PCA computation

Suppose the random variables X_1, X_2, and X_3 have the covariance matrix:

$$C = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Calculate (by hand, show detail work) the eigenvalue-eigenvector pairs and the principal components $Y_{1,}$ Y_2 and Y_3.

ANSWER

Find the ~~engine~~ eigenvalue & eigenvectors of $C = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

1. Start from forming a new matrix by subtracting λ from the diagonal entries of the given matrix:

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

2. Find the determinant of the obtained matrix:

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 4\lambda + (1-\lambda)(2-\lambda)(5-\lambda) - 8$$

This is a characteristic polynomial.

Solve the equation $4\lambda + (1-\lambda)(2-\lambda)(5-\lambda) - 8 = 0$

$\Rightarrow$ The roots are: $\begin{cases} \lambda_1 = 2\sqrt{2} + 3 = 5,8284271 \\ \lambda_2 = 2 \\ \lambda_3 = 3 - 2\sqrt{2} = 0,1715729 \end{cases}$

$\Rightarrow$ These are the eigenvalues.

(Detail solution of determinant:

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0.(-1)^{3+1} \begin{vmatrix} -2 & 0 \\ 5-\lambda & 0 \end{vmatrix} + 0.(-1)^{3+2} \begin{vmatrix} 1-\lambda & 0 \\ -2 & 0 \end{vmatrix}$$

$$+ (2-\lambda).(-1)^{3+3}. \begin{vmatrix} 1-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix}$$

The determinant of a 2×2 matrix is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = (1-\lambda).(5-\lambda) - (-2).(-2) = \lambda^2 - 6\lambda + 1$$

Finally, $(2-\lambda)(\lambda^2 - 6\lambda + 1) = -(\lambda - 2)(\lambda^2 - 6\lambda + 1)$)

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2 - 6\lambda + 1) = 4\lambda + (1-\lambda)(2-\lambda)(5-\lambda) - 8$$

Next, find the eigenvectors:

ⓐ. $\lambda = 2\sqrt{2} + 3$

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -2\sqrt{2}-2 & -2 & 0 \\ -2 & 2-2\sqrt{2} & 0 \\ 0 & 0 & -2\sqrt{2}-1 \end{bmatrix}$$ (rref calculator) - reduced row echelon form calculator

$$= \begin{bmatrix} 1 & \frac{-2}{2\sqrt{2}-2} & \cancel{\ } & 0 \\ 0 & 0 & \cancel{\ } & 1 \\ 0 & -4+(2-2\sqrt{2})(-2\sqrt{2}-2) & & 0 \end{bmatrix}$$

Solve the matrix equation:

$$\begin{bmatrix} 1 & -\frac{2}{-2\sqrt{2}-2} & 0 \\ 0 & 0 & 1 \\ 0 & -4+(2-2\sqrt{2})(-2\sqrt{2}-2) & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Take $v_2 = t \Rightarrow v_1 = t(1-\sqrt{2}),\ v_2 = t,\ v_3 = 0$

$$\Rightarrow \begin{bmatrix} t(1-\sqrt{2}) \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{bmatrix} t$$

ⓑ. $\lambda = 2$

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Perform row operations to obtain the rref of the matrix:

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Make zeros in column 1 except the entry at row 1, column 1 (pivot entry)
- Subtract row 1 multiplied by 2 from row 2 $(R_2 = R_2 - (2)R_1)$:

$$\begin{bmatrix} -1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Make zeros in column 2 except the entry at row 2, column 2 (pivot entry)

$$\Rightarrow \text{rref}(A) = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, solve the matrix equation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Take $v_3 = t \Rightarrow v_1 = 0,\ v_2 = 0,\ v_3 = t$

$$\Rightarrow v = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$$

ⓒ $\lambda = 3 - 2\sqrt{2}$

$$\begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -2+2\sqrt{2} & -2 & 0 \\ -2 & 2+2\sqrt{2} & 0 \\ 0 & 0 & -1+2\sqrt{2} \end{bmatrix}$$

