Monitored long-range interacting spin systems

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Abstract

In this report, we focus on the continuous monitoring with the homodyne detection scheme on the long-range interaction of the power-law spin model. This monitored scheme leads to the equation of motion of the quantum system interacts with the device or environment with an additional dissipative term, all of them covers in the Lindblad master equation. The simplest version of this physical model is the infinite-long range interaction and single dissipative Lindblad operator, will be the showcase of our spin-wave method. With less complexity, the spin-wave method can capture the dynamics of the quantum trajectories, and shows the phase transition from stationary phase to time crystal phase, which can be predicted from mean-field theory, and can be seen in the SWQT method and Euler-Maruyama method (Exact method). On top of that, at the transition point, we observe a special phenomena, where the half-chain entanglement entropy exhibits its maximum point. Futhermore, our method is comparable to the stochastic behaviors of quantum trajectory from single to average ensemble with the exact trajectory.

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Stay Hungry. Stay Foolish.
by ***Steve Jobs***

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Chapter 1

Introduction

Long-range interacting systems have been extensively studied theoretically and experimentally [2, 3]. The motivation for the study of long-range interacting systems is that these systems allow highly entangled or correlated dynamical states to be realized. These properties are essential for the development of quantum computing and quantum simulation, which cannot be found in short-range interacting systems. In these systems, the interaction potential typically takes the form

$$V(r) \sim 1/r^{\alpha}. \tag{1.1}$$

Such interactions are characteristic of trapped ions [4], Bose-Einstein condensates in cavities [5], dipolar [6] and Rydberg atoms [7], or driven ultra-cold atomic gases [8], where α can be tunable. Long-range interactions are divided into two main categories, including:

- Weak long-range interactions: Infinite-range interactions with power-law behavior 1.1 for large r and for α such that $d < \alpha < \alpha^*$.
- Strong long-range interactions: Infinite-range interactions with power-law behavior 1.1 for large r and for $\alpha < d$.

Here, d is the dimension of the system, and α^* is the exponent value at which the critical behavior takes the form of a short-range interaction.

In these quantum long-range interacting system, the interaction with the environment or the controlling or monitored with the measurement equipment is inevitable. It necessitates to have a theory capable of describing open quantum systems [9, 10]. This equation of motion must satisfy the Markovian properties and describe its dissipative nature of open system. All of these dynamics can be accurately encoded within the Lindblad master equation.

Futhermore, with the development of experimental technologies, it's now possible to directly control the quantum systems, such as single trapped ions or cavity field modes [9]. However, when describing the equation of motion for an ensemble of these systems, we cannot capture phenomena like quantum jumps of single particles. It leads us to more profound quantum realm of particular quantum trajectories. By examining these trajectories in many-body systems, we find that quantum trajectories contain richer information than the ensemble-averaged dynamics. For instance, they reveal quantum phases and measurement-induced phase transitions, which are described by non-linear functions of the trajectories, such as entanglement probes or trajectory correlation functions [11–13].

In general, the theoretical study of many-particle, long-range interacting systems presents significant challenges. This has led researchers to explore numerical methods. However, state-of-the-art techniques like matrix-product states [14] or tensor networks [15], which are effective for one-dimensional and low-entropy systems, struggle to capture the correlations in non-equilibrium dynamics and are difficult to extend to higher dimensions. One promising alternative simulation method is the spin-wave method [1]. This approach was first inspired by the spin-wave theory developed by Bloch [16] in 1930 for ferromagnetic spin systems in equilibrium, and later by Holstein and Primakoff [17]. Whereas the author of [1], they focus on using the spin-wave method for heterodyne detection, in this work, we discuss mainly on the simulation of the homodyne detection scheme.

My thesis structure is organized at follows. In chapter 2, I focus in the average dynamics of open quantum systems, I derive from mathematical and physical point of view to construct the well-known

Lindblad master equation [18, 19]; the following chapter, we will zoom in particular quantum trajectories [10, 20–23] to see their quantum dynamics, which can be summarized inside the stochastic master equation, with a particular focus on homodyne detection. Chapter 4 will reveal the spin-wave theory for quantum trajectories and the long-range quantum spin system in the strong long-range interaction regime. Followed by the result of my simulation on chapter 5. Moreover, we have the Appendix A part to showcase the connection of different perspective of POVM and Lindblad master equation, the mean field calculation, and the Monte Carlo Wavefunction Method [24], which is the method that we use for simulating the Lindblad master equation dynamics.

Chapter 2

Lindblad Master Equation

2.1 Derivation of the Lindblad equation from microscopic dynamics

The Lindblad master equation presents the dynamics of the quantum subsystem that having the interaction with the environment. We work with the Hamiltonian that has weak interaction between our subsystem and the environment, the total system is a closed system. We start with the von Neumann equation of the total system

$$\frac{dp_T(t)}{dt} = -i[H_T, p_T(t)], \tag{2.1}$$

with $H_T = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + H_I$. The interaction term can be decomposed as

$$H_I = \sum_\alpha S_\alpha \otimes E_\alpha, \tag{2.2}$$

where S_α and E_α are Hermitian operator.

Revised Interaction picture

Assume, we work with the time-dependent Hamiltonian

$$\mathcal{H}(t) = H_0 + H_I(t). \tag{2.3}$$

Time-dependent Schrodinger equation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = [H_0 + H_I(t)]|\Psi(t)\rangle. \tag{2.4}$$

Set $A_I(t) = e^{iH_0(t-t_0)/\hbar}Ae^{-iH_0(t-t_0)/\hbar}$. Equation of motion of A_I (only true for time-dependent operator A

$$\frac{dA_I(t)}{dt} = \frac{i}{\hbar}H_0A_I - \frac{i}{\hbar}A_IH_0 = -\frac{i}{\hbar}[A_I(t), H_0] \tag{2.5}$$

Also, we define new state where $|\Psi(t)\rangle = e^{-iH_0(t-t_0)/\hbar}|\Psi_I(t)\rangle$. Now substitute back into time dependent Schrödinger equation, we have

$$\begin{aligned}[H_0 + H_I(t)]e^{-iH_0(t-t_0)/\hbar}|\Psi_I(t)\rangle &= i\hbar\frac{d}{dt}e^{-iH_0(t-t_0)/\hbar}|\Psi_I(t)\rangle \\ &= H_0e^{-iH_0(t-t_0)/\hbar}|\Psi_I(t)\rangle + i\hbar e^{-iH_0(t-t_0)/\hbar}\frac{\partial}{\partial t}|\Psi_I(t)\rangle.\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\implies e^{iH_0(t-t_0)/\hbar}H_I(t)e^{-iH_0(t-t_0)/\hbar}|\Psi_I(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle \\ &\implies H_I^I(t)\frac{\partial}{\partial t}|\Psi_I(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle.\end{aligned} \tag{2.7}$$

Now, let's apply these calculation in our system. Notice that $H_0 = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E$, we have

$$\begin{aligned}\frac{d\rho_T^I}{dt} &= \frac{d}{dt}\left[|\Psi_I(t)\rangle\langle\Psi_I(t)|\right] = \frac{d}{dt}|\Psi_I(t)\rangle\langle\Psi_I(t)| + |\Psi_I(t)\rangle\frac{d}{dt}\langle\Psi_I(t)|, \\ &= -\frac{i}{\hbar}H_I^I(t)\rho_T^I + \frac{i}{\hbar}\rho_T^I H_I^I(t) \overset{\hbar=1}{=} -i[H_I^I(t), p_T^I(t)].\end{aligned} \tag{2.8}$$

We integrate out this equation one time to have

$$\rho_T^I(t) = \rho_T^I(0) - i\int_0^t ds[H_I^I(s), \rho_T^I(s)]. \tag{2.9}$$

Introduce equation 2.9 into 2.8, we have

$$\begin{aligned}\frac{d}{dt}\rho_T^I(t) &= -i\alpha[H_I^I(t), \rho_T^I(0) - i\int_0^t ds[H_I^I(s)], \\ &= -i[H_I^I(t), \rho_T^I(0)] - [H_I^I(t), \int_0^t ds[H_I^I(s), \rho_T^I(s)]].\end{aligned} \tag{2.10}$$

Trace our the environment part of this equation, we have

$$\frac{d}{dt}\rho_S^I(t) = -\int_0^t \mathrm{Tr}_E[H_I^I(t), [H_I^I(s), \rho_T^I(s)]]. \tag{2.11}$$

Here, we use the Born approximation, assume the coupling between the environment and the system is weak or time scale of the correlation and relaxation time of the environment is much more smaller than typical time scale of the system. So we can decompose the total density matrix in terms of the system and environment density matrix. The system density matrix is always in the thermal state

$$\rho_T^I(t) = \rho_S^I(t) \otimes \rho_E^I(0), \tag{2.12}$$

with $\rho_E = \dfrac{\exp\{-H_E/T\}}{\mathrm{Tr}[\exp\{-H_E/T\}]}$. This is the well-known **Redfield equation**. We substitute s by t - s, and let's the upper limit to infinity. This is permissible provided the integrand disappears sufficiently fast for $s \gg \tau_B$ The Markov approximation is therefore justified if the time scale τ_S over which the state of the system varies appreciably is large compared to the time scale τ_B over which the reservoir correlation functions decay. We obtain the Markovian quantum master equation

$$\frac{d}{dt}\rho_S^I(t) = -\int_0^\infty ds\mathrm{Tr}_E[H_I^I(t), [H_I^I(t-s), \rho_S^I(t) \otimes \rho_E^I(0)]]. \tag{2.13}$$

Then, let's work with the Schrodinger picture of the interaction term, we define the operators

$$S_\alpha(\omega) = \sum_{\epsilon'-\epsilon=\omega} P_\epsilon S_\alpha P_{\epsilon'}, \tag{2.14}$$

where $P_\epsilon = |\psi_\epsilon\rangle\langle\psi_\epsilon|$ is the quantum projector of the eigenstate of Hamiltonian system, corresponds to eigenvalue ϵ. We realize the following properties

$$[H_S, S_\alpha(\omega)] = H_S\sum_{\epsilon'-\epsilon=\omega} P_\epsilon S_\alpha P_{\epsilon'} - \sum_{\epsilon'-\epsilon=\omega} P_\epsilon S_\alpha P_{\epsilon'} H_S = \epsilon S_\alpha(\omega) - \epsilon' S_\alpha(\omega) = -\omega S_\alpha(\omega), \tag{2.15}$$

$$[H_S, S_\alpha^\dagger(\omega)] = H_S\sum_{\epsilon'-\epsilon=\omega} P_{\epsilon'} S_\alpha P_\epsilon - \sum_{\epsilon'-\epsilon=\omega} P_{\epsilon'} S_\alpha P_\epsilon H_S = \epsilon S_\alpha(\omega') - \epsilon S_\alpha(\omega) = \omega S_\alpha^\dagger(\omega). \tag{2.16}$$

We have the completeness relation

$$\sum_\omega S_\alpha(\omega) = \sum_\omega S_\alpha^\dagger(\omega) = S_\alpha. \tag{2.17}$$

The Schrodinger

$$H_I = \sum_{\alpha,\omega} S_\alpha(\omega) \otimes E_\alpha = \sum_{\alpha,\omega} S_\alpha^\dagger(\omega) \otimes E_\alpha^\dagger. \tag{2.18}$$

