Bài tập về nhà 1

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VẬT LÝ LÝ THUYẾT

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Câu 1:

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin x,$$

với điều kiện biên $u(x, 0) = \cos x$.

Câu 2:

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = u^2,$$

với $u = 4$, dọc theo $y = 2x - 1$.

Câu 3:

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = x,$$

với điều kiện biên $u(y, 0) = -2y, \forall x \in \mathbb{R}$.

Bài làm

Câu 1: The Lagrange - Charpit equations for the PDE are:

$$\frac{dx}{1} = \frac{dy}{y} = \frac{du}{\sin x}$$

$$\Rightarrow \begin{cases} \dfrac{dy}{dx} = y \\ \dfrac{du}{dy} = \dfrac{\sin x}{y} \\ \dfrac{du}{dx} = \dfrac{\sin x}{1} \end{cases}$$

which leads to

$$
\begin{aligned}
&\frac{dy}{y} = dx \Rightarrow \int \frac{dy}{y} = \int dx \Rightarrow y = Ae^{x} \\
&\frac{du}{dy} = \frac{\sin x}{y} \Rightarrow
\end{aligned}
$$

Câu 2:

The Lagrange - Charpit equations for the PDE are:

$$
\begin{gathered}
\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{u^2} \\
\Rightarrow \begin{cases} \dfrac{dy}{dx} = -1 \\ \dfrac{du}{dy} = -u^2 \\ \dfrac{du}{dx} = u^2 \end{cases},
\end{gathered}
$$

this leads to:

$$
\begin{aligned}
dy &= -dx \Rightarrow y = -x + A \\
-\frac{du}{u^2} &= dy \Rightarrow \frac{1}{u} = y + B \Rightarrow u = \frac{1}{y+B}
\end{aligned}
$$

By applying the initial conditions.

$$
u(x,y) = u(x, 2x-1) = 4,
$$

we set

$$
\begin{cases} (2x-1)\Big|_{x=\xi} = -x\Big|_{x=\xi} + A \\ u = \dfrac{1}{2x-1+B}\Big|_{x=\xi} = 4 \end{cases} \Rightarrow \begin{cases} A = 3\xi - 1 \\ B = -2\xi + \dfrac{5}{4} \end{cases} \Rightarrow \begin{cases} y = -x + 3\xi - 1 \\ u = \dfrac{1}{y - 2\xi + \frac{5}{4}} \end{cases}
$$

Eliminating ξ from the second equation by using the first, this yield

$$
u(x,y) = \frac{1}{y - \frac{2}{3}(y+x+1) + \frac{5}{4}}
$$

is solution of the PDE.

Câu 3: The value of function u at an arbitrary ξ on the initial curve ($s = 0$) is given by $u(y,0) = -2y$. By replacing dummy variables $y \equiv \xi$, of courses, the initial condition now is $u(\xi, 0) = -2\xi$.

At $s = 0$, the curve starts at the initial point $(x = 0, y = \xi)$.

$$\frac{dy}{ds} = u \qquad \text{with } y(s=0) = \xi, \tag{1}$$
$$\frac{dx}{ds} = 1 \qquad \text{with } x(s=0) = 0, \tag{2}$$
$$\frac{du}{ds} = x \qquad \text{with } u(s=0) = -2\xi. \tag{3}$$

Consider equation (2)

$$dx = ds \Rightarrow x = s + A. \tag{4}$$

Next is equation (3)

$$du = xds \Rightarrow u = (s + A)ds \Rightarrow u = \frac{s^2}{2} + sA + C \tag{5}$$

Plugging u into (1), this leads to

$$dy = uds = \left(\frac{s^2}{2} + sA + C\right) ds \Rightarrow y = \frac{s^3}{6} + \frac{s^2 A}{2} + Cs + B \tag{6}$$

Apllying initial condition to (4),(5),(6)

$$\begin{cases} x(s=0) = 0 = A \\ y(s=0) = \xi = B \\ u(s=0) = -2\xi = C \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = \xi \\ C = -2\xi \end{cases}$$

We have

$$\begin{aligned} x &= s \\ y &= \frac{s^3}{6} - 2\xi s + \xi = \frac{s^3}{6} - (2s+1)\xi \\ u &= \frac{s^2}{2} - 2\xi \end{aligned}$$

Eliminating ξ, this leads to

$$\begin{aligned} x &= s \\ \xi &= \frac{\frac{x^3}{6} - y}{2s + 1} \\ u &= \frac{x^2}{2} - 2\left(\frac{\frac{x^3}{6} - y}{2 + 1}\right) \end{aligned} \tag{7}$$

Symplyfing the Eqn.(7), we have

$$u(x,y) = \frac{x^2}{2} - \frac{x^3}{6x+3} + \frac{2y}{2x+1}$$

which is the solution of the PDE