Problem. *Given any **real** numbers, such that*

$$D = xa^2 + yb^2 + zc^2 + abc - 4xyz \leqslant 0,$$

and at least one condition below is true

$$x > 0, \quad 4xy \geqslant c^2, \quad \textbf{or} \quad y > 0, \quad 4yz \geqslant a^2, \quad \textbf{or} \quad z > 0, \quad 4zx \geqslant b^2.$$

Prove that

$$xu^2 + yv^2 + zw^2 \geqslant a \cdot vw + b \cdot wu + c \cdot uv. \tag{1}$$

(Nguyen Van Huyen)

Proof. (*Nguyen Van Huyen*) Assume that $x > 0,\ 4xy \geqslant c^2$. Denote

$$f(u) = xu^2 - (bw + cv)u + yv^2 - avw + zw^2.$$

The quadratic $f(u)$ has the discriminant

$$\begin{aligned}
\Delta_u &= (bw + cv)^2 - 4x(yv^2 - avw + zw^2) \\
&= -\left[(4xy - c^2)v^2 - 2w(2ax + bc)v + w^2(4zx - b^2)\right] \\
&= -f(v).
\end{aligned}$$

The quadratic $f(v)$ has the discriminant

$$\begin{aligned}
\Delta_v &= [w(2ax + bc)]^2 - (4xy - c^2) \cdot w^2(4zx - b^2) \\
&= 4xw^2(xa^2 + yb^2 + zc^2 + abc - 4xyz) \leqslant 0.
\end{aligned}$$

Since $4xy - c^2 \geqslant 0$ and $\Delta_v \leqslant 0$, we get $f(v) \geqslant 0$. Therefore $\Delta_u \leqslant 0$, which gives $f(u) \geqslant 0$.

The proof is completed. $\square$