The interaction picture of the interaction Hamiltonian

$$\begin{aligned} H_I^I(t) &= e^{i(H_S\otimes \mathbb{1}_E + \mathbb{1}_S\otimes H_E)t} \sum_{\alpha,\omega} S_\alpha(\omega) \otimes E_\alpha e^{-i(H_S\otimes \mathbb{1}_E + \mathbb{1}_S\otimes H_E)t} = \sum_{\alpha,\omega} e^{-i\omega t} S_\alpha(\omega) \otimes E_\alpha(t), \\ &= e^{i(H_S\otimes \mathbb{1}_E + \mathbb{1}_S\otimes H_E)t} \sum_{\alpha,\omega} S_\alpha^\dagger(\omega) \otimes E_\alpha^\dagger e^{-i(H_S\otimes \mathbb{1}_E + \mathbb{1}_S\otimes H_E)t} = \sum_{\alpha,\omega} e^{i\omega t} S_\alpha^\dagger(\omega) \otimes E_\alpha^\dagger(t). \end{aligned} \tag{2.19}$$

Now, we insert the interaction picture of interaction term into our equation of motion of our density matrix to have

$$\begin{aligned} \frac{d}{dt}\rho_S^I(t) &= -\int_0^\infty ds \mathrm{Tr}_E \left[H_I^I(t), \left[H_I^I(t-s), \rho_S^I(t) \otimes \rho_E^I(0) \right] \right] \\ &= -\int_0^\infty \mathrm{Tr}_E \left[H_I^I(t) H_I^I(t-s) \rho_S^I(s) \otimes \rho_E^I(0) - H_I^I(t) \rho_S^I(t) \otimes \rho_E^I(0) H_I^I(t-s) \right. \\ &\quad \left. - H_I^I(t-s) \rho_S^I(t) \otimes \rho_E^I(0) H_I(t) + \rho_S^I(t) \otimes \rho_E^I(0) H_I^I(t-s) H_I^I(t) \right] \\ &= \int_0^\infty \mathrm{Tr}_E \left[H_I^I(t-s) \rho_S^I(t) \otimes \rho_E^I(0) H_I(t) - H_I^I(t) H_I^I(t-s) \rho_S^I(t) \otimes \rho_E^I(0) \right] + \text{h.c.} \\ &= \int_0^\infty \mathrm{Tr}_E \left[\sum_{\beta,\omega} e^{-i\omega(t-s)} S_\beta(\omega) \otimes E_\beta(t-s) \rho_S^I(t) \otimes \rho_E^I(0) \sum_{\alpha,\omega'} e^{i\omega' t} S_\alpha^\dagger(\omega') \otimes E_\alpha^\dagger(t) \right. \\ &\quad \left. - \sum_{\alpha,\omega'} e^{i\omega' t} S_\alpha^\dagger(\omega') \otimes E_\alpha^\dagger(t) \sum_{\beta,\omega} e^{-i\omega(t-s)} S_\beta(\omega) \otimes E_\beta(t-s) \rho_S^I(t) \otimes \rho_E^I(0) \right] + \text{h.c.} \\ &= \sum_{\omega,\omega'} \sum_{\alpha,\beta} e^{i(\omega'-\omega)t} \int_0^\infty ds e^{i\omega s} \mathrm{Tr}_E \left[E_\alpha^\dagger(t) E_\beta(t-s) \rho_E^I(0) \right] S_\beta(\omega) \rho_S^I(t) S_\alpha^\dagger(\omega') \\ &\quad - \sum_{\omega,\omega'} \sum_{\alpha,\beta} e^{i(\omega'-\omega)t} \int_0^\infty ds e^{i\omega s} \mathrm{Tr}_E \left[E_\alpha^\dagger(t) E_\beta(t-s) \rho_E^I(0) \right] S_\alpha^\dagger(\omega') S_\beta(\omega) \rho_S^I(t) \\ &= \sum_{\omega,\omega'} \sum_{\alpha,\beta} e^{i(\omega'-\omega)t} \Gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega) \rho_S^I(t) S_\alpha^\dagger(\omega') - S_\alpha^\dagger(\omega') S_\beta(\omega) \rho_S^I(t) \right], \end{aligned} \tag{2.20}$$

where $\Gamma_{\alpha\beta}(\omega)$ is the Fourier transform of the reservoir correlation functions.

$$\frac{d}{dt}\rho_S^I(t) = \sum_{\omega} \sum_{\alpha,\beta} \Gamma_{\alpha\beta}(\omega) \left[S_\beta(\omega) \rho_S^I(t) S_\alpha^\dagger(\omega') - S_\alpha^\dagger(\omega') S_\beta(\omega) \rho_S^I(t) \right]. \tag{2.21}$$

Let's decompose $\Gamma_{\alpha\beta}$ to the complex and real part

$$\Gamma_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + i\pi_{\alpha\beta}(\omega) \qquad \text{with} \qquad \begin{cases} \pi_{\alpha\beta}(\omega) & = \frac{1}{2i}\left(\Gamma_{\alpha\beta}(\omega) - \Gamma_{\beta\alpha}^*(\omega) \right), \\ \gamma_{\alpha\beta} & = \Gamma_{\alpha\beta}(\omega) + \Gamma_{\beta\alpha}^*(\omega). \end{cases} \tag{2.22}$$

Again, we substitute back to equation 2.22 to have

$$
\begin{aligned}
\frac{d}{dt}\rho_S^I(t) &= \sum_\omega \sum_{\alpha\beta} \left[\frac{1}{2}\gamma_{\alpha\beta}(\omega) + i\pi_{\alpha\beta}(\omega)\right] \left[S_\beta(\omega)\rho_S^I(t)S_\alpha^\dagger(\omega) - S_\alpha^\dagger(\omega)S_\beta(\omega)\rho_S^I(t)\right] \\
&+ \sum_\omega \sum_{\alpha\beta} \left[\frac{1}{2}\gamma_{\alpha\beta}(\omega) - i\pi_{\alpha\beta}(\omega)\right] \left[S_\beta(\omega)\rho_S^I(t)S_\alpha^\dagger(\omega) - \rho_S^I(t)S_\alpha^\dagger(\omega)S_\beta(\omega)\right] \\
&= \sum_\omega \sum_{\alpha\beta} \gamma_{\alpha\beta}(\omega)S_\beta(\omega)\rho_S^I(t)S_\alpha^\dagger(\omega) - i\left[\underbrace{\sum_\omega \sum_{\alpha\beta} \pi_{\alpha\beta}(\omega)S_\alpha^\dagger(\omega)S_\beta(\omega)}_{H_{LS}}, \rho_S^I(t)\right] \\
&- \frac{1}{2}\sum_\omega \sum_{\alpha\beta} \gamma_{\alpha\beta}(\omega)\left\{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S^I(t)\right\} \\
&= -i[H_{LS}, \rho_S^I(t)] + D(p_S^I(t)), \text{ with } D(\rho_S^I) \\
&= \sum_\omega \sum_{\alpha\beta} \gamma_{\alpha\beta}(\omega)\left[S_\beta(\omega)\rho_S^I(t)S_\alpha^\dagger(\omega) - \frac{1}{2}\left\{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S^I(t)\right\}\right].
\end{aligned}
\tag{2.23}
$$

We notice the relationship between the density matrix in Schrodinger and interaction picture

$$
\begin{aligned}
\frac{d}{dt}\rho_S(t) &= \frac{d}{dt}\left[e^{-iH_0t}\rho_S^I(t)e^{iH_0t}\right] = \frac{d}{dt}\left[e^{-iH_St}\rho_S^I(t)e^{iH_St}\right] \\
&= -iH_se^{-iH_St}e^{iH_St}\rho_S(t)e^{-iH_St}e^{iH_St} + e^{-iH_St}\frac{d}{dt}\rho_S^I(t)e^{iH_St} + ie^{-iH_St}e^{iH_St}\rho_S(t)e^{-iH_St}e^{iH_St}H_S \\
&= -i[H_S, \rho_S(t)] + e^{-iH_St}\left\{-i[H_{LS}, \rho_S^I(t)] + D(\rho_S^I)\right\}e^{iH_St}.
\end{aligned}
\tag{2.24}
$$

Commutation properties

Before further derivation, we must notice that

$$
\begin{aligned}
[H_S, S_\alpha^\dagger(\omega)S_\beta(\omega)] &= [H_S, S_\alpha^\dagger(\omega)]S_\beta(\omega) + S_\alpha^\dagger(\omega)[H_S, S_\beta(\omega)] \\
&\overset{2.19}{=} \omega S_\alpha^\dagger(\omega)S_\beta(\omega) - \omega S_\alpha^\dagger(\omega)S_\beta(\omega) = 0,
\end{aligned}
\tag{2.25}
$$

, followed by the commutable of $[H_S, H_{LS}] = 0$. Also based on the definition of $S_\alpha(\omega)$ and $S_\alpha^\dagger(\omega)$, we have

$$
e^{iH_St}S_\alpha(\omega)e^{-iH_St} = e^{iH_St}\sum_{\epsilon'-\epsilon=\omega} P_\epsilon S_\alpha P_{\epsilon'}e^{-iH_St} = e^{-i\omega t}S_\alpha(\omega), \tag{2.26}
$$

$$
e^{iH_St}S_\alpha^\dagger(\omega)e^{-iH_St} = e^{iH_St}\sum_{\epsilon'-\epsilon=\omega} P_{\epsilon'} S_\alpha^\dagger P_{\epsilon}e^{-iH_St} = e^{+i\omega t}S_\alpha^\dagger(\omega). \tag{2.27}
$$

Apply these intuition back to equation 2.25, we have

$$
\begin{aligned}
\frac{d}{dt}\rho_S(t) &= -i[H_S, \rho_S(t)] - i[H_{LS}, \rho_S(t)] + D(\rho_S(t)) = -i[H_S + H_{LS}, \rho_S(t)] + D(\rho_S(t)), \\
&= -i[H_S + H_{LS}, \rho_S(t)] + \sum_\omega \sum_{\alpha\beta} \gamma_{\alpha\beta}(\omega)\left[S_\beta(\omega)\rho_S(t)S_\alpha^\dagger(\omega) - \frac{1}{2}\left\{S_\alpha^\dagger(\omega)S_\beta(\omega), \rho_S(t)\right\}\right].
\end{aligned}
\tag{2.28}
$$

Because $\gamma_{\alpha\beta}$ is positive, we can diagonalize this matrix $\gamma_{\alpha\beta} = \sum_i O^\dagger_{\alpha i} D_{ii}(\omega) O_{i\beta}$. We have

$$\begin{aligned}
\frac{d}{dt}\rho_S(t) = -i[H_S + H_{LS}, \rho_S(t)] \\
&+ \sum_\omega \sum_{\alpha\beta} \sum_i O^\dagger_{\alpha i} D_{ii}(\omega) O_{i\beta} \left[S_\beta(\omega)\rho_S(t)S^\dagger_\alpha(\omega) - \frac{1}{2}\left\{S^\dagger_\alpha(\omega)S_\beta(\omega), \rho_S(t)\right\}\right] \\
&= -i[H_S + H_{LS}, \rho_S(t)] + \sum_{i,\omega} \left\{ \left[\sum_\beta O_{i\beta}S_\beta(\omega)\right] \rho_S(t) \left[\sum_\beta S^\dagger_\alpha(\omega) O^\dagger_{\alpha i}\right] \right. \\
&\left. -\frac{1}{2}\left\{\left[\sum_\beta \sqrt{D_{ii}(\omega)} S^\dagger_\alpha(\omega) O^\dagger_{\alpha i} - \right]\left[\sum_\beta \sqrt{D_{ii}(\omega)} O_{i\beta} S_\beta(\omega)\right], \rho_S(t)\right\}\right\} \\
&= -i[H_S + H_{LS}, \rho_S(t)] + \sum_{i,\omega} \left[L_i(\omega)\rho_S(t) L^\dagger_i(\omega) - \frac{1}{2}\left\{ L^\dagger_i(\omega) L_i(\omega), \rho_S(t)\right\}\right],
\end{aligned} \tag{2.29}$$

where L_i are usually referred to jump operators. This is the well-known Lindblad (or Lindblad-Gorini-Kossakowski-Sudarshan) master equation.

