Analysis of time-series data

(based on the lecture of Prof. Vicent Lefieux, VIASM 2016)

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VIASM

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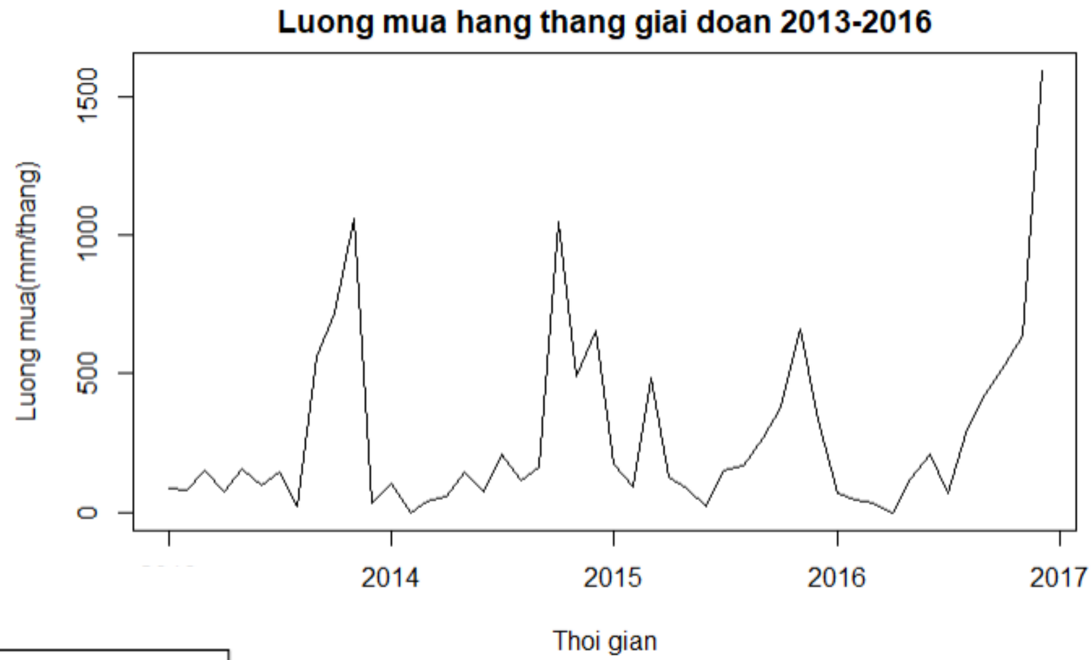
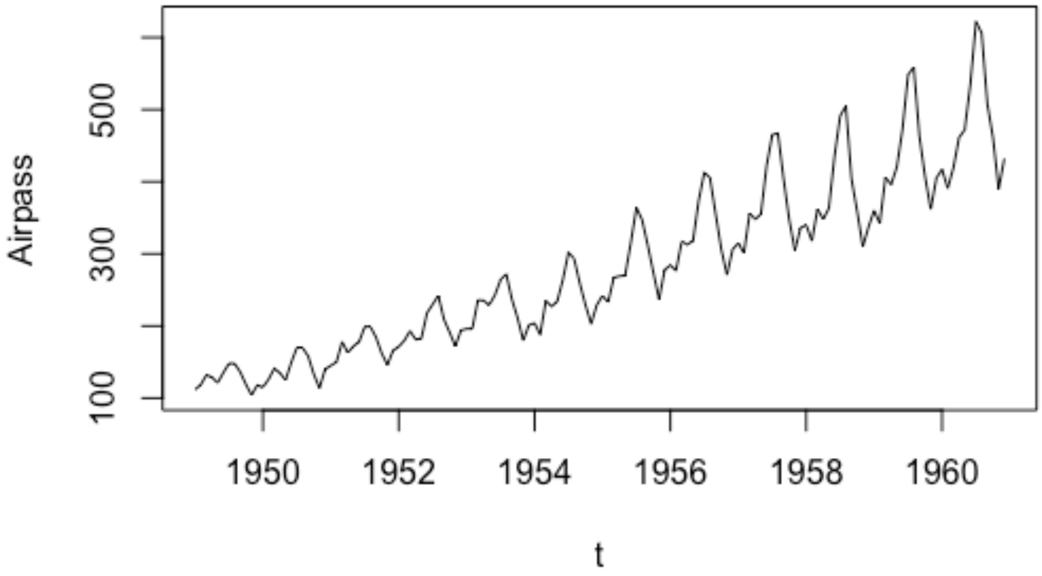
The object of study

