52nd—24th INTERNATIONAL-RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS
Problem 15

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We consider a thin pipe with a regular N-angle base with radius of the inscribed circle is R and wall thickness is δ and Electrical resistivity γ. The mass of the magnet is m and its magnetic dipole is p. For a long enough tube, the velocity of the magnet is v. We choose cylindrical coordinate system (ρ, θ, z) with the height $z = 0$ is the coordinate of the magnet at $t = 0$ (We only consider this proplem when the velocity of magnet is constant, so $t = 0$ can be any time we want to choose).
The magnetic vector potential at a position (ρ, z) from the location of a point dipole is:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{p}\times\vec{r}}{r^3} = \frac{\mu_0 m\rho}{4\pi\left[\rho^2 + (z+vt)^2\right]^{\frac{3}{2}}}\hat{\theta}.$$

The electric field of that point will be:

$$\vec{E} = -\frac{d\vec{A}}{dt} = \frac{3\mu_0 mv\rho\,(z+vt)}{4\pi\left[\rho^2 + (z+vt)^2\right]^{\frac{5}{2}}}\hat{\theta}.$$

With the rectangle wall of pipe, we will divide into 2 components: parallel to the surface and perpendicular to the surface. With the paralled component, the current density is the same as cylindrical tubes, we have: $j_t = \frac{E_t}{\gamma}$. But with the perpendicular component, when the magnet is falling, the current density will create the surface charge density σ at two surface of tube. This surface charge density at two surface will create the new electric field $\frac{\sigma}{\varepsilon_0}$, so the paralled component of current density is $j_n = \frac{1}{\gamma}\left(E_n - \frac{\sigma}{\varepsilon_0}\right)$.

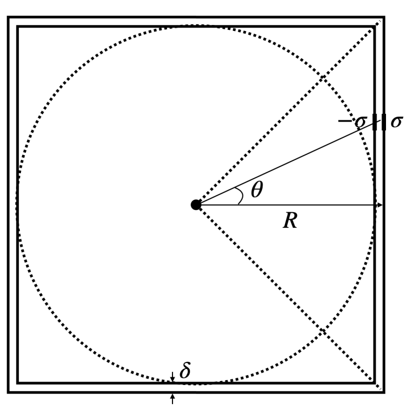

Figure 1: The pipe $N = 4$.

In the regular N-angle, we consider a edge from $\theta = -\frac{\pi}{N}$ to $\theta = \frac{\pi}{N}$. The radius of a point at θ is $\rho = \frac{R}{\cos\theta}$. Therefore, we have two equations of the current density at a point (θ, z):

$$j_t(\theta, z) = \frac{E_t}{\gamma} = \frac{3\mu_0 mvR\,(z+vt)}{4\pi\gamma\left[\left(\frac{R}{\cos\theta}\right)^2 + (z+vt)^2\right]^{\frac{5}{2}}}. \tag{1}$$

$$j_n = \frac{1}{\gamma}\left(E_n - \frac{\sigma}{\varepsilon_0}\right). \tag{2}$$

Derivatives with respect to time with equation (2), we have:

$$\frac{dj_n}{dt} = \frac{1}{\gamma}\frac{dE_n}{dt} - \frac{1}{\gamma\varepsilon_0}j_n$$

$$\Leftrightarrow \frac{d}{dt}\left(j_n e^{\frac{t}{\gamma\varepsilon_0}} - \frac{E_n}{\gamma}e^{\frac{t}{\gamma\varepsilon_0}}\right) = -\frac{E_n}{\gamma^2\varepsilon_0}e^{\frac{t}{\gamma\varepsilon_0}}$$

$$\Rightarrow j_n = \frac{E_n}{\gamma} - e^{-\frac{t}{\gamma\varepsilon_0}}\int \frac{E_n}{\gamma^2\varepsilon_0}e^{\frac{t}{\gamma\varepsilon_0}}\,dt.$$

At $t = -\infty$, $j_n = 0$ and $E_n = 0$, so we will have:

$$j_n(\theta, z) = \frac{3\mu_0 mvR\,(z+vt)\tan\theta}{4\pi\gamma\left[\left(\frac{R}{\cos\theta}\right)^2 + (z+vt)^2\right]^{\frac{5}{2}}} - e^{-\frac{t}{\gamma\varepsilon_0}}\int_{-\infty}^{t}\frac{3\mu_0 mvR\,(z+vt)\tan\theta}{4\pi\gamma^2\varepsilon_0\left[\left(\frac{R}{\cos\theta}\right)^2 + (z+vt)^2\right]^{\frac{5}{2}}}e^{\frac{t}{\gamma\varepsilon_0}}\,dt.$$

