1. Consider a small blocked design with 6 experimental units arranged in 2 blocks of 3 units each, to study 2 experimental treatments. In each block, two units are to be assigned to treatment 1 and one to treatment 2.

(a) Given two physical blocks of 3 units each, how many actual assignments of units to treatments are possible for this design? (In application, each of these possible arrangements should receive equal probability in the randomization process.)

3 assignments of a unit to trt 2 in block 1
× 3 " " " " " " " " " 2

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(b) Using the parameterization we've developed in class, compute $\mathbf{H}_1$ and $\mathbf{H}_1\mathbf{X}_2$.

$X_1 = \begin{pmatrix} 1 & \\ 1 & \\ 1 & \\ & 1 \\ & 1 \\ & 1 \end{pmatrix}$, $(X_1'X_1)^{-1} = \frac{1}{3}I_2$, $H_1 = \begin{pmatrix} \frac{1}{3}J_{3\times3} & \\ & \frac{1}{3}J_{3\times3} \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 & \\ 1 & \\ & 1 \\ 1 & \\ 1 & \\ & 1 \end{pmatrix}$, $H_1X_2 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{pmatrix}$

(c) Using your answer to the previous part of this question, show why this design is or is not "Condition E - equivalent" to a CRD in which $n_1 = 4$ and $n_2 = 2$. X_2 is the same

For CRD, $H_1 = \frac{1}{6}J_{6\times6}$, $H_1X_2 = \frac{1}{6}\begin{pmatrix} 4 & 2 \\ 4 & 2 \\ 4 & 2 \\ 4 & 2 \\ 4 & 2 \\ 4 & 2 \end{pmatrix}$ = same, so yes

(d) Compute the quadratic form Q (i.e. the numerator of the noncentrality parameter) for this design; your answer should be entirely numeric except for τ_1 and τ_2, and reduced to relatively simple terms.

Because (c) is yes,

$Q = 4(\tau_1 - \bar{\tau}_w)^2 + 2(\tau_2 - \bar{\tau}_w)^2$ where $\bar{\tau}_w = \frac{2\tau_1 + \tau_2}{3}$

$= \frac{4}{3}(\tau_1 - \tau_2)^2$

2. Consider a small blocked design with 6 experimental units arranged in 2 blocks of 3 units each, to study 2 experimental treatments. In one of the blocks, two units are to be assigned to treatment 1 and one to treatment 2; in the other block, two units are to be assigned to treatment 2 and one to treatment 1.

(a) Given two physical blocks of 3 units each, how many actual assignments of units to treatments are possible for this design? (In application, each of these possible arrangements should receive equal probability in the randomization process.)

3 assignments of a unit to trt 2 in "one block"
× 3 " " " " " " 1 in "other block"
× 2 assignments of block 1/2 to "one/other"

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(b) Using the parameterization we've developed in class, compute $\mathbf{H}_1$ and $\mathbf{H}_1\mathbf{X}_2$.

$$H_1 = \begin{pmatrix} \frac{1}{3}J_{3\times3} & \\ & \frac{1}{3}J_{3\times3} \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & \\ 1 & \\ & 1 \\ 1 & \\ & 1 \\ & 1 \end{pmatrix}, \quad H_1X_2 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$$

(as in prob #1)

(c) Using your answer to the previous part of this question, show why this design is or is not "Condition E - equivalent" to a CRD in which $n_1 = n_2 = 3$. ← X_2 is the same

For CRD, $H_1 = \frac{1}{6}J_{6\times6}$, $H_1X_2 = \frac{1}{2}J_{6\times2}$ = different, so no

(d) Compute the quadratic form Q (i.e. the numerator of the noncentrality parameter) for this design; your answer should be entirely numeric except for τ_1 and τ_2, and reduced to relatively simple terms.

