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Plenary

All the proofs and results I don't want to write twice.

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Definition 1.1.1 (Supremum/Infimum) Let $X \subset \mathbb{R}$ be a non-empty. The supremum of X is a real number $M =: \sup X$ that satisfies

(i) M is an upper bound of X, and

(ii) if M' is an upper bound of X, then $M' \geq M$

that is, M is the least upper bound of X. The infimum of X is the greatest lower bound of X.

Definition 1.1.2 (Subsequential limit) Let $\{x_n \in \mathbb{R}\}$ sequence. $\bar{x} \in \mathbb{R}$ is called a **subsequential limit** of $\{x_n\}$ if $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to $\bar{x}$.

Definition 1.1.3 (Limit superior/limit inferior) Let $\{x_n \in \mathbb{R}\}$ sequence, and let $S(x_n)$ be the set of all subsequential limits of $\{x_n\}$.

Then we define the **limit superior** of $\{x_n\}$ to be

$$\limsup x_n := \sup S(x_n)$$

and the **limit inferior** of $\{x_n\}$ to be

$$\liminf x_n := \inf S(x_n)$$

Alternatively, we can also define them by

$$\limsup x_n := \lim_{n\to\infty} \sup \{x_k \mid k \geq n\}$$
$$\liminf x_n := \lim_{n\to\infty} \inf \{x_k \mid k \geq n\}$$

Definition 1.1.4 (Cluster point) Let S be a subset of a topological space X. A point x in X is a cluster point of the set S if every neighborhood of x contains at least one point of S different from x itself.

A cluster point is also called a limit point or accumulation point.

In real analysis, $c \in \mathbb{R}$ is a cluster point of a non-empty set $A \subseteq \mathbb{R}$ if for every $\varepsilon > 0$ there exists a point $x \in A \setminus \{c\}$ such that $x \in (c - \varepsilon, c + \varepsilon)$.

In complex analysis, $c \in \mathbb{C}$ is a cluster point of a non-empty set $A \subseteq \mathbb{C}$ if for every $\varepsilon > 0$ there exists a point $z \in A \setminus \{c\}$ such that $z \in B_\varepsilon(c)$.

Definition 1.1.5 (Dense) Informally, a subset A of a topological space X is said to be **dense** in X if every point of X either belongs to A or else is arbitrarily "close" to a member of A.

A subset A if a topological space X is said to be a dense subset of X if any of the following equivalent conditions are satisfied:

(i) The smallest closed subset of X containing A is X itself.

(ii) The closure of A in X is equal to X. ($\mathrm{cl}_X A = X$).

(iii) Every point in X either belongs to A or is a cluster point of A.

Definition 1.1.6 (Point of closure) For S as a subset of a Euclidean space, x is a point of closure of S if every open ball centered at x contains a point of S (this point can be x itself).

Definition 1.1.7 (Closure) The closure of a subset S of points in a topological space can be defined using any of the following equivalent definitions:

(i) $\mathrm{cl}\, S$ is the set of all points of closure of S.

(ii) $\mathrm{cl}\, S$ is the set S together with all of its limit points.

(iii) $\mathrm{cl}\, S$ is the intersection of all closed sets containing S.

(iv) $\mathrm{cl}\, S$ is the smallest closed set containing S.

(v) $\mathrm{cl}\, S$ is the union of S and its boundary ∂S

Definition 1.1.8 (Open sets) A subset U of a metric space (M, d) is called open if for any point x in U, there exists a real number $\varepsilon > 0$ such that any point $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to U.

Equivalently, U is open if every point U has a neighborhood contained in U.

An example of a metric space is $(\mathbb{R}^2, \|\cdot\|)$.

Definition 1.1.9 (Closed sets) A subset A of a topological space (X, τ) is closed if its complement $X \setminus A$ is an open subset of (X, τ)

A set A is closed in X if and only if it is equal to its closure $\mathrm{cl}\, A$ in X.

Yet another equivalent definition is that a set is closed if and only if it contains all of its boundary points.

Definition 1.2.1 (Monotone sequences) A sequence $\{x_n\}$ is said to be **increasing** if $x_0 \leq x_1 \leq x_2 \leq \ldots$ and **decreasing** if $x_0 \geq x_1 \geq x_2 \geq \ldots$ and **monotone** if it is either increasing or decreasing.

Theorem 1.2.2 (Monotone convergence theorem) If $\{x_n\}$ is monotone and bounded, then $\{x_n\}$ converges.

$$\lim_{n\to\infty} = \begin{cases} \sup\{x_n : n \in \mathbb{N}\} & \text{if } \{x_n\} \text{ is increasing} \\ \inf\{x_n : n \in \mathbb{N}\} & \text{if } \{x_n\} \text{ is decreasing} \end{cases}$$

Theorem 1.2.3 (Monotone subsequence theorem) Every sequence has a monotone subsequence.

Proof. Let $\{x_n\}$ be a sequence. We call a term x_p a **peak term** of $\{x_n\}$ if

$$x_p \geq x_n \quad (\forall n \geq p)$$

That is, all terms after x_p never go above x_p again. Then there are only two cases:

Case 1: $\{x_n\}$ has infinitely many peak terms.

Then the subsequence formed by all the peak terms form a decreasing subsequence of $\{x_n\}$.

Case 2: $\{x_n\}$ has finitely many peak terms.

Let $x_{p_1}, x_{p_2}, \ldots, x_{p_j}$ be **all** the peak terms.

Let $n_1 = p_j + 1$ be the first term after the last peak term.

Since x_{n_1} is not a peak term. $\implies \exists\, n_2 > n_1$ such that $x_{n_1} < x_{n_2}$.

Since x_{n_2} is not a peak term, $\implies \exists\, n_3 > n_2$ such that $x_{n_2} < x_{n_3}$.

Continuing indefinitely, we can form an increasing subsequence $\{x_{n_k}\}$. $\square$

Theorem 1.2.4 (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.

Proof. Let $\{x_n\}$ be a bounded sequence. By the monotone subsequence theorem, $\{x_n\}$ has a monotone subsequence $\{x_{n_k}\}$.

Since $\{x_n\}$ is bounded, so is $\{x_{n_k}\}$.

Since $\{x_{n_k}\}$ is both monotone and bounded, it follows from the monotone convergence theorem that $\{x_{n_k}\}$ converges. $\square$

Theorem 1.2.5 (Monotone seq. with a convergent subseq. is convergent) Let $\{x_n\}$ be a monotone sequence with a subsequence $\{x_{n_k}\}$ that converges to L. Then $\{x_n\}$ converges to L.

Proof. WLOG, assume that $\{x_n\}$ is decreasing. Given any $\varepsilon > 0$, we want to find a $N_\varepsilon \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \quad \forall (n \geq N_\varepsilon)$$

Since $\{x_{n_k}\}$ is decreasing and converges to L, we can find (and fix) a k_ε such that

$$0 < x_{n_k} - L < \varepsilon \quad \forall (k \geq k_\varepsilon) \tag{$*$}$$

So we take $N_\varepsilon = n_{k_\varepsilon}$. Then since $\{x_n\}$ is decreasing,

$$x_n \leq x_{N_\varepsilon} = x_{n_{k_\varepsilon}} \quad \forall (n \geq N_\varepsilon)$$

Moreover, $L \leq x_n \leq x_{n_{k_\varepsilon}}$, and hence

$$0 \leq x_n - L \leq x_{n_{k_\varepsilon}} - L$$

and from $(*)$, we have that this entire inequality $< \varepsilon$, and hence

$$0 \leq x_n - L < \varepsilon$$

and finally

$$|x_n - L| < \varepsilon$$

$\square$

Theorem 1.2.6 (Mean value theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous on the $[a, b]$, and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Generalized to multiple variables, the mean value theorem can be written as:

Let $f : [a, b] \to \mathbb{R}$, where $a, b \in \mathbb{R}^n$, and $[a, b]$ refers to the line segment connecting a and b, namely

$$[a, b] := \{\lambda a + (1 - \lambda)b \mid \lambda \in [0, 1]\}$$

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in [a, b]$ such that

$$\nabla f(c)^T (b - a) = f(b) - f(a)$$

In some arguments, we use $f : [x, x + td] \to \mathbb{R}$ and write that there exists $\eta \in [x, x + td]$ such that

$$\nabla f(\eta)^T d = \frac{f(x + td) - f(x)}{t}$$

Result 1.2.7 (Preprocessed limits) Let $k, \ell \in \mathbb{N}$ and $a, b, c \in \mathbb{R}$ be fixed.

(a) $\lim_{n\to\infty} \frac{1}{n^k} = 0$

(b) $\lim_{n\to\infty} b^n = 0 \quad \text{if} \quad |b| < 1$

(c) $\lim_{n\to\infty} c^{\frac{1}{n}} = 1 \quad \text{if} \quad c > 0$

(d) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

(e) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

(f) $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$

(g) $\lim_{n\to\infty} \frac{n^k}{c^n} = 0 \quad \text{if} \quad c > 0$

if $k < \ell$ and $1 < a < b$, we have

$$n^k << n^\ell << a^n << b^n << n!$$

Theorem 1.2.8 (Bernoulli's inequality)

$$(1 + x)^r \geq 1 + rx$$

This holds under any of the following conditions:

- $r \in \mathbb{Z}, r \geq 1$ and $x \in \mathbb{R}, x \geq -1$ (inequality is strict if $x \neq 0$ and $r \geq 2$)
- $r \in \mathbb{Z}, r \geq 0$ and $x \in \mathbb{R}, x \geq -2$
- $r \in \mathbb{Z}$, r is even and $x \in \mathbb{R}$
- $r \in \mathbb{R}, r \geq 1$ and $x \in \mathbb{R}, x \geq -1$ (inequality is strict if $x \neq 0$ and $r \neq 1$)

and separately,

$$(1 + x)^r \leq 1 + rx$$

for every $r \in \mathbb{R}, 0 \leq r \leq 1$ and $x \geq -1$.

Result 1.2.9 (Limit to infinity of a rational function) Let P, Q be polynomial functions, where Q is of a higher degree. Then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0$$

Compute. Consider the example of

$$\lim_{x\to\infty} \frac{x^2 - 3x}{x^3 + 2x + 5}$$

We can divide both numerator and denominator by x^2 to obtain

$$\lim_{x\to\infty} \frac{1 - \frac{3}{x}}{x + \frac{2}{x} + \frac{5}{x^2}}$$

And we can see that the numerator $\to 1$ while the denominator $\to \infty$. $\square$

Result 1.2.10 (Limit of $\frac{e^x}{x}$ as $x \to \infty$)

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty$$

Proof. Since e^x can be written as a Taylor series

$$e^x = 1 + x + \frac{x^2}{2} + \dots$$

We have $e^x \geq 1 + x + x^2$ and hence

$$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{x} &\geq \lim_{x\to\infty} \frac{1 + x + \frac{x^2}{2}}{x} \\ &= \lim_{x\to\infty} \frac{1}{x} + 1 + \frac{x}{2} \\ &= \infty \end{aligned}$$

$\square$

Result 1.2.11 (Limit of $\frac{\ln x}{x}$ as $x \to \infty$)

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

Proof. Given any ε, we have to find a $N \in \mathbb{N}$ such that

$$n \geq N \implies \frac{\ln x}{x} < \varepsilon$$

But, if you've been paying attention,

$$\frac{\ln x}{x} < \varepsilon \iff \frac{e^{\varepsilon x}}{\varepsilon x} > \frac{1}{\varepsilon}$$

And since $\varepsilon x \to +\infty$, using Result 1.2.10 with εx as the limiting variable tells us that indeed there exists such an N, hence completing the proof. $\square$

Result 1.2.12 (Limit of a polynomial divided by an exponential) Let $a, b \in \mathbb{R}$ be fixed, with $b > 1$. Then we have

$$\lim_{x\to\infty} \frac{x^a}{b^x} = 0$$

Proof. Given any ε we want to find a $N \in \mathbb{N}$ such that

$$n \geq N \implies \frac{x^a}{b^x} < \varepsilon$$

But this is equivalent to

$$a \ln x - x \ln b < \ln \varepsilon$$

So it suffices to prove that

$$a \ln x - x \ln b \to -\infty.$$

Rewriting, we have

$$\begin{aligned} a \ln x - x \ln b &= x \left(a \cdot \frac{\ln x}{x} - \ln b \right) \\ &= \infty(-\ln b) \qquad \because \frac{\ln x}{x} \to 0 \\ &= -\infty \end{aligned}$$

This completes the proof. ☐

Definition 1.2.13 (Norm properties) Given a vector space X over a subfield F of the complex numbers $\mathbb{C}$, a **norm** on X is a real-valued function $p : X \to \mathbb{R}$ with the following properties, where $|k|$ denotes the absolute value of a scalar k.

(N1) *(Positive definiteness)* For all $x \in X$, if $p(x) = 0$ then $x = 0$.

(N2) *(Absolute homogeneity)* $p(kx) = |k|p(x)$ for all $x \in X$ and scalars k.

(N3) *(Subadditivity/Triangle inequality)* $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$

Theorem 1.2.14 (Limit and limit superior/inferior) Let $\{x_n\}$ be a bounded sequence. Then $\{x_n\}$ converges to $\bar{x}$ if and only if

$$\limsup x_n = \liminf x_n = \bar{x}$$

In short,

$$\lim_{n\to\infty} x_n = \bar{x} \text{ (exists)} \iff \limsup x_n = \liminf x_n = \bar{x}$$

Sidenote: all convergent sequences are bounded, so the boundedness can be taken for free once convergence is established.

Result 1.2.15 (Limit of a polynomial divided by its successor) Let P be a polynomial. Show that

$$\lim_{x\to\infty} \frac{P(x)}{P(x+1)} = 1$$

Proof. We will write $P(x)$ as

$$P(x) := \sum_{i=0}^{n} a_i x^i$$

where n is the degree of polynomial P.

$$\begin{aligned}
P(x+1) &= P\big(x(1+\tfrac{1}{x})\big) \\
&= a_0 + a_1 x\left(1+\frac{1}{x}\right) + a_2x^2\left(1+\frac{1}{x}\right)^2 + \ldots + a_n x^n\left(1+\frac{1}{x}\right)^n \\
&= x^n\left[\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}}\left(1+\frac{1}{x}\right) + \frac{a_2}{x^{n-2}}\left(1+\frac{1}{x}\right)^2 + \ldots + a_n\left(1+\frac{1}{x}\right)^n\right] \\
P(x) &= x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + \ldots + a_n\right) \\
\implies \frac{P(x)}{P(x+1)} &= \frac{\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}}\left(1+\frac{1}{x}\right) + \frac{a_2}{x^{n-2}}\left(1+\frac{1}{x}\right)^2 + \ldots + a_n\left(1+\frac{1}{x}\right)^n}{\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + \ldots + a_n} \\
\implies \lim_{x\to\infty}\frac{P(x)}{P(x+1)} &= \frac{a_n}{a_n} = 1
\end{aligned}$$

□

Result 1.2.16 (Rational times irrational is irrational) Let $a \in \mathbb{R}\setminus\mathbb{Q}$ and $b \in \mathbb{Q}$. Then $ab \in \mathbb{R}\setminus\mathbb{Q}$.

Proof. Clearly $ab \in \mathbb{R}$. Suppose $ab \in \mathbb{Q}$. Then there exists $n, m, s, t \in \mathbb{Z}$ such that

$$\frac{n}{m} = ab \quad \text{and} \quad \frac{s}{t} = b$$

Then we have

$$a = \frac{n}{m}\cdot\frac{1}{b} = \frac{n}{m}\cdot\frac{t}{s} \in \mathbb{Q}$$

which is a contradiction. Hence we must have $ab \in \mathbb{R}\setminus\mathbb{Q}$. □

Theorem 2.1.1 (Fundamental theorem of calculus)

First part Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $F : [a, b] \to \mathbb{R}$ be defined by

$$F(x) = \int_a^x f(t)\,dt$$

Then F is uniformly continuous on $[a, b]$ and differentiable on (a, b), and

$$F'(x) = f(x)$$

on (a, b) so F is an antiderivative of f.

Corollary

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Second part Let $f : [a, b] \to \mathbb{R}$. Let $F : [a, b] \to \mathbb{R}$ be continuous and also the antiderivative of f in (a, b). If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

This is stronger than the corollary because it does not assume that f is continuous.

Definition 3.1.1 (Affine functions) An affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ is of the form

$$f(x) = Ax - b \quad (A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m)$$

Definition 3.1.2 (Coercive functions) A function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive if

$$\lim_{\|x\| \to \infty} f(x) = +\infty$$

Definition 3.1.3 (Supercoercive functions) A function $f : \mathbb{R}^n \to \mathbb{R}$ is supercoercive if

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty$$

Remark 4.1.1 (Thinking about matrix dimensions) Let $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. We can validly write

$$Ax = b$$

So A is a gadget that takes a n-dim vector and returns a m-dim vector.

(A has m rows and n columns)

Definition 4.1.2 (Positive (semi)definiteness) A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if

$$x^T Ax > 0 \quad \forall (x \in \mathbb{R}^n)$$

and positive semidefinite if

$$x^T Ax \geq 0 \quad \forall (x \in \mathbb{R}^n)$$

Definition 4.1.3 (Inner product space) An inner product space is a vector space V over the field F together with an *inner product*.

An inner product is a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies the following for all $x, y, z \in V$ and $a, b \in F$:

(I1) *(Positive definiteness)* If x is non-zero, then

$$\langle x, x \rangle > 0$$

(I2) *(Linearity in the first argument)*

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$$

(I3) *(Conjugate symmetry)*

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

Definition 5.0.0 (Absolute basics of boolean algebra)

(a) Literal: a boolean variable x or $\neg x$ (or $\bar{x}$)

(b) Conjunction: $\wedge$ (and)

(c) Disjunction: $\vee$ (or)

(d) Clause: a disjunction of **distinct** literals

Definition 5.1.1 (Conjunctive normal form) This is a conjunction of one or more clauses.

$$(A \vee B) \wedge (C \vee D \vee E)$$

Definition 5.1.2 (Disjunctive normal form) This is a disjunction of one or more conjunctions.

$$(A \wedge B) \vee (C \wedge D \wedge E)$$

Proposition 5.1.3 (Extending a CNF to 3 variables) Given a 1-variable or 2-variable CNF, we want to write a logically equivalent 3-variable CNF. (Useful for 3-SAT problems). Here's how:

2-var CNF. Say we have the expression $(x \vee y)$. This is logically equivalent to

$$(x \vee y \vee z) \wedge (x \vee y \vee \bar{z})$$

Notice that if z is TRUE then we can drop the left branch because it's true and hence

$$(x \vee y \vee z) \wedge (x \vee y \vee \bar{z}) \equiv (x \vee y \vee \bar{z}) \equiv (x \vee y)$$

Similarly if z is FALSE then we drop the right branch and get

$$(x \vee y \vee z) \wedge (x \vee y \vee \bar{z}) \equiv (x \vee y \vee z) \equiv (x \vee y)$$

1-var CNF. Now consider the expression x. Instead of adding just one variable we now add two and get the logically equivalent expression

$$(x \vee y \vee z) \wedge (x \vee y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee \bar{y} \vee \bar{z})$$

If $(y, z) = (\text{TRUE}, \text{TRUE})$ we can drop all clauses containing y or z, leaving us with

$$(x \vee \bar{y} \vee \bar{z})$$

but then $(\bar{y}, \bar{z}) = (\text{FALSE}, \text{FALSE})$ and hence it is logically equivalent to just x. Repeating this logic for all combinations of (y, z), we can see that $(*)$ is logically equivalent to x.

Definition 9.1.1 (Gamma function) The gamma function is defined via a convergent improper integral:

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} \, dt \qquad (\mathrm{Re}(z) > 0)$$

Note that "$\int_0^\infty$" is a shorthand for "$\lim_{k\to\infty} \int_0^k$".

Observe that $\Gamma(1) = 1$.

$$\int_0^\infty e^{-t} \, dt = \Big[-e^{-t} \Big]_0^\infty = 1$$

And that $\Gamma(n+1) = n\Gamma(n)$.

$$\begin{aligned} \int_0^\infty e^{-t} t^n \, dt &= \Big[-e^{-t} \cdot t^n \Big]_0^\infty - \int_0^\infty -e^{-t} \cdot nt^{n-1} \, dt \\ &= 0 + \int_0^\infty e^{-t} \cdot nt^{n-1} \, dt \qquad (\text{by Result 1.2.12}) \\ &= n\Gamma(n) \end{aligned}$$

Definition 9.1.2 (Language reductions) If problem A is reducible to problem B, we write $A \leq B$.

Reducing A to B by a **Many-one reduction** is to find a function f which converts inputs x of A into inputs $f(x)$ of B, such that $A(x) = B(f(x))$ under all values of x.

Reducing A to B by a **Turing reduction** is to find a function which mimics the behavior of A using an oracle of B. i.e., $A(x) = \text{TRUE} \iff B(f(x)) = \text{TRUE}$.

A being reducible to B means solving A cannot be harder than the combined difficulty of solving B and executing the reduction. In particular, if the reduction runs in constant-time, A cannot be harder than B. In order words, $\leq$ is referring to hardness.

Definition 9.1.3 (Everything P-, NP-related) This is a compilation of everything P- and NP-related. For in-depth definitions, refer to each link below.

A problem L is in P if it runs in polynomial time.

A problem L is in NP if has a polynomial-time verifier.

We say that $L_1 \leq_{\mathsf{P}} L_2$ if there is a polynomial-time reduction from L_1 to L_2.

A problem L is NP-complete when $L \in \mathsf{NP}$, and every problem L' in NP has a polynomial-time reduction to it:

$$\forall L' \in \mathsf{NP} : L' \leq_{\mathsf{P}} L$$

A problem H is NP-hard when for every $L \in \mathsf{NP}$, there is a polynomial-time reduction from L to H:

$$\forall L \in \mathsf{NP} : L \leq_{\mathsf{P}} H \tag{$*$}$$

The only difference between NP-complete and NP-hard is that NP-complete has the extra constraint of having to be in NP.

$(*)$, based on a previous remark, also implies that H is at least as hard as the hardest problem in NP.

Theorem 9.1.4 (Cauchy-Schwarz inequality) For all vectors u and v of an inner product space,

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \cdot \langle v, v\rangle$$

This gives the following corollaries:

(a) Let $u_i, v_i \in \mathbb{R}$ for $i = 1, \ldots, n$ for any integer n. Then

$$\left(\sum u_i v_i\right)^2 \leq \left(\sum u_i^2\right)\left(\sum v_i^2\right)$$

(b) Let $u_k, v_k \in \mathbb{C}$ for $k = 1, \ldots, n$ for any integer n. Then

$$\left|\sum u_i v_i\right|^2 \leq \left(\sum |u_i|^2\right)\left(\sum |v_i|^2\right)$$

Proof. To prove (a), we observe that $\mathbb{R}^n$ equipped with the standard dot product is an inner product space. We can build vectors $u, v \in \mathbb{R}^n$ by arranging u_i for $i = 1, \ldots, n$ into a column vector and do the same for v_i to get v.

Then applying the Cauchy-Schwarz inequality with the standard dot product, we have

$$|u \cdot v|^2 \leq (u \cdot u)(v \cdot v)$$

Which gives the statement in (a) exactly.

To prove (b), instead of the inner product space constructed from $\mathbb{R}^n$ and the standard dot product, we use $\mathbb{C}^n$ and the complex inner product defined by

$$\langle u, w\rangle := u_1\bar{w}_1 + \ldots + u_n\bar{w}_n$$

Then by the Cauchy-Schwarz inequality, for all $u, w \in \mathbb{C}^n$,

$$\begin{aligned}
|\langle u, w\rangle|^2 &= \left|\sum u_k \bar{w}_k\right|^2 \\
&\leq \langle u, u\rangle \cdot \langle w, w\rangle \\
&= \left(\sum u_k \bar{u}_k\right)\left(\sum w_k \bar{w}_k\right) \\
&= \left(\sum |u_k|^2\right)\left(\sum |w_k|^2\right)
\end{aligned}$$

That is

$$|u_1\bar{w}_1 + \ldots + u_n\bar{w}_n|^2 \leq \left(|u_1|^2 + \ldots + |u_n|^2\right)\left(|\bar{w}_1|^2 + \ldots + |\bar{w}_n|^2\right)$$

But since $|z|^2 = |\bar{z}|^2$ for all $z \in \mathbb{C}$, we can define a collection $v_1, \ldots, v_n$ such that $v_k = \bar{w}_k$, then we can rewrite the above inequality as

$$|u_1 v_1 + \ldots + u_n v_n|^2 \leq \left(|u_1|^2 + \ldots + |u_n|^2\right)\left(|v_1|^2 + \ldots + |v_n|^2\right)$$

And finally since the collection w_k were arbitrarily chosen, so can the collection v_k. $\square$

Real Analysis

Definition 1.1.1 (Number systems)

(i) $\mathbb{N} :=$ set of all natural numbers $\{1, 2, 3, \ldots\}$

(ii) $\mathbb{Z} :=$ set of all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

(iii) $\mathbb{Q} :=$ set of all rational numbers $\left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}$

(iv) $\mathbb{R} :=$ set of all real numbers

We have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

The set of irrational numbers is denoted by $\mathbb{R} \setminus \mathbb{Q}$.

Theorem 1.1.2 ($\sqrt{2}$ is an irrational number)

Proof. Suppose $\sqrt{2}$ is rational. Then we can write

$$\sqrt{2} = \frac{a}{b}$$

where a and b are integers with no common factor other than 1. Then

$$2 = \frac{a^2}{b^2}$$

and

$$2b^2 = a^2$$

This says that a^2 is even. So a is also even, and $a = 2k$ for some integer k. Then we get

$$2b^2 = 4k^2$$

So

$$b^2 = 2k^2$$

But this says that b^2 is even, so b is even. It follows that 2 is a common factor for a and b. This contradicts our assumption of a and b, and hence $\sqrt{2}$ is not rational. $\square$

Principle 1.2.1 (Well-ordering Property of $\mathbb{N}$) Every non-empty subset S of $\mathbb{N}$ has a *least (or minimum)* element. Formally,

$$\exists\, m \in S : \ \forall s \in S, \ m \leq s$$

Note that S may not have a largest element.

Theorem 1.2.2 (Induction on natural numbers) Let $S \subseteq \mathbb{N}$. If we have

(i) $1 \in S$, and

(ii) for every $k \in \mathbb{N}$, $k \in S \implies k + 1 \in S$.

Then $S = \mathbb{N}$.

Proof. Suppose that $S \neq \mathbb{N}$. Then its complement $\mathbb{N} \setminus S \neq \emptyset$

By the well-ordering property of $\mathbb{N}$, there exists a least element $m \in \mathbb{N} \setminus S$.

By (i), we have $m \neq 1$ and hence $m \geq 2$. Thus, $m - 1 \in \mathbb{N}$. Since m is the smallest natural number *not* in S, we have $m - 1 \in S$. But by (ii), $m = (m-1) + 1 \in S$, which is a contradition to $m \in \mathbb{N} \setminus S$. □

Theorem 1.2.3 (Principle of Mathematical Induction) For each $n \in \mathbb{N}$, let $P(n)$ be a statement about n. Suppose that

(i) $P(1)$ is true, and

(ii) for every $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true.

Observe that 1 can be replaced with any natural number n_0, but we would have only proved that P is true for all natural numbers $\geq n_0$.

Proof. Apply induction on natural numbers on the set

$$\{n \in \mathbb{N} : P(n) \text{ is true}\}$$

□

Remark 1.3.1 (Field properties of $\mathbb{R}$) The binary operation **addition** on the set $\mathbb{R}$ satisfies the following properties, for all $a, b, c \in \mathbb{R}$:

(**A1**) *(Commutativity)* $a + b = b + a$

(**A2**) *(Associativity)* $(a + b) + c = a + (b + c)$

(**A3**) *(Existence of additive identity)* $\exists 0 \in \mathbb{R} : \ a + 0 = 0 + a = a$

(**A4**) *(Existence of additive inverse)* $\forall x \in \mathbb{R}, \ \exists -x \in \mathbb{R} : \ x + (-x) = (-x) + x = 0$

The binary operation **multiplication** on $\mathbb{R}$ satisfies the following properties, for all $a, b, c \in \mathbb{R}$:

(**M1**) *(Commutativity)* $a \cdot b = b \cdot a$

(**M2**) *(Associativity)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(**M3**) *(Existence of multiplicative identity)* $\exists 1 \in \mathbb{R} : \ a \cdot 1 = 1 \cdot a = a$

(**M4**) *(Existence of multiplicative inverse)*
$\forall x \in \mathbb{R} \setminus \{0\}, \ \exists 1/x \in \mathbb{R} : \ x \cdot (1/x) = (1/x) \cdot x = 1$

In addition, the two binary operations satisfy the following property:

(**D**) *(Distributivity of multiplication over addition)*

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall a, b, c \in \mathbb{R}$$

Because of (**A1**)-(**A4**), (**M1**)-(**M4**), and (**D**), we say that $(\mathbb{R}, +, \cdot)$ forms a **field**.

