Problem. *Let a, b, c, d be real numbers. Prove that*

$$a^2+b^2+c^2+d^2 \geqslant 3(a+b)(c+d)-(a+c)(b+d)-(a+d)(b+c).$$

Solution. (*Nguyen Van Huyen*) Since Euler's four-square identity[1], we have

$$4(a^2+b^2+c^2+d^2)=(a+b+c+d)^2+(a-b+c-d)^2+(a-b-c+d)^2+(a+b-c-d)^2.$$

Now, using the AM–GM inequality, we obtain

$$\begin{aligned} 4(a^2+b^2+c^2+d^2) &\geqslant (a+b+c+d)^2+(a-b+c-d)^2+(a-b-c+d)^2 \\ &\geqslant 4(a+b)(c+d)+4(a-d)(c-b)+4(a-c)(d-b) \quad . \\ &= 4[(a+b)(c+d)+(a-d)(c-b)+(a-c)(d-b)]. \end{aligned}$$

But

$$(a-d)(c-b)+(a-c)(d-b)=2(a+b)(c+d)-(c+a)(b+d)-(a+d)(b+c).$$

Therefore

$$a^2+b^2+c^2+d^2 \geqslant 3(a+b)(c+d)-(c+a)(b+d)-(a+d)(b+c).$$

The equality holds when $a+b=c+d$. □

[1] https://en.wikipedia.org/wiki/Euler%27s_four-square_identity