# 10 SCATTERING

## 10.1 INTRODUCTION

### 10.1.1 Classical Scattering Theory

Imagine a particle incident on some scattering center (say, a marble bouncing off a bowling ball, or a proton fired at a heavy nucleus). It comes in with energy $E$ and **impact parameter** $b$, and it emerges at some **scattering angle** $\theta$—see Figure 10.1. (I'll assume for simplicity that the target is symmetrical about the $z$ axis, so the trajectory remains in one plane, and that the target is very heavy, so its recoil is negligible.) The essential problem of classical scattering theory is this: *Given the impact parameter, calculate the scattering angle*. Ordinarily, of course, the smaller the impact parameter, the greater the scattering angle.

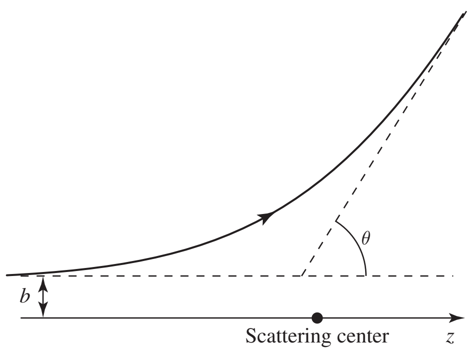


**Figure 10.1:** The classical scattering problem, showing the impact parameter $b$ and the scattering angle $\theta$.

**Example 10.1**

**Hard-sphere scattering.** Suppose the target is a billiard ball, of radius $R$, and the incident particle is a BB, which bounces off elastically (Figure 10.2). In terms of the angle $\alpha$, the impact parameter is $b = R\sin\alpha$, and the scattering angle is $\theta = \pi - 2\alpha$, so

$$b = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right). \tag{10.1}$$

Evidently

$$\theta = \begin{cases} 2\cos^{-1}(b/R), & (b \le R), \\ 0, & (b \ge R). \end{cases} \tag{10.2}$$

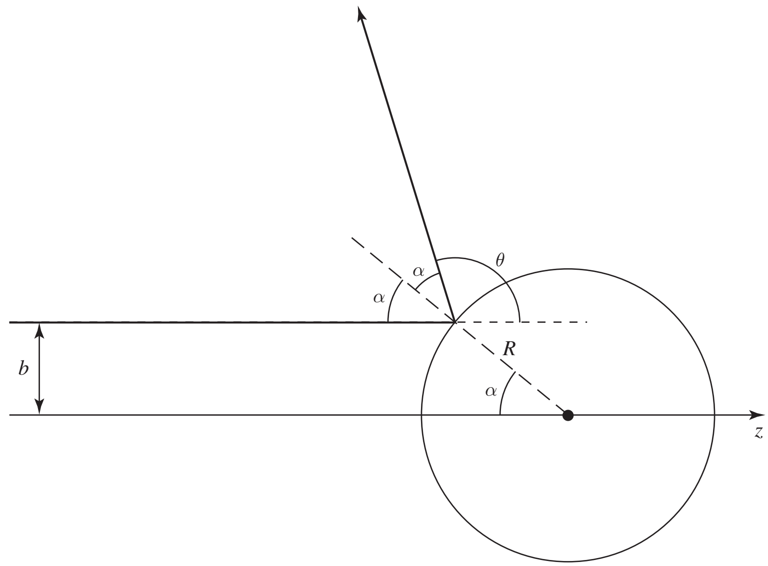


**Figure 10.2:** Elastic hard-sphere scattering.

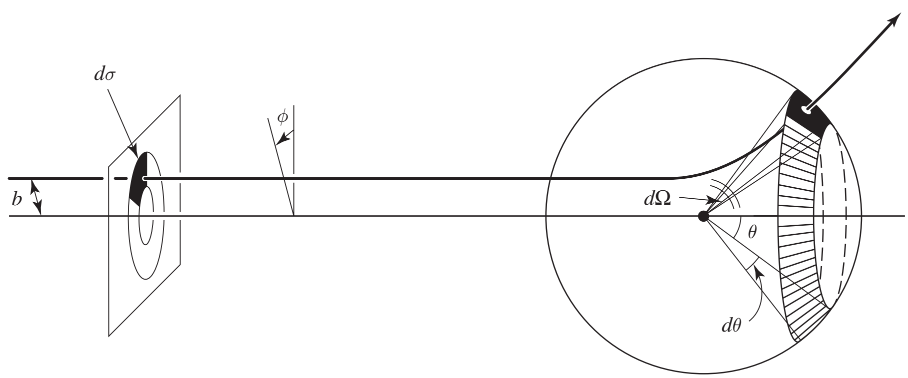


**Figure 10.3:** Particles incident in the area $d\sigma$ scatter into the solid angle $d\Omega$.

More generally, particles incident within an infinitesimal patch of cross-sectional area $d\sigma$ will scatter into a corresponding infinitesimal solid angle $d\Omega$ (Figure 10.3). The larger $d\sigma$ is, the bigger $d\Omega$ will be; the proportionality factor, $D(\theta) \equiv d\sigma/d\Omega$, is called the **differential (scattering) cross-section**:[1]

$$\boxed{d\sigma = D(\theta)\, d\Omega.} \tag{10.3}$$

[1] This is terrible language: $D$ isn't a *differential*, and it isn't a cross-section. To my ear, the words "differential cross-section" would attach more naturally to $d\sigma$. But I'm afraid we're stuck with this terminology. I should also warn you that the notation $D(\theta)$ is nonstandard—most people just call it $d\sigma/d\Omega$ (which makes Equation 10.3 look like a tautology). I think it will be less confusing if we give the differential cross-section its own symbol.

In terms of the impact parameter and the azimuthal angle $\phi$, $d\sigma = b\,db\,d\phi$ and $d\Omega = \sin\theta\,d\theta\,d\phi$, so

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|. \tag{10.4}$$

(Since $\theta$ is typically a *decreasing* function of $b$, the derivative is actually negative—hence the absolute value sign.)

**Example 10.2**

**Hard-sphere scattering (continued).** In the case of hard-sphere scattering (Example 10.1)

$$\frac{db}{d\theta} = -\frac{1}{2}R\sin\left(\frac{\theta}{2}\right), \tag{10.5}$$

so

$$D(\theta) = \frac{R\cos(\theta/2)}{\sin\theta}\left(\frac{R\sin(\theta/2)}{2}\right) = \frac{R^2}{4}. \tag{10.6}$$

This example is unusual, in that the differential cross-section is independent of $\theta$.

The **total cross-section** is the *integral* of $D(\theta)$, over all solid angles:

$$\boxed{\sigma \equiv \int D(\theta)\,d\Omega;} \tag{10.7}$$

roughly speaking, it is the total area of incident beam that is scattered by the target. For example, in the case of hard-sphere scattering,

$$\sigma = \left(R^2/4\right)\int d\Omega = \pi R^2, \tag{10.8}$$

which is just what we would expect: It's the cross-sectional area of the sphere; BB's incident within this area will hit the target, and those farther out will miss it completely. But the virtue of the formalism developed here is that it applies just as well to "soft" targets (such as the Coulomb field of a nucleus) that are *not* simply "hit-or-miss".

Finally, suppose we have a *beam* of incident particles, with uniform intensity (or **luminosity**, as particle physicists call it)

$$\mathcal{L} \equiv \text{number of incident particles per unit area, per unit time.} \tag{10.9}$$

The number of particles entering area $d\sigma$ (and hence scattering into solid angle $d\Omega$), per unit time, is $dN = \mathcal{L}\,d\sigma = \mathcal{L}\,D(\theta)\,d\Omega$, so

$$D(\theta) = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega}. \tag{10.10}$$

This is sometimes taken as the *definition* of the differential cross-section, because it makes reference only to quantities easily measured in the laboratory: If the detector subtends a solid angle $d\Omega$, we simply count the *number* recorded per unit time (the **event rate**, $dN$), divide by $d\Omega$, and normalize to the luminosity of the incident beam.

*** **Problem 10.1 Rutherford scattering.** An incident particle of charge $q_1$ and kinetic energy $E$ scatters off a heavy stationary particle of charge $q_2$.

**(a)** Derive the formula relating the impact parameter to the scattering angle.[2] *Answer:* $b = (q_1 q_2/8\pi\epsilon_0 E)\cot(\theta/2)$.

