Problem. *Let $a,\ b,\ c$ are positive real numbers such that $a^3+b^3+c^3=5abc$. Prove that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geqslant 10.$$

Solution. (*Nguyen Van Huyen*) From the condition, we have that $a,\ b,\ c$ are the lengths of the sides of a triangle.

Indeed, suppose $a,\ b,\ c$ are **not** the lengths of the sides of a triangle, example $a>b+c$. Then, there exists a real number $x>0$ such that $a=x+b+c$. At this point

$$\begin{aligned} 0 &= 5abc-a^3-b^3-c^3 \\ &= 5(x+b+c)bc-(x+b+c)^3-b^3-c^3 \\ &= -[x^3+3(b+c)x^2+(3b^2+bc+3c^2)x+2(b+c)(b-c)^2] \\ &< 0 \quad \text{(contradiction).} \end{aligned}$$

Therefore, $a,\ b,\ c$ must be the lengths of the sides of a triangle.

Now, we have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-5-\frac{a^3+b^3+c^3}{abc}=\frac{(a+b-c)(b+c-a)(c+a-b)}{abc}\geqslant 0.$$

Thus

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geqslant 10.$$

The equality holds for $a=b=\frac{c}{2}$ (or any cyclic permutation). $\square$