-x^2\rho_{1.4}\rho_{1.2})b_2}{((1-x^2\rho_{1.2}^2)f(x))^{1/2}};\ \rho_{3.4}\end{array}\right]dx\\
&+\rho_{1.3}\int_0^1 \frac{\exp\left(-\frac{b_1^2-2x\rho_{1.3}b_1b_3+b_3^2}{2(1-x^2\rho_{1.3}^2)}\right)}{2\pi\sqrt{1-x^2\rho_{1.3}^2}} N_2\left[\begin{array}{l}\frac{b_2(1-x^2\rho_{1.3}^2)-(\rho_{1.2}-\rho_{2.3}\rho_{1.3})xb_1-(\rho_{2.3}-x^2\rho_{1.2}\rho_{1.3})b_3}{((1-x^2\rho_{1.3}^2)f(x))^{1/2}},\\ \frac{b_4(1-x^2\rho_{1.3}^2)-(\rho_{1.4}-\rho_{3.4}\rho_{1.3})xb_1-(\rho_{3.4}-x^2\rho_{1.4}\rho_{1.3})b_3}{((1-x^2\rho_{1.3}^2)f(x))^{1/2}};\ \rho_{2.4}\end{array}\right]dx\\
&+\rho_{1.4}\int_0^1 \frac{\exp\left(-\frac{b_1^2-2x\rho_{1.4}b_1b_4+b_4^2}{2(1-x^2\rho_{1.4}^2)}\right)}{2\pi\sqrt{1-x^2\rho_{1.4}^2}} N_2\left[\begin{array}{l}\frac{b_2(1-x^2\rho_{1.4}^2)-(\rho_{1.2}-\rho_{2.4}\rho_{1.4})xb_1-(\rho_{2.4}-x^2\rho_{1.2}\rho_{1.4})b_4}{((1-x^2\rho_{1.4}^2)f(x))^{1/2}},\\ \frac{b_3(1-x^2\rho_{1.4}^2)-(\rho_{1.3}-\rho_{3.4}\rho_{1.4})xb_1-(\rho_{3.4}-x^2\rho_{1.3}\rho_{1.4})b_4}{((1-x^2\rho_{1.4}^2)f(x))^{1/2}};\ \rho_{2.3}\end{array}\right]dx
\end{aligned} \tag{27}$$

where:

$$\begin{aligned}
f(x) = {} & 1-x^2\rho_{1.2}^2-x^2\rho_{1.3}^2-x^2\rho_{1.4}^2-\rho_{2.3}^2-\rho_{2.4}^2-\rho_{3.4}^2+x^2\rho_{1.2}^2\rho_{3.4}^2\\
&+x^2\rho_{1.3}^2\rho_{2.4}^2+x^2\rho_{1.4}^2\rho_{2.3}^2+2x^2\rho_{1.2}\rho_{1.3}\rho_{2.3}+2x^2\rho_{1.2}\rho_{1.4}\rho_{2.4}+2x^2\rho_{1.3}\rho_{1.4}\rho_{3.4}\\
&+2\rho_{2.3}\rho_{2.4}\rho_{3.4}-2x^2\rho_{1.2}\rho_{1.3}\rho_{2.4}\rho_{3.4}-2x^2\rho_{1.2}\rho_{1.4}\rho_{2.3}\rho_{3.4}-2x^2\rho_{1.3}\rho_{1.4}\rho_{2.3}\rho_{2.4}
\end{aligned} \tag{28}$$

After some testing, however, this scheme was not pursued further as it led to significant inaccuracies.

4. Proof of main results

4.1. Proof of Proposition 1

By definition of conditional probability, for any $n \in \mathbb{N}$, the joint density function of n correlated standard normal random variables $f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n)$ can be written as the following product:

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1)\, f_{X_2|X_1}(x_2\,|x_1) \ldots f_{X_n|X_1,X_2,\ldots,X_{n-1}}(x_n\,|x_1, x_2, \ldots, x_{n-1}) \tag{29}$$

To obtain the required conditional density functions in (29), one can notice that each random variable X_k, $\forall k \in \{1, \ldots, n\}$, in an n-dimensional multivariate standard normal random vector $[X_1, \ldots, X_n]$, admits an orthogonal decomposition as a linear combination of k pairwise independent standard normal random variables, i.e., we have

$$X_k = \alpha_1 X_1 + \alpha_2 Y_2 + \cdots + \alpha_k Y_k \tag{30}$$

where each scalar α_j, $\forall j \in \{1, \ldots, k\}$, is real-valued and the set $\{X_1, Y_2, \ldots, Y_n\}$ forms an orthogonal basis of independent standard normal variables for the vector space of n correlated standard normal variables $X_1, X_2, \ldots, X_n$. The scalars α_j, $\forall j \in \{1, \ldots, k-1\}$, are the solutions of the equations:

$$\rho_{i.j} = \frac{\operatorname{cov}\left[X_i, X_j\right]}{\sigma_i \sigma_j} = \operatorname{cov}\left[X_i, \alpha_i X_1 + \cdots + \alpha_j X_j\right], \quad i \in \left\{1, \ldots, j-1\right\} \tag{31}$$

