4 Coherent states of the Harmonic oscillator

Coherent states are quantum states that exhibit some sort of classical behavior. We will introduce them and explore their properties. To begin our discussion we introduce translation operators.

4.1 Translation operator

Let us construct unitary *translation* operators T_{x_0} that acting on states moves them (or translates them) by a distance x_0, where x_0 is a real constant with units of length:

$$\boxed{\text{Translation operator:} \quad T_{x_0} \;\equiv\; e^{-\frac{i}{\hbar}\hat{p}\,x_0}\,.} \tag{4.1}$$

This operator is unitary because it is the exponential of an antihermitian operator ($\hat{p}$ is hermitian, and $i\hat{p}$ antihermitian). The multiplication of two such operators is simple:

$$T_{x_0}T_{y_0} \;=\; e^{-\frac{i}{\hbar}\hat{p}\,x_0}e^{-\frac{i}{\hbar}\hat{p}\,y_0} \;=\; e^{-\frac{i}{\hbar}\hat{p}\,(x_0+y_0)}\,, \tag{4.2}$$

since the exponents commute ($e^Ae^B = e^{A+B}$ if $[A,B]=0$). As a result

$$T_{x_0}T_{y_0} \;=\; T_{x_0+y_0}\,. \tag{4.3}$$

The translation operators form a group: the product of two translation is a translation. There is a unit element $T_0 = I$ corresponding to $x_0 = 0$, and each element T_{x_0} has an inverse T_{-x_0}. Note that the group multiplication rule is commutative.

It follows from the explicit definition of the translation operator that

$$(T_{x_0})^\dagger \;=\; e^{\frac{i}{\hbar}\hat{p}\,x_0} \;=\; e^{-\frac{i}{\hbar}\hat{p}\,(-x_0)} \;=\; T_{-x_0} \;=\; (T_{x_0})^{-1}\,. \tag{4.4}$$

confirming again that the operator is unitary. In the following we denote $(T_{x_0})^\dagger$ simply by $T^\dagger_{x_0}$.

We say that T_{x_0} translates by x_0 because of its action[2] on the operator $\hat{x}$ is as follows:

$$T^\dagger_{x_0}\,\hat{x}\,T_{x_0} \;=\; e^{\frac{i}{\hbar}\hat{p}\,x_0}\,\hat{x}\,e^{-\frac{i}{\hbar}\hat{p}\,x_0} \;=\; \hat{x} + \frac{i}{\hbar}\,[\hat{p},\hat{x}]x_0 \;=\; \hat{x} + x_0\,, \tag{4.5}$$

where we used the formula $e^ABe^{-A} = B + [A,B] + \ldots$ and the dots vanish in this case because $[A,B]$ is a number (check that you understand this!).

To see physically why the above is consistent with intuition, consider a state $|\psi\rangle$ and the expectation value of $\hat{x}$ on this state

$$\langle\hat{x}\rangle_\psi \;=\; \langle\psi|\,\hat{x}\,|\psi\rangle \tag{4.6}$$

[2] The action of a unitary operator $\mathcal{U}$ on an operator $\mathcal{O}$ is defined as $\mathcal{O} \to \mathcal{U}^\dagger\mathcal{O}\mathcal{U}$.

Now we ask: What is the expectation value of $\hat{x}$ on the state $T_{x_0}|\psi\rangle$? We find

$$\langle \hat{x} \rangle_{T_{x_0}\psi} = \langle \psi | T_{x_0}^\dagger \, \hat{x} \, T_{x_0} |\psi\rangle \tag{4.7}$$

The right-hand side explains why $T_{x_0}^\dagger \, \hat{x} \, T_{x_0}$ is the natural thing to compute! Indeed using our result for this

$$\langle \hat{x} \rangle_{T_{x_0}\psi} = \langle \psi | (\hat{x} + x_0) | \psi \rangle = \langle \hat{x} \rangle_\psi + x_0 \,. \tag{4.8}$$

The expectation value of $\hat{x}$ on the displaced state is indeed equal to the expectation value of $\hat{x}$ in the original state plus x_0, confirming that *we should view $T_{x_0}|\psi\rangle$ as the state $|\psi\rangle$ displaced a distance x_0.*