Theorem 1.3.2 (Results from the field properties of $\mathbb{R}$) For any $a, b, c \in \mathbb{R}$,

(i) *(Uniqueness of additive inverse)* If $a + b = 0$, then $b = -a$

(ii) *(Uniqueness of multiplicative inverse)* If $a \cdot b = 1$ and $a \neq 0$, then $b = \frac{1}{a}$.

(iii) If $a + b = b$, then $a = 0$.

(iv) If $b \neq 0$ and $a \cdot b = b$, then $a = 1$.

(v) $a \cdot 0 = 0$.

(vi) If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

(vii) *(Cancellative property)* If $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$.

Proof. Let $a, b, c \in \mathbb{R}$.
Proof of (i): Suppose $a + b = 0$. Then

$$\begin{aligned} a + b + (-a) &= 0 + (-a) \\ a - (-a) + b &= -a \quad \text{(commutativity and additive identity)} \\ 0 + b &= -a \quad \text{(additive inverse)} \\ b &= -a \quad \text{(additive identity)} \end{aligned}$$

Proof of (ii): Suppose $a \cdot b = 1$ and $a \neq 0$. Then

$$\begin{aligned} a \cdot b \cdot (1/a) &= 1 \cdot (1/a) \\ a \cdot (1/a) \cdot b &= 1/a \quad \text{(commutativity and multiplicative identity)} \\ 1 \cdot b &= 1/a \quad \text{(multiplicative inverse)} \\ b &= 1/a \quad \text{(multiplicative identity)} \end{aligned}$$

Proof of (iii): Suppose $a + b = b$. Then

$$\begin{aligned} a + b + (-b) &= b + (-b) \\ a + 0 &= 0 \quad \text{(additive inverse)} \\ a &= 0 \quad \text{(additive identity)} \end{aligned}$$

Proof of (iv): Suppose $a \cdot b = b$ and $b \neq 0$. Then

$$\begin{aligned} a \cdot b \cdot (1/b) &= b \cdot (1/b) \\ a \cdot 1 &= 1 \quad \text{(multiplicative inverse)} \\ a &= 1 \quad \text{(multiplicative identity)} \end{aligned}$$

Proof of (v):

$$\begin{aligned} a \cdot 0 &= a \cdot 0 + 0 \quad \text{(additive identity)} \\ &= a \cdot 0 + [(a \cdot 0) + (-(a \cdot 0))] \quad \text{(additive inverse)} \\ &= (a \cdot 0 + a \cdot 0) + (-(a \cdot 0)) \quad \text{(associativity)} \\ &= a \cdot (0 + 0) + (-(a \cdot 0)) \quad \text{(distributivity)} \\ &= a \cdot 0 + (-(a \cdot 0)) \quad \text{(additive identity)} \\ &= 0 \quad \text{(additive inverse)} \end{aligned}$$

Proof of (vi): Suppose $a \cdot b = 0$. Now if $a = 0$ then we are done. So suppose that $a \neq 0$. Then $1/a$ exists.

$$\begin{aligned}
a \cdot b \cdot (1/a) &= 0 \cdot (1/a) \\
a \cdot b \cdot (1/a) &= 0 \quad \text{(result (v))} \\
a \cdot (1/a) \cdot b &= 0 \quad \text{(commutativity)} \\
1 \cdot b &= 0 \quad \text{(multiplicative inverse)} \\
b &= 0 \quad \text{(multiplicative identity)}
\end{aligned}$$

Proof of (vii): Suppose that $a \neq 0$ and $a \cdot b = a \cdot c$. By (M4) that $1/a$ exists. Then

$$\begin{aligned}
(1/a) \cdot (a \cdot b) &= (1/a) \cdot (a \cdot c) \\
((1/a) \cdot a) \cdot b &= ((1/a) \cdot a) \cdot c \quad \text{(commutativity)} \\
1 \cdot b &= 1 \cdot c \quad \text{(multiplicative inverse)} \\
b &= c \quad \text{(multiplicative identity)}
\end{aligned}$$

□

Remark 1.3.3 (Order properties of $\mathbb{R}$) There is a binary relation $>$ on $\mathbb{R}$ which has the following properties (with $a, b, c \in \mathbb{R}$):

(**O1**) If $a > b$, then $a + c > b + c$.

(**O2**) If $a > 0$ and $b > 0$, then $a \cdot b > 0$.

(**O3**) *(Trichotomy Property)* If $a, b \in \mathbb{R}$, then exactly one of the following holds:

$$a > b, \quad a = b, \quad b > a$$

(**O4**) *(Transitive Property)* If $a > b$ and $b > c$, then $a > c$.

Theorem 1.3.7 If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for every positive number ε, then $a = 0$.

Proof. Since $a \geq 0$, by definition either $a > 0$ or $a = 0$. Suppose $a > 0$. Then $\frac{a}{2} > 0$. Now let $\varepsilon := \frac{a}{2}$. Then by assumption, $a < \varepsilon = \frac{a}{2}$

$$\begin{aligned}
a < \frac{a}{2} &\implies 2 \cdot a < 2 \cdot \frac{a}{2} = a \\
&\implies 2a - a < 0 \\
&\implies a < 0
\end{aligned}$$

This contradict that $a > 0$. Hence we must have $a = 0$. □

Theorem 1.6.2 (Triangle inequality for $\mathbb{R}$) For all $a, b \in \mathbb{R}$, we have

$$|a + b| \leq |a| + |b|$$

Proof. We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. Adding, we have

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

which implies that $|a + b| \leq |a| + |b|$. □

Corollary 1.6.3 (Corollaries of triangle inequality for $\mathbb{R}$)

(i) $\big||a| - |b|\big| \leq |a - b|$

(ii) $|a - b| \leq |a| + |b|$

(iii) $\big||a| - |b|\big| \leq |a + b|$

Proof. By the triangle inequality,

$$|a| = |(a - b) + b| \leq |a - b| + |b|$$

So

$$|a| - |b| \leq |a - b| \tag{$*$}$$

By symmetry, we also have

$$|b| - |a| \leq |b - a|$$

which can be rewritten as

$$-(|a| - |b|) \leq |a - b| \tag{$**$}$$

With $(*)$ and $(**)$, we have (i).
(ii) is obtained by using $-b$ in the place of b in the triangle inequality
(iii) follows from using $-b$ in the place of b in (i). $\square$

Definition 2.1.1 (Boundedness) A non-empty set $S \subseteq \mathbb{R}$ is said to be **bounded above** if there exists some $M \in \mathbb{R}$ such that

$$x \leq M, \quad \forall x \in S.$$

Such an M is called an **upper bound** of S.
On the other hand, S is said to be **bounded below** if there exists some $m \in \mathbb{R}$ such that

$$m \leq x, \quad \forall x \in S.$$

Such an m is called a **lower bound** of S.
If S is both bounded above and bounded below, then we simply call it **bounded**.
Equivalently, a set S is bounded if there exists $M \geq 0$ such that

$$|x| \leq M, \quad \forall x \in S$$

Definition 2.2.1 (Maximum and minimum of a subset of $\mathbb{R}$) For a non-empty $S \subseteq \mathbb{R}$, one defines the maximum of S to be the (necessarily unique) number M such that

(i) $M \in S$, and

(ii) $x \leq M$ for all $x \in S$.

Similarly, the **minimum** of S is the (necessarily unique) number m such that

(i) $m \in S$, and

(ii) $m \leq x$ for all $x \in S$.

Definition 2.3.1 (Supremum) Let $E \subseteq \mathbb{R}$ be non-empty. A real number $M \in \mathbb{R}$ is called the **supremum** of E (we write $\sup E$) if

(i) M is an upper bound of E, and

(ii) if M' is an upper bound of E, then $M \leq M'$.

Lemma 2.3.2 Let $E \subseteq \mathbb{R}$ be non-empty. Then $M = \sup E$ if and only if M is an upper bound of E and for every $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that $M - \varepsilon < x_\varepsilon$.

Proof. ($\implies$) Suppose $M = \sup E$. Let $\varepsilon > 0$. Then $M - \varepsilon < M$. Since M is the least upper bound of E by definition, $M - \varepsilon$ cannot be an upper bound for E. Hence there exists $x_\varepsilon \in E$ such that $M - \varepsilon < x_\varepsilon$.

($\impliedby$) Suppose M is an upper bound for E and that there exists $x_\varepsilon \in E$ such that $M - \varepsilon < x_\varepsilon$. Let M' be an upper bound of E. Suppose on the contrary that $M' < M$. Then we let $\varepsilon := M - M' > 0$. Then there exists $x_\varepsilon \in E$ such that

$$M' = M - (M - M') = M - \varepsilon < x_\varepsilon$$

This contradicts the assumption that M' is an upper bound for E. Hence we must have $M \leq M'$, making M the least upper bound of E. $\square$

Lemma 2.3.3 If $A \subseteq B \subseteq \mathbb{R}$ and both $\sup A$ and $\sup B$ exist, then $\sup A \leq \sup B$.

Proof. $\sup B$ is an upper bound for B, but since $A \subseteq B$, $\sup B$ is an upper bound for A as well. Since $\sup A$ is the least upper bound of A, we have $\sup A \leq \sup B$. $\square$

Definition 2.3.4 (Infimum) Let $E \subseteq \mathbb{R}$ be non-empty. A real number $m \in \mathbb{R}$ is called the **infimum** of E (we write $\inf E$) if

(i) m is a lower bound of E, and

(ii) if m' is a lower bound of E, then $m' \leq m$.

This is an analog to Definition 2.3.1.

Lemma 2.3.5 Let $E \subseteq \mathbb{R}$ be non-empty. Then $m = \inf E$ if and only if m is a lower bound of E and for every $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that $x_\varepsilon < m + \varepsilon$.

Proof. Exercise (similar to proof of Lemma 2.3.2) $\square$

Principle 2.3.6 (Completeness/supremum property of $\mathbb{R}$) Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.

In other words, if $E \subseteq \mathbb{R}$ is non-empty, then $\sup E$ exists.

Remark 2.3.7 This marks the end of our assumptions, which are:

(i) the field properties of $\mathbb{R}$ (**A1**)-(**A4**), (**M1**)-(**M4**), and (**D**).

(ii) the order properties of $\mathbb{R}$ (**O1**)-(**O4**).

(iii) the completeness property of $\mathbb{R}$.

With these we will build up other properties of $\mathbb{R}$.

Theorem 2.3.8 (The infimum property of $\mathbb{R}$) Every non-empty subuset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.

Proof. Let $E \subseteq \mathbb{R}$ be non-empty and bounded below by $b \in \mathbb{R}$. Let $A := \{-x : x \in E\}$. Then $A \subseteq \mathbb{R}$ and is non-empty. For all $x \in E$, $b \leq x$ and hence $-x \leq -b$. And so $-b$ is an upper bound for A.

Since A is non-empty and bounded above, by the supremum property of $\mathbb{R}$, A has a supremum $M \in \mathbb{R}$. We claim that

$$\inf E = -\sup A = -M. \tag{$*$}$$

Since M is an upper bound for A,

$$\begin{aligned} -x &\leq M, \quad \forall -x \in A \\ -x &\leq M, \quad \forall x \in E \\ -M &\leq x, \quad \forall x \in E \end{aligned}$$

Hence $-M$ is a lower bound for E.

Now let m be another lower bound of E. Then $-m$ is an upper bound of A. Since $M = \sup A$, we have $M \leq -m$. So $m \leq -M$. Hence $-M$ is indeed the greatest lower bound of E. This proves $(*)$. □

Result 2.3.9 Let $A, B \subseteq \mathbb{R}$ be non-empty sets, and let

$$C := \{a + b : a \in A,\ b \in B\}$$

Then $\sup C = \sup A + \sup B$.

Proof. Let $c \in C$. Then $c = a + b$ for some $a \in A$ and $b \in B$. Now since $a \leq \sup A$ and $b \leq \sup B$, we have

$$c = a + b \leq \sup A + \sup B.$$

Hence $\sup A + \sup B$ is an upper bound of C.

Next, let M be an upper bound of C. Then for all $a \in A$ and $b \in B$,

$$a + b \leq M$$

and thus $a \leq M - b$. So then for each $b \in B$, $M - b$ is an upper bound for A. Consequently, $\sup A \leq M - b$, and we have

$$b \leq M - \sup A \quad \forall b \in B.$$

Which now implies that $M - \sup A$ is an upper bound for B, so

$$\sup B \leq M - \sup A$$

and thus

$$\sup A + \sup B \leq M$$

Showing that $\sup A + \sup B$ is indeed the least upper bound for C. Hence $\sup C = \sup A + \sup B$. $\square$

Theorem 2.4.1 (Archimedean property of $\mathbb{R}$) For any $x \in \mathbb{R}$, there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

Alternatively, any $x \in \mathbb{R}$ is not an upper bound for $\mathbb{N}$.

In other words, $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Proof. Suppose on the contrary that the Archimedean property of $\mathbb{R}$ does not hold.

Then there exists some $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. That is, the non-empty set $\mathbb{N}$ is bounded above.

By the completeness property of $\mathbb{R}$, $M = \sup \mathbb{N}$ exists.

By Lemma 2.3.2 (using $\varepsilon := 1$), there exists $\bar{n} \in \mathbb{N}$ such that $M - 1 < \bar{n}$. Then $M < \bar{n} + 1$. But $\bar{n} + 1 \in \mathbb{N}$. This contradicts that M is an upper bound of $\mathbb{N}$. $\square$

Corollary 2.4.2 Let $A \subseteq \mathbb{R}$ be given by $A := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then

(i) $\inf A = 0$, and

(ii) given any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $0 < \frac{1}{n_\varepsilon} < \varepsilon$.

Note that (ii) is significant because it claims that for any positive real number, there exists a *rational* number between it and zero.

Proof. Proving (i): For any $x \in A$, $x = \frac{1}{n}$ for some $n \in \mathbb{N}$, and thus $x > 0$. Thus 0 is a lower bound of A. Now suppose m' is another lower bound for A. Then

$$m' \leq \frac{1}{n} \quad \forall n \in \mathbb{N} \tag{$*$}$$

If $m' > 0$, then $1/m' > 0$. By the Archimedean property of $\mathbb{R}$, there exists $\bar{n} \in \mathbb{N}$ such that

$$\frac{1}{m'} < \bar{n} \implies \frac{1}{\bar{n}} < m'$$

which contradicts ($*$). Hence we must have $m' \leq 0$, which implies that 0 is the greatest lower bound of A.

Proving (ii): from (i) since $\varepsilon > 0 = \inf A$, ε is not a lower bound of A. That is, there is an element of A smaller than ε:

$$\exists n_\varepsilon \in \mathbb{N} : \frac{1}{n_\varepsilon} < \varepsilon$$

This completes the proof. $\square$

Corollary 2.4.3 (Existence of the floor of a real number) Let $x \in \mathbb{R}$. Then there exists a unique $m \in \mathbb{Z}$ such that

$$m \leq x < m+1$$

We denote m by $\lfloor x \rfloor$.

Proof. For this proof, we shall consider two cases:

Case 1: $x \geq 1$. Consider the set

$$S := \{n \in \mathbb{N} : n > x-1\} \subseteq \mathbb{N}$$

We claim that $\lfloor x \rfloor$ is minimum of this set.

By the Archimedean property of $\mathbb{R}$, we know that S is non-empty. Then by the well-ordering property of $\mathbb{N}$, it follows that S has a minimum element, which we denote by m. Since $m \in S$, it follows that $m \in \mathbb{N}$ and

$$m > x-1 \implies x < m+1 \tag{$*$}$$

Next we show that $m \leq x$. Suppose on the contrary that $m > x$. Then

$$\begin{aligned} m > x \geq 1 &\implies m-1 > 0 \quad \text{and} \quad m-1 > x-1 \\ &\implies m-1 \in \mathbb{N} \quad \text{and} \quad m-1 > x-1 \\ &\implies m-1 \in S \end{aligned}$$

But this contradicts that $m = \min S$. Hence $m \leq x$. Together with $(*)$, we have

$$m \leq x < m+1.$$

Case 2: $x < 1$. It follows from the Archimedean property of $\mathbb{R}$ that there exists $k \in \mathbb{N}$ such that

$$1 - x < k$$

which implies that $x + k > 1$. Then from Case 1 applied to $x+k$, there exists $m' \in \mathbb{Z}$ such that

$$m' \leq x+k < m'+1$$

which is then

$$m' - k \leq x < m' - k + 1$$

Let $m := m' - k \in \mathbb{Z}$. then we have $m \leq x < m+1$. Thus we have proved existence.

Next, on to uniqueness. Let $m_1, m_2 \in \mathbb{Z}$ be such that

$$m_1 \leq x < m_1 + 1 \quad \text{and} \quad m_2 \leq x < m_2 + 1$$

Then we have

$$m_1 \leq x < m_2 + 1 \implies m_1 - m_2 < 1$$

and by symmetry, $m_2 - m_1 < 1$. Hence

$$-1 < m_1 - m_2 < 1$$

But since $m_1, m_2 \in \mathbb{Z}$, we have $m_1 - m_2 \in \mathbb{Z}$ and thus $m_1 - m_2 = 0$. Hence $m_1 = m_2$ and this completes the proof for uniqueness. $\square$

Lemma 2.4.3 There exists a unique positive real number a such that $a^2 = 2$, without assuming the existence of $\sqrt{2} \in \mathbb{R}$.

Proof. *(Existence)* Consider the set

$$S := \left\{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\right\} \subseteq \mathbb{R}$$

We claim that $(\sup S)^2 = 2$.

S is non-empty since $1 \in S$. Also, S is bounded above (by 2, for instance). Hence by the completeness property of $\mathbb{R}$, $\sup S$ exists in $\mathbb{R}$. Let $a := \sup S \in \mathbb{R}$.

We know that $a > 0$ since $1 \in S$, and hence a is positive, as desired.

It remains to show that $a^2 = 2$. By the trichotomy property of $\mathbb{R}$, we just have to exclude the possibilities

$$\text{Case 1: } a^2 < 2 \quad \text{and} \quad \text{Case 2: } a^2 > 2$$

Case 1: $a^2 < 2$. We will argue that there exists some $n \in \mathbb{N}$ such that $(a + \frac{1}{n})^2 < 2$, which implies that $(a + \frac{1}{n})^2 \in S$, which then implies that $a = \sup S$ is not an upper bound of S.

Observe that

$$\left(a + \frac{1}{n}\right)^2 = a^2 + \frac{2a}{n} + \frac{1}{n^2} \leq a^2 + \frac{2a}{n} + \frac{1}{n} = a^2 + \frac{2a+1}{n} \tag{1}$$

since $\dfrac{1}{n^2} \leq \dfrac{1}{n}$ for any $n \in \mathbb{N}$. As $a^2 < 2$, we have

$$a^2 + \frac{2a+1}{n} < 2 \iff n > \frac{2a+1}{2-a^2} \tag{2}$$

Since $a^2 < 2$, we have $\dfrac{2a+1}{2-a^2} \in \mathbb{R}$. Thus by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ satisfying

$$n > \frac{2a+1}{2-a^2}.$$

Fixing this n, and together with (2) and then (1), we have $(a + \frac{1}{n})^2 < 2$. Hence Case 1 is not possible.

Case 2: $a^2 > 2$. We claim that there exists some $n \in \mathbb{N}$ such that $a - \frac{1}{n}$ is an upper bound of S, breaking the fact that a is the least upper bound of S. We proceed by

(i) Find $n \in \mathbb{N}$ such that $(a - \frac{1}{n})^2 > 2$.

(ii) Show that $x \leq a - \frac{1}{n}$ for all $x \in S$.

Step (i): Note that

$$\left(a - \frac{1}{n}\right)^2 = a^2 - \frac{2a}{n} + \frac{1}{n^2} > a^2 - \frac{2a}{n} \tag{3}$$

On the other hand, we have

$$a^2 - \frac{2a}{n} > 2 \iff \frac{1}{n} < \frac{a^2-2}{2a} \tag{4}$$

Since $a^2 > 2$ and $a > 0$, we have $\dfrac{a^2-2}{2a} > 0$, and by Corollary 2.4.2(ii), there exists $n \in \mathbb{N}$ such that $\dfrac{1}{n} < \dfrac{a^2-2}{2a}$.

Fixing this n, and together with (4) and then (3), we have $(a - \frac{1}{n})^2 > 2$

Step (ii): For all $x \in S$, we have $x \geq 0$ and $x^2 < 2$. Thus,

$$\left(a - \frac{1}{n}\right)^2 - x^2 > 2 - 2 = 0 \implies \left(a - \frac{1}{n} + x\right)\left(a - \frac{1}{n} - x\right) > 0$$

Note that $a > 1$, $\frac{1}{n} \leq 1$, $x > 0$, and thus $a - \dfrac{1}{n} + x > 0$. Hence we must have

$$a - \frac{1}{n} - x > 0$$

and thus $x < a - \frac{1}{n}$. This completes the contradiction of Case 2.

Hence we must have $a^2 = 2$.

(Uniqueness) Suppose $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$ such that $a^2 = 2$ and $b^2 = 2$. Then

$$a^2 - b^2 = 2 - 2 = 0 \implies (a-b)(a+b) = 0$$

Since $a > 0$ and $b > 0$, it follows that $a + b > 0$ and in particular $a + b \neq 0$. Hence we must have $a - b = 0$, which means that $a = b$. $\square$

Theorem 2.4.5 (Existence of the positive k-th root of a positive real number) Let $c > 0$ and $k \in \mathbb{N}$. Then there exists a unique $a \in \mathbb{R}$ with $a^k = c$.

Proof. The proof is similar to the square root case. Let

$$S := \left\{t \in \mathbb{R} : t > 0 \text{ and } t^k < c\right\}$$

Then one can show that $1 \in S$ if $c > 1$, and $\frac{c}{2} \in S$ if $c \leq 1$ (hence S is non-empty). Moreover, c is an upper bound of S if $c > 1$, and 1 is an upper bound of S if $c \leq 1$ (hence S is bounded above). By the supremum property of $\mathbb{R}$, $a = \sup S$ exists. We claim that $a^k = c$. To justify this claim, one shows that it is impossible to have $a^k < c$ or $a^k > c$. Again, refer to the square root case for inspiration. $\square$

Theorem 2.4.6 (Density Theorem) For any $x, y \in \mathbb{R}$ satisfying $x < y$, there exists a $r \in \mathbb{Q}$ such that

$$x < r < y$$

Proof. Since $x < y$, we have $y - x > 0$ and thus by Corollary 2.4.2(ii), there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} y - x > \frac{1}{n} &\implies ny - nx > 1 \\ &\implies nx + 1 < ny \end{aligned} \tag{$*$}$$

Then by Corollary 2.4.3, the floor $\lfloor nx \rfloor \in \mathbb{Z}$ exists and it satisfies

$$\lfloor nx \rfloor \leq nx < \lfloor nx \rfloor + 1 \implies nx < \lfloor nx \rfloor \leq nx + 1$$

Together with $(*)$, we have

$$nx < \lfloor nx \rfloor + 1 < ny$$

and thus

$$x < \frac{\lfloor nx \rfloor + 1}{n} < y$$

Hence by setting $r := \dfrac{\lfloor nx \rfloor + 1}{n}$, we have

$$r \in \mathbb{Q} \text{ and } x < r < y$$

$\square$

Example 2.4.6 Let $E := \{x \in \mathbb{Q} : x < \sqrt{3}\}$. Then $\sup E = \sqrt{3}$.

Proof. By definition of E, $x \leq \sqrt{3}$ for all $x \in E$. Thus, E is bounded above (by $\sqrt{3}$). Also, since $0 \in E$, E is non-empty. Thus by the completeness property of $\mathbb{R}$, $\sup E$ exists in $\mathbb{R}$.

Since $\sqrt{3}$ is an upper bound of E, we must have $\sup E \leq \sqrt{3}$.

Suppose that $\sup E \neq \sqrt{3}$. Then $\sup E < \sqrt{3}$. By the Density Theorem, there exists $r \in \mathbb{Q}$ such that

$$\sup E < r < \sqrt{3} \tag{$*$}$$

Since $r \in \mathbb{Q}$ and $r < \sqrt{3}$, it follows that $r \in E$. But this and $(*)$ contradicts the fact that $\sup E$ is an upper bound for E.

Hence we must have $\sup E = \sqrt{3}$. $\square$

Corollary 2.4.6 Let $\alpha \in \mathbb{R}$, and let

$$E_\alpha := \{x \in \mathbb{Q} : x < \alpha\} \subseteq \mathbb{Q}$$

Then $\sup E_\alpha = \alpha$.

Proof. Exercise. (Similar to Example 2.4.6) $\square$

Corollary 2.4.7 If $a, b \in \mathbb{R}$ such that $a < b$, then there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < x < b$.

Proof. If $a < b$, then $a < \dfrac{a+b}{2} < b$ and thus

$$\frac{a}{\sqrt{2}} < \frac{a+b}{2\sqrt{2}} < \frac{b}{\sqrt{2}}$$

By the density theorem, there exist $r_1, r_2 \in \mathbb{Q}$ such that

$$\frac{a}{\sqrt{2}} < r_1 < \frac{a+b}{2\sqrt{2}} < r_2 < \frac{b}{\sqrt{2}}$$

At least one of r_1, r_2 is non-zero. Call it r. Then we have $r \in \mathbb{Q} \setminus \{0\}$ and

$$\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$$

Hence we have $a < r\sqrt{2} < b$. $r\sqrt{2}$ is because $\sqrt{2}$ is irrational (by Theorem 1.1.2), and by a known result, the product of a rational number and an irrational number is irrational. $\square$

Corollary 2.4.8 If an interval $I \subset \mathbb{R}$ has at least two elements, then I contains infinitely many rational numbers and infinitely many irrational numbers.

Proof. Assume that I contains finitely many rational numbers. Enumerate all of them by $x_1, \ldots, x_n \in I$ in order of increasing value:

$$x_1 < x_2 < \ldots < x_n.$$

Also, by assumption we have $n \geq 2$.

By the density theorem, there exists $r \in \mathbb{Q}$ such that

$$x_1 < r < x_2$$

Note that since I is an interval, we have $r \in I$. But clearly r is not equal to any of the $x_1, \ldots, x_n$ previously identified. This contradicts the assumption that $x_1, \ldots, x_n$ are all the numbers in I.

Hence I must contain infinitely many rational numbers.

The case with irrational numbers is completely analog to this, but instead of the density theorem we use Corollary 2.4.7. $\square$

Definition 2.4.9 (Dense sets) The set $D \subseteq \mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for any $a, b \in \mathbb{R}$ with $a < b$, we have $D \cap (a, b) \neq \emptyset$.

In other words, $\exists\, x \in D$ such that $a < x < b$.

Remark 2.4.10

(i) By the density theorem, $\mathbb{Q}$ is dense in $\mathbb{R}$.

(ii) By Corollary 2.4.7, $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 2.5.1 (Intervals) An **interval** is a subset I of $\mathbb{R}$ with the following (equivalent) properties:

- if $x \leq t \leq y$ and $x, y \in I$, then $t \in I$.
- if $x, y \in I$ and $x \leq y$, then $[x, y] \subseteq I$.

Definition 3.1.1 (Sequences) A **sequence** in $\mathbb{R}$ is a function $X : \mathbb{N} \to \mathbb{R}$.

The numbers $\{X(n) : n \in \mathbb{N}\}$ are called the **terms** of the sequence. For each $n \in \mathbb{N}$, $X(n)$ is called the n-th term of the sequence.

Notation. We usually write x_n for $X(n)$ and denote the sequence X by any one of

$$\{x_n\},\ \{x_n\}_{n=1}^{\infty},\ \{x_n\}_{n\in\mathbb{N}},\ \{x_n\}_{\mathbb{N}}$$

Definition 3.1.2 (Constant sequence) A constant sequence is of the form

$$\{c, c, c, \ldots\}$$

for some constant $c \in \mathbb{R}$.

Definition 3.1.3 (Neighborhoods) Let $a \in \mathbb{R}$ and $\varepsilon > 0$. The **ε-neighborhood of a** is the set

$$B_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$$

or alternatively, $(a - \varepsilon, a + \varepsilon)$.

