LATIN SQUARE DESIGNS (LSD)

- Like CBD, but arranged with 2 KINDS of blocks, crossed with each other
- Example:
  - machined parts are made t different ways (treatments)
  - b_1 "machinists" may also have an effect (type 1 blocks)
  - b_2 "lathes" may also have an effect (type 2 blocks)
- Might consider an all-possible-combinations experiment, but
  - $N = tb_1b_2$, (as CBD w/ b_1b_2 blks) too large in many cases
  - each machinist would need to make tb_2 parts ...
  - tb_1 parts would need to be made on each lathe ...
  - note, even THEN there would be no true replication

- A smaller alternative:

machinist

lathe	1	2	3	...	b_1
1	*	*	*	...	*
2	*	*	*	...	*
3	*	*	*	...	*
...	...	...	...	...	...
b_2	*	*	*	...	*

- Each "*" stands for one treatment (rather than all)
- Unlike CBD, no groups of units are assumed to be "as much alike as possible" ... possible systematic differences exist between every pair, even before treatments are applied.

- Would result in $N = b_1 b_2$ rather than $N = b_1 b_2 t$
- Requires a rule to assign a treatment to each "*"
- The general form of this is called a "row-column design" ... we will consider only a special case ...
- Assign units to treatments so that the design is:
  - a CBD with lathes as blocks if machinists are ignored
  - a CBD with machinists as blocks if lathes are ignored
- Implications:
  - $b_1 = b_2 = t$
  - each treatment is used once in each row and once in each column

EXAMPLE

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

- This is a "standard form" Latin Square (ordered symbols in first row and column), one of 4 unique standard form Latin Squares of order 4
- Restricted randomization of units to treatments:
  - randomly select a unique Latin Square
  - randomly permute the rows
  - randomly permute the columns

- As with CBD's, block-treatment structure does not allow estimation of interactions, leading to an assumption of additivity:
- $y_{i,j,k} \quad = \quad \alpha + \beta_i + \gamma_j + \tau_k + \epsilon_{i,j,k}$
- $i = 1...t, \quad j = 1...t, \quad k = 1...t$
- All possible pair (i,j), (i,k), (j,k) appears once, and for each, only one value of the 3rd index appears
- Ordering observations so that those in the same physical row are taken together, model matrix for α, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$:

$$\mathbf{X}_1 = \left(\begin{array}{c|ccccc|c} \mathbf{1} & \mathbf{1} & & ... & & \mathbf{I} \\ \mathbf{1} & & \mathbf{1} & ... & & \mathbf{I} \\ ... & ... & ... & ... & ... & ... \\ \mathbf{1} & & & ... & \mathbf{1} & \mathbf{I} \end{array} \right)$$

- i.e. $\mathbf{X}_1$ is the same as the ENTIRE $\mathbf{X}$ for a CBD with $b = t$
- Again, the first column can be ignored

- Model matrix for $\boldsymbol{\tau}$:

$$\mathbf{X}_2 = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \ldots \\ \mathbf{P}_t \end{pmatrix}$$

- Each $\mathbf{P}_i$ is a *permutation matrix*:
  - i.e., a single 1 in each row and each column, all other entries 0
  - $\sum \mathbf{P}_i = \mathbf{J}_{t \times t}$
  - "Standard Form" if:

$$\mathbf{P}_1 = \mathbf{I},\ \{\mathbf{P}_2\}_{1,2} = 1,\ \{\mathbf{P}_3\}_{1,3} = 1,\ \ldots\ ,\ \{\mathbf{P}_t\}_{1,t} = 1$$

Reduced Normal Equations for $\boldsymbol{\tau}$:

$$\mathbf{X}_2'(\mathbf{I} - \mathbf{H}_1)\mathbf{X}_2\hat{\boldsymbol{\tau}} = \mathbf{X}_2'(\mathbf{I} - \mathbf{H}_1)\mathbf{y}$$

- For simplicity, consider $\mathbf{X}_1$ model without α

- As before, the key is the matrix product:

$\mathbf{H}_1$	$\mathbf{X}_2$
$\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-$	$\mathbf{X}_1'\mathbf{X}_2$
(g-inv now needed)	(counts trt's in each row or col)
	("incidence matrix")
	$\mathbf{J}_{2t\times t}$
	$\mathbf{X}_1' \frac{1}{t}\mathbf{J}_{t^2\times t}$
	(leading matrix is $2t$ by t^2, each row with t ones)
$\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-\mathbf{X}_1'$	$\frac{1}{t}\mathbf{J}$
(projects into $\mathbf{X}_1$-space)	(already in $\mathbf{X}_1$-space)

