

A solution on Mathematic Methods

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Problem sheet 1

Problem 1 David Skinner, 1B Methods, Page 1

- Given $f(\theta) = (\theta^2 - \pi^2)^2$, where $\theta \in [-\pi, \pi)$.
- Given $f(\theta) = e^\theta$, where $\theta \in [-\pi, \pi)$.
- Given $f(\theta) = \theta e^{i\theta}$, where $\theta \in [-\pi, \pi)$.

Find the Fourier series.

Solution: (a) $f(\theta)$ is even function. Therefore, $b_n = 0$, the Fourier coefficient a_n can be written as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta) f(\theta) d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta) (\theta^2 - \pi^2)^2 d\theta. \end{aligned} \quad (1)$$

Let

$$\begin{cases} u = (\theta^2 - \pi^2)^2 \\ dv = \cos(n\theta) d\theta \end{cases} \Rightarrow \begin{cases} du = (4\theta^3 - 4\pi^2\theta) d\theta \\ v = \frac{\sin(n\theta)}{n} \end{cases} \quad (2)$$

we get

$$a_n = \frac{(\theta^2 - \pi^2)^2 \sin n\theta}{2\pi n} \Big|_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin n\theta}{n} (4\theta^3 - 4\pi^2\theta) d\theta, \quad (3)$$

let

$$\begin{cases} u = (4\theta^3 - 4\pi^2\theta) \\ dv = -\frac{\sin(n\theta)}{n}d\theta \end{cases} \Rightarrow \begin{cases} du = (12\theta^2 - 4\pi^2) d\theta \\ v = \frac{\cos n\theta}{n^2} \end{cases} \quad (4)$$

we get

$$a_n = \cancel{(4\theta^3 - 4\pi^2\theta)\frac{\cos n\theta}{n^2}} \Big|_{-\pi}^{\pi} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos n\theta}{n^2} (12\theta^2 - 4\pi^2) d\theta, \quad (5)$$

let

$$\begin{cases} u = (12\theta^2 - 4\pi^2) \\ dv = -\frac{\cos n\theta}{n^2}d\theta \end{cases} \Rightarrow \begin{cases} du = 24\theta d\theta \\ v = -\frac{\sin n\theta}{n^3} \end{cases} \quad (6)$$

we get

$$a_n = -(12\theta^2 - 4\pi^2)\frac{\sin n\theta}{n^3} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\sin n\theta}{n^3} 24\theta d\theta, \quad (7)$$

let

$$\begin{cases} u = 24\theta \\ dv = \frac{\sin n\theta}{n^3} \end{cases} \Rightarrow \begin{cases} du = 24d\theta \\ v = -\frac{\cos n\theta}{n^4} \end{cases} \quad (8)$$

the a_n can be written as

$$\begin{aligned} a_n &= -\frac{24}{\pi} \theta \frac{\cos n\theta}{n^4} \Big|_{-\pi}^{\pi} + \frac{24}{\pi} \int_{-\pi}^{\pi} \frac{\cos n\theta}{n^4} \\ &= -\frac{24}{n^4\pi} (\pi \cos n\pi - (-\pi \cos(-n\pi))) \\ &= -\frac{48}{n^4} (-1)^n, \end{aligned} \quad (9)$$

and a_0 is

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \\ &= \frac{2}{\pi} \int_0^{\pi} (\theta^2 - \pi^2)^2 d\theta \\ &= \frac{16\pi^4}{15}. \end{aligned} \quad (10)$$

The Fourier series for f can be written as

$$f(\theta) = \frac{8\pi^4}{15} - \sum_{n=1}^{\infty} \frac{48}{n^4} (-1)^n \cos n\theta. \quad (11)$$

The $f(\theta)$ is continuous on $[-\pi, \pi)$, therefore, for arbitrary of θ , this Fourier series converge to $f(\theta)$.¹

Solution: (b) $f(\theta)$ is neither even nor odd, the Fourier coefficients a_n, b_n can be written as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta) f(\theta) d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\theta) e^{\theta} d\theta, \end{aligned} \quad (12)$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\theta) f(\theta) d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\theta) e^{\theta} d\theta. \end{aligned} \quad (13)$$

Proof: $\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$. Let

$$I = e^{ax} \sin bx,$$

and

$$\begin{cases} u = \sin bx \\ dv = e^{ax} dx \end{cases} \Rightarrow \begin{cases} du = b \cos bx \\ v = \frac{e^{ax}}{a} \end{cases}$$

One can get

$$\frac{e^{ax}}{a} \sin bx - \int \frac{e^{ax}}{a} b \cos bx dx.$$

Let

$$J = \frac{e^{ax}}{a} b \cos bx dx,$$

and

$$\begin{cases} u = b \cos bx \\ dv = \frac{e^{ax}}{a} dx \end{cases} \Rightarrow \begin{cases} du = -b \sin bx \\ v = \frac{e^{ax}}{a^2} \end{cases}$$

¹If the function is continuous, the series converges to the function; otherwise, it converges to the midpoint of the jump, i.e, the left and right hand limits.