**(b)** Determine the differential scattering cross-section. *Answer:*

$$D(\theta) = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2. \tag{10.11}$$

**(c)** Show that the total cross-section for Rutherford scattering is *infinite*.

### 10.1.2 Quantum Scattering Theory

In the quantum theory of scattering, we imagine an incident plane wave, $\psi(z) = Ae^{ikz}$, traveling in the $z$ direction, which encounters a scattering potential, producing an outgoing *spherical* wave (Figure 10.4).[3] That is, we look for solutions to the Schrödinger equation of the generic form

$$\boxed{\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}, \quad \text{for large } r.} \tag{10.12}$$

(The spherical wave carries a factor of $1/r$, because this portion of $|\psi|^2$ must go like $1/r^2$ to conserve probability.) The **wave number** $k$ is related to the energy of the incident particles in the usual way:

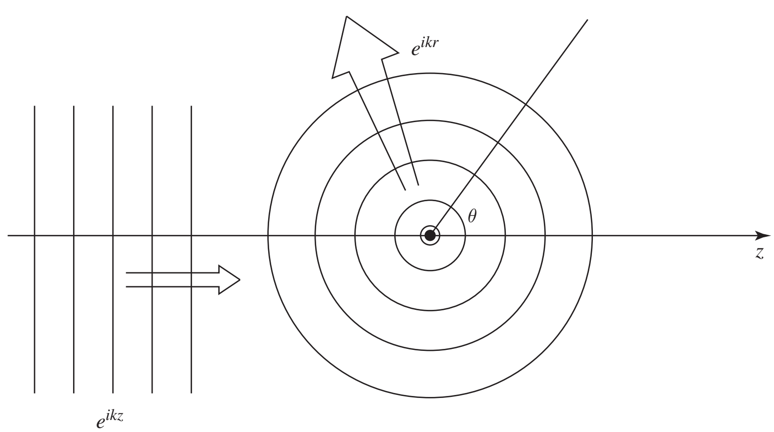


**Figure 10.4:** Scattering of waves; an incoming plane wave generates an outgoing spherical wave.

[2] This isn't easy, and you might want to refer to a book on classical mechanics, such as Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics of Particles and Systems*, 4th edn, Saunders, Fort Worth, TX (1995), Section 9.10.

[3] For the moment, there's not much *quantum* mechanics in this; what we're really talking about is the scattering of *waves*, as opposed to *particles*, and you could even think of Figure 10.4 as a picture of water waves encountering a rock, or (better, since we're interested in three-dimensional scattering) sound waves bouncing off a basketball.

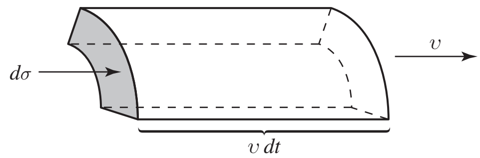


**Figure 10.5:** The volume $dV$ of incident beam that passes through area $d\sigma$ in time $dt$.

$$k \equiv \frac{\sqrt{2mE}}{\hbar}. \tag{10.13}$$

(As before, I assume the target is azimuthally symmetrical; in the more general case $f$ would depend on $\phi$ as well as $\theta$.)

The whole problem is to determine the **scattering amplitude** $f(\theta)$; it tells you the *probability of scattering in a given direction* $\theta$, and hence is related to the differential cross-section. Indeed, the probability that the incident particle, traveling at speed $v$, passes through the infinitesimal area $d\sigma$, in time $dt$, is (see Figure 10.5)

$$dP = |\psi_{\text{incident}}|^2\, dV = |A|^2\,(v\,dt)\; d\sigma.$$

But this is equal to the probability that the particle scatters into the corresponding solid angle $d\Omega$:

$$dP = |\psi_{\text{scattered}}|^2\, dV = \frac{|A|^2\,|f|^2}{r^2}\,(v\,dt)\,r^2\,d\Omega,$$

from which it follows that $d\sigma = |f|^2\, d\Omega$, and hence

$$\boxed{D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2\,.} \tag{10.14}$$

Evidently the differential cross-section (which is the quantity of interest to the experimentalist) is equal to the absolute square of the scattering amplitude (which is obtained by solving the Schrödinger equation). In the following sections we will study two techniques for calculating the scattering amplitude: **partial wave analysis** and the **Born approximation**.

**Problem 10.2** Construct the analogs to Equation 10.12 for one-dimensional and two-dimensional scattering.

## 10.2 PARTIAL WAVE ANALYSIS

### 10.2.1 Formalism

As we found in Chapter 4, the Schrödinger equation for a spherically symmetrical potential $V(r)$ admits the separable solutions

$$\psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi), \tag{10.15}$$

where $Y_\ell^m$ is a spherical harmonic (Equation 4.32), and $u(r) = rR(r)$ satisfies the radial equation (Equation 4.37):

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu. \tag{10.16}$$

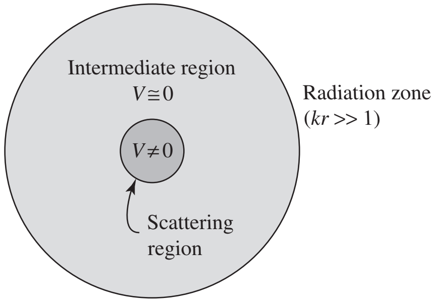


**Figure 10.6:** Scattering from a localized potential: the scattering region (dark), the intermediate region, where $V = 0$ (shaded), and the radiation zone (where $kr \gg 1$).

At *very* large $r$ the potential goes to zero, and the centrifugal contribution is negligible, so

$$\frac{d^2u}{dr^2} \approx -k^2 u.$$

The general solution is

$$u(r) = Ce^{ikr} + De^{-ikr};$$

the first term represents an *outgoing* spherical wave, and the second an *incoming* one—for the scattered wave we want $D = 0$. At very large $r$, then,

$$R(r) \sim \frac{e^{ikr}}{r},$$

as we already deduced (on physical grounds) in the previous section (Equation 10.12).

That's for *very* large $r$ (more precisely, for $kr \gg 1$; in optics it would be called the **radiation zone**). As in one-dimensional scattering theory, we assume that the potential is "localized," in the sense that exterior to some finite scattering region it is essentially zero (Figure 10.6). In the intermediate region (where $V$ can be ignored but the centrifugal term cannot),[4] the radial equation becomes

$$\frac{d^2u}{dr^2} - \frac{\ell(\ell+1)}{r^2}u = -k^2 u, \tag{10.17}$$

and the general solution (Equation 4.45) is a linear combination of spherical Bessel functions:

$$u(r) = Arj_\ell(kr) + Brn_\ell(kr). \tag{10.18}$$

However, neither $j_\ell$ (which is somewhat like a sine function) nor $n_\ell$ (which is a sort of generalized cosine function) represents an outgoing (or an incoming) wave. What we need are the linear combinations analogous to $e^{ikr}$ and $e^{-ikr}$; these are known as **spherical Hankel functions**:

$$h_\ell^{(1)}(x) \equiv j_\ell(x) + in_\ell(x); \quad h_\ell^{(2)}(x) \equiv j_\ell(x) - in_\ell(x). \tag{10.19}$$

[4] What follows does not apply to the Coulomb potential, since $1/r$ goes to zero more slowly than $1/r^2$, as $r \to \infty$, and the centrifugal term does *not* dominate in this region. In this sense the Coulomb potential is not localized, and partial wave analysis is inapplicable.

Table 10.1: *Spherical Hankel functions,* $h_\ell^{(1)}(x)$ *and* $h_\ell^{(2)}(x)$.

| | |
|---|---|
| $h_0^{(1)} = -i\dfrac{e^{ix}}{x}$ | $h_0^{(2)} = i\dfrac{e^{-ix}}{x}$ |
| $h_1^{(1)} = \left(-\dfrac{i}{x^2} - \dfrac{1}{x}\right)e^{ix}$ | $h_1^{(2)} = \left(\dfrac{i}{x^2} - \dfrac{1}{x}\right)e^{-ix}$ |
| $h_2^{(1)} = \left(-\dfrac{3i}{x^3} - \dfrac{3}{x^2} + \dfrac{i}{x}\right)e^{ix}$ | $h_2^{(2)} = \left(\dfrac{3i}{x^3} - \dfrac{3}{x^2} - \dfrac{i}{x}\right)e^{-ix}$ |

$$\left.\begin{aligned} h_\ell^{(1)} &\to \frac{1}{x}\,(-i)^{\ell+1}e^{ix} \\ h_\ell^{(2)} &\to \frac{1}{x}\,(i)^{\ell+1}e^{-ix} \end{aligned}\right\} \text{for } x \gg 1$$

The first few spherical Hankel functions are listed in Table 10.1. At large $r$, $h_\ell^{(1)}(kr)$ (the **Hankel function of the first kind**) goes like $e^{ikr}/r$, whereas $h_\ell^{(2)}(kr)$ (the **Hankel function of the second kind**) goes like $e^{-ikr}/r$; for outgoing waves, then, we need spherical Hankel functions of the first kind:

$$R(r) \sim h_\ell^{(1)}(kr). \tag{10.20}$$

The exact wave function, in the exterior region (where $V(r) = 0$), is

$$\psi(r, \theta, \phi) = A\left\{e^{ikz} + \sum_{\ell,m} C_{\ell,m}\, h_\ell^{(1)}(kr)\, Y_\ell^m(\theta, \phi)\right\}. \tag{10.21}$$

The first term is the incident plane wave, and the sum (with expansion coefficients $C_{\ell,m}$) is the scattered wave. But since we are assuming the potential is spherically symmetric, the wave function cannot depend on $\phi$.[5] So only terms with $m = 0$ survive (remember, $Y_\ell^m \sim e^{im\phi}$). Now (from Equations 4.27 and 4.32)

$$Y_\ell^0(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}}\, P_\ell(\cos\theta), \tag{10.22}$$

where $P_\ell$ is the $\ell$th Legendre polynomial. It is customary to redefine the expansion coefficients $\left(C_{\ell,0} \equiv i^{\ell+1}k\sqrt{4\pi\,(2\ell+1)}\,a_\ell\right)$:

$$\boxed{\psi(r, \theta) = A\left\{e^{ikz} + k\sum_{\ell=0}^{\infty} i^{\ell+1}\,(2\ell+1)\, a_\ell\, h_\ell^{(1)}\,(kr)\;P_\ell(\cos\theta)\right\}.} \tag{10.23}$$

You'll see in a moment why this peculiar notation is convenient; $a_\ell$ is called the $\ell$th **partial wave amplitude**.

