QUANTUM MECHANICS EXERCISES

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1 CHAPTER 10: SCATTERING

Problem 10.3 Prove Equation 10.33, starting with Equation 10.32. *Hint:* Exploit the orthogonality of the Legendre polynomials to show that the coefficients with different values of ℓ must separately vanish.

From equation (10.32), we get:

$$\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(ka)+ika_l h_l^1(ka)\right] P_l(cos\theta)=0$$

Because Legendre Polynomial has an orthogonal basis, hence the co-efficents have to be zero:

$$[i^l(2l+1)\left[j_l(ka)+ika_l\cdot h_l^{(1)}(ka)\right]=0$$

$$\Rightarrow a_l=\frac{ij_l}{k\cdot h_l^{(1)}(ka)}$$

** **Problem 10.4** Consider the case of low-energy scattering from a spherical delta-function shell:

$$V(r)=\alpha\delta(r-a),$$

where α and a are constants. Calculate the scattering amplitude, $f(\theta)$, the differential cross-section, $D(\theta)$, and the total cross-section, σ. Assume $ka \ll 1$, so that only the $\ell=0$ term contributes significantly. (To simplify matters, throw out all $\ell \neq 0$ terms right from the start.) The main problem, of course, is to determine C_0. Express your answer in terms of the dimensionless quantity $\beta \equiv 2ma\alpha/\hbar^2$. *Answer:* $\sigma=4\pi a^2\beta^2/(1+\beta)^2$.

From equation (10.29), we get:

$$\begin{aligned}
\psi_{\mathbf{ext}}(r,\theta) &= A\cdot\left[j_0(kr)+ika_0\cdot h_0^{(1)}(kr)\right]P_0(cos\theta)\\
&= A\left[\frac{sinkr}{kr}+ika_0\cdot\left(-i\frac{e^{ikr}}{kr}\right)\right]\\
&= A\left(\frac{sin(kr)}{kr}+a_0\frac{e^{ikr}}{r}\right)
\end{aligned}$$

From equation (10.18) we get:

$$\psi_{\mathbf{int}} = \frac{u(r)}{r}\cdot Y_0^0(\theta,\phi) = B\frac{sinkr}{kr}$$

Apply the first boundary condition, means that wave function is continuous at a:

$$A(\frac{sinka}{ka}+a_0\frac{e^{ika}}{a}) = B\frac{sinka}{ka}$$

and the second condition by integrating the radial equation across the delta function gives:

$$\int_{a-\epsilon}^{a+\epsilon} -\frac{\hbar^2}{2m}\frac{d^2u}{dx^2}dx + \alpha\int_{a-\epsilon}^{a+\epsilon}\delta(x)u(x)dx = E\int_{a-\epsilon}^{a+\epsilon}u(x)dx$$

Take the limit of ϵ to ∞, we get;

$$\int_{a-\epsilon}^{a+\epsilon}u(x)dx = 0$$

$$lim_{\epsilon\to 0}\frac{du}{dx}\bigg|_{a-\epsilon}^{a+\epsilon} = \frac{2m\alpha}{\hbar}u(a)$$

$$\leftrightarrow \frac{du_{ext}}{dx}\bigg|_{r=a} - \frac{du_{int}}{dx}\bigg|_{r=a} = \frac{2m\alpha}{\hbar}u(a)$$

$$\begin{aligned}
A\left[Coska+ia_0k\cdot e^{ika}\right] &= B\cdot Cos(ka)+\beta B\cdot\frac{Sinka}{ka}\\
coska+ia_0k\cdot e^{ika} &= coska+\frac{\beta}{ka}sinka+a_0k\cdot cotka\cdot e^{ika}+\beta\frac{a_0}{a}\cdot e^{ika}\\
ia_0k\cdot e^{ika}\left[1+icotka+i\frac{\beta}{ka}\right] &= \frac{\beta}{ka}sinka \text{ But } ka<<1 \text{ , so } sinka\approx ka \text{ and } cotka\approx\frac{1}{ka}\\
ia_0k(1+ika)\left[1+\frac{i}{ka}(1+\beta)\right] &= \beta\\
ia_0\left[\frac{i}{ka}(1+\beta)\right] &= \beta
\end{aligned}$$

$$\Rightarrow a_0 = -\frac{a\beta}{1+\beta}, \text{ with } \beta = \frac{2m\alpha a}{\hbar^2}$$

The scattering amplitude, $f(\theta)$:

$$f(\theta)\approx a_0 = -\frac{a\beta}{1+\beta}$$

The differential cross-section, $D(\theta)$:

$$D(\theta) = |f(\theta)|^2 - (\frac{a\beta}{1+\beta})^2$$

The total cross-section, σ:

$$\sigma = 4\pi D = 4\pi(\frac{a\beta}{1+\beta})^2$$

Problem 10.5 A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0, & (x < -a), \\ -V_0, & (-a \leq x \leq 0), \\ \infty, & (x > 0). \end{cases}$$

(a) If the incoming wave is Ae^{ikx} (where $k = \sqrt{2mE}/\hbar$), find the reflected wave. *Answer:*

$$Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right]e^{-ikx}, \text{ where } k' = \sqrt{2m(E+V_0)}/\hbar.$$

(b) Confirm that the reflected wave has the same amplitude as the incident wave.

(c) Find the phase shift δ (Equation 10.40) for a very deep well ($E \ll V_0$). *Answer:* $\delta = -ka$.

(a) In the region to the left

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad (x \leq -a).$$

In the region $-a < x < 0$, the Schrödinger equation gives

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \quad \Longrightarrow \quad \frac{d^2\psi}{dx^2} = -(k')^2\psi$$

where $k' = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$. The general solution is

$$\psi = C\sin(k'x) + D\cos(k'x)$$

But $\psi(0) = 0$ implies $D = 0$, so

$$\psi(x) = C\sin(k'x) \quad (-a \leq x \leq 0).$$

The continuity of $\psi(x)$ **and** $\psi'(x)$ at $x = -a$ says

$$Ae^{-ika} + Be^{ika} = -C\sin(k'a), \quad ikAe^{-ika} - ikBe^{ika} = k'C\cos(k'a).$$

Divide and solve for B:

$$\frac{ikAe^{-ika} - ikBe^{ika}}{Ae^{-ika} + Be^{ika}} = -k' \cot\big(k'a\big),$$

$$ikAe^{-ika} - ikBe^{ika} = -Ae^{-ika}k' \cot\big(k'a\big) - Be^{ika}k' \cot\big(k'a\big),$$

$$Be^{ika}[-ik + k' \cot\big(k'a\big)] = Ae^{-ika}[-ik - k' \cot\big(k'a\big)].$$

$$B = Ae^{-2ika}\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}.$$

(b)

$$|B|^2 = |A|^2 \left[\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}\right] \cdot \left[\frac{k + ik' \cot(k'a)}{k - ik' \cot(k'a)}\right] = |A|^2.$$

(c) From part (a) the wave function for $x < -a$ is

$$\psi(x) = Ae^{ikx} + Ae^{-2ika}\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)}e^{-ikx}.$$

But by definition of the phase shift (Eq. 10.40)

$$\psi(x) = A\left[e^{ikx} - e^{i(2\delta - kx)}\right],$$

so

$$e^{-2ika}\frac{k - ik' \cot(k'a)}{k + ik' \cot(k'a)} = e^{-2i\delta}.$$

$$\Rightarrow -e^{-2ika}\left[\frac{ik + k' \cot(k'a)}{ik - k' \cot(k'a)}\right] = e^{2i\delta}$$

$$\Rightarrow \ln\left\{-e^{-2ika}\left[\frac{ik + k' \cot(k'a)}{ik - k' \cot(k'a)}\right]\right\} = 2i\delta$$

$$\therefore \delta = \frac{1}{2i}\ln\left\{-e^{-2ika}\left[\frac{ik + k' \cot(k'a)}{ik - k' \cot(k'a)}\right]\right\}$$

In this situation, we know that for a very deep well:

$$\frac{\sqrt{2mE}}{\hbar} \ll \frac{\sqrt{2m(E + V_0)}}{\hbar} \Rightarrow k \ll k'$$

This amounts to the factor k being negligible thus:

$$\delta \approx \frac{1}{2i}\ln\left\{-e^{-2ika}\left[\frac{i \cdot 0 + k' \cot(k'a)}{i \cdot 0 - k' \cot(k'a)}\right]\right\}$$

$$= \frac{1}{2i}\ln\left\{-e^{-2ika}\left[\frac{k'\cot(k'a)}{-k'\cot(k'a)}\right]\right\}$$

$$= \frac{1}{2i}\ln\left\{-e^{-2ika}[-1]\right\} = \frac{1}{2i}\ln\left(e^{-2ika}\right)$$

$$= \frac{1}{2i}(-2ika) = -ka \quad \therefore \delta \approx -ka$$

Problem 10.6 What are the partial wave phase shifts (δ_ℓ) for hard-sphere scattering (Example 10.3)?

From Eq. 10.46, $a_\ell = \frac{1}{k}e^{i\delta_\ell}\sin\delta_\ell,$ *and* $Eq.10.33,$ $a_\ell = i\frac{j_\ell(ka)}{kh_\ell^{(1)}(ka)},$ *it follows that* $e^{i\delta_\ell}\sin\delta_\ell = i\frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}.$

But (Eq. 10.19) $h_\ell^{(1)}(x) = j_\ell(x) + in_\ell(x),$ *so*

$$e^{i\delta_\ell}\sin\delta_\ell = \frac{i\,j_\ell(ka)}{j_\ell(ka) + in_\ell(ka)} = \frac{i}{1+i(n/j)} = \frac{i(1-i(n/j))}{1+(n/j)^2} = \frac{(n/j)+i}{1+(n/j)^2},$$

(writing (n/j) as shorthand for $n_\ell(ka)/j_\ell(ka)$). Equating the real and imaginary parts:

$$\cos\delta_\ell\sin\delta_\ell = \frac{(n/j)}{1+(n/j)^2}; \quad \sin^2\delta_\ell = \frac{1}{1+(n/j)^2}.$$

Dividing the second by the first, I conclude that

$$\tan\delta_\ell = \frac{1}{(n/j)}, \quad or \quad \delta_\ell = \tan^{-1}\left[\frac{j_\ell(ka)}{n_\ell(ka)}\right].$$

Problem 10.7 Find the S-wave ($\ell = 0$) partial wave phase shift $\delta_0(k)$ for scattering from a delta-function shell (Problem 10.4). Assume that the radial wave function $u(r)$ goes to 0 as $r \to 0$. *Answer:*

$$-\cot^{-1}\left[\cot(ka) + \frac{ka}{\beta\sin^2(ka)}\right], \quad \text{where} \quad \beta \equiv \frac{2m\alpha a}{\hbar^2}.$$

$$-\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + V(r)u(r) = eu(r)$$

$$\frac{d^2u(r)}{dr^2} = -k^2u(r) + \frac{2m}{\hbar^2}V(r)u(r)\ , \text{as } k = \frac{\sqrt{2mE}}{\hbar}$$

For r > a and r < a, we have:

$$\frac{d^2u(r)}{dr^2} = -k^2u(r)$$

$$\Rightarrow u(r) = Asin(kr) + Bcos(kr)$$

Wavefunction goes to 0 as r $\to$ 0, we have:

$$u(0) = Asin(k \cdot 0) + Bcos(k \cdot 0) = A \cdot 0 + B \cdot 1 \Rightarrow B = 0$$

When dealing with scattering, it is common to express the wave function in terms of phase shift delta Kronecker , the solution of $sin(kr)$ describe the free particle wave, but when the potential the particle scatters the wave get shifted due to the interaction with the potential, and the shift is encoded in the phase factor

$$u(r) = \begin{cases} A\sin(kr + \delta_0), & r > a \\ B\sin(kr), & r < a \end{cases}$$

$$Asin(ka + \delta_0) = Bsin(ka) \Rightarrow A \cdot \frac{sin(ka + \delta_0)}{sinka} = B$$

This modify the wave function into the form:

$$u(r) = \begin{cases} A\sin(kr + \delta_0), & r > a \\ A \cdot \frac{sin(ka+\delta_0)}{sinka} \cdot \sin(kr), & r < a \end{cases}$$

Let's take a look at the radial wave function, it has a discontinuity at a , this can be seen by integrating the Schrodinger equation across a small interval around a:

$$\int_{a-\epsilon}^{a+\epsilon} \left[\frac{d^2u(r)}{dr^2} + k^2 u(r) \right] dr = \frac{2m}{\hbar^2} \int_{a-\epsilon}^{a+\epsilon} \alpha\delta(r-a)u(r)dr$$

$$\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u(r)}{dr^2} dr + k^2 \int_{a-\epsilon}^{a+\epsilon} u(r)dr = \frac{2m\alpha}{\hbar^2} \int_{a-\epsilon}^{a+\epsilon} \delta(r-a)u(r)dr$$

$$\Rightarrow \Delta \left(\frac{du(r)}{dr} \right) \Bigg|_{r=a} = \frac{2m\alpha}{\hbar^2} u(a) = \frac{\beta}{a} u(a)$$

$$\Rightarrow \left(\frac{du(r)}{dr} \Bigg|_{r>a} \right)_{r=a} - \left(\frac{du(r)}{dr} \Bigg|_{r<a} \right)_{r=a} = \frac{\beta}{a} u(a)$$

$$\Rightarrow \left\{ A\cos(kr + \delta_0)k \right\}_{r=a} - \left\{ A\frac{\sin(ka + \delta_0)}{\sin(ka)} \cos(kr)k \right\}_{r=a}$$

$$= \frac{\beta}{a} A\sin(ka + \delta_0)$$

$$\cos(ka + \delta_0) - \frac{\sin(ka + \delta_0)}{\sin(ka)} \cos(ka) = \frac{\beta}{ka} \sin(ka + \delta_0)$$

$$\sin(ka)\cos(ka + \delta_0) - \sin(ka + \delta_0)\cos(ka) = \frac{\beta}{ka} \sin(ka + \delta_0)\sin(ka).$$

Here, we need the sum/difference identities:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \rightarrow \text{RHS}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \quad \rightarrow \text{LHS}$$

$$\Rightarrow \sin(ka)\cos(ka+\delta_0) - \cos(ka)\sin(ka+\delta_0)$$

$$= \frac{\beta}{ka}\sin(ka)\sin(ka+\delta_0) \Rightarrow \sin\big(ka - \{ka+\delta_0\}\big)$$

$$= \frac{\beta}{ka}\sin(ka)\left[\sin(ka)\cos(\delta_0) + \cos(ka)\sin(\delta_0)\right]$$

$$\sin(ka-ka-\delta_0) = \frac{\beta}{ka}\sin(ka)\left[\sin(ka)\cos(\delta_0) + \frac{\sin(ka)\cos(ka)}{\sin(ka)}\sin(\delta_0)\right]$$

$$\Rightarrow \sin(-\delta_0) = \frac{\beta}{ka}\sin(ka)\sin(ka)\left[\cos(\delta_0) + \frac{\cos(ka)}{\sin(ka)}\sin(\delta_0)\right]$$

$$\Rightarrow -\sin(\delta_0) = \frac{\beta}{ka}\sin^2(ka)\left[\cos(\delta_0) + \cot(ka)\sin(\delta_0)\right]$$

$$\Rightarrow -1 = \frac{\beta}{ka}\frac{\sin^2(ka)}{\sin(\delta_0)}\left[\cos(\delta_0) + \cot(ka)\sin(\delta_0)\right]$$

$$\Rightarrow -1 = \frac{\beta\sin^2(ka)}{ka}\left[\frac{\cos(\delta_0)}{\sin(\delta_0)} + \frac{\cot(ka)\sin(\delta_0)}{\sin(\delta_0)}\right]$$

$$\Rightarrow -1 = \frac{\beta\sin^2(ka)}{ka}\left[\cot(\delta_0) + \cot(ka)\right].$$

$$\Rightarrow -\frac{ka}{\beta\sin^2(ka)} = \cot(\delta_0) + \cot(ka)$$

$$\Rightarrow -\frac{ka}{\beta\sin^2(ka)} - \cot(ka) = \cot(\delta_0)$$

$$\Rightarrow -\left[\frac{ka}{\beta\sin^2(ka)} + \cot(ka)\right] = \cot(\delta_0)$$

$$\therefore \cot(\delta_0) = -\left[\cot(ka) + \frac{ka}{\beta\sin^2(ka)}\right] with/\beta = \frac{2ma\alpha}{\hbar^2}$$

$$\therefore \delta_0 = -\cot^{-1}\left[\cot(ka) + \frac{ka}{\beta\sin^2(ka)}\right]$$

Problem 10.8 Check that Equation 10.65 satisfies Equation 10.52, by direct substitution. *Hint:* $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$.[13]

Note that, $\nabla^2 G(r) = \nabla\cdot[\nabla G(r)]$, then we observe:

$$\nabla G(r) = \nabla\left[-\frac{e^{ikr}}{4\pi r}\right] = -\frac{1}{4\pi}\nabla\left[\frac{e^{ikr}}{r}\right] = -\frac{1}{4\pi}\nabla\left[r^{-1}e^{ikr}\right]$$

$$= -\frac{1}{4\pi}\left[\left(\nabla r^{-1}\right)e^{ikr} + r^{-1}\left(\nabla e^{ikr}\right)\right]$$

$$\Rightarrow \nabla\cdot\left\{-\frac{1}{4\pi}\left[e^{ikr}\nabla\frac{1}{r} + \frac{1}{r}\nabla e^{ikr}\right]\right\} = -\frac{1}{4\pi}\nabla\cdot\left[e^{ikr}\nabla\frac{1}{r} + \frac{1}{r}\nabla e^{ikr}\right]$$

$$\Rightarrow -\frac{1}{4\pi}\left(\nabla e^{ikr}\cdot\nabla\frac{1}{r} + e^{ikr}(\nabla\cdot\nabla\frac{1}{r}) + (\nabla\frac{1}{r})\nabla e^{ikr} + \frac{1}{r}\cdot(\nabla\cdot\nabla e^{ikr})\right)$$

$$= -\frac{1}{4\pi}\left[2\left(\nabla\frac{1}{r}\right)\cdot\left(\nabla e^{ikr}\right) + e^{ikr}\left(\nabla^2\frac{1}{r}\right) + \frac{1}{r}\left(\nabla^2 e^{ikr}\right)\right]$$

$$= -\frac{1}{4\pi}\left[2\left(\nabla r^{-1}\right)\cdot\left(\nabla e^{ikr}\right) + e^{ikr}\left(\nabla^2\frac{1}{r}\right) + \frac{1}{r}\left(\nabla^2 e^{ikr}\right)\right].$$

$$\nabla f(r) = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\right)f(r) = \left(\frac{\partial f(r)}{\partial r}, 0, 0\right) = \frac{\partial f(r)}{\partial r}\hat{r}.$$

$$\nabla^2 f(r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f(r)}{\partial r}) + \frac{1}{r^2\sin\phi}\frac{\partial^2 f(r)}{\partial\theta^2} + \frac{1}{r^2\sin^2\phi}\frac{\partial}{\partial\phi}(sin\phi\frac{\partial f(r)}{\partial\phi})$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial^2 f(r)}{\partial rs}\right)$$