- A time series is set of observations recorded in time order.
  - history (e.g. industrial revolution),
  - geography (e.g. migratory flows),
  - demography (e.g. growth of a population),
  - economics (e.g. rate of inflation),
  - finance (e.g. stock price),
  - meteorology (e.g. temperatures),
  - medicine (e.g. electrocardiogram),
  - epidemiology (e.g. spread of a disease),
  - geophysics (e.g. earthquakes),
  - communication (e.g. digital television),
  - energy (e.g. load curve, wind and solar generation),
  - . . .

Samples: Rainfall vs AirPassengers





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Course aims
(Time-series analysis in practice)

- understand the stationarity of time dependent data,
- use R statistics tools to detect seasonality,
- be able to estimate model's parameters of time series,
- and forecast.

References

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[3] Peter J. Brockwell and Richard A. Davis. *Time series: Theory and methods*. Springer, Second edition, 2009.

[4] Michael Friendly. A brief history of data visualization. In Chun-houh Chen, Wolfgang Hardle, and Antony Unwin, editors, *Handbook of data visualization*, chapter II.1, pages 15{56. Springer, 2008.

[5] Alan Pankratz. *Forecasting with dynamic regression models*. Wiley, 1991.

[6] Robert H. Shumway and David S. Stoer. *Time series analysis and its applications. With R examples*. Springer Texts in Statistics. Springer, 3 edition, 2011.

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Methodology

Probabilistic model

Let $(x_t)_{t\in\mathcal{T}}$ be a sequence of observations (for example in the fields of economics, life sciences, physics. . .).

Each observation x_t, in $\mathbb{R}^d$, is recorded at a specific time $t \in \mathcal{T}$.

$(x_t)_{t\in\mathcal{T}}$ is called time series.

- Observation x_t is considered as the realization of a random variable X_t.
- Time series $(x_t)_{t\in\mathcal{T}}$ is considered as the realization of a stochastic process $(X_t)_{t\in\mathcal{T}}$.

Methodology

Stationary

- To obtain parsimony in a time series model we often assume some form of distributional invariance over time, or stationarity.
- For observed time series:
  - Fluctuations appear random.
  - However, same type of stochastic behavior holds from one time period to the next.
- For example, returns on stocks or changes in interest rates:
  - Individually, very different from the previous year.
  - But mean, standard deviation, and other statistical properties are often similar from one year to the next.

Methodology

Stationary processes

$(X_t)_{t\in\mathbb{Z}}$ is said to be strictly stationary if the joint distribution of $(X_{t_1},\dots,X_{t_k})$ is equal to the distribution of $(X_{t_1+h},\dots,X_{t_k+h})$, for $k\in\mathbb{N}^\star$, $(t_1,\dots,t_k)\in\mathbb{Z}^k$ and $h\in\mathbb{Z}$.

Methodology

(Weakly) Stationary processes

$(X_t)_{t\in\mathbb{Z}}$ is said to be a second-order process if:

$$\forall t \in \mathbb{Z} : \mathbb{E}\left(X_t^2\right) < +\infty.$$

A second-order process $(X_t)_{t\in\mathbb{Z}}$ is weakly stationary, if the expectation $\mathbb{E}\left(x_t\right)$ and the (auto)covariances $\mathrm{Cov}(X_s, X_t)$ are time-shifted invariant:

- $\forall t \in \mathbb{Z} : \mathbb{E}\left(x_t\right) = \mu$
- $\forall \left(s, t\right) \in \mathbb{Z}^2, \forall h \in \mathbb{Z} :$

$$\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(X_{s+h}, X_{t+h}).$$

In this case we have:

$$\mathrm{Cov}\left(X_s, X_t\right) = \gamma\left(t - s\right).$$

Methodology

Weakly Stationary

Weakly stationary is also referred to covariance stationary.

- The mean and variance do not change with time
- The covariance between two observations depends only on the lag, the time distance $|t - s|$ between observations, not on the indices t or s.

