Vietnam Academy of Science and Technology
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ERGODICITY ECONOMICS

Cooperation game and finding optimal strategy for rebalanced portfolio

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Chapter 1

Introduction and some preliminaries

1.1 Non - cooperation

Suppose that there are n individuals having their own business. As non-cooperators, each individual business will grow its wealth following the dynamics

$$\Delta x_i(t) = x_i(t)\left[\mu\Delta t + \sigma\sqrt{\Delta t}\xi_i\right], \tag{1.1}$$

$$x_i(t+\Delta t) = x_i(t) + \Delta x_i(t) \tag{1.2}$$

where ξ_i are standard normal random variates, $\xi_i \sim \mathcal{N}(0,1)$.
Commonly, the expectation value is usually used to evaluate future prospects; however, it may be sometimes unsuitable to be used to do so in the short-time. The expectation values of a non-cooperator is called $\langle x_i(t)\rangle$.

1.2 Cooperation game

The idea of the cooperation game is that, although the dynamic of each individual's business fluctuates around a certain growth rate, cooperation will reduce noise and lead to a faster growth. For that reason, to decrease risk in business, owners have an intention to cooperate with the others. In the discrete-time, there are two steps for this strategy which includes: phase (1) of each individual business' growth and phase (2) of both pooling and sharing of resources. This is illustrated via the figure 1.1.

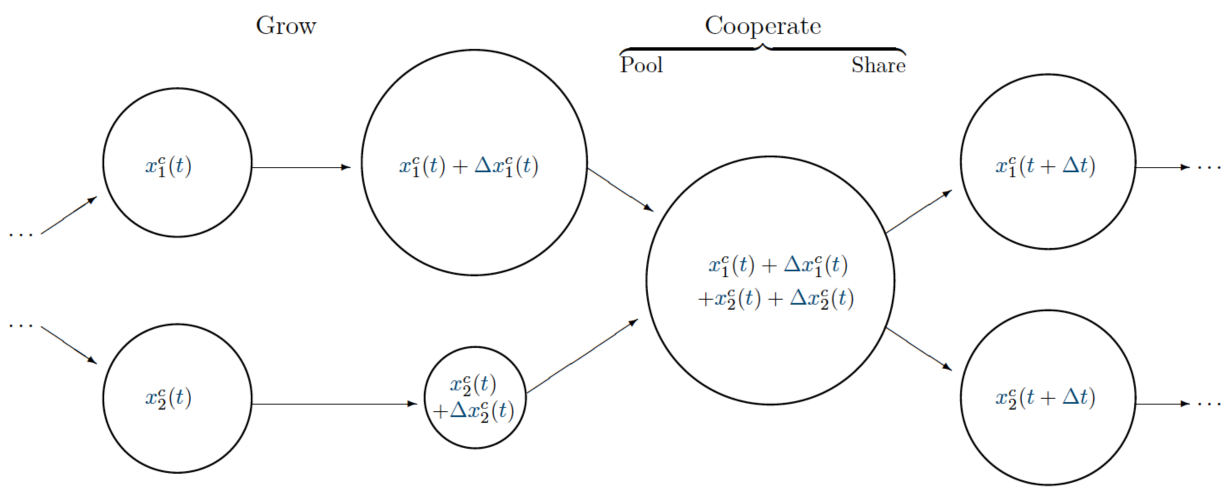


Figure 1.1: Cooperation dynamics [1]

After cooperation, the wealth is split into $x_1^c(t)$ and $x_2^c(t)$ respectively with $x_1^c = x_2^c$ such that:

$$x_1 \oplus x_2 = x_1^c + x_2^c. \tag{1.3}$$

Since each cooperator increases its resources by the dynamics

$$\Delta x_i^c(t) = x_i^c(t) \left[\mu \Delta t + \sigma \sqrt{\Delta t} \xi_i \right], \tag{1.4}$$

it implies that:

$$\frac{\Delta x_i(t)}{x_i(t)} = \mu \Delta t + \sigma \sqrt{\Delta t} \xi_i.$$

The limiting equation for this discrete version is

$$dx_i(t) = x_i(t) \Big[\mu dt + \sigma dW_i(t) \Big], \tag{1.5}$$

which can be solved explicitly (using Ito's calculus) as

$$x_i(t) = x_i(0) \exp \left\{ \Big[(\mu - \frac{\sigma^2}{2})t + \sigma W_i(t) \Big] \right\}. \tag{1.6}$$