2.2 Derivation of the Lindblad master equation as a CPT generator

Another approach for deriving the Lindblad master equation from mathematical point of view, where we define the type of operators that satisfy the complete positive and trace-preserving properties.

Theorem 1 (Choi's theorem)**.** *A linear map V: B(H) $\rightarrow$ B(H) is completely positive iff it can*

$$V\rho = \sum_i V_i \rho V_i^\dagger, \tag{2.30}$$

with $V_i \in B(H)$.

We can prove the "if" condition easily, because the density matrix is the positive operator so we can express it as $\rho = A^\dagger A$,

$$\langle\psi|V\rho|\psi\rangle = \langle\psi|\sum_i V_i\rho V_i^\dagger|\psi\rangle = \sum_i \langle\psi|V_i A^\dagger A V_i^\dagger|\psi\rangle = \sum_i ||AV_i^\dagger|\psi\rangle||^2 \geq 0. \tag{2.31}$$

The following calculation need to prove the "only if" condition, we define an unnormalized maximally pure state

$$|\Gamma\rangle = \sum_{i=0}^{d-1} |i\rangle_A \otimes |i\rangle, \tag{2.32}$$

where $i\rangle_A$, $|i\rangle$ are the orthonormal base of H and H_A. Let's consider the complete positive operator $V_2 = \mathbb{1}_{B(H_A)} \otimes V$. We have

$$V_2|\Gamma\rangle\langle\Gamma| = \left(\mathbb{1}_{B(H_A)} \otimes V\right) \sum_{i,j=0}^{d-1} (|i\rangle_A \otimes |i\rangle)(\langle j|_A \otimes \langle j|) = \sum_{i,j=0}^{d-1} |i\rangle\langle j|_A \otimes V|i\rangle\langle j|. \tag{2.33}$$

We define the arbitrary vector $|\psi\rangle = \sum_{i,j=0}^{d-1} \alpha_{ij}|i\rangle_A \otimes |j\rangle$, we define the transformation $\mathbb{1} \otimes V_{|\psi\rangle} = \mathbb{1}_A \otimes \sum_{i,j=0}^{d-1} \alpha_{ij}|j\rangle\langle i|$ that converts $|\Gamma\rangle$ to $|\psi\rangle$.

$$\left[\mathbb{1}_A \otimes \sum_{i,j=0}^{d-1} \alpha_{ij}|j\rangle\langle i|\right] \sum_{l=0}^{d-1} |l\rangle_A \otimes |l\rangle = \sum_{i,j=0}^{d-1} \alpha_{ij}|i\rangle_A \otimes |j\rangle. \tag{2.34}$$

Because of the maximally entangled properties of the state $|\Gamma\rangle$, we can relate the vector in the extended space $H_A \otimes H$ with the operator that acts locally in the Hilbert space H. Now, because V_2 is a positive operator, we have

$$V_2|\Gamma\rangle\langle\Gamma| = \sum_{i=0}^{d^2-1} |v_l\rangle\langle v_l|, \tag{2.35}$$

with $|v_l\rangle = (\mathbb{1} \otimes V_l)|\Gamma\rangle$. We observe that

$$\begin{aligned}
&(\langle i|_A \otimes \mathbb{1})\, V_2|\Gamma\rangle\langle\Gamma|\,(|j\rangle_A \otimes \mathbb{1}) = (\langle i|_A \otimes \mathbb{1}) \sum_{i=0}^{d^2-1} |v_l\rangle\langle v_l|\,(|j\rangle_A \otimes \mathbb{1})\,,\\
\Leftrightarrow\ &(\langle i|_A \otimes \mathbb{1}) \sum_{k,l=0}^{d-1} |k\rangle\langle l|_A \otimes V|k\rangle\langle l|\,(|j\rangle_A \otimes \mathbb{1}) = (\langle i|_A \otimes \mathbb{1}) \sum_{i=0}^{d^2-1} (\mathbb{1} \otimes V_l)|\Gamma\rangle\langle\Gamma|(\mathbb{1} \otimes V_l^\dagger)\,(|j\rangle_A \otimes \mathbb{1})\\
\Leftrightarrow\ &V|i\rangle\langle j| = (\langle i|_A \otimes \mathbb{1}) \sum_{l=0}^{d^2-1} (\mathbb{1} \otimes V_l) \sum_{m,n=0}^{d-1} (|m\rangle_A \otimes |m\rangle)\,(\langle n|_A \otimes \langle n|)\,(\mathbb{1} \otimes V_l^\dagger)\,(|j\rangle_A \otimes \mathbb{1})\\
\Leftrightarrow\ &V|i\rangle\langle j| = \sum_{l=0}^{d^2-1} \sum_{m,n=0}^{d-1} \delta_{im}\delta_{jn} V_l|m\rangle\langle n|V_l^\dagger = \sum_{l=0}^{d^2-1} V_l|i\rangle\langle j|V_l^\dagger.
\end{aligned} \tag{2.36}$$

Because any operators can be construct from the basis of $|i\rangle\langle j|$, we directly prove that

$$V\rho = V_l\rho V_l^\dagger. \tag{2.37}$$

Theorem 2 (Choi-Krauss' theorem)**.** *A linear map $\mathcal{V} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is completely positive and trace-preserving iff it can be expressed as*

$$V\rho = \sum_l V_l\rho V_l^\dagger, \tag{2.38}$$

with $V_l \in B(H)$ fulfilling

$$\sum_l V_l^\dagger V_l = \mathbb{1}_H. \tag{2.39}$$

We will now prove the trace-preserving part of the theorem. Specifically, we will prove that the map $V\rho$ is trace-preserving, if and only if the operators $\{V_l\}$ satisfy the condition $\sum_l V_l^\dagger V_l = \mathbb{1}_H$. The "if" condition can be proved directly by the following calculation

$$\mathrm{Tr}[V\rho] = \mathrm{Tr}\left[\sum_l V_l^\dagger \rho V_l\right] = \mathrm{Tr}\left[\sum_l V_l^\dagger V_l \rho\right] = \mathrm{Tr}[\rho]. \tag{2.40}$$

Now, we need to prove the "only-if" condition. If our operators satisfy the trace-preserving condition, that means

$$\mathrm{Tr}\,[V|i\rangle\langle j|] = \mathrm{Tr}\,[|i\rangle\langle j|] = \langle j|i\rangle = \delta_{ij}. \tag{2.41}$$

Another perspective that we can follow is

$$\mathrm{Tr}\,[V|i\rangle\langle j|] = \mathrm{Tr}\left[\sum_l V_l|i\rangle\langle j|V_l^\dagger\right] = \mathrm{Tr}\left[\sum_l V_l^\dagger V_l|i\rangle\langle j|\right] = \langle j|\sum_l V_l^\dagger V_l|i\rangle, \tag{2.42}$$

and therefore in order to satisfy the trace-conserving condition, we must have $\sum_l V_l^\dagger V_l = \mathbb{1}_H$.

2.3 Lindblad master equation derivation

While the structure of the CPT-maps is well established, however still we don't have the continuous set of differential equation. Our approach begins by selecting the orthonormal basis of the bounded space B(H)

$$\langle\langle F_i, F_j\rangle\rangle = \mathrm{Tr}\left(F_i^\dagger F_j\right) = \delta_{ij}. \tag{2.43}$$

Without loss of generality, we set $F_{d^2} = \frac{1}{\sqrt{d}}\mathbb{1}_H$

$$\mathrm{Tr}\left[F_i\right] = \mathrm{Tr}\left[F_i\sqrt{d}F_{d^2}\right] = 0 \quad (d^2 \neq i). \tag{2.44}$$

The closure relation of this basis $\mathbb{1}_H = \sum_i |F_i\rangle\rangle\langle\langle F_i|$. We can express the Krauss operators V_l in the Fock-Liouville notation

$$V_l(t) = \sum_{i=1}^{d^2}\langle\langle F_i|V_\epsilon(t)\rangle\rangle|F_i\rangle\rangle. \tag{2.45}$$

We have

$$V(t)\rho = \sum_l\left[\sum_{i=1}^{d^2}\langle\langle F_i|V_l(t)\rangle\rangle|F_i\rangle\rangle\rho\sum_{j=1}^{d^2}\langle\langle F_j|\langle\langle V_l(t)|F_j\rangle\rangle\right] \tag{2.46}$$

$$\overset{|F_i\rangle\rangle\equiv F_i}{=}\sum_{i,j=1}^{d^2}\sum_l\langle\langle F_i|V_l(t)\rangle\rangle|F_i\rangle\rangle\langle\langle F_j|\langle\langle V_l(t)|F_j\rangle\rangle F_i\rho F_j^\dagger = \sum_{i,j=1}^{d^2}c_{i,j}(t)F_i\rho F_j^\dagger. \tag{2.47}$$

We define the equation of motion of the density matrix $\dfrac{d}{dt}\rho(t) = L\rho$. The CPT-map could be expressed as $V(t) = e^{Lt}$. We have

$$\begin{aligned}
\frac{d\rho}{dt} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[V(t+\Delta t)\rho - \rho\right] = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\sum_{i,j=1}^{d^2}c_{i,j}(\Delta t)F_i\rho F_j^\dagger - \rho\right)\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\sum_{i,j=1}^{d^2-1}c_{ij}(\Delta t)F_i\rho F_j^\dagger + \sum_{i=1}^{d^2-1}c_{id^2}(\Delta t)F_i\rho F_{d^2}^\dagger\right.\\
&\qquad\left. + \sum_{j=1}^{d^2-1}c_{d^2j}(\Delta t)F_{d^2}\rho F_j^\dagger + c_{d^2d^2}(\Delta t)F_{d^2}\rho F_{d^2}^\dagger - \rho\right]\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\sum_{i,j=1}^{d^2-1}c_{ij}(\Delta t)F_i\rho F_j^\dagger + \frac{1}{\sqrt{d}}\sum_{i=1}^{d^2-1}c_{id^2}(\Delta t)F_i\rho\right.\\
&\qquad\left. + \frac{1}{\sqrt{d}}\sum_{j=1}^{d^2-1}c_{d^2j}(\Delta t)\rho F_j^\dagger + \frac{1}{d}c_{d^2d^2}(\Delta t)\rho - \rho\right].
\end{aligned} \tag{2.48}$$