Autocovariance function

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
The autocovariance function of X is:

$$\forall h \in \mathbb{Z} : \gamma(h) = \mathrm{Cov}\left(X_t, X_{t-h}\right).$$

- $\gamma(0) \geq 0$
- $\forall h \in \mathbb{Z} : |\gamma(h)| \leq \gamma(0)$.
- γ is even:
$$\forall h \in \mathbb{Z} : \gamma(-h) = \gamma(h).$$
- γ is a nonnegative definite function:
$$\forall n \in \mathbb{N}^*, \forall (a_i)_{i\in\{1,\dots,n\}} \in \mathbb{R}^n : \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma(i-j) \geq 0.$$

Autocorrelation function

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
We call (simple) autocorrelation function of X the following function ρ:

$$\forall h \in \mathbb{Z} : \rho(h) = \mathrm{Corr}(X_t, X_{t-h}) = \frac{\gamma(h)}{\gamma(0)}.$$

We have $\rho(0) = 1$.

Autocorrelation matrix

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
The autocorrelation matrix of $(X_t, \dots, X_{t-h+1})$ is:

$$R_h = \begin{bmatrix} 1 & \rho(1) & \dots & \rho(h-1) \\ \rho(1) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho(1) \\ \rho(h-1) & \dots & \rho(1) & 1 \end{bmatrix}.$$

Notice that:

$$R_h = \left[\begin{array}{ccc|c} & & & \rho(h-1) \\ & R_{h-1} & & \vdots \\ & & & \rho(1) \\ \hline \rho(h-1) & \dots & \rho(1) & 1 \end{array}\right]$$

Estimation of the autocorrelation functions

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Based on $(X_1,\ldots,X_T)$, $\overline{X}_T$ is a consistent and unbiased estimator of $\mathbb{E}(X)=\mu$:

$$\overline{X}_T=\frac{1}{T}\sum_{t=1}^{T}X_t.$$

$\forall h\in\{1,\ldots,T-1\}$:

$$\widehat{\rho}(h)=\frac{\sum_{t=h+1}^{T}\left(X_t-\overline{X}_T\right)\left(X_{t-h}-\overline{X}_T\right)}{\sum_{t=1}^{T}\left(X_t-\overline{X}_T\right)^2}.$$

Test of randomness

Portmanteau test

Portmanteau test

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary process.
Consider the test:

$$\begin{cases} H_0 : (X_t)_{t\in\mathbb{Z}} \text{ is a white noise} \\ H_1 : (X_t)_{t\in\mathbb{Z}} \text{ isn't a white noise} \end{cases} .$$

Based on $(X_1, \dots, X_T)$, the Portmanteau statistic is:

$$Q_k = T \sum_{h=1}^{k} \widehat{\rho}^2(h)$$

Under H_0: $Q_k \xrightarrow{d} \chi_k^2$.
So we reject the null hypothesis at the α level if $Q_k > \chi_k^2(1-\alpha)$.

Test of randomness

Shapiro-Wilk test

$$\begin{cases} H_0 : (X_1, \ldots, X_n) \text{ comes from a normal distribution} \\ H_1 : (X_1, \ldots, X_n) \text{ doesn't come from a normal distribution} \end{cases}$$

The Shapiro-Wilk statistic is:

$$W = \frac{\left(\sum_{i=1}^{n} a_i X_{(i)}\right)^2}{\sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2}$$

where $X_{(i)}$ is the i-th order statistic. Coefficients $(a_i)_{i\in\{1,\ldots,n\}}$ are constants generated from the means, variances and covariances of the order statistics of a sample of size n from a normal distribution.
We reject the null hypothesis at the α level if:

$$W < W_{n,\alpha}^{threshold}.$$

$W^{threshold}$ is obtained with Monte Carlo simulations

AR process

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise of variance σ^2.

$(X_t)_{t\in\mathbb{Z}}$ is said to be an autoregressive process or a AR process of order p, written AR(p), if:

- $(X_t)_{t\in\mathbb{Z}}$ is stationary,
- $$\forall t \in \mathbb{Z} : X_t = \sum_{i=1}^{p} \varphi_i X_{t-i} + \varepsilon_t$$

  where $(\varphi_1, \dots, \varphi_p) \in \mathbb{R}^p$ and $\varphi_p \neq 0$.

We generally use the notation $\Phi(B) X_t = \varepsilon_t$ where:

$$\Phi(B) = I - \sum_{i=1}^{p} \varphi_i B^i.$$

Note that:

- Sometimes we find $\Phi(B) = I + \sum_{i=1}^{p} \varphi_i B^i$.
- If $\Phi(B)$ has a root on the unit circle then the process $(X_t)_{t\in\mathbb{Z}}$ isn't stationary, thus it isn't an AR process.