- So yet again, $\mathbf{X}_2'\mathbf{H}_1 = \frac{1}{t}\mathbf{J}$, and $\hat{\boldsymbol{\tau}}$ is as with CRD or CBD with same treatment group sizes

- Solutions:
  - $\bar{\hat{\tau}} = 0 \quad \rightarrow \quad \hat{\tau}_k = \bar{y}_{.k} - \bar{y}_{..}$
  - $\hat{\tau}_t = 0 \quad \rightarrow \quad \hat{\tau}_k = \bar{y}_{.k} - \bar{y}_{.t}$
  - ...
- Estimable functions are $\sum c_k \tau_k$ such that $\sum c_k = 0$
  - $Var[\sum \widehat{c_k \tau_k}] = \sigma^2 \sum c_k^2 \frac{1}{t}$
  - $\sigma_{LSD}/\sigma_{CRD} \; < \; t(1 - \frac{\alpha}{2}, N - t)/t(1 - \frac{\alpha}{2}, N - 3t + 2)$?
- Power of F-test for $\tau_1 = \tau_2 = ... = \tau_t$
  - Q as with CRD and CBD
  - $F'(t-1, N-3t+2, Q/\sigma^2_{LSD}) \; : \; F'(t-1, N-t, Q/\sigma^2_{CRD})$

A NOTE ABOUT INTERACTIONS

- Think of a Latin Square dataset as a two-way table, corresponding to the two kinds of blocks:

Two-Way		LSD	
source	df	source	df
rows	$t-1$	rows	$t-1$
cols	$t-1$	cols	$t-1$
row-col interaction	$(t-1)^2$	trts	$t-1$
or resid		*and* resid	$(t-1)(t-2)$

- Treatments

2	1	4	3
4	3	2	1
1	4	3	2
3	2	1	4

- R-by-C patterns belonging to LSD treatments, and LSD residuals:

8	8	5	5
5	5	8	8
8	5	5	8
5	8	8	5

5	8	8	5
5	8	8	5
8	5	5	8
8	5	5	8

REPLICATED LATIN SQUARES

- Strength of LSD that it can accommodate 2 simultaneous kinds of block structure in a small study
- Weakness is that the design is sometimes *too* small

	d.f. for SSE(Hyp$_A$)
t	$t^2 - 3t + 2$
3	2
4	6
5	12
...	...

- One solution is to use multiple (replicated) Latin Squares
- For example, think of machined parts problem, blocks representing "machinist" and "lathe" ... Ref: Montgomery, "Design and Analysis of Exp's" Wiley

- Perform Latin Square experiment r times, using t machinists crossed with t lathes each time, to test t methods
- y_{mijk} = measurement:
  - from rep $m = 1...r$,
  - from machinist $i = 1...t$,
  - from lathe $j = 1...t$, and
  - using method $k = 1...t$, (treatments of interest):
    * but only one value of k per (m, i, j) combination
    * restricted randomized assignment of k to (i, j) should ordinarily be carried out independently in each rep

Source	*df*	*SS*
Replicate Squares	$r-1$	$\sum t^2(\bar{y}_{m\ldots} - \bar{y}_{\ldots.})^2$
Machinists	–	–
Lathes	–	–
Treatments	$t-1$	$\sum rt(\bar{y}_{\ldots k} - \bar{y}_{\ldots.})^2$
Error	(subtraction)	(subtraction)
Corr Tot	rt^2-1	$\sum(y_{mijk} - \bar{y}_{\ldots})^2$

Version 1: Same Machinists and Lathes in each Rep

- $y_{mijk} = \alpha + \rho_m + \beta_i + \gamma_j + \tau_k + \epsilon_{mijk}$

  (putting treatment terms and subscripts last)

- Machinists and Lathes are the same physical entities within each rep

Source	*df*	*SS*
Machinists	$t-1$	$\sum rt(\bar{y}_{.i..} - \bar{y}_{....})^2$
Lathes	$t-1$	$\sum rt(\bar{y}_{..j.} - \bar{y}_{....})^2$

