Problem (Tran Nam Dung, Viet Nam TST 2009). *Let a, b, c are positive real numbers. Find all values of the real number k, such that*

$$\left(k+\frac{a}{b+c}\right)\left(k+\frac{b}{c+a}\right)\left(k+\frac{c}{a+b}\right) \geqslant \left(k+\frac{1}{2}\right)^3. \tag{1}$$

Solution. (*Nguyen Van Huyen*) We have a lemma [1]

Lemma. *Given six **real** numbers a, b, c, x, y, z satisfying*

$$x+y+z>0, \quad xy+yz+zx \geqslant 0, \quad a+b+c=0.$$

Prove that

$$xa^2+yb^2+zc^2 \geqslant \frac{(xy+yz+zx)(a^2+b^2+c^2)}{2(x+y+z)}.$$

Let $a=b=1$, $c \to 0$ the inequality (1) becomes

$$4k^2+2k-1 \geqslant 0.$$

We will prove that $4k^2+2k-1 \geqslant 0$, which is the condition we are looking for. We have

$$\frac{1}{8(a+b)(b+c)(c+a)}\sum[4k^2(a+b)+(2k-1)c](a-b)^2 \geqslant 0.$$

Let

$$f(a,b,c)=\sum[4k^2(a+b)+(2k-1)c](a-b)^2,$$

then

$$f(a,b,c) \geqslant \frac{\left[8k^2(2k^2+2k-1)\sum a^2+(48k^4+16k^3-4k^2-4k+1)\sum bc\right]\sum(a^2-bc)}{[3(2k^2+2k-1)+2(k-1)^2](a+b+c)}.$$

□

[1] https://nguyenhuyenag.wordpress.com/2018/05/12/the-sos-theorem/