Problem. *Let a, b, c be three positive real numbers. Prove that*

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc}\geqslant 2.$$

(Nguyen Van Huyen)

Solution. (*Nguyen Van Huyen*) We rewrite the inequality as

$$(a+b+c-3)(ab+bc+ca-3abc)+a^2b^2c^2+1\geqslant 2abc.$$

Group the inequality as a quadratic in terms of a

$$(b^2c^2+b+c-3bc)a^2-[b^2+10bc+c^2-3(b+c)(1+bc)]a+bc(b+c)-3bc+1\geqslant 0.$$

By the AM-GM inequality, we have

$$b^2c^2+b+c\geqslant 3\sqrt[3]{b^2c^2\cdot b\cdot c}=3bc,$$
$$bc(b+c)+1\geqslant 3\sqrt[3]{b^2c\cdot bc^2\cdot 1}=3bc.$$

We consider two cases

(1) If $b^2+10bc+c^2>3(b+c)(1+bc)$, then it is obvious.

(2) If $b^2+10bc+c^2\leqslant 3(b+c)(1+bc)$, we calculate the discriminant

$$\begin{aligned}\Delta &=\left[b^2+10bc+c^2-3(b+c)(1+bc)\right]^2-4(b^2c^2+b+c-3bc)[bc(b+c)-3bc+1]\\ &=-(b-1)^2(c-1)^2[4(b+c)(bc+1)-b^2-14bc-c^2]\end{aligned}$$

Since

$$\begin{aligned}&4(b+c)(bc+1)-b^2-14bc-c^2\\ &\qquad=\frac{4[3(b+c)(1+bc)-b^2-10bc-c^2]}{3}+\frac{(b-c)^2}{3}\geqslant 0.\end{aligned}$$

We obtain $\Delta\leqslant 0$.

The proof completed. □

References

[1] https://artofproblemsolving.com/community/c6h3460563p33432469