Homework 5 solutions notes

3.

(a) The residual sum of squares for a model containing only block effects is 1.0199, while that for a model containing both blocks and treatments is 0.1468. So for these data, an ANOVA decomposition is:

source	d.f.	sum of squares
blocks (chamber sessions)	7	0.5194 = 1.5393 - 1.0199
treatments after blocks	3	0.8731 = 1.0199 - 0.1468
residuals	13	0.1468
corrected total	23	1.5393

(b)

block	A-B	A-C	A-D	B-C	B-D	C-D
1		0.16	0.29			0.13
2	0.48		0.73		0.25	
3	0.49	0.08		-0.41		
4				-0.32	0.04	0.36
5		0.14	0.28			0.14
6	0.47		0.75		0.28	
7	0.46	0.07		-0.39		
8				-0.20	0.06	0.26

A appears to have higher response than B, C, or D. D appears to have higher response than A, B, or C. C appears to have higher response than B. This suggests the ordering A > C > B > D.

7. For this BIBD, $r = 8$ and $\lambda = 2$.

(a) $Var(\widehat{\tau_1 - \tau_2}) = \frac{k}{\lambda t}\mathbf{c}'\mathbf{c}\sigma^2 = \frac{2}{2\times 5}2\sigma^2 = \frac{2}{5}\sigma^2$.

(b) $t_{0.975}(16) = 2.120$.

(c) $\frac{\lambda t}{k}\frac{\sum(\tau_i - \bar{\tau})^2}{\sigma^2} = \frac{2\times 5}{2}\frac{4}{2} = 10$.

6.

(a) Using R (for instance) with a model matrix as defined in class for both blocks (first 8 columns) and treatments (last 4):

```
fit2<-lsfit(X,y,intercept=F)
fit2$coefficients
      X1       X2       X3       X4       X5       X6
5.682917 6.110208 6.112500 6.181875 5.922917 6.176875
      X7       X8       X9      X10      X11      X12
6.075833 5.955208 0.498125 0.061250 0.333125 0.000000
```

So the intra-block estimate of $\tau_1 - \tau_2 = 0.498125 - 0.061250 = 0.436875$, and MSE can be computed as:

```
MSE2<-sum(fit2$residuals^2)/(24-11) ... 0.01129199
```

(b) Partitioning X above into $X1$ and $X2$ representing blocks and treatments, respectively:

```
z<-t(X1)%*%y
U<-t(X1)%*%X2
fit3<-lsfit(U,z)
fit3$coefficients
Intercept        X1        X2        X3        X4
   17.480     0.575     0.935     0.185     0.000
```

So the inter-block estimate of $\tau_1 - \tau_2 = 0.575 - 0.937 = -0.362$, and MSE for this fit is:

```
MSE3<-sum(fit3$residuals^2)/(8-4) ... 0.1291125
```

(c) Estimated weights are proportional to:
$\hat{w}_1 \propto \frac{1}{k}\texttt{MSE3}/(r-\lambda) = \frac{1}{3}0.1291125/(6-4) = 0.02151875.$
$\hat{w}_2 \propto \texttt{MSE2}/(t\lambda) = 0.01129199/(4 \times 4) = 0.0007057493.$
Hence the weights are $\hat{w}_1 = 0.96824$ and $\hat{w}_2 = 0.03176$, so the combined estimate of $\tau_1 - \tau_2$ is:

$$0.43688 \times 0.96824 - 0.36200 \times 0.03176 = 0.41151.$$

(d)

Intra-block analysis: $\sqrt{\mathbf{c}'\mathbf{c} \times k \times \texttt{MSE2}/(\lambda t)} = \sqrt{2 \times 0.002117248} = 0.065073.$

Inter-block analysis: $\sqrt{\mathbf{c}'\mathbf{c} \times \texttt{MSE3}/(r-\lambda)} = \sqrt{2 \times 0.06455625} = 0.3593223.$

Combined: $\sqrt{\hat{w}_1^2 Var_{intra} + \hat{w}_2^2 Var_{inter}} = \sqrt{0.96824^2 \times 0.065073^2 + -.03176^2 \times 0.35932^2} = 0.064031.$

7.

(a) $\mathbf{H}_1 = \frac{1}{2}\text{diag}(\mathbf{J}_{2\times2}, \mathbf{J}_{2\times2}, \mathbf{J}_{2\times2}, ..., \mathbf{J}_{2\times2}),$

$$\mathbf{X}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{X}_{2|1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The resulting reduced normal equation, $\mathbf{X}_2(\mathbf{I} - \mathbf{H}_1)\mathbf{X}_2\hat{\boldsymbol{\tau}} = \mathbf{X}_2(\mathbf{I} - \mathbf{H}_1)\mathbf{y}$, are:

$$\begin{array}{c} \frac{3}{2}\hat{\tau}_1 - \frac{3}{2}\hat{\tau}_2 = \frac{1}{2}\sum_{i=1}^{3} y_{i1} - \frac{1}{2}\sum_{i=1}^{3} y_{i2} \\ -\frac{3}{2}\hat{\tau}_1 + \frac{3}{2}\hat{\tau}_2 = -\frac{1}{2}\sum_{i=1}^{3} y_{i1} + \frac{1}{2}\sum_{i=1}^{3} y_{i2} \end{array}.$$

Either, when simplified, leads to $\widehat{\tau_1 - \tau_2} = \frac{1}{3}(\sum_{i=1}^{3} y_{i1} - \sum_{i=1}^{3} y_{i2})$.
Residual degrees of freedom are 17(corrected total) − 8(blocks) − 1(treatments) = 8.

(b) Using a model for block totals denoted by $\mathbf{z}$:

$$\mathbf{X}_1'\mathbf{X}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 0 \\ 2 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 2 \\ 0 & 2 \end{pmatrix}, \quad (\mathbf{I} - \tfrac{1}{9}\mathbf{J})\mathbf{X}_1'\mathbf{X}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ -1 & 1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The resulting reduced normal equation, $\mathbf{X}_2'\mathbf{X}_1(\mathbf{I} - \frac{1}{9}\mathbf{J})\mathbf{X}_1'\mathbf{X}_2\hat{\boldsymbol{\tau}} = \mathbf{X}_2'\mathbf{X}_1(\mathbf{I} - \frac{1}{9}\mathbf{J})\mathbf{z}$, are:

$$\begin{aligned} 6\hat{\tau}_1 - 6\hat{\tau}_2 &= \textstyle\sum_{i=4}^{6} y_{i.} - \sum_{i=7}^{9} y_{i.} \\ -6\hat{\tau}_1 + 6\hat{\tau}_2 &= -\textstyle\sum_{i=4}^{6} y_{i.} + \sum_{i=7}^{9} y_{i.} \end{aligned}.$$

Either, when simplified, leads to $\widehat{\tau_1 - \tau_2} = \frac{1}{6}(\sum_{i=4}^{6} y_{i.} - \sum_{i=7}^{9} y_{i.})$.
Residual degrees of freedom are 8(corrected total) $-$ 1(treatments) $= 7$.