A solution on Mathematatic Methods

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Problem sheet 1

Problem 1 David Skinner, 1B Methods, Page 1

- Given $f(\theta) = (\theta^2 - \pi^2)^2$, where $\theta \in [-\pi, \pi)$.
- Given $f(\theta) = e^{\theta}$, where $\theta \in [-\pi, \pi)$.
- Given $f(\theta) = \theta e^{i\theta}$, where $\theta \in [-\pi, \pi)$.

Find the Fourier series.

Solution: (a) $f(\theta)$ is even function. Therefore, $b_n = 0$, the Fourier coefficient a_n can be written as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta) f(\theta) d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta)(\theta^2 - \pi^2)^2 d\theta. \end{aligned} \tag{1}$$

Let

$$\begin{cases} u = (\theta^2 - \pi^2)^2 \\ dv = \cos(n\theta)\, d\theta \end{cases} \Rightarrow \begin{cases} du = \left(4\theta^3 - 4\pi^2\theta\right) d\theta \\ v = \dfrac{\sin(n\theta)}{n} \end{cases} \tag{2}$$

we get

$$a_n = \cancelto{0}{\frac{\left(\theta^2 - \pi^2\right)^2}{2\pi}\frac{\sin n\theta}{n}\Bigg|_{-\pi}^{\pi}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin n\theta}{n}\left(4\theta^3 - 4\pi^2\theta\right) d\theta, \tag{3}$$

let

$$
\begin{cases} u = (4\theta^3 - 4\pi^2\theta) \\ dv = -\dfrac{\sin(n\theta)}{n} d\theta \end{cases} \Rightarrow \begin{cases} du = \left(12\theta^2 - 4\pi^2\right) d\theta \\ v = \dfrac{\cos n\theta}{n^2} \end{cases} \tag{4}
$$

we get

$$
a_n = \cancelto{0}{(4\theta^3 - 4\pi^2\theta)\frac{\cos n\theta}{n^2}\bigg|_{-\pi}^{\pi}} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos n\theta}{n^2}\left(12\theta^2 - 4\pi^2\right) d\theta, \tag{5}
$$

let

$$
\begin{cases} u = \left(12\theta^2 - 4\pi^2\right) \\ dv = -\dfrac{\cos n\theta}{n^2} d\theta \end{cases} \Rightarrow \begin{cases} du = 24\theta d\theta \\ v = -\dfrac{\sin n\theta}{n^3} \end{cases} \tag{6}
$$

we get

$$
a_n = -(12\theta^2 - 4\pi^2)\frac{\sin n\theta}{n^3}\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\sin n\theta}{n^3} 24\theta d\theta, \tag{7}
$$

let

$$
\begin{cases} u = 24\theta \\ dv = \dfrac{\sin n\theta}{n^3} \end{cases} \Rightarrow \begin{cases} du = 24 d\theta \\ v = -\dfrac{\cos n\theta}{n^4} \end{cases} \tag{8}
$$

the a_n can be written as

$$
\begin{aligned}
a_n &= -\frac{24}{\pi}\theta\frac{\cos n\theta}{n^4}\bigg|_{-\pi}^{\pi} + \frac{24}{\pi}\int_{-\pi}^{\pi} \frac{\cos n\theta}{n^4} \\
&= -\frac{24}{n^4\pi}(\pi\cos n\pi - (-\pi\cos(-n\pi))) \\
&= -\frac{48}{n^4}(-1)^n,
\end{aligned} \tag{9}
$$

and a_0 is

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) d\theta \\
&= \frac{2}{\pi}\int_{0}^{\pi} (\theta^2 - \pi^2)^2 d\theta \\
&= \frac{16\pi^4}{15}.
\end{aligned} \tag{10}
$$

The Fourier series for f can be written as

$$
f(\theta) = \frac{8\pi^4}{15} - \sum_{n=1}^{\infty} \frac{48}{n^4}(-1)^n \cos n\theta. \tag{11}
$$

The $f(\theta)$ is continuous on $[-\pi, \pi)$, therefore, for arbitrary of θ, this Fourier series converge to $f(\theta)$. [1]

Solution: (b) $f(\theta)$ is neither even nor odd, the Fourier coefficients a_n, b_n can be written as

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta) f(\theta) d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\theta) e^{\theta} d\theta, \end{aligned} \tag{12}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta) f(\theta) d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\theta) e^{\theta} d\theta. \end{aligned} \tag{13}$$

