Bài tập 1

TRẦN KHÔI NGUYÊN
VẬT LÝ LÝ THUYẾT

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Giải:

Cho:

$$\begin{cases} \phi(\vec{x}) & = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}} e^{\alpha} + a^\dagger_{\vec{k}} e^{-\alpha} \right] \\ \pi(\vec{x}) & = (-i) \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left[a_{\vec{k}} e^{\alpha} + a^\dagger_{\vec{k}} e^{-\alpha} \right] \end{cases}$$

trong đó $\alpha = -i\omega_{\vec{k}} t + i\vec{k} \cdot \vec{x}$.

Phép biến đổi Fourier 4D cho $\phi(\vec{x})$ và $\pi(\vec{x})$:

$$\begin{aligned} * \int d^3x \phi(\vec{x}) e^{i\vec{k'}\cdot\vec{x}} &= \int d^3x \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}} e^{\alpha} + a^\dagger_{\vec{k}} e^{-\alpha} \right] e^{i\vec{k'}\cdot\vec{x}} \\ &= \int d^3x \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}} e^{\alpha} e^{i\vec{k'}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{-\alpha} e^{i\vec{k'}\cdot\vec{x}} \right] \\ &= \int d^3x \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} e^{i\vec{k'}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} e^{i\vec{k'}\cdot\vec{x}} \right] \\ &= (2\pi)^3 \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t} \delta(\vec{k} + \vec{k'}) + a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t} \delta(\vec{k'} - \vec{k}) \right], \end{aligned}$$

do $\int d^3x e^{i(\vec{k'}+\vec{k})\cdot x} = (2\pi)^3 \delta(\vec{k} + \vec{k'})$ và $\int d^3x e^{i(\vec{k'}-\vec{k})\cdot x} = (2\pi)^3 \delta(\vec{k} - \vec{k'})$

$$\Rightarrow \int d^3x \phi(\vec{x}) e^{i\vec{k'}\cdot\vec{x}} = \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{2\omega_{\vec{k'}}}} \left[a_{-\vec{k'}} e^{-i\omega_{\vec{k'}} t} + a^\dagger_{\vec{k'}} e^{i\omega_{-\vec{k'}} t} \right] = \frac{(2\pi)^{\frac{3}{2}}}{\sqrt{2\omega_{\vec{q}}}} \left[a_{-\vec{q}} e^{-i\omega_{\vec{q}} t} + a^\dagger_{\vec{q}} e^{i\omega_{-\vec{q}} t} \right]$$

$$\Rightarrow \frac{\sqrt{2\omega_{\vec{q}}}}{(2\pi)^{\frac{3}{2}}} \int d^3x \phi(\vec{x}) e^{i\vec{k'}\cdot\vec{x}} = \left[-a_{\vec{q}} e^{-i\omega_{\vec{q}} t} + a^\dagger_{\vec{q}} e^{i\omega_{-\vec{q}} t} \right] \tag{1}$$

$$\begin{aligned} * \int d^3x \pi(\vec{x}) e^{i\vec{k'}\cdot\vec{x}} &= \int d^3x (-i) \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left[a_{\vec{k}} e^{\alpha} - a^\dagger_{\vec{k}} e^{-\alpha} \right] e^{i\vec{k'}\cdot\vec{x}} \\ &= \int d^3x (-i) \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left[a_{\vec{k}} e^{\alpha} e^{i\vec{k'}\cdot\vec{x}} - a^\dagger_{\vec{k}} e^{-\alpha} e^{i\vec{k'}\cdot\vec{x}} \right] \\ &= (-i) \int d^3x \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}} e^{i\vec{k'}\cdot\vec{x}} - a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t - i\vec{k}\cdot\vec{x}} e^{i\vec{k'}\cdot\vec{x}} \right] \\ &= -i(2\pi)^3 \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left[a_{\vec{k}} e^{-i\omega_{\vec{k}} t} \delta(\vec{k} + \vec{k'}) - a^\dagger_{\vec{k}} e^{i\omega_{\vec{k}} t} \delta(\vec{k'} - \vec{k}) \right], \end{aligned}$$

do $\int d^3x e^{i(\vec{k'}+\vec{k})\cdot x} = (2\pi)^3\delta(\vec{k}+\vec{k'})$ và $\int d^3x e^{i(\vec{k'}-\vec{k})\cdot x} = (2\pi)^3\delta(\vec{k}-\vec{k'})$

$$\Rightarrow \int d^3x\pi(\vec{x})e^{i\vec{k'}\cdot\vec{x}} = -i(2\pi)^{\frac{3}{2}}\sqrt{\frac{\omega_{\vec{k}}}{2}}\left[a_{-\vec{k'}}e^{-i\omega_{\vec{k'}}t} - a^\dagger_{\vec{k'}}e^{i\omega_{-\vec{k'}}t}\right]$$

