

BTVN1

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Ngày 22 tháng 10 năm 2024

Problem 3.4: $\mathcal{L}y = y'' - \lambda^2 y = f(x)$. $\begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases}$.

Giải: Phương trình thuần nhất:

$$\begin{aligned} y'' - \lambda^2 y &= 0 \\ \Rightarrow y'' &= \lambda^2 y. \end{aligned}$$

Nghiệm tổng quát có dạng:

$$y_1 = A \cosh(\lambda x) + B \sinh(\lambda x). \quad (1)$$

$$y_2 = A \cosh(1 - \lambda x) + B \sinh(1 - \lambda x). \quad (2)$$

Áp dụng điều kiện biên cho (1):

$$\begin{cases} y_1(0) = A \cosh(0) + B \sinh(0) = 0 \Rightarrow A = 0, \\ y_2(1) = B \sinh(1 - \lambda) = 0 \Rightarrow \sinh(1 - \lambda) = 0 \Rightarrow \lambda = 1, \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = \sinh(x) \\ y_2 = \sinh(1 - x). \end{cases} \quad (3)$$

Wronskian của y_1 và y_2 :

$$\begin{aligned} \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} \sinh(x) & \sinh(1 - x) \\ \cosh(x) & -\cosh(1 - x) \end{vmatrix} \\ &= -\sinh(x) \cosh(1 - x) - \cosh(x) \sinh(1 - x) \\ &= -\sinh 1. \end{aligned}$$

$\alpha(\xi) = 1$.

Hàm Green có dạng:

$$\begin{aligned} G(x, \xi) &= \frac{1}{\alpha(\xi)W(\xi)} [\Theta(\xi - x)y_1(x)y_2(1 - \xi) + \Theta(x - \xi)y_2(1 - x)y_1(\xi)] \\ &= -\frac{1}{\sinh 1} [\Theta(\xi - x) \sinh(x) \sinh(1 - \xi) + \Theta(x - \xi) \sinh(1 - x) \sinh(\xi)]. \quad (4) \end{aligned}$$

Nghiệm của $\mathcal{L}y = f$ là:

$$\begin{aligned}
y &= \int_a^b G(x, \xi) f(\xi) d\xi \\
&= y_2(x) \int_a^x \frac{y_1(\xi)}{\alpha W} f(\xi) d\xi + y_1(x) \int_x^b \frac{y_2(\xi)}{\alpha W} f(\xi) d\xi \\
&= \sinh(1-x) \int_a^x \frac{\sinh(\xi)}{\alpha W} f(\xi) d\xi + \sinh(x) \int_x^b \frac{\sinh(1-\xi)}{\alpha W} f(\xi) d\xi
\end{aligned} \tag{5}$$

với $\alpha = 1, W = -\sinh 1$

Tính tích phân: $g_n(\xi) = 2 \int_0^1 G(x, \xi) \sin(n\pi x) dx$.

Trong đó :

$$G = \Theta(x - \xi) \sin \xi \cos x + \Theta(\xi - x) \cos \xi \sin x - \cot 1 \sin \xi \sin x,$$

thay vào $g_n(\xi)$ ta được,

$$RHS = 2 \int_0^1 \left[\underbrace{\Theta(x - \xi) \sin \xi \cos x}_A + \underbrace{\Theta(\xi - x) \cos \xi \sin x}_B - \underbrace{\cot 1 \sin \xi \sin x}_C \right] \sin(n\pi x) dx$$

