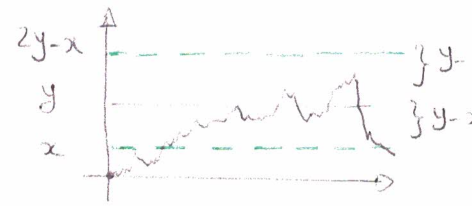
Reflection Principle (classic case)

Thm if $(W_t : t \geq 0)$ is a Brownian Motion and $M_t^W := \max_{0 \leq u \leq t} W_u$

then for $0 \vee x \leq y$, one has $\mathbb{P}(W_t \leq x, M_t^W \geq y) = \mathbb{P}(W_t \geq 2y - x)$

Proof Define $T_y := \inf\{t \geq 0 : W_t \geq y\}$



$$\Rightarrow \begin{cases} T_y \text{ is an } \mathbb{F}^W\text{- stopping time} \\ W_{T_y} = y \\ \{T_y \leq t\} \Leftrightarrow \{M_t^W \geq y\} \end{cases}$$

Thus, $\mathbb{P}(W_t \leq x, M_t^W \geq y) = \mathbb{P}(W_t - W_{T_y} \leq x - y, T_y \leq t)$

iterative expectation
$$\stackrel{\downarrow}{=} \mathbb{E}\left[\mathbb{1}_{T_y \leq t}\, \mathbb{P}(W_t - W_{T_y} \leq x - y \mid T_y)\right]$$

Strong Markov + Properties of Wiener process
$$\stackrel{\downarrow}{=} \mathbb{E}\left[\mathbb{1}_{T_y \leq t}\, \mathbb{P}(W_t - W_{T_y} \geq y - x \mid T_y)\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{T_y \leq t}\, \mathbb{P}(W_t \geq 2y - x \mid T_y)\right]$$

$$= \mathbb{P}(W_t \geq 2y - x, M^W_t \geq y) = \mathbb{P}(W_t \geq 2y - x)$$

since $x \leq y$ $\quad 2y - x \geq 2y - y = y$

thus $W_t \geq 2y - x$ implies $W_t \geq y$ and thus $M_t^W \geq y$

$\Rightarrow$ on $x \leq y$ $\quad \{W_t \geq y\} \subseteq \{M_t^W \geq y\}$

Corollary. The joint distribution of (W, M^W) is

$$\mathbb{P}(W_t \leq x, M_t^W \leq y) = \begin{cases} \mathcal{N}\left(\frac{x}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{x - 2y}{\sqrt{t}}\right) & \max\{0, x\} \leq y \\ \mathcal{N}\left(\frac{y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{-y}{\sqrt{t}}\right) & 0 \leq y \leq x \\ 0 & y \leq 0 \end{cases}$$

Summary of formulae :

$$0 \vee x \le y \implies \mathbb{P}(W_t \le x,\ M_t^W \ge y) = \mathbb{P}(W_t \ge 2y - x)$$

MAX

$$\mathbb{P}(W_t \le x,\ M_t^W \le y) = \begin{cases} \mathbb{P}(W_t \le x) - \mathbb{P}(W_t \le x,\ M_t^W \ge y) & 0 \vee x \le y \\ \mathbb{P}(M_t^W \le y) & 0 \le y \le x \\ 0 & y \le 0 \end{cases}$$

Joint density function

$$\mathbb{P}(W_t \in dx,\ M_t^W \in dy) = \mathbb{1}_{y \ge 0}\, \mathbb{1}_{y \ge x}\, \frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left(- \frac{(2y - x)^2}{2t} \right) dx\, dy .$$

$(M_t, W_t\ ;\ t \ge 0)$ is a Markov process.

Min

If we set $B \equiv -W$. Then

$$m_t^W = \inf_{0 \le u \le t} W_u = - \sup_{0 \le u \le t} (-W_u) = - \sup_{0 \le u \le t} B_u = - M_t^B$$

$\Rightarrow$

$$\mathbb{P}(W_t \ge x,\ m_t^W \ge y) = \begin{cases} \mathbb{P}(W_t \ge x) - \mathbb{P}(W_t \ge x,\ m_t^W \le y) & y \le x \wedge 0 \\ \mathbb{P}(m_t^W \ge y) & x \le y \le 0 \\ 0 & y \ge 0 \end{cases}$$

Reflection Principle for min

$$\mathbb{P}(W_t \ge x,\ m_t^W \le y) = \mathbb{P}(W_t \le 2y - x) \qquad \text{for } y \le x \wedge 0$$

