Problem. *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$\sqrt{\frac{8a(b+c)+9bc}{(2b+c)(2c+b)}}+\sqrt{\frac{8b(c+a)+9ca}{(2c+a)(2a+c)}}+\sqrt{\frac{8c(a+b)+9ab}{(2a+b)(2b+a)}}\geqslant 5.$$

(Vo Quoc Ba Can)

Solution. (*Nguyen Van Huyen*) We will show that

$$\sqrt{\frac{8a(b+c)+9bc}{(2b+c)(2c+b)}}\geqslant \frac{1}{3}\cdot\frac{17a^2-b^2-c^2+20(ab+bc+ca)}{a^2+b^2+c^2+4(ab+bc+ca)}. \tag{1}$$

Indeed, setting

$$X=17a^2-b^2-c^2+20(ab+bc+ca).$$

If $X\leqslant 0$, then (1) holds. Otherwise

$$\begin{aligned}&9\big[8a(b+c)+9bc\big]\big[a^2+b^2+c^2+4(ab+bc+ca)\big]^2-(2b+c)(2c+b)X^2\\
&=\big(3X+91a^2+b^2+c^2+58ab+18bc+58ca\big)(b-c)^2(b+c-a)^2\\
&+2\big[2bc(15b^2+58bc+15c^2)+15ab(c+a-b)^2+15ac(a+b-c)^2\big](b-c)^2\\
&+72a\big[c(a-b)^2(a+b-c)^2+b(a-c)^2(c+a-b)^2\big]\\
&+18bc\big[2a^2-b^2-c^2-a(b+c)+2bc\big]^2\geqslant 0.\end{aligned}$$

Therefore, inequality (1) is true. Which gives

$$\sum\sqrt{\frac{8a(b+c)+9bc}{(2b+c)(2c+b)}}\geqslant\frac{1}{3}\sum\frac{17a^2-b^2-c^2+20a(b+c)+20bc}{a^2+b^2+c^2+4(ab+bc+ca)}=5.$$

The proof is completed. $\square$