Problem. *Let a, b, c be non-negative real numbers such that $ab + bc + ca > 0$. Prove that*

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \geqslant \frac{9}{4(ab+bc+ca)}.$$

(Ji Chen, Iran TST 1996)

Solution. (*Nguyen Van Huyen*) We have the following estimate

$$\frac{ab+bc+ca}{(b+c)^2} \geqslant \frac{3}{4} \cdot \frac{10a^2 - 2(b^2+c^2) + 4a(b+c) + 7bc}{2(a^2+b^2+c^2) + 5(ab+bc+ca)}. \tag{1}$$

We rewrite the estimate (1) as a polynomial in the variable a as follows

$$8(b+c)a^3 - 2(5b^2+6bc+5c^2)a^2 - 4(b+c)(b^2-4bc+c^2)a + (6b^2+11bc+6c^2)(b-c)^2 \geqslant 0.$$

Using the AM-GM inequality, we have

$$(b+c)a^3 + \frac{a(b^2+c^2)^2}{b+c} \geqslant 2a^2(b^2+c^2).$$

It remains to prove that

$$\left[16a^2(b^2+c^2) - 2(5b^2+6bc+5c^2)a^2\right] - \left[\frac{8a(b^2+c^2)^2}{b+c} + 4(b+c)(b^2-4bc+c^2)a\right]$$
$$+(6b^2+11bc+6c^2)(b-c)^2 \geqslant 0,$$

equivalent to

$$6a^2(b-c)^2 - \frac{4a(3b^2+4bc+3c^2)(b-c)^2}{b+c} + (6b^2+11bc+6c^2)(b-c)^2 \geqslant 0,$$

or

$$\frac{(b-c)^2}{b+c}[6(b+c)a^2 - 4(3b^2+4bc+3c^2)a + (b+c)(6b^2+11bc+6c^2)] \geqslant 0.$$

Let
$$f(a) = 6(b+c)a^2 - 4(3b^2+4bc+3c^2)a + (b+c)(6b^2+11bc+6c^2).$$
The quadratic polynomial $f(a) \geqslant 0$ holds because its discriminant is

$$\begin{aligned}\Delta_a &= [2(3b^2+4bc+3c^2)]^2 - 6(b+c)^2(6b^2+11bc+6c^2)\\ &= -2bc(21b^2+34bc+21c^2) \leqslant 0.\end{aligned}$$

The proof is completed. □

Remark. The estimate (1) was proposed by *Dinh Binh Duong*.