Proof. Define $T_y = \inf\{t \ge 0 : W_t \le y\} \implies \begin{cases} W_{T_y} = y \\ T_y \le t \iff m_t^W \le y \\ T_y \text{ is an } \mathbb{F}^W\text{-stopping time} \end{cases}$

$$\mathbb{P}(W_t \ge x,\ m_t^W \le y) = \mathbb{P}(W_t - W_{T_y} \ge x - y,\ T_y \le t)$$

$$= \mathbb{E}\left[\mathbb{1}_{T_y \le t}\ \mathbb{P}(W_t - W_{T_y} \ge x - y \mid T_y) \right]$$

$$= \mathbb{E}\left[\mathbb{1}_{T_y \le t}\ \mathbb{P}(W_t - W_{T_y} \le -(x - y) \mid T_y) \right]$$

$$= \mathbb{P}(W_t \le 2y - x,\ m_t^W \le y)$$

$$= \mathbb{P}(W_t \le 2y - x)$$

$$\mathbb{P}(m_t^W \geq y) = \mathbb{P}(m_t^W \geq y,\ W_t \geq y) = \mathbb{P}(W_t \geq y) - \mathbb{P}(W_t \leq y) \qquad y \leq 0$$

$$= \mathcal{N}\left(\frac{-y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{y}{\sqrt{t}}\right)$$

$$\mathbb{P}(m_t^W \leq y) = \begin{cases} 1 - \mathbb{P}(W_t \geq y) + \mathbb{P}(W_t \leq y) = 2\mathcal{N}\left(\frac{y}{\sqrt{t}}\right) & y \leq 0 \\ 1 & y > 0 \end{cases}$$

$$\mathbb{P}(W_t \leq x, m_t^W \leq y) = \mathbb{P}(m_t^W \leq y) - \mathbb{P}(W_t \geq x, m_t^W \leq y)$$

$$= \mathbb{P}(m_t^W \leq y) - \left\{ \mathbb{P}(W_t \geq x) - \mathbb{P}(W_t \geq x, m_t^W \geq y) \right\}$$

$$= \mathbb{P}(m_t^W \leq y) - \mathbb{P}(W_t \geq x) + \mathbb{P}(W_t \geq x, m_t^W \geq y)$$

$$\mathbb{P}(W_t \leq x, m_t^W \leq y) = \begin{cases} \mathcal{N}\left(\frac{x}{\sqrt{t}}\right) & y \geq 0 \text{ or } x \leq y \leq 0 \\ 2\mathcal{N}\left(\frac{y}{\sqrt{t}}\right) - \mathcal{N}\left(\frac{2y-x}{\sqrt{t}}\right) & y \leq x \wedge 0 \end{cases}$$

$$\boxed{\mathbb{P}(W_t \leq x, m_t^W \leq y)} = \begin{cases} \mathbb{P}(W_t \leq x) & y \geq 0 \\ 2\mathbb{P}(W_t \leq y) - \mathbb{P}(W_t \leq 2y - x) & y \leq x \wedge 0 \\ \mathbb{P}(W_t \leq x) & x \leq y \leq 0 \end{cases}$$

Drifted Brownian Motion for Minimum

- Cameron-Martin Theorem. $\mathbb{E}[f(\mu t + W_t\,;\ t \le T)]$

$$= \mathbb{E}\left[e^{\mu W_T - \mu^2/2\, T} f(W_t\,;\ t\le T)\right]$$

Let $X_t = \mu t + W_t$ and $m_t^X := \min_{0\le u\le t} X_u$

$$p := \mathbb{P}(X_t \ge x,\ m_t^X \le y) = \mathbb{E}\left[e^{\mu W_t - \mu^2/2\, t}\ \mathbb{1}_{W_t \ge x}\ \mathbb{1}_{m_t^W \le y}\right]$$

Now from reflection principle

$$\mathbb{P}(W_t \ge x,\ m_t^W \le y) = \mathbb{P}(W_t \le 2y - x,\ m_t^W \le y)$$

$$= \mathbb{P}(2y - W_t \ge x,\ m_t^W \le y)$$

Thus;

$$p = \mathbb{E}\left[e^{\mu(2y - W_t) - \mu^2/2\, t}\ \mathbb{1}_{2y - W_t \ge x}\ \mathbb{1}_{m_t^W \le y}\right]$$

$$= e^{2\mu y}\ \mathbb{E}\left[e^{-\mu W_t - \mu^2/2\, t}\ \mathbb{1}_{2y - W_t \ge x}\right]$$

C.M.Thm

$$\overset{\downarrow}{=} e^{2\mu y}\ \mathbb{E}\left[\mathbb{1}_{W_t - \mu t \le 2y - x}\right]$$

$$= e^{2\mu y}\ \mathbb{P}(W_t \le 2y - x + \mu t)$$

Notation $\xi_t = \mu t + W_t \qquad m_t^\xi := \min_{0\le u\le t} \xi_u$

GBM

$$\mathbb{P}(\xi_t \ge x,\ m_t^\xi \le y) = e^{2\mu y}\ \mathbb{P}(W_t \le 2y - x + \mu t)$$

$X_t = X_0 e^{\sigma \xi_t}$ with $\mu = \dfrac{r - \sigma^2/2}{\sigma}$ Then $\begin{cases} ① & X_t = X_0 e^{(r - \sigma^2/2)t + \sigma W_t} \\ ② & m_t^X = \min_{0\le u\le t} X_u = X_0 e^{\sigma m_t^\xi} \end{cases}$

