

Problem. Let a, b, c are positive real numbers such that $a^3 + b^3 + c^3 = 5abc$. Prove that

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 10.$$

Solution. (Nguyen Van Huyen) From the condition, we have that a, b, c are the lengths of the sides of a triangle.

Indeed, suppose a, b, c are **not** the lengths of the sides of a triangle, example $a > b + c$. Then, there exists a real number $x > 0$ such that $a = x + b + c$. At this point

$$\begin{aligned} 0 &= 5 - abc - a^3 - b^3 - c^3 \\ &= 5(x + b + c)bc - (x + b + c)^3 - b^3 - c^3 \\ &= -[x^3 + 3(b + c)x^2 + (3b^2 + bc + 3c^2)x + 2(b + c)(b - c)^2] \\ &< 0 \quad (\text{contradiction}). \end{aligned}$$

Therefore, a, b, c must be the lengths of the sides of a triangle.

Now, we have

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) - 5 - \frac{a^3 + b^3 + c^3}{abc} = \frac{(a + b - c)(b + c - a)(c + a - b)}{abc} \geq 0.$$

Thus

$$(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 10.$$

The equality holds for $a = b = \frac{c}{2}$ (or any cyclic permutation). □