Through the above formula, we get the expression:

$$\nabla^2 G(\mathbf{r}) = -\frac{1}{4\pi}\left[2\left(\frac{\partial r^{-1}}{\partial r}\hat{r}\right)\cdot\left(\frac{\partial e^{ikr}}{\partial r}\hat{r}\right) + e^{ikr}\left(\nabla^2\frac{1}{r}\right) + \frac{1}{r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial e^{ikr}}{\partial r}\right)\right)\right]$$

$$= -\frac{1}{4\pi}\left[2\left(-r^{-2}\hat{r}\right)\cdot\left(ike^{ikr}\hat{r}\right) + e^{ikr}\left(-4\pi\delta^3(\mathbf{r})\right) + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^2 ike^{ikr}\right)\right]$$

$$= -\frac{1}{4\pi}\left[-\frac{2ik}{r^2}e^{ikr}\hat{r}\cdot\hat{r} - 4\pi e^{ikr}\delta^3(\mathbf{r}) + \frac{ik}{r^3}\frac{\partial}{\partial r}\left(r^2 e^{ikr}\right)\right]$$

$$= -\frac{1}{4\pi}\left[-\frac{2ik}{r^2}e^{ikr}(1) - 4\pi e^{ikr}\delta^3(\mathbf{r}) + \frac{ik}{r^3}\left(2re^{ikr} + ikr^2 e^{ikr}\right)\right]$$

$$= -\frac{1}{4\pi}\left[-\frac{2ik}{r^2}e^{ikr} - 4\pi e^{ikr}\delta^3(\mathbf{r}) + \frac{ik}{r^3}\left(2re^{ikr}\right) + \frac{ik}{r^3}(ikr^2e^{ikr})\right]$$

$$= -\frac{1}{4\pi}e^{ikr}\left[-\frac{2ik}{r^2} - 4\pi\delta^3(\mathbf{r}) + \frac{2ik}{r^2} + \frac{i^2k^2}{r}\right] = -\frac{1}{4\pi}e^{ikr}\left[-4\pi\delta^3(\mathbf{r}) + \frac{(-1)k^2}{r}\right]$$

$$= -\frac{1}{4\pi}e^{ikr}\left[-4\pi\delta^3(\mathbf{r})\right] - \frac{1}{4\pi}e^{ikr}\left[\frac{-k^2}{r}\right] = e^{ikr}\delta^3(\mathbf{r}) - k^2\left[-\frac{e^{ikr}}{4\pi r}\right]$$

$$= e^{ik\cdot 0}\delta^3(\mathbf{r}) - k^2G(\mathbf{r}) = 1\cdot\delta^3(\mathbf{r}) - k^2G(\mathbf{r}) = \delta^3(\mathbf{r}) - k^2G(\mathbf{r}).$$

$$\Rightarrow (\nabla^2 + k^2)G(\mathbf{r}) = \delta^3(\mathbf{r})$$

$$\Rightarrow \nabla^2G(\mathbf{r}) + k^2G(\mathbf{r}) = \delta^3(\mathbf{r})$$

$$\Rightarrow \delta^3(\mathbf{r}) - k^2G(\mathbf{r}) + k^2G(\mathbf{r}) = \delta^3(\mathbf{r})$$

$$\boxed{\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})\checkmark}$$

Problem 10.9 Show that the ground state of hydrogen (Equation 4.80) satisfies the integral form of the Schrödinger equation, for the appropriate V and E (note that E is *negative*, so $k = i\kappa$, where $\kappa \equiv \sqrt{-2mE}/\hbar$).

$$\psi_0(r) = 0, \quad k = i\kappa = \frac{i}{a(1)},$$

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}\cdot 1 = -\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}\cdot\frac{\frac{4\pi\varepsilon_0\hbar^2}{me^2}}{a} = -\frac{1}{r}\frac{\hbar^2}{ma}.$$

$$\Rightarrow \psi(r) = 0 - \frac{m}{2\pi\hbar^2}\int\frac{e^{i\frac{i}{a}|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r}-\mathbf{r_0}|}\left(-\frac{1}{r_0}\cdot\frac{\hbar^2}{ma}\right)\left[\frac{1}{\sqrt{\pi a^3}}e^{-\frac{r_0}{a}}\right]d^3\mathbf{r_0},$$

$$= \frac{m}{2\pi\hbar^2}\left(\frac{\hbar^2}{ma}\right)\cdot\frac{1}{\sqrt{\pi a^3}}\int\frac{e^{\frac{i^2}{a}|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r}-\mathbf{r_0}|}\frac{e^{-\frac{r_0}{a}}}{r_0}d^3\mathbf{r_0}.$$

$$\Rightarrow \psi(r) = \frac{1}{2\pi a\sqrt{\pi a^3}}\int\frac{e^{-\frac{|r-r_0|}{a}}}{|r-r_0|}\frac{1}{r_0}e^{-\frac{r_0}{a}}\,d^3r_0.$$

Law of cosines: $|r - r_0| = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta}$,

Differential element: $d^3r_0 = r_0^2\sin\theta\,dr_0\,d\theta\,d\phi$.

$$\Rightarrow \psi(r) = \frac{1}{2\pi a\sqrt{\pi a^3}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{e^{-\frac{\sqrt{r^2+r_0^2-2rr_0\cos\theta}}{a}}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}\cdot r_0}e^{-\frac{r_0}{a}}r_0^2\sin\theta\,dr_0\,d\theta\,d\phi.$$

$$= \frac{1}{2\pi a\sqrt{\pi a^3}} 2\pi \int_0^\infty r_0 e^{-\frac{r_0}{a}} \left[\int_0^\pi \frac{e^{-\frac{1}{a}\sqrt{r^2+r_0^2-2rr_0\cos\theta}}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} \sin\theta\, d\theta \right] dr_0.$$

$$u = -\frac{1}{a}\sqrt{r^2+r_0^2-2rr_0\cos\theta},$$

$$du = -\frac{1}{a} \cdot \frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} \cdot r \cdot r_0 \sin\theta\, d\theta$$

$$\Rightarrow -\frac{a}{r\cdot r_0} du = \frac{1}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} \sin\theta\, d\theta$$

$$\Rightarrow u(\pi) = -\frac{1}{a}\sqrt{r^2+r_0^2-2rr_0\cos\pi} = -\frac{1}{a}(r+r_0).$$

$$\Rightarrow u(0) = -\frac{1}{a}\sqrt{r^2+r_0^2-2rr_0\cos 0} = -\frac{1}{a}|r-r_0|.$$

$$\Rightarrow \psi(\mathbf{r}) = \frac{1}{a\sqrt{\pi a^3}} \int_0^\infty r_0 e^{-\frac{r_0}{a}} \left[\int_{-\frac{1}{a}|r-r_0|}^{-\frac{1}{a}(r+r_0)} e^u \left(-\frac{a}{rr_0}\right) du \right] dr_0$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \int_0^\infty e^{-\frac{r_0}{a}} \left[\int_{-\frac{1}{a}(r-r_0)}^{-\frac{1}{a}|r+r_0|} e^u du \right] dr_0$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \int_0^\infty e^{-\frac{r_0}{a}} \left[e^{-\frac{1}{a}(r+r_0)} - e^{-\frac{1}{a}|r-r_0|} \right] dr_0$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \int_0^\infty \left[e^{-\frac{r_0}{a}-\frac{1}{a}(r+r_0)} - e^{-\frac{r_0}{a}-\frac{1}{a}|r-r_0|} \right] dr_0$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \left[\int_0^\infty e^{-\frac{r_0}{a}-\frac{1}{a}(r+r_0)} dr_0 - \int_0^\infty e^{-\frac{r_0}{a}-\frac{1}{a}|r-r_0|} dr_0 \right]$$

$$|r-r_0| = \begin{cases} +(r-r_0), & r_0 \le r \\ -(r-r_0), & r_0 \ge r \end{cases}$$

$$\psi(r) = -\frac{1}{r\sqrt{\pi a^3}} \left[\int_0^\infty e^{-\frac{r_0}{a}-\frac{r}{a}-\frac{r_0}{a}} dr_0 - \int_0^r e^{-\frac{r_0}{a}-\frac{1}{a}(+(r-r_0))} dr_0 - \int_r^\infty e^{-\frac{r_0}{a}-\frac{1}{a}(-(r-r_0))} dr_0 \right]$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \left[\int_0^\infty e^{-\frac{2r_0}{a}} e^{-\frac{r}{a}} dr_0 - \int_0^r e^{-\frac{r}{a}} dr_0 - \int_r^\infty e^{-\frac{2r_0}{a}} e^{\frac{r}{a}} dr_0 \right]$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \left[e^{-\frac{r}{a}} \int_0^\infty e^{-\frac{2r_0}{a}} dr_0 - e^{-\frac{r}{a}} \int_0^r dr_0 - e^{\frac{r}{a}} \int_r^\infty e^{-\frac{2r_0}{a}} dr_0 \right]$$

$$= -\frac{1}{r\sqrt{\pi a^3}} \left[e^{-\frac{r}{a}} \left\{ -\frac{a}{2} e^{-\frac{2r_0}{a}} \right\} \Bigg|_0^\infty - e^{-\frac{r}{a}} r_0 \Bigg|_0^r - e^{\frac{r}{a}} \left\{ -\frac{a}{2} e^{-\frac{2r_0}{a}} \right\} \Bigg|_r^\infty \right]$$

$$= -\frac{1}{r\sqrt{\pi a^3}}\left[-\frac{a}{2}e^{-\frac{r}{a}}\{e^{-\infty} - e^0\} - e^{-\frac{r}{a}}\{r-0\} + \frac{a}{2}e^{\frac{r}{a}}\left(e^{-\infty} - e^{-\frac{2r}{a}}\right)\right]$$
$$= -\frac{1}{r\sqrt{\pi a^3}}\left[-\frac{a}{2}e^{-\frac{r}{a}}\{0-1\} - re^{-\frac{r}{a}} + \frac{a}{2}e^{\frac{r}{a}}\{0 - e^{-\frac{2r}{a}}\}\right]$$
$$= -\frac{1}{r\sqrt{\pi a^3}}\left[\frac{a}{2}e^{-\frac{r}{a}}\{1\} - re^{-\frac{r}{a}} + \frac{a}{2}e^{\frac{r}{a}}\left\{-e^{-\frac{2r}{a}}\right\}\right]$$
$$= -\frac{1}{r\sqrt{\pi a^3}}\left[\frac{a}{2}e^{-\frac{r}{a}} - re^{-\frac{r}{a}} - \frac{a}{2}e^{\frac{r}{a}-\frac{2r}{a}}\right]$$
$$= -\frac{1}{r\sqrt{\pi a^3}}\left[\frac{a}{2}e^{-\frac{r}{a}} - re^{-\frac{r}{a}} - \frac{a}{2}e^{-\frac{r}{a}}\right]$$
$$= \frac{1}{r\sqrt{\pi a^3}}\left[re^{-\frac{r}{a}}\right] = \frac{1}{\sqrt{\pi a^3}}e^{-\frac{r}{a}} = \psi(r).$$

Problem 10.10 Find the scattering amplitude, in the Born approximation, for soft-sphere scattering at arbitrary energy. Show that your formula reduces to Equation 10.82 in the low-energy limit.

Potential of a soft-sphere: $V(r) = \begin{cases} V_0, & r \le a \\ 0, & r > a \end{cases}$

$$f(\theta) \approx -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(kr)\,dr$$
$$= -\frac{2m}{\hbar^2\kappa}\left[\int_0^a rV_0\sin(kr)\,dr + \int_a^\infty r(0)\sin(kr)\,dr\right]$$
$$= -\frac{2mV_0}{\hbar^2\kappa}\int_0^a r\sin(kr)\,dr.$$

$$u = r \quad \Rightarrow \quad du = dr, \quad dv = \sin(kr) \quad \Rightarrow \quad v = -\frac{1}{\kappa}\cos(kr)$$

$$f(\theta) \approx -\frac{2mV_0}{\hbar^2\kappa}\left[\left(r\left(-\frac{1}{\kappa}\cos(kr)\right)\right)\Bigg|_0^a - \int_0^a\left(-\frac{1}{\kappa}\cos(kr)\right)dr\right]$$
$$= -\frac{2mV_0}{\hbar^2\kappa}\left[-\frac{1}{\kappa}\left(r\cos(kr)\right)\Bigg|_0^a + \frac{1}{\kappa}\int_0^a \cos(kr)dr\right]$$
$$= -\frac{2mV_0}{\hbar^2\kappa}\left[-\frac{1}{\kappa}\left(a\cos(ka) - 0\cdot\cos(k0)\right) + \frac{1}{\kappa}\left(\frac{1}{k}\sin(kr)\right)\Bigg|_0^a\right]$$
$$= -\frac{2mV_0}{\hbar^2\kappa}\left[-\frac{1}{\kappa}\left(a\cos(ka)\right) + \frac{1}{k^2}\left(\sin(ka) - \sin(k0)\right)\right]$$
$$= -\frac{2mV_0}{\hbar^2\kappa}\left[\frac{1}{\kappa}\left(-a\cos(ka)\cdot\frac{\kappa}{\kappa} + \frac{1}{\kappa^2}\sin(ka)\right)\right]$$

$$= -\frac{2mV_0}{\hbar^2\kappa} \cdot \frac{1}{\kappa^2}\left[-\kappa a\cos(ka) + \sin(ka)\right]$$

$$\therefore f(\theta) \approx -\frac{2mV_0}{\hbar^2\kappa^3}\left[\sin(ka) - \kappa a\cos(ka)\right]$$

$$\kappa = 2k\sin\left(\frac{\theta}{2}\right). \quad \text{Low-energy scattering:} \quad ka \ll 1.$$

$$\sin(ka) \approx ka - \frac{(ka)^3}{3!}, \quad \& \quad \cos(ka) \approx 1 - \frac{(ka)^2}{2!}.$$

$$f(\theta) \approx -\frac{2mV_0}{\hbar^2\kappa^3}\left[\kappa a - \frac{(\kappa a)^3}{3!} - \kappa a\left(1 - \frac{(\kappa a)^2}{2!}\right)\right]$$

$$= \frac{-2mV_0}{\hbar^2\kappa^3}\left[\kappa a - \frac{(\kappa a)^3}{6} - \kappa a + \frac{(\kappa a)^3}{2}\right] = -\frac{2mV_0}{\hbar^2\kappa^3}\left[-\frac{(\kappa a)^3}{6} + \frac{(\kappa a)^3}{2}\right]$$

$$= -\frac{2mV_0}{\hbar^2\kappa^3}\left[-\frac{a^3\kappa^3}{6} + \frac{a^3\kappa^3}{2}\right] = -\frac{2mV_0a^3}{\hbar^2}\left[-\frac{1}{6} + \frac{3}{6}\right]$$

$$= -\frac{2mV_0a^3}{\hbar^2}\left[\frac{1}{3}\right] = -\frac{2mV_0a^3}{3\hbar^2}$$

$$\therefore f(\theta) \approx -\frac{2mV_0a^3}{3\hbar^2} = -\frac{m}{2\pi\hbar^2}V_0\left(\frac{4}{3}\pi a^3\right)$$

Problem 10.11 Evaluate the integral in Equation 10.91, to confirm the expression on the right.

$$e^{+i\kappa r} = \cos(\kappa r) + i\sin(\kappa r) \quad \text{and} \quad e^{-i\kappa r} = \cos(\kappa r) - i\sin(\kappa r).$$

$$\Rightarrow \quad e^{+i\kappa r} - e^{-i\kappa r} = \cos(\kappa r) + i\sin(\kappa r) - \cos(\kappa r) + i\sin(\kappa r)$$

$$\Rightarrow \quad [e^{+i\kappa r} - e^{-i\kappa r}] = 2i\sin(\kappa r) \Rightarrow \frac{1}{2i}[e^{+i\kappa r} - e^{-i\kappa r}] = \sin(\kappa r).$$

$$\Rightarrow \int_0^\infty e^{-\mu r}\sin(\kappa r)\,dr = \int_0^\infty e^{-\mu r}\left\{\frac{1}{2i}[e^{+i\kappa r} - e^{-i\kappa r}]\right\}dr$$

$$= \frac{1}{2i}\int_0^\infty e^{-ur}\left[e^{+i\kappa r} - e^{-i\kappa r}\right]dr$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{-\mu r}e^{+i\kappa r} - e^{-\mu r}e^{-i\kappa r}\right]dr$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{-(\mu - i\kappa)r} - e^{-(\mu + i\kappa)r}\right]dr$$

$$u = -(\mu - i\kappa)r, \quad du = -(\mu - i\kappa)dr, \quad \Rightarrow \quad \frac{du}{-(\mu - i\kappa)} = dr;$$

$$u(\infty) = -(\mu - i\kappa) \cdot \infty = -\infty, \quad u(0) = -(\mu - i\kappa) \cdot 0 = 0.$$

$$v = -(\mu + i\kappa)r, \quad dv = -(\mu + i\kappa)dr, \quad \Rightarrow \quad \frac{dv}{-(\mu + i\kappa)} = dr;$$

$$v(\infty) = -(\mu + i\kappa) \cdot \infty = -\infty, \quad v(0) = -(\mu + i\kappa) \cdot 0 = 0.$$

$$= \frac{1}{2i} \left[\int_0^\infty e^{-(\mu - i\kappa)r}\, dr - \int_0^\infty e^{-(\mu + i\kappa)r}\, dr \right].$$

$$\Rightarrow \quad \frac{1}{2i} \left[\int_0^{-\infty} e^u \frac{du}{-(\mu - i\kappa)} - \int_0^{-\infty} e^v \frac{dv}{-(\mu + i\kappa)} \right]$$

$$= \frac{1}{2i} \left[-\frac{1}{(\mu - i\kappa)} \int_0^{-\infty} e^u\, du + \frac{1}{(\mu + i\kappa)} \int_0^{-\infty} e^v\, dv \right]$$

$$= \frac{1}{2i} \left[-\frac{1}{(\mu - i\kappa)} \left\{ e^u \right\}_0^{-\infty} + \frac{1}{(\mu + i\kappa)} \left\{ e^v \right\}_0^{-\infty} \right]$$

$$= \frac{1}{2i} \left[-\frac{1}{(\mu - i\kappa)} \left\{ e^{-\infty} - e^0 \right\} + \frac{1}{(\mu + i\kappa)} \left\{ e^{-\infty} - e^0 \right\} \right]$$

$$= \frac{1}{2i} \left[\frac{1}{(\mu - i\kappa)}(-1) + \frac{1}{(\mu + i\kappa)}(-1) \right]$$

$$= \frac{1}{2i} \left[\frac{1}{(\mu - i\kappa)} \cdot \frac{(\mu + i\kappa)}{(\mu + i\kappa)} - \frac{1}{(\mu + i\kappa)} \cdot \frac{(\mu - i\kappa)}{(\mu - i\kappa)} \right]$$

$$= \frac{1}{2i} \left[\frac{(\mu + i\kappa) - (\mu - i\kappa)}{(\mu - i\kappa)(\mu + i\kappa)} \right] = \frac{1}{2i} \left[\frac{\mu + i\kappa - \mu + i\kappa}{\mu^2 + \kappa^2} \right]$$

$$= \frac{1}{2i} \left[\frac{2i\kappa}{\mu^2 + \kappa^2} \right] = \frac{\kappa}{\mu^2 + \kappa^2}.$$

$$f(\theta) \approx -\frac{2m\beta}{\hbar^2 \kappa} \int_0^\infty e^{-\mu r} \sin(\kappa r)\, dr = -\frac{2m\beta}{\hbar^2} \frac{1}{\mu^2 + \kappa^2}$$

