<u>Remind:</u>
Residues and Improper integrals

1. Zeros and Poles

A. Zeros

- If f $(z_0) = 0$, then the point z_0 is said to be a zero of the function f (z).
- If f (z) is analytic at z_0, then we can expand it in a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

- If f $(z_0) = 0$, clearly $a_0 = 0$.
  - If $a_1 \neq 0$, then z_0 is said to be a simple zero.
  - If both a_0 and a_1 are zero and $a_2 \neq 0$, then z_0 is a zero of order two, and so on.
- If f $(z) \neq 0$ for $0 < |z - z_0| < \delta$, z_0 is said to be an isolated zero of f (z)

B. Isolated Singularities

- A point z_0 is said to be an isolated singularity of f (z) if there exists a neighborhood of z_0 in which z_0 is the only singular point of f (z)

C. Poles

- If f(z) has an isolated singular point at z_0, f(z) can be expanded in a Laurent series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{a_{-1}}{(z-z_0)}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-3}}{(z-z_0)^3}+\cdots$$

- There exists an integer m such that

$$a_{-m}\neq 0 \quad \text{and} \quad a_{-(m+1)}=a_{-(m+2)}=\cdots=0.$$

- The expansion takes the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{a_{-1}}{(z-z_0)}+\frac{a_{-2}}{(z-z_0)^2}+\cdots+\frac{a_{-m}}{(z-z_0)^m},$$

- The isolated singular point z_0 is called a pole of order m

2. Residues

- If z_0 is an isolated singular point of f(z), in the immediate neighborhood of z_0, f(z) can be expanded in a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

with

$$a_{-1} = \frac{1}{2\pi \mathrm{i}} \oint_C f(z)\, \mathrm{d}z$$

- a_{-1} is called the residue of f (z) at the isolated singular point z_0

$$a_{-1} = \mathrm{Res}_{z=z_0}\left[f\left(z\right)\right]$$

- The reason for the name "residue" is that if we integrate the Laurent series term by term over a circular contour, the only term which survives the integration process is the a_{-1} term.

3. Methods of finding residues

A. Laurent Series

If it is easy to write down the Laurent series for f (z) about $z = z_0$ that is valid in the immediate neighborhood of z_0, then the residue is just the coefficient a_{-1} of the term $1/(z - z_0)$

For example:

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2}\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^6} + \cdots .$$

There is no 1/z term, therefore the residue is equal to zero!

B. Simple Pole

If f(z) has a simple, or first-order, pole at $z = z_0$, the residue can be determined by

$$\boxed{a_{-1} = \lim_{z \to a} (z - z_0) f(z).}$$

Example 1:

$$f(z) = \frac{4 - 3z}{z(z-1)(z-2)},$$

C. Multiple-Order Pole

If f(z) has a pole of order m at $z = z_0$, then the residue at z_0

$$\boxed{\operatorname{Res}_{z=z_0}\left[f(z)\right] = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left[(z-a)^m f(z)\right].}$$

Example 2:

$$f(z) = \frac{1}{z(z-2)^4}$$

D. Derivatives of the denominator

If $p(z)$ and $q(z)$ are analytic functions, and $q(z)$ has a simple zero at z_0 and $p(z_0) \neq 0$, then

$$f(z) = \frac{p(z)}{q(z)}$$ has a simple pole at z_0

The residue at z_0 is

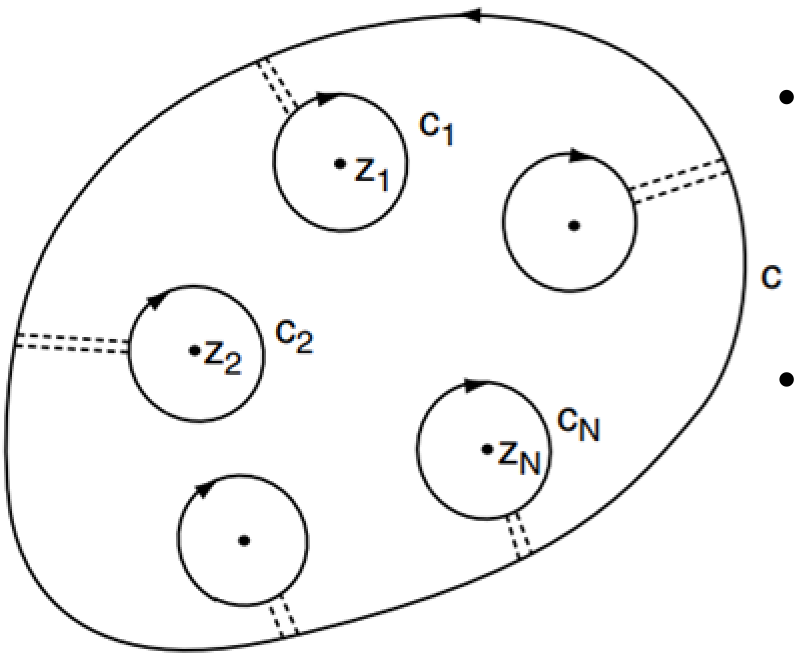
$$\begin{aligned}
\operatorname{Res}_{z=z_0}[f(z)] &= \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} \\
&= \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q'(z_0)(z - z_0) + \frac{q''(z_0)}{2!}(z - z_0)^2 + \cdots} \\
&= \frac{p(z_0)}{q'(z_0)}.
\end{aligned}$$

The most efficient way of finding the residue!