For *very large* $r$, the Hankel function goes like $(-i)^{\ell+1}\,e^{ikr}/kr$ (Table 10.1), so

$$\psi(r, \theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}, \tag{10.24}$$

[5] There's nothing wrong with $\theta$ dependence, of course, because the incoming plane wave defines a $z$ direction, breaking the spherical symmetry. But the *azimuthal* symmetry remains; the incident plane wave has no $\phi$ dependence, and there is nothing in the scattering process that could introduce any $\phi$ dependence in the outgoing wave.

where

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1)\ a_\ell\ P_\ell(\cos\theta). \tag{10.25}$$

This confirms more rigorously the general structure postulated in Equation 10.12, and tells us how to compute the scattering amplitude, $f(\theta)$, in terms of the partial wave amplitudes ($a_\ell$). The differential cross-section is

$$D(\theta) = |f(\theta)|^2 = \sum_{\ell} \sum_{\ell'} (2\ell + 1) \left(2\ell' + 1\right)\ a_\ell^*\, a_{\ell'}\ P_\ell(\cos\theta)\ P_{\ell'}(\cos\theta), \tag{10.26}$$

and the total cross-section is

$$\sigma = 4\pi \sum_{\ell=0}^{\infty} (2\ell + 1)\ |a_\ell|^2. \tag{10.27}$$

(I used the orthogonality of the Legendre polynomials, Equation 4.34, to do the angular integration.)

### 10.2.2 Strategy

All that remains is to determine the partial wave amplitudes, $a_\ell$, for the potential in question. This is accomplished by solving the Schrödinger equation in the *interior* region (where $V(r)$ is *not* zero), and matching it to the exterior solution (Equation 10.23), using the appropriate boundary conditions. The only problem is that as it stands my notation is hybrid: I used *spherical* coordinates for the scattered wave, but *cartesian* coordinates for the incident wave. We need to rewrite the wave function in a more consistent notation.

Of course, $e^{ikz}$ satisfies the Schrödinger equation with $V = 0$. On the other hand, I just argued that the *general* solution to the Schrödinger equation with $V = 0$ can be written in the form

$$\sum_{\ell,m} \left[A_{\ell,m}\ j_\ell(kr) + B_{\ell,m}\ n_\ell(kr)\right] Y_\ell^m(\theta, \phi).$$

In particular, then, it must be possible to express $e^{ikz}$ in this way. But $e^{ikz}$ is finite at the origin, so no Neumann functions are allowed in the sum ($n_\ell(kr)$ blows up at $r = 0$), and since $z = r\cos\theta$ has no $\phi$ dependence, only $m = 0$ terms occur. The resulting expansion of a plane wave in terms of spherical waves is known as **Rayleigh's formula**:[6]

$$e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell\ (2\ell + 1)\ j_\ell(kr) P_\ell(\cos\theta). \tag{10.28}$$

Using this, the wave function in the exterior region (Equation 10.23) can be expressed entirely in terms of $r$ and $\theta$:

$$\psi(r, \theta) = A \sum_{\ell=0}^{\infty} i^\ell\ (2\ell + 1) \left[j_\ell(kr) + ik\, a_\ell\, h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta). \tag{10.29}$$

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[6] For a guide to the proof, see George Arfken and Hans-Jurgen Weber, *Mathematical Methods for Physicists*, 7th edn, Academic Press, Orlando (2013), Exercises 15.2.24 and 15.2.25.

**Example 10.3**

**Quantum hard-sphere scattering.** Suppose

$$V(r) = \begin{cases} \infty, & (r \le a)\,, \\ 0, & (r > a)\,. \end{cases} \tag{10.30}$$

The boundary condition, then, is

$$\psi(a, \theta) = 0, \tag{10.31}$$

so

$$\sum_{\ell=0}^{\infty} i^{\ell}\,(2\ell+1)\left[j_\ell(ka) + ik\,a_\ell\,h_\ell^{(1)}(ka)\right] P_\ell(\cos\theta) = 0 \tag{10.32}$$

for all $\theta$, from which it follows (Problem 10.3) that

$$a_\ell = i\,\frac{j_\ell(ka)}{k\,h_\ell^{(1)}(ka)}. \tag{10.33}$$

In particular, the total cross-section (Equation 10.27) is

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left|\frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}\right|^2. \tag{10.34}$$

That's the *exact* answer, but it's not terribly illuminating, so let's consider the limiting case of *low-energy scattering*: $ka \ll 1$. (Since $k = 2\pi/\lambda$, this amounts to saying that the wavelength is much greater than the radius of the sphere.) Referring to Table 4.4, we note that $n_\ell(z)$ is much larger than $j_\ell(z)$, for small $z$, so

$$\begin{aligned} \frac{j_\ell(z)}{h_\ell^{(1)}(z)} &= \frac{j_\ell(z)}{j_\ell(z) + i n_\ell(z)} \approx -i\,\frac{j_\ell(z)}{n_\ell(z)} \\ &\approx -i\,\frac{2^\ell \ell! z^\ell / (2\ell+1)!}{-(2\ell)! z^{-\ell-1}/2^\ell \ell!} = \frac{i}{2\ell+1}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 z^{2\ell+1}, \end{aligned} \tag{10.35}$$

and hence

$$\sigma \approx \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\left[\frac{2^\ell \ell!}{(2\ell)!}\right]^4 (ka)^{4\ell+2}\,.$$

But we're assuming $ka \ll 1$, so the higher powers are negligible—in the low-energy approximation the scattering is dominated by the $\ell = 0$ term. (This means that the differential cross-section is independent of $\theta$, just as it was in the classical case.) Evidently

$$\sigma \approx 4\pi a^2, \tag{10.36}$$

for low energy hard-sphere scattering. Surprisingly, the scattering cross-section is *four times* the geometrical cross-section—in fact, $\sigma$ is the *total surface area of the sphere*. This "larger effective size" is characteristic of long-wavelength scattering (it would be true in optics, as well); in a sense, these waves "feel" their way around the whole sphere, whereas classical *particles* only see the head-on cross-section (Equation 10.8).

**Problem 10.3** Prove Equation 10.33, starting with Equation 10.32. *Hint:* Exploit the orthogonality of the Legendre polynomials to show that the coefficients with different values of $\ell$ must separately vanish.

** **Problem 10.4** Consider the case of low-energy scattering from a spherical delta-function shell:

$$V(r) = \alpha\delta(r-a),$$

where $\alpha$ and $a$ are constants. Calculate the scattering amplitude, $f(\theta)$, the differential cross-section, $D(\theta)$, and the total cross-section, $\sigma$. Assume $ka \ll 1$, so that only the $\ell = 0$ term contributes significantly. (To simplify matters, throw out all $\ell \neq 0$ terms right from the start.) The main problem, of course, is to determine $C_0$. Express your answer in terms of the dimensionless quantity $\beta \equiv 2ma\alpha/\hbar^2$. *Answer:* $\sigma = 4\pi a^2\beta^2/(1+\beta)^2$.

## 10.3 PHASE SHIFTS

Consider first the problem of *one*-dimensional scattering from a localized potential $V(x)$ on the half-line $x < 0$ (Figure 10.7). I'll put a "brick wall" at $x = 0$, so a wave incident from the left,

$$\psi_i(x) = Ae^{ikx} \qquad (x < -a) \tag{10.37}$$

is entirely reflected

$$\psi_r(x) = Be^{-ikx} \qquad (x < -a). \tag{10.38}$$

Whatever happens in the interaction region ($-a < x < 0$), the amplitude of the reflected wave has *got* to be the same as that of the incident wave ($|B| = |A|$), by conservation of probability. But it need not have the same *phase*. If there were no potential at all (just the wall at $x = 0$), then $B = -A$, since the total wave function (incident plus reflected) must vanish at the origin:

$$\psi(x) = A\left(e^{ikx} - e^{-ikx}\right) \qquad (V(x) = 0). \tag{10.39}$$

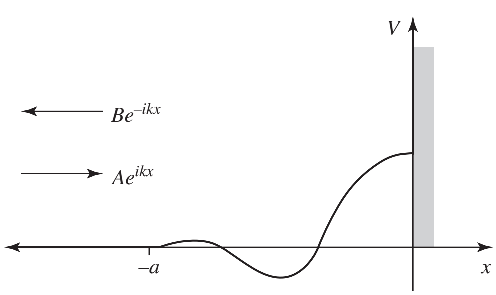


**Figure 10.7:** One-dimensional scattering from a localized potential bounded on the right by an infinite wall.

If the potential is *not* zero, the wave function (for $x < -a$) takes the form

$$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right) \qquad (V(x) \neq 0). \tag{10.40}$$

The whole theory of scattering reduces to the problem of calculating the **phase shift**[7] $\delta$ (as a function of $k$, and hence of the energy $E = \hbar^2k^2/2m$), for a specified potential. We do this, of course, by solving the Schrödinger equation in the scattering region ($-a < x < 0$), and imposing appropriate boundary conditions (see Problem 10.5). The advantage of working with the phase shift (as opposed to the complex number $B$) is that it exploits the physics to simplify the mathematics (trading a complex quantity—two real numbers—for a single real quantity).