Definition 3.1.4 (Limit) We say that $\bar{x}$ is the **limit** of $\{x_n\}$ if for every $\varepsilon > 0$, there exist $K = K(\varepsilon) \in \mathbb{N}$ such that

$$n \geq K \implies |x_n - \bar{x}| < \varepsilon$$

or equivalently,

$$\forall n \geq K, \ |x_n - \bar{x}| < \varepsilon$$

or,

$$\forall n \geq K, \ x_n \in B_\varepsilon(\bar{x})$$

Remark Here we write $K = K(\varepsilon)$ to signify that K depends on ε.

Definition 3.1.5 (Convergence)

(i) If $\bar{x}$ is the limit of $\{x_n\}$, then we also say that $\{x_n\}$ **converges** to $\bar{x}$, and we write

$$\lim_{n\to\infty} x_n = \bar{x}$$

or " $x_n \to \bar{x}$ as $n \to \infty$ " or simply " $x_n \to \bar{x}$ ".

(ii) We say that a sequence $\{x_n\}$ **converges** if it converges to a (finite) limit $\bar{x} \in \mathbb{R}$; and that it **diverges** if it does not converge (to a finite limit).

Theorem 3.1.6 (Uniqueness of limit) If $\{x_n\}$ converges, then it has exactly one limit.

Proof. Suppose x and x' are limits of $\{x_n\}$. Let $\varepsilon > 0$ be arbitrarily given, and let $\bar{\varepsilon} = \varepsilon/2$.
Since $x_n \to x$, there exists $K_1 \in \mathbb{N}$ such that

$$n \geq K_1 \implies |x_n - x| < \bar{\varepsilon}$$

Similarly since $x_n \to x'$, there exists $K_2 \in \mathbb{N}$ such that

$$n \geq K_2 \implies |x_n - x'| < \bar{\varepsilon}$$

Then let $K := \max\{K_1, K_2\} \in \mathbb{N}$. Then for all $n \geq K$,

$$\begin{aligned} |x - x'| &= |(x - x_n) - (x_n - x')| \\ &\leq |x - x_n| - |x_n - x'| \quad \text{(triangle inequality)} \\ &< \bar{\varepsilon} + \bar{\varepsilon} \\ &= \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, by Theorem 1.3.7 we have $|x - x'| = 0$ and hence $x = x'$. $\square$

Definition 3.2.1 (Bounded sequence) The boundedness of a sequence $\{x_n\}$ is determined by the set

$$\{x_n : n \in \mathbb{N}\}$$

and the definitions stated here.

Theorem 3.2.2 (Convergence implies boundedness) Every convergent sequence is bounded.

Proof. Let $\{x_n\}$ be a convergent sequence and $\lim_{n\to\infty} x_n = \bar{x}$. Put $\varepsilon := 1$. Then there exists $K \in \mathbb{N}$ such that

$$|x_n - \bar{x}| < 1, \quad \forall n \geq K$$

Thus when $n \geq K$,

$$\begin{aligned} |x_n| &= |(x_n - \bar{x}) + \bar{x}| \\ &\leq |x_n - \bar{x}| + |\bar{x}| \quad \text{(triangle inequality)} \\ &\leq 1 + |\bar{x}| \end{aligned}$$

Let $M = \max\{x_1, \dots, x_{K-1}, 1 + |\bar{x}|\}$. Then

$$|x_n| \leq M, \quad \forall n \in \mathbb{N}$$

So $\{x_n\}$ is bounded. □

Corollary 3.2.3

(i) Every unbounded sequence is divergent (contrapositive of Theorem 3.2.2)

(ii) Boundedness does not imply convergence. Consider the bounded sequence $\{x_n\}$ defined by $x_n = (-1)^n$.

Theorem 3.2.2 (Limit arithmetic) If $\lim\limits_{n\to\infty} x_n = \bar{x}$ and $\lim\limits_{n\to\infty} y_n = \bar{y}$, then

(i) $\lim\limits_{n\to\infty} (x_n + y_n) = \bar{x} + \bar{y}$

(ii) $\lim\limits_{n\to\infty} (x_n - y_n) = \bar{x} - \bar{y}$

(iii) $\lim\limits_{n\to\infty} (x_n \cdot y_n) = \bar{x} \cdot \bar{y}$

(iv) $\lim\limits_{n\to\infty} (x_n / y_n) = \bar{x}/\bar{y}$, provided $y_n \neq 0$ for all $n \in \mathbb{N}$, and $\bar{y} \neq 0$.

In short, the arithmetic operators $+$, $-$, $\times$, $\div$ are preserved upon taking limits.

Note that these require both $\{x_n\}$ and $\{y_n\}$ to converge.

Corollary 3.2.3 If $\{x_n\}$ converges and $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} (x_n)^k = \left(\lim_{n\to\infty} x_n\right)^k$$

Theorem 3.2.4 (Squeeze Theorem) If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then

$$\lim_{n\to\infty} y_n = a$$

Note that we can weaken the condition on n to just $n \in \mathbb{N},\ n \geq K_0$ for some fixed K_0.

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to a$, there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$,

$$|x_n - a| < \varepsilon \implies -\varepsilon < x_n - a \tag{$*$}$$

Since $z_n \to a$, there exists $K_1 \in \mathbb{N}$ such that for all $n \geq K_1$,

$$|z_n - a| < \varepsilon \implies z_n - a < \varepsilon \tag{$**$}$$

Let $K := \max\{K_1, K_2\} \in \mathbb{N}$. If we used the weaker condition, put $K := \max\{K_0, K_1, K_2\}$. Then for all $n \geq K$, we have

$$\begin{aligned} x_n \leq y_n \leq z_n &\implies x_n - a \leq y_n - a \leq z_n - a \\ &\implies -\varepsilon < y_n - a < \varepsilon \quad \text{(from $(*)$ and $(**)$)} \\ &\implies |y_n - a| < \varepsilon \end{aligned}$$

Hence we also have $\lim_{n\to\infty} y_n = a$. □

Theorem 3.2.5 If $|x_n| \to 0$, then $x_n \to 0$.

Proof. Let $\varepsilon > 0$ be given. Since $|x_n| \to 0$, it follows that there exists $K \in \mathbb{N}$ such that

$$n \geq K \implies \big||x_n| - 0\big| < \varepsilon$$

But $\big||x_n| - 0\big| = |x_n - 0|$, and hence we have

$$n \geq K \implies |x_n - 0| = \big||x_n| - 0\big| < \varepsilon$$

Hence $x_n \to 0$. □

Theorem 3.2.6 For a fixed $b \in \mathbb{R}$ satisfying $0 \leq b < 1$, we have

$$\lim_{n\to\infty} b^n = 0$$

Proof. There are two cases to consider:

Case 1: $b = 0$. Let $\varepsilon > 0$ be given. Take $K = 1$. Then for all $n \geq K$,

$$|b^n - 0| = |0 - 0| = 0 < \varepsilon$$

Therefore we have $\lim_{n\to\infty} b^n = 0$.

Case 1: $0 < b < 1$**.** Let $a := \frac{1}{b} - 1$. Then $a > 0$, and $b = \frac{1}{1+a}$. By Bernoulli's inequality, we have $(1+a)^n \geq 1 + na$ for all $n \in \mathbb{N}$, hence

$$0 < b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} \leq \frac{1}{na} \quad (\forall n \in \mathbb{N})$$

Now

$$\lim_{n\to\infty} 0 = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{na} = 0$$

So by Squeeze Theorem, we have $\lim\limits_{n\to\infty} b^n = 0$. $\square$

Calculus

Remark 1.1.1 (Trigonometric identities)

$$\sin 2x = 2\sin x\cos x \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos 2x = 1 - 2\sin^2 x \qquad \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$= 2\cos^2 x - 1$$

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\sin A + \sin B = 2\sin(\tfrac{A+B}{2})\cos(\tfrac{A-B}{2}) \qquad 2\sin A\cos B = \sin(\tfrac{A+B}{2}) + \sin(\tfrac{A-B}{2})$$
$$\sin A - \sin B = 2\cos(\tfrac{A+B}{2})\sin(\tfrac{A-B}{2}) \qquad 2\cos A\sin B = \sin(\tfrac{A+B}{2}) - \sin(\tfrac{A-B}{2})$$
$$\cos A + \cos B = 2\cos(\tfrac{A+B}{2})\cos(\tfrac{A-B}{2}) \qquad 2\cos A\cos B = \cos(\tfrac{A+B}{2}) + \cos(\tfrac{A-B}{2})$$
$$\cos A - \cos B = -2\sin(\tfrac{A+B}{2})\sin(\tfrac{A-B}{2}) \qquad -2\sin A\sin B = \cos(\tfrac{A+B}{2}) - \cos(\tfrac{A-B}{2})$$

Result 1.1.2 (Basic trigonometric constants) Trigonometric preprocessing to finish homework in $O(1)$ time.

	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	-

$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\tan(-x) = -\tan(x)$$

Remark 2.1.1 (Differentiation identities)

Product rule $(uv)' = u'v + uv'$

Quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

(can be derived from product rule using u and $\frac{1}{v}$)

Remark 2.1.2 (Integration identities)

$f(x)$	$\int f(x)\,dx$	
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$	
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$	$(\lvert x\rvert > a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{x-a}{x+a}\right)$	$(x > a)$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left(\dfrac{a+x}{a-x}\right)$	$(\lvert x\rvert > a)$
$\tan x$	$\ln(\sec x)$	$(\lvert x\rvert > \dfrac{\pi}{2})$
$\cot x$	$\ln(\sin x)$	$(0 > x > \pi)$
$\sec x$	$-\ln(\sec x + \tan x)$	$(\lvert x\rvert > \dfrac{\pi}{2})$
$\csc x$	$-\ln(\csc x + \cot x)$	$(0 > x > \pi)$

Remark 2.1.3 (Chain rule) In all the following scenarios, let $h := f \circ g$.

When f takes a scalar Let $f, g : \mathbb{R} \to \mathbb{R}$. Then $h : \mathbb{R} \to \mathbb{R}$ and we have

$$h'(t) = f'(g(t)) \cdot g'(t)$$

And $f', g' : \mathbb{R} \to \mathbb{R}$.

When f takes a vector Let $g : \mathbb{R} \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. Then $h : \mathbb{R} \to \mathbb{R}$ and we have

$$h'(t) = \nabla f(g(t))^T g'(t)$$

Note that $\nabla f(g(t)) \in \mathbb{R}^n$ and $g'(t) \in \mathbb{R}^n$.

When f takes a complex number Let $g : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{R}$. Then $h : \mathbb{R} \to \mathbb{R}$.
In particular, we write $g(t) = g_1(t) + ig_2(t)$ and $f : x + iy \mapsto f(x + iy)$.
Interestingly, we still have

$$(f \circ g)'(t) = f_x(g(t)) \cdot {g_1}'(t) + f_y(g(t)) \cdot {g_2}'(t)$$

Note the lack of i terms on the term with ${g_2}'$. This is intentional.
Remember anyway that $f \circ g : \mathbb{R} \to \mathbb{R}$, and so we must have $(f \circ g)' : \mathbb{R} \to \mathbb{R}$.

Definition 2.1.4 (Differentiability) In single-variable calculus, $f : \mathbb{R} \to \mathbb{R}$ is differentiable if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists (and if so, is denoted as $f'(a)$).

In multivariable calculus, $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable if there exists a **linear map** $J : \mathbb{R}^n \to \mathbb{R}$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - J(h)}{\|h\|} = 0$$

Then from this perspective, differentiability of single-variable complex functions can be written as: $f : \mathbb{C} \to \mathbb{C}$ is differentiable if there is a linear map $J : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{|f(z+h) - f(z) - J(h)|}{|h|} = 0$$

Comment All of these cases are equivalent to saying that there exists a $k \in \mathbb{R}$ such that.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = k$$

Essentially, that there exists a local **linearization** to the function.

Definition 2.1.5 (Directional derivative) Let $D \subset \mathbb{R}$ be open. $f : D \to \mathbb{R}$ is directionally differentiable at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ if

$$\lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(x)}{t}$$

exists. This limit is denoted by $f'(x; d)$ and is called the directional derivative of f at $\bar{x}$ in the direction d.

If f is directionally differentiable at $\bar{x}$ in every direction $d \in \mathbb{R}^n$, we call f directionally differentiable at $\bar{x}$.

If f is directionally differentiable at every $\bar{x} \in \mathbb{R}^n$, we call it directionally differentiable.

Remark 2.1.6 (Gradient) Only scalar-valued functions can have gradients.

The **gradient** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the vector-valued function ∇f whose value at p gives the direction and rate of fastest increase. Further, ∇f can be written as

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}$$

The gradient of f is defined as the unique vector field whose dot product with any vector d at each point x is the directional derivative of f along d. That is,

$$\nabla f(x)^T d = f'(x; d)$$

Remark 2.1.7 (Jacobian) The Jacobian of a vector-valued function in several variables generalizes the gradient of a scalar-valued function in several variables.

In other word, the Jacobian of a scalar-valued function is its gradient.

So let $f : \mathbb{R}^n \to \mathbb{R}^m$. In particular,

$$f(x) := \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$

Then the Jacobian of f is an $m \times n$ matrix

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^T f_1 \\ \vdots \\ \nabla^T f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \nabla f(x)^T$$

Note that by defintion of the gradient, we have

$$\nabla f(x)^T = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \mathbf{J}$$

and hence we say that the Jacobian is the transpose of the gradient.

Algorithm Design

Definition 0.0.1 (Vertex cover) Let $G = (V, E)$ be an undirected graph. A vertex cover $U \subseteq V$ satisfies

$$(u, v) \in E \implies u \in U \vee u \in U.$$

In other words, every edge in E has at least one endpoint in the vertex cover U. Such a set is said to *cover* the edges of G.

Definition 1.1 (Flow network) A *flow network* is a directed graph $G = (V, E)$ with a single *source node* s and a single *target node* t, as well as a positive number $c(e)$ for each edge $e \in E$, called the capacity of e.

Definition 1.3 (Flow) Let G be a flow network. A *flow* on G is given by a positive number $f(e)$ for each edge e in G satisfying the following two constraints:

- **Capacity constraints.** For each edge $e \in E$, we have $0 \leq f(e) \leq c(e)$
- **Flow conservation.** For each vertex $v \in V$ that is not the source or target vertex,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ leaving } v} f(e)$$

The *value* of flow f is all of the flow leaving s:

$$\mathrm{val}(f) := \sum_{e \text{ leaving } s} f(e)$$

where s is the source node of G.

Problem 1 (Max Flow) Input: A flow network G with source s, target t, and positive edge capacities $c(e)$ for $e \in E$.

Output: A flow f with the maximum value.

Definition 1.4 (Residual graph) Let G be a flow network and let f be a flow on G. The *residual graph* of G and f, denoted by G_f, is the directed graph defined as follows:

The vertices of G_f are the same as the vertices of G.

For each edge $e = (u, v)$ in G, if $f(e) < c(e)$ then we add the edge (u, v) to G_f, labelled with the number $c(e) - f(e)$. If $f(e) > 0$, then we also add the edge (v, u) to G_f, labelled with the number $f(e)$.

All paths from s to t in the residual graph correspond to a sequence where flow can be re-routed to increase its value.

Definition 3.0 (Binary representation) Let

$$\{0, 1\}^* := \{\varepsilon, 0, 1, 00, 01, 10, 11, 100, 101, \ldots\}$$

be the set of all finite binary strings. (where ε is the empty string.)

Definition 3.1 (Decision problem) A decision problem L is a subset of $\{0, 1\}^*$. The computational task corresponding to L is “Given a string $x \in \{0, 1\}^*$, is $x \in L$?”

Problem 8 **(L-membership problem)** **Input:** A boolean string x.
Output: Decide if $x \in L$.

Example 3.1.1 **(Rewriting problems as L-membership problems)**

- *Graph Connectivity.* Given a graph $G = (V, E)$, is it connected?

$$\begin{aligned} L &= \{x \in \{0,1\}^* \mid x \text{ encodes a connected graph}\} \\ &= \{G \mid G \text{ is a connected graph}\} \end{aligned}$$

- *Max Flow (Decision Version).* Given a flow network G and a postive integer k, does the max flow on G have value $\geq k$?

$$\begin{aligned} L &= \{x \in \{0,1\}^* \mid x \text{ encodes a } (G,k) \text{ such that } \mathrm{val}(G) \geq k\} \\ &= \{(G,k) \mid \mathrm{val}(G) \geq k\} \end{aligned}$$

- *Sum.* Given $a, b, c \in \mathbb{Z}$, does $a + b = c$?

$$\begin{aligned} L &= \{x \in \{0,1\}^* \mid x \text{ encodes } (a,b,c) \text{ such that } a+b=c\} \\ &= \{(a,b,c) \mid a+b=c\} \end{aligned}$$

Choice of encoding is important and definitely affects runtime. However, for discussion we will assume that the most natural and succint encoding is chosen.

Definition 3.2 **(Polynomial-time algorithms)** An algorithm A runs in *polynomial time* if $\exists\, c \in \mathbb{R}$ s.t. $\forall x \in \{0,1\}^*$, A terminates after $O(|x|^c)$ computation steps.

A decision problem L is *polynomial-time computable* if there exists a polynomial-time A s.t. $\forall x \in \{0,1\}^*$, $x \in L \iff A(x) = \text{Yes}$.

We define

$$\mathsf{P} := \{L \subseteq \{0,1\}^* \mid L \text{ is polynomial-time computable}\}$$

The complexity class P is our proxy for *efficiently computable languages.* ("language" is another way to refer to L, in addition to "problem".)

Definition 3.3.1 **(Vertex Cover Problem)**

$$\mathsf{VC} := \{(G,k) \mid G \text{ is a graph with a vertex cover of size} \leq k\}$$

Definition 3.3.2 **(Satisfiability Problem)**

$$\mathsf{FORMSAT} := \{F \mid F \text{ is a satisfiable boolean formula}\}$$

Definition 3.3 **(Nondeterministic polynomial-time algorithms)** A decision problem L has a polynomial-time verifier if there is a polynomial time algorithm B taking two strings x, y as input, and a polynomial $p(n)$ such that

$$x \in L \iff \exists\, y \in \{0,1\}^*, |y| \leq p(|x|) : B(x,y) = \text{Yes}$$

The complexity class

$$\mathsf{NP} := \{L \subseteq \{0,1\}^* \mid L \text{ has a polynomial-time verifier}\}$$

- A polynomial-time verifier for VC would take a graph (G, k) and a proposed vertex cover U and check if $|U| \leq k$ and that U is a vertex cover.
- A polynomial-time verifier for SAT would take in a boolean formula F and a proposed assignment x and check if $F(x) = \text{Yes}$.

Observe that $\mathsf{P} \subseteq \mathsf{NP}$. If L has a polynomial-time algorithm, then it also has a polynomial-time verifier.

Proof. If $L \in \mathsf{P}$, then by definition there exists a polynomial-time algorithm A with

$$x \in L \iff A(x) = \text{Yes}$$

Then, following the defintion of NP, we need to find a polynomial-time verifier B such that

$$x \in L \iff \exists y \in \{0,1\}^*, |y| \leq p(|x|) : B(x, y) = \text{Yes}$$

But we can simply use $B(x, y) := A(x)$. □

Definition 3.4 (Complement of a decision problem) The complement of a decision problem L is defined as

$$\overline{L} = \{0,1\}^* \setminus L := \{x \in \{0,1\}^* \mid x \notin L\}$$

Note that $\{0,1\}^* = L \cup \overline{L}$ for any decision problem L.

Exercise 3.5 Prove that if $L \in \mathsf{P}$ then $\overline{L} \in \mathsf{P}$.

Proof. If $L \in \mathsf{P}$, then by definition there exists a polynomial-time algorithm A with

$$x \in L \iff A(x) = \text{Yes}$$

Then, following the defintion of P, we need to find a polynomial-time algorithm B such that

$$x \in \overline{L} \iff B(x) = \text{Yes}$$

But we can simply use $B(x) := \neg A(x)$. □

Definition 3.6 (coNP) The complexity class coNP is defined as

$$\mathsf{coNP} := \{L \mid \overline{L} \in \mathsf{NP}\}$$

For example, recall that $\mathsf{SAT} = \{F \mid F \text{ is a satisfiable boolean formula}\}$. Then

$$\overline{\mathsf{SAT}} = \left\{ x \in \{0,1\}^* \;\middle|\; \begin{array}{l} x \text{ is an invalid encoding of a formula, or} \\ x \text{ encodes an unsatisfiable boolean formula} \end{array} \right\}$$

But given $x \in \{0,1\}^*$, it is easy to test its validity as a boolean formula, hence we focus on the second constraint:

$$\mathsf{CONT} := \{F \mid F \text{ is an unsatisfiable boolean formula}\}$$

Note that since $\overline{\mathsf{CONT}} = \mathsf{SAT} \in \mathsf{NP}$, we have that $\overline{\mathsf{CONT}} \in \mathsf{NP}$.

However, is $\mathsf{CONT} \in \mathsf{NP}$? Observe that $F \in \mathsf{CONT}$ if and only if *for every* assignment x to the variables of F, we have that $F(x) = \text{No}$. Since there are 2^n assigments to check, it is not clear how to encode this checking procedure into a single polynomial-sized certificate. For this reason, many researchers conjecture that $\mathsf{NP} \neq \mathsf{coNP}$.

Remark 3.6.1 (The complexity class $\mathsf{NP} \cap \mathsf{coNP}$) We can show that $\mathsf{P} \subseteq \mathsf{coNP}$ by

$$L \in \mathsf{P} \implies \overline{L} \in \mathsf{P} \implies \overline{L} \in \mathsf{NP} \implies L \in \mathsf{coNP}$$

And since $\mathsf{P} \subseteq \mathsf{NP}$, we have

$$\mathsf{P} \subseteq \mathsf{NP} \cap \mathsf{coNP}$$

Complex Analysis

Definition 0.0.0 (General terminology)

Entire function is a complex-valued function that is holomorphic on $\mathbb{C}$.

A **real-valued** function is any function $f : X \to \mathbb{R}$.

A **complex-valued** function is any function $f : X \to \mathbb{C}$.

A subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ is called **compact** if it is closed and bounded.

The C^n notation:

- C^0 : continuous
- C^1 : continuously differentiable
- C^2 : twice continuously differentiable

Theorem 0.0.1 (Conventional notation) For this chapter on Complex Analysis.

Let $U \subseteq \mathbb{C}$ be an open set.

Let $D(P, r)$ be the open disc centered at P with radius r. Then

(i) $\partial D(P, r)$ is the (closed) curve at the border of $D(P, r)$

(ii) $\overline{D}(P, r)$ is the closed disc centered at P with radius r.

Result 0.0.2 (Basic complex arithmetic)

$$\begin{aligned} |z|^2 &= z\bar{z} \\ |zw|^2 &= |z|^2|w|^2 \\ |z+w|^2 &= |z|^2 + |w|^2 + \mathrm{Re}(z \cdot \bar{w}) \\ |z+w|^2 + |z-w|^2 &= 2|z|^2 + 2|w|^2 \end{aligned}$$

Proof. Let $z := x + iy$, and $w := u + iv$.

$$|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}$$

$$\begin{aligned} |zw|^2 &= (xu - yv)^2 + (xv + yu)^2 \\ &= (x^2u^2 - 2xyvu + y^2v^2) + (x^2v^2 + 2xyvu + y^2u^2) \\ &= x^2u^2 + y^2v^2 + x^2v^2 + y^2u^2 \\ &= (x^2 + y^2)(u^2 + v^u) \\ &= |z|^2|w|^2 \end{aligned}$$

$\square$

Theorem 0.0.3 (Complex differentiability) A complex function $f(z) := u(z) + iv(z)$ is complex-differentiable at z_0 if and only if u and v satisfy the Cauchy-Riemann Equations at z_0.

To say a function is **holomorphic** is much stronger, since a holomorphic function is complex-differentiable at every point of some open subset of the complex plane $\mathbb{C}$.

Definition 1.1.3 (Complex Partials)

$$\frac{\partial f}{\partial z} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f$$

$$\frac{\partial f}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

Definition 1.4.1 (Holomorphic functions) Let $U \subseteq \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be in $C^1(U)$. f is said to be *holomorphic* if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

Properties of holomorphic functions If f and g are holomorphic in a domain U, then so are $f+g$, $f-g$, fg, and $f \circ g$.

Additionally, if g has no zeros in U, then f/g is holomorphic too.

Examples of holomorphic functions Here are some building blocks to get started (remember that you can use these with the properties above to show that other more complicated functions are holomorphic too):

(i) $f(z) = 1/z$ on $\mathbb{C} \setminus \{0\}$

(ii) $f(z) = 1/p(z)$ on $\mathbb{C}$ where $p(z) \neq 0$

(iii) $f(z) = z$ on $\mathbb{C}$

All these can be proved using a destructuring of $z := x + iy$ and using Definition 1.1.3.

Here are some functions that are not holomorphic:

(i) $f(z) = \bar{z}$

(ii) $f(z) = |z|$

Showing that a function is holomorphic If we can write $f \equiv u + iv$, and u and v have **continuous** first partial derivatives and satisfy the Cauchy-Riemann equations, then f is holomorphic.

Definition 1.4.2 (Cauchy-Riemann Equations) If $f(z) = u(z) + iv(z)$ is holomorphic, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proposition 1.4.3 If $f : U \to \mathbb{C}$ is C^1 and f satisfies the Cauchy-Riemann equations, then

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

Definition 1.4.4 (Harmonic functions) Let $U \subseteq \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be in $C^2(U)$. f is said to be *harmonic* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

The operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the *Laplace operator*, or *Laplacian*, and is denoted by Δ. We write

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Theorem 1.5.1 Let $f, g \in C^1(U)$ where

$$U := \{(x, y) \in \mathbb{R}^2 : |x - a| < \delta,\ |y - b| < \varepsilon\}$$

and let $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ on U. Then there exists a function $h \in C^2(U)$ such that

$$\frac{\partial h}{\partial x} = f \quad \text{and} \quad \frac{\partial h}{\partial y} = g$$

on U. If f and g are real-valued, then we may take h to be real-valued also.

Theorem 1.5.3 Let $U \subseteq \mathbb{C}$ be either an open rectangle or open disc, and let F be holomorphic on U. Then there exists a holomorphic function H on U such that

$$\frac{\partial H}{\partial z} = F$$

on U.

Definition 2.1.1 (Bounded C^1 functions) A function $\phi : [a, b] \to \mathbb{R}$ is continuously differentiable (and we write $\phi \in C^1([a, b])$) if

(a) ϕ is continuous on $[a, b]$

(b) ϕ' exists on (a, b)

(c) ϕ' has a continuous extension to $[a, b]$

In other words, for (c) we require that

$$\lim_{t \to a^+} \phi'(t) \quad \text{and} \quad \lim_{t \to b^-} \phi'(t)$$

both exist.

The motivation for this definition is so if $\phi \in C^1([a, b])$, then we have

$$\begin{aligned}
\phi(b) - \phi(a) &= \lim_{\varepsilon \to 0^+} \big(\phi(b - \varepsilon) - \phi(a + \varepsilon)\big) \\
&= \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b-\varepsilon} \phi'(t)\, dt \\
&= \int_a^b \phi'(t)\, dt
\end{aligned}$$

and hence have the fundamental theorem of calculus hold for $\phi \in C^1([a, b])$.

Definition 2.1.2 (Continuous complex curve) Let $\gamma : [a, b] \to \mathbb{C}$ be defined by $\gamma(t) := \gamma_1(t) + i\gamma_2(t)$.

Then γ is said to be continuous on $[a, b]$ if both γ_1 and γ_2 are.

The curve γ is $C^1([a, b])$ if γ_1 and γ_2 are continuously differentiable on $[a, b]$. Under these circumstances we will write

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d\gamma_1}{dt} + i\frac{d\gamma_2}{dt}$$

Definition 2.1.3 (Complex integration) Let $\psi : [a, b] \to \mathbb{C}$ be continous on $[a, b]$. Write $\psi(t) = \psi_1(t) + i\psi_2(t)$. Then we define

$$\int_a^b \psi(t)\,dt := \int_a^b \psi_1(t)\,dt + i\int_a^b \psi_2(t)\,dt$$

Using this definition along with Definitions 2.1.1 and 2.1.2, we have that if $\gamma \in C^1([a, b])$ is complex-valued, then

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\,dt$$

Proposition 2.1.4 Let $U \subseteq \mathbb{C}$ be open and let $\gamma : [a, b] \to U$ be a C^1 curve. If $f : U \to \mathbb{R}$ and $f \in C^1(U)$ and we write

$$\begin{gathered} f : x + iy \mapsto f(x + iy) \\ \gamma(t) = \gamma_1(t) + i\gamma_2(t) \end{gathered}$$

then

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b (f \circ \gamma)'(t)\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}(\gamma(t)) \cdot \frac{d\gamma_1}{dt} + \frac{\partial f}{\partial y}(\gamma(t)) \cdot \frac{d\gamma_2}{dt} \right) dt \\ &= \int_a^b f_x(\gamma(t)) \cdot \gamma_1'(t) + f_y(\gamma(t)) \cdot \gamma_2'(t)\,dt \end{aligned}$$

This follows from Definition 2.1.3 and the chain rule.