As an example we look at position states. We claim that on position states the translation operator does what we expect:

$$\boxed{T_{x_0}|x_1\rangle = |x_1 + x_0\rangle \,.} \tag{4.9}$$

We can prove (4.9) by acting on the above left-hand side an arbitrary momentum bra $\langle p|$:

$$\langle p | T_{x_0} | x_1 \rangle = \langle p | e^{-\frac{i}{\hbar}\hat{p}x_0} | x_1 \rangle = e^{-\frac{i}{\hbar}\, p x_0} \frac{e^{-\frac{i}{\hbar} p x_1}}{\sqrt{2\pi\hbar}} = \langle p | x_1 + x_0 \rangle \,, \tag{4.10}$$

proving the desired result, given that $\langle p|$ is arbitrary. It also follows from unitarity and (4.9) that

$$T_{x_0}^\dagger |x_1\rangle = T_{-x_0}|x_1\rangle = |x_1 - x_0\rangle \,. \tag{4.11}$$

Taking the Hermitian conjugate we find

$$\boxed{\langle x_1 | T_{x_0} = \langle x_1 - x_0 | \,.} \tag{4.12}$$

In terms of arbitrary states $|\psi\rangle$, we can also discuss the action of the translation operator by introducing the wavefunction $\psi(x) = \langle x|\psi\rangle$. Then the "translated" state $T_{x_0}|\psi\rangle$ has a wavefunction

$$\langle x | T_{x_0} | \psi \rangle = \langle x - x_0 | \psi \rangle = \psi(x - x_0) \,. \tag{4.13}$$

Indeed, $\psi(x - x_0)$ is the function $\psi(x)$ translated by the distance $+x_0$. For example, the value that $\psi(x)$ takes at $x = 0$ is taken by the function $\psi(x - x_0)$ at $x = x_0$.

4.2 Definition and basic properties of coherent states

We now finally introduce a coherent state $|\tilde{x}_0\rangle$ of the simple harmonic oscillator. The state is labeled by x_0 and the tilde is there to remind you that it is *not* a position state.[3] Here is the definition

$$\boxed{\text{Coherent state:} \quad |\tilde{x}_0\rangle \;\equiv\; T_{x_0}|0\rangle \;= e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle\,,} \tag{4.14}$$

where $|0\rangle$ denotes the ground state of the oscillator. Do not confuse the coherent state with a position state. The coherent state is simply the translation of the ground state by a distance x_0. This state has no time dependence displayed, so it may be thought as the state of the system at $t = 0$. As t increases the state will evolve according to the Schrödinger equation, and we will be interested in this evolution, but not now. Note that the coherent state is well normalized

$$\langle\tilde{x}_0|\tilde{x}_0\rangle \;= \langle 0|T^\dagger_{x_0}T_{x_0}|0\rangle \;=\; \langle 0|0\rangle \;= 1\,. \tag{4.15}$$

This had to be so because T_{x_0} is unitary.

To begin with let us calculate the wavefunction associated to the state:

$$\psi_{x_0}(x) \;\equiv\; \langle x|\tilde{x}_0\rangle \;=\; \langle x|T_{x_0}|0\rangle \;=\; \langle x - x_0|0\rangle \;=\; \psi_0(x - x_0)\,, \tag{4.16}$$

where we used (4.12) and we denoted $\langle x|0\rangle = \psi_0(x)$, as the ground state wavefunction. So, as expected the wavefunction for the coherent state is just the ground state wavefunction displaced x_0 to the right. This is illustrated in Figure 2.

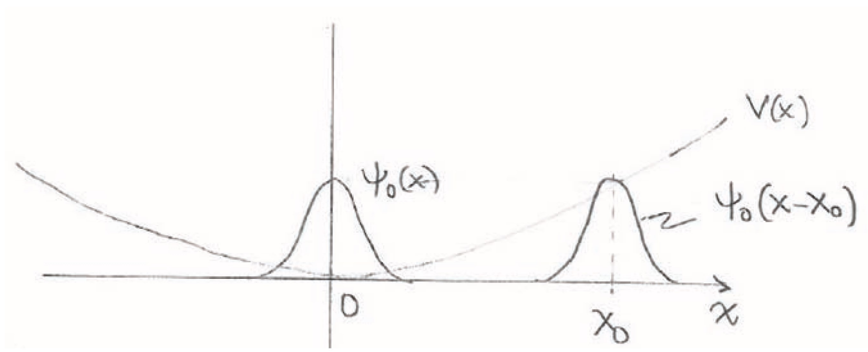

Figure 2: The ground state wavefunction $\psi_0(x)$ displaced to the right a distance x_0 is the wavefunction $\psi_0(x - x_0)$. The corresponding state, denoted as $|\tilde{x}_0\rangle$, is the simplest example of a coherent state.

Let us now do a few sample calculations to understand better these states.

[3]This is not great notation, but it is better than any alternative I have seen!