Example 3:

$$f(z) = \frac{z}{z^4 + 4}$$

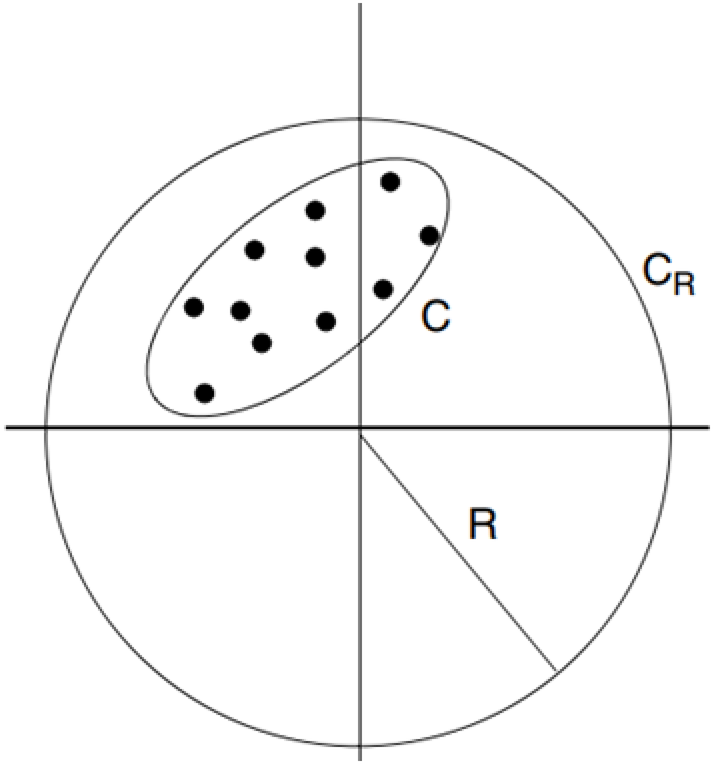
4. Cauchy's Residue Theorem



- Consider a simple closed curve C containing in its interior a number of isolated singular points, z_1, z_2, ..., of a function f(z).
- The circles C_1, C_2,..., C_N enclosing, respectively, the singular points z_1, z_2, ..., z_N within a simple closed curve

$$\boxed{\oint_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\left\{\mathrm{Res}_{z=z_1}[f(z)] + \mathrm{Res}_{z=z_2}[f(z)] + \cdots + \mathrm{Res}_{z=z_n}[f(z)]\right\}}$$

5. Second Residue Theorem



If the number of singularities enclosed in C is too large, then it is more convenient to replace the contour C with a large circular contour C_R centered at the origin

$$\boxed{\oint_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]}$$

Example 4: Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = \frac{5z-2}{z(z-1)},$$

where C is along the circle $|z| = 2$ in the counterclockwise direction. (a) Use the Cauchy residue theorem. (b) Use the second residue theorem.

Example 5: Find the value of the integral

$$\oint_C \frac{dz}{z^3 (z+4)}$$

taken counterclockwise around the circle (a) $|z| = 2$, (b) $|z+2| = 3$.

Example 6: Find the value of the integral

$$\oint_C \tan \pi z \mathrm{d}z$$

taken counterclockwise around the unit circle $|z| = 1$.

Example 7: Evaluate the integral $\oint_C f(z)\,\mathrm{d}z$ for

$$f(z) = z^2 \exp\left(\frac{1}{z}\right),$$

where C is counterclockwise around the unit circle $|z| = 1$.

Solution 3.4.4. The function $f(z)$ has an essential singularity at $z = 0$. Thus

$$\oint_C z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}_{z=0}[f(z)].$$

The residue is simply the coefficient of the z^{-1} term in the Laurent series about $z = 0$,

$$\begin{aligned} z^2 \exp\left(\frac{1}{z}\right) &= z^2\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \cdots\right) \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^2} + \cdots. \end{aligned}$$

Therefore

$$\mathrm{Res}_{z=0}[f(z)] = \frac{1}{3!} = \frac{1}{6}.$$

Hence

$$\oint_C z^2 \exp\left(\frac{1}{z}\right) \mathrm{d}z = 2\pi\mathrm{i}\frac{1}{6} = \frac{\pi}{3}\mathrm{i}.$$