Now let's return to the three-dimensional case. The incident plane wave $\left(Ae^{ikz}\right)$ carries no angular momentum in the $z$ direction (Rayleigh's formula contains no terms with $m \neq 0$), but it includes all values of the *total* angular momentum ($\ell = 0, 1, 2, \ldots$). Because angular momentum is conserved (by a spherically symmetric potential), each **partial wave** (labelled by a particular $\ell$) scatters independently, with (again) no change in amplitude[8]—only in phase.

If there is no potential at all, then $\psi = Ae^{ikz}$, and the $\ell$th partial wave is (Equation 10.28)

$$\psi^{(\ell)} = Ai^{\ell}\,(2\ell+1)\; j_\ell(kr)\, P_\ell(\cos\theta) \qquad (V(r) = 0). \tag{10.41}$$

But (from Equation 10.19 and Table 10.1)

$$j_\ell(x) = \frac{1}{2}\left[h_\ell^{(1)}(x) + h_\ell^{(2)}(x)\right] \approx \frac{1}{2x}\left[(-i)^{\ell+1}\, e^{ix} + i^{\ell+1} e^{-ix}\right] \quad (x \gg 1). \tag{10.42}$$

So for large $r$

$$\psi^{(\ell)} \approx A\frac{(2\ell+1)}{2ikr}\left[e^{ikr} - (-1)^{\ell}\, e^{-ikr}\right] P_\ell(\cos\theta) \qquad (V(r) = 0). \tag{10.43}$$

The second term inside the square brackets represents an incoming spherical wave; it comes from the incident plane wave, and is unchanged when we now introduce a potential. The first term is the outgoing wave; it picks up a phase shift (due to the scattering potential):

$$\psi^{(\ell)} \approx A\frac{(2\ell+1)}{2ikr}\left[e^{i(kr+2\delta_\ell)} - (-1)^{\ell}\, e^{-ikr}\right] P_\ell(\cos\theta) \qquad (V(r) \neq 0). \tag{10.44}$$

Think of it as a converging spherical wave (the $e^{-ikr}$ term, due exclusively to the $h_\ell^{(2)}$ component in $e^{ikz}$), which is phase shifted an amount $\delta_\ell$ on the way in, and again $\delta_\ell$ on the way out (hence the 2), emerging as an outgoing spherical wave (the $e^{ikr}$ term, due to the $h_\ell^{(1)}$ part of $e^{ikz}$ plus the scattered wave).

In Section 10.2.1 the whole theory was expressed in terms of the partial wave amplitudes $a_\ell$; now we have formulated it in terms of the phase shifts $\delta_\ell$. There must be a connection between the two. Indeed, comparing the asymptotic (large $r$) form of Equation 10.23

$$\psi^{(\ell)} \approx A\left\{\frac{(2\ell+1)}{2ikr}\left[e^{ikr} - (-1)^{\ell}\, e^{-ikr}\right] + \frac{(2\ell+1)}{r}\, a_\ell\, e^{ikr}\right\} P_\ell(\cos\theta) \tag{10.45}$$

[7] The 2 in front of $\delta$ is conventional. We think of the incident wave as being phase shifted once on the way in, and again on the way out; $\delta$ is the "one way" phase shift, and the *total* is $2\delta$.

[8] One reason this subject can be so confusing is that practically everything is called an "amplitude": $f(\theta)$ is the "scattering amplitude", $a_\ell$ is the "partial wave amplitude", but the first is a function of $\theta$, and both are complex numbers. I'm *now* talking about "amplitude" in the original sense: the (*real*, of course) height of a sinusoidal wave.

with the generic expression in terms of $\delta_\ell$ (Equation 10.44), we find[9]

$$a_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) = \frac{1}{k}e^{i\delta_\ell}\sin(\delta_\ell). \tag{10.46}$$

It follows in particular (Equation 10.25) that

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{i\delta_\ell}\sin(\delta_\ell)\,P_\ell(\cos\theta) \tag{10.47}$$

and (Equation 10.27)

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell). \tag{10.48}$$

Again, the advantage of working with phase shifts (as opposed to partial wave amplitudes) is that they are easier to interpret physically, and simpler mathematically—the phase shift formalism exploits conservation of angular momentum to reduce a complex quantity $a_\ell$ (two real numbers) to a single real one $\delta_\ell$.

**Problem 10.5** A particle of mass $m$ and energy $E$ is incident from the left on the potential

$$V(x) = \begin{cases} 0, & (x < -a)\,, \\ -V_0, & (-a \le x \le 0)\,, \\ \infty, & (x > 0)\,. \end{cases}$$

**(a)** If the incoming wave is $Ae^{ikx}$ (where $k = \sqrt{2mE}/\hbar$), find the reflected wave. *Answer:*

$$Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right]e^{-ikx}, \text{ where } k' = \sqrt{2m\,(E+V_0)}/\hbar.$$

**(b)** Confirm that the reflected wave has the same amplitude as the incident wave.

**(c)** Find the phase shift $\delta$ (Equation 10.40) for a very deep well ($E \ll V_0$). *Answer:* $\delta = -ka$.

**Problem 10.6** What are the partial wave phase shifts ($\delta_\ell$) for hard-sphere scattering (Example 10.3)?

**Problem 10.7** Find the $S$-wave ($\ell = 0$) partial wave phase shift $\delta_0(k)$ for scattering from a delta-function shell (Problem 10.4). Assume that the radial wave function $u(r)$ goes to 0 as $r \to 0$. *Answer:*

$$-\cot^{-1}\left[\cot(ka) + \frac{ka}{\beta\sin^2(ka)}\right], \quad \text{where} \quad \beta \equiv \frac{2m\alpha a}{\hbar^2}.$$

[9] Although I used the asymptotic form of the wave function to draw the connection between $a_\ell$ and $\delta_\ell$, there is nothing approximate about the result (Equation 10.46). Both of them are *constants* (independent of $r$), and $\delta_\ell$ *means* the phase shift in the asymptotic region (where the Hankel functions have settled down to $e^{\pm ikr}/kr$).

## 10.4 THE BORN APPROXIMATION

### 10.4.1 Integral Form of the Schrödinger Equation

The time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi, \tag{10.49}$$

can be written more succinctly as

$$\left(\nabla^2 + k^2\right)\psi = Q, \tag{10.50}$$

where

$$k \equiv \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad Q \equiv \frac{2m}{\hbar^2}V\psi. \tag{10.51}$$

This has the superficial appearance of the **Helmholtz equation**; note, however, that the "inhomogeneous" term ($Q$) *itself* depends on $\psi$. Suppose we could find a function $G(\mathbf{r})$ that solves the Helmholtz equation with a *delta function* "source":

$$\left(\nabla^2 + k^2\right)G(\mathbf{r}) = \delta^3(\mathbf{r}). \tag{10.52}$$

Then we could express $\psi$ as an integral:

$$\psi(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}_0)Q(\mathbf{r}_0)\,d^3\mathbf{r}_0. \tag{10.53}$$

For it is easy to show that this satisfies Schrödinger's equation, in the form of Equation 10.50:

$$\begin{aligned}\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) &= \int \left[\left(\nabla^2 + k^2\right)G(\mathbf{r} - \mathbf{r}_0)\right]Q(\mathbf{r}_0)\,d^3\mathbf{r}_0 \\ &= \int \delta^3(\mathbf{r} - \mathbf{r}_0)Q(\mathbf{r}_0)\,d^3\mathbf{r}_0 = Q(\mathbf{r}).\end{aligned}$$

$G(\mathbf{r})$ is called the **Green's function** for the Helmholtz equation. (In general, the Green's function for a linear differential equation represents the "response" to a delta-function source.)