1. Calculate the expectation value of $\hat{x}$ in a coherent state.

$$\langle\tilde{x}_0|\hat{x}\,|\tilde{x}_0\rangle \;=\; \langle 0|T^\dagger_{x_0}\,\hat{x}\,T_{x_0}|0\rangle \;=\; \langle 0|(\hat{x}+x_0)|0\rangle\,, \tag{4.17}$$

where we used (4.5). Recalling now that $\langle 0|\hat{x}|0\rangle = 0$ we get

$$\langle\tilde{x}_0|\hat{x}\,|\tilde{x}_0\rangle \;=\; x_0\,. \tag{4.18}$$

Not that surprising! The position is essentially x_0.

2. Calculate the expectation value of $\hat{p}$ in a coherent state. Since $\hat{p}$ commutes with T_{x_0} we have

$$\langle\tilde{x}_0|\hat{p}\,|\tilde{x}_0\rangle \;=\; \langle 0|T^\dagger_{x_0}\,\hat{p}\,T_{x_0}|0\rangle \;=\; \langle 0|\,\hat{p}\,T^\dagger_{x_0}\,T_{x_0}|0\rangle \;=\; \langle 0|\,\hat{p}\,|0\rangle \;=\; 0\,, \tag{4.19}$$

The coherent state has no (initial) momentum. It has an initial position (as seen in 1. above)

3. Calculate the expectation value of the energy in a coherent state. Note that the coherent state is not an energy eigenstate (nor a position eigenstate, nor a momentum eigenstate!). With H the Hamiltonian we have

$$\langle\tilde{x}_0|H|x_0\rangle \;=\; \langle 0|T^\dagger_{x_0}HT_{x_0}|0\rangle\,. \tag{4.20}$$

We now compute

$$\begin{aligned} T^\dagger_{x_0}HT_{x_0} \;&=\; T^\dagger_{x_0}\Big(\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2\Big)T_{x_0} \;=\; \frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2(\hat{x}+x_0)^2 \\ &=\; H + m\omega^2 x_0\hat{x} + \frac{1}{2}m\omega^2 x_0^2\,. \end{aligned} \tag{4.21}$$

where we recall that T_{x_0} commutes with $\hat{p}$ and used eqn. (4.5). Back in (4.20) we have

$$\langle\tilde{x}_0|H|x_0\rangle \;=\; \langle 0|H|0\rangle + m\omega^2 x_0\langle 0|\hat{x}|0\rangle + \frac{1}{2}m\omega^2 x_0^2\,. \tag{4.22}$$

Recalling that the ground state energy is $\hbar\omega/2$ and that in the ground state $\hat{x}$ has no expectation value we finally get

$$\langle\tilde{x}_0|H|x_0\rangle \;=\; \frac{1}{2}\hbar\omega + \frac{1}{2}m\omega^2 x_0^2\,. \tag{4.23}$$

This is reasonable: the total energy is the zero-point energy plus the potential energy of a particle at x_0. The coherent state $|\tilde{x}_0\rangle$ is the quantum version of a point particle on a spring held stretched to $x = x_0$.

4.3 Time evolution and uncertainties

Evolving the coherent states in time is a somewhat involved procedure that will be explained later. We can discuss time evolution quite easily using the Heisenberg picture, since we have already calculated in (3.44) the time-dependent Heisenberg operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$.

If we have at time equal zero the coherent state $|\tilde{x}_0\rangle$ then at time t we write the time-evolved state as $|\tilde{x}_0, t\rangle$. We now ask what is the (time-dependent) expectation value of $\hat{x}$ on this state:

$$\langle\, \hat{x}\rangle(t) \;=\; \langle \tilde{x}_0, t|\, \hat{x}\, |\tilde{x}_0, t\rangle \;=\; \langle \tilde{x}_0|\, \hat{x}_H(t)\, |\tilde{x}_0\rangle\,. \tag{4.24}$$

Using (3.44) we get

$$\langle\, \hat{x}\rangle(t) \;=\; \langle \tilde{x}_0|\, \Big(\hat{x}\, \cos\omega t + \frac{1}{m\omega}\,\hat{p}\, \sin\omega t\Big)\, |\tilde{x}_0\rangle\,. \tag{4.25}$$

Finally, using (4.18) and (4.19) we get

$$\langle\, \hat{x}\rangle(t) \;=\; \langle \tilde{x}_0|\, \hat{x}_H(t)\, |\tilde{x}_0\rangle \;=\; x_0 \cos\omega t\,. \tag{4.26}$$

The expectation value of $\hat{x}$ is performing oscillatory motion! This confirms the classical interpretation of the coherent state. For the momentum the calculation is quite similar,

$$\langle\, \hat{p}\rangle(t) \;=\; \langle \tilde{x}_0|\, \hat{p}_H(t)\, |\tilde{x}_0\rangle \;=\; \langle \tilde{x}_0|\, \Big(\hat{p}\cos\omega t - m\omega\, \hat{x}\, \sin\omega t\Big)\, |\tilde{x}_0\rangle \tag{4.27}$$

and we thus find

$$\langle\, \hat{p}\rangle(t) \;=\; \langle \tilde{x}_0|\, \hat{p}_H(t)\, |\tilde{x}_0\rangle \;=\; -m\omega\, x_0\, \sin\omega t\,, \tag{4.28}$$

which is the expected result as it is equal to $m\frac{d}{dt}\langle\hat{x}\rangle(t)$.

