Problem. *Let $a,\ b,\ c$ are positive real numbers such that $a^3+b^3+c^3=5abc$. Prove that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant 10.$$

Solution. (*Nguyen Van Huyen*) From the condition, we have that $a,\ b,\ c$ are the lengths of the sides of a triangle.

Indeed, suppose $a,\ b,\ c$ are **not** the lengths of the sides of a triangle, example $a > b+c$. Then, there exists a real number $x>0$ such that $a = x+b+c$. At this point

$$\begin{aligned} 0 &= 5abc - a^3 - b^3 - c^3 \\ &= 5(x+b+c)bc - (x+b+c)^3 - b^3 - c^3 \\ &= -[x^3 + 3(b+c)x^2 + (3b^2+bc+3c^2)x + 2(b+c)(b-c)^2] \\ &< 0 \quad \text{(contradiction).} \end{aligned}$$

Therefore, $a,\ b,\ c$ must be the lengths of the sides of a triangle.

Now, we have

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 5 - \frac{a^3+b^3+c^3}{abc} = \frac{(a+b-c)(b+c-a)(c+a-b)}{abc} \geqslant 0.$$

Thus

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant 10.$$

The equality holds for $a = b = \frac{c}{2}$ (or any cyclic permutation). □