These equations, deriving from the definition of a correlation coefficient, must be solved iteratively in ascending order from $i = 1$ to $i = j - 1$, for each j. Each α_1, for $j = 1$ to k, is the correlation coefficient between X_1 and X_j, i.e., the number $\rho_{1.j}$. Each α_i, for $i = 2$ to j and for $j = 1$ to k, is the partial correlation coefficient between X_i and X_j conditional on the sequence $X_1, \ldots, X_{i-1}$.

Once the scalars α_j, $\forall j \in \{1, \ldots, k-1\}$, have been determined, the scalars α_k, $\forall k \in \{1, \ldots, n\}$, can be obtained as the solutions of the following equations:

$$\sigma^2_{X_k} = 1 = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_k^2 \tag{32}$$

These equations derive from the stability of the normal distribution under addition. They must be solved iteratively in the ascending order from $k = 1$ to n. Each α_k, $\forall k \in \{1, \ldots, n\}$, is the standard deviation of X_k conditional on $X_1, \ldots, X_{k-1}$, hence only the positive roots $\alpha_k = \sqrt{1 - \sum_{j=1}^{k-1} \alpha_j^2}$ are considered.

Applying this method to the case $n = 5$, we obtain the following orthogonal decompositions of X_2, X_3, X_4 and X_5:

$$X_2 = \rho_{1.2} X_1 + \sigma_{2|1} Y_2 \tag{33}$$

$$X_3 = \rho_{1.3} X_1 + \rho_{2.3|1} Y_2 + \sigma_{3|1.2} Y_3 \tag{34}$$

$$X_4 = \rho_{1.4} X_1 + \rho_{2.4|1} Y_2 + \rho_{3.4|1.2} Y_3 + \sigma_{4|1.2.3} Y_4 \tag{35}$$

$$X_5 = \rho_{1.5} X_1 + \rho_{2.5|1} Y_2 + \rho_{3.5|1.2} Y_3 + \rho_{4.5|1.2.3} Y_4 + \sigma_{5|1.2.3.4} Y_5 \tag{36}$$

where the coefficients are as given by Proposition 1

Thus, the following conditional distributions hold:

$$X_2\,|X_1 \sim \mathcal{N}\left(\rho_{1.2} X_1; \sigma_{2|1}\right) \tag{37}$$

$$X_3 \left| X_1, X_2 \sim \mathcal{N}\left(\rho_{1.3}X_1 + \rho_{2.3|1}\frac{X_2 - \rho_{1.2}X_1}{\sigma_{2|1}}; \sigma_{3|1.2}\right)\right. \tag{38}$$

$$X_4 \left| X_1, X_2, X_3 \sim \mathcal{N}\left(\rho_{1.4}X_1 + \rho_{2.4|1}\frac{X_2 - \rho_{1.2}X_1}{\sigma_{2|1}} + \rho_{3.4|1.2}\frac{X_3 - \rho_{1.3}X_1 - \rho_{2.3|1}\frac{X_2-\rho_{1.2}X_1}{\sigma_{2|1}}}{\sigma_{3|1.2}}; \sigma_{4|1.2.3}\right)\right. \tag{39}$$

$$X_5 \left| X_1, X_2, X_3, X_4 \sim \mathcal{N}\left(\begin{array}{l}\rho_{1.5}X_1 + \rho_{2.5|1}\frac{X_2-\rho_{1.2}X_1}{\sigma_{2|1}} + \rho_{3.5|1.2}\frac{X_3-\rho_{1.3}X_1-\rho_{2.3|1}\frac{X_2-\rho_{1.2}X_1}{\sigma_{2|1}}}{\sigma_{3|1.2}} \\ +\rho_{4.5|1.2.3}\frac{X_4-\rho_{1.4}X_1-\rho_{2.4|1}\frac{X_2-\rho_{1.2}X_1}{\sigma_{2|1}}-\rho_{3.4|1.2}\frac{X_3-\rho_{1.3}X_1-\rho_{2.3|1}\frac{X_2-\rho_{1.2}X_1}{\sigma_{2|1}}}{\sigma_{3|1.2}}}{\sigma_{4|1.2.3}}; \sigma_{5|1.2.3.4}\end{array}\right)\right. \tag{40}$$

where $\mathcal{N}(a, b)$ refers to the normal distribution with expectation a and standard deviation b.