Our first task[10] is to solve Equation 10.52 for $G(\mathbf{r})$. This is most easily accomplished by taking the Fourier transform, which turns the *differential* equation into an *algebraic* equation. Let

$$G(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\mathbf{s}\cdot\mathbf{r}}g(\mathbf{s})\,d^3\mathbf{s}. \tag{10.54}$$

Then

$$\left(\nabla^2 + k^2\right)G(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}}\int \left[\left(\nabla^2 + k^2\right)e^{i\mathbf{s}\cdot\mathbf{r}}\right]g(\mathbf{s})\,d^3\mathbf{s}.$$

But

$$\nabla^2 e^{i\mathbf{s}\cdot\mathbf{r}} = -s^2 e^{i\mathbf{s}\cdot\mathbf{r}}, \tag{10.55}$$

and (see Equation 2.147)

$$\delta^3(\mathbf{r}) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{s}\cdot\mathbf{r}}\,d^3\mathbf{s}, \tag{10.56}$$

[10] *Warning:* You are approaching two pages of heavy analysis, including contour integration; if you wish, skip straight to the answer, Equation 10.65.

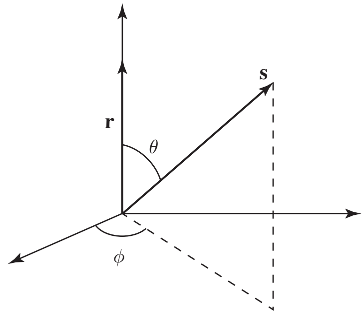


**Figure 10.8:** Convenient coordinates for the integral in Equation 10.58.

so Equation 10.52 says

$$\frac{1}{(2\pi)^{3/2}} \int \left(-s^2 + k^2\right) e^{i\mathbf{s}\cdot\mathbf{r}} g\,(\mathbf{s})\ d^3\mathbf{s} = \frac{1}{(2\pi)^3} \int e^{i\mathbf{s}\cdot\mathbf{r}}\, d^3\mathbf{s}.$$

It follows[11] that

$$g(\mathbf{s}) = \frac{1}{(2\pi)^{3/2}\left(k^2 - s^2\right)}. \tag{10.57}$$

Putting this back into Equation 10.54, we find:

$$G(\mathbf{r}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{s}\cdot\mathbf{r}} \frac{1}{\left(k^2 - s^2\right)}\, d^3\mathbf{s}. \tag{10.58}$$

Now, $\mathbf{r}$ is *fixed*, as far as the $\mathbf{s}$ integration is concerned, so we may as well choose spherical coordinates $(s, \theta, \phi)$ with the polar axis along $\mathbf{r}$ (Figure 10.8). Then $\mathbf{s}\cdot\mathbf{r} = sr\cos\theta$, the $\phi$ integral is trivial $(2\pi)$, and the $\theta$ integral is

$$\int_0^\pi e^{isr\cos\theta} \sin\theta\, d\theta = -\frac{e^{isr\cos\theta}}{isr}\bigg|_0^\pi = \frac{2\sin(sr)}{sr}. \tag{10.59}$$

Thus

$$G(\mathbf{r}) = \frac{1}{(2\pi)^2}\frac{2}{r}\int_0^\infty \frac{s\sin(sr)}{k^2 - s^2} ds = \frac{1}{4\pi^2 r}\int_{-\infty}^\infty \frac{s\sin(sr)}{k^2 - s^2} ds. \tag{10.60}$$

The remaining integral is not so simple. It pays to revert to exponential notation, and factor the denominator:

$$\begin{aligned} G(\mathbf{r}) &= \frac{i}{8\pi^2 r}\left\{\int_{-\infty}^\infty \frac{se^{isr}}{(s-k)\,(s+k)} ds - \int_{-\infty}^\infty \frac{se^{-isr}}{(s-k)\,(s+k)} ds\right\} \\ &= \frac{i}{8\pi^2 r}\left(I_1 - I_2\right). \end{aligned} \tag{10.61}$$

These two integrals can be evaluated using **Cauchy's integral formula**:

$$\oint \frac{f(z)}{(z - z_0)} dz = 2\pi i f(z_0), \tag{10.62}$$

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[11] This is clearly *sufficient*, but it is also *necessary*, as you can easily show by combining the two terms into a single integral, and using Plancherel's theorem, Equation 2.103.

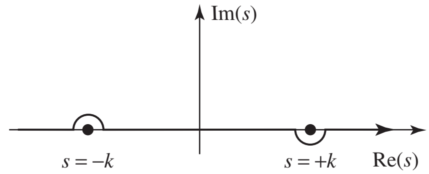


**Figure 10.9:** Skirting the poles in the contour integral (Equation 10.61).

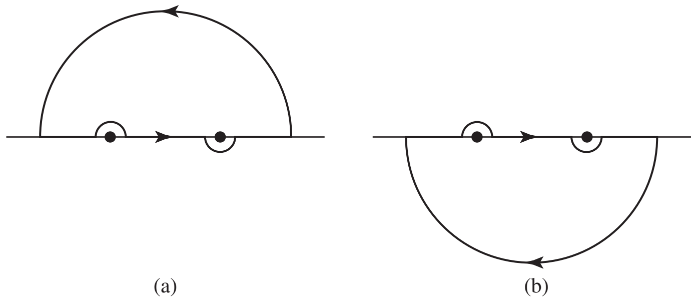


**Figure 10.10:** Closing the contour in Equations 10.63 and 10.64.

if $z_0$ lies within the contour (otherwise the integral is zero). In the present case the integration is along the real axis, and it passes *right over* the pole singularities at $\pm k$. We have to decide how to skirt the poles—I'll go *over* the one at $-k$ and *under* the one at $+k$ (Figure 10.9). (You're welcome to choose some *other* convention if you like—even winding seven times around each pole—you'll get a different Green's function, but, as I'll show you in a minute, they're all equally acceptable.)[12]

For each integral in Equation 10.61 I must "close the contour" in such a way that the semicircle at infinity contributes nothing. In the case of $I_1$, the factor $e^{isr}$ goes to zero when $s$ has a large *positive* imaginary part; for this one I close *above* (Figure 10.10(a)). The contour encloses only the singularity at $s=+k$, so

$$I_1 = \oint \left[\frac{se^{isr}}{s+k}\right]\frac{1}{s-k}\,ds = 2\pi i\left[\frac{se^{isr}}{s+k}\right]\Bigg|_{s=k} = i\pi e^{ikr}. \tag{10.63}$$

In the case of $I_2$, the factor $e^{-isr}$ goes to zero when $s$ has a large *negative* imaginary part, so we close *below* (Figure 10.10(b)); this time the contour encloses the singularity at $s=-k$ (and it goes around in the *clockwise* direction, so we pick up a minus sign):

$$I_2 = -\oint \left[\frac{se^{-isr}}{s-k}\right]\frac{1}{s+k}\,ds = -2\pi i\left[\frac{se^{-isr}}{s-k}\right]\Bigg|_{s=-k} = -i\pi e^{ikr}. \tag{10.64}$$

*Conclusion:*

$$G(\mathbf{r}) = \frac{i}{8\pi^2 r}\left[\left(i\pi e^{ikr}\right) - \left(-i\pi e^{ikr}\right)\right] = -\frac{e^{ikr}}{4\pi r}. \tag{10.65}$$

[12] If you are unfamiliar with this technique you have every right to be suspicious. In truth, the integral in Equation 10.60 is simply ill-defined—it does not converge, and it's something of a miracle that we can make sense of it at all. The root of the problem is that $G(\mathbf{r})$ doesn't really have a legitimate Fourier transform; we're exceeding the speed limit, here, and just hoping we won't get caught.

This, finally, is the Green's function for the Helmholtz equation—the solution to Equation 10.52. (If you got lost in all that analysis, you might want to *check* the result by direct differentiation—see Problem 10.8.) Or rather, it is *a* Green's function for the Helmholtz equation, for we can add to $G(\mathbf{r})$ any function $G_0(\mathbf{r})$ that satisfies the *homogeneous* Helmholtz equation:

$$\left(\nabla^2 + k^2\right) G_0(\mathbf{r}) = 0; \tag{10.66}$$

clearly, the result $(G + G_0)$ still satisfies Equation 10.52. This ambiguity corresponds precisely to the ambiguity in how to skirt the poles—a different choice amounts to picking a different function $G_0(\mathbf{r})$.

Returning to Equation 10.53, the general solution to the Schrödinger equation takes the form

$$\boxed{\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\, d^3\mathbf{r}_0,} \tag{10.67}$$

where $\psi_0$ satisfies the *free*-particle Schrödinger equation,

$$\left(\nabla^2 + k^2\right)\psi_0 = 0. \tag{10.68}$$

Equation 10.67 is the **integral form of the Schrödinger equation**; it is entirely equivalent to the more familiar differential form. At first glance it *looks* like an explicit *solution* to the Schrödinger equation (for any potential)—which is too good to be true. Don't be deceived: There's a $\psi$ under the integral sign on the right hand side, so you can't do the integral unless you already know the solution! Nevertheless, the integral form can be very powerful, and it is particularly well suited to scattering problems, as we'll see in the following section.