MA Process

Let $(\varepsilon_t)_{t\in\mathbb{Z}}$ be a white noise of variance σ^2.

$(X_t)_{t\in\mathbb{Z}}$ is said to be a MA process of order q, written MA(q), if:

$$\forall t \in \mathbb{Z} : X_t = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$

where $(\theta_1, \ldots, \theta_q) \in \mathbb{R}^q$ and $\theta_q \neq 0$.

Autocovariance and Autocorrelation

The function γ *(gamma) is called the* *autocovariance function.*

Note that $\gamma(h) = \gamma(-h)$. *Why?*

Assuming weak stationarity:

Correlation between Y_t *and* Y_{t+h} *is denoted by* $\rho(h)$.

The function ρ *(rho) is called the* *autocorrelation function.*

Note:

- $\gamma(0) = \sigma^2$ *(variance)*
- $\gamma(h) = \sigma^2 \rho(h)$ *(autocovariance)*
- $\rho(h) = \gamma(h)/\sigma^2 = \gamma(h)/\gamma(0)$ *(autocorrelation)*

Estimating Parameters of a Stationary Process

Suppose we observe $Y_1, \ldots, Y_n$ from a weakly stationary process.

Estimate the mean μ and variance σ^2 using:

- the sample mean $\overline{y}$ and sample variance s^2.

Estimate the autocovariance function using

- the sample autocovariance function

$$\widehat{\gamma}(h) = n^{-1} \sum_{t=1}^{n-h} (Y_{t+h} - \overline{y})(Y_t - \overline{y}) = n^{-1} \sum_{t=h+1}^{n} (Y_t - \overline{y})(Y_{t-h} - \overline{y}).$$

Estimating Autocorrelations of a Stationary Process

To estimate $\rho(\cdot)$, we use the sample autocorrelation function (sample ACF) defined as

$$\widehat{\rho}(h) = \frac{\widehat{\gamma}(h)}{\widehat{\gamma}(0)},$$

for each lag h.

ACF Plot

R will plot a sample ACF with test bounds.

- Bounds test the null hypothesis that an autocorrelation coefficient is 0.
- The null hypothesis is rejected if the sample autocorrelation is outside the bounds.
- The usual level of the test is $\alpha = 0.05$
- We expect 1 out of 20 sample autocorrelations outside the test bounds simply by chance.

ACF plot example

Inflation rates and changes in the inflation rate—sample ACF plots

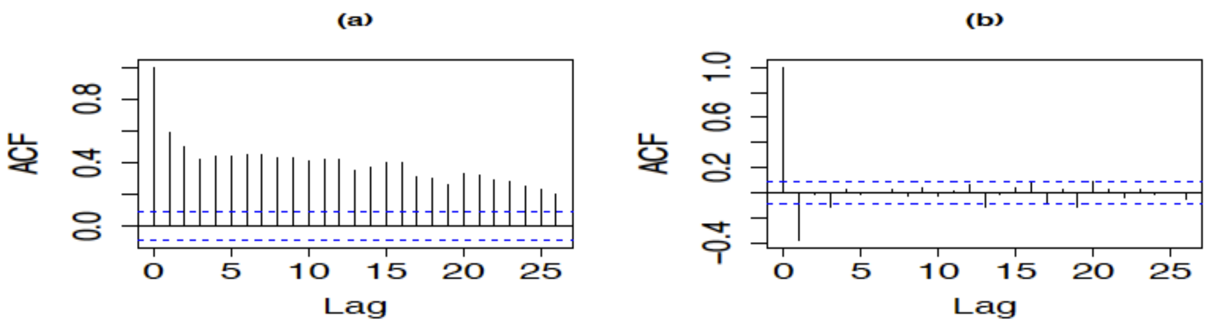

Figure: *Sample ACF plots of the one-month inflation rate (a) and changes in the inflation rate (b).*

```
data(Mishkin, package = "Ecdat")
y = as.vector(Mishkin[,1])
par(mfrow=c(1,2))
acf(y)
acf(diff(y))
```

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Autoregressive (AR) processes

Let $\epsilon_1, \epsilon_2, \ldots$ be White Noise(0,σ_ϵ^2) innovations, with variance σ_ϵ^2