Problem 10.12 Calculate the total cross-section for scattering from a Yukawa potential, in the Born approximation. Express your answer as a function of E.

$$D(\theta) = \left| \frac{-\frac{2m\beta}{\hbar^2}}{\mu^2 + \kappa^2} \right|^2 = \left(\frac{2m\beta}{\hbar^2} \right)^2 \frac{1}{(\mu^2 + \kappa^2)^2}$$

$$D(\theta) = \left(\frac{2m\beta}{\hbar^2} \right)^2 \frac{1}{\left[\mu^2 + \left(2k \sin\left(\frac{\theta}{2}\right) \right)^2 \right]^2}$$

$$= \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\left[\mu^2 + \frac{\mu^2}{\mu^2}\cdot(2k)^2\sin^2\left(\frac{\theta}{2}\right)\right]^2}$$

$$= \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{[\mu^2]^2}\left[1 + \left(\frac{2k}{\mu}\right)^2\sin^2\left(\frac{\theta}{2}\right)\right]^2$$

$$\therefore D(\theta) = \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4}\frac{1}{\left[1 + \left(\frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right)\right)^2\right]^2}$$

$$\Rightarrow D(\theta) = \frac{d\sigma}{d\Omega} \implies \int D(\theta)d\Omega = \sigma$$

$$\sigma_{tot} = \int_0^{2\pi}\int_0^{\pi}\left\{\left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4}\frac{1}{\left[1 + \left(\frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right)\right)^2\right]^2}\right\}\sin\theta\, d\theta\, d\phi$$

$$= \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4}\int_0^{2\pi} d\phi \int_0^{\pi}\frac{1}{\left[1 + \left(\frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right)\right)^2\right]^2}\sin\theta\, d\theta$$

$$= \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4}(2\pi)\int_0^{\pi}\frac{1}{\left[1 + \left(\frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right)\right)^2\right]^2}\left\{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right\}d\theta$$

$$u = \frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right), \quad \& \quad \frac{\mu}{k}u = 2\sin\left(\frac{\theta}{2}\right),$$

$$du = \frac{2k}{\mu}\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}d\theta = \frac{k}{\mu}cos(\frac{\theta}{2})d\theta \quad \& \quad \frac{\mu}{k}du = \cos\left(\frac{\theta}{2}\right)d\theta.$$

Changing the limits as well yields:

$$u(\pi) = \frac{2k}{\mu}\sin\left(\frac{\pi}{2}\right) = \frac{2k}{\mu}\cdot 1 = \frac{2k}{\mu},$$

$$u(0) = \frac{2k}{\mu}\sin\left(\frac{0}{2}\right) = \frac{2k}{\mu}\cdot 0 = 0.$$

$$= 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4}\int_0^{\frac{2k}{\mu}}\frac{1}{[1+(u)^2]^2}\left\{\frac{\mu}{k}u\right\}\frac{\mu}{k}du$$

$$= 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2k^2}\int_0^{\frac{2k}{\mu}}\frac{u}{[1+u^2]^2}du = 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2k^2}\int_1^{1+\frac{4k^2}{\mu^2}}\frac{1}{t^2}\cdot\frac{1}{2}dt$$

$$= \pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2k^2}\left[-t^{-1}\right]_1^{1+\frac{4k^2}{\mu^2}}$$

$$= -\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2k^2}\left[\frac{1}{t}\right]_1^{1+\frac{4k^2}{\mu^2}}$$

$$= -\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2k^2}\left[\frac{1}{1+\frac{4k^2}{\mu^2}}-1\right]$$

$$= -\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2k^2}\left(\frac{1-(1+\frac{4k^2}{\mu^2})}{1+\frac{4k^2}{\mu^2}}\right)$$

$$t = 1+u^2, \quad dt = 2udu, \quad \Rightarrow \frac{1}{2}dt = udu.$$

$$t\left(\frac{2k}{\mu}\right) = 1+\left(\frac{2k}{\mu}\right)^2 = 1+\frac{4k^2}{\mu^2}, \quad \& \quad t(0) = 1+(0)^2 = 1.$$

$$= -\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\cdot\frac{1}{\mu^2k^2}\left[\frac{-\frac{4k^2}{\mu^2}}{1+\frac{4k^2}{\mu^2}}\right] = \pi\left(\frac{2m\beta}{\hbar^2}\right)^2\cdot\frac{1}{\mu^2k^2}\left[\frac{\frac{4k^2}{\mu^2}}{1+\frac{4k^2}{\mu^2}}\right]$$

$$= \pi\left(\frac{2m\beta}{\hbar^2}\right)^2\cdot\frac{4k^2}{\mu^2}\cdot\frac{1}{k^2\left(1+\frac{4k^2}{\mu^2}\right)} = \pi(\frac{2m\beta}{\hbar^2})^2\cdot\frac{2^2k^2}{\mu^2}\cdot\frac{1}{\mu^2k^2\left(1+\frac{4k^2}{\mu^2}\right)}$$

$$= \pi\left(\frac{2\cdot 2m\beta}{\mu\cdot\hbar^2}\right)^2\cdot\frac{1}{\mu^2\left(1+\frac{4k^2}{\mu^2}\right)} = \pi\cdot\left(\frac{4m\beta}{\mu\hbar^2}\right)^2\cdot\frac{1}{\mu^2+\mu^2\frac{4k^2}{\mu^2}}$$

$$\therefore \sigma_{\text{total}} = \pi\left(\frac{4m\beta}{\mu\hbar^2}\right)^2\cdot\frac{1}{\mu^2+\left(\frac{8mE}{\hbar^2}\right)}$$

Problem 10.13 For the potential in Problem 10.4,

(a) calculate $f(\theta)$, $D(\theta)$, and σ, in the low-energy Born approximation;

(b) calculate $f(\theta)$ for arbitrary energies, in the Born approximation;

(c) show that your results are consistent with the answer to Problem 10.4, in the appropriate regime.

(a) In the low energy approximation:

$$f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int \alpha\delta(r-a)d^3\mathbf{r}$$

$$= -\frac{m\alpha}{2\pi\hbar^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\delta(r-a)r^2\sin\theta\,drd\theta d\phi$$

$$= -\frac{m\alpha}{2\pi\hbar^2}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^{\infty}\delta(r-a)r^2\,dr$$

$$= \frac{-m\alpha}{2\pi\hbar^2}\left[\phi\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi}\left[r^2\right]_{r=a}$$

$$= \frac{m\alpha}{2\pi\hbar^2}\left[2\pi-0\right]\left[\cos\pi-\cos 0\right]\left[a^2\right]$$

$$= \frac{m\alpha}{2\pi\hbar^2}\left[2\pi\right]\left[-1-1\right]\left[a^2\right] = \frac{m\alpha}{\hbar^2}[-2]a^2$$

$$\therefore f(\theta,\phi) = -\frac{2m\alpha}{\hbar^2}a^2$$

$$D(\theta) = \left|-\frac{2m\alpha}{\hbar^2}a^2\right|^2 \quad \Rightarrow \quad D(\theta) = \left(\frac{2m\alpha}{\hbar^2}a^2\right)^2$$

$$D(\theta) = \frac{d\sigma}{d\Omega} \Rightarrow \int_0^{2\pi}\int_0^{\pi}D(\theta)\sin\theta\,d\theta d\phi = \sigma_{\text{total}}$$

$$\Rightarrow \sigma = \int_0^{2\pi}\int_0^{\pi}\left(\frac{2m\alpha}{\hbar^2}a^2\right)^2\sin\theta\,d\theta d\phi$$

$$= \left(\frac{2m\alpha}{\hbar^2}a^2\right)^2\left[2\pi-0\right]\left[-\cos\pi+\cos 0\right]$$

$$= \left(\frac{2m\alpha}{\hbar^2}a^2\right)^2\left[2\pi\right]\big[-(-1)+1\big] = \left(\frac{2m\alpha}{\hbar^2}a^2\right)^2 2^2\pi$$

$$\therefore \sigma = \left(\frac{2\cdot 2m\alpha}{\hbar^2}a^2\right)^2\pi = \pi\left(\frac{4m\alpha}{\hbar^2}a^2\right)^2$$

(b) Arbitrary energy levels – Eq. 10.88:

$$f(\theta) \approx -\frac{2m}{\hbar^2\kappa}\int_0^{\infty}rV(r)\sin(\kappa r)\,dr$$

$$= -\frac{2m}{\hbar^2\kappa}\int_0^{\infty}r\{a\delta(r-a)\}\sin(kr)\,dr$$

$$= -\frac{2ma\alpha}{\hbar^2\kappa}[r\sin(kr)]_{r=a} = -\frac{2ma\alpha}{\hbar^2\kappa}\big[a\sin(\kappa a)\big]\,.$$

$$(c) \quad \kappa = 2k\sin\left(\frac{\theta}{2}\right) \Rightarrow \kappa a = 2ka\sin\left(\frac{\theta}{2}\right) \Rightarrow \kappa a \ll 1.$$

$$f_b(\theta) \approx -\frac{2ma\alpha}{\hbar^2\kappa}\big[a(\kappa a)\big] = -\frac{ma\alpha}{\hbar^2}a^2 = f_a(\theta,\phi).$$

$$f_{10.4}(\theta) = -\frac{a\beta}{1+\beta} \approx -\frac{a\beta}{1} = -a\beta = -a\cdot\frac{2ma\alpha a}{\hbar^2} = \frac{2ma^2\alpha}{\hbar^2} = f_a(\theta,\phi).$$

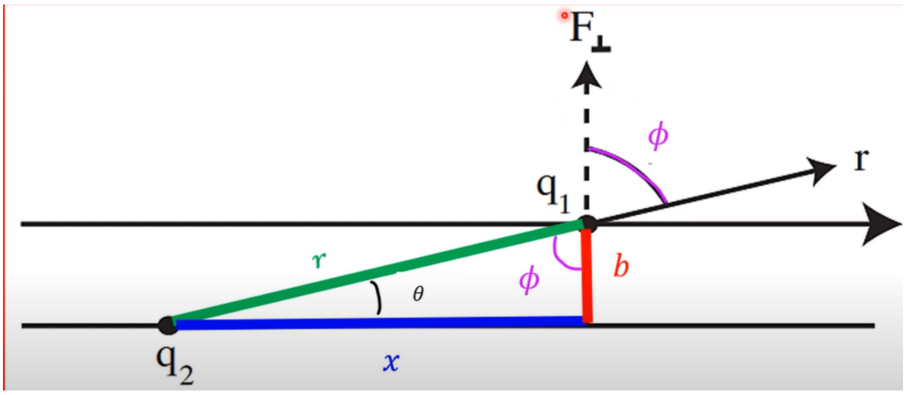
Problem 10.14 Calculate θ (as a function of the impact parameter) for Rutherford scattering, in the impulse approximation. Show that your result is consistent with the exact expression (Problem 10.1(a)), in the appropriate limit.

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\hat{r} \Rightarrow F_\perp = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}[\cos\phi]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\left[\frac{b}{r}\right] = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2b}{r^3}.$$

$$r^2 = x^2 + b^2 \Rightarrow r = \sqrt{x^2+b^2} = (x^2+b^2)^{1/2}$$

$$\Rightarrow r^3 = \left[(x^2+b^2)^{1/2}\right]^3 = (x^2+b^2)^{3/2}.$$



$$I_\perp = \int_{-\infty}^{+\infty}\{F_\perp\}[dt] = \int_{-\infty}^{+\infty}\left\{\frac{1}{4\pi\epsilon_0}\frac{q_1q_2b}{(x^2+b^2)^{3/2}}\right\}\left[\frac{dx}{v}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q_1q_2b}{v}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+b^2)^{3/2}}$$

$$= 2\frac{1}{4\pi\epsilon_0}\frac{q_1q_2b}{v}\int_0^{+\infty}\frac{dx}{(x^2+b^2)^{3/2}}$$

$$= \frac{1}{\frac{1}{2}}\frac{1}{4\pi\epsilon_0}\frac{q_1q_2b}{v}\int_0^{\frac{\pi}{2}}\frac{b\sec^2\theta\,d\theta}{(b^2\tan^2\theta+b^2)^{3/2}}.$$

$$x = b\tan\theta, \quad +\infty = b\tan\theta = \frac{b\sin\theta}{\cos\theta} \to \theta = \frac{\pi}{2},$$

$$x^2 = b^2\tan^2\theta, \quad \Rightarrow dx = b\sec^2\theta\,d\theta;$$

$$0 = b\tan\theta = \frac{b\sin\theta}{\cos\theta} \to \theta = 0.$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}\nu} \int_0^{\frac{\pi}{2}} \frac{b^2 \sec^2\theta \, d\theta}{\left(b^2[\tan^2\theta + 1]\right)^{3/2}}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}\nu} \int_0^{\frac{\pi}{2}} \frac{b^2 \sec^2\theta \, d\theta}{(b^2 \sec^2\theta)^{3/2}}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}\nu} \int_0^{\frac{\pi}{2}} \frac{b^2 \sec^2\theta \, d\theta}{\left((b \sec\theta)^2\right)^{3/2}}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}v} \int_0^{\frac{\pi}{2}} \frac{b^2 \sec^2\theta \, d\theta}{b^3 \sec^3\theta} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}v} \int_0^{\frac{\pi}{2}} \frac{d\theta}{b \sec\theta}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}vb} \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}vb} \left[\sin\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}vb} \left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}vb}$$

$$\theta \approx \tan^{-1}\left(\frac{I_\perp}{p}\right) = \tan^{-1}\left(\frac{\frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{\frac{1}{2}vb}}{mv}\right)$$

$$= \tan^{-1}\left(\frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{\frac{1}{2}mv^2 b}\right) = \tan^{-1}\left(\frac{q_1 q_2}{4\pi\epsilon_0 E b}\right)$$

$$\therefore \theta = \tan^{-1}\left(\frac{q_1 q_2}{4\pi\epsilon_0 E b}\right), \quad \text{or} \quad \tan\theta = \frac{q_1 q_2}{4\pi\epsilon_0 E b}.$$

Now in order to compare this with the results from Problem 10.1a, let's first solve for the impact parameter:

$$\tan\theta = \frac{q_1 q_2}{4\pi\epsilon_0 E b} \implies b = \frac{q_1 q_2}{4\pi\epsilon_0 E} \frac{1}{\tan\theta} \cdot 1$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0 E} \cot\theta \cdot \frac{2}{2} = \frac{q_1 q_2}{8\pi\epsilon_0 E}(2\cot\theta)$$

So in the comparison we observe that:

$$b = \frac{q_1 q_2}{8\pi\epsilon_0 E}(2\cot\theta) \quad \text{vs.} \quad b = \frac{q_1 q_2}{8\pi\epsilon_0 E} \cot\left(\frac{\theta}{2}\right).$$

It is now evident that we must show that:

$$2\cot\theta = \cot\left(\frac{\theta}{2}\right)$$

$$2\cot\theta = 2 \cdot \frac{\cos\theta}{\sin\theta} \approx 2 \cdot \frac{1}{\theta} = \frac{2}{\theta}$$

$$\cot\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \approx \frac{1}{\frac{\theta}{2}} = \frac{2}{\theta}$$

Problem 10.18 Use the one-dimensional Born approximation (Problem 10.17) to compute the transmission coefficient ($T = 1 - R$) for scattering from a delta function (Equation 2.117) and from a finite square well (Equation 2.148). Compare your results with the exact answers (Equations 2.144 and 2.172).

