Problem. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

$$\frac{a(b+c)}{a^2+2bc+3}+\frac{b(c+a)}{b^2+2ca+3}+\frac{c(a+b)}{c^2+2ab+3}\leqslant 1.$$

Solution. *(Nguyen Van Huyen)* Using the Cauchy-Schwarz inequality and AM-GM inequality, we have

$$a^2+\frac{(a+b+c)^2}{3}\geqslant\frac{(a+a+b+c)^2}{1+3}=\frac{(2a+b+c)^2}{4}\geqslant 2a(b+c).$$

So

$$2bc+a^2+\frac{(a+b+c)^2}{3}\geqslant 2(ab+bc+ca),$$

or

$$\frac{a(b+c)}{a^2+2bc+\frac{(a+b+c)^2}{3}}\leqslant\frac{a(b+c)}{2(ab+bc+ca)}.$$

Therefore

$$\sum\frac{a(b+c)}{a^2+2bc+3}\leqslant\sum\frac{a(b+c)}{2(ab+bc+ca)}=1.$$

The proof is completed. □