Problem (The S.O.S Theorem). *Given six **real** numbers a, b, c, x, y, z satisfying*

$$x + y + z > 0, \quad xy + yz + zx \geqslant 0, \quad a + b + c = 0.$$

Prove that

$$xa^2 + yb^2 + zc^2 \geqslant 0.$$

Solution. (*Nguyen Van Huyen*) Denoting $m = xy + yz + zx$, we have

$$\begin{aligned} & 2(x+y+z)(xa^2+yb^2+zc^2) \\ = & \sum(2x^2+m)a^2 - 2(xyab+yzbc+zxca) + (a+b+c)\sum xy(a+b-c) \\ = & \sum(2x^2+m)a^2 - 2(xyab+yzbc+zxca) \end{aligned}$$

We need to prove

$$(2x^2+m)a^2 + (2y^2+m)b^2 + (2z^2+m)c^2 \geqslant 2(xyab+yzbc+zxca).$$

This inequality follows from adding this inequality and two similar inequalities

$$(2x^2+m)a^2 + (2y^2+m)b^2 \geqslant 4xyab.$$

Indeed, noting that $m \geqslant 0$ and using the AM-GM inequality, we have

$$(2x^2+m)a^2 + (2y^2+m)b^2 \geqslant 2(x^2a^2+y^2b^2) \geqslant 4xyab.$$

The proof is completed. $\square$

Note. In addition, with

$$\sum(ax-by)^2 = (ax-by)^2 + (by-cz)^2 + (cz-ax)^2.$$

We obtain

$$\begin{aligned} & 2(x+y+z)(xa^2+yb^2+zc^2) \\ = & (xy+yz+zx)(a^2+b^2+c^2) + \sum(ax-by)^2 + (a+b+c)\sum xy(a+b-c) \\ \geqslant & (xy+yz+zx)(a^2+b^2+c^2). \end{aligned}$$

This gives

$$xa^2 + yb^2 + zc^2 \geqslant \frac{(xy+yz+zx)(a^2+b^2+c^2)}{2(x+y+z)}.$$