$$\therefore 2\cot\theta = \cot\left(\frac{\theta}{2}\right).$$

$$R_d \approx \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{+\infty} e^{2ikx} V(x)\,dx\right|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{+\infty} e^{2ikx}\{-\alpha\delta(x)\}\,dx\right|^2$$

$$= \left(\frac{m}{\hbar^2 k}\right)^2 \left|-\alpha\int_{-\infty}^{+\infty} e^{2ikx}\delta(x)\,dx\right|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \left|-\alpha e^{2ik\cdot 0}\right|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \alpha^2 = \frac{m^2}{(\hbar^2)^2 k^2}\alpha^2 = \frac{m\alpha^2}{2\hbar^2 E}$$

$$T_d = 1 - R_d = 1 - \frac{m\alpha^2}{2\hbar^2 E}$$

Từ phương trình 2.144 ta có

$$T_d = \frac{1}{1 + \frac{m\alpha^2}{2\hbar^2 E}} = \left[1 + \frac{m\alpha^2}{2\hbar^2 E}\right]^{-1} \approx 1 - \frac{m\alpha^2}{2\hbar^2 E'}$$

$$\text{provided that } \frac{m\alpha^2}{2\hbar^2 E} \ll 1 \implies \frac{m\alpha^2}{2\hbar^2} \ll E.$$

$$R_w \approx \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{+\infty} e^{2ikx} V(x)\,dx\right|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-a}^{+a} e^{2ikx}\{-V_0\}\,dx\right|^2$$

$$= \left(\frac{m}{\hbar^2 k}\right)^2 \left|-V_0\int_{-a}^{+a} e^{2ikx}\,dx\right|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \left|-V_0\left(\frac{e^{2ikx}}{2ik}\right)_{-a}^{+a}\right|^2$$

$$= \left(\frac{m}{\hbar^2 k}\right)^2 \left|-\frac{V_0}{k}\left(\frac{e^{2ika} - e^{-2ika}}{2i}\right)\right|^2 = \left(\frac{m}{\hbar^2 k}\right)^2 \left|-\frac{V_0}{k}\sin(2ka)\right|^2$$

$$= \frac{m^2 V_0^2}{(\hbar^2)^2 k^2 k^2}\sin^2(2ka) = \frac{V_0^2}{(2E)^2}\sin^2\left(\frac{2a}{\hbar}\sqrt{2mE}\right) = \left(\frac{V_0}{2E}\sin\frac{2a}{\hbar}\sqrt{2mE}\right)^2$$

$$\Rightarrow T_w = 1 - R_w = 1 - \left(\frac{V_0}{2E}\sin\left(\frac{2a}{\hbar}\sqrt{2mE}\right)\right)^2$$

Phương trình 2.172

$$T_w = \frac{1}{1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)}$$

$$\approx 1 - \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right).$$

$$\text{Provided that } V_0 \ll E, \text{ we see:}$$

$$T_w \approx 1 - \frac{V_0^2}{4E(E)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E)}\right) = 1 - \frac{V_0^2}{2^2E^2}\sin^2\left(\frac{2a}{\hbar}\sqrt{2mE}\right).$$

$$\therefore T_w = 1 - \left[\frac{V_0}{2E}\sin\left(\frac{2a}{\hbar}\sqrt{2mE}\right)\right]^2.$$

Problem 10.19 Prove the **optical theorem**, which relates the total cross-section to the imaginary part of the forward scattering amplitude:

$$\sigma = \frac{4\pi}{k}\mathrm{Im}\left[f(0)\right]. \tag{10.104}$$

Hint: Use Equations 10.47 and 10.48.

$$f(0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell)P_\ell(\cos 0) \text{ với } P_\ell(1) = 1$$

$$= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin(\delta_\ell) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\{\cos(\delta_\ell)+i\sin(\delta_\ell)\}\sin(\delta_\ell)$$

$$= \frac{1}{k}\sum_{\ell=0}^{\infty}\{(2\ell+1)\cos(\delta_\ell)\sin(\delta_\ell) + i(2\ell+1)\sin(\delta_\ell)\sin(\delta_\ell)\}$$

$$= \sum_{\ell=0}^{\infty}\left\{\frac{1}{k}(2\ell+1)\cos(\delta_\ell)\sin(\delta_\ell) + i\frac{1}{k}(2\ell+1)\sin^2(\delta_\ell)\right\}.$$

$$\mathrm{Re}[f(0)] = \sum_{\ell=0}^{\infty}\left(\frac{1}{k}(2\ell+1)\cos(\delta_\ell)\sin(\delta_\ell)\right),$$

$$\mathrm{Im}[f(0)] = \sum_{\ell=0}^{\infty}\left(\frac{1}{k}(2\ell+1)\sin^2(\delta_\ell)\right).$$

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell)$$

$$= \frac{4\pi}{k}\sum_{\ell=0}^{\infty}\frac{1}{k}(2\ell+1)\sin^2(\delta_\ell) = \frac{4\pi}{k}\mathrm{Im}[f(0)].$$

$$\therefore \sigma = \frac{4\pi}{k}\mathrm{Im}[f(0)].$$

Problem 10.20 Use the Born approximation to determine the total cross-section for scattering from a gaussian potential

$$V(\mathbf{r}) = V_0 e^{-\mu r^2/a^2}.$$

Express your answer in terms of the constants V_0, a, and m (the mass of the incident particle), and $k \equiv \sqrt{2mE}/\hbar$, where E is the incident energy.

$$f(\theta) \approx -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)sin(\kappa r)dr = -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV_0e^{-\frac{\mu}{a^2}r^2}sin(\kappa r)dr = -\frac{2mV_0}{\hbar^2\kappa}\int_0^\infty re^{-\frac{\mu}{a^2}r^2}sin(\kappa r)dr$$

$$\frac{d}{dr}\left(e^{-\frac{\mu}{a^2}r^2}\right) = e^{-\frac{\mu}{a^2}r^2}(-\frac{\mu}{a^2}2r) = -\frac{2\mu}{a^2}re^{-\frac{\mu}{a^2}r^2} \Rightarrow -\frac{a^2}{2\mu}\frac{d}{dr}\left(e^{-\frac{\mu}{a^2}r^2}\right) = re^{-\frac{\mu}{a^2}r^2}$$

$$\Rightarrow f(\theta) = -\frac{2mV_0}{\hbar^2\kappa}\int_0^\infty\left[-\frac{a^2}{2\mu}\frac{d}{dr}(e^{-\frac{\mu}{a^2}r^2})\right]\sin\kappa rdr = \frac{mV_0a^2}{\mu\hbar^2\kappa}\int_0^\infty\frac{d}{dr}(e^{-\frac{\mu}{a^2}r^2})\sin\kappa rdr$$

$$= \frac{mV_0a^2}{\mu\hbar^2\kappa}\left[\left(\sin(\kappa r)e^{-\frac{\mu}{a^2}r^2}\right)\Big|_0^\infty - \int_0^\infty e^{-\frac{\mu}{a^2}r^2}(\kappa\cos\kappa r)dr\right]$$

$$= \frac{mV_0a^2}{\mu\hbar^2\kappa}\left[\sin\infty.e^{-\infty} - sin(0)e^0 - \kappa\int_0^\infty e^{-\frac{\mu}{a^2}r^2}\cos\kappa rdr\right]$$

$$= -\frac{mV_0a^2}{\mu\hbar^2}\left[\frac{1}{2}\sqrt{\frac{\pi}{\frac{\mu}{a^2}}}e^{-\frac{\kappa^2}{4\frac{\mu}{a^2}}}\right] = -\frac{mV_0a^2}{\mu\hbar^2}\left[\frac{a}{2}\sqrt{\frac{\pi}{\mu}}e^{-\frac{\kappa^2a^2}{4\mu}}\right]$$

$$\Rightarrow f(\theta) = -\frac{mV_0a^3}{2\mu\hbar^2}\sqrt{\frac{\pi}{\mu}}e^{-\frac{\kappa^2a^2}{4\mu}} = -\frac{mV_0a^3}{2\mu\hbar^2}\sqrt{\frac{\pi}{\mu}}e^{-\frac{(2\kappa\sin\frac{\theta}{2})^2a^2}{4\mu}}$$

Tính tiết diện tán xạ toàn phần:

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 \Rightarrow d\sigma = \left|f(\theta)\right|^2d\Omega \Rightarrow d\sigma\int\left|f(\theta)\right|^2d\Omega$$

$$\sigma_{tot} = \int_0^{2\pi}\int_0^\pi\left|-\frac{mV_0a^3}{2\mu\hbar^2}\sqrt{\frac{\pi}{\mu}}e^{-\frac{(2\kappa\sin\frac{\theta}{2})^2a^2}{4\mu}}\right|^2\sin\theta d\theta d\phi = \int_0^{2\pi}d\phi\int_0^\pi\left|\frac{m^2V_0^2a^6}{2^2\mu^2\hbar^4}\frac{\pi}{\mu}e^{-\frac{\kappa^2a^2\sin^2\frac{\theta}{2}}{\mu}2}\right|\sin\theta d\theta$$

$$= \frac{m^2V_0^2a^6\pi}{2^2\mu^3\hbar^4}\cdot 2\pi\int_0^\pi e^{-\frac{2\kappa^2a^2\sin^2\frac{\theta}{2}}{\mu}}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$$= \frac{m^2V_0^2a^6\pi}{2\mu^3\hbar^4}\cdot 2\pi\int_0^\pi e^{-\frac{2k^2a^2\sin^2\left(\frac{\theta}{2}\right)}{\mu}}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)d\theta.$$

$$x = \sin^2\left(\frac{\theta}{2}\right) = \left[\sin\left(\frac{\theta}{2}\right)\right]^2 \quad \& \quad dx = 2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}d\theta,$$

$$x(\pi) = \sin^2\left(\frac{\pi}{2}\right) = (1)^2 = 1 \quad \& \quad x(0) = \sin^2\left(\frac{0}{2}\right) = (0)^2 = 0.$$

$$\Rightarrow \sigma_{\text{tot}} = \frac{m^2V_0^2a^6\pi^2}{\mu^3\hbar^4}\int_0^1 e^{-\frac{2k^2a^2}{\mu}x}dx.$$

$$t = -\frac{2k^2a^2}{\mu}x, \quad dt = -\frac{2k^2a^2}{\mu}dx, \quad \Rightarrow -\frac{\mu}{2k^2a^2}dt = dx,$$

$$t(1) = -\frac{2k^2a^2}{\mu}(1) = -\frac{2k^2a^2}{\mu} \quad \& \quad t(0) = -\frac{2k^2a^2}{\mu}(0) = 0.$$

$$\Rightarrow \sigma_{\text{tot}} = \frac{m^2V_0^2a^6\pi^2}{\mu^3\hbar^4}\int_0^{e^{\frac{-2k^2a^2}{\mu}t}} e^t\left\{-\frac{\mu}{2k^2a^2}dt\right\}.$$

$$= \frac{m^2 V_0^2 a^6 \pi^2}{\mu^3 \hbar^4} \left\{ \frac{\mu}{2k^2 a^2} \right\} \int_{-\frac{2k^2 a^2}{\mu}}^{0} e^t dt.$$

$$= \frac{m^2 V_0^2 a^4 \pi^2}{2\mu^2 \hbar^4 k^2} \left\{ e^t \right\}_{-\frac{2k^2 a^2}{\mu}}^{0} = \frac{m^2 V_0^2 a^4 \pi^2}{2\mu^2 \hbar^4 k^2} \left\{ e^0 - e^{-\frac{2k^2 a^2}{\mu}} \right\}.$$

$$\therefore \sigma_{\text{tot}} = \frac{\pi^2 m^2 V_0^2 a^4}{2\hbar^4 \mu^2 k^2} \left(1 - e^{-\frac{2k^2 a^2}{\mu}} \right).$$

2 CHAPTER 11: QUANTUM DYNAMICS

Problem 11.1 Why isn't it *trivial* to solve the time-dependent Schrödinger equation (11.1), in its dependence on t? After all, it's a first-order differential equation.

(a) How would you solve the equation

$$\frac{df}{dt} = k\, f$$

(for $f(t)$), if k were a constant?

(b) What if k is itself a function of t? (Here $k(t)$ and $f(t)$ might also depend on other variables, such as $\mathbf{r}$—it doesn't matter.)

(c) Why not do the same thing for the Schrödinger equation (with a time-dependent Hamiltonian)? To see that this doesn't work, consider the simple case

$$\hat{H}(t) = \begin{cases} \hat{H}_1, & (0 < t < \tau)\,, \\ \hat{H}_2, & (t > \tau)\,, \end{cases}$$

where $\hat{H}_1$ and $\hat{H}_2$ are themselves time-independent. If the solution in part (b) held for the Schrödinger equation, the wave function at time $t > \tau$ would be

$$\Psi(t) = e^{-i\left[\hat{H}_1\tau + \hat{H}_2(t-\tau)\right]/\hbar}\,\Psi(0),$$

but of course we could also write

$$\Psi(t) = e^{-i\hat{H}_2(t-\tau)/\hbar}\,\Psi(\tau) = e^{-i\hat{H}_2(t-\tau)/\hbar}e^{-i\hat{H}_1\tau/\hbar}\,\Psi(0).$$

Why are these generally *not* the same? [This is a subtle matter; if you want to pursue it further, see Problem 11.23.]

Nhắc lại phương trình Schrodinger phụ thuộc thời gian

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}(t)\Psi(t)$$

Nếu $\hat{H}$ không phụ thuộc t : $\Psi(t) = e^{-\frac{i\hat{H}t}{\hbar}}\psi_0$ Ta xét phương trình vi phân sau

$$\frac{df}{dt} = kf$$

k = const

$$\int_{f_0}^{f}\frac{df'}{f'} = \int_{t_0}^{t} k dt' \Rightarrow \ln\left(\frac{f}{f_0}\right) = k(t-t_0)$$

$$\rightarrow f = f_0 e^{k(t-t_0)}$$

$\mathbf{k} \in \mathbf{t}$

$$\int_{f_0}^{f} \frac{df'}{f'} = \int_{t_0}^{t} k(t')dt' \Rightarrow f = f_0 e^{\int_{t_0}^{t} k(t')dt'}$$

Bây giờ áp dụng k là toán tử $k \approx \hat{H}(t) = \begin{cases} \hat{H}_1, & 0 < t < \tau \\ \hat{H}_2, & t > \tau \end{cases}$

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\hat{H}(t)\psi(t)$$

$$\begin{aligned} \psi(t) &= \exp\left(\int_0^t -\frac{i}{\hbar}\hat{H}(t')dt'\right)\psi(0) \\ &= \exp\left(\int_0^\tau -\frac{i}{\hbar}\hat{H}_1 dt' + \int_\tau^t -\frac{i}{\hbar}\hat{H}_2 dt'\right)\psi(0) \\ &= \exp\left(-\frac{i}{\hbar}H_1\tau - \frac{i}{\hbar}H_2(t-\tau)\right)\psi(0) \end{aligned}$$

Ta có thể viết lại như sau

$$\psi(\tau) = \exp\left(-\frac{i}{\hbar}\int_0^\tau H_1 dt'\right)\psi(0) = \exp\left(-\frac{i}{\hbar}H_1\tau\right)\psi(0)$$

$$\psi(t) = \exp\left(-\frac{i}{\hbar}\int_\tau^t H_2 dt'\right)\psi(\tau) = \exp\left(-\frac{i}{\hbar}H_2(t-\tau)\right)\exp\left(-\frac{i}{\hbar}H_1\tau\right)\psi(0)$$

Problem 11.2 A hydrogen atom is placed in a (time-dependent) electric field $\mathbf{E} = E(t)\hat{k}$. Calculate all four matrix elements H'_{ij} of the perturbation $\hat{H}' = eEz$ between the ground state ($n = 1$) and the (quadruply degenerate) first excited states ($n = 2$). Also show that $H'_{ii} = 0$ for all five states. *Note:* There is only one integral to be done here, if you exploit oddness with respect to z; only one of the $n = 2$ states is "accessible" from the ground state by a perturbation of this form, and therefore the system functions as a two-state configuration—assuming transitions to higher excited states can be ignored.

$$H'_{ij} = \langle i| H' |j\rangle \ \ , H' = -qEz = eEz (e < 0) \ , H'_{ij} = eE \langle i| z |j\rangle$$

Ta có các trạng thái khả dĩ

$$|i\rangle = |100\rangle \ \ ; \ |j\rangle = |200\rangle \,, |210\rangle \,, |211\rangle \,, |21-1\rangle$$

$$|\psi_{nlm}\rangle = R_{nl}(r)Y_l^m(\theta,\phi)$$

$$|100\rangle = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$$|200\rangle = A.e^{-r/2a}(1-\frac{r}{2a})$$

$$|210\rangle = A.\frac{r}{a}e^{-r/2a}\cos\theta$$

$$|21-1\rangle = \frac{r}{a}e^{-r/2a}\sin\theta e^{-i\phi}$$

$$|211\rangle = \frac{r}{a}e^{-r/2a}\sin\theta e^{i\phi}$$

Thành phần Hamiltonian thứ ii là

$$H'_{ii} = eE\int_{-\infty}^{\infty} z|\psi_i|^2 dxdydz = 0$$

Thành phần Hamiltonian thứ ij là

$$H'_{ij} = eE\int_{-\infty}^{\infty} \psi_i^* z\psi_j dxdydz$$

Do tính chất hàm lẻ dưới tích phân nên ta có

$$H'_{1,2a} = H'_{1,2c} = H'_{1,2d} = 0$$

$$H'_{1,2b} = eE\int_{-\infty}^{\infty} z^2 f(r)d^3r = eE\int const\cdot\frac{r^2}{a}e^{-\frac{3r}{2a}}\cos^2\theta r^2 dr\sin\theta d\theta d\phi = const\int_0^{\infty} r^4 e^{-\frac{3r}{2a}} \approx \Gamma(5) = 4!$$

Problem 11.3 Solve Equation 11.17 for the case of a *time-independent* perturbation, assuming that $c_a(0) = 1$ and $c_b(0) = 0$. Check that $|c_a(t)|^2 + |c_b(t)|^2 = 1$. *Comment:* Ostensibly, this system oscillates between "pure ψ_a" and "some ψ_b." Doesn't this contradict my general assertion that no transitions occur for time-independent perturbations? No, but the reason is rather subtle: In this case ψ_a and ψ_b are not, and never were, eigenstates of the Hamiltonian—a measurement of the energy *never* yields E_a or E_b. In time-dependent perturbation theory we typically contemplate turning *on* the perturbation for a while, and then turning it *off* again, in order to examine the system. At the beginning, and at the end, ψ_a and ψ_b are eigenstates of the exact Hamiltonian, and only in this context does it make sense to say that the system underwent a transition from one to the other. For the present problem, then, assume that the perturbation was turned on at time $t = 0$, and off again at time T—this doesn't affect the *calculations*, but it allows for a more sensible interpretation of the result.

Problem 11.8 Consider a perturbation to a two-level system with matrix elements

$$H'_{ab} = H'_{ba} = \frac{\alpha}{\sqrt{\pi}\,\tau}\, e^{-(t/\tau)^2}, \qquad H'_{aa} = H'_{bb} = 0\,.$$

where τ and α are positive constants with the appropriate units.

(a) According to first-order perturbation theory, if the system starts off in the state $c_a = 1, c_b = 0$ at $t = -\infty$, what is the probability that it will be found in the state b at $t = \infty$?

