52nd—24th INTERNATIONAL-RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS
Problem 2

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We can see that the Lagrangian will be:

$$L = r^2 \sin^2 \theta \dot{\varphi}^2 + r^2 \dot{\theta}^2 + \dot{r}^2.$$

If we let $x = r$, $y = \sin\theta$ (If $y^2 > 1$, θ may be a complex number) and $z = \varphi$. If we multiphy this Lagrangian by a constant $\frac{m}{2}$, this exactly is the Lagrangian of a free particle, with mass m, moving non-relativistic in the spherical coordinate system. Therefore, in the same way, we can build a coordinate system (x_1, x_2, x_3) in order to the Lagrangian be quadratic form with:

$$\begin{aligned} x_1 &= x\sqrt{1-y^2}\cos(z) \\ x_2 &= x\sqrt{1-y^2}\sin(z) \\ x_3 &= xy. \end{aligned}$$

The new Lagrangian will be:

$$L = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2.$$

The Euler-Lagrange equation with x_1:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1} \Rightarrow 2\frac{d\dot{x}_1}{dt} = 0 \Rightarrow \dot{x}_1 = C_{1a} \Rightarrow x_1 = C_{1a}t + C_{1b}.$$

Where C_{1a} and C_{1b} is a constant. Similar, we have $x_2 = C_{2a}t + C_{2b}$ and $x_3 = C_{3a}t + C_{3b}$.
Now, we write again x, y, z and receive the general solution of Lagrangian:

$$\begin{aligned} x &= \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{(C_{1a}t + C_{1b})^2 + (C_{2a}t + C_{2b})^2 + (C_{3a}t + C_{3b})^2}. \\ y &= \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{C_{3a}t + C_{3b}}{\sqrt{(C_{1a}t + C_{1b})^2 + (C_{2a}t + C_{2b})^2 + (C_{3a}t + C_{3b})^2}}. \\ z &= \arctan\left(\frac{x_2}{x_1}\right) = \arctan\left(\frac{C_{2a}t + C_{2b}}{C_{1a}t + C_{1b}}\right). \end{aligned}$$

With the values of x, $\dot{x}$, y, $\dot{y}$, z, $\dot{z}$ at any time, we have 6 equations to find 6 constant: C_{1a}, C_{1b}, C_{2a}, C_{2b}, C_{3a}, C_{3b}.

Using the result we have found, if we choose (x_1, x_2, x_3) is a Canonical basis (or Descartes coordinate for three dimensions), (x, y, z) will be a new coordinate system as shown below (in Figure 1).

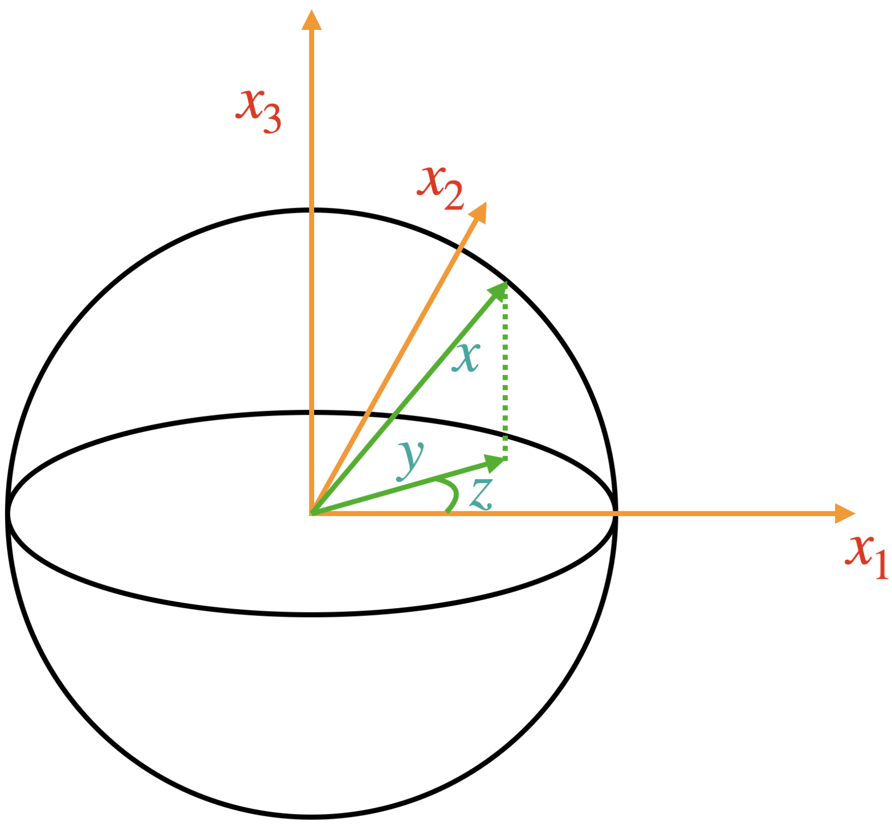

Figure 1: The coordinate system of this problems

Remember, in this problem, the Lagragian does not have any explicit dependence on z, so we only can find $\dot{z}$ and when we performance z, may be we need a constant ϕ so as to: $x_1 = x\sqrt{1-y^2}\cos(z-\phi)$, $x_2 = x\sqrt{1-y^2}\sin(z-\phi)$. But it doesn't changes our general solution cause:

$$z = \arctan\left(\frac{C_{2a}t + C_{2b}}{C_{1a}t + C_{1b}}\right) + \phi = \arctan\left[\frac{(C_{1a}\tan\phi + C_{2a})\,t + (C_{1b}\tan\phi + C_{2b})}{(C_{1a} - C_{2a}\tan\phi)\,t + (C_{1b} - C_{2b}\tan\phi)}\right].$$

For the new coordinate when we change z to be $z - \phi$, the x_1 axis will rotate clockwise an angle around the x_3 axis (an fixed axis).

After all, we can conclude the Lagrangian in this problems is the Lagrangian of a free partical moving non-relativitic on the coordinate (x, y, z) as we built.