Problem. *Let a, b, c be non-negative real numbers. Prove that*

$$a^3+b^3+c^3+2(ab^2+bc^2+ca^2) \geqslant 3(a^2b+b^2c+c^2a). \tag{1}$$

Solution 1. Note that

$$a^2b+b^2c+c^2a-3abc = c(a-b)^2+a(a-c)(b-c),$$

$$a^3+b^3+c^3-3abc = (a+b+c)[(a-b)^2+(a-c)(b-c)].$$

Write the inequality (1) as

$$a^3+b^3+c^3-3abc+2(ab^2+bc^2+ca^2-3abc) \geqslant 3(a^2b+b^2c+c^2a-3abc).$$

or

$$(a+b)(a-b)^2+(c+3b-2a)(a-c)(b-c) \geqslant 0.$$

Suppose c is between a and b, then $(a-c)(c-b) \geqslant 0$. By the AM-GM inequality, we have

$$(a-b)^2 = [(a-c)+(c-b)]^2 \geqslant 4(a-c)(c-b).$$

So, we need to show that

$$4(a+b)(a-c)(c-b)+(c+3b-2a)(a-c)(b-c) \geqslant 0.$$

or

$$(a-c)(c-b)(6a+b-c) \geqslant 0. \tag{2}$$

Now

- If $a \geqslant c$, then $6a+b-c = (a-c)+5a+b \geqslant 0$.
- If $b \geqslant c$, then $6a+b-c = (b-c)+6a \geqslant 0$.

Therefore, (2) is always true. The proof is complete. □

Note. In addition, we also have

$$c(6a+b-c) = (a-c)(c-b)+a(b+5c) \geqslant 0.$$

Solution 2. Suppose $c = \min\{a,b,c\}$. Let $a = x+c$, $b = y+c$ for $x, y \geqslant 0$.

The inequality (1) become

$$2(x^2-xy+y^2)c+x^3-3x^2y+2xy^2+y^3 \geqslant 0. \tag{3}$$

But

$$x^3-3x^2y+2xy^2+y^3 = x(x-2y)^2+y(x-y)^2 \geqslant 0.$$

Therefore, (3) is always true. □