Problem (Darij Grinberg). *Let $a,\ b,\ c$ be non-negative real numbers. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2}\geqslant 2.$$

Solution. (*Nguyen Van Huyen*) Since

$$\sum\frac{8a^2-b^2-c^2+2(ab+bc+ca)}{3(a^2+b^2+c^2+ab+bc+ca)}=2.$$

It suffices to prove that

$$\frac{a(b+c)}{b^2+bc+c^2}\geqslant\frac{8a^2-b^2-c^2+2(ab+bc+ca)}{3(a^2+b^2+c^2+ab+bc+ca)}. \tag{1}$$

Indeed, let

$$X=8a^2-b^2-c^2+2(ab+bc+ca).$$

Then, we have

$$(b+c)\left[3a(b+c)\sum(a^2+bc)-X(b^2+bc+c^2)\right]=3a[b^2+c^2-a(b+c)]^2+(b-c)^2\cdot f(a),$$

with

$$f(a)=(b+c)a^2-(2b^2+bc+2c^2)a+(b+c)(b^2+bc+c^2).$$

Since $f(a)$ is a quadratic polynomial in a, it suffices to check that $f(a)\geqslant 0$, which is true because its discriminant is

$$\begin{aligned}\Delta_a&=(2b^2+bc+2c^2)^2-4(b+c)^2(b^2+bc+c^2)\\&=-bc(8b^2+7bc+8c^2)\leqslant 0.\end{aligned}$$

Thus, the inequality (1) is proved. $\square$