Problem. *Given a triangle ABC with side lengths $BC = a,\ CA = b,\ AB = c$ and median lengths m_a, m_b, m_c drawn from vertices $A,\ B$ and C respectively*

(1) Prove that

$$(a+b+c)(a^2b^2+b^2c^2+c^2a^2) \geqslant 3abc(a^2+b^2+c^2). \tag{1}$$

(Nguyen Van Huyen)

(2) Prove that

$$\frac{m_am_b}{m_c}+\frac{m_cm_a}{m_b}+\frac{m_bm_c}{m_a} \geqslant \frac{3(m_a^2+m_b^2+m_c^2)}{m_a+m_b+m_c}.$$

(Nguyen Van Quy)

Solution (1).

First Proof. (*Nguyen Van Huyen*) Note that

$$3abc(a^2+b^2+c^2)-3abc(ab+bc+ca) = 3abc[(a-b)^2+(a-c)(b-c)],$$

and

$$\begin{aligned}&(a+b+c)(a^2b^2+b^2c^2+c^2a^2)-3abc(ab+bc+ca)\\ &\qquad = [c^3+(a^2+3ab+b^2)c-ab(a+b)](a-b)^2+(a^3+b^3+2abc)(a-c)(b-c).\end{aligned}$$

Thus, the inequality (1) can be written as

$$\left[c^3+(a^2+b^2)c-ab(a+b)\right](a-b)^2+(a^3+b^3-abc)(a-c)(b-c) \geqslant 0.$$

Asumme that $c = \max\{a,b,c\}$, then

$$(a^2+b^2)c-ab(a+b) \geqslant 2abc-ab(a+b) = ab(2c-a-b) \geqslant 0,$$

and

$$\begin{aligned}a^3+b^3-abc &= [a^3+b^3-ab(a+b)]+[ab(a+b)-abc]\\ &= (a+b)(a-b)^2+ab(a+b-c) \geqslant 0.\end{aligned}$$

The equality holds for $a=b=c$. The proof is completed. □

Second Proof. (*Pham Huu Hoai*) The inequality is equivalent to each of the following inequalities

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \geqslant \frac{3(a^2+b^2+c^2)}{a+b+c},$$

$$\left(\frac{ab}{c}-a-b+c\right)+\left(\frac{bc}{a}+\frac{ca}{b}-2c\right) \geqslant \frac{3(a^2+b^2+c^2)}{a+b+c}-(a+b+c),$$

$$\frac{(a-c)(b-c)}{c}+\frac{c(a-b)^2}{ab} \geqslant \frac{2(a-b)^2+2(a-c)(b-c)}{a+b+c},$$

$$\left(\frac{c}{ab}-\frac{2}{a+b+c}\right)(a-b)^2+\left(\frac{1}{c}-\frac{2}{a+b+c}\right)(a-c)(b-c)\geqslant 0.$$

Suppose that $c=\max\{a,b,c\}$, then

$$a+b+c-2c=a+b-c>0,$$

and

$$c(a+b+c)-2ab=a(c-b)+b(c-a)+c^2>0.$$

The proof is completed. □

Solution (2). We will prove that $m_a,\ m_b,\ m_c$ are also three sides of a triangle.

Indeed, we will prove that

$$m_a+m_b\geqslant m_c.$$

Which is equivalent to

$$\sqrt{2(b^2+c^2)-a^2}+\sqrt{2(c^2+a^2)-b^2}\geqslant\sqrt{2(a^2+b^2)-c^2},$$

or

$$2\sqrt{[2(b^2+c^2)-a^2][2(c^2+a^2)-b^2]}\geqslant a^2+b^2-5c^2.$$

It remains to show that

$$4[2(b^2+c^2)-a^2][2(c^2+a^2)-b^2]\geqslant(a^2+b^2-5c^2)^2.$$

This is true because

$$\begin{aligned}&4[2(b^2+c^2)-a^2][2(c^2+a^2)-b^2]-(a^2+b^2-5c^2)^2\\&\qquad=9\left[2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4\right]\\&\qquad=9(a+b+c)(a+b-c)(b+c-a)(c+a-b)\geqslant 0.\end{aligned}$$

Now, from (1) we get

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geqslant\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

We replace $(a,\ b,\ c)\to(m_a,\ m_b,\ m_c)$ and the inequality becomes

$$\frac{m_am_b}{m_c}+\frac{m_cm_a}{m_b}+\frac{m_bm_c}{m_a}\geqslant\frac{3(m_a^2+m_b^2+m_c^2)}{m_a+m_b+m_c}.$$

The proof is completed. □