

**Problem (Vasile Cirtoaje).** Let  $a, b, c$  be three positive real numbers. Prove that

$$(a + b + c - 3) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 3 \right) + abc + \frac{1}{abc} \geq 2.$$

**Solution.** (Nguyen Van Huyen) We rewrite the inequality as

$$(a + b + c - 3)(ab + bc + ca - 3abc) + a^2b^2c^2 + 1 \geq 2abc.$$

Group the inequality as a quadratic in terms of  $a$

$$(b^2c^2 + b + c - 3bc)a^2 + [b^2 + 10bc + c^2 - 3(b + c)(1 + bc)]a + bc(b + c) - 3bc + 1 \geq 0.$$

By the AM-GM inequality, we have

$$b^2c^2 + b + c \geq 3\sqrt[3]{b^2c^2 \cdot b \cdot c} = 3bc,$$

$$bc(b + c) + 1 \geq 3\sqrt[3]{b^2c \cdot bc^2 \cdot 1} = 3bc.$$

We consider two cases

- (1) If  $b^2 + 10bc + c^2 > 3(b + c)(1 + bc)$ , then it is obvious.
- (2) If  $b^2 + 10bc + c^2 \leq 3(b + c)(1 + bc)$ , we calculate the discriminant

$$\begin{aligned} \Delta &= [b^2 + 10bc + c^2 - 3(b + c)(1 + bc)]^2 - 4(b^2c^2 + b + c - 3bc)[bc(b + c) - 3bc + 1] \\ &= -[4(b + c)(bc + 1) - b^2 - 14bc - c^2](b - 1)^2(c - 1)^2 \\ &= -f(b, c) \cdot (b - 1)^2(c - 1)^2. \end{aligned}$$

Since

$$f(b, c) = \frac{4}{3}[3(b + c)(1 + bc) - b^2 - 10bc - c^2] + \frac{(b - c)^2}{3} \geq 0.$$

We obtain  $\Delta \leq 0$ .

The proof completed. □