Problem (Ji Chen, Iran TST 1996). *Let a, b, c be non-negative real numbers, no two of which are zero. Prove that*

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \geqslant \frac{9}{4}.$$

Solution 1. (*Nguyen Van Huyen*) Asumme that $c=\min\{a,b,c\}$. The inequality results from adding the two inequalities below

$$\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \geqslant \frac{2(a^2+b^2)+(a+b)^2}{(a+b)^2(b+c)(c+a)},$$

$$\frac{1}{(a+b)^2}+\frac{2(a^2+b^2)+(a+b)^2}{(a+b)^2(b+c)(c+a)} \geqslant \frac{9}{4(ab+bc+ca)}.$$

Indeed, we have

$$\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}-\frac{2(a^2+b^2)+(a+b)^2}{(a+b)^2(b+c)(c+a)}=\frac{(a-b)^2[(a+b)^2-(a+c)(b+c)]}{(a+b)^2(b+c)^2(c+a)^2} \geqslant 0.$$

and

$$\begin{aligned}&\frac{1}{(a+b)^2}+\frac{2(a^2+b^2)+(a+b)^2}{(a+b)^2(b+c)(c+a)}-\frac{9}{4(ab+bc+ca)}\\ &=\frac{(3ab+c^2)(a-b)^2+c(3b+a)(b-c)^2+c(3a+b)(c-a)^2}{4(ab+bc+ca)(a+b)^2(b+c)(c+a)} \geqslant 0.\end{aligned}$$

The proof is completed. □

Solution 2. (*Nguyen Van Huyen*) Since

$$\sum \frac{10a^3+11(b+c)a^2+8(b^2+c^2)a-(b+c)(5b^2-13bc+5c^2)}{12[ab(a+b)+bc(b+c)+ca(c+a)]}=\frac{9}{4}.$$

We have the following estimate

$$\frac{ab+bc+ca}{(b+c)^2} \geqslant \frac{10a^3+11(b+c)a^2+8(b^2+c^2)a-(b+c)(5b^2-13bc+5c^2)}{12[ab(a+b)+bc(b+c)+ca(c+a)]}.$$

Indeed, let

$$X=10a^3+11(b+c)a^2+8(b^2+c^2)a-(b+c)(5b^2-13bc+5c^2),$$

then

$$12(ab+bc+ca)\sum bc(b+c)-X(b+c)^2=2a[b^2+c^2-a(b+c)]^2+(b-c)^2\cdot f(a),$$

with

$$f(a)=5(b+c)a^2-2(5b^2+6bc+5c^2)a+(b+c)(5b^2+7bc+5c^2).$$

The quadratic polynomial $f(a) \geqslant 0$ holds because its discriminant is

$$\begin{aligned}\Delta_a&=(5b^2+6bc+5c^2)^2-5(b+c)\cdot(b+c)(5b^2+7bc+5c^2)\\ &=-bc(25b^2+34bc+25c^2) \leqslant 0.\end{aligned}$$

The estimate is proved. □

Note. Another result can be proved similarly

$$\frac{ab+bc+ca}{(b+c)^2} \geqslant \frac{4a^3+26a^2(b+c)-(4b^2-27bc+4c^2)a-(b+c)(2b^2-7bc+2c^2)}{12(a+b+c)(ab+bc+ca)}.$$