(b) In the limit that $\tau \to 0$, $H'_{ab} = \alpha\,\delta(t)$. Compute the $\tau \to 0$ limit of your expression from part (a) and compare the result of Problem 11.4.

(c) Now consider the opposite extreme: $\omega_0\tau \gg 1$. What is the limit of your expression from part (a)? *Comment*: This is an example of the adiabatic theorem (Section 11.5.2).

$$H'_{ab} = H'_{ba} = \frac{\alpha}{\sqrt{\pi}\tau}e^{-(t/\tau)^2},\ H'_{aa} = H'_{bb} = 0$$

a/

$$c_a = 1, c_b = 0 \text{ tại } t = -\infty$$

Từ eq (11.17):

$$\dot{c_a} = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}c_b$$
$$\dot{c_b} = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a$$

Bậc 0: $c_a^{(0)}(t) = 0,\ c_b^{(b)} = 0$

Bậc 1: $\dot{c_a} = 0 \to c_a^{(1)} = 1 \to c_b^{(1)} = -\frac{i}{\hbar}H'_{ba}e^{-i\omega_0 t} = \frac{dc_b}{dt}$

$$\to \int_{-\infty}^{+\infty} dc_b = \int_{-\infty}^{+\infty} -\frac{i}{\hbar}\frac{\alpha}{\sqrt{\pi}\tau}e^{-(t/\tau)^2}e^{i\omega t}dt = \frac{-i\alpha}{\hbar\sqrt{\pi}\tau}\int_{-\infty}^{+\infty} e^{-(t/\tau)^2+i\omega t}$$

Biến đổi thành phần mũ:

$$-\left(\frac{t}{\tau}\right)^2 + i\omega t = -\left(\frac{t}{\tau} - i\frac{\omega_0\tau}{2}\right)^2 - \left(\frac{\omega_0}{2}\right)^2$$

Đặt $u = \frac{t}{\tau}$ Từ đó, ta tính được:

$$c_b^{(1)}(\infty) - c_(-\infty) = c_b(\infty) - 0 = \frac{-i\alpha}{\hbar\sqrt{\pi}\tau}e^{-\left(\omega_0\tau/2\right)^2}\tau\int_{-\infty}^{+\infty} e^{-u^2}du = \frac{-i\alpha}{\hbar\sqrt{\pi}}e^{-\left(\omega_0\tau/2\right)^2}\sqrt{\pi} = \frac{-i\alpha}{\hbar}e^{-\left(\omega_0\tau/2\right)^2/4}$$

$$P_{a\to b} \approx |c_b^{(1)}|^2 = \left(\frac{\alpha}{\hbar}\right)^2 \exp\left[\frac{-1}{2}(\omega_0\tau)^2\right]$$

b/ $\tau \to \infty$

$$P_{a\to b} \approx \left(\frac{\alpha}{\hbar}\right)^2$$

Ý nghĩa: Bật nhiễu loạn lên rất chậm $\to$ nó chính là biến đổi đoạn nhiệt!

Problem 11.9 The first term in Equation 11.32 comes from the $e^{i\omega t}/2$ part of $\cos(\omega t)$, and the second from $e^{-i\omega t}/2$. Thus dropping the first term is formally equivalent to writing $\hat{H}' = (V/2)\, e^{-i\omega t}$, which is to say,

$$H'_{ba} = \frac{V_{ba}}{2} e^{-i\omega t}, \quad H'_{ab} = \frac{V_{ab}}{2} e^{i\omega t}. \tag{11.36}$$

(The latter is required to make the Hamiltonian matrix hermitian—or, if you prefer, to pick out the dominant term in the formula analogous to Equation 11.32 for $c_a(t)$.) Rabi noticed that if you make this so-called **rotating wave approximation** at the *beginning* of the calculation, Equation 11.17 can be solved exactly, with no need for perturbation theory, and no assumption about the strength of the field.

(a) Solve Equation 11.17 in the rotating wave approximation (Equation 11.36), for the usual initial conditions: $c_a(0) = 1$, $c_b(0) = 0$. Express your results ($c_a(t)$ and $c_b(t)$) in terms of the **Rabi flopping frequency**,

$$\omega_r \equiv \frac{1}{2}\sqrt{(\omega - \omega_0)^2 + (|V_{ab}|/\hbar)^2}. \tag{11.37}$$

(b) Determine the transition probability, $P_{a\to b}(t)$, and show that it never exceeds 1. Confirm that $|c_a(t)|^2 + |c_b(t)|^2 = 1$.

(c) Check that $P_{a\to b}(t)$ reduces to the perturbation theory result (Equation 11.35) when the perturbation is "small," and state precisely what small *means* in this context, as a constraint on V.

(d) At what time does the system first return to its initial state?

$$H' = \frac{V}{2}e^{-i\omega t} \to H'_{ab} = \frac{V_{ab}}{2}e^{i\omega t},\ H'_{ba} = \frac{V_{ba}}{2}e^{-i\omega t}$$

Điều kiện biên: $\begin{cases} c_a(0) = 1 \\ c_b(0) = 0 \end{cases}$ Từ eq (11.17):

$$\dot{c_a} = -\frac{i}{\hbar}\frac{V_{ab}}{2}e^{i(\omega-\omega_0)t}c_b$$
$$\dot{c_b} = -\frac{i}{\hbar}\frac{V_{ba}}{2}e^{-i(\omega-\omega_0)t}c_a$$

Lấy đạo hàm bậc 2 theo t của c_b, ta được:

$$\begin{aligned}\ddot{c_b} &= -i\frac{V_ba}{2\hbar}\left[i(\omega_0-\omega)e^{i(\omega-\omega_0)t}c_a + e^{i(\omega_0-\omega)t}\right] \\ &= i(\omega_0-\omega)\left[-i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}c_a\right] - i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}\left[-i\frac{V_{ba}}{2\hbar}e^{-i(\omega_0-\omega)t}c_b\right] \\ &= i(\omega_0-\omega)\dot{c_b} - \frac{|V_{ab}|^2}{(2\hbar)^2}c_b\end{aligned}$$

$$\to \ddot{c_b} + i(\omega-\omega_0)\dot{c_b} + \frac{|V_{ab}|^2}{(2\hbar)^2}c_b = 0$$

Phương trình đặc trưng có dạng:

$$m^2 + i(\omega - \omega_0)m + \frac{|V_{ab}|^2}{4\hbar^2} = 0$$

$$\Delta = -(\omega - \omega_0)^2 - \frac{|V_{ab}|^2}{\hbar^2} = -4\omega_r^2 \rightarrow m = \frac{1}{2}\left[-i\left(\omega - \omega_0\right) \pm \sqrt{-4\omega_r^2}\right] = i\left[-\frac{(\omega - \omega_0)}{2} \pm \omega_r\right]$$

$$\rightarrow c_b = e^{-i(\omega-\omega_0)t/2}\left(C_1 e^{i\omega_r t} + C_2 e^{-i\omega_r t}\right)$$

$$= e^{-i(\omega-\omega_0)t/2}\left(A\cos(\omega_r t) + B\sin(\omega_r t)\right)$$

$$c_b(0) = 0 \rightarrow A = 0$$

$$c_b(t) = Be^{-i(\omega-\omega_0)t/2}\sin(\omega_r t)$$

$$\rightarrow \dot{c}_b = B\left[i\left(\frac{\omega - \omega_0}{2}\right)e^{-i(\omega-\omega_0)t/2}\sin(\omega_r t) + \omega_r t e^{-i(\omega-\omega_0)t/2}\cos(\omega_r t)\right]$$

$$\rightarrow c_a = i\frac{2\hbar}{V_{ba}}e^{-i(\omega-\omega_0)t/2}B\left[i\left(\frac{\omega - \omega_0}{2}\right)\sin(\omega_r t) + \omega_r t\cos(\omega_r t)\right]$$

$$c_a(0) = 1 \rightarrow 1 = i\frac{2\hbar}{V_{ba}}B\omega_r \rightarrow B = \frac{-iV_{ba}}{2\hbar\omega_r}$$

$$P_{a\rightarrow b} = |c_b|^2 = |B|^2\sin^2(\omega_r t) = \frac{|V_{ba}|^2}{4\hbar^2\omega_r^2}\sin^2(\omega_r t)$$

$$|c_a|^2 + |c_b|^2 = 1$$

c/

Khi nhiễu loạn rất nhỏ $|V_{ab}|^2 \ll \hbar^2(\omega - \omega_0)^2 \rightarrow \omega_r \approx \frac{1}{2}|\omega - \omega_0|$

$$P_{a\rightarrow b} \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2\left(\frac{\omega - \omega_0}{2}t\right)}{(\omega - \omega_0)^2}$$

Problem 11.13 Calculate the lifetime (in seconds) for each of the four $n = 2$ states of hydrogen. *Hint:* You'll need to evaluate matrix elements of the form $\langle\psi_{100}\,|x|\,\psi_{200}\rangle$, $\langle\psi_{100}\,|y|\,\psi_{211}\rangle$, and so on. Remember that $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. Most of these integrals are zero, so inspect them closely before you start calculating. *Answer:* 1.60×10^{-9} seconds for all except ψ_{200}, which is infinite.

$$\tau = \frac{1}{A} = \frac{1}{\omega_0^3|\mathbf{p}|^2}\frac{3\pi\epsilon_0\hbar c^3}{\omega_0^3|\mathbf{p}|^2}$$

$$\tau = \frac{3\pi\epsilon_0\hbar c^3}{\omega_0^3 q^2|\langle\psi_b|\mathbf{r}|\psi_a\rangle|^2}$$

$$\langle\psi_b|\mathbf{r}|\psi_a\rangle = \langle\psi_b|x|\psi_a\rangle\hat{\mathbf{i}} + \langle\psi_b|y|\psi_a\rangle\hat{\mathbf{j}} + \langle\psi_b|z|\psi_a\rangle\hat{\mathbf{k}}$$

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}}, \quad \psi_{200} = \frac{1}{\sqrt{8\pi a^3}} \left(1 - \frac{r}{2a}\right) e^{-\frac{r}{2a}},$$

$$\psi_{210} = \frac{1}{\sqrt{32\pi a^3}} \frac{r}{a} e^{-\frac{r}{2a}} \cos\theta, \quad \psi_{21\pm1} = \pm \frac{1}{\sqrt{64\pi a^3}} \frac{r}{a} e^{-\frac{r}{2a}} \sin\theta\, e^{\pm i\phi}.$$

$$\psi_{100}\psi_{200} = \frac{1}{2\sqrt{2}\pi a^3} \left(1 - \frac{r}{2a}\right) e^{-\frac{3}{2a} r},$$

$$= \frac{1}{2\sqrt{2}\pi a^3} \left(1 - \frac{\sqrt{x^2+y^2+z^2}}{2a}\right) e^{-\frac{3}{2a}\sqrt{x^2+y^2+z^2}} \quad \text{(even in } x,\, y,\, z\text{)},$$

$$\psi_{100}\psi_{210} = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}} \cdot \frac{1}{\sqrt{32\pi a^3}} \frac{r}{a} e^{-\frac{r}{2a}} \cos\theta,$$

$$= \frac{1}{4\sqrt{2}\pi a^3} \frac{r}{a} e^{-\frac{3}{2a} r} \cos\theta = \frac{z}{4\sqrt{2}\pi a^4} e^{-\frac{3}{2a}\sqrt{x^2+y^2+z^2}} \quad \text{(even in } x,\, y\text{)}.$$

$$\psi_{100}\psi_{21\pm1} = \frac{1}{\sqrt{\pi a^3}} e^{-\frac{r}{a}} \cdot \frac{1}{\sqrt{64\pi a^3}} \frac{r}{a} e^{-\frac{r}{2a}} \sin\theta e^{\pm i\phi}$$

$$= \pm \frac{1}{8\pi a^4} e^{-\frac{3}{2a} r} \{r \sin\theta e^{\pm i\phi}\} = \pm \frac{1}{8\pi a^4} e^{-\frac{3}{2a} r} \{x \pm iy\} \to \text{(odd in } x\text{)}, \text{(odd in } y\text{)}.$$

$$\langle \psi_{100} | x | \psi_{200} \rangle = 0, \quad \langle \psi_{100} | y | \psi_{200} \rangle = 0, \quad \langle \psi_{100} | x | \psi_{210} \rangle = 0, \quad \langle \psi_{100} | y | \psi_{210} \rangle = 0.$$

$$\langle \psi_{100} | x | \psi_{21\pm1} \rangle = \iiint_{\text{All Space}} \psi_{100}^* x \psi_{21\pm1} dV = \iiint_{\text{All Space}} \{x\}\{\psi_{100}\psi_{21\pm1}\} dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \Big\{ r\cos\phi \sin\theta \Big\} \Big\{ \pm \frac{1}{8\pi a^4} r e^{-\frac{3}{2a} r} \sin\theta e^{\pm i\phi} \Big\} \Big\{ r^2 \sin\theta dr d\theta d\phi \Big\}$$

$$= \pm \frac{1}{8\pi a^4} \int_0^{2\pi} \Big\{ \cos^2\phi \pm i \cos\phi \sin\phi \Big\} d\phi \int_0^{\pi} \sin^3\theta d\theta \int_0^{\infty} r^4 e^{-\frac{3}{2a} r} dr.$$

$$= \mp \frac{1}{8\pi a^4} \Big[\pi \pm i \cdot 0 \Big] \left[\frac{4}{3} \cdot 4! \cdot \left(\frac{2a}{3}\right)^5 \right] = \mp \frac{\pi}{8\pi a^4} \left[\frac{4}{3} \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \frac{2^5 a^5}{3^5} \right]$$

$$= \mp \frac{1}{a^4} \left[2^2 \cdot \frac{2^5 a^5}{3^5} \right] = \mp \frac{2^7}{3^5} a.$$

$$\langle \psi_{100} | y | \psi_{21\pm1} \rangle = \iiint_{\text{All Space}} \psi_{100}^* y \psi_{21\pm1} dV = \iiint_{\text{All Space}} \{y\}\{\psi_{100}\psi_{21\pm1}\} dV.$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \Big\{ r\sin\phi \sin\theta \Big\} \Big\{ \pm \frac{1}{8\pi a^4} r e^{-\frac{3}{2a} r} \sin\theta e^{\pm i\phi} \Big\} \Big\{ r^2 \sin\theta dr d\theta d\phi \Big\}.$$

$$= \pm \frac{1}{8\pi a^4} \int_0^{2\pi} \Big\{ \sin\phi \cos\phi \pm i \sin^2\phi \Big\} d\phi \int_0^{\pi} \sin^3\theta d\theta \int_0^{\infty} r^4 e^{-\frac{3}{2a} r} dr.$$

$$= \mp\frac{1}{8\pi a^4}\left[0 \pm i\cdot\pi\right]\left[\frac{4}{3}\cdot 4!\cdot\left(\frac{2a}{3}\right)^5\right].$$

$$= \mp\frac{1}{8\pi a^4}\left[\pm i\pi\right]\left[\frac{4}{3}\cdot 4\cdot 3\cdot 2\cdot 1\cdot\frac{2^5a^5}{3^5}\right].$$

$$= \mp\frac{1}{a^4}[\pm i]\left[2^2\cdot\frac{2^5a^5}{3^5}\right] = -i\frac{2^7}{3^5}a.$$

$$\langle\psi_{100}|\mathbf{r}|\psi_{200}\rangle = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}} = 0.$$

$$\langle\psi_{100}|\mathbf{r}|\psi_{210}\rangle = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \frac{2^8}{3^5\sqrt{2}}a\hat{\mathbf{k}} = \frac{2^8}{3^5\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}a\hat{\mathbf{k}} = \frac{2^7\sqrt{2}}{3^5}a\hat{\mathbf{k}}.$$

$$\langle\psi_{100}|\mathbf{r}|\psi_{21\pm1}\rangle = \pm\frac{2^7}{3^5}a\hat{\mathbf{i}} - i\frac{2^7}{3^5}a\hat{\mathbf{j}} + 0\hat{\mathbf{k}} = \frac{2^7}{3^5}a(\mp\hat{\mathbf{i}} - i\hat{\mathbf{j}}).$$

$$|\langle\psi_{100}|\mathbf{r}|\psi_{200}\rangle|^2 = |0|^2 = 0.$$

$$|\langle\psi_{100}|\mathbf{r}|\psi_{210}\rangle|^2 = \left|\frac{2^7\sqrt{2}}{3^5}a\hat{\mathbf{k}}\right|^2 = \frac{2^{15}}{3^{10}}a^2.$$

$$|\langle\psi_{100}|\mathbf{r}|\psi_{21\pm1}\rangle|^2 = \left|\frac{2^7}{3^5}a(\mp\hat{\mathbf{i}} - i\hat{\mathbf{j}})\right|^2 = \frac{2^{14}}{3^{10}}a^2\left[(\mp1)^2 + (-i)i\right] = \frac{2^{15}}{3^{10}}a^2.$$

For all three of the $\ell = 1$ states, we have:

$$|\mathbf{p}|^2 = q^2\frac{2^{15}}{3^{10}}a^2 = q^2a^2\frac{2^{15}}{3^{10}}.$$

So the lifetime of these states, with $q = e$, we have:

$$\tau_{\ell=0} = \frac{3\pi\epsilon_0\hbar c^3}{\omega_0^3e^2\cdot 0} = \frac{3\pi\epsilon_0\hbar c^3}{0} = \infty;$$

$$\tau_{\ell=1} = \frac{3\pi\epsilon_0\hbar c^3}{\omega_0^3e^2a^2\frac{2^{15}}{3^{10}}} = \frac{3\cdot 3^{10}\pi\epsilon_0\hbar c^3}{2^{15}\omega_0^3e^2a^2} = \frac{3^{11}\pi\epsilon_0\hbar c^3}{2^{15}\omega_0^3e^2a^2}.$$

Transition $|2XX\rangle \to |100\rangle$:

$$\omega_0 = \frac{E_2 - E_1}{\hbar} = \frac{\frac{E_1}{4} - E_1}{\hbar} = \frac{E_1}{\hbar}\left(\frac{1}{4} - 1\right) = \frac{E_1}{\hbar}\left(-\frac{3}{4}\right) = -\frac{3E_1}{4\hbar}.$$

$$\implies \tau_{\ell=1} = \frac{3^{11}\pi\epsilon_0\hbar c^3}{2^{15}\left(-\frac{3E_1}{2\hbar}\right)^3e^2a^2} = \frac{3^{11}\pi\epsilon_0\hbar c^3}{2^{15}\left(-\frac{3}{2}\frac{E_1}{\hbar}\right)^3e^2a^2}.$$

$$= \frac{3^{11-3}\pi\epsilon_0\hbar^4c^3}{2^{15-6}\left(-E_1^3\right)e^2a^2} = \frac{3^8\pi\epsilon_0\hbar^4c^3}{2^9\left(-E_1\right)^3e^2a^2}.$$

$$= \frac{3^8 \pi \epsilon_0 \hbar^2 c^3 m_e}{2^8 E_1^2 e^2} \quad \left(\text{since } a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \text{ for Bohr Radius}\right).$$

$$= \frac{3^8 c^3 m_e a m_e}{2^8 E_1^2 2^2} = \frac{3^8 a c^3 m_e^2}{2^{8+2} E_1^2} = \frac{3^8 a}{2^{10} c}\left(\frac{m_e c^2}{E_1}\right)^2.$$

$$\therefore \tau_{\ell=1} = \frac{3^8\left(0.529 \times 10^{-10}\right)}{2^{10}\left(3.00 \times 10^8\right)}\left(\frac{0.511 \times 10^6}{13.6}\right)^2 = 1.595 \times 10^{-9}\,\text{s}.$$

Problem 11.14 From the commutators of L_z with x, y, and z (Equation 4.122):

$$\left[L_z, x\right] = i\hbar y, \quad \left[L_z, y\right] = -i\hbar x, \quad \left[L_z, z\right] = 0, \tag{11.77}$$

obtain the selection rule for Δm and Equation 11.76. *Hint:* Sandwich each commutator between $\langle n'\ell'm'|$ and $|n\ell m\rangle$.

