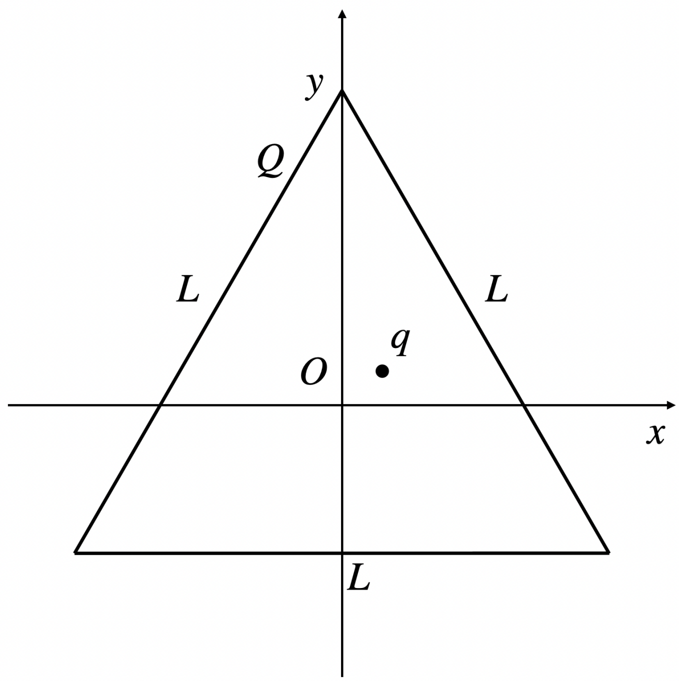
52nd—24th INTERNATIONAL-RUDOLF ORTVAY PROBLEM SOLVING CONTEST IN PHYSICS
Problem 13

Nguyen Thanh Long

March 20, 2022

Consider a regular triangle with total charge is Q and length of edges are L. The charge of massive point at center of the triangle is q and its mass is m.



So using the Coulomb law, we can calculate the voltage of the massive point is:

$$\begin{aligned}
\varphi(x,y) =& \frac{Q}{12\pi\varepsilon_0 L}\left[\sinh^{-1}\left(\sqrt{3}\frac{1+\frac{2x}{L}}{1+\frac{2\sqrt{3}y}{L}}\right)+\sinh^{-1}\left(\sqrt{3}\frac{1-\frac{2x}{L}}{1+\frac{2\sqrt{3}y}{L}}\right)\right] \\
&+\frac{Q}{12\pi\varepsilon_0 L}\left[\sinh^{-1}\left(\sqrt{3}\frac{1+\frac{-x+\sqrt{3}y}{L}}{1+\frac{-3x-\sqrt{3}y}{L}}\right)+\sinh^{-1}\left(\sqrt{3}\frac{1-\frac{-x+\sqrt{3}y}{L}}{1+\frac{-3x-\sqrt{3}y}{L}}\right)\right] \\
&+\frac{Q}{12\pi\varepsilon_0 L}\left[\sinh^{-1}\left(\sqrt{3}\frac{1+\frac{-x-\sqrt{3}y}{L}}{1+\frac{3x-\sqrt{3}y}{L}}\right)+\sinh^{-1}\left(\sqrt{3}\frac{1-\frac{-x-\sqrt{3}y}{L}}{1+\frac{3x-\sqrt{3}y}{L}}\right)\right].
\end{aligned}$$

The Lagrangian of this massive point is:

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - q\varphi(x, y).$$

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \Rightarrow m\ddot{x} = -q\frac{\partial \varphi}{\partial x} \approx -\frac{3\sqrt{3}}{2}\frac{qQ}{\pi\varepsilon_0 L^3}x.$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y} \Rightarrow m\ddot{y} = -q\frac{\partial \varphi}{\partial y} \approx -\frac{3\sqrt{3}}{2}\frac{qQ}{\pi\varepsilon_0 L^3}y.$$

Therefore, if $qQ > 0$, the massive point will oscillate with frequency $f = \frac{1}{2\pi}\sqrt{\frac{3\sqrt{3}}{2}\frac{qQ}{\pi\varepsilon_0 m L^3}}$. This oscillation doesn't depend on the direction of displacement on the plane of the triangle.
With three dimensions, when the massive point have a small displacement, there is a force effect on it along the z axis:

$$F(z) \approx 3\sqrt{3}\frac{qQ}{\pi\varepsilon_0 L^3}z.$$

So, we can see that in this case, $z = 0$ isn't stable equilibrium.
Similarly, if $qQ < 0$, $z = 0$ is a stable equilibrium, but the center of triangle is not. Therefore, we can't make any frame made of insulating wire from which the charge cannot 'escape'!