Plugging (37)–(40) into (29) yields:

$$\begin{aligned}
f_{X_1,X_2,X_3,X_4,X_5}(x_1, x_2, x_3, x_4, x_5) &= \frac{1}{(2\pi)^{\frac{5}{2}}\sigma_{2|1}\sigma_{3|1.2}\sigma_{4|1.2.3}\sigma_{5|1.2.3.4}} \\
&\times \exp\left(-\frac{x_1^2}{2} - \frac{1}{2}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)^2 - \frac{1}{2}\left(\frac{x_3 - \rho_{1.3}x_1}{\sigma_{3|1.2}} - \frac{\rho_{2.3|1}}{\sigma_{3|1.2}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)\right)^2\right) \\
&\times \exp\left(-\frac{1}{2}\left(\frac{x_4 - \rho_{1.4}x_1}{\sigma_{4|1.2.3}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2.3}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right) - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2.3}}\left(\frac{x_3 - \rho_{1.3}x_1}{\sigma_{3|1.2}}\right.\right.\right. \\
&\left.\left.\left. - \frac{\rho_{2.3|1}}{\sigma_{3|1.2}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)\right)\right)^2\right) \\
&\times \exp\left(-\frac{1}{2}\left(\frac{x_5 - \rho_{1.5}x_1}{\sigma_{5|1.2.3.4}} - \frac{\rho_{2.5|1}}{\sigma_{5|1.2.3.4}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)\right.\right. \\
&- \frac{\rho_{3.5|1.2}}{\sigma_{5|1.2.3.4}}\left(\frac{x_3 - \rho_{1.3}x_1}{\sigma_{3|1.2}} - \frac{\rho_{2.3|1}}{\sigma_{3|1.2}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)\right) \\
&- \frac{\rho_{4.5|1.2.3}}{\sigma_{5|1.2.3.4}}\left(\frac{x_4 - \rho_{1.4}x_1}{\sigma_{4|1.2.3}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2.3}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)\right. \\
&\left.\left.\left.\left. - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2.3}}\left(\frac{x_3 - \rho_{1.3}x_1}{\sigma_{3|1.2}} - \frac{\rho_{2.3|1}}{\sigma_{3|1.2}}\left(\frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}}\right)\right)\right)\right)^2\right)
\end{aligned} \tag{41}$$

The density function $f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4)$ follows immediately by dividing $f_{X_1,X_2,X_3,X_4,X_5}(x_1, x_2, x_3, x_4, x_5)$ by $f_{X_5|X_1,X_2,X_3,X_4}(x_5|x_1, x_2, x_3, x_4)$.

The five-dimensional multivariate standard normal integral is thus given by:

$$\begin{aligned}
&P(X_1 \leq b_1, X_2 \leq b_2, X_3 \leq b_3, X \leq b_4, X \leq b_5) \\
&\quad = \int_{\Delta} f_{X_1,X_2,X_3,X_4,X_5}(x_1, x_2, x_3, x_4, x_5)\, dx_5 dx_4 dx_3 dx_2 dx_1
\end{aligned} \tag{42}$$

where $\Delta =]-\infty, b_1] \times]-\infty, b_2] \times]-\infty, b_3] \times]-\infty, b_4] \times]-\infty, b_5]$

Eq. (1) in Proposition 1 can then be obtained through the following steps:

(i) Substitute the variable

$$y_2 = \frac{x_2 - \rho_{1.2}x_1}{\sigma_{2|1}} \tag{43}$$

(ii) Notice that

$$\sigma_{3|1.2} = \sigma_{3|1}\sqrt{1 - \frac{\rho_{2.3|1}^2}{\sigma_{3|1}^2}} \tag{44}$$

(iii) Substitute the variable

$$y_3 = \frac{x_3 - \rho_{1.3}x_1}{\sigma_{3|1}} \tag{45}$$

(iv) Notice that

$$\sigma_{4|1.2.3} = \sigma_{4|1.2}\sqrt{1 - \frac{\rho_{3.4|1.2}^2}{\sigma_{4|1.2}^2}} \tag{46}$$

(v) Substitute the variable

$$y_4 = \frac{x_4 - \rho_{1.4}x_1}{\sigma_{4|1.2}} \tag{47}$$

(vi) Substitute the variable

$$z_3 = \left(y_3 - \frac{\rho_{2.3|1}}{\sigma_{3|1}}y_2\right) \Bigg/ \sqrt{1 - \left(\frac{\rho_{2.3|1}}{\sigma_{3|1}}\right)^2} \tag{48}$$

(vii) Substitute the variable

$$z_4 = \left(y_4 - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}y_2 - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}z_3\right) \Bigg/ \sqrt{1 - \frac{\rho_{3.4|1.2}^2}{\sigma_{4|1.2}^2}} \tag{49}$$