Proof: $\displaystyle\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$**.** Let

$$I = e^{ax}\sin bx,$$

and

$$\begin{cases} u = \sin bx \\ dv = e^{ax}dx \end{cases} \Rightarrow \begin{cases} du = b\cos bx \\ v = \dfrac{e^{ax}}{a} \end{cases}$$

One can get

$$\frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a} b\cos bx dx.$$

Let

$$J = \frac{e^{ax}}{a} b\cos bx dx,$$

and

$$\begin{cases} u = b\cos bx \\ dv = \dfrac{e^{ax}}{a}dx \end{cases} \Rightarrow \begin{cases} du = -b^2\sin bx \\ v = \dfrac{e^{ax}}{a^2} \end{cases}$$

[1] If the function is continuous, the series converges to the function; otherwise, it converges to the midpoint of the jump, i.e, the left and right hand limits.

One can get

$$J = \frac{e^{ax}}{a^2} b\cos bx + \frac{b^2}{a^2}\int e^{ax}\sin bx dx$$
$$= \frac{e^{ax}}{a^2} b\cos bx + \frac{b^2}{a^2}\int I dx,$$

We can see this pattern is repeat. In particular,

$$\int I dx = \frac{e^{ax}}{a}\sin bx - \frac{e^{ax}}{a^2} b\cos bx - \frac{b^2}{a^2}\int I dx$$
$$\Rightarrow (a^2+b^2)\int I dx = e^{ax}(a\sin bx - b\cos bx).$$

This leads to

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

□

Note:

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2},$$
$$\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx - b\sin bx)}{a^2+b^2}.$$

One can get

$$\begin{aligned} a_n &= \frac{1}{\pi}\left.\frac{e^{\theta}(\cos n\theta - n\sin n\theta)}{1+n^2}\right|_{-\pi}^{\pi} \\ &= \frac{1}{\pi(1+n^2)}\left(e^{\pi}\cos(n\pi) - e^{-\pi}\cos(-n\pi)\right) \\ &= \frac{(-1)^n}{2\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right), \end{aligned} \tag{14}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\left.\frac{e^{\theta}(\sin n\theta - n\cos n\theta)}{1+n^2}\right|_{-\pi}^{\pi} \\ &= \frac{n}{\pi(1+n^2)}\left(-e^{\pi}\cos(n\pi) + e^{-\pi}\cos(-n\pi)\right) \\ &= -\frac{n(-1)^n}{2\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right). \end{aligned} \tag{15}$$

a_0 can be written as

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{\theta}d\theta \\ &= \frac{1}{\pi}(e^{\pi} - e^{-\pi}). \end{aligned} \tag{16}$$

The Fourier series of $f(\theta) = e^{\theta}$ can be written as

$$\begin{aligned} e^{\theta} = \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) + \sum_{n=1}^{\infty} \Bigg[\frac{(-1)^n}{\pi(1+n^2)} \left(e^{\pi} - e^{-\pi}\right) \cos n\theta \\ - \frac{n(-1)^n}{\pi(1+n^2)} \left(e^{\pi} - e^{-\pi}\right) \sin n\theta \Bigg], \end{aligned} \tag{17}$$

where

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{1+n^2} &= \frac{1}{2}(\pi \coth \pi - 1), \\ a_0 &= \frac{1}{\pi}(e^{\pi} - e^{-\pi}), \\ a_n &= \frac{(-1)^n}{\pi(1+n^2)} \left(e^{\pi} - e^{-\pi}\right) \cos n\theta, \\ b_n &= -\frac{n(-1)^n}{\pi(1+n^2)} \left(e^{\pi} - e^{-\pi}\right) \sin n\theta. \end{aligned}$$

Solution: (c) $f(\theta) = \theta e^{i\theta}$ is neither even nor odd, one can obtain the Fourier series by

$$\theta e^{i\theta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \tag{18}$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta e^{i\theta} d\theta, \tag{19}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos n\theta e^{i\theta} d\theta, \tag{20}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin n\theta e^{i\theta} d\theta. \tag{21}$$

Note:

$$\int x e^{ax} \sin bx dx = \frac{xe^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} - \frac{e^{ax}\left[(a^2-b^2)\sin bx - 2ab \cos bx\right]}{(a^2+b^2)^2},$$

$$\int x e^{ax} \cos bx dx = \frac{xe^{ax}(a \cos bx - b \sin bx)}{a^2+b^2} - \frac{e^{ax}\left[(a^2-b^2)\cos bx + 2ab \sin bx\right]}{(a^2+b^2)^2}.$$

