Problem (Darij Grinberg). *Let $a,\ b,\ c$ be non-negative real numbers. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ca+a^2}+\frac{c(a+b)}{a^2+ab+b^2}\geqslant 2.$$

Solution. (*Nguyen Van Huyen*) Since

$$\sum\frac{8a^2-b^2-c^2+2(ab+bc+ca)}{3(a^2+b^2+c^2+ab+bc+ca)}=2.$$

It suffices to prove that

$$\frac{a(b+c)}{b^2+bc+c^2}\geqslant\frac{8a^2-b^2-c^2+2(ab+bc+ca)}{3(a^2+b^2+c^2+ab+bc+ca)}. \tag{1}$$

Indeed, we rewrite the inequality (1) as follows

$$3a(b+c)(a^2+b^2+c^2+ab+bc+ca)\geqslant(b^2+bc+c^2)[8a^2-b^2-c^2+2(ab+bc+ca)].$$

We group the inequality as a quadratic in terms of a

$$3(b+c)a^3-(5b^2+2bc+5c^2)a^2+(b+c)(b^2+bc+c^2)a+(b^2+bc+c^2)(b-c)^2\geqslant 0.$$

Using the AM-GM inequality, we have

$$(b+c)a^3+\frac{a(b^2+c^2)^2}{b+c}\geqslant 2a^2(b^2+c^2).$$

It remains to prove that

$$6a^2(b^2+c^2)-\frac{3a(b^2+c^2)^2}{b+c}-(5b^2+2bc+5c^2)a^2+(b+c)(b^2+bc+c^2)a+(b^2+bc+c^2)(b-c)^2\geqslant 0.$$

Which is equivalent to

$$a^2(b-c)^2-\frac{a(2b^2+bc+2c^2)^2(b-c)^2}{b+c}+(b^2+bc+c^2)(b-c)^2\geqslant 0,$$

or

$$\frac{(b-c)^2}{b+c}[(b+c)a^2-(2b^2+bc+2c^2)a+(b+c)(b^2+bc+c^2)]\geqslant 0.$$

Let

$$f(a)=(b+c)a^2-(2b^2+bc+2c^2)a+(b+c)(b^2+bc+c^2).$$

Since $f(a)$ is a quadratic polynomial in a, it suffices to check that $f(a)\geqslant 0$, which is true because its discriminant is

$$\begin{aligned}\Delta_a&=(2b^2+bc+2c^2)^2-4(b+c)^2(b^2+bc+c^2)\\&=-bc(8b^2+7bc+8c^2)\leqslant 0.\end{aligned}$$

Thus, the inequality (1) is proven. □