Problem. *Let a, b, c be positive numbers and x, y, z be the lengths of the sides of a triangle. Prove that*

$$a(y+z-x)+b(z+x-y)+c(x+y-z) \geqslant \frac{(x+y+z)(xbc+yca+zab)}{ax+by+cz}.$$

(Nguyen Van Huyen)

Proof. Using Ravi's substitution, we need to prove that

$$ax+by+cz \geqslant \frac{(x+y+z)[ab(x+y)+bc(y+z)+ca(z+x)]}{a(y+z)+b(x+z)+c(x+y)}, \tag{1}$$

where x, y and z are positive numbers.

We write the inequality (1) as

$$f(c) = z(x+y)c^2 - 2z(xa+yb)c + x(y+z)a^2 - 2xyab + y(x+z)b^2 \geqslant 0.$$

The quadratic $f(c)$ has the discriminant

$$\begin{aligned}\Delta_c &= z(xa+yb)^2 - (x+y)[x(y+z)a^2 - 2xyab + y(x+z)b^2] \\ &= -xy(x+y+z)a^2 - xy(x+y+z)b^2 + 2abxy(x+y+z) \\ &= -xy(x+y+z)(a-b)^2 \leqslant 0.\end{aligned}$$

Thus, $f(c) \geqslant 0$. The proof is completed. $\square$

Remark 1. We have

$$\begin{aligned}&a(y+z-x)+b(z+x-y)+c(x+y-z) - \frac{(x+y+z)(xbc+yca+zab)}{ax+by+cz} \\ &= \frac{[z^2-(x-y)^2](a-b)^2 + [x^2-(y-z)^2](b-c)^2 + [y^2-(z-x)^2](c-a)^2}{2(ax+by+cz)} \geqslant 0\end{aligned}$$

Remark 2.

(1) If $a=y$, $b=z$, $c=x$ the inequality becomes

$$x^2y(x-y)+y^2z(y-z)+z^2x(z-x) \geqslant 0.$$

This is problem 6 in the IMO 1983.

(2) If $a=x$, $b=y$, $c=z$ the inequality becomes

$$2(xy+yz+zx) \geqslant x^2+y^2+z^2 + \frac{3xyz(x+y+z)}{x^2+y^2+z^2}.$$