Problem. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{a(b+c)}{a^2+2bc}+\frac{b(c+a)}{b^2+2ca}+\frac{c(a+b)}{c^2+2ab}\leqslant 2.$$

(Vo Quoc Ba Can, Vasile Cirtoaje)

Solution. (*Nguyen Van Huyen*) If $0<x\leqslant y$ and $m>0$, then

$$\frac{x}{y}\leqslant\frac{x+m}{y+m}.$$

Since

$$a^2+2bc-a(b+c)=bc-(a-c)(b-a)\geqslant|a-c||b-a|-(a-c)(b-a)\geqslant 0.$$

We have

$$\frac{a(b+c)}{a^2+2bc}\leqslant\frac{a(b+c)+(b-c)^2}{a^2+2bc+(b-c)^2}=\frac{b^2+c^2+a(b+c)-2bc}{a^2+b^2+c^2}.$$

Therefore

$$\sum\frac{a(b+c)}{a^2+2bc}\leqslant\sum\frac{b^2+c^2+a(b+c)-2bc}{a^2+b^2+c^2}=\frac{2(a^2+b^2+c^2)}{a^2+b^2+c^2}=2.$$

The proof is completed. $\square$