

**Problem (Ji Chen, Iran TST 1996).** Let  $a, b, c$  be non-negative real numbers, no two of which are zero. Prove that

$$(ab + bc + ca) \left[ \frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \right] \geq \frac{9}{4}.$$

**Solution 1.** (Nguyen Van Huyen) Assume that  $c = \min\{a, b, c\}$ . The inequality results from adding the two inequalities below

$$\begin{aligned} \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} &\geq \frac{2(a^2 + b^2) + (a+b)^2}{(a+b)^2(b+c)(c+a)}, \\ \frac{1}{(a+b)^2} + \frac{2(a^2 + b^2) + (a+b)^2}{(a+b)^2(b+c)(c+a)} &\geq \frac{9}{4(ab + bc + ca)}. \end{aligned}$$

Indeed, we have

$$\frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} - \frac{2(a^2 + b^2) + (a+b)^2}{(a+b)^2(b+c)(c+a)} = \frac{(a-b)^2[(a+b)^2 - (a+c)(b+c)]}{(a+b)^2(b+c)^2(c+a)^2} \geq 0.$$

and

$$\begin{aligned} &\frac{1}{(a+b)^2} + \frac{2(a^2 + b^2) + (a+b)^2}{(a+b)^2(b+c)(c+a)} - \frac{9}{4(ab + bc + ca)} \\ &= \frac{(3ab + c^2)(a-b)^2 + c(3b+a)(b-c)^2 + c(3a+b)(c-a)^2}{4(ab + bc + ca)(a+b)^2(b+c)(c+a)} \geq 0. \end{aligned}$$

The proof is completed. □

**Solution 2.** (Nguyen Van Huyen) Since

$$\sum \frac{10a^3 + 11(b+c)a^2 + 8(b^2 + c^2)a - (b+c)(5b^2 - 13bc + 5c^2)}{12[ab(a+b) + bc(b+c) + ca(c+a)]} = \frac{9}{4}.$$

We have the following estimate

$$\frac{ab + bc + ca}{(b+c)^2} \geq \frac{10a^3 + 11(b+c)a^2 + 8(b^2 + c^2)a - (b+c)(5b^2 - 13bc + 5c^2)}{12[ab(a+b) + bc(b+c) + ca(c+a)]}.$$

Indeed, let

$$X = 10a^3 + 11(b+c)a^2 + 8(b^2 + c^2)a - (b+c)(5b^2 - 13bc + 5c^2),$$

then

$$12(ab + bc + ca) \sum bc(b+c) - X(b+c)^2 = 2a[b^2 + c^2 - a(b+c)]^2 + (b-c)^2 \cdot f(a),$$

with

$$f(a) = 5(b+c)a^2 - 2(5b^2 + 6bc + 5c^2)a + (b+c)(5b^2 + 7bc + 5c^2).$$

The quadratic polynomial  $f(a) \geq 0$  holds because its discriminant is

$$\begin{aligned} \Delta_a &= (5b^2 + 6bc + 5c^2)^2 - 5(b+c) \cdot (b+c)(5b^2 + 7bc + 5c^2) \\ &= -bc(25b^2 + 34bc + 25c^2) \leq 0. \end{aligned}$$

The estimate is proved. □

**Note.** Another result can be proved similarly

$$\frac{ab + bc + ca}{(b + c)^2} \geq \frac{4a^3 + 26a^2(b + c) - (4b^2 - 27bc + 4c^2)a - (b + c)(2b^2 - 7bc + 2c^2)}{12(a + b + c)(ab + bc + ca)}.$$

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