(the lack of an i term is intentional. Remember that $f \circ \gamma : \mathbb{R} \to \mathbb{R}$)

Definition 2.1.5 (Complex line integral) Let $U \subseteq \mathbb{C}$ open, $F : U \to \mathbb{C}$ continuous on U, and let $\gamma : [a, b] \to U$ be a C^1 curve. Then we define the complex line integral

$$\oint_\gamma F(z)\,dz := \int_a^b F(\gamma(t)) \cdot \frac{d\gamma}{dt}\,dt$$

Proposition 2.1.6 (Holomorphic line integral) Let $U \subseteq \mathbb{C}$ open, $F : U \to \mathbb{C}$ continuous on U, and let $\gamma : [a, b] \to U$ be a C^1 curve. If f is a holomorphic function on U, then

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma \frac{\partial f}{\partial z}(z)\,dz$$

Definition 2.1.6a (Complex antiderivative) A function f has an antiderivative F if and only if, for every $\gamma : [a, b] \to \mathbb{C}$,

$$\oint_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$$

This comes from using a holomorphic function on Proposition 2.1.4, and then applying Definition 2.1.5.

Proposition 2.1.7 (Moving || into integral) Let $\phi : [a, b] \to \mathbb{C}$ be continuous. Then

$$\left| \int_a^b \phi(t)\, dt \right| \leq \int_a^b |\phi(t)|\, dt$$

Proposition 2.1.8 (Upper bound of line integral) Let $U \subseteq \mathbb{C}$ be open and $f \in C^0(U)$. Let $\gamma : [a, b] \to U$ be a C^1 curve, and let $\ell(\gamma)$ be given by

$$\ell(\gamma) := \int_a^b \left| \frac{d\gamma}{dt}(t) \right| dt$$

Then we have

$$\left| \oint_\gamma f(z)\, dz \right| \leq \Big(\sup_{t \in [a,b]} |f(\gamma(t))| \Big) \cdot \ell(\gamma)$$

(Note that $\ell(\gamma)$ is the length of γ.)

Proposition 2.1.9 (Parameterization-independence of line integrals) Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a continuous function. Let $\gamma : [a, b] \to U$ be a C^1 curve. Suppose that $\phi : [c, d] \to [a, b]$ is a bijective increasing C^1 with a C^1 inverse.

Let $\tilde{\gamma} = \gamma \circ \phi$. Then

$$\oint_{\tilde{\gamma}} f(z)\, dz = \oint_\gamma f(z)\, dz$$

The proof involves the standard change of variable formula from calculus.

Theorem 2.2.1 (Existence of f' on holomorphic f) Let $U \subseteq \mathbb{C}$ be open and let f be holomorphic on U. Then f' exists at each point of U and

$$f'(z) = \frac{\partial f}{\partial z}$$

for all $z \in U$.

As a result of this theorem, we often will write $f' = \dfrac{\partial f}{\partial z}$ when f is holomorphic.

Theorem 2.2.2 (Holomorphic by existence of derivative) Let $U \subseteq \mathbb{C}$ be open. If $f \in C^1(U)$ and f has a complex derivative at each point of U, then f is holomorphic on U.

In other words, if a continuous, complex-valued function f on U has a complex derivative at each point and if f' is continuous on U, then f is holomorphic on U.

Theorem 2.2.3 (Holomorphism and directional derivatives) Let f be holomorphic in a neighborhood $P \in \mathbb{C}$. Let $w_1, w_2 \in \mathbb{C}$ have unit modulus. Consider the directional derivatives

$$D_{w_1} f(P) := \lim_{t \to 0} \frac{f(P + tw_1) - f(P)}{t}$$
$$D_{w_2} f(P) := \lim_{t \to 0} \frac{f(P + tw_2) - f(P)}{t}$$

Then

(a) $|D_{w_1} f(P)| = |D_{w_2} f(P)|$

(b) if $|f'(P)| \neq 0$, then the directed angle from w_1 to w_2 equals the directed angle from $D_{w_1} f(P)$ to $D_{w_2} f(P)$.

Note:

- 2.2.3(a) alone implies that f is holomorphic.
- 2.2.3(b) alone implies that f is holomorphic.

Lemma 2.3.1 Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an open interval and let $H, F : (\alpha, \beta) \to \mathbb{R}$ be continuous functions. Let $p \in (\alpha, \beta)$ and suppose that dH/dx exists and equals $F(x)$ for all $x \in (\alpha, \beta) \setminus \{p\}$. Then $(dH/dx)(p)$ exists and $(dH/dx)(x) = F(x)$ for all $x \in (\alpha, \beta)$.

$$\forall_{x \in (\alpha, \beta) \setminus \{p\}} : \frac{dH}{dx}(x) = F(x) \implies \forall_{x \in (\alpha, \beta)} : \frac{dH}{dx}(x) = F(x)$$

It's as if the continuity fills in the gap at p.

Theorem 2.3.2 Let $U \subseteq \mathbb{C}$ be either an open rectangle or an open disc and let $P \in U$. Let f and g be continuous, real-valued functions on U which are continuously differentiable on $U \setminus \{P\}$. Suppose further that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{on } U \setminus \{P\}$$

Then there exists a C^1 function $h : U \to \mathbb{R}$ such that

$$\frac{\partial h}{\partial x} = f, \quad \frac{\partial h}{\partial y} = g$$

at every point of U (including P).

Theorem 2.3.3 (Existence of holomorphic antiderivative) Let $U \subseteq \mathbb{C}$ be either an open rectangle or an open disc. Let $P \in U$ be fixed. Suppose that F is continuous on U and holomorphic on $U \setminus \{P\}$. Then there is a holomorphic H on U such that $\partial H / \partial z = F$.

Note that since H is holomorphic, by Theorem 2.2.1, we can write $H' = F$.

Lemma 2.4.1 Let γ be the boundary of a disc $D(z_0, r)$ in the complex plane, equipped with the counterclockwise orientation. Let z be a point inside the circle $\partial D(z_0, r)$. Then

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - z} \, d\zeta = 1$$

The proof involves considering the function

$$I(z) := \oint_\gamma \frac{1}{\zeta - z}\, d\zeta$$

and showing that $I(z)$ is independent of z, and that $I(z_0) = 2\pi i$.

Theorem 2.4.2 (Cauchy integral formula) Suppose that $U \subseteq \mathbb{C}$ is open and that f is a holomorphic function on U. Let $z_0 \in U$ and let $r > 0$ such that $\overline{D}(z_0, r) \subseteq U$. Let $\gamma : [0, 1] \to \mathbb{C}$ be the C^1 curve $\gamma(t) = z_0 + r\cos(2\pi t) + ir\sin(2\pi t)$. Then, for each $z \in D(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

The converse of this theorem is true too: if f is given by the Cauchy integral formula, then f is holomorphic.

Example 2.4.2a (Examples with Cauchy integral formula) Here's some ground-truth computations to get started. (Almost all problems in MATH 466 can be re-routed back to these)

$$\oint_\gamma \zeta^k\, d\zeta = \begin{cases} 0 & \text{if } k \neq -1 \\ 2\pi i & \text{if } k = -1 \end{cases} \qquad (k \in \mathbb{Z})$$

Theorem 2.4.3 (Cauchy integral theorem) If f is a holomorphic function on an open disc $U \subseteq \mathbb{C}$, and if $\gamma : [a, b] \to U$ is a C^1 curve in U with $\gamma(a) = \gamma(b)$, then

$$\oint_\gamma f(z)\, dz = 0$$

Note that this implies that the Cauchy integral formula gives a zero whenever z does not lie in the contour γ, since the integrand is holomorphic. (Integrand is holomorphic because numerator is assumed to be holomorphic, and the denominator is never zero.)

Proof. By Theorem 1.5.3, there is a holomorphic function $G : U \to \mathbb{C}$ with $G' = f$ on U. Since $\gamma(a) = \gamma(b)$, we have that

$$0 = G(\gamma(b)) - G(\gamma(a))$$

By Proposition 2.1.6, this equals

$$\oint_\gamma G'(z)\, dz = \oint_\gamma f(z)\, dz$$

(Reminder that since G is holomorphic, $G' = \dfrac{\partial G}{\partial z}$ by Theorem 2.2.1) □

Definition 2.6.1 (Piecewise C^1 curve) A piecewise C^1 curve $\gamma : [a, b] \to \mathbb{C}$ is a continuous function such that there exists a finitne set of numbers $a_1 \le a_2 \le \ldots \le a_k$ satisfying $a_1 = a$ and $a_k = b$, and with the property that for every $i \le j \le k - 1$, $\gamma|_{[a_j, a_{j+1}]}$ is a C^1 curve.

γ is a piecewise C^1 curve in an open set U if $\gamma([a, b]) \subseteq U$.

Note that while joining $C^1(\mathbb{R})$ curves may not lead to a piecewise $C^1(\mathbb{R})$ curve, doing it in $\mathbb{C}$ somehow works.

Definition 2.6.2 (Integrating over a piecewise C^1 curve) If $U \subseteq \mathbb{C}$ is open and $\gamma : [a,b] \to U$ is a piecewise C^1 curve in U and if $f : U \to \mathbb{C}$ is a continuous function on U, then

$$\oint_\gamma f(z)\, dz := \sum_{j=1}^{k} \oint_{\gamma|_{[a_j, a_{j+1}]}} f(z)\, dz$$

where $a_1, a_2, \ldots, a_k$ are as in Definition 2.6.1.

Lemma 2.6.3 Let $U \subseteq \mathbb{C}$ be open. Let $\gamma : [a,b] \to U$ be a piecewise C^1 curve. Let $\phi : [c,d] \to [a,b]$ be a piecewise C^1 strictly monotone increasing function with $\phi(c) = a$ and $\phi(d) = b$. Let $f : U \to \mathbb{C}$ be a continuous function on U. Then the function $\gamma \circ \phi : [c,d] \to U$ is a piecewise C^1 curve and

$$\oint_\gamma f(z)\, dz = \oint_{\gamma \circ \phi} f(z)\, dz$$

(Really, $\{\gamma(t) \mid t \in [a,b]\} = \{(\gamma \circ \phi)(s) \mid s \in [c,d]\}$, and there are no added crossovers on the parameterization of $\gamma \circ \phi$ because ϕ is strictly monotone increasing.)

Lemma 2.6.4 Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ a holomorphic function and $\gamma : [a,b] \to U$ a piecewise C^1 curve. Then

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma f'(z)\, dz$$

(This is really just Proposition 2.1.6 restated with a piecewise C^1 version of γ)

Proposition 2.6.5 If $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is a holomorphic function, and if γ_r describes the circle of radius r around 0, tranversed once around counterclockwise, then, for any two positive numbers $r_1 < r_2$,

$$\oint_{\gamma_{r_1}} f(z)\, dz = \oint_{\gamma_{r_2}} f(z)\, dz$$

Proposition 2.6.6 Let $0 < r < R < \infty$ and define the annulus $\mathcal{A} := \{z \in \mathbb{C} : r < |z| < R\}$. Let $f : \mathcal{A} \to \mathbb{C}$ be a holomorphic function. If $r < r_1 < r_2 < R$ and if for each j the curve γ_{r_j} describes the circle of radius r_j around 0, traversed once counterclockwise, then we have

$$\oint_{\gamma_{r_1}} f(z)\, dz = \oint_{\gamma_{r_2}} f(z)\, dz$$

(On this annulus (donut), integrating a holomorphic f along any two circles centered at zero will yield the same value.)

Theorem 2.6.7 (Cauchy integral formula and theorem: general form) Let $U \subseteq \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be holomorphic. Then

$$\oint_\gamma f(z)\, dz = 0$$

for any piecewise C^1 closed curve γ in U that can be deformed in U through closed curves to a closed curve lying entirely in a disc contained in U.

In addition, suppose that $\overline{D}(z, r) \subseteq U$. Then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z)$$

for any piecewise C^1 closed curve γ in $U \setminus \{z\}$ that can be continuously deformed in $U \setminus \{z\}$ to $\partial D(z, r)$ equipped with counterclockwise orientation.

Theorem 3.1.1 (Analyticity of holomorphic functions) Let $U \subseteq \mathbb{C}$ be open and let f be a holomorphic on U. Then $f \in C^\infty(U)$. Moreover, if $\overline{D}(P, r) \subseteq U$ and $z \in D(P, r)$, then

$$\left(\frac{\partial}{\partial z}\right)^k f(z) = \frac{k!}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

for all $k \in \mathbb{N}_0$.

Corollary 3.1.2 (Derivative of a holomorphic function is holomorphic) Let $U \subseteq \mathbb{C}$ be open. If $f : U \to \mathbb{C}$ is holomorphic, then $f' : U \to \mathbb{C}$ is holomorphic.

Theorem 3.1.3 If ϕ is a continuous function on $\{\zeta : |\zeta - P| = r\}$, then the function f given by

$$f(z) := \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{\phi(\zeta)}{\zeta - z}\, d\zeta$$

is defined and holomorphic on $D(P, r)$.

This theorem induces a very strong way to create a holomorphic function. Instead of differentiability, we only need a continuous ϕ to build a holomorphic f.

Theorem 3.1.4 (Morera's Theorem) Let $U \subseteq \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be a continuous function on a connected open subset U of $\mathbb{C}$. Suppose that for every closed, piecewise C^1 curve $\gamma : [0, 1] \to U$, with $\gamma(0) = \gamma(1)$, we have

$$\oint_\gamma f(\zeta)\, d\zeta = 0$$

Then f is holomorphic on U.

Lemma 3.2.1 The sequence $\{a_k \in \mathbb{C}\}$ converges to a limit if and only if for each $\varepsilon > 0$ there is an N_0 such that $j, k \geq N_0$ implies that $|a_j - a_k| < \varepsilon$.

Definition 3.2.2 (Complex power series) Let $P \in \mathbb{C}$ be fixed. A *complex power series* (centered at P) is an expression of the form

$$\sum_{k=0}^{\infty} a_k (z - P)^k$$

where a_k for $k = 0, \ldots, \infty$ are complex constants.

Note that this power series expansion is only a formal expression. It may or may not converge.

A *necessary* condition for $\sum a_k (z - P)^k$ to converge is that $a_k (z - P)^k \to 0$.

Lemma 3.2.3 (Abel's Theorem) If $\sum_{k=0}^{\infty} a_k(z-P)^k$ converges at some z, then the series converges at each $w \in D(P,r)$, where $r = |z-P|$.

Definition 3.2.4 (Radius of convergence of power series) Let $\sum_{k=0}^{\infty} a_k(z-P)^k$ be a power series. Then

$$r := \sup\left\{ |w-P| \;\middle|\; \sum_{k=0}^{\infty} a_k(w-P)^k \text{ converges} \right\}$$

is called the *radius of convergence* of the power series. We will call $D(P,r)$ the disc of convergence.

Lemma 3.2.5 If $\sum_{k=0}^{\infty} a_k(z-P)^k$ is a power series with radius of convergence r, then the series converges for each $w \in D(P,r)$ and diverges for each w such that $|w-P| > r$.

Note that the convergence or divergence question for $|w-P| = r$ is left open.

Lemma 3.2.6 (Computing radius of convergence) Using

$$\ell : \limsup_{k \to +\infty} |a_k|^{1/k},$$

the radius of convergence r of the power series $\sum_{k=0}^{\infty} a_k(z-P)^k$ is given by

$$r = \begin{cases} 1/\ell & \text{if } \ell > 0 \\ +\infty & \text{if } \ell = 0 \end{cases}$$

Definition 3.2.7 (Uniform convergence of complex functions) A series $\sum_{k=0}^{\infty} f_k(z)$ of functions $f_k(z)$ converges uniformly on a set E to the function $g(z)$ if for each $\varepsilon > 0$ there is an N_0 such that if $N \geq N_0$, then

$$\left| g(z) - \sum_{k=0}^{N} f_k(z) \right| < \varepsilon \quad \forall z \in E$$

The point is that N_0 does not depend on $z \in E$: There is, for each ε, an N_0 depending on ε (but not on z) that works for all $z \in E$.

Definition 3.2.8 (Uniformly Cauchy series) Let $\sum_{k=0}^{\infty} f_k(z)$ be a series of functions on a set E. The series is said to be *uniformly Cauchy* if, for any $\varepsilon > 0$, there is a positive integer N_0 such that if $m \geq j \geq N_0$, then

$$\left| \sum_{k=j}^{m} f_k(z) \right| < \varepsilon \quad \forall z \in E$$

If a series is uniformly Cauchy on a set E, then it converges uniformly on E to some limit function. From this it follows that if $\sum |f_k(z)|$ is uniformly convergent, then $\sum f_k(z)$ is uniformly convergent (to some limit function).

Proposition 3.2.9 Let $\sum_{k=0}^{\infty} a_k(z-P)^k$ be a power series with radius of convergence r. Then, for any number R with $0 \leq R < r$, the series $\sum_{k=0}^{\infty} |a_k(z-P)^k|$ converges uniformly on $\overline{D}(P,R)$.

In particular, the series $\sum_{k=0}^{+\infty} a_k(z-P)^k$ converges uniformly and absolutely on $\overline{D}(P,R)$.

Lemma 3.2.10 If a power series

$$\sum_{j=0}^{\infty} a_j (z-P)^j \tag{$*$}$$

has a radius of convergence $r > 0$, then the series defines a C^∞ function $f(z)$ on $D(P,r)$. The function f is holomorphic on $D(P,r)$. The series obtained by termwise differentiation k times of $(*)$,

$$\sum_{j=k}^{\infty} \Big[j(j-1)\ldots(j-k+1)\Big] a_j (z-P)^{j-k}$$

converges on $D(P,r)$, and its sum is $[\partial/\partial z]^k f(z)$ for each $z \in D(P,r)$.

Proposition 3.2.11 If both series $\sum_{j=0}^{\infty} a_j(z-P)^j$ and $\sum_{j=0}^{\infty} b_j(z-P)^j$ converge on a disc $D(P,r)$, $r > 0$, and if

$$\sum_{j=0}^{\infty} a_j (z-P)^j = \sum_{j=0}^{\infty} b_j (z-P)^j$$

on $D(P,r)$, then $a_j = b_j$ for every j.

Theorem 3.3.1 (Power series of a holomorphic function) Let $U \subseteq \mathbb{C}$ be open and let f be holomorphic on U. Let $P \in U$ and suppose that $D(P,r) \subseteq U$. Then the complex power series

$$\sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k f}{\partial z^k}(P)\right](z-P)^k$$

has radius of convergence at least r. It converges to $f(z)$ on $D(P,r)$.

Theorem 3.4.1 (The Cauchy estimates) Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Let $P \in U$ and assume that the closed disc $\overline{D}(P,r)$, $r > 0$, is contained in U. Set

$$M := \sup_{z \in \overline{D}(P,r)} |f(z)|.$$

Then for $k = 1, 2, 3\ldots$ we have

$$\left|\frac{\partial^k f}{\partial z^k}(P)\right| \leq \frac{M k!}{r^k}$$

Lemma 3.4.2 Let $U \subseteq \mathbb{C}$ be open and connected and $f : U \to \mathbb{C}$ be holomorphic. If $\partial f/\partial z = 0$ on U, then f is constant on U.

Proof. Since f is holomorphic, $\partial f/\partial \bar{z} = 0$. But we have assumed that $\partial f/\partial z = 0$. Thus $\partial f/\partial x = \partial f/\partial y = 0$. So f is constant. $\square$

Theorem 3.4.3 (Liouville's Theorem) A bounded entire function is constant.

Proof. Let f be entire and assume that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix a $P \in \mathbb{C}$ and let $r > 0$. We apply the Cauchy estimate for $k = 1$ on $\overline{D}(P, r)$. The result is

$$\left| \frac{\partial f}{\partial z}(P) \right| \leq \frac{M}{r}$$

Since this inequality holds for all $r > 0$, we can blow it up to $+\infty$ and conclude that

$$\frac{\partial f}{\partial z}(P) = 0$$

But since P is arbitrary, we conclude that

$$\frac{\partial f}{\partial z} \equiv 0$$

By Lemma 3.4.2, the proof is complete. □

Theorem 3.4.4 If $f : \mathbb{C} \to \mathbb{C}$ is an entire function and if for some real number C and some positive integer k it holds that

$$|f(z)| \leq C|z|^k$$

for all $z \in \mathbb{C}$ with $|z| > 1$, then f is a polynomial in z of degree at most k.

Theorem 3.4.5 Let $p(z)$ be a non-constant (holomorphic) polynomial. Then p has a root. That is, there exists an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

This is in fact the fundamental theorem of algebra, and one of the most elegant applications of Liouville's Theorem.

Proof. Suppose there isn't an $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$. Then

$$g(z) := \frac{1}{p(z)}$$

is entire. Notice that as $|z| \to \infty$, $|p(z)| \to +\infty$. Thus $1/|p(z)| \to 0$ as $|z| \to \infty$ and hence g is bounded. By Liouville's Theorem, g is constant; hence p is constant. Contradiction! □

Corollary 3.4.6 If $p(z)$ is a holomorphic polynomial of degree k, then there are k complex numbers $\alpha_1, \ldots, \alpha_k$ (not necessarily distinct) and a non-zero constant C such that

$$p(z) = C(z - \alpha_1) \ldots (z - \alpha_k)$$

Theorem 3.5.1 Let $U \subseteq \mathbb{C}$ be an open set. Let $f_j : U \to \mathbb{C}$, $j = 1, 2, 3 \ldots$ be a sequence of holomorphic functions. Suppose that there is a function $f : U \to \mathbb{C}$ such that, for each compact subset E of U, the sequence $f_j|_E$ converges uniformly to $f|_E$. Then f is holomorphic on U. (In particular, $f \in C^\infty(U)$)

Corollary 3.5.2 If f_j, f, U are as defined in Theorem 3.5.1, then for any integer $k \in \{0, 1, 2 \ldots\}$ we have

$$\left(\frac{\partial}{\partial z}\right)^k f_j(z) \to \left(\frac{\partial}{\partial z}\right)^k f(z)$$

uniformly on compact sets.

Theorem 3.6.1 Let $U \subseteq \mathbb{C}$ be a connected open set and let $f : U \to \mathbb{C}$ be holomorphic. Let $\mathbf{Z} := \{z \in U \mid f(z) = 0\}$. If there is a $z_0 \in \mathbf{Z}$ and a sequence $\{z_j\} \subseteq \mathbf{Z} \setminus \{z_0\}$ such that $z_j \to z_0$, then $f \equiv 0$.

Corollary 3.6.2 Let $U \subseteq \mathbb{C}$ be connected and open, and $D(P, r) \subseteq U$. If f is holomorphic on U and $f|_{D(P,r)} \equiv 0$, then $f \equiv 0$ on U.

Note the strength of this statement. As long as f is holomorphic, if it's zero on just a tiny $D(P, r)$, then it is zero on the entire domain.

Corollary 3.6.3 Let $U \subseteq \mathbb{C}$ be connected and open. Let f, g be holomorphic on U. If $\{z \in U \mid f(z) = g(z)\}$ has an accumulation point in U, then $f \equiv g$.

Corollary 3.6.4 Let $U \subseteq \mathbb{C}$ be connected and open and let f, g be holomorphic on U. If $f \cdot g \equiv 0$ on U, then either $f \equiv 0$ on U or $g \equiv 0$ on U.

Corollary 3.6.5 Let $U \subseteq \mathbb{C}$ be connected and open and let f be holomorphic on U. If there is a $P \in U$ such that

$$\left(\frac{\partial}{\partial z}\right)^j f(P) = 0$$

for every j, then $f \equiv 0$.

Corollary 3.6.6 If f and g are entire holomorphic functions and if $f(x) = g(x)$ for all $x \in \mathbb{R} \subseteq \mathbb{C}$, then $f \equiv g$.

Definition 4.1.0 (Types of singularities) Let $U \subseteq \mathbb{C}$ be open and $P \in U$. Suppose that $f : U \setminus \{P\} \to \mathbb{C}$ is holomorphic.

There are three possibilities for the behavior of f near P:

(i) *(Removable singularity)* $|f(z)|$ is bounded on $D(P, r) \setminus \{P\}$ for some $r > 0$ with $D(P, r) \subseteq U$.

(ii) *(Pole)* $\lim_{z\to P} |f(z)| = +\infty$.

(iii) *(Essential singularity)* Neither (i) nor (ii) applies.

Theorem 4.1.1 (The Riemann removable singularities theorem) Let $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ be holomorphic and bounded. Then

(a) $\lim_{z\to P} f(z)$ exists

(b) the function $\hat{f} : D(P, r) \to \mathbb{C}$ defined by

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \neq P \\ \lim\limits_{\zeta \to P} f(\zeta) & \text{if } z = P \end{cases}$$

is holomorphic

Notice that, a priori, it is not even clear that $\lim_{z\to P} f(z)$ exists, or, even if it does, that the function $\hat{f}$ has any regularity at P beyond just continuity.

Theorem 4.1.4 (Casorati-Weierstrass) If $f : D(P, r_0) \setminus \{P\} \to \mathbb{C}$ is holomorphic and P is an essential singularity of f, then $f(D(P,r) \setminus \{P\})$ is dense in $\mathbb{C}$ for any $0 < r < r_0$.

Definition 4.2.0 (Laurent series) A Laurent series on $D(P, r)$ is a (formal) expression of the form

$$\sum_{j=-\infty}^{+\infty} a_j (z-P)^j$$

where j are integer indices.

We say that the infinite series $\sum_{j=-\infty}^{+\infty} \alpha_j$ converges if $\sum_{j=0}^{+\infty} \alpha_j$ and $\sum_{j=1}^{+\infty} \alpha_{-j}$ converge. In this case, we set

$$\sum_{j=-\infty}^{+\infty} \alpha_j = \left(\sum_{j=0}^{+\infty} \alpha_j\right) + \left(\sum_{j=1}^{+\infty} \alpha_{-j}\right)$$

This doubly infinite series converges to a complex number σ if and only if for each $\varepsilon > 0$ there is an $N > 0$ such that if $\ell \geq N$ and $k \geq N$, then $|\sigma - \sum_{j=-k}^{\ell} \alpha_j| < \varepsilon$.

It is important to realize that ℓ and k are independent here. In particular, the existence of the limit $\lim_{k\to+\infty} \sum_{j=-k}^{+k} \alpha_j$ does not imply in general that $\sum_{j=-\infty}^{+\infty} \alpha_j$ converges.

Lemma 4.2.1 This is the analogue for Laurent series for Lemma 3.2.3.

If $\sum_{j=-\infty}^{+\infty} a_j (z-P)^j$ converges at $z_1 \neq P$ and at $z_2 \neq P$ and if $|z_1 - P| < |z_2 - P|$, then the series converges for all z with $|z_1 - P| < |z - P| < |z_2 - P|$.

Lemma 4.2.2 This is the analogue for Laurent series for Lemma 3.2.5. Let

$$\sum_{j=-\infty}^{+\infty} a_j (z-P)^j$$

converge at (at least) one point z_0. There are unique non-negative numbers r_1 and r_2 (r_1 or r_2 may be $+\infty$) such that the series converges absolutely for all z with

$$r_1 < |z-P| < r_2$$

and diverges for all z with

$$|z-P| < r_1 \quad \text{or} \quad r_2 < |z-P|$$

Proposition 4.2.4 (Uniqueness of Laurent expansion) Let $0 \leq r_1 < r_2 \leq \infty$. If the Laurent series $\sum_{j=-\infty}^{+\infty} a_j (z-P)^j$ converges on $D(P, r_2) \setminus \overline{D}(P, r_1)$ to a function f, then, for any $r \in (r_1, r_2)$ and each $j \in \mathbb{Z}$, we have

$$a_j = \frac{1}{2\pi i} \oint_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - P)^{j+1}} \, d\zeta$$

In particular, the a_j's are uniquely determined by f.

Theorem 4.3.1 (The Cauchy integral formula for an annulus) Suppose that $0 \leq r_1 < r_2 \leq +\infty$ and that $f : D(P, r_2) \setminus \overline{D}(P, r_1) \to \mathbb{C}$ is holomorphic. Then, for each s_1, s_2 such that $r_1 < s_1 < s_2 < r_2$ and each $z \in D(P, s_2) \setminus \overline{D}(P, s_1)$, it holds that

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - P| = s_2} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - P| = s_1} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

Theorem 4.3.2 (The existence of Laurent expansions) If $0 \le r_1 < r_2 \le +\infty$ and $f : D(P, r_2) \setminus \overline{D}(P, r_1) \to \mathbb{C}$ is holomorphic, then there exist complex numbers a_j such that

$$\sum_{j=-\infty}^{+\infty} a_j (z-P)^j$$

converges on $D(P, r_2) \setminus \overline{D}(P, r_1)$ to f. If $r_1 < s_1 < s_2 < r_2$, then the series converges absolutely and uniformly on $D(P, s_2) \setminus \overline{D}(P, s_1)$.