A term:

$$\begin{aligned}
&\int_0^1 \Theta(x - \xi) \sin \xi \cos x \sin(n\pi x) dx \\
&= \int_\xi^1 \sin \xi \cos x \sin(n\pi x) dx \\
&= \sin \xi \left[\int_\xi^1 \cos x \sin(n\pi x) dx \right] \\
&= \frac{\sin \xi}{2} \left[\int_\xi^1 \sin[(n\pi + 1)x] + \sin[(n\pi - 1)x] dx \right] \\
&= -\frac{\sin \xi}{2} \left[\frac{\cos[(n\pi + 1)x]}{n\pi + 1} \Big|_\xi^1 + \frac{\cos[(n\pi - 1)x]}{n\pi - 1} \Big|_\xi^1 \right] \\
&= -\frac{\sin \xi}{2} \frac{\cos(n\pi + 1)x - \cos(n\pi - 1)x - n\pi[\cos(n\pi + 1)x + \cos(n\pi - 1)x]}{n^2\pi^2 - 1} \Big|_\xi^1 \\
&= -\frac{\sin \xi - 2 \sin x \sin(n\pi x) - 2n\pi \cos x \cos n\pi x}{n^2\pi^2 - 1} \Big|_\xi^1 \\
&= \sin \xi \left[-\frac{\sin 1 \sin n\pi + n\pi \cos 1 \cos n\pi}{n^2\pi^2 - 1} + \frac{\sin \xi \sin n\pi\xi + n\pi \cos \xi \cos n\pi\xi}{n^2\pi^2 - 1} \right] \\
&= \sin \xi \left[-\frac{n\pi \cos 1 \cos n\pi}{n^2\pi^2 - 1} + \frac{\sin \xi \sin n\pi\xi + n\pi \cos \xi \cos n\pi\xi}{n^2\pi^2 - 1} \right].
\end{aligned}$$

B term:

$$\begin{aligned}
& \int_0^1 \Theta(\xi - x) \cos \xi \sin x \sin(n\pi x) dx \\
&= \int_0^\xi \cos \xi \sin x \sin(n\pi x) dx \\
&= \cos \xi \int_0^\xi \sin x \sin(n\pi x) dx \\
&= -\frac{\cos \xi}{2} \int_0^\xi \cos [(n\pi + 1)x] - \cos [(n\pi - 1)x] dx \\
&= -\frac{\cos \xi}{2} \left[\frac{\sin [(n\pi + 1)x]}{n\pi + 1} - \frac{\sin [(n\pi - 1)x]}{n\pi - 1} \right] \Big|_0^\xi \\
&= -\frac{\cos \xi}{2} \left[\frac{(n\pi - 1) \sin [(n\pi + 1)x] - (n\pi + 1) \sin [(n\pi - 1)x]}{n^2\pi^2 - 1} \right] \Big|_0^\xi \\
&= -\frac{\cos \xi}{2} \left[\frac{-\sin[(n\pi + 1)x] - \sin[(n\pi - 1)x] + n\pi [\sin[(n\pi + 1)x] - \sin[(n\pi - 1)x]]}{n^2\pi^2 - 1} \right] \Big|_0^\xi \\
&= -\frac{\cos \xi}{2} \left[\frac{-2 \sin(n\pi x) \cos x + n\pi [2 \cos(n\pi x) \sin x]}{n^2\pi^2 - 1} \right] \Big|_0^\xi \\
&= \frac{\cos \xi}{2} \left[\frac{2 \sin(n\pi \xi) \cos \xi - n\pi [2 \cos(n\pi \xi) \sin \xi]}{n^2\pi^2 - 1} \right] \\
&= \left[\frac{\sin(n\pi \xi) \cos^2 \xi - n\pi \cos(n\pi \xi) \sin \xi \cos \xi}{n^2\pi^2 - 1} \right].
\end{aligned}$$

