Problem. *Let a, b, c are positive real numbers. Prove that*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geqslant 3\sqrt[6]{\frac{a^6+b^6+c^6}{3}}.$$

(Vo Quoc Ba Can)

Proof. (*Nguyen Van Huyen*) We write the inequality as

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^6\geqslant 243(a^6+b^6+c^6).$$

WLOG, suppose $a+b+c=1$, setting $q=\frac{1-t^2}{3}$ with $0\leqslant t<1$, we have

$$\begin{aligned}a^6+b^6+c^6&=3r^2+6p(p^2-2q)r+(p^2-2q)(p^4-4p^2q+q^2)\\&=3r^2+2(2t^2+1)r+\frac{1}{27}(2t^2+1)(t^4+10t^2-2).\end{aligned}$$

In addition, from [1]

$$0<r\leqslant r_0=\frac{p(9q-2p^2)+2(p^2-3q)\sqrt{p^2-3q}}{27},$$

with $q=\frac{1-t^2}{3}$, we get

$$0<r\leqslant r_0=\frac{(1+2t)(1-t)^2}{27}.$$

so

$$243(a^6+b^6+c^6)\leqslant 243\left[3r_0^2+2(2t^2+1)r_0+\frac{1}{27}(2t^2+1)(t^4+10t^2-2)\right]$$

$$\begin{aligned}&=(2t+1)^2(1-t)^4+18(2t^2+1)(2t+1)(1-t)^2+9(2t^2+1)(t^4+10t^2-2)\\&=22t^6+60t^5+90t^4+40t^3+30t^2+1.\end{aligned}$$

We need to prove

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)^6\geqslant 22t^6+60t^5+90t^4+40t^3+30t^2+1,$$

Using the inequality (*lemma* 2) [2]

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geqslant\frac{1+2t+5t^2+4t^3-3t^4}{(2t+1)(1-t^2)},$$

we will show that

$$\left(\frac{1+2t+5t^2+4t^3-3t^4}{(2t+1)(1-t^2)}\right)^6\geqslant 22t^6+60t^5+90t^4+40t^3+30t^2+1,$$

[1] https://wp.me/p7ChCZ-ht
[2] https://wp.me/p7ChCZ-ZS

equivalent to

$$(1+2t+5t^2+4t^3-3t^4)^6 \geqslant (22t^6+60t^5+90t^4+40t^3+30t^2+1)(2t+1)^6(1-t^2)^6,$$

or

$$t^2 \cdot f(t) \geqslant 0,$$

for

$$\begin{aligned} f(t) = {} & 6 - 4t - 48t^2 + 1620t^3 + 14278t^4 + 56208t^5 \\ & + 131769t^6 + 197148t^7 + 177936t^8 + 59148t^9 - 50214t^{10} \\ & - 25188t^{11} + 69318t^{12} + 40416t^{13} - 98436t^{14} - 118368t^{15} - 1320t^{16} \\ & + 46524t^{17} + 25437t^{18} + 21748t^{19} - 1962t^{20} - 13896t^{21} - 679t^{22}. \end{aligned}$$

We will show that $f(t) > 0$. Indeed, setting $t = \frac{1}{x+1} < 1$ with $x > 0$, we have

$$f(t) = f\left(\frac{1}{x+1}\right) = \frac{g(x)}{(x+1)^{22}},$$

with

$$\begin{aligned} g(x) = {} & 6x^{22} + 128x^{21} + 1254x^{20} + 9060x^{19} + 74508x^{18} + 669576x^{17} \\ & + 4975341x^{16} + 27941724x^{15} + 119289936x^{14} + 395553816x^{13} \\ & + 1037590858x^{12} + 2180644272x^{11} + 3699912978x^{10} + 5084994888x^9 \\ & + 5658046506x^8 + 5075026812x^7 + 3637113678x^6 + 2051467236x^5 \\ & + 888549669x^4 + 283986324x^3 + 62710038x^2 + 8503056x + 531441 > 0. \end{aligned}$$

The proof is completed. $\square$