Problem. *Let a, b, c be three positive real numbers. Prove that*

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc} \geqslant 2.$$

(Nguyen Van Huyen)

Solution. (*Nguyen Van Huyen*) We rewrite the inequality as

$$(a+b+c-3)(ab+bc+ca-3abc)+a^2b^2c^2+1 \geqslant 2abc.$$

Group the inequality as a quadratic in terms of a

$$(b^2c^2+b+c-3bc)a^2+[b^2+10bc+c^2-3(b+c)(1+bc)]a+bc(b+c)-3bc+1 \geqslant 0.$$

By the AM-GM inequality, we have

$$b^2c^2+b+c \geqslant 3\sqrt[3]{b^2c^2\cdot b\cdot c}=3bc,$$
$$bc(b+c)+1 \geqslant 3\sqrt[3]{b^2c\cdot bc^2\cdot 1}=3bc.$$

We consider two cases

(1) If $b^2+10bc+c^2>3(b+c)(1+bc)$, then it is obvious.

(2) If $b^2+10bc+c^2 \leqslant 3(b+c)(1+bc)$, we calculate the discriminant

$$\begin{aligned}\Delta &= \left[b^2+10bc+c^2-3(b+c)(1+bc)\right]^2-4(b^2c^2+b+c-3bc)[bc(b+c)-3bc+1]\\ &= -[4(b+c)(bc+1)-b^2-14bc-c^2](b-1)^2(c-1)^2\\ &= -f(b,c)\cdot(b-1)^2(c-1)^2.\end{aligned}$$

Since

$$f(b,c)=\frac{4}{3}[3(b+c)(1+bc)-b^2-10bc-c^2]+\frac{(b-c)^2}{3} \geqslant 0.$$

We obtain $\Delta \leqslant 0$.

The proof completed. □

References

[1] https://artofproblemsolving.com/community/c6h3460563p33432469