Business owners believe that cooperation will have increases growth rate and avoid risk less than non-cooperation. Indeed, resources are pooled and shared equally among the cooperators:

$$\left(\frac{x_1 \oplus x_2}{2}\right)_i (t + \Delta t) = \left(\frac{x_1 \oplus x_2}{2}\right)_i (t) + \Delta(\frac{x_1 \oplus x_2}{2})_i(t) \tag{1.7}$$

$$\Rightarrow \sum_{i=1}^{2} \Delta(\frac{x_1 \oplus x_2}{2})_i(t) = \sum_{i=1}^{2} (\frac{x_1 + x_2}{2})_i(t) \cdot (\mu\Delta t + \sigma\sqrt{\Delta t}\xi_i) \tag{1.8}$$

$$\Rightarrow \Delta(x_1 \oplus x_2)(t) = \frac{x_1 \oplus x_2}{2} \cdot (2\mu\Delta t + \frac{2}{\sqrt{2}}\sigma\sqrt{\Delta t}\frac{\xi_1 + \xi_2}{2}) \tag{1.9}$$

Thus one obtains the cooperation dynamics as follows

$$(x_1 \oplus x_2)(t + \Delta t) = (x_1 \oplus x_2)(t) \left[1 + \left(\mu\Delta t + \sigma\sqrt{\Delta t}\frac{\xi_1 + \xi_2}{2}\right)\right] \tag{1.10}$$

where

$$\xi_{1\oplus 2} = \frac{\xi_1 + \xi_2}{\sqrt{2}}, \tag{1.11}$$

In case $(\xi_1, \xi_2) = 0$ it implies that

$$\mathbb{E}\left(\frac{\xi_1 + \xi_2}{\sqrt{2}}\right)^2 = \frac{\mathbb{E}\xi_1^2 + 2\mathbb{E}\xi_1\xi_2 + \mathbb{E}\xi_2^2}{2} = 1. \tag{1.12}$$

As a result, $\xi_{1\oplus 2} = \frac{\xi_1+\xi_2}{\sqrt{2}}$ follows a normal distribution. Moreover, (1.10) is equivalent to

$$(x_1 \oplus x_2)(t + \Delta t) = (x_1 \oplus x_2)(t) \left[1 + \left(\mu\Delta t + \frac{\sigma}{\sqrt{2}}\sqrt{\Delta t}\frac{\xi_1 + \xi_2}{\sqrt{2}}\right)\right].$$

Therefore,

$$(x_1 \oplus x_2)(t + \Delta t) = (x_1 \oplus x_2)(t) \left[1 + \left(\mu\Delta t + \frac{\sigma}{\sqrt{2}}\sqrt{\Delta t}\xi_{1\oplus 2}\right)\right].$$

The limiting equation for the discrete version is the following Ito stochastic differential equation

$$d(x_1 \oplus x_2)(t) = (x_1 \oplus x_2)(t)\Big[\mu dt + \frac{\sigma}{\sqrt{2}} dW_{1\oplus 2}(t)\Big]. \tag{1.13}$$

Thus it can be solved explicitely as

$$(x_1 \oplus x_2)(t) = (x_1 \oplus x_2)(0) \exp\Big\{\Big[(\mu - \frac{\sigma^2}{4})t + \frac{\sigma}{\sqrt{2}} W_{1\oplus 2}(t)\Big]\Big\}. \tag{1.14}$$

Considering the time-average growth rate, it follows from (1.6) that the non-cooperating entities grow at

$$g_t(x_i) = \mu - \frac{\sigma^2}{2}.$$

Meanwhile, it follows from (1.14) that the cooperating grows at

$$g_t\left(x_1 \oplus x_2\right) = \mu - \frac{\sigma^2}{4},$$

Obviously, we obtain

$$g_t\left(x_1 \oplus x_2\right) > g_t\left(x_i\right). \tag{1.15}$$

The inequality (1.15) shows that the benefits from cooperation when the growth rate of cooperation is higher than those of non-cooperation.