For $B \le H \wedge X_0$

$$\mathbb{P}(X_t \ge H,\ m_t^X \le B) = \mathbb{P}\left(\xi_t \ge \tfrac{1}{\sigma}\ln\left(\tfrac{H}{X_0}\right),\ m_t^\xi \le \tfrac{1}{\sigma}\ln\left(\tfrac{B}{X_0}\right)\right)$$

$$= \left(\frac{B}{X_0}\right)^{\left(\frac{2r - \sigma^2}{\sigma^2}\right)} \mathcal{N}\left(\frac{\ln\left(\frac{B^2}{X_0 H}\right) + (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right)$$

$$= \left(\frac{B}{X_0}\right)^{\frac{2r - \sigma^2}{\sigma^2}} \mathbb{P}\left(W_t \le \frac{1}{\sigma}\left\{\ln\left(\frac{B^2}{X_0 H}\right) + (r - \sigma^2/2)t\right\}\right)$$

for $B \leq H \wedge X_0$

$$\mathbb{P}(X_t \leq H, m_t^X \leq B) = \mathbb{P}(m_t^X \leq B) - \mathbb{P}(X_t \geq H, m_t^X \leq B)$$

$$= \left(\frac{B}{X_0}\right)^{\left(\frac{2r-\sigma^2}{\sigma^2}\right)} \mathbb{P}\left(W_t \leq \frac{1}{\sigma}\left\{\ln\left(\frac{B}{X_0}\right) + (r-\sigma^2/2)t\right\}\right)$$

$$- \left(\frac{B}{X_0}\right)^{\left(\frac{2r-\sigma^2}{\sigma^2}\right)} \mathbb{P}\left(W_t \leq \frac{1}{\sigma}\left\{\ln\left(\frac{B^2}{X_0 H}\right) + (r-\sigma^2/2)t\right\}\right)$$

for $B > X_0$, $\quad \frac{1}{\sigma}\ln\left(\frac{B}{X_0}\right) > 0$ thus $m_t^X \leq B$ a.s.

$$\Rightarrow \quad \mathbb{P}(X_t \leq H, m_t^X \leq B) = \mathbb{P}(X_t \leq H)$$

for $B > H$

Since $m_t^X \leq X_t$, one has $\{X_t \leq H\} \subseteq \{m_t^X \leq B\}$

$$\Rightarrow \quad \mathbb{P}(X_t \leq H, m_t^X \leq B) = \mathbb{P}(X_t \leq H)$$

Summary

$$\mathbb{P}(X_t \leq H, m_t^X \leq B) = \begin{cases} \mathbb{P}(m_t^X \leq B) - \mathbb{P}(X_t \geq H, m_t^X \leq B) & \text{if } B \leq H \wedge X_0 \\ \mathbb{P}(X_t \leq H) & \text{if } B > H \text{ or } B > X_0. \end{cases}$$

Adding other observation dates

$$\mathbb{P}(X_{t_1} \leq H_1 \cdots X_{t_m} \leq H_m, m_{t_m}^X \leq B)$$

$$= \mathbb{E}\left[\mathbb{1}_{X_{t_1} < H_1} \cdots \mathbb{1}_{X_{t_{m-1}} \leq H_{m-1}} \mathbb{P}\left(X_{t_m} \leq H_m, m_{t_m}^X \leq B \mid \mathcal{F}_{t_{m-1}}\right)\right]$$

Drifted Brownian Motion for Maximum

Let $\xi_t = \mu t + W_t$ and $M_t^{\xi} = \max\limits_{0 \le u \le t} \xi_u$

$$p = \mathbb{P}(\xi_t \le x,\ M_t^{\xi} \ge y) = \mathbb{E}\left[e^{\mu W_T - \mu^2/2\, T} \mathbb{1}_{W_t \le x} \mathbb{1}_{M_t^W \ge y} \right]$$

from reflection principle, $\mathbb{P}(W_t \le x,\ M_t^W \ge y) = \mathbb{P}(W_t \ge 2y - x,\ M_t^W \ge y)$

$$= \mathbb{P}(2y - W_t \le x,\ M_t^W \ge y)$$

$$p = \mathbb{E}\left[e^{\mu(2y - W_T) - \mu^2/2\, T} \mathbb{1}_{2y - W_t \le x} \mathbb{1}_{M_t^W \ge y} \right]$$

$$= e^{2\mu y}\, \mathbb{E}\left[e^{-\mu W_T - \mu^2/2\, T} \mathbb{1}_{W_t \ge 2y - x} \mathbb{1}_{M_t^W \ge y} \right]$$

$$= e^{2\mu y}\, \mathbb{P}(-\mu t + W_t \ge 2y - x)$$

$$= e^{2\mu y}\, \mathbb{P}(W_t \ge 2y - x + \mu t)$$

GBM : Maximum

Let $X_t = X_0 e^{\sigma \xi_t}$ with $\begin{cases} \xi_t = \mu t + W_t \\ \mu = \dfrac{r - \sigma^2/2}{\sigma} \end{cases}$

Then $\begin{cases} ① \quad X_t = X_0 e^{(r - \sigma^2/2)t + \sigma W_t} \\ ② \quad M_t^X = \max\limits_{0 \le u \le t} X_u = X_0 e^{\sigma M_t^{\xi}} \end{cases}$