Perform row operations to obtain the rref of the matrix:

$$\begin{bmatrix} -2+2\sqrt{2} & -2 & 0 \\ -2 & 2+2\sqrt{2} & 0 \\ 0 & 0 & -1+2\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{2}{-2+2\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ 0 & -4+(-2+2\sqrt{2})\cdot(2+2\sqrt{2}) & 0 \end{bmatrix}$$

Solve the matrix equation:

$$\begin{bmatrix} 1 & -\frac{2}{-2+2\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ 0 & -4+(-2+2\sqrt{2})(2+2\sqrt{2}) & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Take $v = \begin{bmatrix} t(1+\sqrt{2}) \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{bmatrix} t$

Answer: eigenvalue: $2\sqrt{2}+3$; eigenvector: $\begin{bmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$

2 $\qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$3-2\sqrt{2}$ $\qquad \begin{bmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{bmatrix}$

$$\Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad E(x) = 0$$

with R-p : The eigen values are: $\lambda_1 = 2\sqrt{2}+3$; $\lambda_2 = 2$; $\lambda_3 = 3-2\sqrt{2}$

the eigen vectors are: $e_1^T = [1-\sqrt{2} \quad 1 \quad 0]$

$e_2^T = [0 \quad 0 \quad 1]$

$e_3^T = [1+\sqrt{2} \quad 1 \quad 0]$

So the principle components are:

$Y_1 = e_1^T x = 1-\sqrt{2}\, x_1 + x_2$

$Y_2 = e_2^T x = x_3$

$Y_3 = e_3^T x = 1+\sqrt{2}\, x_2 + x_2$

we can find the eigen values from the principle components:

① $var(Y_1) = var\left(\cancel{(1-\sqrt{2})X_1 + X_2}\right)$

$\cancel{= (1-\sqrt{2})^2 var(X_1) + (1)^2 var(X_2)} + 2(1-\sqrt{2})(1)\, cov(X_1, X_2)$

$\cancel{= 3-2\sqrt{2} \cdot 1 + 1 \cdot 5} + (2-2\sqrt{2})(-2)$

The eigen vectors are the coefficients of the principle components. The eigen vectors when multiplied with vector X yields a scalar value which is Y.

we have X with cov matrix Σ that has eigenpairs $(\lambda_1, e_1), \dots (\lambda_p, e_p)$

$pc's: Y = \begin{bmatrix} e_1^T \\ \vdots \\ e_p^T \end{bmatrix} X$

$cov(X_i, Y_j) = cov(a_i^T X, e_j^T X)$

let $a_i = \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix} \leftarrow i^{th}$ element

$= a_i^T \, cov(X, X)\, e_j$

$= a_i^T \, \Sigma_X \, e_j$

$= a_i^T \, d_j e_j = d_j \, a_i^T e_j = \lambda_j e_{ij}$ where e_{ij} is the i^{th} element of the j^{th} eign vectors.

The variance of the 1^{st} PC is λ, which is the magnitude. And to find the variance of the k^{th} PC, $\lambda_k = \frac{1}{n} \sum_{i=1}^{n} Y_{ki}^2$

- The sample covarience of the Y_i is diagonal.
- The k^{th} pc explains:

$\frac{\lambda_k}{\lambda_1 + \dots + \lambda_p}$ 100% of the percentage total variance in the data.

$corr(X_1, Y_1) = \frac{e_{11}\sqrt{\lambda_1}}{\sigma_{11}} = \frac{(1-\sqrt{2})\sqrt{2\sqrt{2}+3}}{\sqrt{1}} = -1$

The correlation of X_1 with Y_1 is almost large and strong correlation because X_1 has a coefficient $(1-\sqrt{2})$ which is kind of large with negative sign.

$corr(X_1, X_2) = \frac{e_{21}\sqrt{\lambda_2}}{\sqrt{\sigma_{11}}} = \frac{0 \cdot \sqrt{2}}{\sqrt{1}} = 0$

It is uncorrelated because the pc $Y_2 = X_3$ which make sense there is no X_1 and the coefficient of X_1 is 0.