Then, $Y_1, Y_2, \ldots$ is an AR process if for some constants μ and ϕ,

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t$$

- We focus on 1st order case, the simplest AR process

Autoregressive (AR) processes

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t$$

- μ is the mean of the $\{Y_t\}$ process
- If $\phi = 0$, then $Y_t = \mu + \epsilon_t$, such that Y_t is White Noise(μ, σ_ϵ^2)
- If $\phi \neq 0$, then observations Y_t depend on both ϵ_t and Y_{t-1}
- And the process $\{Y_t\}$ is autocorrelated
- If $\phi \neq 0$, then $(Y_{t-1} - \mu)$ is fed forward into Y_t
- ϕ determines the amount of feedback
- Larger values of $|\phi|$ result in more feedback

AR Processes: Properties

If $|\phi| < 1$, then

$$\begin{aligned} E(Y_t) &= \mu \\ \mathrm{Var}(Y_t) &= \sigma_Y^2 = \frac{\sigma_\epsilon^2}{1-\phi^2} \\ \mathrm{Corr}(Y_t, Y_{t-h}) &= \rho(h) = \phi^{|h|} \quad for\ all\ h \end{aligned}$$

- If $\mu = 0$ and $\phi = 1$, then

$$Y_t = Y_{t-1} + \epsilon_t$$

which is a random walk process, and $\{Y_t\}$ is NOT stationary

Simple moving average (MA) processes

$$Y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

- μ is the mean of the $\{Y_t\}$ process
- If $\theta = 0$, then $Y_t = \mu + \epsilon_t$, such that Y_t is White Noise(μ, σ_ϵ^2)
- If $\theta \neq 0$, then observations Y_t depend on both ϵ_t and ϵ_{t-1}
- And the process $\{Y_t\}$ is autocorrelated
- If $\theta \neq 0$, then ϵ_{t-1} is fed forward into Y_t
- θ determines its impact
- Larger values of $|\theta|$ result in greater impact

MA Processes: Autocorrelations

$$\begin{aligned} \text{Corr}(Y_t, Y_{t-1}) &= \rho(1) = \frac{\theta}{1+\theta^2} \\ \text{Corr}(Y_t, Y_{t-h}) &= \rho(h) = 0 \qquad \textit{for all } h > 1 \end{aligned}$$

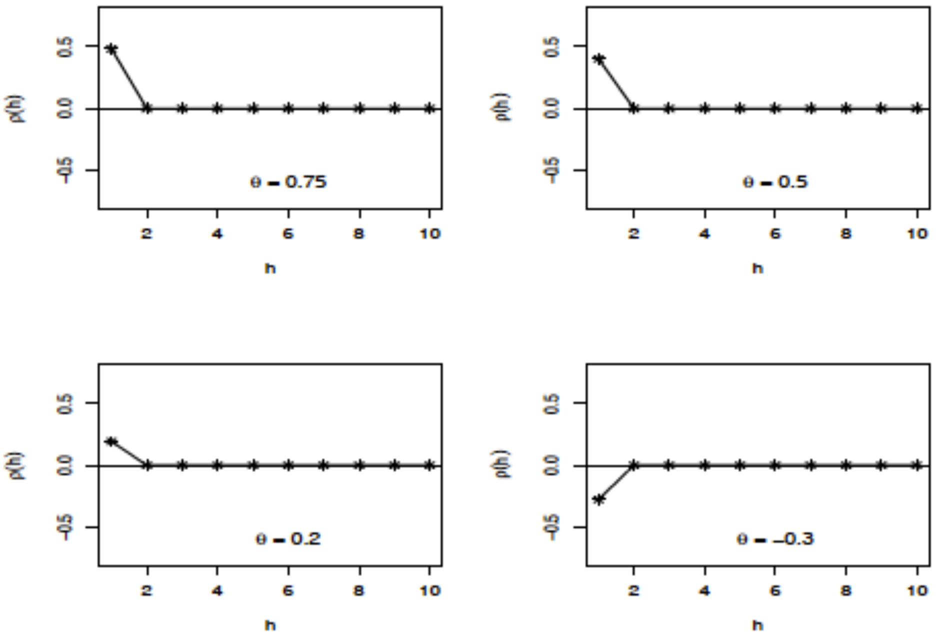

Figure: *Autocorrelation functions of MA processes with* θ *equal to* 0.75, 0.5, 0.2, *and* -0.3.

Time-series analysis with R

- Simulation
- AirPassenger data
- Rainfall data