For $B \ge H \vee X_0$

$$\mathbb{P}(X_t \le H,\ M_t^X \ge B) = \mathbb{P}(X_0 e^{\sigma \xi_t} \le H,\ X_0 e^{\sigma M_t^{\xi}} \ge B)$$

$$= \mathbb{P}\left(\xi_t \le \frac{1}{\sigma} \ln\left(\frac{H}{X_0}\right),\ M_t^{\xi} \ge \frac{1}{\sigma} \ln\left(\frac{B}{X_0}\right)\right)$$

$$= \left(\frac{B}{X_0}\right)^{\left(\frac{r - \sigma^2/2}{\sigma^2/2}\right)} \mathbb{P}\left(W_t \ge \frac{\ln\left(\frac{B^2}{X_0 H}\right) + (r - \sigma^2/2)t}{\sigma} \right)$$

$$= \left(\frac{B}{X_0}\right)^{\left(\frac{r - \sigma^2/2}{\sigma^2/2}\right)} \mathcal{N}\left(- \frac{\ln\left(\frac{B^2}{X_0 H}\right) + (r - \sigma^2/2)t}{\sigma \sqrt{t}} \right)$$

In particular if $B = H$;

$$\mathbb{P}(M_t^X \ge B) = \left(\frac{B}{X_0}\right)^{\left(\frac{r - \sigma^2/2}{\sigma^2/2}\right)} \mathbb{P}\left(W_t \ge \frac{\ln\left(\frac{B}{X_0}\right) + (r - \sigma^2/2)t}{\sigma} \right)$$

Reflection Principle : Two observation dates : case Min of BM

Let $s<t$ and $y \leq 0 \wedge x$ then

$$p := \mathbb{P}(W_s \geq \xi,\ W_t \geq x,\ m_t^W \leq y) = \mathbb{E}\left[1\!\!1_{W_s \geq \xi}\, 1\!\!1_{m_s^W > y}\, \mathcal{N}\left(\frac{2y-x-W_s}{\sqrt{t-s}}\right)\right]$$

$$+ \mathbb{E}\left[1\!\!1_{W_s \geq \xi}\, 1\!\!1_{m_s^W \leq y}\, \mathcal{N}\left(\frac{x-W_s}{\sqrt{t-s}}\right)\right]$$

<u>Proof</u> By iterated expectation

$$p = \mathbb{E}\Big[1\!\!1_{W_s \geq \xi}\ \underbrace{\mathbb{P}(W_t \geq x,\ m_t^W \leq y \mid \mathcal{F}_s^W)}_{A_s}\Big]$$

Now, consider two cases

① if $m_s^W > y$ then $\{m_t^W \leq y\} = \{\min_{s<u\leq t} W_u \leq y\}$

② if $m_s^W \leq y$ then $m_t^W \leq y$ a.s.

Thus,

$$A_s = 1\!\!1_{m_s^W > y}\ \mathbb{P}\big(W_t - W_s \geq x - W_s,\ \min_{s<u\leq t} W_u - W_s \leq y - W_s \mid \mathcal{F}_s^W\big)$$

$$+ 1\!\!1_{m_s^W \leq y}\ \mathbb{P}\big(W_t - W_s \geq x - W_s \mid \mathcal{F}_s^W\big)$$

Now, since $W_s \in \mathcal{F}_s^W$ and $W_u - W_s \perp\!\!\!\perp \mathcal{F}_s^W$ for $u>s$, one has

$$A_s = 1\!\!1_{m_s^W > y}\ \mathbb{P}(B_{t-s} \geq 2y - x - \theta)\Big|_{\theta = W_s} + 1\!\!1_{m_s^W \leq y}\ \mathbb{P}(B_{t-s} > x - \theta)\Big|_{\theta=}$$

Note that on $m_s^W > y$, one has
$W_s \geq m_s^W > y$ thus $y - W_s \leq 0$ $\Big\} \Rightarrow y - W_s \leq \underbrace{(x - W_s)\wedge 0}_{\uparrow}$

from hypothesis.

Reflection Principle : Two observation dates : case of GBM.

Let $t < t_1 < t_2$

$$p := \mathbb{P}\left(X_{t_1} \leqslant H_1,\ X_{t_2} \leqslant H_2,\ m^X_{t_2} \leq B \,\middle|\, \mathcal{F}_t \right) = \begin{cases} \mathbb{P}\left(X_{t_1} \leq H_1,\ X_{t_2} \leq H_2 \,|\, \mathcal{F}_t\right) & \text{if } B > H_2 \wedge \text{[illegible]} \\ \mathbb{E}\left[\mathbb{1}_{X_{t_1} \leq H_1} A_{t_1} \,|\, \mathcal{F}_t \right] & \text{otherwise} \end{cases}$$

where
$$A_{t_1} = \mathcal{N}\left(\frac{\ln\left(\frac{H_2}{X_{t_1}}\right) - (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right) \mathbb{1}_{m^X_{t_1} \leq B} + \mathcal{N}\left(\frac{\ln\left(\frac{B}{X_{t_1}}\right) - (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right) * \mathbb{1}_{m^X_{t_1} \text{[illegible]}}$$
$$- \mathbb{1}_{m^X_{t_1} > B}\ B^{\left(\frac{r - \sigma^2/2}{\sigma^2/2}\right)}\ \mathcal{N}\left(\frac{\ln\left(\frac{B^2}{X_{t_1}^2 H_2}\right) + (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right)$$