The next step is to eliminate the explicit time dependence but set up some constants

$$\begin{cases}
g_{ij} = \lim\limits_{\Delta t\to 0}\dfrac{c_{ij}(\Delta t)}{\Delta t} & (i,j < d^2),\\
g_{id^2} = \lim\limits_{\Delta t\to 0}\dfrac{c_{id^2}(\Delta t)}{\Delta t} & (i < d^2),\\
g_{d^2j} = \lim\limits_{\Delta t\to 0}\dfrac{c_{d^2j}(\Delta t)}{\Delta t} & (j < d^2),\\
g_{d^2d^2} = \lim\limits_{\Delta t\to 0}\dfrac{c_{d^2d^2} - d}{\Delta t},
\end{cases}$$

to have

$$\frac{d\rho}{dt}=\sum_{i,j=1}^{d^2-1}g_{i,j}F_i\rho F_j^\dagger+\frac{1}{\sqrt{d}}\sum_{i=1}^{d^2-1}g_{i,d^2}F_i\rho+\frac{1}{\sqrt{d}}\sum_{j=1}^{d^2-1}g_{d^2,j}\rho F_j^\dagger+\frac{g_{d^2,d^2}}{d}\rho. \tag{2.49}$$

We define the equation $F\equiv\frac{1}{\sqrt{d}}\sum_{i=1}^{d^2-1}g_{i,d^2}F_i.$, substitute back to the previous equation, we have

$$\frac{d\rho}{dt}=\sum_{i,j=1}^{d^2-1}g_{i,j}F_i\rho F_j^\dagger+F\rho+\rho F^\dagger+\frac{g_{d^2,d^2}}{d}\rho. \tag{2.50}$$

We split operator F to the Hermitian and non-Hermitian part $F=\frac{F+F^\dagger}{2}+i\frac{F-F^\dagger}{2i}=G-iH$, we have

$$\begin{aligned}\frac{d\rho}{dt}&=\sum_{i,j=1}^{d^2-1}g_{i,j}F_i\rho F_j^\dagger+(G-iH)\,\rho+\rho\,(G+iH)+\frac{g_{d^2,d^2}}{d}\rho\\&=\sum_{i,j=1}^{d^2-1}g_{ij}F_i\rho F_j^\dagger+\left[\left(G+\frac{g_{d^2,d^2}}{2d}\right)\rho+\rho\left(G+\frac{g_{d^2,d^2}}{2d}\right)\right]-i[H,\rho]\\&=\sum_{i,j=1}^{d^2-1}g_{ij}F_i\rho F_j^\dagger+\{G_2,\rho\}-i[H,\rho].\end{aligned} \tag{2.51}$$

Now let's use trace preserving property

$$\mathrm{Tr}\left[\frac{d\rho}{dt}\right]=\mathrm{Tr}\left[\sum_{i,j=1}^{d^2-1}F_i\rho F_j^\dagger+2G_2\rho+0\right]=0. \tag{2.52}$$

$$\implies G_2=-\frac{1}{2}\sum_{i,j=1}^{d^2-1}F_j^\dagger F_i g_{ij}. \tag{2.53}$$

Substitute back to equation 2.51,

$$\begin{aligned}\frac{d\rho}{dt}&=-i[H,\rho]+\sum_{i,j=1}^{d^2-1}g_{ij}F_i\rho F_j^\dagger+\left\{-\frac{1}{2}\sum_{i,j=1}^{d^2-1}F_j^\dagger F_i g_{ij},\rho\right\}\\&=-i[H,\rho]+\sum_{i,j=0}^{d^2-1}g_{ij}\left[F_i\rho F_j^\dagger-\frac{1}{2}\left\{F_j^\dagger F_i,\rho\right\}\right].\end{aligned} \tag{2.54}$$

We can arrange the coefficient g_{ij} to form the Hermitian matrix, so that we can diagonalize $g_{ij}=\sum_k M_{ik}^\dagger\Gamma_k M_{kj}$

$$\begin{aligned}\frac{d\rho}{dt}&=-i[H,\rho]+\sum_{i,j=0}^{d^2-1}M_{ik}^\dagger\Gamma_k M_{kj}\left[F_i\rho F_j^\dagger-\frac{1}{2}\left\{F_j^\dagger F_i,\rho\right\}\right]\\&=-i[H,\rho]\\&\quad+\sum_k\left[\sum_{i=0}^{d^2-1}\sqrt{\Gamma_k}F_{ik}M_{ik}^\dagger\,\rho\sum_{j=0}^{d^2-1}\sqrt{\Gamma_k}M_{kj}F_j^\dagger-\frac{1}{2}\left\{\sum_{j=0}^{d^2-1}\sqrt{\Gamma_k}M_{kj}F_j^\dagger\sum_{i=0}^{d^2-1}\sqrt{\Gamma_k}F_{ik}M_{ik}^\dagger,\rho\right\}\right]\\&=-i[H,\rho]+\sum_k\left[L_k\rho L_k^\dagger-\frac{1}{2}\left\{L_k^\dagger L_k,\rho\right\}\right].\end{aligned} \tag{2.55}$$

Again, we receive the Lindblad master equation.

Chapter 3

Stochastic Master Equation of Homodyne detection

We start with the standard form of the deterministic Lindblad master equation [10]

$$d\rho = -idt[\hat{H},\rho] + dt\mathcal{D}[\hat{L}]\rho, \tag{3.1}$$

where

$$\mathcal{D}[\hat{L}] = \hat{L} \bullet \hat{L}^\dagger - \frac{1}{2}\left\{\hat{L}^\dagger\hat{L},\bullet\right\}. \tag{3.2}$$

We notice this equation is invariant under the following transformation

$$\hat{L} \to \hat{L} + \gamma; \qquad \hat{H} \to \hat{H} - \frac{i}{2}\left(\gamma^*\hat{L} - \gamma\hat{L}^\dagger\right), \tag{3.3}$$

where γ is an arbitrary complex number.

The scheme of homodyne detection is the detection scheme that has the interference of the output field of the cavity and only a small amount of the strong signal from the local oscillator with the same frequency, through the low-reflectivity beam-splitter. Under this type of measurement, the measurement operators will be changed under the equation 3.3, with the real γ parameter indicates the strength of the local oscillator,

$$\hat{M}_1(dt) = \sqrt{dt}\hat{L} \;\; \to \;\; \hat{M}_1(dt) = \sqrt{dt}(\hat{L}+\gamma), \tag{3.4}$$

$$\hat{M}_0(dt) = \mathbb{1} - dt\left[i\hat{H} + \frac{1}{2}\hat{L}^\dagger\hat{L}\right] \;\; \to \;\; \hat{M}_0(dt) = \mathbb{1} - dt\left[i\hat{H} + \frac{1}{2}(\hat{L}-\hat{L}^\dagger)\gamma + \frac{1}{2}(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\right]. \tag{3.5}$$

In here, $\hat{M}_1$ and $\hat{M}_0$ are corresponding to a detection and no-detection operator respectively. Let's consider how our state changes with these measurement operators

$$\frac{M_1|\psi(t)\rangle}{\sqrt{\langle M_1^\dagger M_1\rangle}} = \frac{\sqrt{dt}(\hat{L}+\gamma)|\psi(t)\rangle}{\sqrt{dt\langle\psi(t)|(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)|\psi(t)\rangle}} = \frac{(\hat{L}+\gamma)|\psi(t)\rangle}{\sqrt{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle}}. \tag{3.6}$$

We have

$$\sqrt{\langle\psi(t)|\hat{M}_0^\dagger(dt)\hat{M}_0(dt)|\psi(t)\rangle}^{-1} = 1 + \frac{dt}{2}\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle, \tag{3.7}$$

that leads

$$\begin{aligned}\frac{M_0|\psi(t)\rangle}{\sqrt{\langle M_0^\dagger M_0\rangle}} &= \left\{\mathbb{1} + dt\left[\frac{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle}{2} - i\hat{H} - \frac{1}{2}(\hat{L}-\hat{L}^\dagger)\gamma - \frac{1}{2}(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\right]\right\}|\psi(t)\rangle \\ &= \left\{\mathbb{1} + dt\left[\frac{\langle\hat{L}^\dagger\hat{L}\rangle}{2} - \frac{\hat{L}^\dagger\hat{L}}{2} + \gamma\cdot\frac{\langle\hat{L}^\dagger+\hat{L}\rangle}{2} - \gamma\hat{L} - iH\right]\right\}|\psi(t)\rangle.\end{aligned} \tag{3.8}$$

In the end, we have the quantum trajectory of simple homodyne detection

$$
\begin{aligned}
d|\psi(t)\rangle = dN & \left[\frac{(\hat{L}+\gamma)|\psi(t)\rangle}{\sqrt{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle}} - \mathbb{1}\right]|\psi\rangle \\
& + dt\left[\frac{\langle\hat{L}^\dagger\hat{L}\rangle}{2} - \frac{\hat{L}^\dagger\hat{L}}{2} + \gamma\cdot\frac{\langle\hat{L}^\dagger+\hat{L}\rangle}{2} - \gamma\hat{L} - iH\right]|\psi\rangle,
\end{aligned} \tag{3.9}
$$

where N(t) is the number of detection up to time t, the stochastic increment of dN(t) obeys

$$
dN(t)^2 = dN(t), \tag{3.10}
$$

$$
\mathrm{E}[dN(t)] = \left\langle \hat{M}_1^\dagger(dt)\hat{M}_1(dt)\right\rangle = dt\left\langle\psi(t)\middle|\hat{L}^\dagger\hat{L}\middle|\psi(t)\right\rangle, \tag{3.11}
$$

which is an example of point process. We have the following stochastic master equation

$$
\begin{aligned}
d\rho &= d|\psi\rangle\langle\psi| + |\psi\rangle d\langle\psi| + d|\psi\rangle d\langle\psi| \\
&= dN\left[\frac{\hat{L}+\gamma}{\sqrt{\langle\hat{L}^\dagger+\gamma\rangle\langle\hat{L}+\gamma\rangle}} - \mathbb{1}\right]\rho + dt\left[\frac{\langle\hat{L}^\dagger\hat{L}\rangle}{2} + \frac{\hat{L}^\dagger\hat{L}}{2} + \frac{\langle\hat{L}^\dagger\gamma+\gamma\hat{L}\rangle}{2} - \gamma\hat{L} - iH\right]\rho \\
&\quad + \rho dN\left[\frac{\hat{L}^\dagger+\gamma}{\sqrt{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle}} - \mathbb{1}\right] + dt\rho\left[\frac{\langle\hat{L}^\dagger\hat{L}\rangle}{2} - \frac{\hat{L}^\dagger\hat{L}}{2} + \frac{\langle\hat{L}^\dagger\gamma+\gamma\hat{L}\rangle}{2} - \gamma\hat{L}^\dagger + iH\right] \\
&\quad + dN\left[\frac{\hat{L}+\gamma}{\sqrt{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle}} - \mathbb{1}\right]\rho\left[\frac{\hat{L}^\dagger+\gamma}{\sqrt{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle}} - \mathbb{1}\right] \\
&= dN\left[\frac{(\hat{L}+\gamma)\rho(\hat{L}^\dagger+\gamma)}{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle} - \rho\right] \\
&\quad + dt\left\{-i[H,\rho] - \gamma\left(\hat{L}\rho+\rho\hat{L}^\dagger\right) - \frac{1}{2}\left\{\hat{L}^\dagger\hat{L},\rho\right\} + \gamma\langle\hat{L}+\hat{L}^\dagger\rangle\rho + \langle\hat{L}^\dagger\hat{L}\rangle\right\}.
\end{aligned} \tag{3.12}
$$