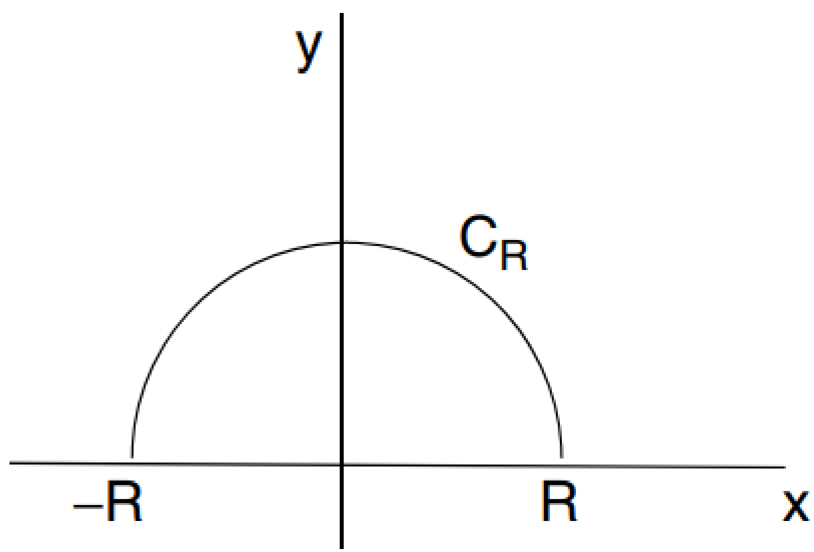
6. Evaluation of improper integrals using residue theorem

- We consider the real **improper** integrals

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x)\mathrm{d}x.$$

- If f(x) is a rational function: f(x) = p(x)/q(x) an $\lim_{z\to\infty} z f(z) = 0,$

$$\boxed{\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty} \left[\int_{-R}^{R} f(x)\mathrm{d}x + \int_{C_R} f(z)\,\mathrm{d}z\right] = \oint_{\text{u.h.p}} f(z)\,\mathrm{d}z,}$$



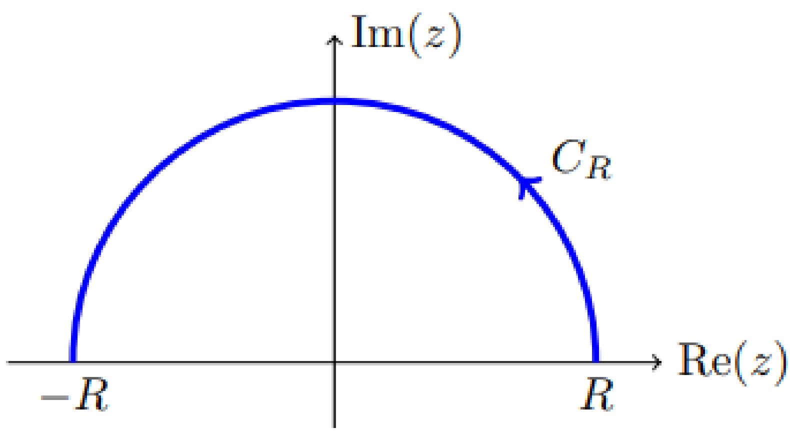
See pages 144-145 for the proof!

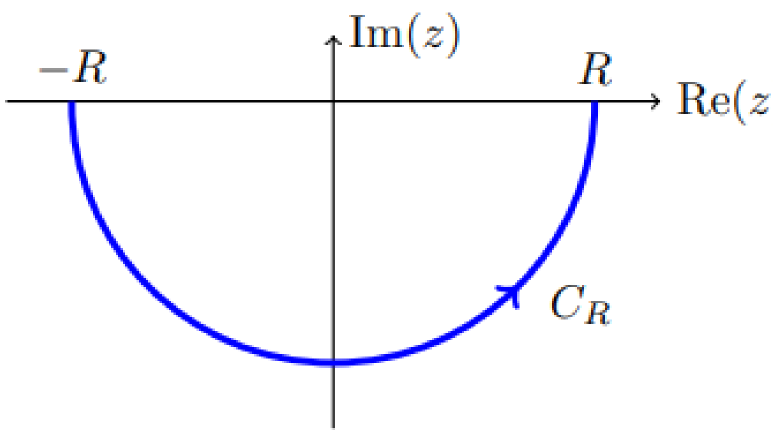
- As $R \to \infty$, all the poles of $f(z)$ in the upper half-plane will be inside the contour. Hence:

$$\boxed{\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 2\pi\mathrm{i}\left(\text{sum of residues of } f(z) \text{ in the upper half-plane}\right)}$$

- By the same token, we can, of course, close the contour in the lower half-plane

$$\boxed{\begin{aligned}\int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \oint_{\mathrm{l.h.p}} f(z)\,\mathrm{d}z \\ &= -2\pi\mathrm{i}\left(\text{sum of residues of } f(z) \text{ in the lower half-plane}\right)\end{aligned}}$$





Example 8: Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, \mathrm{d}x.$$

Example 9: Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)}\,\mathrm{d}x.$$

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