$$= -i(2\pi)^{\frac{3}{2}}\sqrt{\frac{\omega_{\vec{q}}}{2}}\left[a_{-\vec{q}}e^{-i\omega_{\vec{q}}t} - a^\dagger_{-\vec{q}}e^{i\omega_{-\vec{q}}t}\right]$$

$$\Rightarrow i\sqrt{\frac{2}{\omega_{\vec{q}}}}\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\pi(\vec{x})e^{i\vec{k'}\cdot\vec{x}} = \left[a_{-\vec{q}}e^{-i\omega_{\vec{q}}t} - a^\dagger_{\vec{q}}e^{i\omega_{-\vec{q}}t}\right] \tag{2}$$

Từ (1) và (2):

$$\begin{cases} a_{-\vec{q}}e^{-i\omega_{\vec{q}}t} + a^\dagger_{\vec{q}}e^{i\omega_{-\vec{q}}t} = \dfrac{\sqrt{2\omega_{\vec{q}}}}{(2\pi)^{\frac{3}{2}}}\displaystyle\int d^3x\phi(\vec{x})e^{i\vec{k'}\cdot\vec{x}}, \\ a_{-\vec{q}}e^{-i\omega_{\vec{q}}t} - a^\dagger_{\vec{q}}e^{i\omega_{-\vec{q}}t} = i\sqrt{\dfrac{2}{\omega_{\vec{q}}}}\dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int d^3x\pi(\vec{x})e^{i\vec{k'}\cdot\vec{x}}, \end{cases}$$

lấy (1) + (2):

$$2a_{-\vec{q}}e^{-i\omega_{\vec{q}}t} = \frac{\sqrt{2\omega_{\vec{q}}}}{(2\pi)^{\frac{3}{2}}}\int d^3x\phi(\vec{x})e^{i\vec{k'}\cdot\vec{x}} + i\sqrt{\frac{2}{\omega_{\vec{q}}}}\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\pi(\vec{x})e^{i\vec{k'}\cdot\vec{x}}$$

$$\Rightarrow a_{-\vec{q}} = \frac{1}{2(2\pi)^{\frac{3}{2}}}\int d^3x\left[\sqrt{2\omega_{\vec{q}}}\phi(\vec{x}) + i\sqrt{\frac{2}{\omega_{\vec{q}}}}\pi(\vec{x})\right]e^{i\vec{k'}\cdot\vec{x}}e^{i\omega_{\vec{q}}t}, \tag{3}$$

lấy (1) - (2):

$$2a_{-\vec{q}}e^{i\omega_{\vec{q}}t} = \frac{\sqrt{2\omega_{\vec{q}}}}{(2\pi)^{\frac{3}{2}}}\int d^3x\phi(\vec{x})e^{i\vec{k'}\cdot\vec{x}} - i\sqrt{\frac{2}{\omega_{\vec{q}}}}\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x\pi(\vec{x})e^{i\vec{k'}\cdot\vec{x}}$$

$$\Rightarrow a^\dagger_{-\vec{q}} = \frac{1}{2(2\pi)^{\frac{3}{2}}}\int d^3x\left[\sqrt{2\omega_{\vec{q}}}\phi(\vec{x}) - i\sqrt{\frac{2}{\omega_{\vec{q}}}}\pi(\vec{x})\right]e^{i\vec{k'}\cdot\vec{x}}e^{-i\omega_{\vec{q}}t}, \tag{4}$$

từ (3) và (4):

$$\begin{cases} a^\dagger_{\vec{q}} = \dfrac{1}{2(2\pi)^{\frac{3}{2}}}\displaystyle\int d^3x\left[\sqrt{2\omega_{\vec{q}}}\phi(\vec{x}) + i\sqrt{\frac{2}{\omega_{\vec{q}}}}\pi(\vec{x})\right]e^{i\vec{k'}\cdot\vec{x}}e^{i\omega_{\vec{q}}t} \\ a^\dagger_{\vec{q}} = \dfrac{1}{2(2\pi)^{\frac{3}{2}}}\displaystyle\int d^3x\left[\sqrt{2\omega_{\vec{q}}}\phi(\vec{x}) - i\sqrt{\frac{2}{\omega_{\vec{q}}}}\pi(\vec{x})\right]e^{i\vec{k'}\cdot\vec{x}}e^{-i\omega_{\vec{q}}t} \end{cases}$$

thay $k' = -k'$ cho (3)

$$\Rightarrow \begin{cases} a_{\vec{q}} = \dfrac{1}{2(2\pi)^{\frac{3}{2}}}\displaystyle\int d^3x\left[\sqrt{2\omega_{\vec{q}}}\phi(\vec{x}) + i\sqrt{\frac{2}{\omega_{\vec{q}}}}\pi(\vec{x})\right]e^{i(\omega_{\vec{q}}t - \vec{k'}\cdot\vec{x})} \\ a^\dagger_{\vec{q}} = \dfrac{1}{2(2\pi)^{\frac{3}{2}}}\displaystyle\int d^3x\left[\sqrt{2\omega_{\vec{q}}}\phi(\vec{x}) - i\sqrt{\frac{2}{\omega_{\vec{q}}}}\pi(\vec{x})\right]e^{-i(\omega_{\vec{q}}t - \vec{k'}\cdot\vec{x})} \end{cases}$$