Proof. The case $B > H_2 \wedge X_{t_1}$ is trivial. Assume that $B \leq H_2 \wedge X_{t_1}$ then

$$p = \mathbb{E}\Big[\mathbb{1}_{X_{t_1} \leq H_1} \underbrace{\mathbb{P}\left(X_{t_2} \leq H_2,\ m^X_{t_2} \leq B \,|\, \mathcal{F}_{t_1} \right)}_{=: A_{t_1}} \,\Big|\, \mathcal{F}_t \Big]$$

Note that $Y_u := \frac{X_u}{X_{t_1}}$ $u \geqslant t_1$ is a GBM $\perp\!\!\!\perp \mathcal{F}_{t_1}$. Two cases:

(1) $\{m^X_{t_1} \leq B\}$ then $m^X_{t_2} \leq B$ a.s. $\Rightarrow A_{t_1} = \mathbb{P}\left(\frac{X_{t_2}}{X_{t_1}} \leqslant \frac{H_2}{X_{t_1}} \,\middle|\, \mathcal{F}^X_{t_1} \right)$

(2) $\{m^X_{t_1} > B\}$ then $\{m^X_{t_2} \leq B\} = \{\min\limits_{t_1 < u \leq t_2} X_u \leq B\}$

$$\Rightarrow A_{t_1} = \mathbb{P}\left(\frac{X_{t_2}}{X_{t_1}} \leq \frac{H}{X_{t_1}},\ \min_{t_1 < u \leq t_2} \frac{X_u}{X_{t_1}} \leq \frac{B}{X_{t_1}} \,\middle|\, \mathcal{F}_{t_1} \right)$$

cf. page 5 from the drafts

$$= \underbrace{\mathbb{P}\left(m^Y_{t_2} \leqslant \frac{B}{X_{t_1}} \,\middle|\, \mathcal{F}_{t_1} \right)}_{= \mathbb{P}\left(Y_{t_2} \leq \frac{B}{X_{t_1}} \,\middle|\, \mathcal{F}_{t_1} \right)} - \underbrace{\mathbb{P}\left(Y_{t_2} \geqslant \frac{H_2}{X_{t_1}},\ m^Y_{t_2} \leqslant \frac{B}{X_{t_1}} \,\middle|\, \mathcal{F}_{t_1} \right)}_{= \mathbb{P}\left(\cdots \,|\, \mathcal{F}_{t_1} \right) \text{ reflection}}$$

Thus,

$$A_{t_1} = \mathbb{1}_{m^X_{t_1} \leq B}\ \mathcal{N}\left(\frac{\ln\left(\frac{H_2}{X_{t_1}}\right) - (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right)$$
$$+ \mathbb{1}_{m^X_{t_1} > B}\ \mathcal{N}\left(\frac{\ln\left(\frac{B}{X_{t_1}}\right) - (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right)$$
$$- \mathbb{1}_{m^X_{t_1} > B}\ \mathcal{N}\left(\frac{\ln\left(\frac{B^2}{X_{t_1} H_2}\right) + (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right) \left(\frac{B}{X_{t_1}} \right)^{\left(\frac{r - \sigma^2/2}{\sigma^2/2} \right)}$$

Corollary. For $t < t_1 < t_2$,

$$\mathbb{P}(X_{t_1} \le H_1,\ X_{t_2} \le H_2,\ m^X_{t_2} \le B \mid \mathcal{F}_t)$$

$$= \begin{cases} \mathbb{P}(X_{t_1} \le H_1,\ X_{t_2} \le H_2 \mid \mathcal{F}_t) & \text{if } B > H_2 \wedge X_{t_1} \\ \mathbb{E}[\mathbb{1}_{X_{t_1} \le H_1} \mathbb{1}_{m^X_{t_1} \le B}\, h(X_{t_1}) \mid \mathcal{F}_t] & \text{otherwise} \\ \quad + \mathbb{E}[\mathbb{1}_{X_{t_1} \le H_1} \mathbb{1}_{m^X_{t_1} \ge B}\, g(X_{t_1}) \mid \mathcal{F}_t] & \end{cases}$$

in which
$$h(x) = \mathcal{N}\left(\frac{\ln\left(\frac{H_2}{x}\right) - (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right)$$

$$g(x) = \mathcal{N}\left(\frac{\ln(B/x) - (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right)$$
$$- \left(\frac{B}{x}\right)^{\zeta} \mathcal{N}\left(\frac{\ln\left(\frac{B^2}{x H_2}\right) + (r - \sigma^2/2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} \right)$$