One can get

$$\begin{aligned} J &= \frac{e^{ax}}{a^2} b \cos bx + \frac{b^2}{a^2} \int e^{ax} \sin bx dx \\ &= \frac{e^{ax}}{a^2} b \cos bx + \frac{b^2}{a^2} \int I dx, \end{aligned}$$

We can see this pattern is repeat. In particular,

$$\begin{aligned} \int I dx &= \frac{e^{ax}}{a} \sin bx - \frac{e^{ax}}{a^2} b \cos bx - \frac{b^2}{a^2} \int I dx \\ \Rightarrow (a^2 + b^2) \int I dx &= e^{ax} (a \sin bx - b \cos bx). \end{aligned}$$

This leads to

$$\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}.$$

□

Note:

$$\begin{aligned} \int e^{ax} \sin bx dx &= \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}, \\ \int e^{ax} \cos bx dx &= \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}. \end{aligned}$$

One can get

$$\begin{aligned} a_n &= \frac{1}{\pi} \frac{e^\theta (\cos n\theta - n \sin n\theta)}{1 + n^2} \Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi(1 + n^2)} (e^\pi \cos(n\pi) - e^{-\pi} \cos(-n\pi)) \\ &= \frac{(-1)^n}{\pi(1 + n^2)} (e^\pi - e^{-\pi}), \end{aligned} \tag{14}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi} \frac{e^\theta (\sin n\theta - n \cos n\theta)}{1 + n^2} \Big|_{-\pi}^{\pi} \\ &= \frac{n}{\pi(1 + n^2)} (-e^\pi \cos(n\pi) + e^{-\pi} \cos(-n\pi)) \\ &= -\frac{n(-1)^n}{\pi(1 + n^2)} (e^\pi - e^{-\pi}). \end{aligned} \tag{15}$$

a_0 can be written as

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^\theta d\theta \\ &= \frac{1}{\pi} (e^\pi - e^{-\pi}). \end{aligned} \tag{16}$$

The Fourier series of $f(\theta) = e^\theta$ can be written as

$$e^\theta \simeq \frac{1}{2\pi}(e^\pi - e^{-\pi}) + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\pi(1+n^2)} (e^\pi - e^{-\pi}) \cos n\theta - \frac{n(-1)^n}{\pi(1+n^2)} (e^\pi - e^{-\pi}) \sin n\theta \right], \quad (17)$$

where

$$\begin{aligned} a_0 &= \frac{1}{\pi}(e^\pi - e^{-\pi}), \\ a_n &= \frac{(-1)^n}{\pi(1+n^2)} (e^\pi - e^{-\pi}) \cos n\theta, \\ b_n &= -\frac{n(-1)^n}{\pi(1+n^2)} (e^\pi - e^{-\pi}) \sin n\theta. \end{aligned}$$

At discontinuity, function $f(\theta) = e^\theta$ converges to the average of the left and right limits. In particular,

$$S_n = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{e^\pi + e^{-\pi}}{2} = \cosh \pi \quad (18)$$

Therefore, one can rewrite Eq. (17) with $\theta = \pi$ as

$$\begin{aligned} \cosh \pi &= \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos n\pi - \frac{n(-1)^n}{(1+n^2)} \sin \pi \\ &= \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} (-1)^n \\ &\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1). \end{aligned} \quad (19)$$

Solution: (c) $f(\theta) = \theta e^{i\theta}$ is neither even nor odd, one can obtain the Fourier series by

$$\theta e^{i\theta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \quad (20)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta e^{i\theta} d\theta, \quad (21)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos n\theta e^{i\theta} d\theta, \quad (22)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin n\theta e^{i\theta} d\theta. \quad (23)$$

Note:

$$\int x e^{ax} \sin bx dx = \frac{x e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} - \frac{e^{ax} [(a^2 - b^2) \sin bx - 2ab \cos bx]}{(a^2 + b^2)^2},$$

$$\int x e^{ax} \cos bx dx = \frac{x e^{ax} (a \cos bx - b \sin bx)}{a^2 + b^2} - \frac{e^{ax} [(a^2 - b^2) \cos bx + 2ab \sin bx]}{(a^2 + b^2)^2}.$$

