If we want to dualize a *minimization* linear program, we can first transform it to a maximization linear program by changing the sign of the objective function, and then follow the recipe.

In this way one can also find out that the rules work symmetrically "there" and "back." By this we mean that if we start with some linear program, construct the dual linear program, and then again the dual linear program, we get back to the original (primal) linear program; two consecutive dualizations cancel out. In particular, the linear programs (P) and (D) in the duality theorem are *dual to each other*.

**A physical interpretation of duality.** Let us consider a linear program

$$\text{maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} \le \mathbf{b}.$$

According to the dualization recipe the dual linear program is

$$\text{minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} = \mathbf{c} \text{ and } \mathbf{y} \ge \mathbf{0}.$$

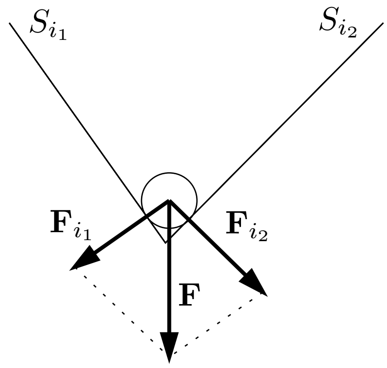
Let us assume that the primal linear program is feasible and bounded, and let $n = 3$. We regard $\mathbf{x}$ as a point in three-dimensional space, and we interpret $\mathbf{c}$ as the gravitation vector; it thus points downward.

Each of the inequalities of the system $A\mathbf{x} \le \mathbf{b}$ determines a half-space. The intersection of these half-spaces is a nonempty convex polyhedron bounded from below. Each of its two-dimensional faces is given by one of the equations $\mathbf{a}_i^T\mathbf{x} = b_i$, where the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are the rows of the matrix $A$, but interpreted as column vectors. Let us denote the face given by $\mathbf{a}_i^T\mathbf{x} = b_i$ by $S_i$ (not every inequality of the system $A\mathbf{x} \le \mathbf{b}$ has to correspond to a face, and so $S_i$ is not necessarily defined for every $i$).

Let us imagine that the boundary of the polyhedron is made of cardboard and that we drop a tiny steel ball somewhere inside the polyhedron. The ball falls and rolls down to the lowest vertex (or possibly it stays on a horizontal edge or face). Let us denote the resulting position of the ball by $\mathbf{x}^*$; thus, $\mathbf{x}^*$ is an optimal solution of the linear program. In this stable position the ball touches several two-dimensional faces, typically 3. Let $D$ be the set of $i$ such that the ball touches the face $S_i$. For $i \in D$ we thus have

$$\mathbf{a}_i^T\mathbf{x}^* = b_i. \tag{6.2}$$

Gravity exerts a force $\mathbf{F}$ on the ball that is proportional to the vector $\mathbf{c}$. This force is decomposed into forces of pressure on the faces touched by the ball. The force $\mathbf{F}_i$ by which the ball acts on face $S_i$ is orthogonal to $S_i$ and it is directed outward from the polyhedron (if we neglect friction); see the schematic two-dimensional picture below:




The forces acting on the ball are in equilibrium, and thus $\mathbf{F} = \sum_{i \in D} \mathbf{F}_i$. The outward normal of the face $S_i$ is $\mathbf{a}_i$; hence $\mathbf{F}_i$ is proportional to $\mathbf{a}_i$, and for some nonnegative numbers $y_i^*$ we have

$$\sum_{i \in D} y_i^* \mathbf{a}_i = \mathbf{c}.$$

If we set $y_i^* = 0$ for $i \notin D$, we can write $\sum_{i=1}^m y_i^* \mathbf{a}_i = \mathbf{c}$, or $A^T \mathbf{y}^* = \mathbf{c}$ in matrix form. Therefore, $\mathbf{y}^*$ is a feasible solution of the dual linear program.

Let us consider the product $(\mathbf{y}^*)^T(A\mathbf{x}^* - \mathbf{b})$. For $i \notin D$ the $i$th component of $\mathbf{y}^*$ equals 0, while for $i \in D$ the $i$th component of $A\mathbf{x}^* - \mathbf{b}$ is 0 according to (6.2). So the product is 0, and hence $(\mathbf{y}^*)^T\mathbf{b} = (\mathbf{y}^*)^T A\mathbf{x}^* = \mathbf{c}^T\mathbf{x}^*$.

We see that $\mathbf{x}^*$ is a feasible solution of the primal linear program, $\mathbf{y}^*$ is a feasible solution of the dual linear program, and $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$. By the weak duality theorem $\mathbf{y}^*$ is an optimal solution of the dual linear program, and we have a situation exactly as in the duality theorem. We have just "physically verified" a special three-dimensional case of the duality theorem.

We remark that the dual linear program also has an economic interpretation. The dual variables are called *shadow prices* in this context. The interested reader will find this nicely explained in Chvátal's textbook cited in Chapter 9.

## 6.3 Proof of Duality from the Simplex Method

The duality theorem of linear programming can be quickly derived from the correctness of the simplex method. To be precise, we will prove the following:

> *If the primal linear program* (P) *is feasible and bounded, then the dual linear program* (D) *is feasible (and bounded as well, by weak duality), with the same optimum value as the primal.*