with $\zeta = \dfrac{r - \sigma^2/2}{\sigma^2/2}$

Put Down & In One observation date only. for $t < t_1$

$$\mathbb{1}_{\tau > t}\ \mathbb{E}[\mathbb{1}_{X_{t_1} \le H_1} (K - X_T)^+ \mathbb{1}_{\min_{0 \le t \le T} X_t \le B} \mid \mathcal{F}_t]$$

$$= \mathbb{1}_{\tau > t}\ \mathbb{E}[\mathbb{1}_{X_{t_1} \le H_1} \mathbb{1}_{\min_{0 \le t \le t_1} X_t > B}\, \varphi(X_{t_1}) \mid \mathcal{F}_t]$$
$$+ \mathbb{1}_{\tau > t}\ \mathbb{E}[\mathbb{1}_{X_{t_1} \le H_1} \mathbb{1}_{\min_{0 \le t \le t_1} X_t \le B}\, \tilde{\varphi}(X_{t_1}) \mid \mathcal{F}_t]$$

$\varphi(x) = \mathbb{E}[(K - X_T)^+ \mathbb{1}_{\min_{t_1 \le t \le T} X_t \le B} \mid X_{t_1} = x] \Rightarrow$ Put Down & In

$\tilde{\varphi}(x) = \mathbb{E}[(K - X_T) \mid X_{t_1} = x] \Rightarrow$ put vanilla

(1) if $\min_{0 \le t \le t_1} X_t \le B \Rightarrow$ barrier already reached $\Rightarrow$ vanilla after t_1

(2) if $\min_{0 \le t \le t_1} X_t > B \Rightarrow \{\min_{0 \le t \le T} X_t \le B\} = \{\min_{t_1 \le t \le T} X_t \le B\}$

$\Rightarrow$ Put Down & In after t_1

Put Down & Out

$$\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}(K-X_T)^+\,\mathbb{1}_{\min\limits_{0\le t\le T} X_t>B}\ \Big|\ \mathcal{F}_t\Big]$$

$$=\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}\,\mathbb{1}_{\min\limits_{0\le t\le t_1} X_t>B}\ \varphi(X_{t_1})\ \Big|\ \mathcal{F}_t\Big]\ +\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}\cdot\mathbb{1}_{\min\limits_{0\le t\le t_1} X_t<B}\ \Big|\ \mathcal{F}_t\Big]\ast\text{cash}$$

where $\varphi(x)=\mathbb{E}\Big[(K-X_T)^+\,\mathbb{1}_{\min\limits_{t_1\le t\le T} X_t>B}\ \Big|\ X_{t_1}=x\Big]$ = put Down & Out

① if $\min\limits_{0\le t\le t_1} X_t<B$ $\Rightarrow$ barrier already reached $\Rightarrow$ price $=0$

② if $\min\limits_{0\le t\le t_1} X_t>B$ $\Rightarrow$ $\Big\{\min\limits_{0\le t\le T} X_t>B\Big\}=\Big\{\min\limits_{t_1\le t\le T} X_t>B\Big\}$

Put Up & In

$$\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}(K-X_T)^+\,\mathbb{1}_{\max\limits_{0\le t\le T} X_t>B}\ \Big|\ \mathcal{F}_t\Big]$$

$$=\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}\,\mathbb{1}_{\max\limits_{0\le t\le t_1} X_t\le B}\ \varphi(X_{t_1})\ \Big|\ \mathcal{F}_t\Big]$$

$$+\ \mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}\,\mathbb{1}_{\max\limits_{0\le t\le t_1} X_t>B}\ \tilde{\varphi}(X_{t_1})\ \Big|\ \mathcal{F}_t\Big]$$

① if $\max\limits_{0\le t\le t_1} X_t\le B$ $\longrightarrow$ $\Big\{\max\limits_{0\le t\le T} X_t\ge B\Big\}=\Big\{\max\limits_{t_1\le t\le T} X_t\ge B\Big\}$ $\longrightarrow$ Put Up & In

② if $\max\limits_{0\le t\le t_1} X_t>B$ $\longrightarrow$ barrier reached before t_1 $\longrightarrow$ vanilla Put option after t_1

$$\varphi(x)=\mathbb{E}\Big[(K-X_T)^+\,\mathbb{1}_{\max\limits_{t_1\le t\le T} X_t>B}\ \Big|\ X_{t_1}=x\Big]=\text{Put Up\&In}$$

$$\tilde{\varphi}(x)=\mathbb{E}\Big[(K-X_T)^+\ \Big|\ X_{t_1}=x\Big]=\text{Put Vanilla.}$$

Put Up & Out

$$\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}(K-X_T)^+\,\mathbb{1}_{\max\limits_{0\le t\le T} X_t\le B}\ \Big|\ \mathcal{F}_t\Big]$$

$$=\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}\,\mathbb{1}_{\max\limits_{0\le t\le t_1} X_t<B}\ \varphi(X_{t_1})\ \Big|\ \mathcal{F}_t\Big]\ +\mathbb{1}_{\tau>t}\ \mathbb{E}\Big[\mathbb{1}_{X_{t_1}\le H_1}\,\mathbb{1}_{\max\limits_{0\le t\le t_1} X_t>B}\ \Big|\ \mathcal{F}_t\Big]\text{cash}$$

① if $\max\limits_{0\le t\le t_1} X_t>B$ $\longrightarrow$ barrier reached $\rightarrow$ out $\longrightarrow$ price $=0$.