In particular,

$$a_0 = -ie^{i\pi}(\pi + i) + ie^{-i\pi}(-\pi + i) = 2i\pi, \quad (24)$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos n\theta e^{i\theta} d\theta, \\ &= \frac{\theta e^{i\theta} (i \cos n\theta - n \sin n\theta)}{n^2 - 1} \Bigg|_{-\pi}^{\pi} - \frac{e^{i\theta} [(-n^2 - 1) \cos n\theta + 2in \sin n\theta]}{(n^2 - 1)^2} \Bigg|_{-\pi}^{\pi} \\ &= -\frac{2\pi i (-1)^n}{n^2 - 1}. \end{aligned} \quad (25)$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin n\theta e^{i\theta} d\theta \\ &= \frac{\theta e^{i\theta} (i \sin n\theta - n \cos n\theta)}{n^2 - 1} \Bigg|_{-\pi}^{\pi} - \frac{e^{i\theta} [(-n^2 - 1) \sin n\theta - 2in \cos n\theta]}{(n^2 - 1)^2} \Bigg|_{-\pi}^{\pi} \\ &= \frac{2\pi n (-1)^n}{n^2 - 1}. \end{aligned} \quad (26)$$

Therefore, one can obtain the Fourier series for $f(\theta) = \theta \cos \theta$ and $f(\theta) = \theta \sin \theta$ by

$$\theta e^{i\theta} = \theta \cos \theta + i\theta \sin \theta. \quad (27)$$

where

$$\theta \cos \theta = \operatorname{Re}\left[\frac{a_0}{2}\right] + \sum_{n=1}^{\infty} \operatorname{Re}[a_n] \cos n\theta + \operatorname{Re}[b_n] \sin n\theta, \quad (28)$$

$$\theta \sin \theta = \operatorname{Im}\left[\frac{a_0}{2}\right] + \sum_{n=1}^{\infty} \operatorname{Im}[a_n] \cos n\theta + \operatorname{Im}[b_n] \sin n\theta. \quad (29)$$

In particular

$$\theta \cos \theta = \sum_{n=1}^{\infty} \frac{2\pi n (-1)^n}{n^2 - 1} \sin n\theta, \quad (30)$$

and

$$\theta \sin \theta = 2\pi - \sum_{n=1}^{\infty} \frac{2\pi n (-1)^n}{n^2 - 1} \cos n\theta. \quad (31)$$

Problem 2 David Skinner, 1B Methods, Page 1

A certain function $\vartheta(x, t)$ obeys the condition

$$\vartheta(x + 1, t) = \vartheta(x, t)$$

$$\vartheta(x + it, t) = e^{\pi t - 2\pi i x} \vartheta(x, t)$$

$$\int_0^1 \vartheta(x, t) dx = 1.$$

- a) Using the first condition. Find the Fourier series with some unknown.
- b) Use the remaining conditions to fix these coefficients. For what range of t does the series converge?
- c) Show that

$$\frac{\partial \vartheta(x, t)}{\partial t} = \frac{1}{4\pi} \frac{\partial^2 \vartheta(x, t)}{\partial x^2}.$$

Solution: (a) The Fourier series of the function $\vartheta(x, t)$ can be written as

$$\vartheta(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx,$$

where

$$a_0 = \int_c^{c+1} \vartheta(x, t) dx,$$

$$a_n = \int_c^{c+1} \vartheta(x, t) \cos(n\pi x) dx,$$

$$b_n = \int_c^{c+1} \vartheta(x, t) \sin(n\pi x) dx.$$

Solution: (b) Using the remaining conditions, one can get

$$\begin{aligned} a_0 &= \int_c^{c+1} \vartheta(x, t) dx \\ &= \int_c^{c+1} \vartheta(x, t) dx \\ &= \int_0^1 \vartheta(x, t) dx \\ &= 1. \end{aligned} \tag{32}$$

In particular, one can obtain the complex Fourier series of $\vartheta(x, t)$ by

$$\vartheta(x, t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{i2\pi n x}, \quad (33)$$

using the normalization condition, we get

$$\begin{aligned} \int_0^1 \vartheta(x, t) dx &= \int_0^1 \sum_{n=-\infty}^{\infty} c_n(t) e^{i2\pi n x} dx \\ &= \sum_{n=-\infty}^{\infty} c_n(t) \int_0^1 e^{2\pi i n x} dx, \end{aligned} \quad (34)$$

with $n = 0$, we can easily get $c_0 = 1$ and with $n \neq 0$, the integral $\int_0^1 e^{2\pi i n x} dx = 0$. In particular

$$\vartheta(x, t) = 1 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n(t) e^{2\pi i n x} \quad (35)$$