Chapter 2

Cooperation with multi-parties and different business growths

In more complicated situation, cooperation clusters of N individuals, there are a large number of individuals pool and share their resources. After cooperation, the wealth is split with fixed proportion of its weath, $p_1 x_1^c, p_2 x_2^c, ...p_n x_n^c$ respectively such that:

$$x_1 \oplus x_2 \oplus ... \oplus x_n = \sum_{i=1}^{n} p_i x_i^c. \tag{2.1}$$

Due to the dynamics of cooperation, each cooperator grows its resources:

$$\Delta x_i^c(t) = p_i x_i^c(t) \left[\mu_i \Delta t + \sigma_i \sqrt{\Delta t} \xi_i \right], \text{for } i = 1, ...n. \tag{2.2}$$

$$\Rightarrow \Delta x_i^c(t) = x_i^c(t) \left[p_i \mu_i \Delta t + p_i \sigma_i \sqrt{\Delta t} \xi_i \right], \text{for } i = 1, ...n. \tag{2.3}$$

There is a system of N individuals whose weaths, $x_i(t)$, to change in accordance with the stochastic differential equation:

$$dx_i(t) = x_i(t) \Big[p_i \mu_i dt + p_i \sigma_i dW_i(t) \Big], \text{for } i = 1, ...n \tag{2.4}$$

which can be also solved explicitly (using Ito's calculus) as

$$x_i(t) = x_i(0) \exp \left\{ \left[(\; p_i \mu_i - \frac{(p_i \sigma_i)^2}{2}) t + p_i \sigma_i W_i(t) \right] \right\}. \tag{2.5}$$

Over a time step, dt, each cooperators puts their asset into a common pot and receives an amount of money back:

$$\sum_{i=1}^{n} \Delta(x_1 \oplus x_2 ... \oplus x_n)_i(t) = \sum_{i=1}^{n} p_i(x_1 + x_2 + ... + x_n)_i(t) \cdot (\mu_i \Delta t + \sigma_i \sqrt{\Delta t}\xi_i) \tag{2.6}$$

$$\Rightarrow \Delta(x_1 \oplus x_2 ... \oplus x_n)(t) = (x_1 \oplus x_2 ... \oplus x_n)(t) \sum_{i=1}^{n} (p_i \mu_i \Delta t + p_i \sigma_i \sqrt{\Delta t}\xi_i) \tag{2.7}$$

The limiting equation for this discrete model is

$$dx_1 \oplus x_2 \oplus ... \oplus x_n(t) = x_1 \oplus x_2 \oplus ... \oplus x_n \Big(\sum_{i=1}^{n} p_i \mu_i dt + \sum_{i=1}^{n} p_i \sigma_i dW_i(t) \Big), \tag{2.8}$$

hence it can be solved by using Ito calculus as

$$x_1 \oplus x_2 \oplus ... \oplus x_n(t) = x_1 \oplus x_2 \oplus ... \oplus x_n(0) \exp \Big\{ \Big[\sum_{i=1}^{n} (p_i \mu_i - \frac{(p_i \sigma_i)^2}{4}) \Big] t + \sum_{i=1}^{n} p_i \sigma_i W_i(t) \Big\}. \tag{2.9}$$

After deriving the similar as previous chapter in this situation, we will have the cooperation growth at:

$$g_t\,(x_1 \oplus x_2 ... \oplus x_n) = \sum_{i=1}^{n} (p_i \mu_i - \frac{(p_i \sigma_i)^2}{4})$$

where μ_i and σ_i is population mean and standard deviation of portfolio i^{th} respectively.

Chapter 3

Results and Discussion

3.1 Rebalanced Portfolio

The asset allocation of a portfolio determines the risk and return characteristics of the portfolio. To maintain its original risk and return characteristics over time, the portfolio must be rebalanced. Rebalancing a portfolio is the process of changing the weightings of assets in an investment portfolio by both purchasing and selling assests to reach desired portfolio composition. Portfolio composition identified at the beginning of individual strategy based on returns assets, risk of portfolio and even personal matters.

