BTVN1

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Problem 3.4: $\mathcal{L}y = y'' - \lambda^2 y = f(x)$. $\begin{cases} y(0) = 0 \\ y(1) = 0 \end{cases}$.

Giải: Phương trình thuần nhất:

$$\begin{aligned} & y'' - \lambda^2 y = 0 \\ \Rightarrow & y'' = \lambda^2 y. \end{aligned}$$

Nghiệm tổng quát có dạng:

$$y_1 = A\cosh(\lambda x) + B\sinh(\lambda x). \tag{1}$$
$$y_2 = A\cosh(1 - \lambda x) + B\sinh(1 - \lambda x). \tag{2}$$

Áp dụng điều kiện biên cho (1):

$$\begin{cases} y_1(0) & = A\cosh(0) + B\sinh(0) = 0 \Rightarrow A = 0, \\ y_2(1) & = B\sinh(1-\lambda) = 0 \Rightarrow \sinh(1-\lambda) = 0 \Rightarrow \lambda = 1, \end{cases}$$

$$\Rightarrow \begin{cases} y_1 & = \sinh(x) \\ y_2 & = \sinh(1-x). \end{cases} \tag{3}$$

Wronskian của y_1 và y_2:

$$\begin{aligned} \begin{vmatrix} y1 & y2 \\ y1' & y2' \end{vmatrix} = \begin{vmatrix} \sinh(x) & \sinh(1-x) \\ \cosh(x) & -\cosh(1-x) \end{vmatrix} \\ &= -\sinh(x)\cosh(1-x) - \cosh(x)\sinh(1-x) \\ &= -\sinh 1. \end{aligned}$$

$\alpha(\xi) = 1$.
Hàm Green có dạng:

$$\begin{aligned} G(x,\xi) &= \frac{1}{\alpha(\xi)W(\xi)}\left[\Theta(\xi - x)y_1(x)y_2(1-\xi) + \Theta(x-\xi)y_2(1-x)y_1(\xi)\right] \\ &= -\frac{1}{\sinh 1}\left[\Theta(\xi - x)\sinh(x)\sinh(1-\xi) + \Theta(x-\xi)\sinh(1-x)\sinh(\xi)\right]. \end{aligned} \tag{4}$$

Nghiệm của $\mathcal{L}y = f$ là:

$$\begin{aligned} y &= \int_a^b G(x,\xi) f(\xi) d\xi \\ &= y_2(x) \int_a^x \frac{y_1(\xi)}{\alpha W} f(\xi) d\xi + y_1(x) \int_x^b \frac{y_2(\xi)}{\alpha W} f(\xi) d\xi \\ &= \sinh(1-x) \int_a^x \frac{\sinh(\xi)}{\alpha W} f(\xi) d\xi + \sinh(x) \int_x^b \frac{\sinh(1-\xi)}{\alpha W} f(\xi) d\xi \end{aligned} \tag{5}$$

với $\alpha = 1, W = -\sinh 1$

Tính tích phân: $g_n(\xi) = 2\displaystyle\int_0^1 G(x,\xi)\sin(n\pi x) dx.$

Trong đó :

$$G = \Theta(x-\xi)\sin\xi\cos x + \Theta(\xi - x)\cos\xi\sin x - \cot 1 \sin\xi\sin x,$$

thay vô $g_n(\xi)$ ta được,

$$RHS = 2\int_0^1 \left[\underbrace{\Theta(x-\xi)\sin\xi\cos x}_{A} + \underbrace{\Theta(\xi-x)\cos\xi\sin x}_{B} - \underbrace{\cot 1\sin\xi\sin x}_{C} \right] \sin(n\pi x) dx$$