Version 2: Different Machinists but Same Lathes in each Rep

- $y_{m,i(m),j,k} = \alpha + \rho_m + \beta_{i(m)} + \gamma_j + \tau_k + \epsilon_{m,i(m),j,k}$
- Machinists are physically different entities within each rep (and so "nested")

Source	*df*	*SS*
Machinists	$r(t-1)$	$\sum t(\bar{y}_{mi..} - \bar{y}_{m...})^2$
Lathes	$t-1$	$\sum rt(\bar{y}_{..j.} - \bar{y}_{....})^2$

Version 3: Different Machinists and Lathes in each Rep

- $y_{m,i(m),j(m),k} = \alpha + \rho_m + \beta_{i(m)} + \gamma_{j(m)} + \tau_k + \epsilon_{m,i(m),j(m),k}$
- Machinists and lathes are physically different entities within each rep (and so "nested")

Source	*df*	*SS*
Machinists	$r(t-1)$	$\sum t(\bar{y}_{mi..} - \bar{y}_{m...})^2$
Lathes	$r(t-1)$	$\sum t(\bar{y}_{m.j.} - \bar{y}_{m...})^2$

As with unreplicated LSDs, versions 1-3 are also "Condition E" equivalent to a CRD of the same size.

GRAECO-LATIN SQUARE DESIGNS

- Now, 3 KINDS of blocks, crossed with each other
- Example:
  - machined parts are made t different ways (treatments)
  - t "machinists" may also have an effect (type 1 blocks)
  - t "lathes" may also have an effect (type 2 blocks)
  - t "batches of stock" may also have an effect (type 3 blocks)
- Construct two *orthogonal Latin Squares*, different squares of order t, using "Latin" letters in one and "Greek" letters in the other, such that when the two squares are superimposed, each pair of letters appears together exactly once.

- Example:

Aα	Bβ	Cγ	Dδ
Bγ	Aδ	Dα	Cβ
Cδ	Dγ	Aβ	Bα
Dβ	Cα	Bδ	Aγ

- Use rows, columns and Greek letters for blocking;
  Latin letters for treatments

- $y_{ijmk} \;=\; \alpha + \beta_i + \gamma_j + \delta_m + \tau_k + \epsilon_{ijmk}$
- (or equivalently, drop the α)

$$\mathbf{X}_1 = \left(\begin{array}{cccc|c|c} \mathbf{1} & & \dots & & \mathbf{I} & \mathbf{P}_1 \\ & \mathbf{1} & \dots & & \mathbf{I} & \mathbf{P}_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \mathbf{1} & \mathbf{I} & \mathbf{P}_t \end{array}\right) \quad \mathbf{X}_2 = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \dots \\ \mathbf{R}_t \end{pmatrix}$$

- Each $\mathbf{P}$ and $\mathbf{R}$ is a permutation matrix:
  - $\sum \mathbf{P}_i = \mathbf{J}_{t\times t}$
  - $\sum \mathbf{R}_i = \mathbf{J}_{t\times t}$
  - $\sum \mathbf{P}_i'\mathbf{R}_i = \mathbf{J}_{t\times t}$

- Once again!

$$\mathbf{X}_1'\mathbf{X}_2 = \mathbf{J}_{3t\times t}$$

- So ...

$$\begin{aligned}
\mathbf{H}_1\mathbf{X}_2 \quad &= [\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-][\mathbf{J}_{3t\times t}] \\
&= [\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^-][\mathbf{X}_1'(\tfrac{1}{t}\mathbf{J}_{N\times t})] \\
&= [\mathbf{H}_1][\tfrac{1}{t}\mathbf{J}_{N\times t}] \\
&= \tfrac{1}{t}\mathbf{J}_{N\times t}
\end{aligned}$$

- So, reduced normal equations for estimation of $\boldsymbol{\tau}$, $Q(\ \boldsymbol{\tau})$, and $Var[\widehat{\mathbf{c}'\boldsymbol{\tau}}]$ are as with CRD's (with t treatments and all sample sizes equal to t), CBD's (with t treatments and t blocks), and LSD's (with t treatments, rows, and columns).

- Note that the same patterns exist here as with LSD's for replication. Each of the "rows", "columns", or "Greek letters" can represent:
  - the SAME physical entities in each rep ("crossed", $t-1$ df)
  - DIFFERENT physical entities ("nested", $r(t-1)$ df)

GLSDs, replicated or not, are also "Condition E" equivalent to a CRD of the same size.