Solution:

$$\Longrightarrow \langle n'\ell'm'|[[L_z, x]]|n\ell m\rangle = \langle n'\ell'm'|L_z x - x L_z|n\ell m\rangle$$

$$= \langle n'\ell'm'|L_z x|n\ell m\rangle - \langle n'\ell'm'|x L_z|n\ell m\rangle$$

$$= \hbar m'\langle n'\ell'm'|x|n\ell m\rangle - \hbar m\langle n'\ell'm'|x|n\ell m\rangle$$

$$= (\hbar m' - \hbar m)\langle n'\ell'm'|x|n\ell m\rangle = \hbar(m' - m)\langle n'\ell'm'|x|n\ell m\rangle.$$

$$\Longrightarrow \langle n'\ell'm'|[[L_z, x]]|n\ell m\rangle = \langle n'\ell'm'|i\hbar y|n\ell m\rangle$$

$$= i\hbar\langle n'\ell'm'|y|n\ell m\rangle.$$

$$\Longrightarrow \langle n'\ell'm'|[[L_z, x]]|n\ell m\rangle = \begin{cases} \hbar(m' - m)\langle n'\ell'm'|x|n\ell m\rangle, \\ i\hbar\langle n'\ell'm'|y|n\ell m\rangle. \end{cases}$$

$$\Longrightarrow \langle n'\ell'm'|[[L_z, y]]|n\ell m\rangle = \langle n'\ell'm'|L_z y - y L_z|n\ell m\rangle$$

$$= \hbar m'\langle n'\ell'm'|y|n\ell m\rangle - \hbar m\langle n'\ell'm'|y|n\ell m\rangle$$

$$= \hbar(m' - m)\langle n'\ell'm'|y|n\ell m\rangle.$$

$$\Longrightarrow \langle n'\ell'm'|[[L_z, y]]|n\ell m\rangle = \langle n'\ell'm'| - i\hbar x|n\ell m\rangle$$

$$= -i\hbar\langle n'\ell'm'|x|n\ell m\rangle.$$

$$\implies \langle n'\ell'm'|[[L_z, y]]|n\ell m\rangle = \begin{cases} \hbar(m'-m)\langle n'\ell'm'|y|n\ell m\rangle, \\ -i\hbar\langle n'\ell'm'|x|n\ell m\rangle. \end{cases}$$

$$\implies \langle n'\ell'm'|[[L_z, z]]|n\ell m\rangle = \langle n'\ell'm'|L_z z - zL_z|n\ell m\rangle$$

$$= \hbar m'\langle n'\ell'm'|z|n\ell m\rangle - \hbar m\langle n'\ell'm'|z|n\ell m\rangle$$

$$= \hbar(m'-m)\langle n'\ell'm'|z|n\ell m\rangle.$$

$$= \hbar(m'-m)\langle n'\ell'm'|z|n\ell m\rangle.$$

$$\implies \langle n'\ell'm'|[[L_z, z]]|n\ell m\rangle = \langle n'\ell'm'|0|n\ell m\rangle = 0.$$

$$\implies \langle n'\ell'm'|[[L_z, z]]|n\ell m\rangle = \begin{cases} \hbar(m'-m)\langle n'\ell'm'|z|n\ell m\rangle, \\ 0. \end{cases}$$

$$\begin{cases} \hbar(m'-m)\langle n'\ell'm'|x|n\ell m\rangle = i\hbar\langle n'\ell'm'|y|n\ell m\rangle, \\ \hbar(m'-m)\langle n'\ell'm'|y|n\ell m\rangle = -i\hbar\langle n'\ell'm'|x|n\ell m\rangle, \\ \hbar(m'-m)\langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

$$\implies \begin{cases} (m'-m)\langle n'\ell'm'|x|n\ell m\rangle = i\langle n'\ell'm'|y|n\ell m\rangle, \\ (m'-m)\langle n'\ell'm'|y|n\ell m\rangle = -i\langle n'\ell'm'|x|n\ell m\rangle, \\ (m'-m)\langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

The first is if $m' = m,$ then we see:

$$\begin{cases} (m-m)\langle n'\ell'm'|x|n\ell m\rangle = i\langle n'\ell'm'|y|n\ell m\rangle, \\ (m-m)\langle n'\ell'm'|y|n\ell m\rangle = -i\langle n'\ell'm'|x|n\ell m\rangle, \\ (m-m)\langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

$$\implies \begin{cases} 0 = i\langle n'\ell'm'|y|n\ell m\rangle, \\ 0 = -i\langle n'\ell'm'|x|n\ell m\rangle, \\ 0 = 0 \end{cases}$$

$$\implies \begin{cases} 0 = \langle n'\ell'm'|y|n\ell m\rangle, \\ 0 = \langle n'\ell'm'|x|n\ell m\rangle, \\ 0 = 0. \end{cases}$$

The second is if $m' = m \pm 1$, then we see:

$$\begin{cases} (m \pm 1 - m)\langle n'\ell'm'|x|n\ell m\rangle = i\langle n'\ell'm'|y|n\ell m\rangle, \\ (m \pm 1 - m)\langle n'\ell'm'|y|n\ell m\rangle = -i\langle n'\ell'm'|x|n\ell m\rangle, \\ (m \pm 1 - m)\langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

$$\begin{cases} (\pm 1)\langle n'\ell'm'|x|n\ell m\rangle = i\langle n'\ell'm'|y|n\ell m\rangle, \\ (\pm 1)\langle n'\ell'm'|y|n\ell m\rangle = -i\langle n'\ell'm'|x|n\ell m\rangle, \\ (\pm 1)\langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

$$\implies \begin{cases} \langle n'\ell'm'|x|n\ell m\rangle = \pm i\langle n'\ell'm'|y|n\ell m\rangle, \\ \langle n'\ell'm'|y|n\ell m\rangle = \pm i\langle n'\ell'm'|x|n\ell m\rangle, \\ \langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

For all other cases we observe:

$$(m' - m)\langle n'\ell'm'|x|n\ell m\rangle = i\langle n'\ell'm'|y|n\ell m\rangle,$$

$$(m' - m) \cdot (m' - m)\langle n'\ell'm'|x|n\ell m\rangle = (m' - m) \cdot i\langle n'\ell'm'|y|n\ell m\rangle,$$

$$(m' - m)^2\langle n'\ell'm'|x|n\ell m\rangle = i \cdot \{(m' - m)\langle n'\ell'm'|y|n\ell m\rangle\}.$$

$$(m' - m)^2\langle n'\ell'm'|x|n\ell m\rangle = i \cdot \{-i\langle n'\ell'm'|x|n\ell m\rangle\}.$$

$$(m' - m)^2\langle n'\ell'm'|x|n\ell m\rangle = 1 \cdot \langle n'\ell'm'|x|n\ell m\rangle.$$

This tells us we are only equal if $(m' - m)^2 = 1$ or if $\langle n'\ell'm'|x|n\ell m\rangle = 0$.

Since $(m' - m)^2 = 1 \implies m' - m = \pm 1 \implies m' = m \pm 1,$

and we know the results for that case, $\langle n'\ell'm'|x|n\ell m\rangle$ has to be 0 for all other m'.

This holds for the y component as well.

$$\begin{cases} \text{if } m' = m, & \text{then } \langle n'\ell'm'|x|n\ell m\rangle = \langle n'\ell'm'|y|n\ell m\rangle = 0, \\ \text{if } m' = m \pm 1, & \text{then } \langle n'\ell'm'|x|n\ell m\rangle = \pm i\langle n'\ell'm'|y|n\ell m\rangle, \\ & \langle n'\ell'm'|z|n\ell m\rangle = 0, \\ \text{otherwise,} & \langle n'\ell'm'|x|n\ell m\rangle = \langle n'\ell'm'|y|n\ell m\rangle = \langle n'\ell'm'|z|n\ell m\rangle = 0. \end{cases}$$

Problem 11.16 An electron in the $n = 3$, $\ell = 0$, $m = 0$ state of hydrogen decays by a sequence of (electric dipole) transitions to the ground state.

(a) What decay routes are open to it? Specify them in the following way:

$$|300\rangle \rightarrow |n\ell m\rangle \rightarrow |n'\ell'm'\rangle \rightarrow \cdots \rightarrow |100\rangle .$$

(b) If you had a bottle full of atoms in this state, what fraction of them would decay via each route?

(c) What is the lifetime of this state? *Hint:* Once it's made the first transition, it's no longer in the state $|300\rangle$, so only the first step in each sequence is relevant in computing the lifetime.

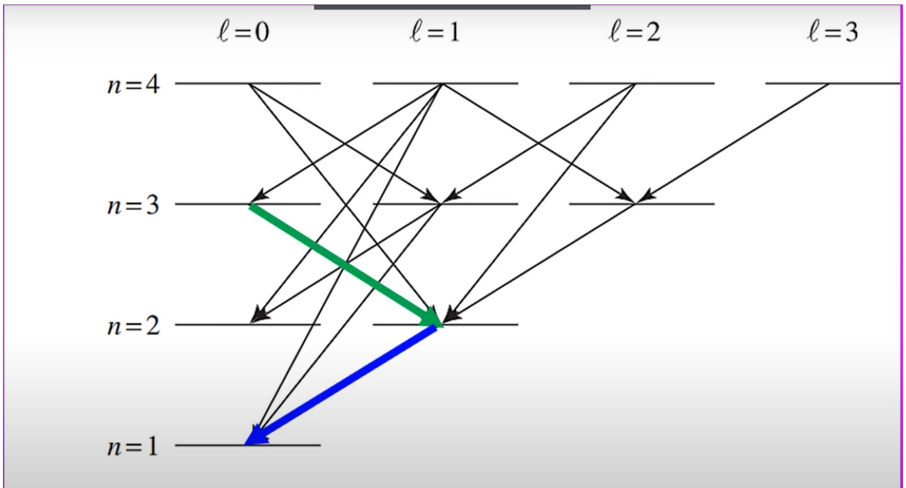


Figure 11.9: Allowed decays for the first four Bohr levels in hydrogen.

$$|300\rangle \to |100\rangle : \quad \Delta\ell = 0-0=0 \quad \times, \quad \Delta m = 0-0=0 \quad \checkmark$$

$$|300\rangle \to |200\rangle : \quad \Delta\ell = 0-0=0 \quad \times, \quad \Delta m = 0-0=0 \quad \checkmark$$

$$|300\rangle \to |210\rangle : \quad \Delta\ell = 1-0=1 \quad \checkmark, \quad \Delta m = 0-0=0 \quad \checkmark$$

$$|300\rangle \to |21+1\rangle : \quad \Delta\ell = 1-0=1 \quad \checkmark, \quad \Delta m = +1-0=1 \quad \checkmark$$

$$|300\rangle \to |21-1\rangle : \quad \Delta\ell = 1-0=1 \quad \checkmark, \quad \Delta m = -1-0=-1 \quad \checkmark$$

$$|210\rangle \to |100\rangle : \quad \Delta\ell = 0-1=-1 \quad \checkmark, \quad \Delta m = 0-0=0 \quad \checkmark$$

$$|21+1\rangle \to |100\rangle : \quad \Delta\ell = 0-1=-1 \quad \checkmark, \quad \Delta m = 0-1=-1 \quad \checkmark$$

$$|21-1\rangle \to |100\rangle : \quad \Delta\ell = 0-1=-1 \quad \checkmark, \quad \Delta m = 0-(-1)=1 \quad \checkmark$$

$$|300\rangle \to \begin{cases} |210\rangle \to |100\rangle, \\ |21+1\rangle \to |100\rangle, \\ |21-1\rangle \to |100\rangle. \end{cases}$$

$$|300\rangle \to |210\rangle : \quad \langle 210|\mathbf{r}|300\rangle = \langle 210|x|300\rangle\hat{\mathbf{i}} + \langle 210|y|300\rangle\hat{\mathbf{j}} + \langle 210|z|300\rangle\hat{\mathbf{k}}.$$

$$= 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \langle 210|z|300\rangle\hat{\mathbf{k}}.$$

$$|300\rangle \to |21\pm1\rangle : \quad \langle 21\pm1|\mathbf{r}|300\rangle = \langle 21\pm1|x|300\rangle\hat{\mathbf{i}} + \langle 21\pm1|y|300\rangle\hat{\mathbf{j}} + \langle 21\pm1|z|300\rangle\hat{\mathbf{k}}.$$

$$= \pm i\langle 21\pm1|y|300\rangle\hat{\mathbf{i}} + \langle 21\pm1|y|300\rangle\hat{\mathbf{j}} + 0\hat{\mathbf{k}}.$$

$$= \langle 21\pm1|y|300\rangle(\pm i\hat{\mathbf{i}} + \hat{\mathbf{j}}).$$

So of the $3\times 3 = 9$ possible triple integrals to be found, we really only need to find 2.

$$|300\rangle = \psi_{300} = R_{30}Y_0^0 = \frac{2}{\sqrt{27a^3}}\left(1 - \frac{2r}{3a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right)e^{-\frac{r}{3a}}\left(\frac{1}{\sqrt{4\pi}}\right)$$

$$= \frac{1}{81\sqrt{3\pi a^3}}\left(27 - 18\frac{r}{a} + 2\left(\frac{r}{a}\right)^2\right)e^{-\frac{r}{3a}}.$$

$$|21\pm1\rangle = \psi_{21\pm1} = R_{21}Y_1^{\pm1} = \frac{1}{\sqrt{24a^3}}\left(\frac{r}{a}\right)e^{-\frac{r}{2a}}\left[\mp\left(\frac{3}{8\pi}\right)^{\frac{1}{2}}\sin\theta e^{\pm i\phi}\right]$$

$$= \mp\frac{1}{8\sqrt{\pi a^3}}\frac{r}{a}e^{-\frac{r}{2a}}\sin\theta e^{\pm i\phi}.$$

$$|210\rangle = \psi_{210} = R_{21}Y_1^0 = \frac{1}{\sqrt{24a^3}}\left(\frac{r}{a}\right)e^{-\frac{r}{2a}}\left[\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\cos\theta\right]$$

$$= \frac{1}{4\sqrt{2\pi a^3}}\frac{r}{a}e^{-\frac{r}{2a}}\cos\theta.$$

The inner products now need to be evaluated.

$$\langle 210|z|300\rangle = \iiint_{\text{All space}} \psi_{210}^*[z]\psi_{300}\,dV$$

$$= \frac{1}{324\sqrt{6}\pi a^4}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^4\left(27 - 18\frac{r}{a} + 2\left(\frac{r}{a}\right)^2\right)e^{-\frac{5r}{6a}}\cos^2\theta\sin\theta\,dr\,d\theta\,d\phi.$$

