A GENERAL INEQUALITY

nguyenhuyenag@gmail.com

Problem. Let $a,\ b,\ c$ be positive real numbers. Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \geqslant \frac{9}{(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2000)

1. A general problem

Problem 1. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\sum \frac{1}{a^2+ab+b^2} \geqslant \frac{(28k+1)(a^2+b^2+c^2)-2(14k-13)(ab+bc+ca)}{3(a+b+c)^2\left[k(a^2+b^2+c^2)+(1-k)(ab+bc+ca)\right]}, \tag{1}$$

for all $0 \leqslant k \leqslant k_0$, with k_0 is a root of the equation $281k^3-2226k^2+780k-88=0$.

(Nguyen Van Huyen, 2020)

For $k \neq \frac{1}{3}$, we can write inequality (1) as

$$\sum \frac{1}{a^2+ab+b^2}+\frac{\frac{k+1}{3(3k-1)}}{k(a^2+b^2+c^2)+(1-k)(ab+bc+ca)} \geqslant \frac{\frac{4(7k-2)}{3k-1}}{(a+b+c)^2}.$$

If $k=0$, we get

Problem 2. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \geqslant \frac{1}{3(ab+bc+ca)}+\frac{8}{(a+b+c)^2}.$$

(Vo Quoc Ba Can, 2010)

Problem 3. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \geqslant \frac{21}{2(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

(Vo Quoc Ba Can, 2010)

If $k = \frac{1}{2}$, we get

Problem 4. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\sum \frac{1}{a^2+ab+b^2} + \frac{2}{a^2+b^2+c^2+ab+bc+ca} \geqslant \frac{12}{(a+b+c)^2}.$$

(Pham Huu Hoai, 2020)

If $k = 1$, we get

Problem 5. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} + \frac{1}{3\left(a^2+b^2+c^2\right)} \geqslant \frac{10}{(a+b+c)^2}.$$

(wya, 2010)

2. Some other stronger results

Problem 6. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2+\frac{(a-b)^2}{23}} + \frac{1}{b^2+bc+c^2+\frac{(b-c)^2}{23}} + \frac{1}{c^2+ca+a^2+\frac{(c-a)^2}{23}} \geqslant \frac{9}{(a+b+c)^2}.$$

(Nguyen Van Huyen, 2019)

Problem 7. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geqslant \frac{31(a+b+c)^2-12(ab+bc+ca)}{3(a+b+c)^4}.$$

(Nguyen Van Huyen, 2019)

3. Similar type

Problem 8. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geqslant \frac{5}{3(ab+bc+ca)} + \frac{4}{3\left(a^2+b^2+c^2\right)}.$$

(Duong Duc Lam)

Problem 9. Let $a,\ b,\ c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} + \frac{2}{a^2+b^2+c^2} \geqslant \frac{10}{a^2+b^2+c^2+ab+bc+ca}.$$

(Pham Huu Hoai, 2020)