Portfolio rebalancing [3] is comcomly considered to be a powerfull risk control strategy. For a long time, the different porfolio selection stategy will produce different returns, portfolio driffs,... For this reason, it is recommended that we should rebalance portfolio. Besides, we should also consider how often an individual rebalances their portfolio because we have to pay tax and fee for every portfolio rebalancing. It is no reason for us to rebalance portfolio when the extra fee is equal higher than expectation returns.

Particularly, an investor who has an intention to invest some portfofio to avoid "put all your eggs in one basket", they need to have a strategy to invest in some stocks over time. Their detailed plan for this investment is portfolio rebalancing: identify the weight of each asset and stay original state of each portfolio by buying and selling portions of their portfolio. We suppose identical expection value and standard deviation to identify the weight of each asset at the beginning at each term:

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{n}(p_i\mu_i - \frac{(p_i\sigma_i)^2}{4}) \\ \text{subject to} \quad & p_i \geq 0 \quad \forall i = 1, 2, ...n \\ & \textstyle\sum_{i=1}^{n} p_i = 1. \end{aligned}$$

where μ_i, σ_i, p_i is population mean, standard deviation, fraction of portfolio i^{th} respectively.

3.2 Data Collection

Data are provided by Vietstock Finance company. We consider three securities FPT, VNM, SSI respectively in this research. FPT is a representative for technology company, VNM is a representative regrading food and beverage, while SSI is on finance service in Vietnam. Data includes attributions such as: open price, close price, date, highest price, lowest price,... and closed price are chosen for this research to visualize fluctuation of stock price from establish to 27/10/2022. Based on both cooperation game and rebalanced porfolio to obtain more and more return, an optimal portfolio selection problem was built and also solved with available data.