Because we work with large γ, we can choose γ^{-1} as the variable to expand our stochastic master equation

$$
\begin{aligned}
\frac{(\hat{L}+\gamma)\rho(\hat{L}^\dagger+\gamma)}{\langle(\hat{L}^\dagger+\gamma)(\hat{L}+\gamma)\rangle} &= \frac{\frac{1}{\gamma^2}(\hat{L}+\gamma)\rho(\hat{L}^\dagger+\gamma)}{1+\frac{\langle\hat{L}^\dagger\hat{L}\rangle}{\gamma^2}+\frac{\langle\hat{L}+\hat{L}^\dagger\rangle}{\gamma}} + \mathrm{O}\left(\gamma^{-3}\right) \\
&= \frac{1}{\gamma^2}(\hat{L}+\gamma)\rho(\hat{L}^\dagger+\gamma)\left(1 - \frac{\langle\hat{L}^\dagger\hat{L}\rangle}{\gamma^2} - \frac{\langle\hat{L}+\hat{L}^\dagger\rangle}{\gamma} + \frac{\langle\hat{L}^\dagger+\hat{L}\rangle^2}{\gamma^2}\right) \\
&= \rho + \frac{1}{\gamma}\left(\rho\hat{L}^\dagger + \hat{L}\rho - \langle\hat{L}^\dagger+\hat{L}\rangle\rho\right) \\
&\quad + \frac{1}{\gamma^2}\left[\hat{L}\rho\hat{L}^\dagger - \rho\langle\hat{L}^\dagger\hat{L}\rangle + \rho\langle\hat{L}+\hat{L}^\dagger\rangle^2 - \langle\hat{L}+\hat{L}^\dagger(\rho\hat{L}^\dagger+\hat{L}\rho)\right].
\end{aligned} \tag{3.13}
$$

In the continuum limit, the infinitesimal number of detection becomes

$$
dN \quad \rightarrow \quad \delta N \equiv \gamma^2\delta t\left[1+\frac{\langle\hat{L}+\hat{L}^\dagger\rangle}{\gamma}\right] + \gamma\delta W. \tag{3.14}
$$

The stochastic master equation of the homodyne detection in the limit of $\gamma \to \infty$, and continuum detection limit, we let δt to dt and δW to dW.

$$
\begin{aligned}
d\rho = &\left\{\gamma^2 dt \left[1 + \frac{\langle \hat{L} + \hat{L}^\dagger \rangle}{\gamma}\right] + \gamma\, dW\right\} \times \\
&\left\{\frac{1}{\gamma}\left(\rho\hat{L}^\dagger + \hat{L}\rho - \langle \hat{L}^\dagger + \hat{L}\rangle\rho\right) + \frac{1}{\gamma^2}\left[\hat{L}\rho\hat{L}^\dagger - \rho\langle \hat{L}^\dagger\hat{L}\rangle + \rho\langle \hat{L} + \hat{L}^\dagger\rangle^2 - \langle \hat{L} + \hat{L}^\dagger(\rho\hat{L}^\dagger + \hat{L}\rho)\right]\right\} \\
&+ dt\left\{-i[H,\rho] - \gamma\left(\hat{L}\rho + \rho\hat{L}^\dagger\right) - \frac{1}{2}\left\{\hat{L}^\dagger\hat{L},\rho\right\} + \gamma\langle \hat{L} + \hat{L}^\dagger\rangle\rho + \langle \hat{L}^\dagger\hat{L}\rangle\right\} \\
= &\, dW\left(\rho\hat{L}^\dagger + \hat{L}\rho - \langle \hat{L}^\dagger + \hat{L}\rangle\rho\right) \\
&+ dt\left[\left(-i[H,\rho] - \frac{1}{2}\left\{\hat{L}^\dagger\hat{L},\rho\right\} + \hat{L}\rho\hat{L}^\dagger\right) + \frac{1}{\gamma}\left(-\hat{L}\rho - \rho\hat{L}^\dagger + \langle \hat{L} + \hat{L}^\dagger\rangle\rho + \rho\hat{L}^\dagger + \hat{L}\rho\langle \hat{L} + \hat{L}^\dagger\rangle\rho\right)\right. \\
&\quad + \frac{1}{\gamma^2}\left(-\langle \hat{L} + \hat{L}^\dagger\rangle(\hat{L}\rho + \rho\hat{L}^\dagger - \langle \hat{L} + \hat{L}^\dagger\rangle\rho) + \langle \hat{L} + \hat{L}^\dagger\rangle(\hat{L}\rho + \rho\hat{L}^\dagger - \langle \hat{L} + \hat{L}^\dagger\rangle\rho)\right) \\
&\quad \left. + \frac{1}{\gamma^2}\left(-\rho\langle \hat{L}^\dagger\hat{L}\rangle + \langle \hat{L}^\dagger\hat{L}\rangle\rho\right)\right] \\
= &\, -i[H,\rho]dt + dt\left(-\frac{1}{2}\left\{\hat{L}^\dagger\hat{L},\rho\right\} + \hat{L}\rho\hat{L}^\dagger\right) + dW\left(\rho\hat{L}^\dagger + \hat{L}\rho - \langle \hat{L}^\dagger + \hat{L}\rangle\rho\right) \\
= &\, -i[H,\rho]dt + dt\,\mathcal{D}[\hat{L}]\rho + dW\,\mathcal{H}[\hat{L}]\rho.
\end{aligned} \tag{3.15}
$$

Summary, we have the stochastic master equation for the homodyne detection

$$
d\rho = -i[H,\rho]dt + dt\,\mathcal{D}[\hat{L}]\rho + dW\,\mathcal{H}[\hat{L}]\rho. \tag{3.16}
$$

Chapter 4

Spin-wave theory for quantum trajectory

4.1 Long-range quantum spin system

We consider the system having the Hamiltonian (illustrated in Fig.4.1)

$$\hat{H} = \omega \hat{S}^x + \frac{2sJ}{\mathcal{N}} \sum_{i,j} \frac{\hat{\sigma}_i^z \hat{\sigma}_j^z}{||\mathbf{r_i} - \mathbf{r_j}||^\alpha}, \tag{4.1}$$

where ω is the amplitude of a collective drive and J is the coupling interaction. The Kac normalzia-tion [25] $\mathcal{N} = \frac{1}{N} \sum_{ij} \frac{1}{||\mathbf{r_i} - \mathbf{r_j}||^\alpha}$ ensures a well-defined thermodynamic limit for the interaction Hamil-tonian. The big spin $\hat{S}^\alpha$ without the site index subscript, is equivalent to $\sum_i \hat{s}_i^\alpha$. These spin operators satisfy the commutation property $\left[\hat{s}_i^\alpha, \hat{s}_j^\beta\right] = i\epsilon^{\alpha\beta\gamma} \hat{s}_j^\gamma \delta_{ij}$. Additionally, we have their relation with the Pauli matrix through the equation $\hat{s}_i^\alpha = \frac{1}{2} \hat{\sigma}_i^\alpha$.

In this report, we focus on the infinite-range interaction case, where $\alpha = 0$, we can approximate our Hamiltonian as

$$\hat{H} = \omega \hat{S}^x + \frac{2J}{S} \left(\hat{S}^z\right)^2, \tag{4.2}$$

where we have all-to-all coupling interaction between different spin sites with no spatial resolution. In this regime, the system can be effectively presented with a single spin $S = N/2$, allowing us to benchmark the spin-wave method against the exact simulation of the dynamics in the Dicke basis.

Furthermore, our quantum spin system is subjected under a collective dissipation (at rate κ and con-tinuous measurement).

4.2 Spin-wave method for quantum trajectory

The spin-wave method is inspired mainly from the Euler–Maruyama method [26] and the spin-wave theory. In spin-wave theory, we treat our spin system in semiclassical regime. It is the first-ordered correction of the mean field theory. The main assumption is our system satisfies a strong collective spin polarization, op top of which spin-wave excitations are bosonized via Holstein-Primakoff expansion truncated to the lowest order:

$$\begin{aligned} \hat{s}_i^{\tilde{z}}(\theta, \phi) &= s - \hat{b}_i^\dagger \hat{b}_i, \\ \hat{s}_i^{\tilde{x}}(\theta, \phi) &\approx \sqrt{\frac{s}{2}} (\hat{b}_i^\dagger + \hat{b}_i), \\ \hat{s}_i^{\tilde{y}}(\theta, \phi) &\approx i\sqrt{\frac{s}{2}} (\hat{b}_i^\dagger - \hat{b}_i). \end{aligned} \tag{4.3}$$

In here, we define the $\hat{s}_i^{\tilde{\alpha}}$ as spin operators that is in rotated frame $O\tilde{x}\tilde{y}\tilde{z}$, illustrated in Figure 4.1. These spin operators can be connected with the lab frame representation via $\hat{s}_i^{\tilde{\alpha}}(\theta, \phi) = \hat{U}(\theta, \phi) \hat{s}_i^\alpha \hat{U}^\dagger(\theta, \phi)$ with $\hat{U}(\theta, \phi) = e^{-i\phi \hat{S}^z} e^{-i\theta \hat{S}^y}$. The $\tilde{z}$ direction of the rotated frame aligns the same with the collective (average) polarization of all the spins, which can be met by imposing $\langle \hat{S}^{\tilde{x}} \rangle = \langle \hat{S}^{\tilde{y}} \rangle = 0$. The $\hat{b}_i$ are

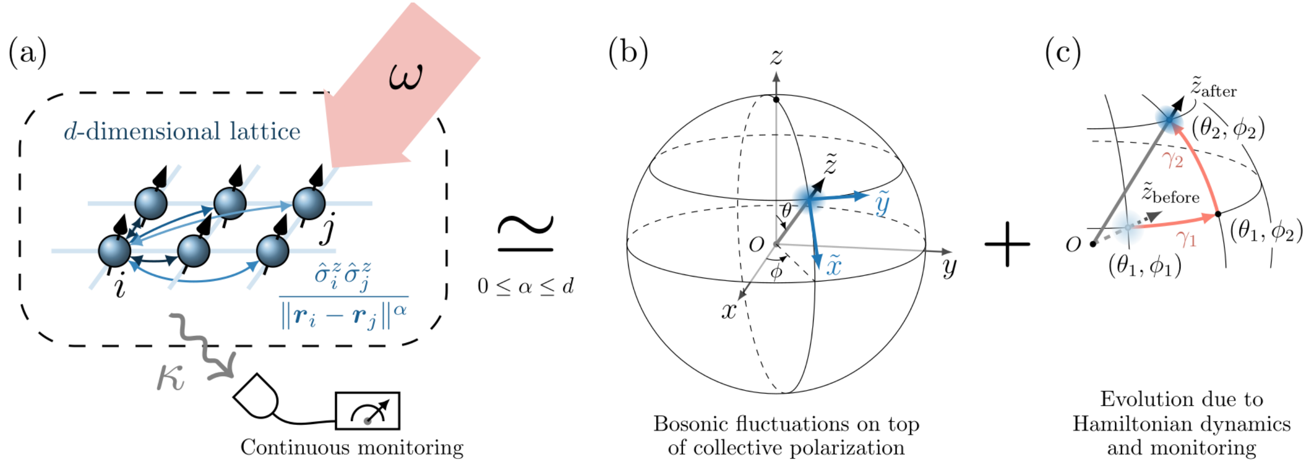

Figure 4.1: Sketch of a long-range quantum spin system (a) and the representation in its monitored dynamics under the spin-wave approximation in panel b and c. From [1] with permission

bosonic operators, they represents the spin fluctuations around the collective magnetization, that follows the bosonic commutative properties $\left[\hat{b}_i, \hat{b}_j^\dagger\right] = \delta_{ij}$. This approximation approximates effectively the Bloch sphere with the tangent plane at the north of the rotated frame, it only becomes exact when $\langle \hat{b}_i^\dagger \hat{b}_i \rangle = 0$, i.e., when the system is in the spin coherent state. Thus, we define the density of bosonic excitations

$$\epsilon = \frac{1}{Ns} \sum_i \langle \hat{b}_i^\dagger \hat{b}_i \rangle, \tag{4.4}$$

also referred to as spin-wave density, serves as a natural control parameter for our approximation [27] .