$*$ Tính các giao hoán tử:

$$\left[a_{-\vec{q}}\,, a^{\dagger}_{\vec{q}}\right] = a_{-\vec{q}}\, a^{\dagger}_{\vec{q}} - a^{\dagger}_{\vec{q}}\, a_{-\vec{q}}$$

Để thuận tiện trong việc đọc các kí hiệu, thì ta sẽ thay đổi một số dummy variables:

$$a_{\vec{k}} = \frac{1}{2(2\pi)^{\frac{3}{2}}} \int d^3x \left[\sqrt{2\omega_{\vec{k}}}\,\phi(\vec{x}) + i\sqrt{\frac{2}{\omega_{\vec{k}}}}\,\pi(\vec{x})\right] e^{i(\omega_{\vec{k}}t - \vec{k}\cdot\vec{x})} \text{ thay } -q = k \text{ và } k' = k,$$

và "tạm" bỏ đi các kí hiệu vector. Giao hoán tử sẽ trở thành:

$$\begin{aligned}
\left[a_{\vec{k}}\,, a^{\dagger}_{\vec{q}}\right] &= a_{\vec{k}}\, a^{\dagger}_{\vec{q}} - a^{\dagger}_{\vec{q}} a_{\vec{k}} \\
&= \frac{1}{2}\int \frac{d^3x d^3y}{(2\pi^3)} \frac{e^{i(\omega_k - \omega_q)t} e^{-i(k\cdot x - q\cdot y)}}{\sqrt{\omega_k \omega_q}} \\
&\times \Big[\omega_k\omega_q \phi(x)\phi(y) - i\omega_k \phi(x)\pi(y) + i\omega_q \pi(x)\phi(y) + \pi(x)\pi(y) \\
&- \omega_k\omega_q \phi(y)\phi(x) + i\omega_k \pi(y)\phi(x) - i\omega_q \phi(y)\pi(x) - \pi(y)\pi(x)\Big] \\
&= \frac{1}{2}\int \frac{d^3x d^3y}{(2\pi^3)} \frac{e^{i(\omega_k - \omega_q)t} e^{-i(k\cdot x - q\cdot y)}}{\sqrt{\omega_k \omega_q}} \Big\{\omega_k\omega_q \left[\phi(x), \phi(y)\right] - i\omega_k \left[\phi(x), \pi(y)\right] \\
&- i\omega_q \left[\phi(y), \pi(x)\right] + \left[\pi(x), \pi(y)\right]\Big\} \\
&= \frac{1}{2}\int \frac{d^3x d^3y}{(2\pi^3)} \frac{e^{i(\omega_k - \omega_q)t} e^{-i(k\cdot x - q\cdot y)}}{\sqrt{\omega_k \omega_q}} \Big\{\cancel{\omega_k\omega_q \left[\phi(x), \phi(y)\right]} - i\omega_k \left[\phi(x), \pi(y)\right] \\
&- i\omega_q \left[\phi(y), \pi(x)\right] + \cancel{\left[\pi(x), \pi(y)\right]}\Big\}
\end{aligned}$$

số hạng thứ 1 và số hạng thứ 4 bị biến mất là do $\left[A(x), A(y)\right] = 0$, nên:

$$\begin{aligned}
&= \frac{1}{2}\int \frac{d^3x d^3y}{(2\pi^3)} \frac{e^{i(\omega_k - \omega_q)t} e^{-i(k\cdot x - q\cdot y)}}{\sqrt{\omega_k \omega_q}} (-i)\left[\omega_k i\delta(x-y) + \omega_q i\delta(y-x)\right] \\
&= \frac{e^{i(\omega_k - \omega_q)t}}{2(2\pi)^3} \frac{\omega_k + \omega_q}{\sqrt{\omega_k\omega_q}} \int d^3x e^{i(q-k)x} \\
&= \frac{e^{i(\omega_k - \omega_q)t}}{2(2\pi)^3} \frac{\omega_k + \omega_q}{\sqrt{\omega_k\omega_q}} (2\pi)^3 \delta(q-k) \\
&= f(k,q)\delta(q-k) = \delta(q-k) \text{ do khi } k = q \text{ thì } f(k,q) \to 1
\end{aligned}$$