Problem. *Let a, b, c are **real** numbers. Prove that*

$$\frac{ab}{a^2+4b^2}+\frac{bc}{b^2+4c^2}+\frac{ca}{c^2+4a^2}\leqslant\frac{3}{5}.$$

Solution. (*Nguyen Van Huyen*) We have the following estimate

$$\frac{bc}{b^2+4c^2}\leqslant\frac{13(a^2+b^2)+4c^2+6b(c+a)-12ca}{50(a^2+b^2+c^2)}. \tag{1}$$

or

$$[13(a^2+b^2)+4c^2+6b(c+a)-12ca](b^2+4c^2)\geqslant 50bc(a^2+b^2+c^2).$$

We rewrite this inequality as a quadratic polynomial in a as follows

$$(13b^2-50bc+52c^2)a^2+6(b-2c)(b^2+4c^2)a+13b^4-44b^3c+56b^2c^2-26bc^3+16c^4\geqslant 0.$$

It is easy to check that $13b^2-50bc+52c^2\geqslant 0$. And by the AM-GM inequality, we have

$$\begin{aligned}13b^4+56b^2c^2+16c^4&=(13b^4+40b^2c^2)+16(b^2c^2+c^4)\\&\geqslant 4\sqrt{130}|b^3c|+32|bc^3|\\&\geqslant 44b^3c+26bc^3.\end{aligned}$$

Therefore, we only need to prove that the discriminant is negative

$$\begin{aligned}\Delta_a&=[3(b-2c)(b^2+4c^2)]^2-(13b^2-50bc+52c^2)(13b^4-44b^3c+56b^2c^2-26bc^3+16c^4)\\&=-2(80b^4-433b^3c+802b^2c^2-532bc^3+128c^4)(b-c)^2\\&=-2f(b,c)\cdot(b-c)^2.\end{aligned}$$

Since

$$f(b,c)=\frac{1}{11}[2(6b^2-29bc+12c^2)^2+(b-2c)^2(808b^2-835bc+280c^2)]\geqslant 0.$$

we conclude that $\Delta_a\leqslant 0$. Hence, the estimate (1) is proven.

Consequently, we obtain

$$\sum\frac{bc}{b^2+4c^2}\leqslant\sum\frac{13(a^2+b^2)+4c^2+6b(c+a)-12ca}{50(a^2+b^2+c^2)}=\frac{3}{5}.$$

This completes the proof. □