Proposition 4.3.3 (Laurent expansion of holomorphic functions) If $f : D(P, r) \setminus \{P\} \to \mathbb{C}$ is holomorphic, then f has a unique Laurent series expansion.

$$f(z) = \sum_{j=-\infty}^{+\infty} a_j (z-P)^j$$

which converges absolutely for $z \in D(P, r) \setminus \{P\}$. The convergence is uniform on compact subsets of $D(P, r) \setminus \{P\}$. The coefficients are given by

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(P,s)} \frac{f(\zeta)}{(\zeta - P)^{j+1}}\, d\zeta$$

for any $0 < s < r$.

There are three mutually exclusive possibilities for the Laurent series of this proposition:

(i) $a_j = 0$ for all $j < 0$;

(ii) for some $k > 0$, $a_j = 0$ for all $-\infty < j < -k$

(iii) neither (i) nor (ii) applies

These three cases correspond exactly to the three types of isolated singularities: (i) $\iff$ P is a removable singularity; (ii) $\iff$ P is a pole; (iii) $\iff$ P is an essential singularity.

Proposition 4.4.1 Let f be holomorphic on $D(P, r) \setminus \{P\}$ and suppose that f has a pole of order k at P. Then the Laurent series coefficients a_j of f expanded about P, for $j = -k, -k+1, -k+2 \ldots$, are given by the formula

$$a_j = \frac{1}{(k+j)!} \left(\frac{\partial}{\partial z}\right)^{k+j} \left((z-P)^k \cdot f\right)\Big|_{z=P}$$

Definition 4.5.1 (Holomorphically simply connected (HSC)) An open set $U \subseteq \mathbb{C}$ is holomorphically simply connected if U is connected and if, for each holomorphic function $f : U \to \mathbb{C}$, there is a holomorphic function $F : U \to \mathbb{C}$ such that $F' \equiv f$.

Lemma 4.5.2 A connected open set U is holomorphically simply connected if and only if for each holomorphic function $f : U \to \mathbb{C}$ and each piecewise C^1 closed curve γ in U,

$$\oint_\gamma f(z)\, dz = 0.$$

Definition 4.5.3a (Residue of a function at a point) The **residue** of a function f at point P is denoted by $\operatorname{Res}_f(P)$, and is the coefficient of $(z-P)^{-1}$ in the Laurent expansion of f about P.

In particular, if f is holomorphic, then $\operatorname{Res}_f(P)$ is given by

$$\operatorname{Res}_f(P) = \frac{1}{2\pi i} \oint_\gamma f(\zeta)\, d\zeta$$

where γ is a counterclockwise simply closed curve around P and not including any other singularities inside the curve.

Theorem 4.5.3 (Residue theorem) Suppose that $U \subseteq \mathbb{C}$ is a HSC open set, and that $P_1, \ldots, P_n$ are distinct points of U. Suppose that $f : U \setminus \{P_1, \ldots, P_n\} \to \mathbb{C}$ is a holomorphic function and γ is a closed, piecewise C^1 curve in $U \setminus \{P_1, \ldots, P_n\}$.

Set R_j to be the coefficient of $(z-P_j)^{-1}$ in the Laurent expansion of f about P_j.

Then

$$\oint_\gamma f(z)\, dz = \sum_{j=1}^{n} R_j \cdot \left(\oint_\gamma \frac{1}{\zeta - P_j}\, d\zeta \right)$$

Using the notation of Res_f and $\operatorname{Ind}_\gamma$, we can rewrite this as

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_f(P_j) \cdot \operatorname{Ind}_\gamma(P_j)$$

Definition 4.5.4 (Index of a curve) Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 closed curve. Suppose $P \notin \gamma([a,b])$. Then the **index** of γ with respect to P, is defined as

$$\operatorname{Ind}_\gamma(P) := \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P}\, d\zeta$$

The index is also sometimes called the “winding number of γ about P”. As we will see later, $\operatorname{Ind}_\gamma(P)$ coincides with the number of times γ winds about P, counting orientation.

Lemma 4.5.5 (Index of a curve is an integer) Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 closed curve. Suppose P is a point not on the image of that curve, then

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - P}\, d\zeta \equiv \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - P}\, dt$$

is an integer.

Proposition 4.5.6 Let f be a function with a pole of order k at P. Then

$$\operatorname{Res}_f(P) = \frac{1}{(k-1)!} \left(\frac{\partial}{\partial z} \right)^{k-1} \left((z-P)^k f(z) \right) \Big|_{z=P}$$

Proof. This is the case $j = -1$ of Proposition 4.4.1. □

Definition 4.7.1 (Discrete sets) A set $S \in \mathbb{C}$ is discrete if and only if for each $z \in S$ there is a positive number r (depending on S and z) such that

$$S \cap D(z,r) = \{z\}$$

We also say in this circumstance that S consists of isolated points.

Definition 4.7.2 (Meromorphic functions) A meromorphic function f on an open set $U \subseteq \mathbb{C}$ with singular (as in "singularity") set S is a function $f : U \setminus S \to \mathbb{C}$ such that

(a) the set S is closed in U and is discrete,

(b) the function f is holomorphic on $U \setminus S$ (note that $U \setminus S$ is necessarily open in $\mathbb{C}$),

(c) for each $z \in S$ and $r > 0$ such that $D(z,r) \subseteq U$ and $S \cap D(z,r) = \{z\}$, the function

$$f|_{D(z,r)\setminus\{z\}}$$

has a (finite order) pole at z.

Lemma 4.7.3 (Reciprocal of a holomorphy with zeros is meromorphic) Let $U \subseteq \mathbb{C}$ be connected and open, and let $f : U \to \mathbb{C}$ be a holomorphic function with $f \not\equiv 0$, then the function

$$F : U \setminus \{z \mid f(z) = 0\} \to \mathbb{C}$$

defined by $F(z) = 1/f(z)$, is a meromorphic function on U with singular set (or pole set) equal to $\{z \in U \mid f(z) = 0\}$.

Definition 4.7.4 Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that for some $R > 0$, we have $\{z \in U : |z| > R\} \subseteq U$. Define $G : \{z \in U : 0 < |z| < 1/R\} \to \mathbb{C}$ by $G(z) = f(1/z)$. Then we say that

(i) f has a removable singularity at ∞ if G has a removable singularity at 0.

(ii) f has a pole at ∞ if G has a pole at 0.

(iii) f has an essential singularity at ∞ if G has an essential singularity at 0.

Remark 4.7.4a (Singularities and Laurent Expansions) Building on Definition 4.7.4, let the Laurent expansion of G around 0 be

$$G(z) = \sum_{-\infty}^{+\infty} a_n z^n$$

Then we have the *Laurent expansion of f around ∞* as

$$f(z) = G(\tfrac{1}{z}) = \sum_{-\infty}^{+\infty} a_{-n} z^n$$

(i) f has a removable singularity at ∞ if and only if the Laurent series has no positive powers of z with non-zero coefficients:

$$f(z) = \left(\sum_{-\infty}^{0} a_{-n} z^n \right) + 0z + 0z^2 + \dots$$

(ii) f has a pole at ∞ if and only if the series has only a finite number of positive powers of z with non-zero coefficients.

(iii) f has an essential singularity at ∞ if and only if the series has infinitely many positive powers.

Theorem 4.7.5 Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Then $\lim_{|z|\to\infty} |f(z)| = +\infty$ (i.e. f has a pole at ∞) if and only if f is a non-constant polynomial.

Then function f has a removable singularity at ∞ if and only if f is a constant.

Definition 4.7.6 Suppose that f is a meromorphic function defined on an open set $U \subseteq \mathbb{C}$ such that, for some $R > 0$, we have $\{z \in U : |z| > R\} \subseteq U$. We say that f is meromorphic at ∞ if the function $G(z) \equiv f(1/z)$ is meromorphic in the usual sense on $\{z : |z| < 1/R\}$.

Theorem 4.7.7 A meromorphic function f on $\mathbb{C}$ which is also meromorphic at ∞ must be a rational function (i.e. a quotient of polynomials in z). Conversely, every rational function is meromorphic on $\mathbb{C}$ and at ∞.

Remark It is conventional to rephrase the theorem by saying that the only functions that are meromorphic in the "extended plane" are rational functions.

Lemma 5.1.1 If f is holomorphic on a neighborhood of a disc $\overline{D}(z_0, r)$ and has a zero of order n at z_0 and no other zeros in the closed disc, then

$$\frac{1}{2\pi i} \oint_{\partial D(z_0,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = n$$

Proposition 5.1.2 Suppose that $f : U \to \mathbb{C}$ is holomorphic on an open set $U \subseteq \mathbb{C}$ and that $\overline{D}(P, r) \subseteq U$. Suppose that f is non-vanishing on $\partial D(P, r)$ and that $z_1, \ldots, z_k$ are the zeros of f in the interior of the disc. Let n_ℓ be the order of the zero of f at z_ℓ. Then

$$\frac{1}{2\pi i} \oint_{|\zeta-P|=r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{\ell=1}^{k} n_\ell$$

Lemma 5.1.3 This is the analogue of Lemma 5.1.1 for a pole.

If $f : U \setminus \{Q\} \to \mathbb{C}$ is a nowhere zero holomorphic function on $U \setminus \{Q\}$ with a pole of order n at Q and $\overline{D}(Q, r) \subseteq U$, then

$$\frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = -n$$

Theorem 5.1.4 (Argument principle for meromorphic functions) This is Lemma 5.1.1 and Lemma 5.1.3 put together.

Suppose f is a meromorphic function on an open set $U \subseteq \mathbb{C}$, that $\overline{D}(P, r) \subseteq U$ and that f has neither poles nor zeros on $\partial D(P, r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \sum_{j=1}^{p} n_j - \sum_{k=1}^{p} m_k$$

where $n_1, \ldots, n_p$ are the multiplicities of the zeros $z_1, \ldots, z_p$ of f in $D(P, r)$, and $m_1, \ldots, m_p$ are the orders of the poles $w_1, \ldots, w_p$ of f in $D(P, r)$.

Theorem 5.2.1 (The open mapping theorem) If $f : U \to \mathbb{C}$ is a non-constant holomorphic function on a connected open set U, then $f(U)$ is an open set in $\mathbb{C}$.

Theorem 5.2.2 Suppose that $f : U \to \mathbb{C}$ is a non-constant holomorphic function on a connected open set U such that $P \in U$ and $f(P) = Q$ with order k. Then there are numbers $\delta, \varepsilon > 0$ such that each $q \in D(Q, \varepsilon) \setminus \{Q\}$ has exactly k distinct preimages in $D(P, \delta)$ and each preimage a simple point of f.

Lemma 5.2.3 Let $f : U \to \mathbb{C}$ be a non-constant holomorphic function on a connected open set $U \subseteq \mathbb{C}$. Then the multiple points of f in U are isolated.

Proof. Since f is non-constant, the holomorphic function f' is not identically zero. But then Theorem 3.6.1 tells us that the zeros of f' are isolated. Since any multiple point p of f has the property that $f'(p) = 0$, it follows that the multiple points are isolated. □

Theorem 5.3.1 (Rouché's Theorem) Suppose that $f, g : U \to \mathbb{C}$ are holomorphic functions on an open set $U \subseteq \mathbb{C}$. Suppose also that $\overline{D}(P, r) \subseteq U$ and that, for each $\zeta \in \partial D(P, r)$,

$$|f(\zeta) - g(\zeta)| < |f(\zeta)| + |g(\zeta)| \tag{$*$}$$

Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta$$

That is, the number of zeros of f in $D(P, r)$ counting multiplicities equals the number of zeros of g in $D(P, r)$ counting multiplicities.

Remark Note that the inequality $(*)$ implies that neither $f(\zeta)$ nor $g(\zeta)$ can vanish on $\partial D(P, r)$. In particular, neither f nor g vanishes identically; moreover, the integral of f'/f and g'/g are defined on $\partial D(P, r)$.

Theorem 5.3.3 (Hurwitz's Theorem) Suppose that $U \subseteq \mathbb{C}$ is a connected open set and that $\{f_j\}$ is a sequence of nowhere vanishing holomorphic functions on U. If the sequence $\{f_j\}$ converges uniformly on compact subsets of U to a (necessarily holomorphic) limit function f, then either f is nowhere vanishing or $f \equiv 0$.

Definition 5.4.1 (Domains) A *domain* in $\mathbb{C}$ is a connected open set. A *bounded domain* is a connected open set U such that there is an $R > 0$ with $|z| < R$ for all $z \in U$.

Theorem 5.4.2 (The maximum modulus principle) Let $U \subseteq \mathbb{C}$ be a domain. Let f be a holomorphic function on U. If there is a point $P \in U$ such that $|f(z)| \leq |f(P)|$ for all $z \in U$, then f is constant.

Proof. Assume that there is such a P. If f is not constant, then $f(U)$ is open by the open mapping principle. Hence there are points ζ of $f(U)$ with $|f(P)| < |\zeta|$. This is a contradiction. Hence f is constant. □

Corollary 5.4.3 (Maximum modulus theorem) Let $U \subseteq \mathbb{C}$ be a bounded domain. Let f be a continuous function on $\overline{U}$ that is holomorphic on U. Then the maximum value of $|f|$ on $\overline{U}$ (which must occur, since $\overline{U}$ is closed and bounded) must occur on ∂U.

Proof. Since $|f|$ is a continuous function on the compact set $\overline{U}$, then it must attain its maximum somewhere.

If f is constant, then the maximum value of $|f|$ occurs at every point, in which case the conclusion clearly holds. If f is not constant, then the maximum value of $|f|$ on the compact set $\overline{U}$ cannot occur at $P \in U$, by Theorem 5.4.2, and hence the maximum occurs on ∂U. □

Theorem 5.4.4 (Maximum modulus theorem, alt) Let $U \subseteq \mathbb{C}$ be a domain and let f be a holomorphic function on U. if there is a point $P \in U$ at which $|f|$ has a local maximum, then f is constant.

Theorem 5.4.5 Let f be holomorphic on a domain $U \subseteq \mathbb{C}$. Assume that f never vanishes. If there is a point $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$, then f is constant.

Proof. Apply the maximum modulus principle to the function $g(z) = 1/f(z)$. □

Proposition 5.5.1 (Schwarz's Lemma) Let f be holomorphic on the unit disc. Assume that

1. $|f(z)| \leq 1$ for all z,
2. $f(0) = 0$.

Then $|f(z)| \leq |z|$ and $f'(0) \leq 1$.

If either $|f(z)| = |z|$ for some $z \neq 0$ or if $|f'(0)| = 1$, then f is a rotation: $f(z) \equiv \alpha z$ for some complex constant α of unit modulus.

Theorem 5.5.2 (Schwarz-Pick) Let f be holomorphic on the unit disc with $|f(z)| \leq 1$ for all $z \in D(0,1)$. Then, for any $a \in D(0,1)$ and with $b \equiv f(a)$, we have the estimate

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}$$

Moreover, if $f(a_1) = b_1$ and $f(a_2) = b_2$, then

$$\left| \frac{b_2 - b_1}{1 - \bar{b}_1 b_2} \right| \leq \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|$$

Definition 6.0.1 (Conformal/biholomorphic maps) Let U, V be open subsets of $\mathbb{C}$. Let $f : U \to V$ be holomorphic and bijective. Such a function f is called a *conformal* or *biholomorphic* mapping. $h^{-1} : V \to U$ is necessarily holomorphic.

Definition 6.0.2 (Conformal equivalence) Let $U, V \subseteq \mathbb{C}$. We say that U, V are conformally equivalent if there exists a conformal mapping f from U to V.

Theorem 6.1.1 A function $f : \mathbb{C} \to \mathbb{C}$ is a conformal mapping if and only if there are complex numbers a, b with $a \neq 0$ such that

$$f(z) = az + b, \quad z \in \mathbb{C}$$

Lemma 6.1.2 The holomorphic function f satisfies

$$\lim_{|z|\to\infty} |f(z)| = \infty$$

That is, given $\varepsilon > 0$, there is a number $C > 0$ sudh that if $|z| > C$, then $|f(z)| > 1/\varepsilon$.

Lemma 6.1.3 There are numbers $B, D > 0$ such that if $|z| > D$, then

$$|f(z)| < B|z|$$

Lemma 6.2.1 A holomorphic function $f : D \to D$ that satisfies $f(0) = 0$ is a conformal mapping of D onto itself if and only if there is a complex number ω with $|\omega| = 1$ such that

$$f(z) \equiv \omega z, \quad \forall z \in D$$

In other words, a conformal self-map of the disc that fixes the origin must be a rotation.

Lemma 6.2.2 (Construction of Möbius transformation) For $a \in \mathbb{C}$ with $|a| < 1$, we define

$$\phi_a(z) := \frac{z - a}{1 - \bar{a}z}$$

Then each ϕ_a is a conformal self-map of the unit disc.

Theorem 6.2.3 Let $f : D \to D$ be a holomorphic function. Then f is a conformal self-map of D if and only if there are complex numbers a, ω with $|\omega| = 1$, $|a| < 1$ such that for all $z \in D$,

$$f(z) = \omega \cdot \phi_a(z)$$

Definition 6.3.1 (Linear fractional transformations) A function of the form

$$z \mapsto \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

is called a *linear fractional transformation.*

Definition 6.3.2 (Linear fractional transformations over the extended plane) A function $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a linear fractional transformation if there exist $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, such that either

(i) $c = 0$, $f(\infty) = \infty$, and $f(z) = (a/d)z + (b/d)$ for all $z \in \mathbb{C}$

(ii) $c \neq 0$, $f(\infty) = a/c$, $f(-d/c) = \infty$, and $f(z) = (az + b)/(cz + d)$ for all $z \in \mathbb{C}, z \neq -d/c$

Definition 6.3.3 A sequence $\{p_k\}$ in $\mathbb{C} \cup \{\infty\}$ converges to $p \in \mathbb{C} \cup \{\infty\}$ if either

1. $p = \infty$ and $\lim_{k\to+\infty} |p_k| = +\infty$ where the limit in this expression is taken for all k such that $p_k \in \mathbb{C}$; or

2. $p \in \mathbb{C}$, all but a finite number of the p_k are in $\mathbb{C}$, and $\lim_{k\to+\infty} |p_k| = p$ in the usual sense of convergence in $\mathbb{C}$.

Theorem 6.3.4 If $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a linear fractional transformation, then f is a bijective continuous function. Its inverse is also a linear fractional transformation.

Theorem 6.3.5 A function ϕ is a conformal self-mapping of $\mathbb{C} \cup \{\infty\}$ to itself if and only if ϕ is linear fractional.

Theorem 6.3.6 (The inverse Cayley transform) The linear fractional transformation $z \mapsto (z - i)/(z + i)$ maps the upper half plane $\{z : \operatorname{Im} z > 0\}$ conformally to the unit disc $D = \{z : |z| < 1\}$.

Theorem 6.3.7 Let $\mathcal{C}$ be the set of subsets of $\mathbb{C} \cup \{\infty\}$ consisting of **(i)** circles and **(ii)** sets of the form $L \cup \{\infty\}$ where L is a line in $\mathbb{C}$. Then every linear fractional transformation ϕ takes elements of $\mathcal{C}$ to elements of $\mathcal{C}$.

Sets consisting of $L \cup \{\infty\}$ are thought of as "generalized circles". Thus the theorem says that linear fractional transformations take circles to circles, in the generalized sense of the word.

Definition 6.4.1 (Homeomorphisms) Two open sets U and V in $\mathbb{C}$ are *homeomorphic* if there is a bijective continuous function $f : U \to V$ with $f^{-1} : V \to U$ also continuous. Such a function f is called a *homeomorphism* from U to V.

Theorem 6.4.2 (Riemann mapping theorem) If $U \subsetneq \mathbb{C}$ is open, and U is homeomorphic to D, then U is conformally equivalent to the unit disc D.

Definition 6.5.1 A sequence of functions f_j on an open set $U \subseteq \mathbb{C}$ is said to converge normally to a limit function f on U if $\{f_j\}$ converges to f uniformly on compact subsets of U.

That is, convergence is normal if for each compact set $K \subseteq U$ and each $\varepsilon > 0$, there is a $J > 0$ (depending on K and ε) such that

$$j \geq J \implies |f_j(z) - f(z)| < \varepsilon \quad \forall z \in K.$$

Theorem 6.5.3 (Montel's theorem, first version) Let $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ be a family of holomorphic functions on an open set $U \subseteq \mathbb{C}$. Suppose that there is a constant $M > 0$ such that, for all $z \in U$, and all $f_\alpha \in \mathcal{F}$,

$$|f_\alpha(z)| \leq M$$

Then, for every sequence $\{f_j\} \subseteq \mathcal{F}$, there is a subsequence $\{f_{j_k}\}$ the converges normally on U to a limit (holomorphic) function f.

Definition 6.5.4 Let $\mathcal{F}$ be a family of functions on an open set $U \subseteq \mathbb{C}$. We say that $\mathcal{F}$ is *bounded on compact sets* if for each compact set $K \subseteq U$ there is a constant $M = M_K$ such that, for all $f \in \mathcal{F}$ and all $z \in K$,

$$|f(z)| \leq M$$

Theorem 6.5.5 (Montel's theorem, second version) Let $U \subseteq \mathbb{C}$ be an open set and let $\mathcal{F}$ be a family of holomorphic functions on U that is bounded on compact sets. Then for every sequence $\{f_j\} \subseteq \mathcal{F}$ there is a subsequence $\{f_{j_k}\}$ that converges normally on U to a limit (necessarily holomorphic) function f.

Proposition 6.5.7 Let $U \subseteq \mathbb{C}$ be any open set. Fix a point $P \in U$. Let $\mathcal{F}$ be a family of holomorphic functions from U into the unit disc D that take P to 0. Then there is a holomorphic function $f_0 : U \to D$ that is the normal limit of a sequence $\{f_j\}$, $f_j \in \mathcal{F}$, such that

$$|f_0'(P)| \geq |f'(P)|, \quad \forall f \in \mathcal{F}$$

Theorem 6.6.3 (Riemann mapping theorem: analytic form) If U is a HSC open set in $\mathbb{C}$, and $U \neq \mathbb{C}$, then U is conformally equivalent to the unit disc.

Lemma 6.6.4 (The holomorphic logarithm lemma) Let U be a HSC open set. If $f : U \to \mathbb{C}$ is holomorphic and nowhere zero on U, then there exists a holomorphic function h on U such that

$$e^h \equiv f \quad \text{on } U$$

Corollary 6.6.5 If U is HSC and $f : U \to \mathbb{C} \setminus \{0\}$ is holomorphic, then there is a function $g : U \to \mathbb{C} \setminus \{0\}$ such that

$$f(z) = [g(z)]^2$$

for all $z \in U$.

Theorem 20.1.1 (Weierstrass M-test) Suppose that $\{f_n\}$ is a sequence of functions with $f_n : U \to \mathbb{C}$ for some open set $U \subseteq \mathbb{C}$. Let there be a sequence of non-negative numbers $\{M_n\}$ such that

- $|f_n(x)| \leq M_n$ for all $n \geq 1$, $x \in U$, and
- $\sum_{n=1}^{\infty} M_n$ converges.

Then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges absolutely and uniformly on U (to a limit function f).

Definition 20.1.3 (Analytic functions) Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$. f is said to be analytic at P if in some open disc centered at P it can be expanded as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - P)^n$$

(this implies that the radius of convergence is positive)

Theorem 20.1.4 (Holomorphic functions are analytic) f is holomorphic at $z_0 \in \mathbb{C}$ if and only it is analytic at z_0.

Corollary 20.1.5 (Singularities and radius of convergence) Corollary of this theorem.

The radius of convergence at a point P is the distance between P and the nearest non-removable singularity.

If there are no singularities (such as when f is an entire function), then the radius of convergence is infinite.

Nonlinear Optimization, Part I: Unconstrained Optimization

Definition 1.1.2 (The argmin set) The set which minimizes values of f over a domain X is denoted by

$$\underset{x \in X}{\operatorname{argmin}} f := \left\{ x \in X \,\middle|\, f(x) = \inf_X f \right\}$$

Definition 1.1.3 (Local vs. global minima) Let $X \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. Then $\bar{x} \in X$ is called a

- global minimizer of f over X if $\bar{x} \in \operatorname{argmin}_X f$, i.e. $\forall x \in X : f(\bar{x}) \leq f(x)$
- local minimizer of f over X if $\exists\, \varepsilon > 0$ such that $\forall x \in X \cap B_\varepsilon(\bar{x}) : f(\bar{x}) \leq f(x)$

For strict global/local minimizers, the above conditions hold with strict inequality.

Definition 1.1.4 (Level sets and Lower level sets) For $f : \mathbb{R}^n \to \mathbb{R}$ the level set for the level $c \in R$ is given by

$$f^{-1}(\{c\}) = \{x \in \mathbb{R}^n \mid f(x) = c\}$$

The lower level set (or *sublevel* set) of f to the level $c \in \mathbb{R}$ is

$$\operatorname{lev}_c f := f^{-1}((-\infty, c]) = \{x \in \mathbb{R}^n \mid f(x) \leq c\}$$

Proposition 1.1.5 Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. If $\exists\, c \in \mathbb{R}$ such that $\operatorname{lev}_c f$ is non-empty and bounded then f takes its minimum over $\mathbb{R}^n$.

Definition 1.2.1 (Convex sets) A set $C \subset \mathbb{R}^n$ is called convex if

$$\lambda x + (1-\lambda) y \in C \quad \forall (x, y \in C,\ \lambda \in (0,1))$$

or simply a set which contains all connecting lines of points from the set.

Definition 1.2.3 (Convex functions) Let $C \subset \mathbb{R}^n$ be convex. Let $\lambda \in (0,1)$ and $x, y \in C$ and let

Then $f : C \to \mathbb{R}$ is said to be

- convex on C if
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$
- strictly convex on C if
$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$
- strongly convex on C if $\exists\, \mu > 0$ such that
$$f(\lambda x + (1-\lambda)y) + \frac{\mu}{2}\lambda(1-\lambda) \|x-y\|^2 \leq \lambda f(x) + (1-\lambda) f(y)$$

This has parallels here and here.

Example 1.2.5 (Convex functions)

(a) $\exp : \mathbb{R} \to \mathbb{R}$ and $-\log : (0, \infty) \to \mathbb{R}$ are convex.

(b) *(Affine functions)* $f : \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$f(x) = Ax - b \quad (A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m)$$

is called **affine** (linear). All affine functions, hence all linear functions ($b = 0$) $\mathbb{R}^n \to \mathbb{R}$ are convex.

(c) *(Norms)* Any norm $\|\cdot\|$ on $\mathbb{R}^n$ is convex.

Proposition 1.2.6 (Convexity preserving operations)

1. (Positive combinations) For $i = 1, \ldots, n$ let $f_i : \mathbb{R}^n \to \mathbb{R}$ be convex and $\lambda_i \geq 0$. Then $\sum_{i=1}^{n} \lambda_i f_i$ is convex.
2. (Composition with affine mapping) $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $g : \mathbb{R}^m \to \mathbb{R}^n$ affine. Then $f \circ G$ is convex.

Theorem 1.2.7 (Taylor's Theorem, $k = 2$) Let $D \subset \mathbb{R}^n$ be open, let $f : D \to \mathbb{R}$ be twice continuously differentiable, and $x, y \in D$ such that $[x, y] \subset D$. Then there exists $\eta \in [x, y]$ such that

$$f(y) - f(x) = \nabla f(x)^T (y - x) + \frac{1}{2} (y - x)^T \nabla^2 f(\eta)(y - x)$$

Theorem 1.2.8 (Schwarz's Theorem) Let $D \in \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ be twice continuously differentiable at $x \in D$. Then $\nabla^2 f(x)$ is symmetric.

Definition 2.1.1 (Directional derivative) Let $D \subset \mathbb{R}$ be open. $f : D \to \mathbb{R}$ is directionally differentiable at $\bar{x} \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ if

$$\lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(x)}{t}$$

exists. This limit is denoted by $f'(x; d)$ and is called the directional derivative of f at $\bar{x}$ in the direction d.

If f is directionally differentiable at $\bar{x}$ in every direction $d \in \mathbb{R}^n$, we call f directionally differentiable at $\bar{x}$.

If f is directionally differentiable at every $\bar{x} \in \mathbb{R}^n$, we call it directionally differentiable.