Đặt:

$$u = \frac{r}{a} \implies r = au, \quad dr = adu; \quad u(\infty) = \frac{\infty}{a} = \infty, \quad u(0) = \frac{0}{a} = 0.$$

$$= \frac{1}{324\sqrt{6}\pi a^4}\int_0^{2\pi}d\phi\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta\int_0^{\infty}a^5u^4\left(27 - 18u + 2u^2\right)e^{-\frac{5}{6}u}\,du.$$

$$= \frac{a}{324\sqrt{6}\pi}\{2\pi\}\left\{\frac{2}{3}\right\}\int_0^{\infty}\left(27u^4e^{-\frac{5}{6}u} - 18u^5e^{-\frac{5}{6}u} + 2u^6e^{-\frac{5}{6}u}\right)du.$$

$$= \frac{a}{81\sqrt{6}\cdot 3}\left[27\cdot 4!\left(\frac{6}{5}\right)^5 - 18\cdot 5!\left(\frac{6}{5}\right)^6 + 2\cdot 6!\left(\frac{6}{5}\right)^7\right].$$

$$= \frac{a}{243\sqrt{6}}\left[27\left\{24\cdot\frac{6^5}{5^5}\right\} - 18\left\{120\cdot\frac{6^{5+1}}{5^{5+1}}\right\} + 2\left\{720\cdot\frac{6^{5+2}}{5^{5+2}}\right\}\right]$$

$$= \frac{a\cdot 6^5}{243\sqrt{6}\cdot 5^5}\left[648 - 2160\cdot\frac{6^1}{5^1} + 1440\cdot\frac{6^2}{5^2}\right] = \frac{a\cdot 6^5}{243\sqrt{6}\cdot 5^5}\left[\frac{648}{5}\right].$$

$$= \frac{a(2\cdot 3)^5[\frac{8}{3}]}{\sqrt{6}\cdot 5^6} = \frac{a\cdot 2^5\cdot 3^5}{\sqrt{6}\cdot 5^6}[\frac{2^3}{3}].$$

$$= \frac{a\cdot 2^8\cdot 3^4}{\sqrt{6}\cdot 5^6}.$$

$$\langle 21\pm 1|y|300\rangle = \iiint_{\text{All space}} \psi_{21\pm 1}^*[y]\psi_{300}\,dV$$

$$= \frac{1}{648\sqrt{3}\pi a^4}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^4\left(27 - 18\frac{r}{a} + 2\left(\frac{r}{a}\right)^2\right)e^{-\frac{5}{6}\frac{r}{a}}\sin^3\theta\sin\phi\,e^{\mp i\phi}\,dr\,d\theta\,d\phi.$$

$$u = \frac{r}{a} \implies r = au, \quad dr = adu; \quad u(\infty) = \frac{\infty}{a} = \infty, \quad u(0) = \frac{0}{a} = 0.$$

$$= \frac{1}{648\sqrt{3}\pi a^4} \int_0^{2\pi} \sin\phi \left\{\cos\phi \mp i\sin\phi\right\} \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{\infty} a^4 u^4 \left(27 - 18u + 2u^2\right) e^{-\frac{5}{6}u} a\, du.$$

$$= \frac{a}{648\sqrt{3}\pi} \int_0^{2\pi} \left\{\sin\phi\cos\phi \mp i\sin^2\phi\right\} \left(\frac{4}{3}\right) \int_0^{\infty} \left(27u^4 e^{-\frac{5}{6}u} - 18u^5 e^{-\frac{5}{6}u} + 2u^6 e^{-\frac{5}{6}u}\right) du.$$

$$= \frac{a}{162\sqrt{3}\pi \cdot 3} \left[0 \mp i\pi\right] \left[27 \cdot 4! \left(\frac{6}{5}\right)^5 - 18 \cdot 5! \left(\frac{6}{5}\right)^6 + 2 \cdot 6! \left(\frac{6}{5}\right)^7\right].$$

$$= \frac{a}{486\sqrt{3}\pi} \left[\mp i\pi\right] \left[27 \cdot 24 \cdot \frac{6^5}{5^5} - 18 \cdot 120 \cdot \frac{6^{5+1}}{5^{5+1}} + 2 \cdot 720 \cdot \frac{6^{5+2}}{5^{5+2}}\right].$$

$$= \mp \frac{ia}{486\sqrt{3}} \frac{6^5}{5^5} \left[648 - 2160 \cdot \frac{6^1}{5^1} + 1440 \cdot \frac{6^2}{5^2}\right] = \mp \frac{ia}{486\sqrt{3}} \frac{(2\cdot 3)^5}{5^5} \left[\frac{648}{5}\right].$$

$$= \mp \frac{ia \cdot 2^5 \cdot 3^5}{\sqrt{3} \cdot 5^{5+1}} \left[\frac{648}{486}\right] = \pm \frac{ia \cdot 2^5 \cdot 3^5}{\sqrt{3} \cdot 5^6} \left[\frac{4}{3}\right] = \pm \frac{ia \cdot 2^7 \cdot 3^4}{\sqrt{3} \cdot 5^6}.$$

Integrals are now complete, let's get back to the electric dipole:

$$\mathbf{p}_{|300\rangle \to |210\rangle} = q\langle 210|\mathbf{r}|300\rangle = q\frac{a}{\sqrt{6}} \frac{2^8 3^4}{5^6} \hat{\mathbf{k}}.$$

$$\implies \left|\mathbf{p}_{|300\rangle \to |210\rangle}\right|^2 = q^2 \frac{a^2}{6} \frac{2^{16} \cdot 3^8}{5^{12}} = q^2 \frac{a^2 \cdot 2^{16} \cdot 3^8}{2 \cdot 3 \cdot 5^{12}} = q^2 a^2 \frac{2^{15} \cdot 3^7}{5^{12}}.$$

$$\mathbf{p}_{|300\rangle \to |21\pm1\rangle} = q\langle 21 \pm 1|\mathbf{r}|300\rangle = \mp \frac{qai}{\sqrt{3}} \frac{2^7 \cdot 3^4}{5^6} (\pm i\hat{\mathbf{i}} + \hat{\mathbf{j}}).$$

$$= \mp \frac{qa}{\sqrt{3}} \frac{2^7 \cdot 3^4}{5^6} (\pm i^2 \hat{\mathbf{i}} + i\hat{\mathbf{j}}) = \mp \frac{qa}{\sqrt{3}} \frac{2^7 \cdot 3^4}{5^6} (+\hat{\mathbf{i}} \mp i\hat{\mathbf{j}}).$$

$$\implies \left|\mathbf{p}_{|300\rangle \to |21\pm1\rangle}\right|^2 = \frac{q^2 a^2}{3} \frac{2^{7\cdot 2} \cdot 3^{4\cdot 2}}{5^{6\cdot 2}} \left[(+1)(+1) + (\pm i)(\mp i)\right].$$

$$= \frac{q^2 a^2}{3} \frac{2^{14} \cdot 3^8}{5^{12}} \left[1 - i^2\right] = \frac{q^2 a^2}{3} \frac{2^{14} \cdot 3^8}{5^{12}} \left[1 - (-1)\right].$$

$$= \frac{q^2 a^2}{3} \frac{2^{14} \cdot 3^8}{5^{12}} \left[1 + 1\right] = \frac{q^2 a^2}{3} \frac{2^{14} \cdot 3^8}{5^{12}} [2] = q^2 a^2 \frac{2^{15} \cdot 3^7}{5^{12}}.$$

What this tells us is that the transition rates for all 3 states are the same number, since they are the same number, the fraction that goes to each route is $\frac{1}{3}$.

$$\text{(c) } n=3\to n=2:$$

$$\omega_0=\frac{E_3-E_2}{\hbar}=\frac{\frac{E_1}{3^2}-\frac{E_1}{2^2}}{\hbar}=\frac{\frac{E_1}{9}-\frac{E_1}{4}}{\hbar}=\frac{-\frac{5E_1}{36}}{\hbar}=-\frac{5E_1}{36\hbar}.$$

$$\tau_{3\to2}=\frac{1}{A_1+A_2+A_3}=\frac{1}{3A}=\frac{1}{3\cdot\omega_0^3|\mathbf{p}|^2/(3\pi\epsilon_0\hbar c^3)}=\frac{\pi\epsilon_0\hbar c^3}{\omega_0^3|\mathbf{p}|^2}.$$

$$=\frac{\pi\epsilon_0\hbar c^3}{\left(\frac{-5E_1}{36\hbar}\right)^3\left(q^2a^2\frac{2^{15}\cdot3^7}{5^{12}}\right)}.$$

$$=\frac{\pi\epsilon_0\hbar c^3}{\left(-\frac{5^3E_1^3}{2^6\cdot3^6\hbar^3}\right)\left(q^2a^2\frac{2^{15}\cdot3^7}{5^{12}}\right)}.$$

$$=\frac{\pi\epsilon_0\hbar^{1+3}c^3}{-E_1^3e^2a^2\frac{2^{15-6+3}\cdot3^7}{5^{12-3}}}.$$

$$=\frac{\pi\epsilon_0\hbar^4c^3}{-E_1^3e^2a^2\frac{2^9\cdot3^1}{5^9}}=\frac{5^9\pi\epsilon_0\hbar^4c^3}{3\cdot2^9\cdot\frac{\hbar^2}{2m_ea^2}E_1^2e^2a^2}.$$

$$=\frac{5^9\pi\epsilon_0\hbar^2c^3m_e}{3\cdot2^8E_1^2e^2}=\frac{5^9c^3m_e}{3\cdot2^8E_1^2}\cdot\frac{\pi\epsilon_0\hbar^2}{e^2}=\frac{5^9c^3m_eam_e}{3\cdot2^8E_1^2\cdot4}.$$

$$=\frac{5^9ac^3m_e^2}{3\cdot2^{8+2}E_1^2}\cdot\frac{c}{c}=\frac{5^9a}{3\cdot2^{10}c}\left(\frac{m_ec^2}{E_1}\right)^2.$$

$$=\frac{5^9(0.529\times10^{-10})}{3\cdot2^{10}(3.00\times10^8)}\left(\frac{0.511\times10^6}{13.6}\right)^2.$$

$$=1.582731437\times10^{-7}\,\text{sec.}$$

$$\Longrightarrow\ \tau_{3\to2}=1.58\times10^{-7}\,\text{sec.}$$

Problem 11.18 A particle of mass *m* is in the ground state of the infinite square well (Equation 2.22). Suddenly the well expands to twice its original size—the right wall moving from a to $2a$—leaving the wave function (momentarily) undisturbed. The energy of the particle is now measured.

(a) What is the most probable result? What is the probability of getting that result?

(b) What is the *next* most probable result, and what is its probability? Suppose your measurement returned this value; what would you conclude about conservation of energy?

(c) What is the *expectation value* of the energy? *Hint:* If you find yourself confronted with an infinite series, try another method.

(a) The sudden, or adiabatic approximation must now be used. Because this wave function is said to be momentarily undisturbed, we can say that at $t = 0$ we have:

$$\Psi(x,0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right);$$

but the new eigenstates, and therefore eigenvalues (energies) for $a \to 2a$ are:

$$\psi_n = \sqrt{\frac{2}{2a}} \sin\left(\frac{n\pi x}{2a}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} = \frac{n^2\pi^2\hbar^2}{2m \cdot 4a^2} = \frac{n^2\pi^2\hbar^2}{8ma^2}$$

$$c_n = \int_0^a \left[\frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right)\right]\left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)\right] dx$$

$$= \sqrt{\frac{1}{a} \cdot \frac{2}{a}} \int_0^a \sin\left(\frac{n\pi x}{2a}\right) \sin\left(\frac{\pi x}{a}\right) dx \quad \text{(sum \& difference identities)}$$

$$= \frac{\sqrt{2}}{a} \int_0^a \left[\cos\left(\frac{n\pi x}{2a} - \frac{\pi x}{a}\right) - \cos\left(\frac{n\pi x}{2a} + \frac{\pi x}{a}\right)\right] \frac{dx}{2}$$

$$= \frac{\sqrt{2}}{2a} \int_0^a \left[\cos\left(\left(\frac{n}{2} - 1\right)\frac{\pi x}{a}\right) - \cos\left(\left(\frac{n}{2} + 1\right)\frac{\pi x}{a}\right)\right] dx$$

$$= \frac{\sqrt{2}}{2a} \int_0^a \cos\left(\left(\frac{n}{2} - 1\right)\frac{\pi x}{a}\right) dx - \int_0^a \cos\left(\left(\frac{n}{2} + 1\right)\frac{\pi x}{a}\right) dx$$

$$= \frac{\sqrt{2}}{2a} \left[\int_0^{\left(\frac{n}{2}-1\right)\pi} \cos(u) \frac{du}{\left(\frac{n}{2} - 1\right)\frac{\pi}{a}} - \int_0^{\left(\frac{n}{2}+1\right)\pi} \cos(t) \frac{dt}{\left(\frac{n}{2} + 1\right)\frac{\pi}{a}}\right]$$

$$= \frac{\sqrt{2}}{2a} \left[\frac{1}{\left(\frac{n}{2} - 1\right)\frac{\pi}{a}} \int_0^{\left(\frac{n}{2}-1\right)\pi} \cos(u)du - \frac{1}{\left(\frac{n}{2} + 1\right)\frac{\pi}{a}} \int_0^{\left(\frac{n}{2}+1\right)\pi} \cos(t)dt\right]$$

$$= \frac{\sqrt{2}}{2a\frac{\pi}{a}} \left[\frac{1}{\left(\frac{n}{2} - 1\right)} \left\{\sin(u)\right\}_0^{\left(\frac{n}{2}-1\right)\pi} - \frac{1}{\left(\frac{n}{2} + 1\right)} \left\{\sin(u)\right\}_0^{\left(\frac{n}{2}+1\right)\pi}\right]$$

$$= \frac{\sqrt{2}}{2\pi} \left[\frac{1}{\left(\frac{n}{2} - 1\right)} \left(\sin\left(\left(\frac{n}{2} - 1\right)\pi\right) - \sin(0)\right) - \frac{1}{\left(\frac{n}{2} + 1\right)} \left(\sin\left(\left(\frac{n}{2} + 1\right)\pi\right) - \sin(0)\right)\right]$$

$$= \frac{\sqrt{2}}{2\pi} \left[\frac{\sin\left(\left(\frac{n}{2} - 1\right)\pi\right)}{\left(\frac{n}{2} - 1\right)} - \frac{\sin\left(\left(\frac{n}{2} + 1\right)\pi\right)}{\left(\frac{n}{2} + 1\right)}\right]$$

$$\rightarrow \text{Holds for } \frac{n}{2} - 1 \neq 0 \Rightarrow n \neq 2$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{\sin\left(\left(\frac{n}{2}-1\right)\pi\right)}{2\left(\frac{n}{2}-1\right)} - \frac{\sin\left(\left(\frac{n}{2}+1\right)\pi\right)}{2\left(\frac{n}{2}+1\right)}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{\sin\left(\frac{n\pi}{2}-\pi\right)}{n-2} - \frac{\sin\left(\frac{n\pi}{2}+\pi\right)}{n+2}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\sin\left(\frac{n\pi}{2}\right)}{n-2} - \frac{-\sin\left(\frac{n\pi}{2}\right)}{n+2}\right]$$

$$= -\frac{\sqrt{2}}{\pi}\sin\left(\frac{n\pi}{2}\right)\left[\frac{1}{n-2} - \frac{1}{n+2}\right]$$

$$= -\frac{\sqrt{2}}{\pi}\sin\left(\frac{n\pi}{2}\right)\left[\frac{n+2}{(n-2)(n+2)} - \frac{n-2}{(n+2)(n-2)}\right]$$

Let's look at the characteristics of these sine functions. Using the sum/difference identities, we see:

$$\sin\left(\frac{n\pi}{2}-\pi\right) = \sin\left(\frac{n\pi}{2}\right)\cos(\pi) - \cos\left(\frac{n\pi}{2}\right)\sin(\pi) = \sin\left(\frac{n\pi}{2}\right)(-1) - \cos\left(\frac{n\pi}{2}\right)(0) = -\sin\left(\frac{n\pi}{2}\right)$$

$$\sin\left(\frac{n\pi}{2}+\pi\right) = \sin\left(\frac{n\pi}{2}\right)\cos(\pi) + \cos\left(\frac{n\pi}{2}\right)\sin(\pi) = \sin\left(\frac{n\pi}{2}\right)(-1) + \cos\left(\frac{n\pi}{2}\right)(0) = -\sin\left(\frac{n\pi}{2}\right)$$

$$= -\frac{\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{\pi}\left[\frac{n+2}{n^2-4} - \frac{n-2}{n^2-4}\right] = -\frac{\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{\pi}\left[\frac{n+2-(n-2)}{n^2-4}\right]$$

$$= -\frac{\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{\pi}\cdot\frac{4}{n^2-4} = \frac{-4\sqrt{2}\sin\left(\frac{n\pi}{2}\right)}{\pi(n^2-4)}$$

$$\Rightarrow c_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \pm\frac{4\sqrt{2}}{\pi(n^2-4)}, & \text{if } n \text{ is odd.} \end{cases}$$

$$c_2 = \frac{\sqrt{2}}{a}\int_0^a \sin\left(\frac{2\pi}{2a}x\right)\sin\left(\frac{\pi}{a}x\right)dx = \frac{\sqrt{2}}{a}\int_0^a \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{a}x\right)dx$$

$$= \frac{\sqrt{2}}{a}\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx \quad \text{(Half angle identity)}$$

$$= \frac{\sqrt{2}}{a}\int_0^a \frac{1-\cos\left(\frac{2\pi}{a}x\right)}{2}dx = \frac{\sqrt{2}}{2a}\int_0^a \left[1-\cos\left(\frac{2\pi}{a}x\right)\right]dx$$

$$= \frac{\sqrt{2}}{2a}\left[\int_0^a 1\,dx - \int_0^a \cos\left(\frac{2\pi}{a}x\right)dx\right]$$

$$= \frac{\sqrt{2}}{2a}\left[x\big|_0^a - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}x\right)\Bigg|_0^a\right]$$

$$= \frac{\sqrt{2}}{2a}\left[a - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}a\right) - \left(0 - \frac{a}{2\pi}\sin\left(\frac{2\pi}{a}\cdot 0\right)\right)\right]$$

$$= \frac{\sqrt{2}}{2a}\left[a - \frac{a}{2\pi}\sin(2\pi) - \frac{a}{2\pi}\sin(0)\right]$$

$$= \frac{\sqrt{2}}{2a}\left[a - \frac{a}{2\pi}(0)\right] = \frac{\sqrt{2}}{2a}\cdot a = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

So now the probabilities for these new states are summarized as:

$$P_n = |c_n|^2 = \begin{cases} \frac{1}{2}, & \text{if } n = 2; \\ \frac{32}{\pi^2(n^2-4)^2}, & \text{if } n \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

From this, it is easily verified that the most probable state is $n = 2$, which yields an energy value of:

$$\boxed{E_2 = \frac{2^2\pi^2\hbar^2}{8ma^2} = \frac{4\pi^2\hbar^2}{8ma^2} = \frac{\pi^2\hbar^2}{2ma^2}, \quad \text{with } P_2 = \frac{1}{2}.}$$

(b) Next most probable state, let's run some calculations:

$$n = 1: \quad \frac{32}{\pi^2(1^2-4)^2} = \frac{32}{\pi^2(1-4)^2} = \frac{32}{\pi^2(-3)^2} = \frac{32}{9\pi^2} \approx 0.36;$$

$$n = 3: \quad \frac{32}{\pi^2(3^2-4)^2} = \frac{32}{\pi^2(9-4)^2} = \frac{32}{\pi^2(5)^2} = \frac{32}{25\pi^2} \approx 0.13;$$

$$n = 4: \boxed{E_1 = \frac{1^2\pi^2\hbar^2}{8ma^2} = \frac{\pi^2\hbar^2}{8ma^2}, \quad \text{with } P_1 = \frac{32}{9\pi^2}.}$$

(c) $\langle\hat{H}\rangle = \langle\Psi|\hat{H}|\Psi\rangle = \int \Psi^*\hat{H}\Psi dx$

$$= \int_0^a \left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)\right]\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi}{a}x\right)\right]\right\}dx.$$

$$-\frac{\hbar^2}{2m}\sqrt{\frac{2}{a}\cdot\frac{2}{a}}\int_0^a \sin\left(\frac{\pi}{a}x\right)\left[\frac{d^2}{dx^2}\sin\left(\frac{\pi}{a}x\right)\right]dx$$

$$= \frac{\hbar^2\pi^2}{ma^3}\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx = \frac{\hbar^2\pi^2}{ma^3}\int_0^a \frac{1-\cos\left(\frac{2\pi}{a}x\right)}{2}dx \quad \text{(Half angle identity)}$$

$$= \frac{\hbar^2\pi^2}{2ma^3}\int_0^a \left[1-\cos\left(\frac{2\pi}{a}x\right)\right]dx = \frac{\hbar^2\pi^2}{2ma^3}\left[x-\frac{a}{2\pi}\sin\left(\frac{2\pi}{a}x\right)\right]_0^a$$

$$= \frac{\hbar^2\pi^2}{2ma^3}\left[a-\frac{a}{2\pi}\sin(2\pi)-\left(0-\frac{a}{2\pi}\sin(0)\right)\right] = \frac{\hbar^2\pi^2}{2ma^3}\cdot a = \frac{\hbar^2\pi^2}{2ma^2}.$$

Problem 11.19 A particle is in the ground state of the harmonic oscillator with classical frequency ω, when suddenly the spring constant quadruples, so $\omega' = 2\omega$, without initially changing the wave function (of course, Ψ will now *evolve* differently, because the Hamiltonian has changed). What is the probability that a measurement of the energy would still return the value $\hbar\omega/2$? What is the probability of getting $\hbar\omega$? *Answer:* 0.943.

