answers_to_hw_01

August 24, 2019

1 Problem 1

The number of ways to partition a set S of n elements into k subsets, $S_1, S_2, \ldots, S_k$, which are mutually exclusive and collectively exhaustive is given by the *Multinomial coefficient*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!}, \tag{1.1}$$

where n_i is the number of elements of subset i and $n_1 + n_2 + \ldots + n_k = n$.

A proof of Eq. (1.1) is as follows.

There are $\binom{n}{n_1}$ ways of choosing subset S_1 from S. For each of the ways of choosing S_1, there are $\binom{n-n_1}{n_2}$ ways of choosing subset S_2 from the remaining elemnents in $S - S_1$, which denotes the set difference. We can reason in the same way until the last set S_k which has only 1 way to choose, given that we have chosen $S_1, S_2, \ldots, S_{k-1}$. So we have

$$\begin{aligned}
\binom{n}{n_1, n_2, \ldots, n_k} &= \binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \ldots \times \binom{n - n_1 - \ldots - n_{k-2}}{n_{k-1}} \times 1 \\
&= \frac{n!}{n_1!\,(n-n_1)!} \times \frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!} \times \ldots \times \frac{(n-n_1-\ldots-n_{k-2})!}{n_{k-1}!\,(n-n_1-\ldots-n_{k-2}-n_{k-1})!} \\
&= \frac{n!}{n_1!} \times \frac{1}{n_2!} \times \ldots \times \frac{1}{n_{k-1}!\,n_k!} \\
&= \frac{n!}{n_1!\,n_2!\,\ldots\,n_k!}.
\end{aligned} \tag{1.2}$$

Q.E.D.

So the number of ways of dividing 15 students into 3 groups of sizes 4, 5 and 6, respectively, is:

$$\binom{15}{4, 5, 5} = \frac{15!}{4!\,5!\,6!} = 630,630. \tag{1.3}$$

$a \quad b$
$c \quad d$

$b = \begin{bmatrix} a \\ c \end{bmatrix}$