Hilbert's Nullstellensatz in algebraic geometry. Computer science also frequently investigates the possibility of deriving some object from given initial objects by certain derivation rules, say in the theory of formal languages.

Proof of the duality theorem from the Farkas lemma. Let us assume that the linear program (P) has an optimal solution $\mathbf{x}^*$. As in the proof of the duality theorem from the simplex method, we show that the dual (D) has an optimal solution as well, and that the optimum values of both programs coincide.

We first define $\gamma = \mathbf{c}^T\mathbf{x}^*$ to be the optimum value of (P). Then we know that the system of inequalities

$$A\mathbf{x} \le \mathbf{b},\ \ \mathbf{c}^T\mathbf{x} \ge \gamma \tag{6.4}$$

has a nonnegative solution, but for any $\varepsilon > 0$, the system

$$A\mathbf{x} \le \mathbf{b},\ \ \mathbf{c}^T\mathbf{x} \ge \gamma + \varepsilon \tag{6.5}$$

has *no* nonnegative solution. If we define an $(m+1)\times n$ matrix $\hat{A}$ and a vector $\hat{\mathbf{b}}_\varepsilon \in \mathbb{R}^m$ by

$$\hat{A} = \begin{pmatrix} A \\ -\mathbf{c}^T \end{pmatrix}, \quad \hat{\mathbf{b}}_\varepsilon = \begin{pmatrix} \mathbf{b} \\ -\gamma - \varepsilon \end{pmatrix},$$

then (6.4) is equivalent to $\hat{A}\mathbf{x} \le \hat{\mathbf{b}}_0$ and (6.5) is equivalent to $\hat{A}\mathbf{x} \le \hat{\mathbf{b}}_\varepsilon$.

Let us apply the variant of the Farkas lemma in Proposition 6.4.3(ii). For $\varepsilon > 0$, the system $\hat{A}\mathbf{x} \le \hat{\mathbf{b}}_\varepsilon$ has no nonnegative solution, so we conclude that there is a nonnegative vector $\hat{\mathbf{y}} = (\mathbf{u}, z) \in \mathbb{R}^{m+1}$ such that $\hat{\mathbf{y}}^T\hat{A} \ge \mathbf{0}^T$ but $\hat{\mathbf{y}}^T\hat{\mathbf{b}}_\varepsilon < 0$. These conditions boil down to

$$A^T\mathbf{u} \ge z\mathbf{c},\ \ \mathbf{b}^T\mathbf{u} < z(\gamma + \varepsilon). \tag{6.6}$$

Applying the Farkas lemma in the case $\varepsilon = 0$ (the system has a nonnegative solution), we see that the very same vector $\hat{\mathbf{y}}$ must satisfy $\hat{\mathbf{y}}^T\hat{\mathbf{b}}_0 \ge 0$, and this is equivalent to

$$\mathbf{b}^T\mathbf{u} \ge z\gamma.$$

It follows that $z > 0$, since $z = 0$ would contradict the strict inequality in (6.6). But then we may set $\mathbf{v} := \frac{1}{z}\mathbf{u} \ge \mathbf{0}$, and (6.6) yields

$$A^T\mathbf{v} \ge \mathbf{c},\ \ \mathbf{b}^T\mathbf{v} < \gamma + \varepsilon.$$

In other words, $\mathbf{v}$ is a feasible solution of (D), with the value of the objective function smaller than $\gamma + \varepsilon$.

By the weak duality theorem, every feasible solution of (D) has value of the objective function at least γ. Hence (D) is a feasible and bounded linear program, and so we know that it has an optimal solution $\mathbf{y}^*$ (Theorem 4.2.3). Its value $\mathbf{b}^T\mathbf{y}^*$ is between γ and $\gamma + \varepsilon$ for every $\varepsilon > 0$, and thus it equals γ. This concludes the proof of the duality theorem. $\square$