Lemma 2.1.2 (Directional derivative and gradient) Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ differentiable at $x \in D$. Then f is directionally differentiable at x with

$$f'(x; d) = \nabla f(x)^T d \quad \forall (d \in \mathbb{R}^n)$$

Where $f'(x; d)$ is the directional derivative of f at x in the direction d.

Lemma 2.1.4 (Basic optimality condition) Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$. If $\bar{x}$ is a local minimizer of f over X and f is directionally differentiable at $\bar{x}$ then

$$f'(x; d) \geq 0 \quad \forall (d \in \mathbb{R}^n)$$

Theorem 2.1.5 (Fermat's rule) Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ differentiable at $\bar{x} \in X$. If $\bar{x}$ is a local minimizer (or maximizer) of f over X then $\nabla f(\bar{x}) = 0$.

Theorem 2.1.6 (Second-order necessary condition) Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ twice continuously differentiable. If $\bar{x}$ is a local minimizer (maximizer) of f over X then $\nabla^2 f(\bar{x})$ is positive (negative) semidefinite.

Proof. We only prove the case in which $\bar{x}$ is a local minimizer. The maximum case follows from it by substituting f for $-f$.

Assume on the contrary that $\nabla^2 f(\bar{x})$ were not positive semidefinite. Then there exists $d \in \mathbb{R}^n$ such that $d^T \nabla^2 f(\bar{x}) d < 0$. By Taylor's Theorem, for $t > 0$ there exists $\eta_t \in [\bar{x}, \bar{x} + td]$ such that

$$f(\bar{x} + td) = f(\bar{x}) + t \nabla f(\bar{x})^T d + \frac{t^2}{2} d^T \nabla^2 f(\eta_t) d$$

But since $\bar{x}$ is a local minimizer, by Fermat's rule we have $\nabla f(\bar{x}) = 0$ and hence

$$f(\bar{x} + td) = f(\bar{x}) + \frac{t^2}{2} d^T \nabla^2 f(\eta_t) d$$

As $t \downarrow 0$, $\eta_t \to \bar{x}$ and hence for some $t > 0$ sufficiently small, we have

$$\frac{t^2}{2} d^T \nabla^2 f(\eta_t) d < 0$$

as $\nabla^2 f$ is continuous by assumption. This yields $f(\bar{x} + td) < f(\bar{x})$ for all $t > 0$ sufficiently small, which contradicts the fact that $\bar{x}$ is a local minimizer of f over X. Hence, $\nabla^2 f(\bar{x})$ must be positive semidefinite. □

Lemma 2.1.7 Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ twice continuously differentiable. If $\bar{x} \in \mathbb{R}^n$ is such that $\nabla^2 f(\bar{x})$ positive definite then $\exists\, \mu, \varepsilon > 0$ such that $B_\varepsilon(\bar{x}) \subset X$ and

$$d^T \nabla^2 f(x) d \geq \mu \left\| d \right\|_2^2 \quad \forall (d \in \mathbb{R}^n,\ x \in B_\varepsilon(\bar{x}))$$

Theorem 2.1.8 (Sufficient optimality condition) Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$ twice continuously differentiable. Moreover, let $\bar{x}$ be a stationary point of f such that $\nabla^2 f(\bar{x})$ is positive definite. Then $\bar{x}$ is a strict local minimizer of f.

Theorem 2.2.1 (First-order characterizations) Let $C \subset \mathbb{R}^n$ be open and convex and let $f : C \to \mathbb{R}$ be differentiable on C. Then the following hold for all $x, \bar{x} \in C$:

(a) f is convex on C iff
$$f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) \tag{$*$}$$

(b) f is strictly convex on C iff $(*)$ holds with strict inequality whenever $x \neq \bar{x}$.

(c) f is strongly convex with modulus $\mu > 0$ on C iff
$$f(x) \geq \left[f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x}) \right] + \frac{\mu}{2} \left\| x - \bar{x} \right\|^2$$

This has parallels here and here.

Corollary 2.2.2 Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Then the following hold:

(a) *(Affine minorization)* There exists an affine function $g : \mathbb{R}^n \to \mathbb{R}$ which minorizes f everywhere, i.e.

$$g(x) \leq f(x) \quad (x \in \mathbb{R}^n)$$

(b) If f is strongly convex then it is strictly convex and coercive (level-bounded).

Corollary 2.2.3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Then the following are equivalent:

(i) $\bar{x}$ is a global minimizer of f, i.e. $\bar{x} \in \operatorname{argmin} f$;

(ii) $\bar{x}$ is a local minimizer;

(iii) $\bar{x}$ is a stationary point of f.

Proof. (i) $\implies$ (ii) is obvious. (ii) $\implies$ (iii) follows from Fermat's Theorem. (iii) $\implies$ (i) follows from Theorem 2.2.1 (a). □

Corollary 2.2.4 (Monotonocity of gradient mappings) Let $C \subset \mathbb{R}^n$ be open and convex and let $f : C \to \mathbb{R}$ be differentiable on C. Then the following hold for all $x, y \in C$

(a) f is convex on C iff

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0 \tag{$*$}$$

(b) f is strictly convex on C iff ($*$) holds with a strict inequality whenever $x \neq y$.

(c) f is strongly convex with modulus $\mu > 0$ on C iff

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu \left\| x - y \right\|^2$$

This has parallels here and here.

Theorem 2.2.5 (Twice differentiable convex functions) Let $C \subset \mathbb{R}^n$ be open and convex and let $f : C \to \mathbb{R}$ be twice continuously differentiable on C. Then the following hold:

(a) f is convex on C iff $\nabla^2 f(x)$ is positive semidefinite $\forall x \in C$.

(b) If $\nabla^2 f(x)$ is positive definite $\forall x \in C$ then f is strictly convex on C.

(c) f is strongly convex with modulus $\mu > 0$ on C iff, $\forall x \in C$, the smallest eigenvalue of $\nabla^2 f(x)$ is bounded by μ from below.

Corollary 2.2.6 (Convexity of quadratic functions) Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $b \in \mathbb{R}^n$, $\gamma \in \mathbb{R}$, and define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \frac{1}{2} x^T A x + b^T x + \gamma$$

Then the following hold:

(a) f is convex if and only if A is positive semidefinite

(b) f is strongly convex if and only if A is positive definite

Proof. In view of Theorem 2.2.5, it suffices to show that f is twice continuously differentiable.

$$\nabla f(x) = Ax + b$$
$$\nabla^2 f(x) = A$$

and we are done. $\square$

Theorem 2.2.7 (Convex optimization) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $X \subset \mathbb{R}^n$ be a non-empty convex set. Consider the convex optimization problem

$$\min f(x) \quad \text{s.t.} \quad x \in X \tag{$*$}$$

Then the following hold:

(a) A point $\bar{x}$ is a global minimizer of $(*)$ if and only if it is a local minimizer of $(*)$.

(b) The solution set $\operatorname{argmin}_X f$ of $(*)$ is convex (possibly empty).

(c) If f is strictly convex, then the solution set has at most one element.

(d) If f is strongly convex and differentiable and X is closed, then $(*)$ has exactly one solution ($\operatorname{argmin}_X f$ is a singleton).

Proposition 2.3.1 (Operator norms) Let $\|\cdot\|_*$ be a (vector) norm on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then for $A \in \mathbb{R}^{m\times n}$,

$$\|A\|_* := \sup_{x\neq 0} \frac{\|Ax\|_*}{\|x\|_*}$$

is a norm on $\mathbb{R}^{m\times n}$ with

$$\|A\|_* = \sup_{\|x\|_*=1} \|Ax\|_* = \sup_{\|x\|_*\leq 1} \|Ax\|_*$$

Proposition 2.3.2 Let $A \in \mathbb{R}^{m\times n}$. Then we have

$$\begin{aligned}
\|A\|_1 &= \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(maximum absolute column sum)}\\
\|A\|_2 &= \sqrt{\lambda_{\max}(A^T A)} \quad \text{(spectral norm)}\\
\|A\|_\infty &= \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(maximum absolute row sum)}
\end{aligned}$$

Proposition 2.3.3 Let $\|\cdot\|_*$ be a norm on $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$ respectively. Then for all $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ the following hold:

- $\|Ax\|_* \leq \|A\|_* \|x\|_*$ for all $x \in \mathbb{R}^n$ *(compatibility)*
- $\|AB\|_* \leq \|A\|_* \|B\|_*$ *(submultiplicativity)*

Proposition 2.3.4 (Banach Lemma) Let $C \in \mathbb{R}^{n\times n}$ with $\|C\| < 1$ where $\|\cdot\|$ is a submultiplicative matrix norm. Then $I + C$ is invertible and we have

$$\left\|(I+C)^{-1}\right\| \leq \frac{1}{1-\|C\|}$$

Corollary 2.3.5 Let $A, B \in \mathbb{R}^{n\times n}$ with $\|I - BA\| < 1$ for some submultiplicative norm $\|\cdot\|$. Then A and B are invertible with

$$\|B^{-1}\| \leq \frac{\|A\|}{1 - \|I - BA\|}$$

Definition 3.1.1 (Descent direction) Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$. A vector $d \in \mathbb{R}^n$ is said to be a *descent direction* of f at x if there exists $\ell > 0$ such that

$$f(x + td) < f(x) \quad (t \in (0, \ell])$$

Proposition 3.1.2 (Sufficient condition for descent direction) Let $f : \mathbb{R}^n \to \mathbb{R}$ be directionally differentiable at $x \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ with

$$f'(x; d) < 0$$

Then d is a descent direction of f at x. In particular, this is true if f is differentiable at x with

$$\nabla f(x)^T d < 0$$

Proof. The first statement follows immediately from the definition of the directional derivative. The second one uses Lemma 2.1.2. □

Corollary 3.1.3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable, $B \in \mathbb{R}^{n\times n}$ be positive definite and $x \in \mathbb{R}^n$ with $\nabla f(x) \neq 0$. Then $-B\nabla f(x)$ is a descent direction of f at x.

Proof. This result follows almost immediately from the definition of a descent direction and the definition of a positive definite matrix. □

Definition 3.1.4 (Step-size rule) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and let $\mathcal{A}_f := \{(x, d) \mid \nabla f(x)^T d < 0\}$. A set-valued mapping

$$T : (x, d) \in \mathcal{A}_f \mapsto T(x, d) \subset \mathbb{R}_{++}$$

is called a step-size rule for f.

We call it well-defined for f if $T(x, d) \neq 0$ for all $(x, d) \in \mathcal{A}_f$.

If the step-size rule is well-defined for all continuously differentiable functions $\mathbb{R}^n \to \mathbb{R}$, we simply call it well-defined.

Definition 3.1.5 (Efficient step-size) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. The step-size rule T is called efficient for f if there exists $\theta > 0$ such that

$$f(x + td) \leq f(x) - \theta \left(\frac{\nabla f(x)^T d}{\|d\|}\right)^2$$

Theorem 3.1.6 (Global convergence of general descent method) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and let $\{x^k\}$, $\{d^k\}$, $\{t_k\}$ be generated by Algorithm 3.1.1. Moreover, assume that the following hold:

(i) *(Angle condition)* There exists $c > 0$ such that

$$-\frac{\nabla f(x^k)^T d^k}{\|\nabla f(x^k)\| \cdot \|d^k\|} \geq c \quad \forall k \in \mathbb{N}$$

i.e. the angle between the gradient vector and the descent direction is at most 90°.

(ii) *(Efficient step-size)* There exists $\theta > 0$ such that

$$f(x^k + t_k d^k) \leq f(x^k) - \theta \left(\frac{\nabla f(x^k)^T d^k}{\|d^k\|} \right)^2 \quad \forall k \in \mathbb{N}$$

Then every cluster point of $\{x^k\}$ is a stationary point of f.

Proof. By (ii), there exists $\theta > 0$ such that

$$f(x^{k+1}) = f(x^k + t_k d^k) \leq f(x^k) - \theta \left(\frac{\nabla f(x^k)^T d^k}{\|d^k\|} \right)^2 \quad \forall k \in \mathbb{N}$$

Putting $k := c^2\theta$, the angle condition implies

$$f(x^{k+1}) \leq f(x^k) - k \left\| \nabla f(x^k) \right\|^2 \tag{$*$}$$

Let $\bar{x}$ be a cluster point of $\{x^k\}$. As $\{f(x^k)\}$ is monotonically decreasing and convergent to $f(\bar{x})$ on a subsequence (since $\{x^k\} \to \bar{x}$ on a subsequence and f is continuous), this implies that the whole sequence $\{f(x^k)\}$ converges to $f(\bar{x})$.

In particular, we have

$$f(x^{k+1}) - f(x^k) \to 0$$

Therefore, $(*)$ implies $\left\| \nabla f(x^k) \right\| \to 0$ by squeezing. □

Definition 3.2.0 (Armijo rule and sufficient decrease) Choose $\beta, \sigma \in (0,1)$. For $x, d \in \mathcal{A}_f$ the Armijo rule T_A is defined by

$$T_A(x,d) = \max_{l \in \mathbb{N}_0} \{ \beta^l \mid f(x^k + \beta^l d^k) \leq f(x^k) + \beta^l \sigma \nabla f(x^k)^T d^k \}$$

The inequality

$$f(x^k + \beta^l d^k) \leq f(x^k) + \beta^l \sigma \nabla f(x^k)^T d^k$$

is called the *Armijo condition*. It ensures a **sufficient decrease** on the objective function.

Example 3.2.1 (Insufficient decrease) Consider the function $f(x) = (x-1)^2 - 1$ with optimal value $f^* = -1$.

The sequence $\{x_k\}$ with $x_k := -\frac{1}{k}$ has $f(x_k) = \frac{1+2k}{k^2}$ and

$$f(x_{k+1}) - f(x_k) = \frac{2k^2 + 4k + 1}{k^2(k+1^2)} < 0$$

Hence we've found a case where the objective value decreases, but $f(x_k)$ converges to a non-optimal value. ($f(x_k) \to 0$, but we want $f(x_k) \to -1$)

Lemma 3.2.3 (Convergence to gradient) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Moreover, let $\{x^k \in \mathbb{R}^n\} \to x$, $\{d^k \in \mathbb{R}^n\} \to d$ and $\{t_k > 0\} \downarrow 0$. Then

$$\lim_{k\to\infty} \frac{f(x^k + t_k d^k) - f(x^k)}{t_k} = \nabla f(x)^T d$$

Proof. By the mean value theorem, for all $k \in \mathbb{N}$, there exists $\eta^k \in [x^k, x^k + t_k d^k]$ such that

$$f(x^k + t_k d^k) - f(x^k) = t_k \nabla f(\eta^k)^T d^k$$

Clearly, $\eta^k \to x$ and hence the continuity of ∇f yields

$$\nabla f(\eta^k)^T d^k \to \nabla f(x)^T d$$

This readily implies

$$\lim_{k\to\infty} \frac{f(x^k + t_k d^k) - f(x^k)}{t_k} = \lim_{k\to\infty} \nabla f(\eta^k)^T d^k = \nabla f(x)^T d$$

□

Theorem 3.2.4 (Global convergence of the gradient method) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Then every cluster point of a sequence generated by the Gradient method with Armijo rule is a stationary point of f.

Proof. Assume on the contrary that $\nabla f(\bar{x}) \neq 0$.

Let $\bar{x}$ be a cluster point of the generated sequence $\{x^k\}$, and let $\{x^k\}_K$ be a subsequence converging to $\bar{x}$. By the continuity of f, $\{f(x^k)\}_K \to f(\bar{x})$.

As $\{f(x^k)\}$ is monotonically decreasing by the Armijo condition and converges on a subsequence to $f(\bar{x})$, by inspection, $\{f(x^k)\}_{\mathbb{N}}$ converges to $f(\bar{x})$.

In particular, we have

$$f(x^k) - f(x^{k+1}) \to 0$$

Substituting $t_k = \beta^l$ and $x^{k+1} = x^k + \beta^l d^k$ into steps **(S2)** and **(S3)** of the algorithm, we have

$$0 \leq t_k \left\|\nabla f(x^k)\right\|^2 = -t_k \nabla f(x^k)^T d^k \leq \frac{f(x^k) - f(x^{k+1})}{\sigma} \to 0$$

Since $\{\nabla f(x^k)\}_K \to \nabla f(\bar{x}) \neq 0$ (by continuity of ∇f), by squeeze theorem on the above inequality, this implies that $\{t_k\}_K \to 0$. Due to **(S3)**, for all $k \in K$ sufficiently large, we have

$$f(x^k + \beta^{l_k - 1} d^k) - f(x^k) > \beta^{l_k - 1} \sigma \nabla f(x^k)^T d^k \tag{$*$}$$

where $\beta^{l_k} = t_k$ and $l_k \in \mathbb{N}$ is the exponent *uniquely* determined by the Armijo rule in **(S3)**. Note that l_k is the smallest value of l that satisfies the Armijo condition, and hence $l_k - 1$ does *not* satisfy the Armijo condition, hence $(*)$.

Passing to the limit on K and using Lemma 3.2.3 gives

$$-\left\|\nabla f(\bar{x})\right\|^2 \geq -\sigma \left\|\nabla f(\bar{x})\right\|^2$$

Which is a contradiction because $\sigma \in (0, 1)$ and $\nabla f(\bar{x}) \neq 0$ by assumption. Hence, $\bar{x}$ is indeed a stationary point of f, completing the proof. □

Proposition 3.2.7 (Kantorovich inequality) Let $A \in \mathbb{R}^{n\times n}$ symmetric positive definite. Then

$$\frac{4\lambda_{\min}\lambda_{\max}}{(\lambda_{\min}+\lambda_{\max})^2} \le \frac{\|d\|^4}{(d^TAd)(d^TA^{-1}d)} \quad \forall (d \in \mathbb{R}^n \setminus \{0\})$$

Theorem 3.2.8 (Gradient method for strongly convex quadratics) Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{2}x^TAx + b^Tx$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $b \in \mathbb{R}^n$. Let $\bar{x} := -A^{-1}b$ be the (unique) global minimizer of f. Assume that $\{x^k\}$ is generated by the gradient method from Algorithm 3.2.1. Then the following hold.

(a) *(Convergence of function values)*

$$f(x^{k+1}) - f(\bar{x}) \le \left(\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}\right)^2 (f(x^k) - f(\bar{x})) \quad \forall (k \in \mathbb{N})$$

i.e. the sequence $\{f(x^k)\}$ converges linearly to $f(\bar{x})$.

(b) *(Convergence of variables)*

$$\|x^k - \bar{x}\| \le \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}\left(\frac{\lambda_{\max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}\right)^k \|x^0 - \bar{x}\| \quad \forall (k \in \mathbb{N})$$

i.e. $\{x^k\}$ converges to $\bar{x}$ for any starting point x^0.

Definition 3.2.9 (Condition number) For a symmetric positive definite matrix A, its *condition number* is given by

$$\mathrm{cond}(A) := \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$$

The condition number of the matrix influences the convergence rate in Theorem 3.2.8. If $\mathrm{cond}(A)$ is very large then convergence can be very slow.

Definition 3.3.1 (Convergence rates) Let $\{x^k \in \mathbb{R}^n\} \to \bar{x}$ and $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Then $\{x^k\}$ converges (at least)

(i) **linearly** to $\bar{x}$ if there exists $c \in (0,1)$ such that

$$\|x^{k+1} - \bar{x}\| \le c\,\|x^k - \bar{x}\| \quad (k \in \mathbb{N} \text{ sufficiently large})$$

(ii) **superlinearly** to $\bar{x}$ if

$$\lim_{k\to\infty} \frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} = 0$$

(iii) **quadratically** to $\bar{x}$ if there exists $C > 0$ such that

$$\|x^{k+1} - \bar{x}\| \le C\,\|x^k - \bar{x}\|^2 \quad \forall (k \in \mathbb{N})$$

Definition 3.3.2 (Landau symbols) Let $\{a_k > 0\}$, $\{b_k > 0\} \downarrow 0$. Then we define

$$
\begin{aligned}
a_k = o(b_k) \quad &\iff \quad \lim_{k\to\infty} \frac{a_k}{b_k} = 0 \\
a_k = O(b_k) \quad &\iff \quad \exists\, C > 0, \forall k \in \mathbb{N} : a_k \leq C b_k
\end{aligned}
$$

Rewriting Definition 3.3.1 using Landau notation, we say $\{x^k\} \to \bar{x}$ converges

(i) superlinearly if and only if

$$\left\|x^{k+1} - \bar{x}\right\| = o(\left\|x^k - \bar{x}\right\|)$$

(ii) quadratically if and only if

$$\left\|x^{k+1} - \bar{x}\right\| = O(\left\|x^k - \bar{x}\right\|^2)$$

Remark 3.3.3a (Newton's method) Our goal is to effectively solve

$$F(x) = 0$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be continuously differentiable. The method we are going to study is called *Newton's method* and its basic idea is shockingly simple and relies on *linearization*, one of the most basic principles in mathematics:

Suppose $\bar{x}$ is a root of F and x^k is our current approximation of it. Then consider a local, linear approximation

$$x \mapsto F_k(x) := F(x^k) + F'(x^k)(x - x^k)$$

of F at x^k. Now, compute x^{k+1} as a root of F_k, and we should move closer to $\bar{x}$.

If $F'(x^k) \in \mathbb{R}^{n\times n}$ is invertible we can write

$$x^{k+1} = x^k - F'(x^k)^{-1} F(x^k)$$

But for numerical reasons one does not explicitly invert a matrix, but instead will compute the *Newton direction* d^k as solution to the *Newton equation*

$$F'(x^k) d = -F(x^k)$$

and then update $x^{k+1} := x^k + d^k$. This yields Algorithm 3.3.1.

Lemma 3.3.3 (Local invertibility) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $\bar{x} \in \mathbb{R}^n$ such that $F'(\bar{x})$ is invertible. Then there exists $\varepsilon > 0$ such that $F'(x)$ is invertible for all $x \in B_\varepsilon(\bar{x})$. Moreover, there exists $c > 0$ such that

$$\left\|F'(x)^{-1}\right\| \leq c \quad (x \in B_\varepsilon(\bar{x}))$$

Remark 3.3.3b (Differentiability in Landau notation) Using the Landau notation, we can express the fact that F is differentiable at $\bar{x} \in \mathbb{R}^n$ if and only if

$$\left\|F(x^k) - F(\bar{x}) - F'(x^k)(x^k - \bar{x})\right\| = o(\left\|x^k - \bar{x}\right\|)$$

for all sequences $\{x^k\} \to \bar{x}$.

Here's how it's expanded:

$$\lim_{x^k \to \bar{x}} \frac{\left\|F(x^k) - F(\bar{x}) - F'(x^k)(x^k - \bar{x})\right\|}{\left\|x^k - \bar{x}\right\|} = 0$$

Definition 3.3.4 (Local Lipschitz) We say that $G : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz (continuous) at $\bar{x} \in \mathbb{R}^n$ if there exists $L = L(\bar{x}) > 0$ such that

$$\|G(x) - G(y)\| \leq L \|x - y\| \quad (x, y \in B_\varepsilon(\bar{x}))$$

Lemma 3.3.5 Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and $\{x^k\}$ such that $\{x^k\} \to \bar{x}$. Then

$$\left\|F(x^k) - F(\bar{x}) - F'(x^k)(x^k - \bar{x})\right\| = o(\left\|x^k - \bar{x}\right\|)$$

If F' is, in addition, locally Lipschitz continuous, we also have

$$\left\|F(x^k) - F(\bar{x}) - F'(x^k)(x^k - \bar{x})\right\| = O(\left\|x^k - \bar{x}\right\|^2)$$

Theorem 3.3.6 (Convergence of local Newton's method for equations) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable and let $\bar{x}$ be a root of F such that $F'(\bar{x})$ is invertible. Then there exists $\varepsilon > 0$ such that for every $x^0 \in B_\varepsilon(\bar{x})$, the following hold:

(a) The local Newton method from Algorithm 3.3.1 is well-defined and generates a sequence $\{x^k\}$ convergent to $\bar{x}$.

(b) The rate of convergence is at least superlinear.

(c) If in addition F' is locally Lipschitz continuous at $\bar{x}$ the rate of convergence is quadratic.

Remark 3.3.6a (Newton's method in optimization) We now want to exploit our study of Newton's method for solving smooth, nonlinear equations to tackle unconstrained optimization problems of the form

$$\min_{x \in \mathbb{R}^n} f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is at least twice continuously differentiable. Recall that a necessary condition for $\bar{x}$ to be a local minimizer of f is

$$\nabla f(\bar{x}) = 0$$

So we can put $F := \nabla f$ and we have $F : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable, and all local minimizers of f are among its roots.

This yields Algorithm 3.3.2.

Theorem 3.3.7 (Convergence of local Newton's method for optimization) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and let $\bar{x}$ be a stationary point of f such that $\nabla^2 f(\bar{x})$ is invertible. Then there exists $\varepsilon > 0$ such that for every $x^0 \in B_\varepsilon(\bar{x})$, the following hold:

(a) Algorithm 3.3.2 is well-defined and generates a sequence $\{x^k\}$ convergent to $\bar{x}$.

(b) The rate of convergence is at least superlinear.

(c) If in addition $\nabla^2 f$ is locally Lipschitz continuous at $\bar{x}$ the rate of convergence is quadratic.

Theorem 3.3.9 (Global convergence of Algorithm 3.3.3) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then every cluster point of a sequence generated by Algorithm 3.3.3 is a stationary point of f.

Lemma 3.3.10 (Moré and Sorensen) Let $\bar{x}$ be an isolated cluster point of $\{x^k \in \mathbb{R}^n\}$ and assume that $\{\|x^{k+1} - x^k\|\}_K \to 0$ for every subsequence $\{x^k\}_K \to \bar{x}$. Then the whole sequence $\{x^k\}$ converges to $\bar{x}$.

Corollary 3.3.11 Let $\bar{x}$ be an isolated cluster point of a sequence $\{x^k\}$ generated by Algorithm 3.3.3. Then the whole sequence $\{x^k\}$ converges to $\bar{x}$.

Proposition 3.3.12 (Acceptance of full step-size in globalized Newton's method) Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable, and $\bar{x} \in \mathbb{R}^n$ such that $\nabla f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x})$ is positive definite. Assume that $\{x^k\} \to \bar{x}$ and that d^k is given by

$$d^k = -\nabla^2 f(x^k)^{-1} \nabla f(x^k)$$

(Newton Equation)

Then there exists $k_0 \in \mathbb{N}$ such that $\forall \sigma \in (0, \frac{1}{2}),\ k \geq k_0$:

$$f(x^k + d^k) \leq f(x^k) + \sigma \nabla f(x^k)^T d^k$$

The significance of this last equation being that f experiences sufficient decrease.

Theorem 3.3.13 Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and let $\{x^k\}$ be generated by Algorithm 3.3.3. If $\bar{x}$ is a cluster point of $\{x^k\}$ such that $\nabla^2 f(\bar{x})$ is positive definite then the following hold:

(a) The whole sequence $\{x^k\}$ converges to $\bar{x}$ and $\bar{x}$ is a strict local minimizer of f.

(b) For all $k \in \mathbb{N}$ sufficiently large, the search direction d^k will be determined through the Newton equation.

(c) For all $k \in \mathbb{N}$ sufficiently large, the full step-size $t_k = 1$ will be accepted

(d) $\{x^k\}$ converges superlinearly to $\bar{x}$

(e) If $\nabla^2 f$ is locally Lipschitz then $\{x^k\}$ converges to $\bar{x}$ quadratically.

Definition 3.3.14a (Quasi-Newton equation) In the context of iterating over $\{x_k\}$ and $\{H_k\}$ where H_k is the approximation of $\nabla^2 f(x^k)$ at x^k, H_{k+1} satisfies the quasi-Newton equation if

$$H_{k+1} s^k = y^k,$$

where $s^k := x^{k+1} - x^k$ and $y^k := \nabla f(x^{k+1}) - \nabla f(x^k)$.

Remark 3.3.14b (Direct Quasi-Newton methods) In order to devise a strategy of how to approximate the Hessian matrix of the underlying function f the current iterate x^k we first need to agree on which properties we would like it to have. To this end, let H_k be an approximation of $\nabla^2 f(x^k)$. We would like for H_{k+1} to satisfy the following criteria:

I. $H_{k+1} = H_{k+1}^T$ is symmetric.

II. H_{k+1} satisfies the quasi-Newton equation.

III. H_{k+1} can be obtained efficiently from H_k.

IV. The resulting method has strong local convergence properties.

Remark 3.3.14 Let x^k be a current iterate for minimizing $f : \mathbb{R}^n \to \mathbb{R}$ that is twice continuously differentiable.

(a) The Hessian $\nabla^2 f(x^k)$ of f at x^k does not necessarily satisfy the quasi-Newton equation.