Because when system reaches steady state we have $\sigma(z,t) = \sigma(z + v\Delta t, t + \Delta t)$, then we can say that total energy of electric field is conservation. So all of work by gravity is used to heat dissipation on the resistor. Therefore, that power is:

$$\begin{aligned} P = mgv &= \int \gamma j^2 dV \\ &= \int_{-\infty}^{\infty}\left[N\int_{-\frac{\pi}{N}}^{\frac{\pi}{N}}\gamma\left(j_t^2 + j_n^2\right)\frac{Rd\theta}{\cos^2\theta}\delta\right]dz. \end{aligned}$$

Remember that P doesn't depend on t, so we choose $t = 0$ to calculate the next part. Let $x = \frac{z}{R}$ We let the function $f(\theta, z)$ is:

$$f(\theta, x) = \int_{-\infty}^{0}\frac{x + \frac{vt}{R}}{\gamma\varepsilon_0\left[\left(\frac{1}{\cos\theta}\right)^2 + \left(x + \frac{vt}{R}\right)^2\right]^{\frac{5}{2}}}e^{\frac{t}{\gamma\varepsilon_0}}\,dt.$$

So we write the equation of power again:

$$mg = \frac{9\mu_0^2 Np^2 v\delta}{16\pi^2\gamma R^4}\int_{-\infty}^{\infty}\int_{-\frac{\pi}{N}}^{\frac{\pi}{N}}\left[\frac{x^2}{\left[\left(\frac{1}{\cos\theta}\right)^2 + x^2\right]^5} + \left(\frac{x\tan\theta}{\left[\left(\frac{1}{\cos\theta}\right)^2 + x^2\right]^{\frac{5}{2}}} - f(\theta, x)\right)^2\right]\frac{d\theta}{\cos^2\theta}dx.$$

Although we can't calculate this intergral by the hand, we can guess that it will be a function of N and $\frac{v\gamma\varepsilon_0}{R}$. Set all the intergral is $F\left(N, \frac{v\gamma\varepsilon_0}{R}\right)$. We can see that our velocity of magnet is the solution of the equation:

$$\frac{16\pi^2\gamma R^4 mg}{9\mu_0^2 p^2 N\delta} = vF\left(N, \frac{v\gamma\varepsilon_0}{R}\right).$$

In reality, the resistivity of a conductive material is $\gamma \sim 10^{-8}\Omega \cdot \mathrm{m}$ and with proper time of this system is $\tau = \gamma\varepsilon_0 \sim 10^{-19}\mathrm{s}$ (very small), so we also can assume that the perpendicular component of current density is very small and we can ignore it. In that case:

$$F(N) = \int_{-\infty}^{\infty}\int_{-\frac{\pi}{N}}^{\frac{\pi}{N}} \frac{x^2}{\left[\left(\frac{1}{\cos\theta}\right)^2 + x^2\right]^5}\frac{d\theta}{\cos^2\theta}dx = \frac{5\pi}{64}\left[\sin\left(\frac{\pi}{N}\right) - \frac{2}{3}\sin^3\left(\frac{\pi}{N}\right) + \frac{1}{5}\sin^5\left(\frac{\pi}{N}\right)\right].$$

Thus, the velocity of magnet is:

$$v(N) = \frac{1024mg\gamma R^4}{45\mu_0^2p^2\delta}\frac{\frac{\pi}{N}}{\sin\left(\frac{\pi}{N}\right) - \frac{2}{3}\sin^3\left(\frac{\pi}{N}\right) + \frac{1}{5}\sin^5\left(\frac{\pi}{N}\right)}.$$

Let's check this solution, with cylindrical tube, $N \Rightarrow \infty$, the velocity of magnet is: $v(\infty) = \frac{1024mg\gamma R^4}{45\mu_0^2p^2\delta}$ (the same as a lot of previous calculations [1]). For the other values, we set a function $g(N)$ so that $v(N) = g(N)v(\infty)$ and we have this tabular:

N	∞	3	4	5	6	7	8
$g(N)$	1	1.974	1.550	1.347	1.238	1.173	1.132

References

[1] Yan Levin, Fernando L. da Silveira, and Felipe B. Rizzato. Electromagnetic braking: A simplequantitative model. *American Journal of Physics*, 74(9):815–817, 2006.