Problem (Vasile Cirtoaje). *If a, b, c are positive real numbers, then*

$$\frac{a^2}{a^2+ab+b^2}+\frac{b^2}{b^2+bc+c^2}+\frac{c^2}{c^2+ca+a^2}\geqslant 1. \tag{1}$$

Solution. (*Nguyen Van Huyen*) Since

$$\sum\frac{a(ab-c^2)}{a+b+c}=0,$$

and

$$\sum\frac{b^2+ab}{a^2+b^2+c^2+ab+bc+ca}=1.$$

We see that the inequality (1) follows by adding the inequality below and two similar inequalities

$$\frac{b^2}{b^2+bc+c^2}\geqslant\frac{b^2+ab+\dfrac{a(ab-c^2)}{a+b+c}}{a^2+b^2+c^2+ab+bc+ca}. \tag{2}$$

Indeed, we can rewrite the inequality (2) as

$$\frac{b^2}{b^2+bc+c^2}-\frac{b^2+ab}{a^2+b^2+c^2+ab+bc+ca}\geqslant\frac{\dfrac{a(ab-c^2)}{a+b+c}}{a^2+b^2+c^2+ab+bc+ca},$$

or

$$\frac{ab(ab-c^2)}{b^2+bc+c^2}\geqslant\frac{a(ab-c^2)}{a+b+c},$$

which is equivalent to

$$\frac{a(ab-c^2)^2}{(a+b+c)(b^2+bc+c^2)}\geqslant 0.$$

Thus, the inequality (2) is proven. □

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