A term:

$$\begin{aligned} &\int_0^1 \Theta(x-\xi)\sin\xi\cos x\sin(n\pi x)dx \\ =&\int_\xi^1 \sin\xi\cos x\sin(n\pi x)dx \\ =&\sin\xi\left[\int_\xi^1 \cos x\sin(n\pi x)dx\right] \\ =&\frac{\sin\xi}{2}\left[\int_\xi^1 \sin[(n\pi+1)x] + \sin[(n\pi-1)x]dx\right] \\ =&-\frac{\sin\xi}{2}\left[\frac{\cos[(n\pi+1)x]}{n\pi+1}\bigg|_\xi^1 + \frac{\cos[(n\pi-1)x]}{n\pi-1}\bigg|_\xi^1\right] \\ =&-\frac{\sin\xi}{2}\frac{\cos(n\pi+1)x - \cos(n\pi-1)x - n\pi[\cos(n\pi+1)x + \cos(n\pi-1)x]}{n^2\pi^2-1}\bigg|_\xi^1 \\ =&-\frac{\sin\xi}{2}\frac{-2\sin x\sin(n\pi x) - 2n\pi\cos x\cos n\pi x}{n^2\pi^2-1}\bigg|_\xi^1 \\ =&\sin\xi\left[\frac{\sin 1\sin n\pi + n\pi\cos 1\cos n\pi}{n^2\pi^2-1} + \frac{\sin\xi\sin n\pi\xi + n\pi\cos\xi\cos n\pi\xi}{n^2\pi^2-1}\right] \\ =&\sin\xi\left[\frac{n\pi\cos 1\cos n\pi}{n^2\pi^2-1} + \frac{\sin\xi\sin n\pi\xi + n\pi\cos\xi\cos n\pi\xi}{n^2\pi^2-1}\right]. \end{aligned}$$

B term:

$$
\begin{aligned}
&\int_0^1 \Theta(\xi - x)\cos\xi \sin x \sin(n\pi x)dx \\
=&\int_0^\xi \cos\xi \sin x \sin(n\pi x)dx \\
=&\cos\xi \int_0^\xi \sin x \sin(n\pi x)dx \\
=&-\frac{\cos\xi}{2}\int_0^\xi \cos\big[(n\pi+1)x\big] - \cos\big[(n\pi-1)x\big]\, dx \\
=&-\frac{\cos\xi}{2}\left[\frac{\sin\big[(n\pi+1)x\big]}{n\pi+1} - \frac{\sin\big[(n\pi-1)x\big]}{n\pi-1}\right]\Bigg|_0^\xi \\
=&-\frac{\cos\xi}{2}\left[\frac{(n\pi-1)\sin\big[(n\pi+1)x\big] - (n\pi+1)\sin\big[(n\pi-1)x\big]}{n^2\pi^2-1}\right]\Bigg|_0^\xi \\
=&-\frac{\cos\xi}{2}\left[\frac{-\sin[(n\pi+1)x] - \sin[(n\pi-1)x] + n\pi\big[\sin[(n\pi+1)x] - \sin[(n\pi-1)x]\big]}{n^2\pi^2-1}\right]\Bigg|_0^\xi \\
=&-\frac{\cos\xi}{2}\left[\frac{-2\sin(n\pi x)\cos x + n\pi\big[2\cos(n\pi x)\sin x\big]}{n^2\pi^2-1}\right]\Bigg|_0^\xi \\
=&\frac{\cos\xi}{2}\left[\frac{2\sin(n\pi\xi)\cos\xi - n\pi\big[2\cos(n\pi\xi)\sin\xi\big]}{n^2\pi^2-1}\right] \\
=&\left[\frac{\sin(n\pi\xi)\cos^2\xi - n\pi\cos(n\pi\xi)\sin\xi\cos\xi}{n^2\pi^2-1}\right].
\end{aligned}
$$