**Problem 10.8** Check that Equation 10.65 satisfies Equation 10.52, by direct substitution. *Hint:* $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$.[13]

** **Problem 10.9** Show that the ground state of hydrogen (Equation 4.80) satisfies the integral form of the Schrödinger equation, for the appropriate $V$ and $E$ (note that $E$ is *negative*, so $k = i\kappa$, where $\kappa \equiv \sqrt{-2mE}/\hbar$).

### 10.4.2 The First Born Approximation

Suppose $V(\mathbf{r}_0)$ is localized about $\mathbf{r}_0 = 0$—that is, the potential drops to zero outside some finite region (as is typical for a scattering problem), and we want to calculate $\psi(\mathbf{r})$ at points *far away* from the scattering center. Then $|\mathbf{r}| \gg |\mathbf{r}_0|$ for all points that contribute to the integral in Equation 10.67, so

$$|\mathbf{r}-\mathbf{r}_0|^2 = r^2 + r_0^2 - 2\mathbf{r}\cdot\mathbf{r}_0 \approx r^2\left(1 - 2\frac{\mathbf{r}\cdot\mathbf{r}_0}{r^2}\right), \tag{10.69}$$

and hence

[13] See, for example, D. Griffiths, *Introduction to Electrodynamics*, 4th edn (Cambridge University Press, Cambridge, UK, 2017), Section 1.5.3.

$$|\mathbf{r} - \mathbf{r}_0| \approx r - \hat{r} \cdot \mathbf{r}_0. \tag{10.70}$$

Let

$$\mathbf{k} \equiv k\hat{r}; \tag{10.71}$$

then

$$e^{ik|\mathbf{r}-\mathbf{r}_0|} \approx e^{ikr} e^{-i\mathbf{k}\cdot\mathbf{r}_0}, \tag{10.72}$$

and therefore

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} \approx \frac{e^{ikr}}{r} e^{-i\mathbf{k}\cdot\mathbf{r}_0}. \tag{10.73}$$

(In the *denominator* we can afford to make the more radical approximation $|\mathbf{r} - \mathbf{r}_0| \approx r$; in the *exponent* we need to keep the next term. If this puzzles you, try including the next term in the expansion of the denominator. What we are doing is expanding in powers of the small quantity $(r_0/r)$, and dropping all but the lowest order.)

In the case of scattering, we want

$$\psi_0(\mathbf{r}) = Ae^{ikz}, \tag{10.74}$$

representing an incident plane wave. For large $r$, then,

$$\psi(\mathbf{r}) \approx Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\, d^3\mathbf{r}_0. \tag{10.75}$$

This is in the standard form (Equation 10.12), and we can read off the scattering amplitude:

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\, d^3\mathbf{r}_0. \tag{10.76}$$

This is *exact*.[14] Now we invoke the **Born approximation**: Suppose the incoming plane wave is *not substantially altered by the potential;* then it makes sense to use

$$\psi(\mathbf{r}_0) \approx \psi_0(\mathbf{r}_0) = Ae^{ikz_0} = Ae^{i\mathbf{k}'\cdot\mathbf{r}_0}, \tag{10.77}$$

where

$$\mathbf{k}' \equiv k\hat{z}, \tag{10.78}$$

inside the integral. (This would be the *exact* wave function, if $V$ were zero; it is essentially a *weak potential* approximation.[15]) In the Born approximation, then,

$$\boxed{f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0} V(\mathbf{r}_0)\, d^3\mathbf{r}_0.} \tag{10.79}$$

(In case you have lost track of the definitions of $\mathbf{k}'$ and $\mathbf{k}$, they both have magnitude $k$, but the former points in the direction of the incident beam, while the latter points toward the detector—see Figure 10.11; $\hbar\left(\mathbf{k}-\mathbf{k}'\right)$ is the **momentum transfer** in the process.)

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[14] Remember, $f(\theta,\phi)$ is by definition the coefficient of $Ae^{ikr}/r$ at *large* $r$.

[15] Typically, partial wave analysis is useful when the incident particle has low energy, for then only the first few terms in the series contribute significantly; the Born approximation is more useful at *high* energy, when the deflection is relatively small.

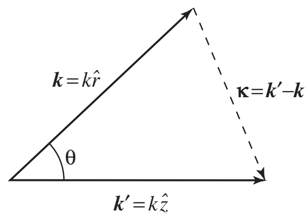


**Figure 10.11:** Two wave vectors in the Born approximation: $\mathbf{k}'$ points in the *incident* direction, $\mathbf{k}$ in the *scattered* direction.

In particular, for **low energy** (long wavelength) **scattering**, the exponential factor is essentially constant over the scattering region, and the Born approximation simplifies to

$$\boxed{f(\theta, \phi) \approx -\frac{m}{2\pi\hbar^2} \int V(\mathbf{r})\, d^3\mathbf{r}, \quad \text{(low energy).}} \tag{10.80}$$

(I dropped the subscript on $\mathbf{r}$, since there is no likelihood of confusion at this point.)

**Example 10.4**

**Low-energy soft-sphere scattering.**[16] Suppose

$$V(\mathbf{r}) = \begin{cases} V_0, & (r \le a), \\ 0, & (r > a). \end{cases} \tag{10.81}$$

In this case the low-energy scattering amplitude is

$$f(\theta, \phi) \approx -\frac{m}{2\pi\hbar^2} V_0 \left(\frac{4}{3}\pi a^3\right), \tag{10.82}$$

(independent of $\theta$ and $\phi$), the differential cross-section is

$$\frac{d\sigma}{d\Omega} = |f|^2 \approx \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2, \tag{10.83}$$

and the total cross-section is

$$\sigma \approx 4\pi \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2. \tag{10.84}$$

For a **spherically symmetrical potential**, $V(\mathbf{r}) = V(r)$—but *not* necessarily at low energy—the Born approximation again reduces to a simpler form. Define

$$\boldsymbol{\kappa} \equiv \mathbf{k}' - \mathbf{k}, \tag{10.85}$$

[16] You can't apply the Born approximation to *hard*-sphere scattering ($V_0 = \infty$)—the integral blows up. The point is that we assumed the potential is *weak*, and doesn't change the wave function much in the scattering region. But a *hard* sphere changes it *radically*—from $Ae^{ikz}$ to *zero*.

and let the polar axis for the $\mathbf{r}_0$ integral lie along $\boldsymbol{\kappa}$, so that

$$\left(\mathbf{k}' - \mathbf{k}\right) \cdot \mathbf{r}_0 = \kappa r_0 \cos\theta_0. \tag{10.86}$$

Then

$$f(\theta) \approx -\frac{m}{2\pi\hbar^2} \int e^{i\kappa r_0 \cos\theta_0} V(r_0) r_0^2 \sin\theta_0 \, dr_0 \, d\theta_0 \, d\phi_0. \tag{10.87}$$

The $\phi_0$ integral is trivial $(2\pi)$, and the $\theta_0$ integral is one we have encountered before (see Equation 10.59). Dropping the subscript on $r$, we are left with

$$\boxed{f(\theta) \approx -\frac{2m}{\hbar^2 \kappa} \int_0^\infty r V(r) \sin(\kappa r) \, dr, \ \text{(spherical symmetry)}.} \tag{10.88}$$

The angular dependence of $f$ is carried by $\kappa$; in Figure 10.11 we see that

$$\kappa = 2k \sin(\theta/2). \tag{10.89}$$

**Example 10.5**

**Yukawa scattering.** The **Yukawa potential** (which is a crude model for the binding force in an atomic nucleus) has the form

$$V(r) = \beta \frac{e^{-\mu r}}{r}, \tag{10.90}$$

where $\beta$ and $\mu$ are constants. The Born approximation gives

$$f(\theta) \approx -\frac{2m\beta}{\hbar^2 \kappa} \int_0^\infty e^{-\mu r} \sin(\kappa r) \, dr = -\frac{2m\beta}{\hbar^2 \left(\mu^2 + \kappa^2\right)}. \tag{10.91}$$

(You get to work out the integral for yourself, in Problem 10.11.)

**Example 10.6**

**Rutherford scattering.** If we put in $\beta = q_1 q_2 / 4\pi\epsilon_0$, $\mu = 0$, the Yukawa potential reduces to the Coulomb potential, describing the electrical interaction of two point charges. Evidently the scattering amplitude is

$$f(\theta) \approx -\frac{2m q_1 q_2}{4\pi\epsilon_0 \hbar^2 \kappa^2}, \tag{10.92}$$

or (using Equations 10.89 and 10.51):

$$f(\theta) \approx -\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}. \tag{10.93}$$

The differential cross-section is the square of this:

$$\frac{d\sigma}{d\Omega} = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2, \tag{10.94}$$

which is precisely the Rutherford formula (Equation 10.11). It happens that for the Coulomb potential classical mechanics, the Born approximation, and quantum field theory all yield the same result. As they say in the computer business, the Rutherford formula is amazingly "robust."

* **Problem 10.10** Find the scattering amplitude, in the Born approximation, for soft-sphere scattering at arbitrary energy. Show that your formula reduces to Equation 10.82 in the low-energy limit.