We approximate the bosonic mode by the Gaussian ansatz, which parametrized by the first and second moments. This allows us to uniquely specify the state of the entire system with the following variational parameters:

$$\theta, \phi, \beta_i \equiv \langle \hat{b}_i \rangle, u_{ij} \equiv \langle \delta_i \delta_j \rangle, v_{ij} \equiv \langle \delta_i^\dagger \delta_j \rangle, \tag{4.5}$$

where $\delta_i = \hat{b}_i - \beta_i$. Consequently, this method can reduce the complexity to present the dynamics of our system from exponentially $O(2^N)$ to only polynomial with only $O(N^2)$ complexity. In particular system, we can reduce furthermore by their symmetric properties.

Now, the stochastic master equation in equation 3.16 then translates into the dynamics of the variational parameters 4.5. We initialize the rotated frame aligns with our lab frame at time t = 0, then we intersperse following two steps to update our parameters:

1. Update the Gaussian parameters β_i, u_{ij}, v_{ij} by calculating their infinitesimal increment, from equation 3.16. It is important to note that the frame angle (ϕ, θ) remains fixed during this process and the $\hat{z}$ axis is no longer the collective spin , as $\sum_i \delta\beta_i \neq 0$ in general.

2. Update the frame angles (θ, ϕ), such that the $\tilde{z}$ axis realign the collective spin, updating all Gaussian parameters accordingly with this new frame. Finally, we increase the time by δt and repeating the process.

In my report, we only consider the infinite-range interaction case and also we consider only one dissipative channel k = 0. The update for each iteration as the following:

$$
\begin{aligned}
\sqrt{\frac{2}{S}}\mathrm{d}\beta = &-i\omega_F \mathrm{d}t\cdot\mathcal{E}^F \\
&+2i\eta\mathrm{d}t\cdot\Big\{-(\mathcal{E}^U C_{\tilde{z}}^{U*}+\mathcal{F}^U C_{\tilde{z}}^{U})(1-\epsilon)+\frac{1}{S}\big[(C_{\tilde{z}}^{U}\mathcal{E}^{U*}+C_{\tilde{z}}^{U*}\mathcal{F}^{U*})u \\
&\qquad +(C_{\tilde{z}}^{U}\mathcal{F}^{U}+C_{\tilde{z}}^{U*}\mathcal{E}^{U})v+C_{\tilde{z}}^{U*}\mathcal{E}^{U}\big]\Big\} \\
&-\frac{\kappa\mathrm{d}t}{2}\cdot\Big\{(\mathcal{F}^D C_{\tilde{z}}^{D}-\mathcal{E}^D C_{\tilde{z}}^{D*})+\frac{1}{S}[(C_{\tilde{z}}^{D}\mathcal{E}^{D*}-C_{\tilde{z}}^{D*}\mathcal{F}^{D*})u-C_{\tilde{z}}^{D*}\mathcal{E}^{D}]\Big\} \\
&+\sqrt{\frac{\kappa}{S}}\cdot\mathrm{d}W\left\{\mathcal{E}^D(\nu+1)+\mathcal{F}^{D*}u+\mathcal{E}^{D*}u+\mathcal{F}^D v\right\},
\end{aligned} \tag{4.6}
$$

$$
\mathrm{d}u = 2i\omega_F\mathrm{d}t\cdot C_{\tilde{z}}^{F}u \tag{4.7}
$$

$$
+2i\eta\mathrm{d}t\cdot\Big\{-\big[(|\mathcal{E}^U|^2+|\mathcal{F}^U|^2)u+\mathcal{E}^U\mathcal{F}^U(2v+1)\big]+4|C_{\tilde{z}}^{U}|^2u
$$

$$
-\frac{2}{S}|C_{\tilde{z}}^{U}|^2[2u(2v+1)+2uv]\Big\} \tag{4.8}
$$

$$
-\frac{\kappa\mathrm{d}t}{2}\cdot\Big\{\mathcal{E}^D\mathcal{F}^D+(|\mathcal{F}^D|^2-|\mathcal{E}^D|^2)u+\frac{4}{S}|C_{\tilde{z}}^{D}|^2u\Big\} \tag{4.9}
$$

$$
-2\sqrt{\frac{\kappa}{S}}\cdot\mathrm{d}W\cdot\left\{C_{\tilde{z}}^{D}(v+1)u+C_{\tilde{z}}^{D*}uv\right\} \tag{4.10}
$$

$$
-\kappa\mathrm{d}t\cdot(\mathcal{E}^{D*}u+\mathcal{F}^D v)[\mathcal{E}^D(v+1)+\mathcal{F}^{D*}u],
$$

$$
\mathrm{d}v = -2\eta\mathrm{d}t\cdot\{2\,\mathrm{Im}[\mathcal{F}^{U*}\mathcal{E}^{U*}u]\} \tag{4.11}
$$

$$
-\frac{\kappa\mathrm{d}t}{2}\cdot\left[\left(|\mathcal{F}^D|^2-|\mathcal{E}^D|^2\right)v-|\mathcal{E}^D|^2\right] \tag{4.12}
$$

$$
-\sqrt{\frac{\kappa}{S}}\cdot 2\cdot\mathrm{d}W\cdot\mathbf{Re}[\cdot C_{\tilde{z}}^{D}](v+v^2+|u|^2) \tag{4.13}
$$

$$
-\frac{\kappa\mathrm{d}t}{2}\cdot\left[\left|\mathcal{E}^{D*}(1+v)+\mathcal{F}^D u^*\right|^2+\left|\mathcal{E}^{D*}u+\mathcal{F}^D v\right|^2\right] \tag{4.14}
$$

and the subsequent angular updates become

$$
\Delta\phi = \frac{1}{\cos\theta}\arctan\left\{\frac{\mathbf{Im}[\beta]}{\sqrt{\frac{S}{2}}\tan\theta+\mathbf{Re}[\beta]}\right\} \tag{4.15}
$$

$$
\Delta\theta = \left(\tan\theta+\sqrt{\frac{2}{S}}\mathbf{Re}[\beta]\right)\cos(\Delta\phi\cos\theta)+\sqrt{\frac{2}{S}}\mathbf{Im}\,[\beta]\sin(\Delta\phi\cos\theta)-\tan\theta, \tag{4.16}
$$

Finally, we update all of Gaussian parameters with the following rules:

$$
\beta \leftarrow 0, \tag{4.17}
$$

$$
u \leftarrow u\cdot e^{-2i\Delta\phi\cos\theta_1}, \tag{4.18}
$$

$$
v \leftarrow v. \tag{4.19}
$$

Chapter 5

Results

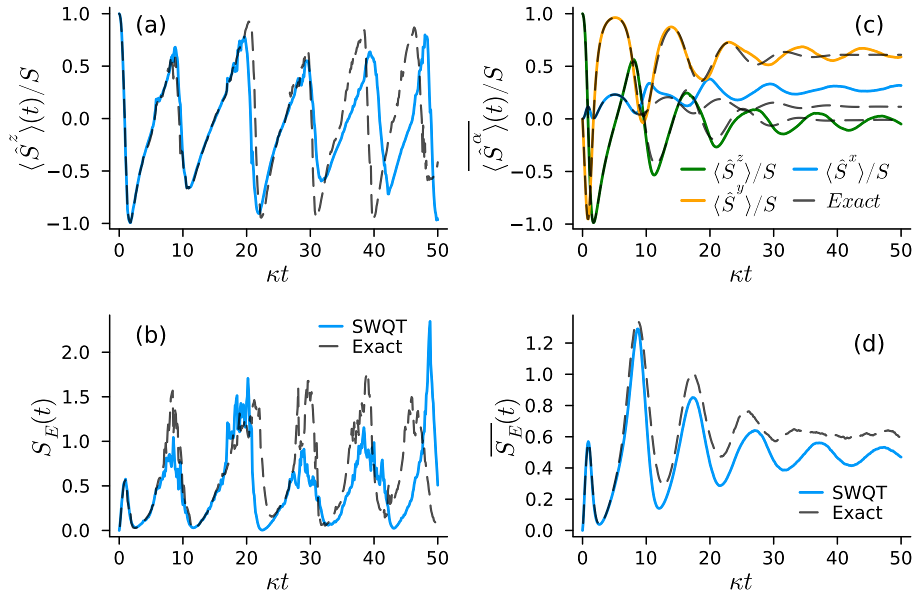

Figure 5.1: Benchmark of spin-wave method against exact method for the time-evolution of the following physical quantities: (a) The magnetization $m_z \equiv \left\langle \hat{S^z} \right\rangle / S$ along one single trajectory; (b) The half-system entanglement entropy S_E along the same trajectories; (c) Trajectory-average of collective spin vector $\overline{\langle \hat{S}^\alpha \rangle}$; (d) Trajectory-average of half-system entanglement entropy. Parameter: $\omega = 1.25\kappa,\ J = 0.1\kappa,\ S = 64,\ k\delta t = 10^{-4}$.

In figure 5.1, for single one quantum trajectory, our spin-wave method in the long region does not fit so well with the exact method for the magnetization and half-system entanglement entropy. This can be explained by the non-Gaussian property of the homodyne trajectory, which is one of the main approximation for our method. At figure 5.1c and 5.1d, in general, the steady expectation value of aforementioned quantities fit quite well.

The mean-field curve is used from our calculation in A.4 for the z-magnetization in the steady state. For figure 5.2a and 5.2b, our method is comparable to the exact method. The spin-wave density for our method in general, in equilibrium regime, it is always around or smaller 0.05, which is acceptable for our method. More detail calculation, you can find at https://github.com/nguyenvulinh666/spin-wave-homodyne-CMP2024.

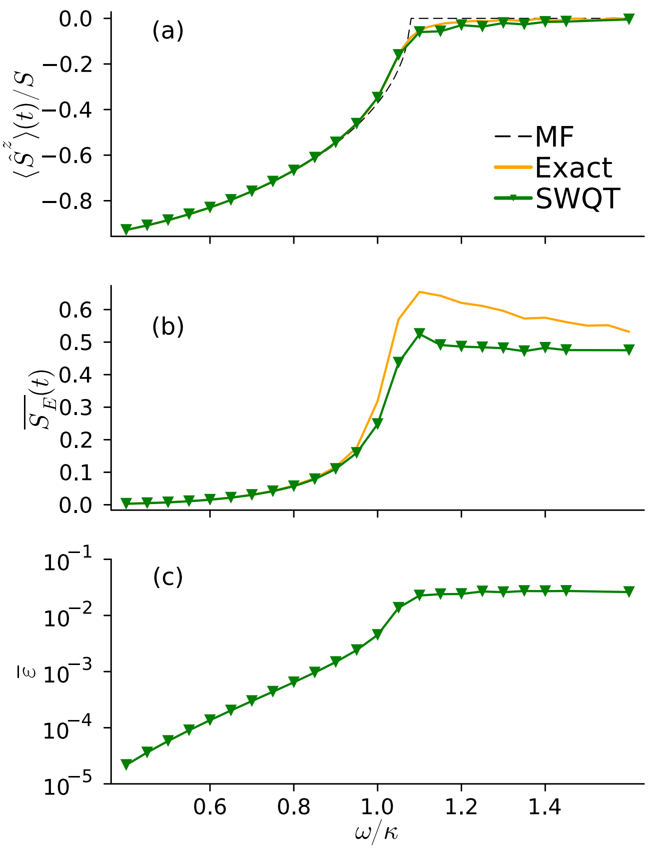

Figure 5.2: Benchmark of the spin-wave method on the steady state of the collective model with S = 64 and $J = 0.1\kappa$:(a) Expectation value of the z-magnetization as a function of ω by the spin-wave method and the exact value. The mean field result illustrates by the dashed line; (b) Trajectory-average of the half system entanglement entropy of the two methods in figure (a); (c) Trajectory-average of the spin-wave density $\bar{\epsilon}$. Parameter: $J = 0.1\kappa$, $S = 64$, $k\delta t = 10^{-4}$.

