

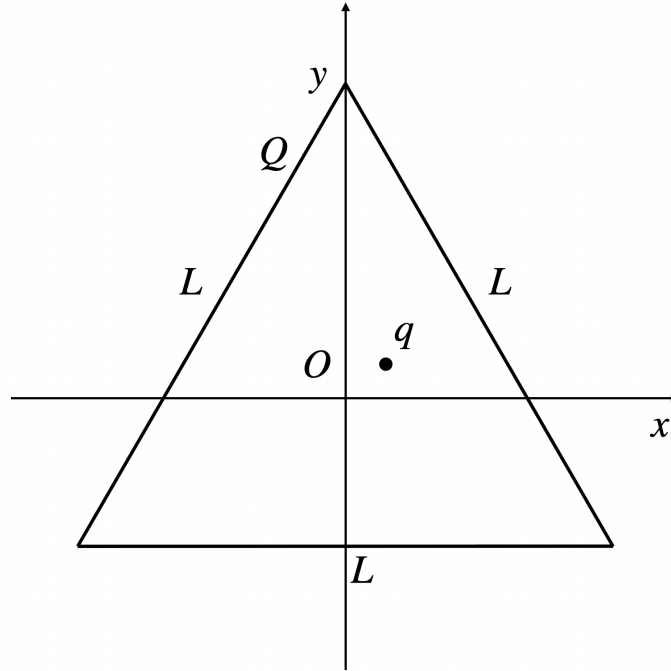
# 52nd—24th INTERNATIONAL-RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS

## Problem 13

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Consider a regular triangle with total charge is  $Q$  and length of edges are  $L$ . The charge of massive point at center of the triangle is  $q$  and its mass is  $m$ .



So using the Coulomb law, we can calculate the voltage of the massive point is:

$$\begin{aligned} \varphi(x, y) = & \frac{Q}{12\pi\epsilon_0 L} \left[ \sinh^{-1} \left( \sqrt{3} \frac{1 + \frac{2x}{L}}{1 + \frac{2\sqrt{3}y}{L}} \right) + \sinh^{-1} \left( \sqrt{3} \frac{1 - \frac{2x}{L}}{1 + \frac{2\sqrt{3}y}{L}} \right) \right] \\ & + \frac{Q}{12\pi\epsilon_0 L} \left[ \sinh^{-1} \left( \sqrt{3} \frac{1 + \frac{-x+\sqrt{3}y}{L}}{1 + \frac{-3x-\sqrt{3}y}{L}} \right) + \sinh^{-1} \left( \sqrt{3} \frac{1 - \frac{-x+\sqrt{3}y}{L}}{1 + \frac{-3x-\sqrt{3}y}{L}} \right) \right] \\ & + \frac{Q}{12\pi\epsilon_0 L} \left[ \sinh^{-1} \left( \sqrt{3} \frac{1 + \frac{-x-\sqrt{3}y}{L}}{1 + \frac{3x-\sqrt{3}y}{L}} \right) + \sinh^{-1} \left( \sqrt{3} \frac{1 - \frac{-x-\sqrt{3}y}{L}}{1 + \frac{3x-\sqrt{3}y}{L}} \right) \right]. \end{aligned}$$

The Lagrangian of this massive point is:

$$L = \frac{1}{2}m (\dot{x}^2 + \dot{y}^2) - q\varphi(x, y).$$

Euler-Lagrange equation:

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) &= \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -q \frac{\partial \varphi}{\partial x} \approx -\frac{3\sqrt{3}}{2} \frac{qQ}{\pi\epsilon_0 L^3} x. \\ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) &= \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = -q \frac{\partial \varphi}{\partial y} \approx -\frac{3\sqrt{3}}{2} \frac{qQ}{\pi\epsilon_0 L^3} y. \end{aligned}$$

Therefore, if  $qQ > 0$ , the massive point will oscillate with frequency  $f = \frac{1}{2\pi} \sqrt{\frac{3\sqrt{3}}{2} \frac{qQ}{\pi\epsilon_0 m L^3}}$ . This oscillation doesn't depend on the direction of displacement on the plane of the triangle.

With three dimensions, when the massive point have a small displacement, there is a force effect on it along the  $z$  axis:

$$F(z) \approx 3\sqrt{3} \frac{qQ}{\pi\epsilon_0 L^3} z.$$

So, we can see that in this case,  $z = 0$  isn't stable equilibrium.

Similarly, if  $qQ < 0$ ,  $z = 0$  is a stable equilibrium, but the center of triangle is not. Therefore, we can't make any frame made of insulating wire from which the charge cannot 'escape'!