We have seen that the harmonic oscillator ground state is a minimum uncertainty state. We will now discuss the extension of this fact to coherent states. We begin by calculating the uncertainties Δx and Δp in a coherent state at $t = 0$. We will see that the coherent state has minimum uncertainty for the product. Then we will calculate uncertainties of the coherent state as a function of time!

We have

$$\langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle \;=\; \langle 0|T^\dagger_{x_0}\hat{x}^2 T_{x_0}|0\rangle \;=\; \langle 0|(\hat{x}+x_0)^2|0\rangle \;=\; \langle 0|\hat{x}^2|0\rangle + x_0^2\,. \tag{4.29}$$

The first term on the right-hand side was calculated in (1.58). We thus find

$$\langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle \;=\; \frac{\hbar}{2m\omega} + x_0^2\,. \tag{4.30}$$

Since $\langle \tilde{x}_0|\hat{x}|\tilde{x}_0\rangle = x_0$ we find the uncertainty

$$(\Delta x)^2 \;=\; \langle \tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle - (\langle \tilde{x}_0|\hat{x}|\tilde{x}_0\rangle)^2 = \frac{\hbar}{2m\omega} + x_0^2 - x_0^2$$

$$\rightarrow \quad (\Delta x)^2 \;=\; \frac{\hbar}{2m\omega}\,, \quad \text{on the state } |\tilde{x}_0\rangle\,. \tag{4.31}$$

For the momentum operator we have, using (1.58),

$$\langle\tilde{x}_0|\hat{p}^2|\tilde{x}_0\rangle \;=\; \langle 0|T_{x_0}^\dagger \hat{p}^2 T_{x_0}|0\rangle \;=\; \langle 0|\hat{p}^2|0\rangle \;=\; \frac{m\hbar\omega}{2}\,. \tag{4.32}$$

Since $\langle\tilde{x}_0|\hat{p}|\tilde{x}_0\rangle = 0$, we have

$$(\Delta p)^2 \;=\; \frac{m\hbar\omega}{2}\,, \quad \text{on the state } |\tilde{x}_0\rangle\,. \tag{4.33}$$

As a result,

$$\Delta x \Delta p \;=\; \frac{\hbar}{2}\,, \quad \text{on the state } |\tilde{x}_0\rangle\,. \tag{4.34}$$

We see that the coherent state has minimum $\Delta x \Delta p$ at time equal zero. This is not surprising because at this time the state is just a displaced ground state.

For the time dependent situation we have

$$\begin{aligned}
(\Delta x)^2(t) \;&=\; \langle\tilde{x}_0, t|\hat{x}^2|\tilde{x}_0, t\rangle - \big(\langle\tilde{x}_0, t|\hat{x}|\tilde{x}_0, t\rangle\big)^2 \\
&=\; \langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle - \big(\langle\tilde{x}_0|\hat{x}_H(t)|\tilde{x}_0\rangle\big)^2 \\
&=\; \langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle - x_0^2\cos^2\omega t\,,
\end{aligned} \tag{4.35}$$

where we used the result in (4.26). The computation of the first term takes a few steps:

$$\begin{aligned}
\langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle \;&=\; \langle\tilde{x}_0|\Big(\hat{x}\,\cos\omega t + \frac{1}{m\omega}\,\hat{p}\,\sin\omega t\Big)^2|\tilde{x}_0\rangle \\
&=\; \langle\tilde{x}_0|\hat{x}^2|\tilde{x}_0\rangle\cos^2\omega t + \langle\tilde{x}_0|\hat{p}^2|\tilde{x}_0\rangle\Big(\frac{\sin\omega t}{m\omega}\Big)^2 + \frac{\cos m\omega\sin m\omega}{m\omega}\langle\tilde{x}_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|\tilde{x}_0\rangle \\
&=\; \Big(\frac{\hbar}{2m\omega} + x_0^2\Big)\cos^2\omega t + \frac{m\hbar\omega}{2}\Big(\frac{\sin\omega t}{m\omega}\Big)^2 + \frac{\cos m\omega\sin m\omega}{m\omega}\langle\tilde{x}_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|\tilde{x}_0\rangle\,.
\end{aligned}$$

