52nd—24th INTERNATIONAL-RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS
Problem 19

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In this problem, we set $4\pi\varepsilon_0 = c = 1$. The coordinate of electron is perfomed by (r, θ) in the polar coordinate system.
The electronic potential of electron is: $V(r) = -\frac{Ze^2}{r}$. So we can write the Lagragian of this electron [1] is:

$$\mathcal{L} = -m\sqrt{1 - \dot{r}^2 - r^2\dot{\theta}^2} + \frac{Ze^2}{r}.$$

This Lagrangian doesn't have any explicit dependence on θ. Therefore, the angular momentum of the electron is a constant L, so we can write:

$$L = \frac{\partial \mathcal{L}}{\partial \theta} = m\frac{r^2\dot{\theta}}{\sqrt{1 - \dot{r}^2 - r^2\dot{\theta}^2}}. \tag{1}$$

Cause our Lagragian doesn't have any explitcit dependence on time t, conservation the energy of the electron:

$$E = \frac{m}{\sqrt{1 - \dot{r}^2 - r^2\dot{\theta}^2}} - \frac{Ze^2}{r}. \tag{2}$$

From (1) and (2), we find $\dot{\theta}$ by r:

$$\dot{\theta} = \frac{L}{Er^2 + Ze^2 r}. \tag{3}$$

Let $r'_\theta = \frac{dr}{d\theta}$. So we can perform $\dot{r}$ by r'_θ:

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta}.$$

We write again (1):

$$\frac{\dot{r}}{\dot{\theta}} = \sqrt{\frac{1}{\dot{\theta}^2} - r^2 - \left(\frac{m}{L}\right)^2 r^4}.$$

And put $\dot{\theta}$ and $\dot{r}$ in our new equation, we will have a relationship of r'_θ and r to find the orbital of the electron:

$$\frac{dr}{d\theta} = \sqrt{\left(\frac{Er^2 - Ze^2 r}{L}\right)^2 - r^2 - \left(\frac{m}{L}\right)^2 r^4}$$

$$\Rightarrow \theta = \int \frac{dr}{\sqrt{\frac{E^2-m^2}{L^2}r^4 - \frac{2EZe^2}{L^2}r^3 + \left[\left(\frac{Ze^2}{L}\right)^2 - 1\right] r^2}}$$

$$\Rightarrow \theta = \frac{1}{\sqrt{1-\left(\frac{Ze^2}{L}\right)^2}} \arccos \frac{\left[1-\left(\frac{Ze^2}{L}\right)^2\right]\frac{1}{r} - \frac{EZe^2}{L^2}}{\sqrt{\left(\frac{E}{L}\right)^2 - \left(\frac{m}{L}\right)^2\left[1-\left(\frac{Ze^2}{L}\right)^2\right]}} + C.$$

Choose the coordinate system with $C = 0$, so the Orbital of the electron is:

$$r = \frac{\frac{L}{E}\left(\frac{L}{Ze^2} - \frac{Ze^2}{L}\right)}{1 + \sqrt{\left(\frac{L}{Ze^2}\right)^2 + \left(\frac{m}{E}\right)^2 - \left(\frac{mL}{EZe^2}\right)^2} \cos\left\{\sqrt{1-\left(\frac{Ze^2}{L}\right)^2}\theta\right\}} = \frac{p}{1+\epsilon\cos(\alpha\theta)}.$$

So, the electron will move in a flower orbital.
In the SI units [1]:

$$r = \frac{\frac{Lc}{E}\left(\frac{4\pi\varepsilon_0 cL}{Ze^2} - \frac{Ze^2}{4\pi\varepsilon_0 cL}\right)}{1 + \sqrt{\left(\frac{4\pi\varepsilon_0 cL}{Ze^2}\right)^2 + \left(\frac{mc^2}{E}\right)^2 - \left(\frac{4\pi\varepsilon_0 c^3 mL}{EZe^2}\right)^2} \cos\left\{\sqrt{1-\left(\frac{Ze^2}{4\pi\varepsilon_0 cL}\right)^2}\theta\right\}}.$$

In reality, with $L \sim \hbar$, $Z \sim 31$, then $\left(\frac{Ze^2}{4\pi\varepsilon_0 cL}\right)^2 \sim 0.05$ and with bigger Z, we can't use the classical approximation or define the semi-axes of ellipse. Therefore, we can't make a new Kepler's 3rd law with the semi-axes. Thus, we will use $p = \frac{L}{E}\left(\frac{L}{Ze^2} - \frac{Ze^2}{L}\right)$, $e = \sqrt{\left(\frac{L}{Ze^2}\right)^2 + \left(\frac{m}{E}\right)^2 - \left(\frac{mL}{EZe^2}\right)^2}$ and $\alpha = \sqrt{1-\left(\frac{Ze^2}{4\pi\varepsilon_0 cL}\right)^2}$ to make a new Kepler's 3rd law.
In this case, the orbiting time is the time T from $\theta = 0$ to $\theta = \frac{2\pi}{\alpha}$.

a) From (3):

$$\begin{aligned}
dt &= \left[\left(\frac{E}{L}\right)r^2 + \left(\frac{Ze^2}{L}\right)r\right]d\theta \\
&= \frac{p}{\sqrt{1-\alpha^2}}\frac{1+(1-\alpha^2)\cos(\alpha\theta)}{[1+\epsilon\cos(\alpha\theta)]^2}d\theta \\
\Rightarrow T &= \frac{p}{\sqrt{1-\alpha^2}}\int_0^{\frac{2\pi}{\alpha}}\frac{1+(1-\alpha^2)\,\epsilon\cos(\alpha\theta)}{[1+\epsilon\cos(\alpha\theta)]^2}d\theta \\
\Leftrightarrow T &= \frac{p}{\alpha\sqrt{1-\alpha^2}}\int_0^{2\pi}\frac{1+(1-\alpha^2)\,\epsilon\cos x}{[1+\epsilon\cos x]^2}dx \\
&= 2\pi p\frac{1-(1-\alpha^2)\,\epsilon^2}{\alpha\sqrt{1-\alpha^2}\,(1-\epsilon^2)^{\frac{3}{2}}}.
\end{aligned}$$

[1] We only use the SI units to calculate some values and compare them, after that, we will back to our units

b) In the electron frame:

$$\begin{aligned}
dt' &= dt\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2} \\
&= \frac{m}{L}r^2 d\theta \\
&= p\alpha\sqrt{\frac{\left(1-\alpha^2\right)\epsilon^2-1}{1-\alpha^2}}\frac{d\theta}{\left[1+\epsilon\cos\left(\alpha\theta\right)\right]^2} \\
\Rightarrow T' &= p\sqrt{\frac{\left(1-\alpha^2\right)\epsilon^2-1}{1-\alpha^2}}\int_0^{2\pi}\frac{dx}{\left[1+\epsilon\cos x\right]^2} \\
&= 2\pi p\frac{\sqrt{1-\left(1-\alpha^2\right)\epsilon^2}}{\sqrt{1-\alpha^2}\left(1-\epsilon^2\right)^{\frac{3}{2}}}.
\end{aligned}$$

References

[1] A. S. Kompaneyets (1978), *A Course Of Theoretical Physics, Vol. 1 Fundamental Laws*, Mir titles.

[2] Lim Yung-Kou (1994), *Problems and Solutions on Mechanics*, WORLD SCIENTIFIC, pp. 738-741.