Much like in the last question, we need to set up our scenario, for the immediate wavefunction, and the new eigenstates and values given that $\omega' = 2\omega$:

$$\Psi(x,0) = \psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2};$$

$$\psi_0'(x) = \left(\frac{m\omega'}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega'}{2\hbar}x^2} = \left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{\hbar}x^2};$$

$$E_n' = \left(n+\frac{1}{2}\right)\hbar\omega' = \left(n+\frac{1}{2}\right)\hbar(2\omega) = (2n+1)\hbar\omega.$$

$$\hbar\omega = (2n+1)\hbar\omega \implies 1 = 2n+1 \implies n = 0.$$

$$c_0 = \int_{-\infty}^{+\infty}[\Psi(x,0)]^*[\psi_0'(x)]dx$$

$$= \int_{-\infty}^{+\infty}\left[\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}\right]\left[\left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{\hbar}x^2}\right]dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar}x^2}e^{-\frac{m\omega}{\hbar}x^2}dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\int_{-\infty}^{+\infty} e^{-\frac{m\omega}{2\hbar}x^2-\frac{m\omega}{\hbar}x^2}dx$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\int_{-\infty}^{+\infty} e^{-\frac{3m\omega}{2\hbar}x^2}dx.$$

$$= 2^{\frac{1}{4}}\sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{+\infty} e^{-\frac{3m\omega}{2\hbar}x^2}dx = 2^{\frac{1}{4}}\sqrt{\frac{m\omega}{\pi\hbar}}\cdot 2\int_{0}^{+\infty} e^{-\frac{3m\omega}{2\hbar}x^2}dx$$

$$= 2^{\frac{1}{4}}\sqrt{\frac{m\omega}{\pi\hbar}}\cdot 2\cdot\sqrt{\pi}\sqrt{\frac{\hbar}{3m\omega\cdot 2}} = 2^{\frac{1}{4}}\sqrt{\frac{m\omega}{\pi\hbar}}\cdot 2\cdot\sqrt{\pi}\sqrt{\frac{\hbar}{3m\omega}}\cdot\frac{1}{\sqrt{2}}$$

$$= 2^{\frac{1}{4}}\sqrt{\frac{m\omega}{\pi\hbar}}\cdot\sqrt{\pi}\cdot\sqrt{\frac{2\hbar}{3m\omega}} = 2^{\frac{1}{4}}\sqrt{\frac{2\hbar}{3m\omega}}$$

$$= 2^{\frac{1}{4}}\cdot\sqrt{\frac{2}{3}} = 2^{\frac{1}{4}}\sqrt{\frac{2}{3}}.$$

Tính xác suất:

$$P_0 = |c_0|^2 = \left(2^{\frac{1}{4}}\sqrt{\frac{2}{3}}\right)^2 = 2^{\frac{1.2}{4}}\cdot\frac{2}{3} = \frac{2}{3}\cdot\sqrt{2}$$

$$\therefore P_0 = \frac{2}{3}\sqrt{2} \approx 0.9428.$$

Problem 11.26 A particle starts out (at time $t = 0$) in the Nth state of the infinite square well. Now the "floor" of the well rises temporarily (maybe water leaks in, and then drains out again), so that the potential inside is uniform but time dependent: $V_0(t)$, with $V_0(0) = V_0(T) = 0$.

(a) Solve for the *exact* $c_m(t)$, using Equation 11.116, and show that the wave function changes *phase*, but no transitions occur. Find the phase change, $\phi(T)$, in terms of the function $V_0(t)$.

(b) Analyze the same problem in first-order perturbation theory, and compare your answers.

Comment: The same result holds *whenever* the perturbation simply adds a constant (constant in x, that is, not in t) to the potential; it has nothing to do with the infinite square well, as such. Compare Problem 1.8.

Solution

(a) For a Multi-level System we see:

$$\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n H'_{mn} e^{i\frac{(E_m-E_n)t}{\hbar}} \tag{1}$$

$$= -\frac{i}{\hbar}\sum_n c_n \langle\psi_m|\hat{H}'|\psi_n\rangle e^{i\frac{(E_m-E_n)t}{\hbar}} \tag{2}$$

$$= -\frac{i}{\hbar}\sum_n c_n \langle\psi_m|V_0(t)|\psi_n\rangle e^{i\frac{(E_m-E_n)t}{\hbar}} \tag{3}$$

$$\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n V_0(t)\langle\psi_m|\psi_n\rangle e^{i\frac{(E_m-E_n)t}{\hbar}} \tag{4}$$

$$= -\frac{i}{\hbar}\sum_n c_n V_0(t)\delta_{mn} e^{i\frac{(E_m-E_n)t}{\hbar}} \tag{5}$$

$$= -\frac{i}{\hbar} c_m V_0(t) e^{i\frac{(E_m-E_m)t}{\hbar}} \tag{6}$$

$$= -\frac{i}{\hbar} c_m V_0(t)\cdot e^{i\cdot 0} \tag{7}$$

$$= -\frac{i}{\hbar} c_m V_0(t). \tag{8}$$

From this, we have:

$$\dot{c}_m = -\frac{i}{\hbar} c_m V_0(t), \quad \text{or equivalently:} \tag{9}$$

$$\frac{dc_m}{dt} = -\frac{i}{\hbar} c_m V_0(t). \tag{10}$$

Rearranging and integrating:

$$\frac{dc_m}{c_m} = -\frac{i}{\hbar} V_0(t)\, dt, \tag{11}$$

$$\int \frac{dc_m}{c_m} = -\frac{i}{\hbar}\int V_0(t)\, dt. \tag{12}$$

$$\ln c_m = -\frac{i}{\hbar}\int V_0(t')\, dt' + \text{constant}, \tag{13}$$

$$c_m = e^{-\frac{i}{\hbar}\int V_0(t')\, dt' + \text{constant}}, \tag{14}$$

$$= e^{-\frac{i}{\hbar}\int V_0(t')\, dt'} e^{\text{constant}}, \tag{15}$$

$$= e^{-\frac{i}{\hbar}\int V_0(t')\, dt'} c_m(0). \tag{16}$$

If we let:

$$\Phi(t) \equiv -\frac{1}{\hbar}\int V_0(t')\, dt', \tag{17}$$

then:

$$c_m = c_m(0) e^{i\Phi(t)}. \tag{18}$$

This tells us that the probability of transition is:

$$P = |c_m|^2, \tag{19}$$

$$= \left|c_m(0) e^{i\Phi(t)}\right|^2, \tag{20}$$

$$= c_m(0)^* c_m(0) = |c_m(0)|^2. \tag{21}$$

The expression for c_m is:

$$c_m^*(0)e^{-i\Phi(t)}c_m(0)e^{i\Phi(t)} = c_m^*(0)c_m(0)e^{-i\Phi(t)+i\Phi(t)} \tag{22}$$

$$= |c_m(0)|^2 e^0 \tag{23}$$

$$= |c_m(0)|^2 \cdot 1 \tag{24}$$

$$= |c_m(0)|^2. \tag{25}$$

Since the probability is the exact same as the initial state, there are **no transitions**; but there is a phase change.

$$\therefore \Phi(t) \equiv -\frac{1}{\hbar}\int_0^T V_0(t)\,dt. \tag{26}$$

(b) Looking through the first-order lens:

$$c_N(t) \approx 1 - \frac{i}{\hbar}\int_0^t H'_{NN}(t')\,dt' \tag{27}$$

$$= 1 - \frac{i}{\hbar}\int_0^t \langle\psi_N|\hat{H}'|\psi_N\rangle\,dt'. \tag{28}$$

$$1 - \frac{i}{\hbar}\int_0^t \langle\psi_N|V_0(t')|\psi_N\rangle\,dt' = 1 - \frac{i}{\hbar}\int_0^t V_0(t')\langle\psi_N|\psi_N\rangle\,dt' \tag{29}$$

$$= 1 - \frac{i}{\hbar}\int_0^t V_0(t')\,dt' \tag{30}$$

$$= 1 + i\Phi(t). \tag{31}$$

For $c_m(t)$, we have:

$$c_m(t) \approx -\frac{i}{\hbar}\int_0^t H'_{mN}(t')e^{i\frac{(E_m-E_N)t'}{\hbar}}\,dt' \tag{32}$$

$$= -\frac{i}{\hbar}\int_0^t \langle\psi_m|\hat{H}'|\psi_N\rangle e^{i\frac{(E_m-E_N)t'}{\hbar}}\,dt' \tag{33}$$

$$= -\frac{i}{\hbar}\int_0^t \langle\psi_m|V_0(t')|\psi_N\rangle e^{i\frac{(E_m-E_N)t'}{\hbar}}\,dt'. \tag{34}$$

Continuing for $c_m(t)$:

$$c_m(t) = -\frac{i}{\hbar}\int_0^t V_0(t')\langle\psi_m|\psi_N\rangle e^{i\frac{(E_m-E_N)t'}{\hbar}}\,dt' \tag{35}$$

$$= -\frac{i}{\hbar}\int_0^t V_0(t')\delta_{mN}e^{i\frac{(E_m-E_N)t'}{\hbar}}\,dt' \tag{36}$$

$$= 0, \quad \text{since } m \neq N. \tag{37}$$

Therefore:

$$\therefore \quad c_N(t) = 1 + i\Phi(t), \quad c_m(t) = 0, \quad \text{for } m \neq N.$$

The exact answer (from Problem 1.8) is:

$$c_N(t) = e^{i\Phi(t)}, \quad \text{and } c_m(t) = 0,$$

which are consistent, since in the first order:

$$e^{i\Phi(t)} \approx 1 + i\Phi(t).$$

Problem 11.27 A particle of mass m is initially in the ground state of the (one-dimensional) infinite square well. At time $t = 0$ a "brick" is dropped into the well, so that the potential becomes

$$V(x) = \begin{cases} V_0, & 0 \leq x \leq a/2, \\ 0, & a/2 < x \leq a, \\ \infty, & \text{otherwise,} \end{cases}$$

where $V_0 \ll E_1$. After a time T, the brick is removed, and the energy of the particle is measured. Find the probability (in first-order perturbation theory) that the result is now E_2.

Equation 11.120 makes our lives incredibly easy so long as we figure out what everything in it requires. Since we are dealing with an infinite square well, we know the wave functions and energy from chapter 2. The two states of interest are $n = 1$ and $n = 2$.

State $n = 1$:

$$E_1 = \frac{1^2\pi^2\hbar^2}{2ma^2} = \frac{\pi^2\hbar^2}{2ma^2}, \tag{38}$$

$$\psi_1 = \sqrt{\frac{2}{a}} \sin\left(\frac{1\pi}{a}x\right) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right). \tag{39}$$

State $n = 2$:

$$E_2 = \frac{2^2\pi^2\hbar^2}{2ma^2} = \frac{4\pi^2\hbar^2}{2ma^2}, \tag{40}$$

$$\psi_2 = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a}x\right). \tag{41}$$

We are looking for a transition from $M(n=1)$ to $N(n=2)$, which tells us the energy difference is:

$$\begin{aligned}
E_N - E_M &= E_2 - E_1 &(42)\\
&= \frac{4\pi^2\hbar^2}{2ma^2} - \frac{\pi^2\hbar^2}{2ma^2} &(43)\\
&= \frac{4\pi^2\hbar^2 - \pi^2\hbar^2}{2ma^2} &(44)\\
&= \frac{(4-1)\pi^2\hbar^2}{2ma^2} &(45)\\
&= \frac{3\pi^2\hbar^2}{2ma^2}. &(46)
\end{aligned}$$

The matrix element is the last thing to calculate:

$$H'_{MN} \equiv \langle \psi_M | \hat{H}' | \psi_N \rangle \tag{47}$$

$$\begin{aligned}
\rightarrow H'_{12} &= \langle \psi_1 | V_0 | \psi_2 \rangle &(48)\\
&= \int_0^{a/2} \left[\sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) \right] V_0 \left[\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi}{a}x\right) \right] dx. &(49)
\end{aligned}$$

$$\begin{aligned}
H'_{12} &= \sqrt{\frac{2}{a}} \cdot \sqrt{\frac{2}{a}} \cdot V_0 \int_0^{a/2} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) dx &(50)\\
&= \frac{2V_0}{a} \int_0^{a/2} \sin\left(\frac{\pi}{a}x\right) \sin\left(\frac{2\pi}{a}x\right) dx. &(51)
\end{aligned}$$

Using the sum/difference identity for sine:

$$\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right],$$

we have:

$$\begin{aligned}
H'_{12} &= \frac{2V_0}{a} \int_0^{a/2} \frac{1}{2} \left[\cos\left(\frac{\pi}{a}x - \frac{2\pi}{a}x\right) - \cos\left(\frac{\pi}{a}x + \frac{2\pi}{a}x\right) \right] dx &(52)\\
&= \frac{V_0}{a} \int_0^{a/2} \left[\cos\left(-\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right) \right] dx. &(53)
\end{aligned}$$

Since $\cos(-x) = \cos(x)$, this simplifies to:

$$H'_{12} = \frac{V_0}{a} \int_0^{a/2} \left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right) \right] dx. \tag{54}$$

$$H'_{12} = \frac{V_0}{a} \int_0^{a/2} \left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right) \right] dx \tag{55}$$

$$= \frac{V_0}{a} \left[\frac{\sin\left(\frac{\pi}{a}x\right)}{\frac{\pi}{a}} - \frac{\sin\left(\frac{3\pi}{a}x\right)}{\frac{3\pi}{a}} \right]_0^a \tag{56}$$

$$= \frac{V_0}{a} \left[\frac{a}{\pi} \sin\left(\frac{\pi}{a}x\right) - \frac{a}{3\pi} \sin\left(\frac{3\pi}{a}x\right) \right]_0^{a/2}. \tag{57}$$

Evaluating at the limits:

$$H'_{12} = \frac{V_0}{a} \left\{ \left[\frac{a}{\pi} \sin\left(\frac{\pi}{2}\right) - \frac{a}{\frac{3\pi}{2}} \sin(3\pi) \right] - \left[\frac{a}{\pi} \sin(0) - \frac{a}{3\pi} \sin(0) \right] \right\} \tag{58}$$

$$= \frac{V_0}{a} \left\{ \frac{a}{\pi} \cdot \sin\frac{\pi}{2} - \frac{a}{3\pi} \cdot \sin\frac{3\pi}{2} - (0 - 0) \right\} \tag{59}$$

$$= \frac{V_0}{a} \left\{ \frac{a}{\pi}(1) - \frac{a}{3\pi}(-1) \right\} \tag{60}$$

$$= \frac{V_0}{a} \left[\frac{a}{\pi} + \frac{a}{3\pi} \right] \tag{61}$$

$$= \frac{V_0}{a} \cdot \frac{4a}{3\pi} \tag{62}$$

$$= \frac{4V_0}{3\pi}. \tag{63}$$

$$H'_{12} = \frac{V_0}{a} \left[\frac{3a}{3\pi} + \frac{a}{3\pi} \right] \tag{64}$$

$$= \frac{V_0}{a} \cdot \frac{4a}{3\pi} \tag{65}$$

$$= \frac{4V_0}{3\pi}. \tag{66}$$

Now that we have all our pieces figured out, all that's left to do is plug them into Equation 11.120 and simplify!

$$P_{1\to 2} = 4|H'_{12}|^2 \frac{\sin^2\left(\frac{(E_2 - E_1)T}{2\hbar}\right)}{(E_2 - E_1)^2}.$$

Substitute the values:

$$H'_{12} = \frac{4V_0}{3\pi}, \quad E_2 - E_1 = \frac{3\pi^2\hbar^2}{2ma^2}.$$

Substitute into $P_{1\to 2}$:

$$P_{1\to 2} = 4 \left| \frac{4V_0}{3\pi} \right|^2 \frac{\sin^2\left(\frac{\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)T}{2\hbar} \right)}{\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)^2} \tag{67}$$

Starting with:

$$P_{1\to 2} = 4\left(\frac{16V_0^2}{9\pi^2}\right)\frac{\sin^2\left(\frac{\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)T}{2\hbar}\right)}{\left(\frac{9\pi^4\hbar^4}{4m^2a^4}\right)},$$

simplify each term:

$$P_{1\to 2} = 4\cdot\frac{16V_0^2}{9\pi^2}\cdot\frac{4m^2a^4}{9\pi^4\hbar^4}\sin^2\left(\frac{3\pi^2\hbar T}{4ma^2}\right).$$

Combine constants:

$$P_{1\to 2} = \frac{64V_0^2\cdot 4m^2a^4}{81\pi^6\hbar^4}\sin^2\left(\frac{3\pi^2\hbar T}{4ma^2}\right).$$

Simplify further:

$$P_{1\to 2} = \frac{256V_0^2m^2a^4}{81\pi^6\hbar^4}\sin^2\left(\frac{3\pi^2\hbar T}{4ma^2}\right).$$

Factorize:

$$P_{1\to 2} = \left[\frac{16V_0ma^2}{9\pi^3\hbar^2}\right]^2\sin^2\left(\frac{3\pi^2\hbar T}{4ma^2}\right).$$

Final expression:

$$P_{1\to 2} = \left[\frac{16ma^2V_0}{9\pi^3\hbar^2}\right]^2\sin^2\left(\frac{3\pi^2\hbar T}{4ma^2}\right).$$