Problem (Pham Kim Hung). *Let $a,\ b,\ c$ be non-negative numbers satisfying $a+b+c=2$. Prove that*

$$\frac{bc}{a^2+1}+\frac{ca}{b^2+1}+\frac{ab}{c^2+1}\leqslant 1. \tag{1}$$

Solution 1. (*Vo Quoc Ba Can, Nguyen Van Huyen*) We have the following estimate

$$\frac{bc}{a^2+1}\leqslant \frac{bc(b+c)}{ab(a+b)+bc(b+c)+ca(c+a)}. \tag{2}$$

If $bc=0$, then (2) is obvious.

If $bc>0$, dividing both sides by bc and homogenizing the inequality becomes

$$[4a^2+(a+b+c)^2](b+c)\geqslant 4[ab(a+b)+bc(b+c)+ca(c+a)].$$

The inequality is quadratic in a, so we can rewrite it as

$$(b+c)a^2-2a(b-c)^2+(b+c)(b-c)^2\geqslant 0,$$

or

$$(b+c)[a^2+(b-c)^2]\geqslant 2a(b-c)^2.$$

Since $b+c\geqslant |b-c|$, using the AM-GM inequality, we have

$$(b+c)[a^2+(b-c)^2]\geqslant (b+c)\cdot 2a\cdot|b-c| = 2a(b+c)|b-c|\geqslant 2a(b-c)^2.$$

Thus

$$\sum\frac{bc}{a^2+1}\leqslant \sum\frac{bc(b+c)}{ab(a+b)+bc(b+c)+ca(c+a)}=2.$$

The equality holds for $a=0,\ b=c=1$ (or any cyclic permutation). □

Solution 2. (*Nguyen Van Huyen*) We have the following estimate

$$\frac{bc}{a^2+1}\leqslant\frac{a^2+b^2+c^2-2a(b+c)+6bc}{3(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

Indeed, since

$$[a^2+b^2+c^2-2a(b+c)+6bc]\left[a^2+\frac{1}{4}(a+b+c)^2\right]-bc[3(a^2+b^2+c^2)+2(ab+bc+ca)]$$

$$=a^2(b+c-a)^2+\frac{1}{4}(a+b-c)^2(c+a-b)^2\geqslant 0.$$

We have

$$\frac{bc}{a^2+\frac{(a+b+c)^2}{4}}\leqslant\frac{a^2+b^2+c^2-2a(b+c)+6bc}{3(a^2+b^2+c^2)+2(ab+bc+ca)},$$

or

$$\frac{bc}{a^2+1}\leqslant\frac{a^2+b^2+c^2-2a(b+c)+6bc}{3(a^2+b^2+c^2)+2(ab+bc+ca)}.$$

□

Note. From this solution, we see that (1) is always true for all ***real*** numbers.