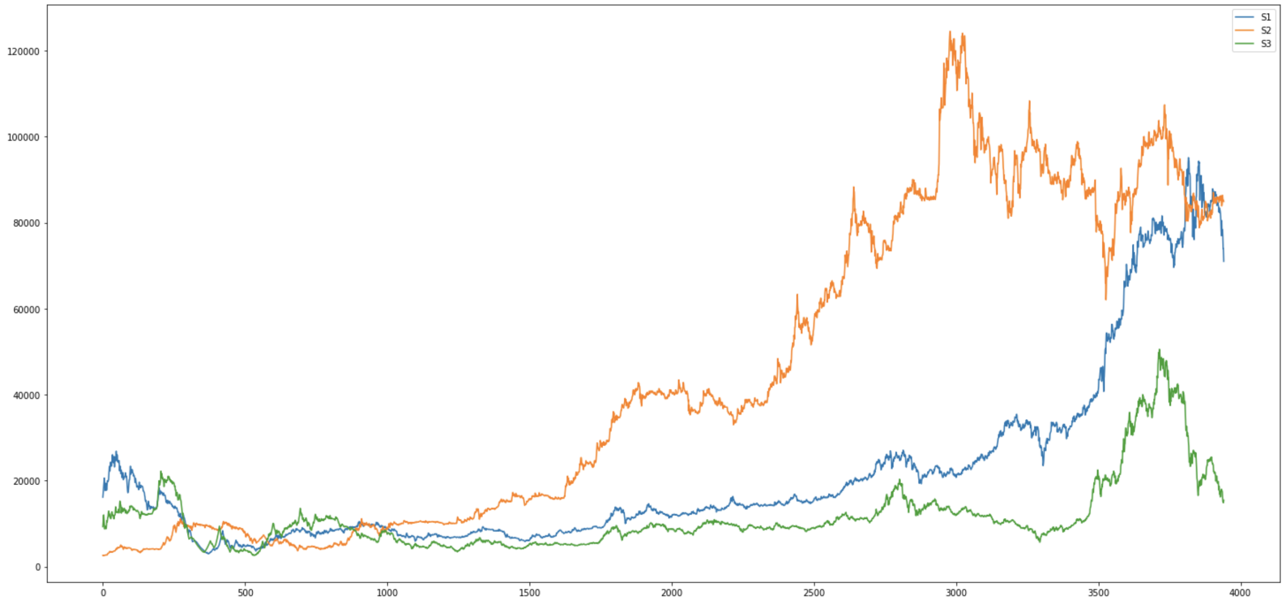


Figure 3.1: Close price of Stock

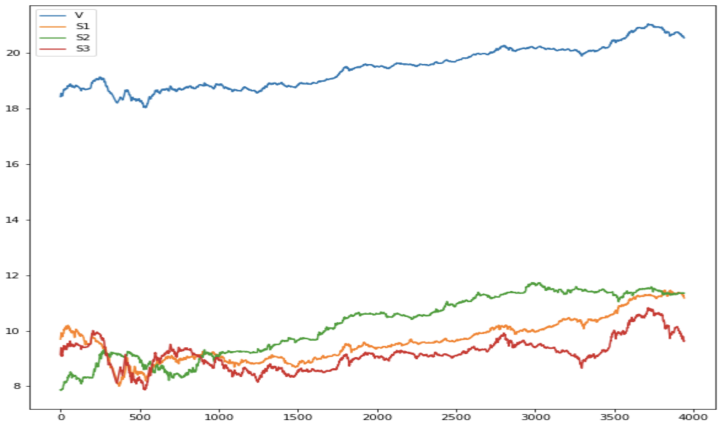
3.3 Cooperation strategy

The portfolio optimization problem is to find a global optimal solution for maximizing returns of strategy of risky securities for given expected return μ and standard deviation σ.

Example 1. *A balanced portfolio V with initial $V_0 = 100M$ (100 millions VND) of 3 stocks: FPT (price A_1), VNM (price A_2), SSI (price A_3) and cash (price A_4). There are 4 systematic parameters (can be changed as in put in the code): $p_i \in (0, 1)$ fixed $p_1 + p_2 + p_3 + p_4 = 1$ such that, everyday in the opening session we re-balance the portfolio that invest p_i amount to A_i.*

Question 1. *Show that whether the cooperation strategy has some positive effect and finding an optimal strategy?*

Dataset Visualization



(a) S_log_series

(b) S_log_rescale_series

Figure 3.2: Stock natural logarithm series visualization

Finding linear regression for each portfolio:

Figure 3.3: Linear Regression

Expectation values μ and standard deviation σ are identical via parameter of linear regression: slope and intercept respectively. All these value of paramaters are displayed on the following table:

	μ	σ
V	0.0006	0.0533
FPT	0.0006	0.1982
VNM	0.0009	0.1100
SSI	0.0003	0.2093

Table 3.1: Expection value and standard deviation

After identifying expection value and standard deviation, we can build a quadratic optimization problem. To solve this problem, we can use available function from cvxopt - a free software package for convex optimization.

Solution 1. *Convex optimization problems of the form*

$$\begin{aligned} \text{minimize} \quad & (1/2)x^T Px + q^T x \\ \textit{subject to} \quad & Gx \preceq h \\ & Ax = b. \end{aligned}$$

can be solved via ***solvers.qp()*** *function.*

Runned by Python progamming language, this problem has a global optimization solution:

	Portion
FPT	2.39e-01
VNM	4.37e-01
SSI	2.24e-01

Table 3.2: Optimal strategy

Then, investors should base on this strategy to get highest returns in risky tolerance.

Chapter 4

Conclusion

In this report, both non-cooperation and cooperation are discussed. Based on research of Ole Peters and Alexander Adamou [2], the benefit of cooperation was showed when there are 2 individuals want to cooperate whole their resources. In other situation, there are a large of number N individuals pool and share their resources, the time-average growth will still closer the expectation-value growth. To illustrate that, an example regarding portfolio rebalancing was given.

Future research includes carrying out portfolio rebalancing with larger dataset by other regression models. Moreover, we will discuss more about to find out some fascinating or unexpected something regardinge ergodicity economics.

Bibliography

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