# HW2 Solution Notes

2. Using the effects model:

$$\mathbf{C} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{C}'\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} N & n_1 & n_2 & n_3 & n_4 & n_5 \\ n_1 & n_1 & 0 & 0 & 0 & 0 \\ n_2 & 0 & n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & n_3 & 0 & 0 \\ n_4 & 0 & 0 & 0 & n_4 & 0 \\ n_5 & 0 & 0 & 0 & 0 & n_5 \end{pmatrix}.$$

One generalized inverse of $\mathbf{X}'\mathbf{X}$ is $\mathrm{diag}(0, n_1^{-1}, n_2^{-1}, n_3^{-1}, n_4^{-1}, n_5^{-1})$. Because all contrasts are estimable, $\mathrm{trace}\mathbf{C}'\mathbf{C}(\mathbf{X}'\mathbf{X})^-$ for *any* generalized inverse is $\frac{1}{n_1} + \frac{2}{n_2} + \frac{2}{n_3} + \frac{2}{n_4} + \frac{1}{n_5}$. The augmented objective function is $h = \frac{1}{n_1} + \frac{2}{n_2} + \frac{2}{n_3} + \frac{2}{n_4} + \frac{1}{n_5} + L(\sum_i n_i - 50)$. $\frac{\partial}{\partial n_1} h = \frac{-1}{n_1^2} + L$, setting this to zero $\rightarrow n_1 = \sqrt{L^{-1}}$, and the same result follows for $n_5$. $\frac{\partial}{\partial n_2} h = \frac{-2}{n_2^2} + L$, setting this to zero $\rightarrow n_2 = \sqrt{2L^{-1}}$, and the same result follows for $n_3$ and $n_4$. So $n_1 = n_5 = \frac{1}{1+\sqrt{2}+\sqrt{2}+\sqrt{2}+1} \times 50 = 8.01$, and $n_2 = n_3 = n_4 = \frac{\sqrt{2}}{1+\sqrt{2}+\sqrt{2}+\sqrt{2}+1} \times 50 = 11.33$. Round to, say, $\mathbf{n} = (8, 11, 11, 12, 8)$.

3.

(a) $\bar{\mu} = 11.4$, $Q = 10(10-11.4)^2 + 10(11-11.4)^2 + 3 \times 10(12-11.4)^2 = 32$. Critical value $F_{.95}(4, 45) = 2.578739$. Power $= 1 - \mathrm{Prob}\{Z < 2.578739\}$ where $Z \sim F'(4, 45, \frac{32}{4})$, $= 0.5540$.

(b) $\bar{\mu} = 11.46$, $Q = 8(10-11.46)^2 + 11(11-11.46)^2 + \ldots = 28.42$. Critical value $F_{.95}(4, 45) = 2.578739$. Power $= 1 - \mathrm{Prob}\{Z < 2.578739\}$ where $Z \sim F'(4, 45, \frac{28.42}{4})$, $= 0.4989$.

(c) $Q$ is a variance-like formula for a "distribution" on the values 10, 8, 8, 8, and 8 (or, just 10 and 8). Hence, it is maximized for any design that assigns half the units to treatment 1:

$$n_1 = 25, \; n_2 + n_3 + n_4 + n_5 = 25.$$

The *power* would be maximized for this value of $Q$ and only one of $n_2$, $n_3$, $n_4$, or $n_5 = 25$ (and the others zero) because this maximizes the denominator degrees of freedom for the test. (But of course, this would not ordinarily be done in practice because it removes three of the five treatments of interest from the experiment!)

8.

(a) A one-way ANOVA of the data yields $MSE = 7.4114$, $F = 5.9112$, with $\mathrm{Prob}(F > 5.9112) = 0.0229$.

(b) Let $N = 3n$, where $n$ is the number of units assigned to each treatment. Then the solution to the problem is the smallest integer value of $n$ such that $2t_{0.975}(3n-3) 2.7224\sqrt{\frac{2}{n}} < 5$; the solution is $n = 10$, so $N = 30$.