C term:

$$-\int_0^1 \cot 1 \sin\xi \sin x \sin(n\pi x)dx$$
$$= -\frac{\cot 1 \sin\xi}{2}\int_0^1 \cos[(n\pi+1)]x - \cos[(n\pi-1)]x dx$$
$$= -\frac{\cot 1 \sin\xi}{2}\left[\frac{\sin[(n\pi+1)]x}{n\pi+1} - \frac{\sin[(n\pi-1)]x}{n\pi-1}\right]\Bigg|_0^1$$
$$= -\frac{\cot 1 \sin\xi}{2}\left[\frac{(n\pi-1)\sin[(n\pi+1)]x - (n\pi+1)\sin[(n\pi-1)]x}{n^2\pi^2-1}\right]\Bigg|_0^1$$
$$= -\frac{\cot 1 \sin\xi}{2}\left[\frac{n\pi\sin[(n\pi+1)]x - \sin[(n\pi+1)]x - n\pi\sin[(n\pi-1)]x - \sin[(n\pi-1)]x}{n^2\pi^2-1}\right]\Bigg|_0^1$$
$$= -\frac{\cot 1 \sin\xi}{2}\left[\frac{-\sin[(n\pi+1)]x - \sin[(n\pi-1)]x - n\pi\big[\sin[(n\pi+1)]x - \sin[(n\pi-1)]x\big]}{n^2\pi^2-1}\right]\Bigg|_0^1$$
$$= -\frac{\cot 1 \sin\xi}{2}\left[\frac{-2\sin n\pi x\cos x - n\pi\big[2\cos n\pi x\sin x\big]}{n^2\pi^2-1}\right]\Bigg|_0^1$$
$$= -\cot 1 \sin\xi\left[\frac{-\sin n\pi x\cos x - n\pi\big[\cos n\pi x\sin x\big]}{n^2\pi^2-1}\right]\Bigg|_0^1$$
$$= -\cot 1 \sin\xi\left[\frac{-\sin n\pi\cos 1 - n\pi\big[\cos n\pi\sin 1\big] + \sin 0\cos 0 - n\pi\big[\cos 0\sin 0\big]}{n^2\pi^2-1}\right]$$
$$= \left[\frac{n\pi\cot 1\sin 1\sin\xi\cos n\pi}{n^2\pi^2-1}\right].$$

Cộng các thành phần lại ta được:

$$g_n(\xi) = 2\Bigg(\sin\xi\left[\frac{-n\pi\cos 1\cos n\pi}{n^2\pi^2-1} + \frac{\sin\xi\sin(n\pi\xi) + n\pi\cos\xi\cos(n\pi\xi)}{n^2\pi^2-1}\right]$$
$$+\left[\frac{\sin(n\pi\xi)\cos^2\xi - n\pi\cos(n\pi\xi)\sin\xi\cos\xi}{n^2\pi^2-1}\right] + \left[\frac{n\pi\cot 1\sin 1\sin\xi\cos n\pi}{n^2\pi^2-1}\right]\Bigg)$$
$$= \frac{2}{n^2\pi^2-1}\Big(-n\pi\sin\xi\cos 1\cos n\pi + \sin^2\xi\sin(n\pi\xi) + n\pi\sin\xi\cos\xi\cos(n\pi\xi)$$
$$+\sin(n\pi\xi)\cos^2\xi - n\pi\cos(n\pi\xi)\sin\xi\cos\xi + n\pi\cot 1\sin 1\sin\xi\cos n\pi\Big)$$
$$= \frac{2}{n^2\pi^2-1}\Big(\cancel{-n\pi\sin\xi\cos 1\cos n\pi} + \sin^2\xi\sin(n\pi\xi) + \cancel{n\pi\sin\xi\cos\xi\cos(n\pi\xi)}$$
$$+\sin(n\pi\xi)\cos^2\xi - \cancel{n\pi\cos(n\pi\xi)\sin\xi\cos\xi} + \cancel{n\pi\cot 1\sin 1\sin\xi\cos n\pi}\Big)$$
$$= \frac{2\sin n\pi\xi}{n^2\pi^2-1}(DPCM)$$