Problem. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{b+c}{\sqrt{a^2+bc}}+\frac{c+a}{\sqrt{b^2+ca}}+\frac{a+b}{\sqrt{c^2+ab}}\geqslant 4.$$

(Pham Kim Hung)

Solution 1. Using the Holder inequality, we obtain

$$\left(\sum\frac{b+c}{\sqrt{a^2+bc}}\right)^2\sum(a^2+bc)(b+c)\geqslant\left[\sum(b+c)\right]^3=8(a+b+c)^3.$$

Therefore, it suffices to show that

$$(a+b+c)^3\geqslant 2\sum(a^2+bc)(b+c),$$

or

$$a^3+b^3+c^3+6abc\geqslant\sum ab(a+b).$$

From Schur's inequality

$$a^3+b^3+c^3+6abc\geqslant a^3+b^3+c^3+3abc\geqslant\sum ab(a+b).$$

The equality holds for $a=b,\ c=0$ (or any cyclic permutation). $\square$

Solution 2. (*Nguyen Van Huyen*) We have

$$\begin{aligned}&(b+c)^2(a^2+b^2+c^2)^2-4(a^2+bc)(b^2+c^2)^2\\&\qquad=(b+c)^2a^4+(b^2+c^2)^2(b-c)^2-2a^2(b^2+c^2)(b-c)^2\end{aligned}$$

Since $b+c\geqslant|b-c|$, using the AM-GM inequality, we have

$$\begin{aligned}(b+c)^2a^4+(b^2+c^2)^2(b-c)^2&\geqslant 2\sqrt{(b+c)^2a^4\cdot(b^2+c^2)^2(b-c)^2}\\&=2a^2(b^2+c^2)(b+c)|b-c|\\&\geqslant 2a^2(b^2+c^2)(b-c)^2.\end{aligned}$$

Which gives

$$(b+c)^2(a^2+b^2+c^2)^2\geqslant 4(a^2+bc)(b^2+c^2)^2,$$

or

$$\frac{b+c}{\sqrt{a^2+bc}}\geqslant\frac{2(b^2+c^2)}{a^2+b^2+c^2}.$$

Thus

$$\sum\frac{b+c}{\sqrt{a^2+bc}}\geqslant 2\sum\frac{b^2+c^2}{a^2+b^2+c^2}=4.$$

$\square$

Solution 3. (*Pham Huu Hoai*) Assume that $c = \min\{a, b, c\}$. Using the AM-GM inequality, we have

$$\begin{aligned}
\frac{b+c}{\sqrt{a^2+bc}} + \frac{c+a}{\sqrt{b^2+ca}} &= \frac{b^2+ca+b(a+c)}{(a+b)\sqrt{a^2+bc}} + \frac{a^2+bc+a(b+c)}{(a+b)\sqrt{b^2+ca}} \\
&\geqslant \frac{2\sqrt{(b^2+ca)\cdot b(a+c)}}{(a+b)\sqrt{a^2+bc}} + \frac{2\sqrt{(a^2+bc)\cdot a(b+c)}}{(a+b)\sqrt{b^2+ca}} \\
&\geqslant \frac{4\sqrt[4]{ab(a+c)(b+c)}}{a+b} \\
&= \frac{4\sqrt[4]{a^2b^2+abc(a+b)+abc^2}}{a+b} \\
&\geqslant \frac{4\sqrt[4]{a^2b^2+2abc^2+c^4}}{a+b} \\
&= \frac{4\sqrt{c^2+ab}}{a+b}.
\end{aligned}$$

Therefore

$$\frac{b+c}{\sqrt{a^2+bc}} + \frac{c+a}{\sqrt{b^2+ca}} + \frac{a+b}{\sqrt{c^2+ab}} \geqslant \frac{4\sqrt{c^2+ab}}{a+b} + \frac{a+b}{\sqrt{c^2+ab}} \geqslant 4.$$

$\square$