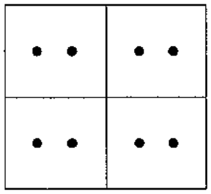
1. Consider a blocked design with structure similar to that of an order-2 Latin Square (i.e. for 2 treatments), except that there are 2 experimental units (rather than 1) in the intersection of any "row block" and "column block". A diagram of the design is given below, where rows and columns denote the two kinds of blocks, respectively, and each • denotes an experimental unit. Ordering these experimental units left-to-right, and then top-to-bottom, and assuming no row-block-by-column-block interaction, the $\mathbf{H}_1$ matrix for this design is also shown:



$$\mathbf{H}_1 = \frac{1}{8}\begin{pmatrix} 3 & 3 & 1 & 1 & 1 & 1 & -1 & -1 \\ 3 & 3 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 3 & 3 & -1 & -1 & 1 & 1 \\ 1 & 1 & 3 & 3 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 3 & 3 & 1 & 1 \\ 1 & 1 & -1 & -1 & 3 & 3 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 3 & 3 \\ -1 & -1 & 1 & 1 & 1 & 1 & 3 & 3 \end{pmatrix}$$

(a) Suppose two treatments are to be compared in this experiment and are assigned to units as:

1 2	1 2
1 2	1 2

Compute $\mathbf{X}_2$ (remembering that the order of rows in model matrices has now been established) and $\mathbf{H}_1\mathbf{X}_2$ for this design. What does this tell you about the form of $\widehat{\tau_2 - \tau_1}$?

$$X_2 = \begin{pmatrix} 1 & \\ & 1 \\ 1 & \\ & 1 \\ 1 & \\ & 1 \\ 1 & \\ & 1 \end{pmatrix} \qquad H_1X_2 = \frac{1}{8}\begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{pmatrix} = \frac{1}{2}J_{8\times 2}$$

"Cond E" is satisfied for this dsn & a CRD w/ 4 units assigned to each of 2 treatments

$\Rightarrow \widehat{\tau_2 - \tau_1} = \bar{y}_{\cdot 2} - \bar{y}_{\cdot 1}$

(b) Suppose two treatments are to be compared in this experiment and are assigned to units as:

1 1	2 2
2 2	1 1

Compute $\mathbf{X}_2$ and $\mathbf{H}_1\mathbf{X}_2$ for this design. What does this tell you about the form of $\widehat{\tau_2 - \tau_1}$?

$$X_2 = \begin{pmatrix} 1 & \\ 1 & \\ & 1 \\ & 1 \\ & 1 \\ & 1 \\ 1 & \\ 1 & \end{pmatrix} \qquad H_1X_2 = \frac{1}{8}\begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{pmatrix}$$

"

(c) The second treatment arrangement shown (in part b) has an important statistical advantage relative to the first one (in part a). What is it?

True replication – multiple units in each block-combination assigned to the same treatment of two

2. Consider two designs:

- a CRD for 3 treatments, with $n_1 = 4$ and $n_2 = n_3 = 2$
- an augmented complete block design for 3 treatments, in which each block assigns two units to treatment 1, and one unit to each of treatments 2 and 3.

(a) For the CRD compute: $\mathbf{H}_1$, $\mathbf{X}_2$, and $\mathbf{H}_1\mathbf{X}_2$.

$$H_1 = \frac{1}{8} J_{8\times 8} \qquad X_2 = \begin{pmatrix} 1 & & \\ 1 & & \\ & 1 & \\ & & 1 \\ 1 & & \\ 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad H_1 X_2 = \frac{1}{8}\begin{pmatrix} 4 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \\ 4 & 2 & 2 \end{pmatrix}$$

(b) For the block design compute: $\mathbf{H}_1$, $\mathbf{X}_2$, and $\mathbf{H}_1\mathbf{X}_2$.

$$H_1 = \begin{pmatrix} \frac{1}{4} J_{4\times 4} & \\ & \frac{1}{4} J_{4\times 4} \end{pmatrix} \qquad X_2 = \qquad H_1 X_2 =$$

(pattern repeats for any number of blocks)

(so the designs are "Cond. E"-equivalent ... you need this in part (d))

(c) For the block design, fill in degrees of freedom below for a standard ANOVA decomposition:

blocks	1	(b-1)
treatments	2	2
residual	4	3b-2
corrected total	7	4b-1

(d) Before the experiment is performed, the investigator says he would not be surprised to find that $\tau_2 = \tau_1 + 1$, $\tau_3 = \tau_1 + 3$, and that $\sigma^2 = 2$. Assuming that these parameter values are actually correct, and using what you showed in previous parts of this problem, what noncentral F distribution would you use to compute the power of a test for equality of treatments. In particular, what are:

- the numerator degrees of freedom?

  2

- the denominator degrees of freedom?

