Problem. *Let a, b, c are non-negative real numbers. Prove that*

$$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab}\geqslant\frac{6}{a+b+c}.$$

(Vasile Cirtoaje)

Lời giải. (*Nguyen Van Huyen*) We need to prove

$$\frac{b+c}{2a^2+bc}\geqslant\frac{2}{a+b+c}+\frac{b+c-2a}{a^2+b^2+c^2}.$$

We rewrite the inequality and group the terms by a

$$3(b+c)a^3-(5b^2+2bc+5c^2)a^2+(b+c)(b^2+bc+c^2)a+(b^2+bc+c^2)(b-c)^2\geqslant 0.$$

Using the AM-GM inequality, we have

$$(b+c)^2a^2+(b^2+c^2)^2\geqslant 2a(b+c)(b^2+c^2),$$

or

$$3(b+c)a^3\geqslant 6a^2(b^2+c^2)-\frac{3a(b^2+c^2)^2}{b+c}.$$

It remains to prove that

$$[6a^2(b^2+c^2)-(5b^2+2bc+5c^2)a^2]+\left[(b+c)(b^2+bc+c^2)a-\frac{3a(b^2+c^2)^2}{b+c}\right]$$
$$+(b^2+bc+c^2)(b-c)^2\geqslant 0,$$

equivalent to

$$a^2(b-c)^2-\frac{a(2b^2+bc+2c^2)(b-c)^2}{b+c}+(b^2+bc+c^2)(b-c)^2\geqslant 0,$$

or

$$\frac{(b-c)^2}{b+c}[(b+c)a^2-(2b^2+bc+2c^2)a+(b+c)(b^2+bc+c^2)]\geqslant 0.$$

We need to prove

$$f(a)=(b+c)a^2-(2b^2+bc+2c^2)a+(b+c)(b^2+bc+c^2)\geqslant 0.$$

The quadratic polynomial $f(a)\geqslant 0$ holds because its discriminant is

$$\begin{aligned}\Delta_a&=(2b^2+bc+2c^2)^2-4(b+c)^2(b^2+bc+c^2)\\&=-bc(8b^2+7bc+8c^2)\leqslant 0.\end{aligned}$$

The proof is completed. $\square$