(b) Condition I. is motivated by Schwarz's Theorem.

(c) The quasi-Newton equation can be motivated by the Mean-Value Theorem in integral form, which yields

$$\begin{aligned} \nabla f(x^{k+1}) - \nabla f(x^k) &= \int_0^1 \nabla^2 f(x^k + t(x^{k+1} - x^k))\, dt \cdot (x^{k+1} - x^k) \\ y^k &= \int_0^1 \nabla^2 f(x^k + t(x^{k+1} - x^k))\, dt \cdot s^k \end{aligned}$$

The term $\int_0^1 \nabla^2 f(x^k + t(x^{k+1} - x^k))\, dt$ can be interpreted as some sort of "averaged Hessian".

Remark x.x.x (Collection of unconstrained minimization methods)

1. Gradient method
2. Globalized Newton's method
3. Globalized BFGS method
4. Globalized inexact Newton's method

Algorithm 3.1.1 (General line-search descent algorithm) Goal is to solve

$$\min_{x \in \mathbb{R}^n} f(x).$$

(S0) *Initialization*: Choose $x^0 \in \mathbb{R}^n$ and put $k := 0$.

(S1) *Termination*: If x^k satisfies a termination criterion: STOP.

(S2) *Search direction*: Determine d^k such that $\nabla f(x^k)^T d^k < 0$.

(S3) *Step-size*: Determine $t_k > 0$ such that $f(x^k + t_k d^k) < f(x^k)$.

(S4) *Update*: Put $x^{k+1} := x^k + t_k d^k$, $k \leftarrow k + 1$ and go to **(S1)**.

Algorithm 3.2.1 (Gradient method with Armijo rule) Goal is to solve

$$\min_{x \in \mathbb{R}^n} f(x).$$

(S0) Choose $x^0 \in \mathbb{R}^n$, $\sigma, \beta \in (0, 1)$, $\varepsilon \geq 0$ and put $k := 0$.

(S1) If $\left\|\nabla f(x^k)\right\| \leq \varepsilon$, STOP.

(S2) Put $d^k := -\nabla f(x^k)$.

(S3) Determine $t_k > 0$ by

$$t_k := \max_{l \in \mathbb{N}_0} \{\beta^l \mid f(x^k + \beta^l d^k) \leq f(x^k) + \beta^l \sigma \nabla f(x^k)^T d^k\}$$

(S4) Put $x^{k+1} := x^k + t_k d^k$, $k \leftarrow k + 1$ and go to **(S1)**.

Algorithm 3.3.1 (Local Newton's method for equations) Goal is to solve

$$F(x) = 0$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ and F is assumed to be continuously differentiable.

(S0) Choose $x^0 \in \mathbb{R}^n$, $\varepsilon \geq 0$ and put $k := 0$.

(S1) If $\left\|F(x^k)\right\| \leq \varepsilon$, STOP.

(S2) Compute d^k as a solution of

$$F'(x^k)d = -F(x^k)$$

(S3) Put $x^{k+1} := x^k + d^k$, $k \leftarrow k+1$ and go to **(S1)**.

Algorithm 3.3.2 (Local Newton's method for unconstrained optimization) Goal is to solve

$$\min_{x \in \mathbb{R}^n} f(x).$$

The following is obtained by substituting F for ∇f in the local Newton's method for equations.

(S0) Choose $x^0 \in \mathbb{R}^n$, $\varepsilon \geq 0$ and put $k := 0$.

(S1) If $\left\|\nabla f(x^k)\right\| \leq \varepsilon$, STOP.

(S2) Compute d^k as a solution of

$$\nabla^2 f(x^k)d = -\nabla f(x^k)$$

(S3) Put $x^{k+1} := x^k + d^k$, $k \leftarrow k+1$ and go to **(S1)**.

Algorithm 3.3.3 (Globalized Newton's method for unconstrained optimization) Goal is to solve

$$\min_{x \in \mathbb{R}^n} f(x).$$

The following is obtained by substituting F for ∇f in the local Newton's method for equations.

(S0) Choose $x^0 \in \mathbb{R}^n$, $\rho > 0$, $p > 2$, $\beta \in (0,1)$, $\sigma \in (0, \frac{1}{2})$, $\varepsilon \geq 0$ and put $k := 0$.

(S1) If $\left\|\nabla f(x^k)\right\| \leq \varepsilon$, STOP.

(S2) Try to compute d^k as a solution of

$$\nabla^2 f(x^k)d = -\nabla f(x^k)$$

If no solution can be found or if

$$\nabla f(x^k)^T d^k > -\rho \left\|d^k\right\|^p$$

(insufficient decrease)

then fall back to $d^k := -\nabla f(x^k)$

(S3) Determine t_k by

$$t_k := \max_{l \in \mathbb{N}_0} \{\beta^l \mid f(x^k + \beta^l d^k) \leq f(x^k) + \beta^l \sigma \nabla f(x^k)^T d^k\}$$

(S4) Put $x^{k+1} := x^k + t_k d^k$, $k \leftarrow k+1$ and go to **(S1)**.

Nonlinear Optimization, Part II: Constrained Optimization

Definition 5.0.0 (Standard Nonlinear Program)

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \begin{aligned} g_i(x) &\leq 0 \quad \forall i \in I \\ h_j(x) &= 0 \quad \forall j \in J \end{aligned} \tag{5.1}$$

Where $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.
We call (5.1) a nonlinear program (NLP) in standard form.
By convention, we let the feasible set of (5.1) be denoted by X, with

$$X := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_i(x) \leq 0 & \forall i \in I \\ h_j(x) = 0 & \forall j \in J \end{array} \right\} \tag{5.2}$$

By the continuity of the constraint functions, X is closed.
We will use $I := \{1, \dots, m\}$ and $J := \{1, \dots, p\}$, and define the **active set** at $\bar{x} \in X$ as

$$I(\bar{x}) := \{ i \in I \mid g_i(\bar{x}) = 0 \}$$

Definition 5.1.1 (Cones) A non-empty set $K \subset \mathbb{R}^n$ is said to be a cone if

$$\lambda v \in K \quad (\lambda \geq 0, \; v \in K)$$

i.e. K is a cone if and only if it is closed under multiplication with non-negative scalars.

Example 5.1.2 (Examples of cones)

(a) *(Non-negative Orthant)* For all $n \in \mathbb{N}$, the non-negative orthan $\mathbb{R}^n_+$ is a convex cone, which is also a polyhedron as

$$\mathbb{R}^n_+ = \left\{ x \in \mathbb{R}^n \;\middle|\; (-e_i)^T x \leq 0, \; \forall i = 1, \dots, n \right\}$$

(b) *(Cone complimentary constraints)* Let $K \subset \mathbb{R}^n$ be a cone. Then the set

$$\Lambda := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x, y \in K, \; \langle x, y \rangle = 0 \}$$

is a cone. A prominent example is $K = \mathbb{R}^n$, in which case Λ is called the complementary constraint set.

(c) *(Positive semidefinite matrices)* For $n \in \mathbb{N}$, the set of positive semidefinite $n \times n$ matrices is a convex cone in the space of $n \times n$ symmetric matrices.

Definition 5.1.3 (Tangent cone) Let $S \subset \mathbb{R}^n$ and $\bar{x} \in S$. Then the set

$$T_S(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \exists \{x^k \in S\} \to \bar{x}, \{t_k\} \downarrow 0 : \frac{x^k - \bar{x}}{t_k} \to d \right\}$$

is called the (Bouligand) tangent cone of S at $\bar{x}$.

Proposition 5.1.4 Let $S \subset \mathbb{R}^n$ and $x \in S$. Then $T_S(x)$ is a closed cone.

Theorem 5.1.5 (Basic first-order optimality condition) Let $\bar{x}$ be a local minimizer of $f \in C^1$ over S. Then the following hold:

(a) $\nabla f(\bar{x})^T d \geq 0 \quad (d \in T_S(\bar{x}))$

(b) If S is convex then

$$\nabla f(\bar{x})^T (x - \bar{x}) \geq 0 \quad (x \in S)$$

Definition 5.1.6 (Projection on a set) Let $S \subset \mathbb{R}^n$ be non-empty and $x \in \mathbb{R}^n$. Then we define the projection of x on S by

$$P_S(x) := \operatorname*{argmin}_{y \in S} \|x - y\|$$

Proposition 5.1.7 (Projection on a closed convex set) Let $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$ be a non-empty, closed, and convex set. Then the following hold:

(a) $P_S(x)$ has exactly one element, i.e. P is a function $\mathbb{R}^n \to S$

(b) $P_S(x) = x$ if and only if $x \in S$

(c) $(P_S(x) - x)^T (y - P_S(x)) \geq 0 \quad (y \in S)$

Proof. (a) follows immediately from Theorem 2.2.7(d) since the function $y \mapsto \frac{1}{2} \|x - y\|^2$ is strongly convex.

(b) is obvious.

(c) follows from Theorem 5.1.5(b) applied to $f : y \mapsto \frac{1}{2} \|x - y\|^2$. □

Lemma 5.1.8 Let $B \in \mathbb{R}^{l \times n}$. Then

$$K := \left\{ B^T x \mid x \geq 0 \right\}$$

is a (non-empty) closed, convex cone.

Theorem 5.1.9 (Farkas' Lemma) Let $B \in \mathbb{R}^{l \times n}$ and $h \in \mathbb{R}^n$. Then the system

$$B^T x = h \quad (x \in \mathbb{R}^l,\ x \geq 0)$$

has a solution if and only if $h^T d \geq 0$ for all $d \in \mathbb{R}^n$ such that $Bd \geq 0$.

Proof. Proving that (1) implies (2):

Let $x \geq 0$ such that $B^T x = h$. Then for any d such that $Bd \geq 0$, we have

$$h^T d = (B^T x)^T d = x^T B d$$

But $x^T B d \geq 0$ because $x \geq 0$ and $Bd \geq 0$.

Proving that (2) implies (1) by contrapositive.

Assume that (1) is false. Then

$$h \notin \{B^T x \mid x \geq 0\} =: K$$

By Lemma 5.1.8, K is a closed convex cone.

Set $\bar{s} := P_K(h)$ and $\bar{d} := \bar{s} - h$. Note that $\bar{s} \in K$, and $h \notin K$, and hence $\bar{d} \neq 0$.

By Proposition 5.1.7(c),

$$\bar{d}^T(s - \bar{s}) \geq 0 \quad \forall (s \in K) \tag{$*$}$$

Substituting $s := 0$ and $s := 2\bar{s}$, we obtain two simultaneous inequalities

$$\bar{d}^T \bar{s} \leq 0 \qquad \text{and} \qquad \bar{d}^T \bar{s} \geq 0$$

And hence $\bar{d}^T \bar{s} = 0$. Using this with $(*)$ gives

$$\bar{d}^T s \geq 0$$

Then by definition of cone K, for all $x \geq 0$,

$$\begin{aligned} \bar{d}^T B^T x &\geq 0 \\ \Rightarrow (B\bar{d})^T x &\geq 0 \end{aligned}$$

Inserting $x := e_i$ (where e_i is the i^{th} component vector) for $i = 1, \ldots, n$ implies $(B\bar{d})^T \geq 0$.

On the other hand (recall $\bar{d}^T \bar{s} = 0$ from above)

$$\begin{aligned} h^T \bar{d} &= (\bar{s} - \bar{d})^T \bar{d} \\ &= \bar{s}^T \bar{d} - \left\|\bar{d}\right\|^2 \\ &= -\left\|\bar{d}\right\|^2 \\ &\leq 0 \end{aligned}$$

But since $\bar{d} \neq 0$, we have the strict inequality $h^T \bar{d} < 0$.

Therefore, $B\bar{d} \geq 0$, but $h^T \bar{d} < 0$, i.e. (2) does not hold. $\square$

Definition 5.1.10 (Karush-Kuhn-Tucker conditions) Consider the standard NLP in (5.1). and let X be the feasible set of (5.1).

1. The function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ defined by

$$\begin{aligned} L(x, \lambda, \mu) &= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) \\ &= f(x) + \lambda^T g(x) + \mu^T h(x) \end{aligned}$$

   is called the Lagrangian (function) of (5.1).

2. The set of conditions

$$\begin{aligned} \nabla_x L(x, \lambda, \mu) &= 0, \\ h(x) &= 0, \\ \lambda \geq 0,\ g(x) \leq 0,\ \lambda^T g(x) &= 0 \end{aligned}$$

   are called the Karush-Kuhn-Tucker conditions for (5.1), where

$$\begin{aligned} \nabla_x L(x, \lambda, \mu) &= \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{j=1}^{p} \mu_j \nabla h_j(x) \\ &= \nabla f(x) + \lambda^T \nabla g(x) + \mu^T \nabla h(x) \end{aligned}$$

3. A triple $(\bar{x}, \bar{\lambda}, \bar{\mu})$ that satisfies the KKT conditions is called a KKT point.
4. Given $\bar{x}$, a feasible point for (5.1), we define

$$M(\bar{x}) := \{(\lambda, \mu) \mid (\bar{x}, \lambda, \mu) \text{ is a KKT point of (5.1)}\}$$

as the set of all KKT multipliers (possibly empty) at $\bar{x}$.

Definition 5.1.10a (Constraint qualification (CQ)) We define a condition about the feasible set X of a standard NLP (5.1) that guarantees that the KKT conditions hold at a local minimizer as a constraint qualification.

If a CQ holds on $\bar{x} \in X$, then KKT is necessary for $\bar{x}$ to be a local minimizer.

If a CQ holds on $\bar{x} \in X$, then $\bar{x}$ being a local minimizer implies that there exists a $(\bar{\lambda}, \bar{\mu})$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a KKT point.

Definition 5.1.11 (Linearized cone) Let X be the feasible set of (5.1). The linearized cone (of X) at $\bar{x} \in X$ is defined by

$$L_X(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T d \leq 0 & \forall i = I(\bar{x}) \\ \nabla h_j(\bar{x})^T d = 0 & \forall j = 1, \ldots, p \end{array} \right\}$$

Definition 5.1.12 (Abadie constraint qualification (ACQ)) We say that the ACQ holds at $\bar{x} \in X$ if

$$T_X(\bar{x}) = L_X(\bar{x})$$

That is, the tangent cone is exactly the linearized cone.

Theorem 5.1.13 (KKT conditions under ACQ) Let $\bar{x} \in X$ be a local minimizer of (5.1) such that ACQ holds at $\bar{x}$. Then there exists $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^p$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a KKT point of (5.1).

Proof. By Theorem 5.1.5,

$$\nabla f(\bar{x})^T d \geq 0 \quad \forall (d \in T_X(\bar{x})) \tag{$*$}$$

Set

$$B := \begin{pmatrix} -\nabla g_i(\bar{x})^T & (i = 1, \ldots, m) \\ -\nabla h_j(\bar{x})^T & (j = 1, \ldots, p) \\ \nabla h_j(\bar{x})^T & (j = 1, \ldots, p) \end{pmatrix} \in \mathbb{R}^{(m+2p) \times n}$$

Then purely from the definition of a linearized cone and how we set B, we have

$$d \in L_X(\bar{x}) \iff Bd \geq 0$$

By ACQ, we have $d \in T_X(\bar{x}) \iff Bd \geq 0$

Combined with $(*)$, we have

$$\nabla f(\bar{x})^T d \geq 0 \quad \forall (d : Bd \geq 0)$$

(Think $h = \nabla f(\bar{x})$ and apply the Farkas Lemma.)

By the Farkas Lemma,

$$\exists\, y = \begin{pmatrix} y^1 \in \mathbb{R}^m \\ y^2 \in \mathbb{R}^p \\ y^3 \in \mathbb{R}^p \end{pmatrix}$$

such that $y \geq 0$, and $B^T y = \nabla f(\bar{x})$
Define $\bar{\lambda} \in \mathbb{R}^n, \bar{\mu} \in \mathbb{R}^p$ by

$$\bar{\lambda}_i = \begin{cases} y_i^1 & \text{if } i = 1, \dots, m \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bar{\mu}_i = \begin{cases} y_j^2 - y_j^3 & \text{if } j = m+1, \dots, m+2p \\ 0 & \text{otherwise} \end{cases}$$

Then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a KKT point.
MORE NOTES

$$\begin{aligned} 0 &= \nabla f(\bar{x}) + \sum_{i=0}^{m} y_i^1 \nabla g_i(\bar{x}) + \sum_{j=m+1}^{m+2p} (y_j^2 - y_j^3) \nabla h_j(\bar{x}) \\ &= \nabla f(\bar{x}) + \sum_{i=0}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j=m+1}^{m+2p} \bar{\mu}_j \nabla h_j(\bar{x}) \end{aligned}$$

and then there is a line with a tick/check next to it:

$$\bar{\lambda}^T g(\bar{x}) = \sum_{i=0}^{m} \bar{\lambda}_i g_i(\bar{x}) = 0$$

□

Definition 5.1.14 (Constraint qualifications) A condition on X (i.e. on g and h) that ensures that the KKT conditions hold at a local minimizer is called a **constraint qualification**.

Definition 5.1.15 (LICQ and MFCQ) Let $\bar{x}$ be feasible for (1). We say that

(a) **(LICQ)** the linear independence constraint qualification holds at $\bar{x}$ if the gradients

$$\begin{aligned} \nabla g_i(\bar{x}) \quad & (i \in I(\bar{x})), \\ \nabla h_j(\bar{x}) \quad & (j \in J) \end{aligned}$$

are linearly indepdendent.

(b) **(MFCQ)** the Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$ if the gradients

$$\nabla h_j(\bar{x}) \quad (j \in J)$$

are linearly indepdendent, and $\exists\, d \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla g_i(\bar{x})^T d < 0 \quad & (i \in I(\bar{x})) \\ \nabla h_j(\bar{x})^T d = 0 \quad & (j \in J) \end{aligned}$$

Proposition 5.1.16 (LICQ implies MFCQ) Let $\bar{x}$ be feasible for (1) such that LICQ holds at $\bar{x}$. Then MFCQ holds.

With $I := 1, \ldots, m$ and $J := 1, \ldots, p$, the Standard NLP is

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \begin{array}{ll} g_i(x) \leq 0 & \forall i \in I \\ h_j(x) = 0 & \forall j \in J \end{array} \tag{5.1}$$

We define the active set $I(\bar{x})$ as

$$I(\bar{x}) := \{i \in I \mid g_i(\bar{x}) = 0\}$$

Let X be the feasible set of the NLP.

TANGENT CONE:

$$T_X(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \exists \{x^k \in X\} \to \bar{x}, \{t_k\} \downarrow 0 : \frac{x^k - \bar{x}}{t_k} \to d \right\}$$

LINEARIZED CONE:

$$L_X(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T d \leq 0 & \forall i \in I(\bar{x}) \\ \nabla h_j(\bar{x})^T d = 0 & \forall j \in J \end{array} \right\}$$

Definition 5.1.22 We say that the affine constraint qualification (Affine CQ) holds for (5.1) if all constraints are affine. That is, there exists

$$\begin{array}{lll} a_i \in \mathbb{R}^n & \alpha_i \in \mathbb{R} & \forall i \in I \\ b_i \in \mathbb{R}^n & \beta_j \in \mathbb{R} & \forall j \in J \end{array}$$

such that for all $x \in \mathbb{R}^n$,

$$\begin{array}{ll} g_i(x) = a_i^T - \alpha_i & \forall i \in I \\ h_j(x) = b_i^T - \beta_j & \forall j \in J \end{array}$$

Proposition 5.1.23 Let the Affine CQ hold for (5.1). Then ACQ holds at every feasible point.

Proof. Let $\bar{x} \in X$. In view of Lemma 5.1.18 we only have to show that $L_X(\bar{x}) \subset T_X(\bar{x})$. Let $d \in L_X(\bar{x})$. Then by defn of the linearized cone we have

$$\begin{array}{ll} \alpha_i^T d \leq 0 & \forall i \in I \\ \beta_j^T d = 0 & \forall j \in J \end{array}$$

Now, let $\{t_k\} \downarrow 0$ and put $x^k := \bar{x} + t_k d$. Then

$$x^k \to \bar{x} \qquad \text{and} \qquad \frac{x^k - \bar{x}}{t_k} \to d$$

Hence, we still need to show that $x^k \in X$ ($\forall k$ suff. large)

For $i \notin I(\bar{x})$ Then $a_i^T \bar{x} < \alpha_i$, hence by continuity

$$a_i^T x^k < \alpha_i \qquad \forall (x \text{ suff. large})$$

For $i \in I(\bar{x})$ Then

$$a_i^T x^k = a_i^T \bar{x} + t_k a_i^T d \le a_i^T \bar{x} = \alpha_i$$

because $a_i^T d \le 0$ by definition of $L_X(\bar{x})$.

For $j \in J$ Then

$$b_j^T x^k = b_j^T \bar{x} + t_k b_j^T d = \beta_j$$

because $b_j^T d = 0$ by definition of $L_X(\bar{x})$.

These three together show that $x^k \in X$ for k sufficiently large. This completes the proof. □

5.1.5 CONVEX PROBLEMS

Consider

$$\min f(x) \quad \text{s.t.} \quad \begin{array}{ll} g_i(x) \le 0 & \forall i \in I \\ h_j(x) = b_j^T x - \beta_j = 0 & \forall j \in J \end{array} \tag{5.2}$$

where $f, g_i, h_j \in \mathbb{R}^n \to \mathbb{R}$ are cont. diff **and convex**, and $b_j \in \mathbb{R}^n, \beta_j \in \mathbb{R}$. Then

$$X = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} g_i(x) \le 0 & \forall i \in I \\ h_j(x) = 0 & \forall j \in J \end{array} \right\}$$

is convex (see Midterms).

Theorem 5.1.24 Let $\bar{x}$ be feasible for (5.2), and consider the following statements:

(a) There exists $(\bar{\lambda}, \bar{\mu}) \in M(\bar{x})$

(b) $\bar{x}$ is a global minimizer of (5.2)

Then (a) implies (b). Hence, if a CQ holds at $\bar{x}$, then (a) iff (b).

Proof. Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be a KKT point of (5.2), and let $\bar{x} \in X$.
Then, by Theorem 2.2.1,

$$f(x) \ge f(\bar{x}) + \nabla f(\bar{x})^T (x - \bar{x})$$

But with the KKT conditions, we can rewrite $\nabla f(\bar{x})$ as

$$\nabla f(\bar{x}) = -\sum_{i \in I(\bar{x})}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) - \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(\bar{x})$$

So then

$$f(x) \ge f(\bar{x}) - \sum_{i \in I(\bar{x})}^{m} \bar{\lambda}_i \nabla g_i(\bar{x})^T (x - \bar{x}) - \sum_{j=1}^{p} \bar{\mu}_j \nabla h_j(\bar{x})^T (x - \bar{x}) \tag{$*$}$$

Again by Theorem 2.2.1, we have

$$\nabla g_i(\bar{x})^T (x - \bar{x}) \le g_i(x) - g_i(\bar{x})$$

Separately,

$$\nabla h_j(\bar{x})^T (x - \bar{x}) = 0$$

Putting everything back to $(*)$ and noting that $g_i(\bar{x}) = 0$ by definition of an active set,

$$f(x) \ge f(\bar{x}) - \sum_{i \in I(\bar{x})}^{m} \bar{\lambda}_i g_i(x)$$

But since $\bar{\lambda}_i \geq 0$ and $g_i(x) \leq 0$, we have

$$f(x) \geq f(\bar{x})$$

and hence $\bar{x}$ is a global minimum of (5.2). Hence shown that (a) implies (b). The converse direction is the definition of a CQ.

And hence (a) iff (b). □

Definition 5.1.25 (Slater constraint qualification) We say that Slater CQ holds for (5.2) if there exists $\hat{x}$ such that

$$\begin{aligned} g_i(\hat{x}) &< 0 \quad \forall i \in I \\ h_j(\hat{x}) &= 0 \quad \forall j \in J \end{aligned}$$

We call such an $\hat{x}$ a Slater point.

Proposition 5.1.26 Let SCQ hold for (5.2). Then ACQ holds at every feasible point.

Proof. Let $\bar{x} \in X$ and set

$$F(\bar{x}) := \left\{ d \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} \nabla g_i(\bar{x})^T \leq 0 & \forall i \in I(\bar{x}) \\ b_j^T d = 0 & \forall j \in J \end{array} \right\}$$

Lemma $(F(\bar{x}) \subset T_X(\bar{x}))$ Let $d \in F(\bar{x})$, take $\{t_k\} \downarrow 0$. Set $x^k := \bar{x} + t_k d$. Then $\dfrac{x^k - \bar{x}}{t_k} \to d$. Moreover,

For $i \notin I(\bar{x})$, $g_i(x^k) < 0$ for k sufficiently large.

For $i \in I(\bar{x})$, $\dfrac{g_i(x^k) - g_i(\bar{x})}{t_k} \to \nabla g_i(\bar{x}) < 0$, and $g_i(\bar{x}) = 0$, so we ahve $g_i(x^k) < 0$ for k sufficiently large.

For $j \in J$, $h_j(x^k) = b_j^T x^k - \beta_j = t_k \nabla h_j(\bar{x})^T d - \beta_j$

$$\begin{aligned} h_j(x^k) &= b_j^T x^k - \beta_j \\ &= t_k \nabla h_j(\bar{x})^T d + \nabla h_j(\bar{x}) \beta_j = 0 \end{aligned}$$

Thus

$$clF(\bar{x}) \subset clT_X(\bar{x}) = T_X(\bar{x})$$

We now show $L_X(\bar{x}) \subset dF(\bar{x})$. To this end, let $d \in L_X(\bar{x})$, and let $\hat{x}$ be a Slater point.

Set $\hat{d}$ to be $\hat{x} - \bar{x}$. Then by Theorem 2.2.1,

$$\nabla g_i(\bar{x})^T \hat{d} \leq g_i(\hat{x}) - g_i(\bar{x}) < 0 \qquad \forall (i \in I(\bar{x})) \tag{$*$}$$

< 0 because of definitions of Slater and Active Set.

Moreover, by the affine-ness of h_j, we have

$$\nabla h_j(\bar{x})^T \hat{d} = h_j(\hat{x}) - h_j(\bar{x}) = 0 \qquad \forall (j \in J) \tag{$**$}$$

because both $h_j(\hat{x})$ and $h_j(\bar{x})$ are zero.

Now we take a small pertubation of d using $\hat{d}$:

$$d(\delta) := d + \delta \hat{d} \qquad (\delta > 0)$$

Then $d(\delta) \in F(\bar{x})$, since

$$\begin{aligned} \nabla g_i(\bar{x})^T d(\delta) &= \nabla g_i(\bar{x})^T d + \delta \nabla g_i(\bar{x})^T \hat{d} < 0 \qquad &\forall (i \in I(\bar{x})) \\ \nabla h_j(\bar{x})^T d(\delta) &= \nabla h_j(\bar{x})^T d + \delta \nabla h_j(\bar{x})^T \hat{d} = 0 \qquad &\forall (j \in J) \end{aligned}$$

< 0 because the 1st term ≤ 0 and 2nd term < 0, because d is in the linearized cone, and $\hat{d}$...?
And $= 0$ because both terms $= 0$.

Hence,

$$d = \lim_{\delta \downarrow 0} d(\delta) \subset clF(\bar{x})$$

□

Consider the standard NLP:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \begin{aligned} g_i(x) &\leq 0 \quad \forall i \in I \\ h_j(x) &= 0 \quad \forall j \in J \end{aligned} \tag{1}$$

But now assume that $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ with no smoothness.

The Lagrangian of (1) is

$$\begin{aligned} L(x, \lambda, \mu) &= f(x) + \sum_{i \in I} \lambda_i g_i(x) + \sum_{j \in J} \mu_j h_j(x) \\ &= f(x) + \lambda^T g(x) + \mu^T h(x) \end{aligned}$$

The Dual Problem

Observe that if x is a feasible point,

$$\sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} = \begin{cases} f(x) & \text{if } x \in X, \\ +\infty & \text{if } x \notin X \end{cases}$$

Therefore the primal problem (1) is equivalent to

$$\min_{x \in \mathbb{R}^n} \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(x, \lambda, \mu)$$

Q. When can we switch min(inf) and sup?

Definition 6.1.1 (Lagrangian dual) The Lagrangian dual of (1) is given by

$$\max d(\lambda, \mu) \quad \text{s.t.} \quad \lambda \geq 0$$

where the dual objective is given by $d : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ and

$$d(\lambda, \mu) := \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$$

The function $p : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ given by

$$p(x) := \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(x, \lambda, \mu)$$

is called the **primal objective**.

