HW3 - STAT 580 - Sp 2015

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1.

(a)

$$\begin{aligned} P(U \leq r(X)) &= \int_{\mathbb{X}} \left(\int_0^{r(x)} du \right) g(x)dx \\ &= \int_{\mathbb{X}} r(x)g(x)dx \\ &= \int_{\mathbb{X}} \frac{q(x)}{\alpha g(x)} g(x)dx \\ &= \frac{1}{\alpha} \int_{\mathbb{X}} q(x)dx. \end{aligned}$$

(b)

$$\begin{aligned} P(X \in A, U \leq r(X)) &= \int_A \left(\int_0^{r(x)} du \right) g(x)dx \\ &= \int_A r(x)g(x)dx \\ &= \int_A \frac{q(x)}{\alpha g(x)} g(x)dx \\ &= \frac{1}{\alpha} \int_A q(x)dx. \end{aligned}$$

Therefore,

$$\begin{aligned} P(Y \in A) &= P(X \in A | U \leq r(X)) \\ &= \frac{P(X \in A, U \leq r(X))}{P(U \leq r(X))} \\ &= \frac{\frac{1}{\alpha}\int_A q(x)dx}{\frac{1}{\alpha}\int_{\mathbb{X}} q(x)dx} \\ &= \frac{\frac{1}{\alpha}\int_A cq(x)dx}{\frac{1}{\alpha}\int_{\mathbb{X}} cq(x)dx} \\ &= \frac{\frac{1}{\alpha}\int_A f(x)dx}{\frac{1}{\alpha}\int_{\mathbb{X}} f(x)dx} \\ &= \int_A f(x)dx \quad (\text{since } \int_{\mathbb{X}} f(x)dx = 1) \end{aligned}$$

2.

(a) Let $g(x) = c(2x^{\theta-1}e^{-x} + x^{\theta-1/2}e^{-x})$. Then

$$\begin{aligned} g(x) &= c\left(2\Gamma(\theta) frac1\Gamma(\theta) e^{\theta-1}e^{-x} + \Gamma(\theta+1/2)\frac{1}{\Gamma(\theta+1/2)} e^{\theta-1/2}e^{-x}\right) \\ &= c(2\Gamma(\theta)\text{Gamma}(\theta,1) + \Gamma(\theta+1/2)\text{Gamma}(\theta+1/2,1)). \end{aligned} \tag{1}$$

Since $\int g dx = 1$, hence

$$c = \frac{1}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}. \tag{2}$$

(b) and from (1), g is the mixture of 2 Gamma distribution Gamma$(\theta, 1)$ and Gamma$(\theta + 1/2, 1)$ with the corresponding weights

$$\frac{2\Gamma(\theta)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}, \quad \frac{\Gamma(\theta + 1/2)}{2\Gamma(\theta) + \Gamma(\theta + 1/2)}$$

(c) A procedure to sample from $g(x)$:

1. Generate $U \sim$ Unif$(0, 1)$, $X_1 \sim$ Gamma$(\theta, 1)$, $X_2 \sim$ Gamma$(\theta+1/2, 1)$ independently, where a procedure to sample from any Gamma$(., 1)$ distribution can be conducted using a rejection sampling procedure following from Theorem 3.3 (page 29 in the link and page 406 in the book) http://www.nrbook.com/devroye/Devroye_files/chapter_nine.pdf
2. If $U \leq \frac{2\Gamma(\theta)}{2\Gamma(\theta)+\Gamma(\theta+1/2)}$ then $X = X_1$, otherwise, $X = X_2$
3. The random variable X has distribution $g(x)$.

(d) Sampling a random variable having distribution f using g as proposed distribution. It is easy to see

$$\sqrt{4 + x} \leq 2 + x^{1/2},$$

hence

$$q(x) \equiv \sqrt{4 + x} x^{\theta-1} e^{-x} \leq (2 + x^{1/2}) x^{\theta-1} e^{-x} \leq \frac{1}{c} g(x), \tag{3}$$

where c is defined as in (2).

1. Generate $X \sim g(x)$ as in part (c), and generate $U \sim$ Unif$(0, 1)$ independently. Set $r(x) = \frac{cq(x)}{g(x)}$, where q is defined as in (3).
2. If $U \leq r(X)$, then accept X, otherwise, repeat step 1.