In addition, with the second condition, we can get

$$\begin{aligned} \vartheta(x + it, t) &= e^{\pi t - 2\pi i x} \vartheta(x, t) \\ &= e^{\pi t - 2\pi i x} \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x} \\ &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i (n-1)x} e^{\pi t} \\ &= \sum_{n=-\infty}^{\infty} c_{n+1}(t) e^{2\pi i n x} e^{\pi t} \end{aligned} \quad (36)$$

however

$$\begin{aligned} \vartheta(x + it, t) &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n (x+it)} \\ &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x} e^{-2\pi n t}, \end{aligned} \quad (37)$$

one can easily obtain the recursive formula of c_n by

$$c_{n+1} = c_n e^{-\pi t (2n+1)}. \quad (38)$$

In particular,

$$\begin{aligned}
c_1 &= c_0 e^{-\pi t}, \\
c_2 &= c_1 e^{-\pi t(2+1)} = c_0 e^{-4\pi t}, \\
c_3 &= c_2 e^{-\pi t(4+1)} = c_0 e^{-9\pi t}, \\
c_4 &= c_3 e^{-\pi t(6+1)} = c_0 e^{-16\pi t}, \\
&\vdots \\
c_k &= c_0 e^{-k^2 \pi t}, \\
c_{k+1} &= c_k e^{-\pi t(2k+1)} = c_0 e^{-k^2 \pi t} e^{-\pi t(2k+1)} = c_0 e^{-(k+1)^2 \pi t}.
\end{aligned} \tag{39}$$

The (38) can be written as

$$c_n = c_0 e^{-n^2 \pi t}, \tag{40}$$

where $c_0 = 1$. The Fourier series of $\vartheta(x, t)$ can be written as

$$\begin{aligned}
\vartheta(x, t) &= \sum_{n=-\infty}^{\infty} c_n(t) e^{2\pi i n x} \\
&= c_0 + \sum_{n=-1}^{-\infty} c_n e^{2\pi i n x} + \sum_{n=1}^{\infty} c_n e^{2\pi i n x} \\
&= 1 + \sum_{n=-1}^{-\infty} c_n e^{2\pi i n x} + \sum_{n=1}^{\infty} c_n e^{2\pi i n x} \\
&= 1 + \sum_{n=1}^{\infty} e^{-n^2 \pi t} (e^{2\pi i n x} + e^{-2\pi i n x}) \\
&= 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi t} \cos 2\pi n x.
\end{aligned} \tag{41}$$

We then investigate whether this sum is converges. Let me consider $t = 0$, this series becomes to

$$\vartheta(x, t) = 1 + 2 \sum_{n=1}^{\infty} \cos 2\pi n x,$$

The series diverges for $t = 0$ because

$$\lim_{n \rightarrow \infty} \cos 2\pi n x \neq 0 \quad \text{for } x \in \mathbb{Z}.$$

For $t > 0$, the exponential factor $e^{-n^2\pi t}$ tends to zero much faster the cosine term, which ensure convergence of the series. In contrast, when $t < 0$, the exponential factor grows exponentially, causing the series to diverge.

Problem 3 Skinner, 1B Methods, Page 1,2

The *sawtooth function* is defined to be the function

$$f(\theta) = \theta,$$

for $\theta \in [-\pi, \pi)$.

- a) Compute the Fourier series of the sawtooth function and comment on its value at $\theta = \pi$.
- b) By applying Parseval's identity to the sawtooth function, show that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solution: (a) The Fourier series of the sawtooth function can be written as

$$f(\theta) = \theta = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta, \quad (42)$$

where

$$a_0 = 0, \quad (43)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \cos(n\theta) d\theta = 0, \quad (44)$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin(n\theta) d\theta \\ &= \frac{2}{\pi} \int_0^{\pi} \theta \sin(n\theta) d\theta \\ &= \frac{2}{\pi} \left(-\theta \frac{\cos(n\theta)}{n} \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos(n\theta)}{n} d\theta \right) \\ &= -\frac{2}{n} \cos n\pi = \frac{2(-1)^{n+1}}{n}. \end{aligned} \quad (45)$$

Solution: (b) One can write the Fourier series of sawtooth function as

$$f(\theta) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} = \sum_{-\infty}^{\infty} \frac{i(-1)^n}{n}. \quad (46)$$

Because $f(\theta) = f(\theta)^*$, by applying the Parseval's identity to the sawtooth function, one can get

$$\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{2\pi^3}{3}, \quad (47)$$

while

$$2\pi \sum_{n \in \mathbb{Z}} |b_n|^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (48)$$

Using Eq. (47) and Eq. (48), one can get

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (49)$$