

Problem. Find the least constant k , such that

$$a + b + c + k \cdot \frac{(a - c)^2}{a + c} \geq \sqrt{3(a^2 + b^2 + c^2)}$$

for all $a \geq b \geq c$ be positive real numbers.

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Solution. (Nguyen Van Huyen) Let $a = 5$, $b = c = 1$. Then we get $k \geq \frac{3}{4}$. We will show that

$$a + b + c + \frac{3}{4} \cdot \frac{(a - c)^2}{a + c} \geq \sqrt{3(a^2 + b^2 + c^2)},$$

or

$$7(c^2 + a^2) + 4b(c + a) + 2ca \geq 4(c + a)\sqrt{3(a^2 + b^2 + c^2)}.$$

Using the AM-GM inequality, we have

$$(c + a)\sqrt{3(a^2 + b^2 + c^2)} \leq \frac{3}{4}(c + a)^2 + (a^2 + b^2 + c^2).$$

We need to prove

$$7(c^2 + a^2) + 4b(c + a) + 2ca \geq 3(c + a)^2 + 4(a^2 + b^2 + c^2).$$

Expand and simplify to

$$4(a - b)(b - c) \geq 0.$$

Which is true. The equality holds for $a = b = c$ or $a = 5b = 5c$. □