Because (c) is "no", can't use "$\sum n(\tau - \bar{\tau})^2$"

$$X_{2|1} = X_2 - H_1X_2 = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ -2 & 2 \\ 2 & -2 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad X_{2|1}'X_{2|1} = \frac{4}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1 \ -1)$$

$$Q = \underline{\tau}' \Downarrow \underline{\tau} = \frac{4}{3}(\tau_1 - \tau_2)^2$$

3. Montgomery (in the experimental design book mentioned in class) describes a particular experiment as follows: *"An experimenter is studying the effects of five different formulations of a rocket propellant used in aircrew escape systems on the observed burning rate. Each formulation is mixed from a batch of raw material that is only large enough for five formulations to be tested. Furthermore, the formulations are prepared by several operators, and there may be substantial differences in the skills and experience of the operators."*

   (a) Compute degrees of freedom for the residual mean square under each of the following designs:

      i. The experiment is carried out as an unrepliacted Latin square, with "batches" and "operators" used as the two blocking systems.

$$(25-1) - \underbrace{(5-1)}_{\text{batch}} - \underbrace{(5-1)}_{\text{operator}} - \underbrace{(5-1)}_{\text{trt}} = 12$$

      ii. The experiment is carried out as a replicated Latin square with 3 replicates, in which the same operators apply treatments to units in each replicate, but 5 different batches of material are used in each replicate.

$$(75-1) - \underbrace{(3-1)}_{\text{rep}} - \underbrace{3(5-1)}_{\text{batch}} - \underbrace{(5-1)}_{\text{operator}} - \underbrace{(5-1)}_{\text{trt}} = 52$$

      iii. The experiment is carried out as a replicated Latin square with 4 replicates, in which different operators and batches are used in each replicate.

$$(100-1) - \underbrace{(4-1)}_{\text{rep}} - \underbrace{4(5-1)}_{\text{batch}} - \underbrace{4(5-1)}_{\text{operator}} - \underbrace{(5-1)}_{\text{trt}} = 60$$

   (b) Complete the following table with numbers indicating treatments, so that the completed table is an unreplicated Latin square of order 5.

e.g.

5	1	2	3	4
3	4	5	1	2
1	2	3	4	5
4	5	1	2	3
2	3	4	5	1

2

4. A completely randomized design will be executed to compare treatments 1, 2, and 3. The two quantities of greatest interest to the experimenter are:

$$\underline{\tau_1 - \tau_2} \quad \text{and} \quad \underline{\tau_2 - \tau_3}$$

and she would like to design the experiment so that the average of the variances of her two estimates is as small as possible. She has acquired all the experimental units she will need (in fact, more than she can use), but faces budget restrictions as follows:

$$\left.\begin{array}{l}\text{The expense of applying treatment 1 to any unit is \$1.}\\ \text{The expense of applying treatment 2 to any unit is \$2.}\\ \text{The expense of applying treatment 3 to any unit is \$4.}\end{array}\right\}$$

Finally, there is a budget of \$100 for the total cost of applying treatments to units in this experiment. Use the Method of Lagrangian Multipliers to determine an optimal allocation (values of n_1, n_2, and n_3) for this study, that meets the required constraint.

$$C = \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \end{pmatrix}, \quad C'C = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad (X'X)^{-1} = \begin{pmatrix} \frac{1}{n_1} & & \\ & \frac{1}{n_2} & \\ & & \frac{1}{n_3} \end{pmatrix}$$

$$\operatorname{tr}(C'C)(X'X)^{-1} \qquad \text{L.M.}$$

$$\left(\frac{1}{n_1} + \frac{2}{n_2} + \frac{1}{n_3}\right) + \lambda(n_1 + 2n_2 + 4n_3 - 100)$$

$$\frac{\partial}{\partial n_1}: \quad \frac{-1}{n_1^2} + \lambda \overset{\text{set}}{=} 0 \Rightarrow 1 = \lambda n_1^2 \Rightarrow n_1 \propto 1$$

$$\frac{\partial}{\partial n_2}: \quad \frac{-2}{n_2^2} + 2\lambda \overset{\text{set}}{=} 0 \Rightarrow 1 = \lambda n_2^2 \Rightarrow n_2 \propto 1$$

$$\frac{\partial}{\partial n_3}: \quad \frac{-1}{n_3^2} + 4\lambda \overset{\text{set}}{=} 0 \Rightarrow 1 = 4\lambda n_3^2 \Rightarrow n_3 \propto \frac{1}{2}$$

$$\begin{pmatrix} 2 \times \$1 \\ 2 \times \$2 \\ 1 \times \$4 \end{pmatrix} \quad \overline{\$10}$$

$$\Rightarrow n_1 = 20, \quad n_2 = 20, \quad n_3 = 10$$

(check: \$20 + \$40 + \$40 = \$100)