  4 (3b-2)

- the noncentrality parameter?

  wlog, let $\tau_1 = 0 \quad \tau_2 = 1 \quad \tau_3 = 3 \Rightarrow \bar{\tau} = (2\cdot 0 + 1 + 3)/4 = 1$

  $Q = 4(0-1)^2 + 2(1-1)^2 + 2(3-1)^2 = 12$

  $Q/\sigma^2 = 12/2 = \underset{\sim}{6} \qquad \underset{\sim}{(3b)}$

3. An experimenter wants to conduct an experiment to compare three treatments, and intends to use a CRD. The operational restrictions are such that only one experimental unit can be processed at a time; the experimenter will include the order in which treatments are applied to units as part of her randomization process to avoid bias in the resulting analysis. The process of applying a treatment and evaluating the unit takes substantial time in this case. Specifically, it takes 1 day to process a unit with treatment 1, 4 days to process a unit with treatment 2, and 4 days to process a unit with treatment 3. Further, there is an overall time constraint of 50 days for the total length of the experiment.

One of the important parametric functions the experimenter wants to estimate is $\boxed{\phi = \frac{1}{2}(\tau_1 + \tau_2) - \tau_3.}$ What are the optimal values of n_1, n_2, and n_3 with respect to minimizing the variance of the least-squares estimate of ϕ, restricted to meet the overall time constraint?

$$\mathrm{Var}(\hat{\phi}) \propto \qquad\qquad \text{constraint}$$

$$\left[\frac{1}{4}\frac{1}{n_1} + \frac{1}{4}\frac{1}{n_2} + \frac{1}{n_3}\right] + \lambda\left[n_1 + 4n_2 + 4n_3 - 50\right]$$

$$\frac{\partial}{\partial n_1} = 0 \Rightarrow n_1 = \frac{1}{2\sqrt{\lambda}}$$

$$\frac{\partial}{\partial n_2} = 0 \Rightarrow n_2 = \frac{1}{4\sqrt{\lambda}} \qquad \text{so } n_1 = n_3 \ \& \ n_2 = \frac{1}{2}n_1$$

$$\frac{\partial}{\partial n_3} = 0 \Rightarrow n_3 = \frac{1}{2\sqrt{\lambda}}$$

$$\begin{array}{ccccccc} n_1 & + & 4n_2 & + & 4n_3 & = & 50 \\ \| & & & & & & \| \\ n_1 & + & 2n_2 & + & 4n_1 & = & 7n_1 \end{array}$$

$\rightarrow$

$n_1 = 7\frac{1}{7}$	7	6	
$n_2 = 7\frac{1}{7}$	7	7	...
$n_3 = 3\frac{4}{7}$	3	4	
	↓ 47 days	↘ 50 days	

4. Recall that in Homework Assignment 1, an experiment was described in which 4 sealant treatments were tested to compare their effectiveness in preventing the growth of a certain pathogen beneath the surface of concrete bricks. Suppose we "rewrite" that problem to say that while the essence of the experimental proceedure does not change (i.e. bricks are made, treated with one of the 4 sealant treatments, submerged in a solution containing the pathogen, and eventually split to evaluate the degree of pathogen growth), some of the operational details are more complicated. Specifically, suppose that:

- There is only enough concrete mix in a batch to make 4 bricks.
- A batch of pathogen solution can only be used to submerge 4 bricks; after that, it is regarded as contaminated and should be discarded.
- During the experiment, 4 laboratory technicians will do the actual testing, and the work schedule for a day's operation makes it convenient for each of them to test 4 bricks.

As a result, the investigator decides to use a Greaco Latin Square design for the experiment, regarding batches of concrete mix, batches of pathogen solution, and technicians, as the 3 overlapping block structures in defining units to compare the 4 treatments.

(a) For a single-replicate GLS as described, compute the residual degrees of freedom under the usual assumption that there are no interactions among any of the 3 kinds of blocks and treatments. (Show your work; show me how you partition the degrees of freedom so as to arrive at your answer.)

$$\underset{\text{c.t.}}{15} - \underset{\text{trt}}{3} - \underset{\text{mix batch}}{3} - \underset{\text{sol batch}}{3} - \underset{\text{tech}}{3} = \underset{\sim}{3}$$

(b) Suppose now that the experimenter decides that the experiment needs to be larger than this, and so decides to run 3 separate replicates of the basic GLS design on 3 separate days. The same 4 technicians will participate in each replicate, but the 4 batches of concrete mix and 4 batches of pathogen solution will need to be different on each day. For this larger version of the experiment, compute the residual degrees of freedom under the usual assumptions that there are no interactions as described above. (Again, show your work ...)

$$\underset{\text{c.t.}}{47} - \underset{\text{rep}}{2} - \underset{\text{trt}}{3} - \underset{\text{mix batch}}{9} - \underset{\text{sol batch}}{9} - \underset{\text{tech}}{3} = \underset{\sim}{21}$$

5. In the following table, let rows represent one blocking system and columns represent another, and write a number (for treatment) and letter (for 3rd blocking system) in each square to form an order-3 GLS:

1 A	2 C	3 B
2 B	3 A	1 C
3 C	1 B	2 A

Why is an unreplicated GLS of order 3 (like you just constructed) an especially bad design choice if the experimental objectives include formal statistical inference to compare the 3 treatments?

$$\underset{\text{c.t.}}{8} - \underset{\text{blk 1}}{2} - \underset{\text{blk 2}}{2} - \underset{\text{blk 3}}{2} - \underset{\text{trt}}{2} = \underset{\sim}{0} \text{ resid DF.}$$