② if $\max\limits_{0\le t\le t_1} X_t<B$ $\Rightarrow$ $\Big\{\max\limits_{0\le t\le T} X_t>B\Big\}=\Big\{\max\limits_{t_1\le t\le T} X_t>B\Big\}$

$$\varphi(x)=\mathbb{E}\Big[(K-X_T)^+\,\mathbb{1}_{\max X_t\le B}\ \Big|\ X_{t_1}=x\Big]\qquad \text{Put Up\&Out}$$

Unconditional joint probabilities for GBM

Down&In

$$\mathbb{P}(X_u \leq H,\ m_u^X \leq B) = \begin{cases} \mathbb{P}(m_u^X \leq B) - \mathbb{P}(X_u \geq H,\ m_u^X \leq B) & \text{if } B \leq H \wedge X_0 \\ \mathbb{P}(X_u \leq H) & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_u \leq H,\ m_u^X > B) = \begin{cases} \mathbb{P}(X_u \leq H) - \mathbb{P}(m_u^X \leq B) + \mathbb{P}(X_u \geq H,\ m_u^X \leq B) & B \leq H \wedge \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_u \leq H,\ M_u^X \leq B) = \begin{cases} \mathbb{P}(X_u \leq H) - \mathbb{P}(X_u \leq H,\ M_u^X > B) & \text{if } B > H \vee X_0 \\ \mathbb{P}(M_u^X \leq B) & \text{if } X_0 < B < H \\ 0 & \text{if } B \leq X_0 \end{cases}$$

$$\mathbb{P}(X_u \leq H,\ M_u^X > B) = \begin{cases} \dots & \text{if } B > H \vee X_0 \\ \mathbb{P}(X_u \leq H) - \mathbb{P}(M_u^X \leq B) & \text{if } X_0 < B \leq H \\ \mathbb{P}(X_u \leq H) & \text{if } B < X_0 \end{cases}$$

Conditional probabilities: $\zeta = \dfrac{r - \sigma^2/2}{\sigma^2/2} \rightarrow \dfrac{2(r - \text{div}) - \sigma^2}{\sigma^2}$

$$\mathbb{P}(X_u \geq H,\ m_u^X \leq B \mid \mathcal{F}_t) = \mathbb{P}\Big(\underbrace{\frac{X_u}{X_t}}_{\perp\!\!\!\perp \mathcal{F}_t^X} \geq \underbrace{\frac{H}{X_t}}_{\in \mathcal{F}_t^X},\ \underbrace{\min_{t<s\leq u} \frac{X_s}{X_t}}_{\perp\!\!\!\perp \mathcal{F}_t^X} \leq \underbrace{\frac{B}{X_t}}_{\in \mathcal{F}_t^X} \ \Big|\ \mathcal{F}_t^X \Big)$$

(1)

$$= \left(\frac{B}{X_t}\right)^{\zeta} \mathcal{N}\left(\frac{\ln\left(\frac{B^2}{X_t H}\right) + (r - \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}} \right)$$

$$\mathbb{P}(X_u \geq H,\ m_u^X \leq B \mid \mathcal{F}_t^X) + \mathbb{P}(X_u \geq H,\ m_u^X > B \mid \mathcal{F}_t^X) = \mathbb{P}(X_u > H \mid \mathcal{F}_t^X)$$

Thus for $H = B$,

$$\mathbb{P}(m_u^X > B \mid \mathcal{F}_t) = \mathbb{P}(X_u \geq B,\ m_u^X > B \mid \mathcal{F}_t^X)$$
$$= \mathbb{P}(X_u > B \mid \mathcal{F}_t^X) - \mathbb{P}(X_u \geq B,\ m_u^X \leq B \mid \mathcal{F}_t^X)$$

$$\Rightarrow \quad \mathbb{P}(m_u^X \leq B \mid \mathcal{F}_t) = \mathbb{P}(X_u \leq B \mid \mathcal{F}_t^X) + \mathbb{P}(X_u \geq B,\ m_u^X \leq B \mid \mathcal{F}_t^X)$$

(2)

$$= \mathcal{N}\left(\frac{\ln(B/X_t) - (r - \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}} \right) + \left(\frac{B}{X_t}\right)^{\zeta} \mathcal{N}\left(\frac{\ln(B/X_t) + (r - \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}} \right)$$

(3) $\mathbb{P}(X_u \leq H \mid \mathcal{F}_t) = \mathcal{N}\left(\dfrac{\ln(H/X_t) - (r - \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}} \right)$

④ $$\mathbb{P}(X_u \le H,\ M_u^X \ge B \mid \mathcal{F}_t) = \mathbb{P}\left(\frac{X_u}{X_t} \le \frac{H}{X_t},\ \max_{t<s\le u} \frac{X_s}{X_t} \ge \frac{B}{X_t} \,\Big|\, \mathcal{F}_t\right)$$

$$= \left(\frac{B}{X_t}\right)^{\zeta} \mathcal{N}\left(-\frac{\ln\left(\frac{B^2}{X_t H}\right) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)$$