**Problem 10.11** Evaluate the integral in Equation 10.91, to confirm the expression on the right.

** **Problem 10.12** Calculate the total cross-section for scattering from a Yukawa potential, in the Born approximation. Express your answer as a function of $E$.

* **Problem 10.13** For the potential in Problem 10.4,

**(a)** calculate $f(\theta)$, $D(\theta)$, and $\sigma$, in the low-energy Born approximation;

**(b)** calculate $f(\theta)$ for arbitrary energies, in the Born approximation;

**(c)** show that your results are consistent with the answer to Problem 10.4, in the appropriate regime.

### 10.4.3 The Born Series

The Born approximation is similar in spirit to the **impulse approximation** in classical scattering theory. In the impulse approximation we begin by pretending that the particle keeps going in a straight line (Figure 10.12), and compute the transverse impulse that would be delivered to it in that case:

$$I = \int F_{\perp}\, dt. \tag{10.95}$$

If the deflection is relatively small, this should be a good approximation to the transverse momentum imparted to the particle, and hence the scattering angle is

$$\theta \approx \tan^{-1}(I/p), \tag{10.96}$$

where $p$ is the incident momentum. This is, if you like, the "first-order" impulse approximation (the *zeroth*-order is what we *started* with: no deflection at all). Likewise, in the zeroth-order Born approximation the incident plane wave passes by with no modification, and what we

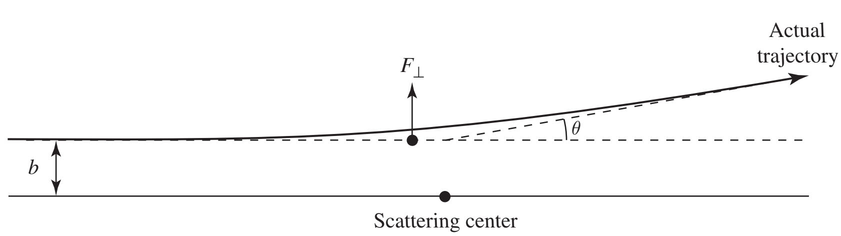


**Figure 10.12:** The impulse approximation assumes that the particle continues undeflected, and calculates the transverse momentum delivered.

explored in the previous section is really the first-order correction to this. But the same idea can be iterated to generate a series of higher-order corrections, which presumably converge to the exact answer.

The integral form of the Schrödinger equation reads

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \int g(\mathbf{r} - \mathbf{r}_0) V(\mathbf{r}_0) \psi(\mathbf{r}_0)\, d^3\mathbf{r}_0, \tag{10.97}$$

where $\psi_0$ is the incident wave,

$$g(\mathbf{r}) \equiv -\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r} \tag{10.98}$$

is the Green's function (into which I have now incorporated the factor $2m/\hbar^2$, for convenience), and $V$ is the scattering potential. Schematically,

$$\psi = \psi_0 + \int gV\psi. \tag{10.99}$$

Suppose we take this expression for $\psi$, and plug it in under the integral sign:

$$\psi = \psi_0 + \int gV\psi_0 + \int\int gVgV\psi. \tag{10.100}$$

Iterating this procedure, we obtain a formal series for $\psi$:

$$\psi = \psi_0 + \int gV\psi_0 + \int\int gVgV\psi_0 + \int\int\int gVgVgV\psi_0 + \cdots. \tag{10.101}$$

In each integrand only the *incident* wave function ($\psi_0$) appears, together with more and more powers of $gV$. The *first* Born approximation truncates the series after the second term, but it is pretty clear how one generates the higher-order corrections.

The Born series can be represented diagrammatically as shown in Figure 10.13. In zeroth order $\psi$ is untouched by the potential; in first order it is "kicked" once, and then "propagates" out in some new direction; in second order it is kicked, propagates to a new location, is kicked again, and then propagates out; and so on. In this context the Green's function is sometimes called the **propagator**—it tells you how the disturbance propagates between one interaction and the next. The Born series was the inspiration for Feynman's formulation of relativistic quantum mechanics, which is expressed entirely in terms of **vertex factors** ($V$) and propagators ($g$), connected together in **Feynman diagrams**.

**Problem 10.14** Calculate $\theta$ (as a function of the impact parameter) for Rutherford scattering, in the impulse approximation. Show that your result is consistent with the exact expression (Problem 10.1(a)), in the appropriate limit.

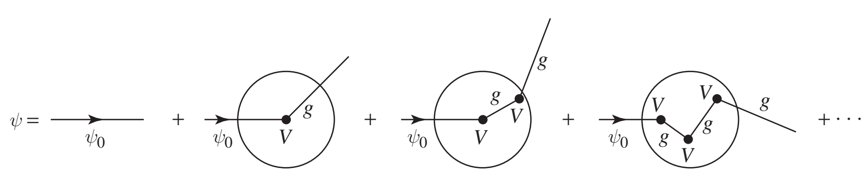


**Figure 10.13:** Diagrammatic interpretation of the Born series (Equation 10.101).

∗∗∗ **Problem 10.15** Find the scattering amplitude for low-energy soft-sphere scattering in the *second* Born approximation. *Answer:*

$$-\left(2mV_0a^3/3\hbar^2\right)\left[1-\left(4mV_0a^2/5\hbar^2\right)\right].$$

## FURTHER PROBLEMS ON CHAPTER 10

∗∗∗ **Problem 10.16** Find the Green's function for the *one*-dimensional Schrödinger equation, and use it to construct the integral form (analogous to Equation 10.66). *Answer:*

$$\psi(x)=\psi_0(x)-\frac{im}{\hbar^2k}\int_{-\infty}^{\infty}e^{ik|x-x_0|}V(x_0)\psi(x_0)\,dx_0. \qquad (10.102)$$

∗∗ **Problem 10.17** Use your result in Problem 10.16 to develop the Born approximation for one-dimensional scattering (on the interval $-\infty<x<\infty$, with no "brick wall" at the origin). That is, choose $\psi_0(x)=Ae^{ikx}$, and assume $\psi(x_0)\approx\psi_0(x_0)$ to evaluate the integral. Show that the reflection coefficient takes the form:

$$R\approx\left(\frac{m}{\hbar^2k}\right)^2\left|\int_{-\infty}^{\infty}e^{2ikx}V(x)\,dx\right|^2. \qquad (10.103)$$

**Problem 10.18** Use the one-dimensional Born approximation (Problem 10.17) to compute the transmission coefficient ($T=1-R$) for scattering from a delta function (Equation 2.117) and from a finite square well (Equation 2.148). Compare your results with the exact answers (Equations 2.144 and 2.172).

**Problem 10.19** Prove the **optical theorem**, which relates the total cross-section to the imaginary part of the forward scattering amplitude:

$$\sigma=\frac{4\pi}{k}\operatorname{Im}\left[f(0)\right]. \qquad (10.104)$$

*Hint:* Use Equations 10.47 and 10.48.

**Problem 10.20** Use the Born approximation to determine the total cross-section for scattering from a gaussian potential

$$V(\mathbf{r})=V_0e^{-\mu r^2/a^2}.$$

Express your answer in terms of the constants $V_0$, $a$, and $m$ (the mass of the incident particle), and $k\equiv\sqrt{2mE}/\hbar$, where $E$ is the incident energy.

**Problem 10.21 Neutron diffraction**. Consider a beam of neutrons scattering from a crystal (Figure 10.14). The interaction between neutrons and the nuclei in the crystal is short ranged, and can be approximated as

$$V(\mathbf{r}) = \frac{2\pi\hbar^2 b}{m} \sum_i \delta^3(\mathbf{r} - \mathbf{r}_i) \;,$$

where the $\mathbf{r}_i$ are the locations of the nuclei and the strength of the potential is expressed in terms of the **nuclear scattering length** $b$.

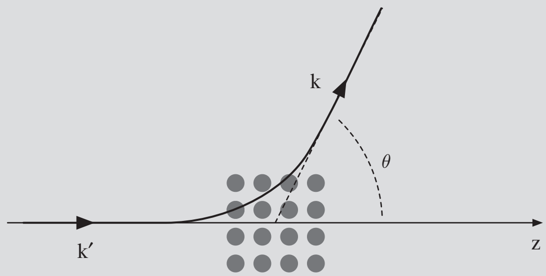


**Figure 10.14:** Neutron scattering from a crystal.

**(a)** In the first Born approximation, show that

$$\frac{d\sigma}{d\Omega} = b^2 \left| \sum_i e^{-i\,\mathbf{q}\cdot\mathbf{r}_i} \right|^2$$

where $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$.

**(b)** Now consider the case where the nuclei are arranged on a cubic lattice with spacing $a$. Take the positions to be

$$\mathbf{r}_i = l\,a\,\hat{\imath} + m\,a\,\hat{\jmath} + n\,a\,\hat{k}$$

where $l$, $m$, and $n$ all range from 0 to $N-1$, so there are a total of $N^3$ nuclei.[17] Show that

$$\frac{d\sigma}{d\Omega} = b^2\,\frac{\sin^2(N\,q_x\,a/2)}{\sin^2(q_x\,a/2)}\,\frac{\sin^2\left(N\,q_y\,a/2\right)}{\sin^2\left(q_y\,a/2\right)}\,\frac{\sin^2(N\,q_z\,a/2)}{\sin^2(q_z\,a/2)} \,.$$

**(c)** Plot

$$\frac{1}{N}\,\frac{\sin^2(N\,q_x\,a/2)}{\sin^2(q_x\,a/2)}$$

as a function of $q_x\,a$ for several values of $N$ ($N = 1, 5, 10$) to show that the function describes a series of peaks that become progressively sharper as $N$ increases.