Example 6.1.2 (LP duality) Consider the standard linear program (LP):

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

The Lagrangian reads

$$\begin{aligned} L(x, \lambda, \mu) &= c^T x - \lambda^T x + \mu^T (b - Ax) \\ &= (c - \lambda - A^T \mu)^T x + b^T u \end{aligned} \tag{$*$}$$

So then

$$\nabla_x L(x, \lambda, \mu) = c - \lambda - A^T \mu$$

The function that takes $x \mapsto L(x, \lambda, \mu)$ is affine (from $(*)$), and in particular it is convex. And hence it takes its minimum if and only if $\nabla_x L(x, \lambda, \mu) = 0$, in which case,

$$\inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu) = b^T \mu$$

otherwise if $\nabla_x L(x, \lambda, \mu) \neq 0$, the infimum must be $-\infty$.

So then

$$d(\lambda, \mu) = \begin{cases} b^T \mu & \text{if } c = A^T u + \lambda, \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, the dual problem reads

$$\max_{\lambda, \mu} d(\lambda, \mu) \quad \text{s.t.} \quad \lambda \geq 0$$

Which is the same as

$$\max_{\lambda, \mu} b^T \mu \quad \text{s.t.} \quad \lambda \geq 0, A^T \mu + \lambda = c$$

and again,

$$\max_{\mu} b^T \mu \quad \text{s.t.} \quad A^T \mu \leq c$$

Theorem 6.2.1 (Weak duality) Let $\hat{x}$ be feasible for (P) and $(\hat{\lambda}, \hat{\mu})$ be feasible for (D). Then

$$p(\hat{x}) \geq d(\hat{\lambda}, \hat{\mu})$$

Proof. We have

$$p(\hat{x}) = f(\hat{x}) \quad (\hat{x} \in X)$$

and hence

$$\begin{aligned} p(\hat{x}) &\geq f(\hat{x}) + \hat{\lambda}^T g(\hat{x}) + \hat{\mu}^T h(\hat{x}) \\ &= L(\hat{x}, \hat{\lambda}, \hat{\mu}) \\ &\geq \inf_{x \in \mathbb{R}^n} L(x, \hat{\lambda}, \hat{\mu}) \\ &= d(\hat{\lambda}, \hat{\mu}) \end{aligned}$$

Remark If $p(\hat{x}) = d(\hat{\lambda}, \hat{\mu})$, then $\hat{x}$ solves (P), and $(\hat{\lambda}, \hat{\mu})$ solves (D). $\square$

From weak duality, if we define

$$\bar{p} := \inf_{x \in \mathbb{R}^n} p(x) \geq \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} d(\lambda, \mu) =: \bar{d}$$

Then $\bar{p} - \bar{d} \geq 0$

Example 6.2.2 (Non-zero duality gap) Consider the following objective function:

$$\min f(x) := \begin{cases} x^2 - 2x & x \geq 0 \\ x & \text{otherwise} \end{cases} \quad \text{s.t.} \quad g(x) := -x \leq 0$$

The Lagrangian reads

$$L(x, \lambda) = \begin{cases} x^2 - (2 + \lambda)x & \text{if } x \geq 0 \\ (1 - \lambda)x & \text{otherwise} \end{cases}$$

A short computation shows that

$$d(\lambda) = \begin{cases} -\frac{(2+\lambda)^2}{4} & \text{if } \lambda \geq 1 \\ -\infty & \text{otherwise} \end{cases}$$

Therefore,

$$\bar{d} = d(1) = -\frac{9}{4} < 1 = \bar{p}$$

Hence the duality gap

$$\bar{p} - \bar{d} = \frac{5}{4} > 0$$

Definition 6.3.1 (Saddle point) The triple $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^p$ is called a saddle point of the Lagrangian L of (P) if

$$L(\bar{x}, \lambda, \mu) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq L(x, \bar{\lambda}, \bar{\mu})$$

Theorem 6.3.2 The following are equivalent:

(i) $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a saddle point of (P)

(ii) $\bar{x}$ solves (P); $(\bar{\lambda}, \bar{\mu})$ solves (D)

Proof.

(i) $\implies$ (ii): If $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a saddle point of (P), then

$$\begin{aligned} L(\bar{x}, \bar{\lambda}, \bar{\mu}) &\overset{\text{S.P.}}{\leq} \inf_x L(x, \bar{\lambda}, \bar{\mu}) \\ &\leq \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} \inf_x L(x, \lambda, \mu) \\ &\leq \inf_x \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(x, \lambda, \mu) \\ &\leq \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(\bar{x}, \lambda, \mu) \\ &\overset{\text{S.P.}}{\leq} L(\bar{x}, \bar{\lambda}, \bar{\mu}) \end{aligned} \tag{$*$}$$

Then,

$$\begin{aligned} d(\bar{\lambda}, \bar{\mu}) &= \inf_x (x, \bar{\lambda}, \bar{\mu}) \\ &\overset{(*)}{=} \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(\bar{x}, \lambda, \mu) \\ &= p(\bar{x}) = \bar{p} < +\infty \end{aligned}$$

Hence if $x \in X$, and by weak duality, $\bar{x}$ solves (P), and $(\bar{\lambda}, \bar{\mu})$ solves (D).

(ii) $\implies$ (i): Observe that

$$\begin{aligned} L(\bar{x}, \bar{\lambda}, \bar{\mu}) &\overset{\bar{x} \in X}{\leq} f(\bar{x}) \\ &\overset{\bar{x} \in X}{=} p(\bar{x}) \\ &\overset{\text{defn. of } p}{=} \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(\bar{x}, \lambda, \mu) \\ &= d(\bar{\lambda}, \bar{\mu}) \\ &= \inf_x (x, \bar{\lambda}, \bar{\mu}) \\ &\leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) \end{aligned}$$

But that's just the original LHS value, and hence all lines are equal. Hence

$$\begin{aligned} L(\bar{x}, \bar{\lambda}, \bar{\mu}) &= \inf_x (x, \bar{\lambda}, \bar{\mu}) \\ &= \sup_{\lambda \in \mathbb{R}^m_+,\ \mu \in \mathbb{R}^p} L(\bar{x}, \lambda, \mu) \end{aligned}$$

And hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a saddle point. □

Consider again

$$\min f(x) \quad \text{s.t.} \quad x \in X \tag{1}$$

and a penalty function

$$P_\alpha^r := f + \alpha r, \quad \alpha > 0 \tag{2}$$

with $r \geq 0$, $r(x) = 0 \iff x \in X^r$

Definition 7.2.1 The penalty function P_α^r is called exact at a local min $\bar{x}$ of (1) if there exists $\bar{\alpha}$ such that $\bar{x}$ is a local min of P_α^r for all $\alpha > \bar{\alpha}$

Consider now the standard NLP

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \begin{aligned} g_i(x) &\leq 0 \quad \forall i \in I \\ h_j(x) &= 0 \quad \forall j \in J \end{aligned} \tag{3}$$

with $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ at least cont. diff.

A whole class of penalty functions in the sense of (2) for problem (3) is defined via

$$r_q(x) := \|(\max\{g(x), 0\}, h(x))\|_q$$

Where the max is interpreted component-wise, and then we're taking the q-norm. The value in $(\cdot)$ is a vector with the first i elements being $\max\{g_i(x), 0\}$ and the last j elements being $h_j(x)$.

and

$$\|z\|_q = \begin{cases} \left(\sum_{i=1}^{\ell} (z_i)^q \right)^{\frac{1}{q}} \dots & \text{if } q \in [1, \infty) \\ \max_{i=1,\dots,\ell} |z_i| & \text{if } q = +\infty \end{cases}$$

we focus on $q = 1$:

$$P_\alpha^1(x) = f(x) + \alpha \sum_{j=1}^{p} |h_j(x)| + \alpha \sum_{i=1}^{m} \max\{g_i(x), 0\}$$

Theorem 7.2.2 Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ be a KKT point of the convex NLP

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \begin{aligned} g_i(x) &\leq 0 \quad \forall i \in I \\ h_j(x) &= 0 \quad \forall j \in J \end{aligned} \tag{4}$$

with $f, g_i : \mathbb{R}^n \to \mathbb{R}$ convex and cont. diff, and $h_j : \mathbb{R}^n \to \mathbb{R}$ affine (and hence convex).

Then $\bar{x} \in \operatorname{argmin}_X P_\alpha^1(x)$, for all $\alpha \geq \left\|(\bar{\lambda}, \bar{\mu})\right\|_\infty$.

In particular, P_α^1 is exact at $\bar{x}$ if a CQ holds.

Proof. By Theorem 5.1.23 (KKT for convex problems), $\bar{x}$ is a global minimizer of (4). Therefore, by Theorem 6.3.2 (Saddle point theorem), $\bar{x}$ is a global minimizer of the Lagrangian $L(\cdot, \bar{\lambda}, \bar{\mu})$.

Therefore, for all $x \in \mathbb{R}^n$ and for all $\alpha \geq \left\|(\bar{\lambda}, \bar{\mu})\right\|_\infty$, we have

$$P_\alpha^1(\bar{x}) = f(\bar{x}) + \alpha \sum_{j=1}^{p} |h_j(\bar{x})| + \alpha \sum_{i=1}^{m} \max\{g_i(\bar{x}), 0\}$$

For reference,

$$\left\|(\bar{\lambda}, \bar{\mu})\right\|_\infty := \max\{|\bar{\lambda}_1|, \dots, |\bar{\lambda}_m|, |\bar{\mu}_1|, \dots, |\bar{\mu}_p|\}$$

But remember that since the point is feasible, the second and third terms are both zero and hence

$$
\begin{aligned}
P^1_\alpha(\bar{x}) &= f(\bar{x}) \\
&\overset{\text{KKT}}{=} f(\bar{x}) + \bar{\lambda}^T g(\bar{x}) + \bar{\mu}^T h(\bar{x}) \\
&= L(\bar{x}, \bar{\lambda}, \bar{\mu}) \\
&\overset{\text{Thm 6.3.2}}{\leq} L(x, \bar{\lambda}, \bar{\mu}) \\
&= f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x) + \sum_{j=1}^{p} \bar{\mu}_j h_j(x) \\
&\leq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i \max\{g_i(x), 0\} + \sum_{j=1}^{p} |\bar{\mu}_j| \cdot |h_j(x)| \\
&\leq f(x) + \alpha \sum_{i=0}^{m} \max\{g_i(x), 0\} + \alpha \sum_{j=1}^{p} |h_j(x)| \\
&= P^1_\alpha(x)
\end{aligned}
$$

Hence $\bar{x}$ is the global minimizer of P^1_α, that is for any $\alpha \geq \left\|(\bar{\lambda}, \bar{\mu})\right\|_\infty$

$$\bar{x} \in \underset{X}{\operatorname{argmin}}\, P^1_\alpha$$

$\square$

SQP Methods Consider

$$\min f(x) \quad \text{s.t.} \quad h_j(x) = 0 \quad \forall j \in J \tag{1}$$

with $f, h_j : \mathbb{R}^n \to \mathbb{R}$ twice cont diff.
Define $\Phi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \times \mathbb{R}^p$ by

$$\Phi(x, \mu) := \begin{bmatrix} \nabla_x L(x, \mu) \\ h(x) \end{bmatrix}$$

where $L : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is the Lagrangian of (1). Then

$$(x, \mu) \text{ is a KKT point of (1)} \iff \Phi(x, \mu) = 0 \tag{$*$}$$

where Φ is C^1.
Idea: Apply Newton's method to $(*)$.

Algorithm 8.1.1 (Lagrange-Newton method)

(S0) Choose $(x^0, \mu^0) \in \mathbb{R}^n \times \mathbb{R}^p$ and set $k := 0$.

(S1) If $\Phi(x^k, \mu^k) = 0$: STOP

(S2) Determine $(\Delta x^k, \Delta \mu^k)$ as solution of

$$\Phi'(x^k, \mu^k) \begin{pmatrix} \Delta x \\ \Delta \mu \end{pmatrix} = -\Phi(x^k, \mu^k)$$

(S3) Put $(x^{k+1}, \mu^{k+1}) := (x^k, \mu^k) + (\Delta x^k, \Delta \mu^k)$, $k \leftarrow k + 1$, and go to (S1).

"Hessian is the Jacobian of the gradient"
Crucial part for well-definedness is to have Φ' be invertible at (x^k, μ^k)

Theorem 8.2.1 Let $(\bar{x}, \bar{\mu}) \in \mathbb{R}^n \times \mathbb{R}^p$ be a KKT point of (1), i.e. a root of the function $\Phi(x, \mu) = 0$ such that:

(i) (LICQ) The vectors $\nabla h_j(x)$ are linearly indepdendent, for $j \in J$

(ii) (2nd order sufficient condition) We have

$$d^T \nabla^2_{xx} L(\bar{x}, \bar{\mu}) d > 0 \quad \forall d \neq 0 \wedge \nabla h_j(\bar{x})^T d = 0$$

Proof. Goal is to show that kernel is trivial.
Sidenote:

$$h'(x)^T = \begin{bmatrix} \nabla h_1(x) & \dots & \nabla h_p(x) \end{bmatrix} \in \mathbb{R}^{n \times p}$$

Observe that

$$\Phi(x, \mu) = \begin{pmatrix} \nabla^2_{xx} L(x, \mu) & h'(x)^T \\ h'(x) & 0 \end{pmatrix} \quad \forall (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^p$$

Hence,

$$\begin{aligned} \Phi'(\bar{x}, \bar{\mu}) \begin{pmatrix} q \\ r \end{pmatrix} &= 0 \\ \iff \nabla^2_{xx} L(\bar{x}, \mu) q + h'(\bar{x})^T r &= 0 \\ \text{and} \qquad h'(\bar{x})^T q &= 0 \end{aligned}$$

Note: $\Phi' \in \mathbb{R}^{(n+p)\times(n+p)}$ $\square$

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Lagrange-Newton Equation Update with $(x^{k+1}, \mu^{k+1}) := (x^k, \mu^k) + (\Delta x^k, \Delta \mu^k)$

$$\begin{aligned} \Phi'(x^k, \mu^k) \begin{pmatrix} \Delta x \\ \Delta \mu \end{pmatrix} &= -\Phi(x^k, \mu^k) \\ \iff \begin{bmatrix} \nabla^2_{xx} L(x^k, \mu^k) & h'(x^k)^T \\ h'(x^k) & 0 \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta \mu \end{pmatrix} &= - \begin{bmatrix} \nabla f(x^k) - h'(x^k)^T \\ h'(x^k) \end{bmatrix} \end{aligned}$$

Ordinary Differential Equations

First order differential equations

Result 1.1.1 (Separable)

$$\frac{dy}{dx} = \frac{p(x)}{q(y)}$$

To solve, we do

$$\int p(x)\,dx = \int q(y)\,dy$$

Result 1.1.2 (Homogeneous)

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

An easier way to check if $\dfrac{dy}{dx}$ satisfies this form is by letting $f(x,y) := \dfrac{dy}{dx}$ and verifying that $f(x,y) = f(kx,ky)$.

To solve, let $u := y/x$ and rewrite the DE in terms of u and x.

Result 1.1.3 (Linear)

$$\frac{dy}{dx} + p(x)y = q(x)$$

To solve, let $\ln u := \int p(x)\,dx$ and jump to

$$\frac{d}{dx}uy = u \cdot q(x)$$

(which is just the product rule of differentiation)

Result 1.1.4 (Bernoulli)

$$\frac{dy}{dx} + p(x)y = q(x) \cdot y^n$$

If $n \in \{0,1\}$, we have the linear case.

Use $u := y^{1-n}$, eliminate all ys, and reduce to a linear DE in u:

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$

Result 1.1.5 (Riccati)

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$$

To solve, first find a basic solution y_1. (By inspection, hopefully. Usually a solution will be given as part of the homework problem)

Then let $y_2 := y_1 + \dfrac{1}{u}$, substitute it into the original DE, and reduce it to a linear DE in u:

$$-u' = (2py_1 + q)u + p$$

Compute. First we obtain $y_2' = y_1' - \dfrac{u'}{u^2}$. Then substitute into the original DE:

$$\begin{aligned}
y_2' &= py_2^2 + qy_2 + r \\
y_1' - \frac{u'}{u^2} &= p\left(y_1 + \frac{1}{u}\right)^2 + q\left(y_1 + \frac{1}{u}\right) + r \\
&= p\left(y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}\right) + q\left(y_1 + \frac{1}{u}\right) + r \\
&= (py_1^2 + qy_1 + r) + p\left(\frac{2y_1}{u} + \frac{1}{u^2}\right) + q\left(\frac{1}{u}\right)
\end{aligned}$$

Since y_1 is a solution to the original DE,

$$\begin{aligned}
-\frac{u'}{u^2} &= p\left(\frac{2y_1}{u} + \frac{1}{u^2}\right) + q\left(\frac{1}{u}\right) \\
-u' &= (2py_1 + q)u + p
\end{aligned}$$

□

Result 1.1.6 (Exact)

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Criteria: $M_y = N_x$.

The idea is to work towards a function F where $F_x = M$, and $F_y = N$. (Because then based on the original DE we'll have $df/dx = 0$)

To solve, integrate to find $F(x,y) = \int M(x,y)\,dx = \int N(x,y)\,dy$.

Then we have $F(x,y) = C$ for some constant C.

Second order differential equations

- $ay'' + by' + cy = g(t)$: Method of undetermined coefficients
- $ax^2y'' + bxy' + cy = 0$: Euler equations
- $y'' + p(x)y' + q(x)y = r(x)$: Variation of parameters for either a particular or complementary solution.

Definition 2.1.1 (Wronskian) The Wronskian of two differentiable functions f and g is $W(f,g) := fg' - gf'$.

More generally, for n complex-valued functions $f_1, \ldots, f_n$ which are $n-1$ times differentiable on an interval I, the Wronskian $W(f_1, \ldots, f_n)$ is a function on $x \in I$ defined by

$$W(f_1,\ldots,f_n)(x) := \det\begin{bmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{bmatrix}$$

Result 2.1.2 (Method of undetermined coefficients)

$$ay'' + by' + cy = g(t)$$

Define $\Sigma_{n,t}(A) := A_0 t^n + A_1 t^{n-1} + \ldots + A_n$ and $P_n(t) := a_0 t^n + a_1 t^{n-1} + \ldots + a_n$.

$g(t)$	$y(t)$
$P_n(t)$	$\Sigma_{n,t}(A)$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t} \cdot \Sigma_{n,t}(A)$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s\left[\Sigma_{n,t}(A)e^{\alpha t}\cos\beta t + \Sigma_{n,t}(B)e^{\alpha t}\sin\beta t\right]$

Here, s is the smallest non-negative integer that ensures that no term in $y(t)$ is a solution of the corresponding homogeneous equation.

Result 2.1.3 (Variation of parameters (particular: y_p)) import Wronskian for W.

$$y'' + p(x)y' + q(x)y = r(x)$$

Suppose we already know that y_1 and y_2 satisfy the corresponding homogeneous equation. Then we can jump to

$$y_p = v_1 y_1 + v_2 y_2$$

where $v_1' := \dfrac{-y_2 r}{W(y_1, y_2)}$ and $v_2' := \dfrac{y_1 r}{W(y_1, y_2)}$.

Result 2.1.4 (Variation of parameters (complementary: y_c))

$$y'' + p(x)y' + q(x)y = r(x)$$

Suppose we already know that y_1 is a solution. Then let

$$y_2 := v y_1$$

Then substituting y_2 back into the original DE, we have

$$v'' y_1 + v'(2y_1' + py_1) = 0$$

Compute.

$$y_2 := vy_1; \quad y_2' = v'y_1 + vy_1'; \quad y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

Substituting this back into the original DE:

$$(v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + q(vy_1) = r$$

But since y_1 is known to be a solution:

$$\begin{aligned}(v''y_1 + 2v'y_1') + p(v'y_1) &= 0\\ v''y_1 + v'(2y_1' + py_1) &= 0\end{aligned}$$

□

Which is a first-order linear equation in v'. Use $u := v'$ to solve for u and then substitute everything back to find y.

Result 2.1.5 (Euler equations)

$$ax^2y'' + bxy' + cy = 0$$

Try $y = x^r$ for some $r \in \mathbb{C}$ to be found.

If two distinct roots: $y := Ax^{r_1} + Bx^{r_2}$.

If one distinct root: $y := Ax^r + B\ln(x)x^r$.

If complex roots ($r = \alpha \pm \beta i$): $y := Ax^\alpha \cos(\beta \ln x) + Bx^\alpha \sin(\beta \ln x)$

Higher order differential equations

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \ldots + p_{n-1}y' + p_n y = q$$

where $p_1, \ldots, p_n, q : \mathbb{R} \to \mathbb{R}$.

For these we split into a few cases:

- Constant coefficients
- Euler equations
- Variation of parameters

Result 3.1.1 (Constant coefficients)

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_{n-1}y' + a_n y = 0 \tag{$*$}$$

where $a_1, \ldots, a_n \in \mathbb{R}$ fixed.

Then try $y = e^{rt}$. Substituting that into $(*)$, we'll obtain

$$a_0 r^n + a_1 r^{n-1} + \ldots + a_n = 0$$

Since $a_1, \ldots, a_n$ are given, we can solve for r.

If all the roots of r are real and no two are equal, then we have n distinct solutions $e^{r_1 t}, \ldots, e^{r_n t}$ of equation $(*)$.

Moreover, if these solutions are linearly independent, then the general solution to $(*)$ is

$$y_g = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$$

One way to establish the linear independence is to evaluate their Wronskian.

Result 3.1.2 (Euler equations)

$$a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_n y = 0 \tag{$*$}$$

where $a_1, \ldots, a_n \in \mathbb{R}$ fixed.

Then try $y = x^r$. Substituting that into $(*)$, we'll obtain

$$\sum_{i=0}^{n} \left(\prod_{j=r-i+1}^{r} j \right) a_i x^r = 0$$

Which is then

$$\sum_{i=0}^{n} \left(\prod_{j=r-i+1}^{r} j \right) a_i = 0$$

Result 3.1.3 (Variation of parameters)

Result 3.1.4 (Radius of convergence of power series) Consider the power series

$$\sum_{k=0}^{n}(ax)^k = 1 + ax + a^2x^2 + a^3x^3 + \dots$$

As $n \to \infty$,

$$\sum_{k=0}^{n}(ax)^k \begin{cases} \text{is convergent on } \dfrac{1}{1-ax} & \text{if } |ax| < 1 \\ \text{is divergent} & \text{otherwise} \end{cases}$$

We use this as a benchmark to check for convergence of an arbitrary power series:

$$\sum_{k=0}^{\infty} b_k x^k$$

where $b_1, b_2, \dots \in \mathbb{R}$.

If for all $k \in \mathbb{N}_0$ we have $|b_k| < a^k$, and the geometric series converges for x, then the arbitrary series converges too.

This is the same as requiring $(b_k)^{1/k} < a$ and $|x| < \frac{1}{|a|}$, and hence we obtain the radius of convergence of the arbitrary power series:

$$R := \frac{1}{\lim\limits_{k\to\infty} (b_k)^{1/k}}$$

The series converges if $|x| < R$, and diverges if $|x| > R$.

Proposition 3.1.5 (Power series at zero) Suppose we have an arbitrary power series that is zero

$$\sum_{k=0}^{n} a_k x^k = 0$$

Then we must have

$$a_k = 0 \quad \forall (k \in \mathbb{N}_0)$$

Definition 3.1.6 (Laplace transform) The Laplace transform of a function $f(t)$, defined for all $t \in \mathbb{R}$, $t > 0$, is the function $F(s)$, a unilateral transform defined by

$$F(s) := \int_0^\infty f(t)e^{-st}\, dt$$

Here's a few fundamental Laplace transforms:

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$te^{\alpha t}$	$\frac{1}{(s-\alpha)^2}$
$\cos(\beta t)$	$\frac{s}{s^2-\beta^2}$
$\sin(\beta t)$	$\frac{\beta}{s^2-\beta^2}$
$e^{\alpha t}\cos(\beta t)$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$
$e^{\alpha t}\sin(\beta t)$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$

Let $\mathcal{L}(f) := \mathcal{L}\{f\}(s)$. Then we have

$$\mathcal{L}(f') = -f(0) + s\mathcal{L}(f)$$

Compute. Let's work through a few examples together.
Let $f(t) := 1$. Then

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st}\,dt = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s}$$

Next, let $f(t) := t$. Then

$$\begin{aligned}\mathcal{L}\{f\}(s) &= \int_0^\infty te^{-st}\,dt \\ &= \left[-\frac{1}{s}e^{-st}\cdot t\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,dt \\ &= \frac{1}{s^2}\end{aligned}$$

$\square$

Exercise 3.1.7 Solve the differential equation

$$y'' + 2ty' - 4y = 5 \qquad y(0) = 9, y'(0) = 0$$

Compute. Let $u(s) := \mathcal{L}(y)$
Let's start by preprocessing some Laplace transforms:

$$\begin{aligned}\mathcal{L}(y') &= -f(0) + su &&= 9 + su \\ \mathcal{L}(y'') &= -f'(0) - sf(0) + s^2u &&= -9s + s^2u\end{aligned}$$

And then,

$$\mathcal{L}(ty') = -\frac{d}{ds}\mathcal{L}(y') = -u - su'$$

So then applying the Laplace transform on both sides:

$$\begin{aligned}(-9s+s^2u)+2(-u-su')-4(u)&=5\\(-2s)u'+(s^2-6)u&=9s+5\\u'+\frac{6-s^2}{2s}u&=-\frac{9}{2}-\frac{5}{2s}\\u'+\left(\frac{3}{s}-\frac{s}{2}\right)u&=-\frac{9}{2}-\frac{5}{2s}\end{aligned} \tag{$*$}$$

This is a first-order linear equation in u.

Let $\ln v := \int \left(\frac{3}{s}-\frac{s}{2}\right)\,ds$

$$\ln v = 3\ln s - \frac{s^2}{4} = \ln s^3 - \ln e^{(s^2/4)} = \ln(s^3e^{-(s^2/4)})$$

$\implies v = s^3e^{-(s^2/4)}$. Substituting back to $(*)$, we have

$$\begin{aligned}uv&=\int \frac{9s+5}{-2s}\cdot s^3e^{-(s^2/4)}\,ds\\&=\int\left(-\frac{9}{2}s^2-\frac{5}{2}s\right)e^{-(s^2/4)}\,ds\end{aligned}$$

$$\begin{aligned}\int xe^{-(x^2/4)}\,dx&=-2e^{-(x^2/4)}\\\int x^2e^{-(x^2/4)}\,dx&=x(-2e^{-(x^2/4)})-\int -2e^{-(x^2/4)}\,dx\\&=x(-2e^{-(x^2/4)})+\int 2e^{-(x^2/4)}\,dx\end{aligned}$$

□

Definition 4.0.0 (Regular singular points) This is prerequisite knowledge for the next section on series solutions.

Consider the following ODE:

$$y''+P(x)y'+Q(x)y=R(x) \tag{$*$}$$

If we have both limits of

$$\lim_{x\to x_0}(x-x_0)P(x) \quad\text{and}\quad \lim_{x\to x_0}(x-x_0)^2Q(x)$$

finite, then x_0 is a regular singular point of the equation $(*)$.

Remark 4.0.1 (Series solutions) The main idea here is to try to write y as a power series:

$$y := \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

for some fixed $x_0, a_0, a_1, \ldots \in \mathbb{R}$.

This can be used to solve these problems:

- $y'' + y = 0$
- *(Airy's Equation)* $y'' - xy = 0$
- *(Hermite Equation)* $y'' - 2xy' + \lambda y = 0$

Note that this substitution is only valid when the x_0 chosen occurs at a regular singular point.

Remark 4.0.2 (Frobenius series) This is when we try the substitution

$$y := \sum_{n=0}^{\infty} a_n x^{r+n}$$

This is useful for solving these problems:

- *(Euler equations)* $x^2y'' + x\alpha y' + \beta y = 0$. Euler equations necessarily have a regular singular point (RSP) at 0.
- $2x^2y'' - x\alpha y' + (1+x)y = 0$. This needs showing that $x = 0$ is a regular singular point before continuing.
- *(Laguerre equation)* $xy'' + (1-x)y' + \lambda y = 0$. (RSP at 0)
- *(Chebyshev equation)* $(1-x^2)y'' - xy' + \lambda^2 y = 0$. (RSP at 1) Being the first non-zero RSP, here's how the series expansion looks like:

$$y := \sum_{n=0}^{\infty} a_n (x-1)^{r+n}$$

- *(Bessel equation)* $x^2y'' + xy' + (x^2 - \lambda^2)y = 0$. (RSP at 0)

Definition 5.0.0 (Dirac delta function) This function is an idealized unit impulse function, denoted by δ. It has the properties

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Since $\delta(t)$ corresponds to a unit impulse at $t = 0$, a unit impulse at an arbitrary point t_0 is given by $\delta(t - t_0)$.