Problem. *Given four positive real numbers a, b, x, y satisfying*

$$\frac{a}{x}+\frac{b}{y}=1.$$

Prove that

$$x+y+\sqrt{x^2+y^2}\geqslant 2a+2b+2\sqrt{2ab}.$$

(Pham Huu Hoai)

Solution 1. (Michael Rozenberg) By the AM-GM inequality, we have

$$\begin{aligned}
x+y+\sqrt{x^2+y^2}-2(a+b) &= \left(x+y+\sqrt{x^2+y^2}\right)\left(\frac{a}{x}+\frac{b}{y}\right)-2(a+b)\\
&= \left(\frac{y+\sqrt{x^2+y^2}}{x}-1\right)a+\left(\frac{x+\sqrt{x^2+y^2}}{y}-1\right)b\\
&\geqslant 2\sqrt{\frac{\left(\sqrt{x^2+y^2}+y-x\right)\left(\sqrt{x^2+y^2}+x-y\right)ab}{xy}}\\
&= 2\sqrt{2ab}.
\end{aligned}$$

The proof is completed. □

Solution 2. (Nguyen Van Huyen) We write the inequality as

$$\left(x+y+\sqrt{x^2+y^2}\right)\left(\frac{a}{x}+\frac{b}{y}\right)\geqslant 2a+2b+2\sqrt{2ab}. \tag{1}$$

Let $a=ux, b=vy$. The inequality (1) become

$$(x+y+\sqrt{x^2+y^2})(u+v)\geqslant 2ux+2vy+2\sqrt{2xyuv},$$

equivalent to

$$(u+v)\sqrt{x^2+y^2}-2\sqrt{2xyuv}\geqslant (x-y)(u-v).$$

It's easy to check $(u+v)\sqrt{x^2+y^2}\geqslant 2\sqrt{2xyuv}$. It remains to prove that

$$\left[(u+v)\sqrt{x^2+y^2}-2\sqrt{2xyuv}\right]^2\geqslant (x-y)^2(u-v)^2,$$

or

$$2uv\left(x^2+y^2\right)+xy(u+v)^2\geqslant 2\sqrt{2}(u+v)\sqrt{xyuv(x^2+y^2)}.$$

By the AM-GM inequality, we have

$$\begin{aligned}
2uv\left(x^2+y^2\right)+xy(u+v)^2 &\geqslant 2\sqrt{2uv\left(x^2+y^2\right)\cdot xy(u+v)^2}\\
&= 2\sqrt{2}(u+v)\sqrt{xyuv(x^2+y^2)}.
\end{aligned}$$

Equality holds for $2uv\left(x^2+y^2\right)=xy(u+v)^2$, or $(ay+bx)^2=2ab(x^2+y^2)$. □