⑤ $$\mathbb{P}(M_u^X \le B \mid \mathcal{F}_t) = \mathbb{P}(X_u \le B \mid \mathcal{F}_t) - \mathbb{P}(X_u \le B,\ M_u^X > B \mid \mathcal{F}_t)$$

$$= \mathcal{N}\left(\frac{\ln(B/X_t) - (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)$$

$$- \left(\frac{B}{X_t}\right)^{\zeta} \mathcal{N}\left(-\frac{\ln\left(\frac{B}{X_t}\right) + (r-\sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}\right)$$

Down & In	$\mathbb{1}_{m_{t_1}^X > B} \underbrace{\varphi(X_{t_1})}_{\text{Down \& In}} + \mathbb{1}_{m_{t_1}^X \le B} \underbrace{\tilde{\varphi}(X_{t_1})}_{\text{vanilla}}$
Down & Out	$\mathbb{1}_{m_{t_1}^X > B} \underbrace{\varphi(X_{t_1})}_{\text{Down \& Out}} + \mathbb{1}_{m_{t_1}^X \le B} \underbrace{\tilde{\varphi}(X_{t_1})}_{\text{airbag}}$
Up & In	$\mathbb{1}_{M_{t_1}^X \le B} \underbrace{\varphi(X_{t_1})}_{\text{Up \& In}} + \mathbb{1}_{M_{t_1}^X > B} \underbrace{\tilde{\varphi}(X_{t_1})}_{\text{vanilla}}$
Up & Out	$\mathbb{1}_{M_{t_1}^X < B} \underbrace{\varphi(X_{t_1})}_{\text{Up \& Out}} + \mathbb{1}_{M_{t_1}^X > B} \underbrace{\tilde{\varphi}(X_{t_1})}_{\text{airbag}}$

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option leg price for continous barrier monitoring with only <u>one</u> observation date

for $t \ge t_1$ option leg price $= \mathbb{1}_{\tau > t}\, \mathbb{E}[\text{option payoff} \mid \mathcal{F}_t]$ (usual Eq.)

for $t < t_1$ option leg price $= \mathbb{1}_{\tau > t}\, \mathbb{E}[\mathbb{1}_{X_{t_1} \le H_1} \text{option payoff} \mid \mathcal{F}_t]$

$$= \mathbb{1}_{\tau > t}\, \mathbb{E}\Big[\mathbb{1}_{X_{t_1} \le H_1} \underbrace{\mathbb{E}[\text{option payoff} \mid \mathcal{F}_{t_1}]}_{\Downarrow} \,\Big|\, \mathcal{F}_t\Big]$$

$$\mathbb{1}_A\, \varphi(X_{t_1}) + \mathbb{1}_{A^c}\, \tilde{\varphi}(X_{t_1})$$

where for each case of barrier, the indicator functions and $\varphi(\cdot)$ and $\tilde{\varphi}(\cdot)$ are as in the above table. In each case, the option type can be call or put, so the corresponding φ and $\tilde{\varphi}$ are call and put.

Conditional expectation computation

We need to compute $\hat{\pi}_t^{\gamma} = \mathbb{E}[\mathbb{1}_{X_{t_1} \leq H_1} \mathbb{1}_A X_{t_1}^{\gamma} \mid \mathcal{F}_t^X]$ for $t < t_1$

where the set A is one of : $m_{t_1}^X \leq B$, $m_{t_1}^X > B$
$M_{t_1}^X \leq B$, $M_{t_1}^X > B$

As in Lemma ..., the idea is change of probability measure.

Recall that, given X is a GBM, X^{γ} is also a GBM

Moreover for $Y_{\gamma,t} := \eta(t) X_t^{\gamma}$ with $\eta(t) = \exp\{-\gamma r t - \gamma(\gamma-1)\frac{\sigma^2}{2} t\}$

$\{Y_{\gamma,t} \, ; \, t \geq 0\}$ is an $(\mathbb{F}, Q)$-martingale.

$\frac{dQ^{\gamma}}{dQ} = Y_{\gamma,T}$ from Bayes' formula $\mathbb{E}^Q[Y_{\gamma,T}\xi \mid \mathcal{F}_t] = Y_{\gamma,t} \mathbb{E}^{Q^{\gamma}}[\xi \mid \mathcal{F}_t]$

Since $dY_{\gamma,t} = \gamma\sigma Y_{\gamma,t} dW_t$; from Girsanov,

$W_t^{\gamma} := W_t - \gamma\sigma t$ is an $(\mathbb{F}, Q^{\gamma})$-Weiner process.

In particular the Q^{γ}-dynamics of X are given by

$$dX_t = (r + \gamma\sigma^2) X_t \, dt + \sigma X_t \, dW_t^{\gamma}$$

This mean that for computing expectation of form $(*)$, one can use Eq ... replacing (r) by $(r + \gamma\sigma^2)$.