(viii) Notice that

$$\sigma_{5|1.2.3.4} = \sigma_{5|1.2.3}\sqrt{1 - \frac{\rho_{4.5|1.2.3}^2}{\sigma_{5|1.2.3}^2}} \tag{50}$$

(ix) Substitute the variable

$$y_5 = \frac{x_5 - \rho_{1.5}x_1 - \rho_{2.5|1}\,y_2 - \rho_{3.5|1.2}\,z_3}{\sigma_{5|1.2.3}} \tag{51}$$

(x) Identify:

$$\int\limits_{z_4=-\infty}^{\frac{\frac{b_4-\rho_{1.4}x_1}{\sigma_{4|1.2}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}y_2 - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}z_3}{\sqrt{1-\frac{\rho_{3.4|1.2}^2}{\sigma_{4|1.2}^2}}}} \int\limits_{y_5=-\infty}^{\frac{b_5-\rho_{1.5}x_1-\rho_{2.5|1}\,y_2-\rho_{3.5|1.2}\,z_3}{\sigma_{5|1.2.3}}} \frac{\exp\left(-\frac{z_4^2}{2} - \frac{1}{2\left(1-\frac{\rho_{4.5|1.2.3}^2}{\sigma_{5|1.2.3}^2}\right)}\left(y_5 - \frac{\rho_{4.5|1.2.3}}{\sigma_{5|1.2.3}}z_4\right)^2\right)}{2\pi\sqrt{1-\frac{\rho_{4.5|1.2.3}^2}{\sigma_{5|1.2.3}^2}}} dy_5 dz_4$$

$$= N_2\left[\frac{\frac{b_4-\rho_{1.4}x_1}{\sigma_{4|1.2}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}y_2 - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}z_3}{\sqrt{1-\frac{\rho_{3.4|1.2}^2}{\sigma_{4|1.2}^2}}}, \frac{b_5-\rho_{1.5}x_1-\rho_{2.5|1}\,y_2-\rho_{3.5|1.2}\,z_3}{\sigma_{5|1.2.3}}; \frac{\rho_{4.5|1.2.3}}{\sigma_{5|1.2.3}}\right] \tag{52}$$

Applying the same procedure from step (i) to step (vii) suffices to obtain Eq. (2) in Proposition 1, as the identification of the function $N[f_5(x_1, x_2, x_3, x_4)]$ becomes obvious at step (vii).

4.2. Proof of Proposition 2

It is shown how to obtain the density function $f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4)$ in the proof of Proposition 1.

To obtain Eq. (11) in Proposition 2, first write down the following integral:

$$\int_{\Delta} f_{X_1,X_2,X_3,X_4}(x_1, x_2, x_3, x_4)\, dx_4 dx_3 dx_2 dx_1 \tag{53}$$

where $\Delta =]-\infty, b_1] \times]-\infty, b_2] \times]-\infty, b_3] \times]-\infty, b_4]$ then apply the steps from (i) to (vi) in the proof of Proposition 1 and identify:

$$\int_{-\infty}^{\frac{\frac{b_3-\rho_{1.3}x_1}{\sigma_{3|1}} - \frac{\rho_{2.3|1}}{\sigma_{3|1}}y_2}{\sqrt{1-\left(\frac{\rho_{2.3|1}}{\sigma_{3|1}}\right)^2}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_3^2}{2}\right) N\left[\frac{\frac{b_4-\rho_{1.4}x_1}{\sigma_{4|1.2}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}y_2 - \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}z_3}{\sqrt{1-\frac{\rho_{3.4|1.2}^2}{\sigma_{4|1.2}^2}}}\right] dz_3$$

$$= N_2\left[\frac{\frac{b_3-\rho_{1.3}x_1}{\sigma_{3|1}} - \frac{\rho_{2.3|1}}{\sigma_{3|1}}y_2}{\sqrt{1-\left(\frac{\rho_{2.3|1}}{\sigma_{3|1}}\right)^2}}, \frac{b_4-\rho_{1.4}x_1}{\sigma_{4|1.2}} - \frac{\rho_{2.4|1}}{\sigma_{4|1.2}}y_2; \frac{\rho_{3.4|1.2}}{\sigma_{4|1.2}}\right] \tag{54}$$

Eq. (12) in Proposition 1 can be obtained upon completing step (vi).

4.3. Proof of Proposition 3

Proposition 3 is a straightforward consequence of the law of total probability, with each function $\Psi(r, s)$ representing the correlation coefficient between X_r and $X_r - X_s$, and each function $\Upsilon(r, s, t)$ representing the correlation coefficient between $X_r - X_s$ and $X_r - X_t$.

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