In particular,

$$a_0 = -ie^{i\pi}(\pi + i) + ie^{-i\pi}(-\pi + i) = 2i\pi, \tag{22}$$

and

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta \cos n\theta e^{i\theta} d\theta, \\
&= \left.\frac{\theta e^{i\theta}(i\cos n\theta - n\sin n\theta)}{n^2-1}\right|_{-\pi}^{\pi} - \left.\frac{e^{i\theta}\left[(-n^2-1)\cos n\theta + 2in\sin n\theta\right]}{(n^2-1)^2}\right|_{-\pi}^{\pi} \qquad (23)\\
&= -\frac{2\pi i(-1)^n}{n^2-1}.
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \theta \sin n\theta e^{i\theta} d\theta \\
&= \left.\frac{\theta e^{i\theta}(i\sin n\theta - n\cos n\theta)}{n^2-1}\right|_{-\pi}^{\pi} - \left.\frac{e^{i\theta}\left[(-n^2-1)\sin n\theta - 2in\cos n\theta\right]}{(n^2-1)^2}\right|_{-\pi}^{\pi} \qquad (24)\\
&= \frac{2\pi n(-1)^n}{n^2-1}.
\end{aligned}
$$

Therefore, one can obtain the Fourier series for $f(\theta) = \theta\cos\theta$ and $f(\theta) = \theta\sin\theta$ by

$$\theta e^{i\theta} = \theta\cos\theta + i\theta\sin\theta. \qquad (25)$$

where

$$\theta\cos\theta = \mathrm{Re}[\frac{a_0}{2}] + \sum_{n=1}^{\infty} \mathrm{Re}[a_n]\cos n\theta + \mathrm{Re}[b_n]\sin n\theta, \qquad (26)$$

$$\theta\sin\theta = \mathrm{Im}[\frac{a_0}{2}] + \sum_{n=1}^{\infty} \mathrm{Im}[a_n]\cos n\theta + \mathrm{Im}[b_n]\sin n\theta. \qquad (27)$$

In particular

$$\theta\cos\theta = \sum_{n=1}^{\infty} \frac{2\pi n(-1)^n}{n^2-1}\sin n\theta, \qquad (28)$$

and

$$\theta\sin\theta = 2\pi - \sum_{n=1}^{\infty} \frac{2\pi n(-1)^n}{n^2-1}\cos n\theta. \qquad (29)$$

Problem 2 David Skinner, 1B Methods, Page 1

A certain function $\vartheta(x,t)$ obeys the condition

$$\vartheta(x+1,t)=\vartheta(x,t)$$

$$\vartheta(x+it,t)=e^{\pi t-2\pi i x}\vartheta(x,t)$$

$$\int_0^1 \vartheta(x,t)dx=1.$$

a) Using the first condition. Find the Fourier series with some unknown.

b) Use the remaining conditions to fix these coefficients. For what range of t does the series converge?

c) Show that

$$\frac{\partial\vartheta(x,t)}{\partial t}=\frac{1}{4\pi}\frac{\partial^2\vartheta(x,t)}{\partial x^2}.$$

Solution: (a) The Fourier series of the function $\vartheta(x,t)$ can be written as

$$\vartheta(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx,$$

where

$$a_0=\int_c^{c+1}\vartheta(x,t)dx,$$

$$a_n=\int_c^{c+1}\vartheta(x,t)\cos(n\pi x)dx,$$

$$b_n=\int_c^{c+1}\vartheta(x,t)\sin(n\pi x)dx.$$

Solution: (b) Using the remaining conditions, one can get

$$\begin{aligned}a_0&=\int_c^{c+1}\vartheta(x,t)dx\\&=\int_c^{c+1}\vartheta(x,t)dx\\&=\int_0^{1}\vartheta(x,t)dx\\&=1.\end{aligned}\tag{30}$$

In particular, one can obtain the complex Fourier series of $\vartheta(x,t)$ by

$$\vartheta(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{i2\pi nx}, \tag{31}$$

using the normalization condition, we get

$$\begin{aligned} \int_0^1 \vartheta(x,t)dx &= \int_0^1 \sum_{n=-\infty}^{\infty} c_n(t) e^{i2\pi nx} dx \\ &= \sum_{n=-\infty}^{\infty} \int_0^1 c_n(t) e^{2\pi inx} dx, \end{aligned} \tag{32}$$

with $n = 0$, we can easily get $c_0 = 1$ and with $n \neq 0$, the integral $\int_0^1 e^{2\pi inx}dx = 0$. In particular

$$\vartheta(x,t) = 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n(t) e^{2\pi inx} dx \tag{33}$$