C term:

$$\begin{aligned}
& - \int_0^1 \cot 1 \sin \xi \sin x \sin(n\pi x) dx \\
&= - \frac{\cot 1 \sin \xi}{2} \int_0^1 \cos[(n\pi + 1)]x - \cos[(n\pi - 1)]x dx \\
&= - \frac{\cot 1 \sin \xi}{2} \left[\frac{\sin[(n\pi + 1)]x}{n\pi + 1} - \frac{\sin[(n\pi - 1)]x}{n\pi - 1} \right] \Big|_0^1 \\
&= - \frac{\cot 1 \sin \xi}{2} \left[\frac{(n\pi - 1) \sin[(n\pi + 1)]x - (n\pi + 1) \sin[(n\pi - 1)]x}{n^2\pi^2 - 1} \right] \Big|_0^1 \\
&= - \frac{\cot 1 \sin \xi}{2} \left[\frac{n\pi \sin[(n\pi + 1)]x - \sin[(n\pi + 1)]x - n\pi \sin[(n\pi - 1)]x - \sin[(n\pi - 1)]x}{n^2\pi^2 - 1} \right] \Big|_0^1 \\
&= - \frac{\cot 1 \sin \xi}{2} \left[\frac{-\sin[(n\pi + 1)]x - \sin[(n\pi - 1)]x - n\pi [\sin[(n\pi + 1)]x - \sin[(n\pi - 1)]x]}{n^2\pi^2 - 1} \right] \Big|_0^1 \\
&= - \frac{\cot 1 \sin \xi}{2} \left[\frac{-2 \sin n\pi x \cos x - n\pi [2 \cos n\pi x \sin x]}{n^2\pi^2 - 1} \right] \Big|_0^1 \\
&= - \cot 1 \sin \xi \left[\frac{-\sin n\pi x \cos x - n\pi [\cos n\pi x \sin x]}{n^2\pi^2 - 1} \right] \Big|_0^1 \\
&= - \cot 1 \sin \xi \left[\frac{-\sin n\pi \cos 1 - n\pi [\cos n\pi \sin 1] + \sin 0 \cos 0 - n\pi [\cos 0 \sin 0]}{n^2\pi^2 - 1} \right] \\
&= \left[\frac{n\pi \cot 1 \sin 1 \sin \xi \cos n\pi}{n^2\pi^2 - 1} \right].
\end{aligned}$$

Cộng các thành phần lại ta được:

$$\begin{aligned}
g_n(\xi) &= 2 \left(\sin \xi \left[\frac{-n\pi \cos 1 \cos n\pi}{n^2\pi^2 - 1} + \frac{\sin \xi \sin(n\pi\xi) + n\pi \cos \xi \cos(n\pi\xi)}{n^2\pi^2 - 1} \right] \right. \\
&\quad \left. + \left[\frac{\sin(n\pi\xi) \cos^2 \xi - n\pi \cos(n\pi\xi) \sin \xi \cos \xi}{n^2\pi^2 - 1} \right] + \left[\frac{n\pi \cot 1 \sin 1 \sin \xi \cos n\pi}{n^2\pi^2 - 1} \right] \right) \\
&= \frac{2}{n^2\pi^2 - 1} \left(-n\pi \sin \xi \cos 1 \cos n\pi + \sin^2 \xi \sin(n\pi\xi) + n\pi \sin \xi \cos \xi \cos(n\pi\xi) \right. \\
&\quad \left. + \sin(n\pi\xi) \cos^2 \xi - n\pi \cos(n\pi\xi) \sin \xi \cos \xi + n\pi \cot 1 \sin 1 \sin \xi \cos n\pi \right) \\
&= \frac{2}{n^2\pi^2 - 1} \left(\cancel{-n\pi \sin \xi \cos 1 \cos n\pi} + \sin^2 \xi \sin(n\pi\xi) + \cancel{n\pi \sin \xi \cos \xi \cos(n\pi\xi)} \right. \\
&\quad \left. + \sin(n\pi\xi) \cos^2 \xi - \cancel{n\pi \cos(n\pi\xi) \sin \xi \cos \xi} + \cancel{n\pi \cot 1 \sin 1 \sin \xi \cos n\pi} \right) \\
&= \frac{2 \sin n\pi\xi}{n^2\pi^2 - 1} (DPCM)
\end{aligned}$$