

Problem. Let a, b, c be three distinct real numbers. Prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2. \quad (1)$$

(Dao Hai Long)

Solution 1. (Nguyen Van Huyen) Let

$$x = \frac{a}{b-c}, \quad y = \frac{b}{c-a}, \quad z = \frac{c}{a-b}.$$

Then $xy + yz + zx = -1$. The inequality (1) becomes

$$x^2 + y^2 + z^2 \geq 2.$$

Since $xy + yz + zx = -1$, we obtain $z = -\frac{1+xy}{x+y}$. We need to prove

$$x^2 + y^2 + \left(\frac{1+xy}{x+y}\right)^2 \geq 2,$$

or

$$(x+y)^2 + \left(\frac{1+xy}{x+y}\right)^2 \geq 2(1+xy).$$

Using the AM-GM inequality, we have

$$(x+y)^2 + \left(\frac{1+xy}{x+y}\right)^2 \geq 2\sqrt{(x+y)^2 \cdot \left(\frac{1+xy}{x+y}\right)^2} = 2|1+xy| \geq 2(1+xy).$$

The proof is completed. □

Solution 2. Note that

$$\sum a(2a-b-c) = 2(a^2 + b^2 + c^2 - ab - bc - ca),$$

and

$$\sum (b-c)^2(2a-b-c)^2 = 2(a^2 + b^2 + c^2 - ab - bc - ca)^2.$$

Therefore, according to the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{(b-c)^2} \geq \frac{[a(2a-b-c) + b(2b-c-a) + c(2c-a-b)]^2}{(b-c)^2(2a-b-c)^2 + (c-a)^2(2b-c-a)^2 + (a-b)^2(2c-a-b)^2} = 2.$$

The proof is completed. □