**(d)** In light of (c), in the limit of large $N$ the differential scattering cross section is negligibly small except at one of these peaks:

$$\mathbf{q} = \mathbf{G}_{\ell mn} = \frac{2\pi}{a}\left(l\,\hat{\imath} + m\,\hat{\jmath} + n\,\hat{k}\right)$$

[17] It makes no difference that this crystal isn't "centered" at the origin: shifting the crystal by $\mathbf{R}$ amounts to adding $\mathbf{R}$ to each of the $\mathbf{r}_i$, and that doesn't affect $d\sigma/d\Omega$. After all, we're assuming an incident plane wave, which extends to $\pm\infty$ in the $x$ and $y$ directions.

for integer $l$, $m$, and $n$. The vectors $\mathbf{G}_{lmn}$ are called **reciprocal lattice vectors**. Find the scattering angles ($\theta$) at which peaks occur. If the neutron's wavelength is equal to the crystal spacing $a$, what are the three smallest (nonzero) angles?

*Comment*: Neutron diffraction is one method used, to determine crystal structures (electrons and x-rays can also be used and the same expression for the locations of the peaks holds). In this problem we looked at a cubic arrangement of atoms, but a different arrangement (hexagonal for example) would produce peaks at a different set of angles. Thus from the scattering data one can infer the underlying crystal structure.

∗∗∗ **Problem 10.22 Two-dimensional scattering theory.** By analogy with Section 10.2, develop partial wave analysis for two dimensions.

**(a)** In polar coordinates $(r, \theta)$ the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{10.105}$$

Find the separable solutions to the (time-independent) Schrödinger equation, for a potential with azimuthal symmetry ($V(r,\theta) \to V(r)$). *Answer:*

$$\psi(r,\theta) = R(r)\, e^{ij\theta}, \tag{10.106}$$

where $j$ is an integer, and $u \equiv \sqrt{r}\, R$ satisfies the radial equation

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\left(j^2 - 1/4\right)}{r^2}\right]u = Eu. \tag{10.107}$$

**(b)** By solving the radial equation for very large $r$ (where both $V(r)$ and the centrifugal term go to zero), show that an outgoing radial wave has the asymptotic form

$$R(r) \sim \frac{e^{ikr}}{\sqrt{r}}, \tag{10.108}$$

where $k \equiv \sqrt{2mE}/\hbar$. Check that an incident wave of the form $Ae^{ikx}$ satisfies the Schrödinger equation, for $V(r) = 0$ (this is trivial, if you use cartesian coordinates). Write down the two-dimensional analog to Equation 10.12, and compare your result to Problem 10.2. *Answer:*

$$\psi(r,\theta) \approx A\left[e^{ikx} + f(\theta)\frac{e^{ikr}}{\sqrt{r}}\right], \quad \text{for large } r. \tag{10.109}$$

**(c)** Construct the analog to Equation 10.21 (the wave function in the region where $V(r) = 0$ but the centrifugal term *cannot* be ignored). *Answer:*

$$\psi(r,\theta) = A\left\{e^{ikx} + \sum_{j=-\infty}^{\infty} c_j H_j^{(1)}(kr)e^{ij\theta}\right\}, \tag{10.110}$$

where $H_j^{(1)}$ is the Hankel function (*not* the *spherical* Hankel function!) of order $j$.[18]

[18] See Mary Boas, *Mathematical Methods in the Physical Sciences,* 3rd edn (Wiley, New York, 2006), Section 12.17.

**(d)** For large $z$,

$$H_j^{(1)}(z) \sim \sqrt{2/\pi}\, e^{-i\pi/4}\, (-i)^j\, \frac{e^{iz}}{\sqrt{z}}. \tag{10.111}$$

Use this to show that

$$f(\theta) = \sqrt{2/\pi k}\, e^{-i\pi/4} \sum_{j=-\infty}^{\infty} (-i)^j\, c_j e^{ij\theta}. \tag{10.112}$$

**(e)** Adapt the argument of Section 10.1.2 to this two-dimensional geometry. Instead of the *area* $d\sigma$, we have a *length*, $db$, and in place of the solid angle $d\Omega$ we have the increment of scattering angle $|d\theta|$; the role of the differential cross-section is played by

$$D(\theta) \equiv \left| \frac{db}{d\theta} \right|, \tag{10.113}$$

and the effective "width" of the target (analogous to the total cross-section) is

$$B \equiv \int_0^{2\pi} D(\theta)\, d\theta. \tag{10.114}$$

Show that

$$D(\theta) = |f(\theta)|^2, \quad \text{and} \quad B = \frac{4}{k} \sum_{j=-\infty}^{\infty} \left| c_j \right|^2. \tag{10.115}$$

**(f)** Consider the case of scattering from a hard disk (or, in three dimensions, an infinite cylinder[19]) of radius $a$:

$$V(r) = \begin{cases} \infty, & (r \leq a), \\ 0, & (r > a). \end{cases} \tag{10.116}$$

By imposing appropriate boundary conditions at $r = a$, determine $B$. You'll need the analog to Rayleigh's formula:

$$e^{ikx} = \sum_{j=-\infty}^{\infty} (i)^j\, J_j(kr)\, e^{ij\theta} \tag{10.117}$$

(where $J_j$ is the Bessel function of order $J$). Plot $B$ as a function of $ka$, for $0 < ka < 2$.

**Problem 10.23 Scattering of identical particles**. The results for scattering of a particle from a fixed target also apply to the scattering of two particles in the center of mass frame. With $\psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\, \psi_r(\mathbf{r})$, $\psi_r(\mathbf{r})$ satisfies

$$-\frac{\hbar}{2\mu} \nabla^2 \psi_r + V(r)\, \psi_r = E_r\, \psi_r \tag{10.118}$$

[19] S. McAlinden and J. Shertzer, *Am. J. Phys.* **84**, 764 (2016).

(see Problem 5.1) where $V(r)$ is the interaction between the particles (assumed here to depend only on their separation distance). This is the *one*-particle Schrödinger equation (with the reduced mass $\mu$ in place of $m$).

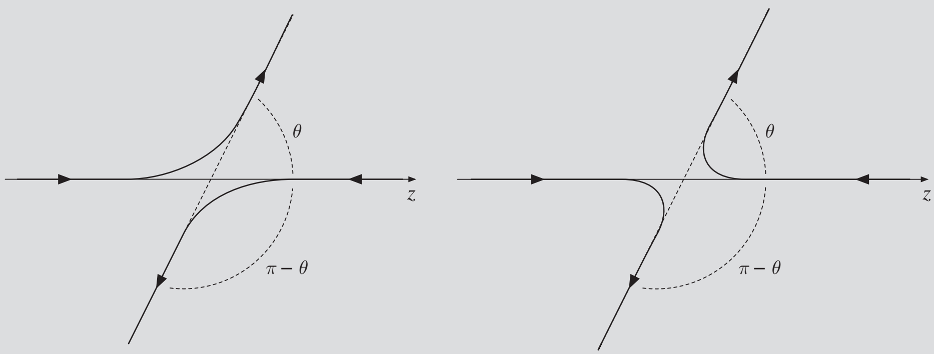


**Figure 10.15:** Scattering of identical particles.

**(a)** Show that if the two particles are identical (spinless) bosons, then $\psi_r(\mathbf{r})$ must be an even function of $\mathbf{r}$ (Figure 10.15).
**(b)** By symmetrizing Equation 10.12 (why is this allowed?), show that the scattering amplitude in this case is

$$f_B(\theta) = f(\theta) + f(\pi - \theta)$$

where $f(\theta)$ is the scattering amplitude of a single particle of mass $\mu$ from a fixed target $V(r)$.
**(c)** Show that the partial wave amplitudes of $f_B$ vanish for all odd powers of $\ell$.
**(d)** How are the results of (a)–(c) different if the particles are identical fermions (in a triplet spin state).
**(e)** Show that the scattering amplitude for identical fermions vanishes at $\pi/2$.
**(f)** Plot the logarithm of the differential scattering cross section for fermions and for bosons in Rutherford scattering (Equation 10.93).[20]

[20] Equation 10.93 was derived by taking the limit of Yukawa scattering (Example 10.5) and the result for $f(\theta)$ is missing a phase factor (see Albert Messiah, *Quantum Mechanics*, Dover, New York, NY (1999), Section XI.7). That factor drops out of the cross-section for scattering from a fixed potential—giving the correct answer in Example 10.6—but would show up in the cross-section for scattering of identical particles.