In addition, with the second condition, we can get

$$\begin{aligned} \vartheta(x+it,t) &= e^{\pi t - 2\pi ix} \vartheta(x,t) \\ &= e^{\pi t - 2\pi ix} \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi inx} \\ &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i(n-1)x} e^{\pi t} \\ &= \sum_{n=\infty}^{\infty} c_{n+1}(t) e^{2\pi in} e^{\pi t} \end{aligned} \tag{34}$$

however

$$\begin{aligned} \vartheta(x+it,t) &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi in(x+it)} \\ &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi inx} e^{-2\pi nt}, \end{aligned} \tag{35}$$

one can easily obtain the recursive formula of c_n by

$$c_{n+1} = c_n e^{-\pi t(2n+1)}. \tag{36}$$

In particular,

$$\begin{aligned}
c_1 &= c_0 e^{-\pi t}, \\
c_2 &= c_1 e^{-\pi t(2+1)} = c_0 e^{-4\pi t}, \\
c_3 &= c_2 e^{-\pi t(4+1)} = c_0 e^{-9\pi t}, \\
c_4 &= c_3 e^{-\pi t(6+1)} = c_0 e^{-16\pi t}, \\
&\vdots \\
c_k &= c_0 e^{-k^2\pi t}, \\
c_{k+1} &= c_k e^{-\pi t(2k+1)} = c_0 e^{-k^2\pi t} e^{-\pi t(2k+1)} = c_0 e^{-(k+1)^2\pi t}.
\end{aligned} \tag{37}$$

The (36) can be written as

$$c_n = c_0 e^{-n^2\pi t}, \tag{38}$$

where $c_0 = 1$. The Fourier series of $\vartheta(x,t)$ can be written as

$$\begin{aligned}
\vartheta(x,t) &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x} \\
&= c_0 + \sum_{n=-1}^{-\infty} c_n e^{2\pi i n x} + \sum_{n=1}^{\infty} c_n e^{2\pi i n x} \\
&= 1 + \sum_{n=-1}^{-\infty} c_n e^{2\pi i n x} + \sum_{n=1}^{\infty} c_n e^{2\pi i n x} \\
&= 1 + \sum_{n=1}^{\infty} e^{-n^2\pi t}(e^{2\pi i n x} + e^{-2\pi i n x}) \\
&= 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi t} \cos 2\pi n x.
\end{aligned} \tag{39}$$

We then investigate whether this sum is converges. Let me consider $t = 0$, this series becomes to

$$\vartheta(x,t) = 1 + 2\sum_{n=1}^{\infty} \cos 2\pi n x,$$

The series diverges for $t = 0$ because

$$\lim_{n\to\infty} \cos 2\pi n x \neq 0 \quad for \quad x \in \mathbb{Z}.$$

For $t > 0$, the exponential factor $e^{-n^2\pi t}$ tends to zero much faster the cosine term, which ensure convergence of the series. In constrast, when $t < 0$. the exponetial factor grows exponentially, causing the series to diverge.

Problem 3 Skinner, 1B Methods, Page 1,2

The *sawtooth function* is defined to be the function

$$f(\theta) = \theta,$$

for $\theta \in [-\pi, \pi)$.

a) Compute the Fourier series of the sawtooth function and comment on its value at $\theta = \pi$.

b) By applying Parseval's identity to the sawtooth function, show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solution: (a) The Fourier series of the sawtooth function can be written as

$$f(\theta) = \theta = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \tag{40}$$

where

$$a_0 = 0, \tag{41}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos(n\theta) d\theta = 0, \tag{42}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin(n\theta) d\theta \\ &= \frac{2}{\pi} \int_{0}^{\pi} \theta \sin(n\theta) d\theta \\ &= \frac{2}{\pi} \left(-\theta \frac{\cos(n\theta)}{n} \bigg|_0^{\pi} + \int_0^{\pi} \frac{\cos(n\theta)}{n} d\theta \right) \\ &= -\frac{2}{n} \cos n\pi = \frac{2(-1)^{n+1}}{n}. \end{aligned} \tag{43}$$

Solution: (b) One can write the Fourier series of sawtooth function as

$$f(\theta) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} = \sum_{-\infty}^{\infty} \frac{i(-1)^n}{n}. \tag{44}$$

Because $f(\theta) = f(\theta)^*$, by applying the Parseval's identity to the sawtooth function, one can get

$$\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{2\pi^3}{3}, \tag{45}$$

while

$$2\pi \sum_{n\in\mathbb{Z}} |b_n|^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{46}$$

Using (45) and (46), one can get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{47}$$