Appendix A

Main Appendix

A.1 2-level Master Equation

We define the set of jump operators L_1, L_2, and L_3

$$L_1 = \sqrt{\gamma_+}\sigma_+, \quad L_2 = \sqrt{\gamma_-}\sigma_-, \quad L_3 = \sqrt{\gamma_z}\sigma_z = 0 \quad (\gamma_z = 0). \tag{A.1}$$

We have Lindblad master equation for 2-state system

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_i \left[L_i\rho(t)L_i^\dagger - \frac{1}{2}L_i^\dagger L_i\rho(t) - \frac{1}{2}\rho(t)L_i^\dagger L_i \right] \quad (1) \tag{A.2}$$

Case 1: Assume $H = 0$ and $\rho(0) = |\uparrow\rangle\langle\uparrow|$.
We have

$$\frac{\partial p}{\partial t} = \mathcal{L}_1\rho\mathcal{L}_1^\dagger + \mathcal{L}_2\rho\mathcal{L}_2^\dagger - \frac{1}{2}\left[\mathcal{L}_1^\dagger\mathcal{L}_1\rho + \rho\mathcal{L}_1^\dagger\mathcal{L}_1 + \mathcal{L}_2^\dagger\mathcal{L}_2\rho + \rho\mathcal{L}_2^\dagger\mathcal{L}_2\right] \tag{A.3}$$

$$= \gamma_+\sigma_+\rho\sigma_- + \gamma_-\sigma_-\rho\sigma_+ - \frac{1}{2}\left[\gamma_+\sigma_-\sigma_+\rho + \gamma_+\rho\sigma_-\sigma_+ + \gamma_-\sigma_+\sigma_-\rho + \gamma_-\rho\sigma_+\sigma_-\right] \tag{A.4}$$

$$= \gamma_+ \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} + \gamma_- \begin{pmatrix} & 0 \\ 1 & \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} & 1 \\ 0 & \end{pmatrix} \tag{A.5}$$

$$-\frac{1}{2}\left[\gamma_+ \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \gamma_+ \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}\right. \tag{A.6}$$

$$\left. +\gamma_- \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} + \gamma_- \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\right] \tag{A.7}$$

$$= \begin{pmatrix} \gamma_+p_{22} - \gamma_-p_{11} & -\frac{1}{2}(\gamma_+ + \gamma_-)p_{12} \\ -\frac{1}{2}(\gamma_+ + \gamma_-)p_{21} & -\gamma_+p_{22} + \gamma_-p_{11} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \tag{A.8}$$

Because, we need to have the $\text{Tr}[\rho] = 1 \implies p_{11} + p_{22} = 1 \implies p_{22} = 1 - p_{11}$

$$\begin{aligned} &\implies \frac{d}{dt}p_{11} = \gamma_+(1 - p_{11}) - \gamma_-p_{11} = -(\gamma_+ + \gamma_-)p_{11} + \gamma_+, \\ &\implies \frac{d\left[p_{11}(\gamma_+ + \gamma_-) - \gamma_+\right]}{(\gamma_+ + \gamma_-)p_{11} - \gamma_+} = -(\gamma_+ + \gamma_-)dt, \\ &\implies \ln[p_{11}(\gamma_+ + \gamma_-) - \gamma_+] = -t(\gamma_+ + \gamma_-) + C. \end{aligned} \tag{A.9}$$

At $t = 0$, we have $p_{11} = 1$, which leads to $\ln[\gamma_+ + \gamma_- - \gamma_+] = -0(\gamma_+ + \gamma_-) + C \implies \ln\gamma_- = C$

$$\begin{aligned} &\implies \ln[p_{11}(\gamma_+ + \gamma_-) - \gamma_+] = -t(\gamma_+ + \gamma_-) + \ln\gamma_- \\ &\implies p_{11}(\gamma_+ + \gamma_-) - \gamma_+ = \gamma_- e^{-t(\gamma_+ + \gamma_-)} \\ &\implies p_{11} = \frac{\gamma_+ + \gamma_- e^{-t(\gamma_+ + \gamma_-)}}{\gamma_+ + \gamma_-} \\ &\implies p_{22} = 1 - p_{11} = \frac{\gamma_-}{\gamma_+ + \gamma_-}\left[1 - e^{-t(\gamma_+ + \gamma_-)}\right] \end{aligned} \tag{A.10}$$

At $t \to \infty$, we have $p_{11} = \dfrac{\gamma_+}{\gamma_+ + \gamma_-}$ and $p_{22} = \dfrac{\gamma_-}{\gamma_+ + \gamma_-}$

Equation of motion of the expectation value

$$\sigma_x \rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} 0 & 0 \\ 0 & \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_{21} & \rho_{22} \\ \rho_{11} & 0 \end{pmatrix} \tag{A.11}$$

$$\sigma_y \rho = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} 0 & -i\rho_{22} \\ i\rho_{11} & 0 \end{pmatrix} \tag{A.12}$$

$$\sigma_z \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ 0 & -\rho_{22} \end{pmatrix} \tag{A.13}$$

$$\langle \sigma_x \rangle = 0, \langle \sigma_y \rangle = 0 \tag{A.14}$$

$$\langle \sigma_z \rangle = \mathrm{Tr}(\sigma_z \rho) = \rho_{11} - \rho_{22} = \frac{1}{\gamma_+ + \gamma_-} \left[\gamma_+ + \gamma_- e^{-t(\gamma_+ + \gamma_-)} - \gamma_- + \gamma_- e^{-t(\gamma_+ + \gamma_-)} \right] \tag{A.15}$$

$$= \frac{1}{\gamma_+ + \gamma_-} \left[\gamma_+ - \gamma_- + 2\gamma_- e^{-t(\gamma_+ + \gamma_-)} \right] \xrightarrow{t \to \infty} \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \tag{A.16}$$

Case 2: Assume $H = 0$ and $\rho(0) = |\downarrow\rangle\langle\downarrow|$.

$$\frac{d\left[-(\gamma_+ + \gamma_-)p_{11} + \gamma_+\right]}{\left[-(\gamma_+ + \gamma_-)p_{11} + \gamma_+\right]} = -(\gamma_+ + \gamma_-)dt \tag{A.17}$$

$$\implies \ln\left[-(\gamma_+ + \gamma_-)p_{11} + \gamma_+\right] = -(\gamma_+ + \gamma_-)t + C \implies C = \ln \gamma_+ \tag{A.18}$$

$$\implies p_{11} = \frac{-\gamma_+ e^{-(\gamma_+ + \gamma_-)t} + \gamma_+}{\gamma_+ + \gamma_-} \xrightarrow{t \to \infty} \frac{\gamma_+}{\gamma_+ + \gamma_-} \tag{A.19}$$

$$\implies p_{22} = 1 - p_{11} = \frac{\gamma_- + \gamma_+ e^{-(\gamma_+ + \gamma_-)t}}{\gamma_+ + \gamma_-} \xrightarrow{t \to \infty} \frac{\gamma_-}{\gamma_+ + \gamma_-} \tag{A.20}$$

We have the same steady state.

Equation of motion of the expectation value

$$\langle \sigma_x \rangle = 0, \langle \sigma_y \rangle = 0 \tag{A.21}$$

$$\langle \sigma_z \rangle = \mathrm{Tr}(\sigma_z \rho) = \rho_{11} - \rho_{22} \tag{A.22}$$

$$= \frac{1}{\gamma_+ + \gamma_-} \left[\gamma_+ - \gamma_- - 2\gamma_+ e^{-t(\gamma_+ + \gamma_-)} \right] \xrightarrow{t \to \infty} \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \tag{A.23}$$

A.2 The connection between the POVM and Lindblad Master equation

Let's define the simple POVM operator $M_0 = \mathbb{1} - idt\left(\mathcal{H} - \frac{i}{2}L^\dagger L\right), \quad M_1 = \sqrt{dt}L$. These operators satisfied the completeness property

$$\mathbb{1} + M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{1} - L^\dagger L dt + L^\dagger L dt = \mathbb{1}. \tag{A.24}$$

The density matrix after applying one of the POVM operators is

$$\rho_f = \frac{M_i \rho M_i^\dagger}{\mathrm{Tr}[M_i \rho M_i^\dagger]} \tag{A.25}$$

with the probability as $P(i) = \text{Tr}[M_i \rho M_i^\dagger]$. The M_0 terms represents the usual quantum evolution with the dissipation operator part $L^\dagger L$. The M_1 operator represents the quantum jump operators where the operator acts at sudden, with the infinitesimal time step $\sqrt{dt}$. In average, our density matrix changes with in the dt time

$$\begin{aligned}
\rho_{ave}(t+dt) &= P(0)\frac{M_0 p M_0^\dagger}{\text{Tr}[M_0\rho_{ave}(t)M_0^\dagger]} + P(1)\frac{M_1\rho_{ave}(t)M_1^\dagger}{\text{Tr}[M_1 p M_1^\dagger]} = M_0 p M_0^\dagger + M_1 p M_1^\dagger, \\
&= \left[\mathbb{1} - idt\left(\mathcal{H} - \frac{i}{2}L^\dagger L\right)\right]\rho_{ave}(t)\left[\mathbb{1} + idt\left(\mathcal{H} + \frac{i}{2}L^\dagger L\right)\right] + dtL\rho_{ave}(t)L^\dagger, \\
&= \rho_{ave}(t) - idt[\mathcal{H}, \rho_{ave}(t)] - \frac{1}{2}\{L^\dagger L, \rho_{ave}\} + dtL\rho_{ave}(t)L^\dagger.
\end{aligned} \tag{A.26}$$

$$\implies \frac{d}{dt}\rho_{ave}(t) = \frac{\rho_{ave}(t+dt) - \rho_{ave}(t)}{dt} = -i[\mathcal{H}, \rho_{ave}(t)] + \left(L\rho_{ave}(t)L^\dagger - \frac{1}{2}\{L^\dagger L, \rho_{ave}\}\right). \tag{A.27}$$

This formalism can generalize with more POVM operators with $M_0 = \mathbb{1} - idt\left(\mathcal{H} - \frac{i}{2}\sum_i L_i^\dagger L_i\right), \quad M_i = \sqrt{dt}L_i$ with $i = 0, 1, \cdots$

A.3 Monte Carlo Wavefunction Method (MCWF)

