# Intro to Fourier transform

# 1. Definition

Fourier series

$$f(x) = \frac{1}{2L} \sum_{n \in \mathbb{Z}} \hat{f}_n \, \mathrm{e}^{\mathrm{i} n \pi x / L}$$

where the period is 2L

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i} n x} \, f(x) \, \mathrm{d}x$$

For a (non-periodic) function $f$ : **R** → **C**

$$\tilde{f}(k) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i} k x} \, f(x) \, \mathrm{d}x \, .$$

# 2. Properties

For any constants $c_1$, $c_2 \in$ C and integrable functions *f, g:*

$$\mathcal{F}[c_1 f + c_2 g] = c_1 \mathcal{F}[f] + c_2 \mathcal{F}[g]\,.$$

And other relations:

$$\mathcal{F}[f(x-a)] = \mathrm{e}^{-\mathrm{i}ka}\tilde{f}(k)$$
$$\mathcal{F}[\mathrm{e}^{\mathrm{i}\ell x}f(x)] = \tilde{f}(k-\ell)$$
$$\mathcal{F}[f(cx)] = \frac{1}{|c|}\tilde{f}(k/c)\,.$$

The convolution

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,\mathrm{d}y$$

$$\longrightarrow \quad \mathcal{F}[f * g(x)] = \mathcal{F}[f]\,\mathcal{F}[g]$$

# 3. Transform differentiation into multiplication

The Fourier transform of the derivative f′ of function f

$$\mathcal{F}[f'(x)] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\, f'(x)\,\mathrm{d}x = -\int_{-\infty}^{\infty} \left(\frac{d}{dx}\mathrm{e}^{-\mathrm{i}kx}\right) f(x)\,\mathrm{d}x = \mathrm{i}k\tilde{f}(k)$$

The assumption that f(x) was integrable means that **it must decay** as |x| → ∞

The Fourier transform of *xf*(*x*) is given by

$$\mathcal{F}[xf(x)] = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\, xf(x)\,\mathrm{d}x = \mathrm{i}\frac{d}{dk}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\, f(x)\,\mathrm{d}x = \mathrm{i}\tilde{f}'(k)$$

Differential operator with constant coefficients $c_r \in \boldsymbol{C}$:

$$\mathcal{L}(\partial) = \sum_{r=0}^{p} c_r \frac{d^r}{dx^r}$$

The Fourier transform:

$$\mathcal{F}[\mathcal{L}(\partial)y] = \mathcal{L}(\mathrm{i}k)\tilde{y}(k)$$

$$\mathcal{L}(\mathrm{i}k) = \sum_{r=0}^{p} c_r (\mathrm{i}k)^r$$

If $y$ obeys some differential equation $\mathcal{L}(\partial)y(x) = f(x)$

$$\tilde{y}(k) = \tilde{f}(k)/\mathcal{L}(\mathrm{i}k)$$

The Fourier inversion theorem

$$f(x) = \frac{1}{2\pi}\int_{\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\,\tilde{f}(k)\,\mathrm{d}k$$

# 4. Example

consider the pde

$$\nabla^2\phi(\mathbf{x}) - m^2\phi(\mathbf{x}) = -\rho(\mathbf{x})$$

the Fourier transform

$$\mathcal{F}[\phi(\mathbf{x})] = \tilde{\phi}(\mathbf{k}) = \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\, \phi(\mathbf{x})\, \mathrm{d}^n x\,,$$

our pde becomes simply

$$\tilde{\phi}(\mathbf{k}) = \frac{\tilde{\rho}(\mathbf{k})}{|\mathbf{k}|^2 + m^2}$$

$$\longrightarrow \quad \phi(\mathbf{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\, \tilde{\rho}(\mathbf{k})}{|\mathbf{k}|^2 + m^2}\, \mathrm{d}^n k\,.$$