We define the quantity $\overline{\sigma}(t)$ that obtained by averaging $\sigma(t) = |\phi(t)\rangle\langle\phi(t)|$. Consider at time $t + \delta t$, we have $1 - \delta p$ probabilities of not having any quantum jump, and δp of having the quantum jump, with the probability of certain jump C_m as $\prod_m = \delta p_m/\delta p$ with

$$\delta p = \delta t\frac{i}{\hbar}\langle\phi(t)|H - H^\dagger|\phi(t)\rangle = \delta t\langle\phi(t)|\sum_m C_m^\dagger C_m|\phi(t)\rangle = \sum_m \delta p_m, \tag{A.28}$$

with $\delta p_m = \delta t\langle\phi(t)|C_m^\dagger C_m|\phi(t)\rangle$ and

$$\overline{\sigma(t+dt)} = (1-\delta p)\frac{|\phi^{(1)}(t+dt)\rangle}{(1-\delta\rho)^{\frac{1}{2}}}\frac{\langle\phi^{(1)}(t+dt)|}{(1-\delta\rho)^{\frac{1}{2}}} + \delta p\sum_m \Pi_m\frac{C_m|\phi(t)\rangle}{(\delta p_m/\delta t)^{\frac{1}{2}}}\frac{\langle\phi(t)\rangle|C_m^\dagger}{(\delta p_m/\delta t)^{\frac{1}{2}}}, \tag{A.29}$$

where $|\phi^{(1)}(t+dt)\rangle$ is the unnormalized unitary evolution of the wave function, the second term represents the average of having the quantum jumps. The following calculation is

$$\begin{aligned}
\overline{\sigma(t+dt)} &= (1 - iH\delta t)|\phi(t)\rangle\langle\phi(t)|(1 + iH\delta t) + \delta p\,\delta t\sum_m \frac{\delta p_m}{\delta p}\frac{C_m\sigma C_m^\dagger}{\delta p_m} \\
&= \sigma - i[H, \sigma]\delta t + \delta t\sum_m C_m\sigma C_m^\dagger.
\end{aligned} \tag{A.30}$$

This equation is equivalent with

$$\frac{\overline{\sigma(t+dt)} - \overline{\sigma(t)}}{\delta t} = i[\overline{\sigma(t)}, H] + \sum_m C_m\overline{\sigma(t)}C_m^\dagger \approx \frac{d\sigma(t)}{dt}. \tag{A.31}$$

A.4 Mean-field equations for the power-law spin model

$$[S^\alpha, S^\beta] = \sum_\gamma i\varepsilon^{\alpha\beta\gamma}s^\gamma = i\varepsilon^{\alpha\beta\gamma}S^\gamma. \tag{A.32}$$

The equation of the magnetization

$$
\begin{aligned}
\frac{dm_x}{dt} &= i\left\langle\left[\frac{2J}{S}\left(S^z\right)^2, \frac{S^x}{S}\right]\right\rangle + \frac{K}{S^2}\left\langle S^+S^xS^- - \frac{1}{2}S^xS^+S^- - \frac{1}{2}S^+S^-S^x\right\rangle \\
&= i\frac{2J}{S^2}\left\langle S^z\cdot iS^y + iS^yS^z\right\rangle + \frac{K}{2S^2}\left\langle S^+\left[S^x, S^x - iS^y\right] + \left[S^x + iS^y, S^x\right]S^-\right\rangle \\
&= -4Jm_zm_y + \frac{K}{2S^2}\left\langle S^+S^z + S^zS^-\right\rangle = -4Jm_zm_y + \frac{K}{2S^2}\left\langle\left(S^x + iS^y\right)S^z + S^z\left(S^x - iS^y\right)\right\rangle \\
&= -4Jm_zm_y + K\frac{1}{S^2}\left\langle S^z\right\rangle\left\langle S^x\right\rangle = -4Jm_zm_y + Km_xm_z.
\end{aligned}
\tag{A.33}
$$

$$
\begin{aligned}
\frac{dm_y}{dt} &= i\left\langle\omega S^x + \frac{2J}{S}\left(S^z\right)^2, \frac{S^y}{S}\right\rangle + \frac{K}{S^2}\left\langle S^+S^yS^- - \frac{1}{2}S^yS^+S^- - \frac{1}{2}S^+S^-S^y\right\rangle \\
&= -\omega m_z + i\frac{2J}{S^2}\left\langle -iS^zS^x - iS^xS^z\right\rangle + \frac{K}{2S^2}\left\langle S^+\left[S^y, S^x - iS^y\right] + \left[S^x + iS^y, S^y\right]S^-\right\rangle \\
&= -\omega m_z + 4Jm_zm_x + \frac{K}{2S^2}\left\langle -iS^+S^z + iS^zS^-\right\rangle \\
&= -\omega m_z + \frac{K}{2S^2}\left\langle -i\left(S^x + iS^y\right)S^z + iS^z\left(S^x - iS^y\right)\right\rangle = -\omega m_z + 4Jm_zm_x + Km_ym_z.
\end{aligned}
\tag{A.34}
$$

$$
\begin{aligned}
\frac{dm_z}{dt} &= i\left\langle\omega S^x + \frac{2J}{S}\left(S^z\right)^2, \frac{S^z}{S}\right\rangle + \frac{K}{S^2}\left\langle S^+S^zS^- - \frac{1}{2}S^zS^+S^- - \frac{1}{2}S^+S^-S^z\right\rangle \\
&= \omega m_y + \frac{K}{2S^2}\left\langle S^+\left[S^z, S^x - iS^y\right] + \left[S^x + iS^y, S^z\right]S^-\right\rangle \\
&= \omega m_y + \frac{K}{2S^2}\left\langle\left(S^x + iS^y\right)\left[iS^y - S^x\right] + \left[-iS^y - S^x\right]\left(S^x - iS^y\right)\right\rangle \\
&= \omega m_y + \frac{K}{2S^2}\left\langle\left(S^x + iS^y\right)\left[iS^y - S^x\right] + \left[-iS^y - S^x\right]\left(S^x - iS^y\right)\right\rangle \\
&= \omega m_y + \frac{K}{2S^2}\left[im_xm_y - m_x^2 - m_y^2 - im_xm_y - im_ym_x - m_y^2 - m_x^2 + im_xm_y\right] \\
&= \omega m_y - K(m_x^2 + m_y^2).
\end{aligned}
\tag{A.35}
$$

We can find the steady-state of the magnetization by imposing the time derivatives to zero, followed by

$$
m_z = -\sqrt{1 - \frac{\omega^2}{16J^2 + k^2}}. \tag{A.36}
$$

Beyond the critical point $\omega_{MF}^{(c)} = \sqrt{16J^2 + K^2}$, the mean-field theory predicts that we have no-longer stable solution, the magnetization of the z direction will fluctuate around zero. This critical point divides our system to different phases, stationary phase and time crystal phase.

A.5 Mathematical perspective of Itô Calculus

We define the infinitesimal Wiener increment

$$
dW(t) = \xi(t)dt, \tag{A.37}
$$

where $\xi(t)$ is called Gaussian white noise, which has two following properties

$$
\mathrm{E}\left[\xi(t)\xi(t')\right] = \delta(t - t'), \tag{A.38}
$$

$$
E\left[\xi(t)\right] = 0. \tag{A.39}
$$

where E denotes as an ensemble average. We call W(t) as a Wiener increment because if we define

$$W(t) = \int_{t_o}^{t} \xi(t')dt', \tag{A.40}$$

then this has all of the properties of the Wiener process. Then if we define the quantity $\Delta W(t) = W(t+\Delta t) - W(t) = \int_t^{t+\Delta t} \xi(t)\, dt$, this quantity is obvious Markovian because it doesn't have any dependence on W(s) with $s < t$. Moreover, we have the following property

$$E[\Delta W(t)^2] = \Delta t, \tag{A.41}$$
$$E[\Delta W(t)] = 0. \tag{A.42}$$

$$\begin{aligned}
\mathrm{E}\left[\int_t^{t+\Delta t}\int_t^{t+\Delta t} \xi(t)\xi(t')\, dt\, dt'\right] &= \int_t^{t+\Delta t}\int_t^{t+\Delta t} \mathrm{E}\left[\xi(t)\xi(t')\right]\, dt\, dt' \\
&= \int_t^{t+\Delta t}\int_t^{t+\Delta t} \delta(t-t')\, dt\, dt' \\
&= \int_t^{t+\Delta t} dt = \Delta t.
\end{aligned} \tag{A.43}$$

$$\mathrm{E}\left[\int_t^{t+\Delta t} \xi(t)\, dt\right] = \int_t^{t+\Delta t} \mathrm{E}\left[\xi(t)\right]\, dt = 0. \tag{A.44}$$

This means $\Delta W(t)$ has the Gaussian distribution with zero mean and variance Δt. Consider a variable

$$\Delta\tau = \sum_{i=0}^{N-1}\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^2. \tag{A.45}$$

Then it follows that

$$\mathrm{E}\left[\Delta\tau\right] = \sum_{i=0}^{N-1}\frac{\Delta t}{N} = N\cdot\frac{\Delta t}{N} = \Delta t. \tag{A.46}$$

And also we have

$$\begin{aligned}
\mathrm{E}\left[\delta\tau^2\right] &= \mathrm{E}\left[\sum_{i=0}^{N-1}\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^2\sum_{j=0}^{N-1}\left[\delta W\left(t_o + j\frac{\Delta t}{N}\right)\right]^2\right] \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\mathrm{E}\left\{\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^2\left[\delta W\left(t_o + j\frac{\Delta t}{N}\right)\right]^2\right\}.
\end{aligned} \tag{A.47}$$

if $i \neq j$, two square of the Wiener increment are statistically independent, we have

$$\begin{aligned}
\mathrm{E}\left\{\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^2\left[\delta W\left(t_o + j\frac{\Delta t}{N}\right)\right]^2\right\} &= \mathrm{E}\left\{\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^2\right\}\cdot\mathrm{E}\left\{\left[\delta W\left(t_o + j\frac{\Delta t}{N}\right)\right]^2\right\} \\
&= \frac{\Delta^2 t}{N^2}.
\end{aligned} \tag{A.48}$$

If i = j, we use one of important properties of Gaussian random variable X of mean 0, $\langle X^4\rangle = 3\langle X^2\rangle^2$.

$$\mathrm{E}\left\{\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^4\right\} = 3\,\mathrm{E}\left\{\left[\delta W\left(t_o + i\frac{\Delta t}{N}\right)\right]^2\right\}^2 = 3\frac{\Delta^2 t}{N^2}. \tag{A.49}$$

In general, we have

$$\mathrm{E}\left\{\left[\delta W\left(t_o+i\frac{\Delta t}{N}\right)\right]^2\left[\delta W\left(t_o+j\frac{\Delta t}{N}\right)\right]^2\right\}=(1+2\delta_{ij})\frac{\Delta^2 t}{N^2}. \tag{A.50}$$

Substitute back to equation A.47, we have

$$\mathrm{E}\left[\delta\tau^2\right]=\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(1+2\delta_{ij})\frac{\Delta^2 t}{N^2}=\Delta^2 t+2\frac{\Delta^2 t}{N}. \tag{A.51}$$

The variance of $\Delta\tau$ is

$$\mathrm{E}\left[\delta\tau^2\right]-\mathrm{E}\left[\delta\tau\right]^2=2\frac{\Delta^2 t}{N}. \tag{A.52}$$

In the limit of N is going to infinity, $\Delta\to d$, the variance vanishes and because this is true for any time interval, we can simply remove average ensemble

$$d\tau\equiv dW=dt. \tag{A.53}$$

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