We now show that the last expectation value vanishes:

$$\begin{aligned}
\langle\tilde{x}_0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|\tilde{x}_0\rangle \;&=\; \langle 0|\big((\hat{x}+x_0)\hat{p} + \hat{p}(\hat{x}+x_0)\big)|0\rangle \\
&=\; \langle 0|\big(\hat{x}\hat{p}+\hat{p}\hat{x}\big)|0\rangle \\
&=\; i\frac{\hbar}{2}\,\langle 0|\big((\hat{a}+\hat{a}^\dagger)(\hat{a}^\dagger-\hat{a}) + (\hat{a}^\dagger-\hat{a})(\hat{a}+\hat{a}^\dagger)\big)|0\rangle \\
&=\; i\frac{\hbar}{2}\,\langle 0|\big(\hat{a}\hat{a}^\dagger + (-\hat{a})\hat{a}^\dagger\big)|0\rangle \;=\; 0\,.
\end{aligned} \tag{4.36}$$

As a result,

$$\begin{aligned}
\langle\tilde{x}_0|\hat{x}_H^2(t)|\tilde{x}_0\rangle \;&=\; \Big(\frac{\hbar}{2m\omega} + x_0^2\Big)\cos^2\omega t + \frac{m\hbar\omega}{2}\Big(\frac{\sin\omega t}{m\omega}\Big)^2 \\
&=\; \frac{\hbar}{2m\omega} + x_0^2\cos^2\omega t\,.
\end{aligned} \tag{4.37}$$

Therefore, finally, back in (4.35) we get

$$(\Delta x)^2(t) \;=\; \frac{\hbar}{2m\omega}\,. \tag{4.38}$$

The uncertainty Δx does not change in time as the state evolves! This suggests, but does not yet prove, that the state does not change shape[4]. It is therefore useful to calculate the time-dependent uncertainty in the momentum:

$$\begin{aligned}(\Delta p)^2(t) \;&=\; \langle \tilde{x}_0, t|\hat{p}^2|\tilde{x}_0, t\rangle - \big(\langle \tilde{x}_0, t|\hat{p}|\tilde{x}_0, t\rangle\big)^2 \\ &=\; \langle \tilde{x}_0|\hat{p}_H^2(t)|\tilde{x}_0\rangle - \big(\langle \tilde{x}_0|\hat{p}_H(t)|\tilde{x}_0\rangle\big)^2 \\ &=\; \langle \tilde{x}_0|\hat{p}_H^2(t)|\tilde{x}_0\rangle - m^2\omega^2\, x_0^2 \sin^2\omega t\,,\end{aligned} \tag{4.39}$$

where we used (4.28). The rest of the computation (recommended!) gives

$$\langle \tilde{x}_0|\hat{p}_H^2(t)|\tilde{x}_0\rangle \;=\; \frac{1}{2}\, m\hbar\omega \;+\; m^2\omega^2\, x_0^2 \sin^2\omega t\,, \tag{4.40}$$

so that we have

$$(\Delta p)^2(t) \;=\; \frac{m\hbar\omega}{2}\,. \tag{4.41}$$

This together with (4.38) gives

$$\Delta x(t)\Delta p(t) \;=\; \frac{\hbar}{2}\,, \qquad \text{on the state } |\tilde{x}_0, t\rangle\,. \tag{4.42}$$

The coherent state remains a minimum $\Delta x\Delta p$ packet for all times. Since only gaussians have such minimum uncertainty, the state remains a gaussian for all times! Since Δx is constant the gaussian does not change shape. Thus the name *coherent state*, the state does not spread out in time, it just moves "coherently" without changing its shape.

In the harmonic oscillator there is a quantum length scale d that can be constructed from $\hbar, m$, and ω. This length scale appears, for example, in the uncertainty Δx in (4.38). We thus define

$$d \;\equiv\; \sqrt{\frac{\hbar}{m\omega}}\,, \tag{4.43}$$

and note that

$$\Delta x(t) \;=\; \frac{d}{\sqrt{2}}\,. \tag{4.44}$$

The length d is typically very small for a macroscopic oscillator. A coherent state with a large x_0 –large compared to d– is classical in the sense that the position uncertainty $\sim d$, is much smaller than the typical excursion x_0. Similarly, the momentum uncertainty

$$\Delta p(t) \;=\; m\omega\,\frac{d}{\sqrt{2}}\,. \tag{4.45}$$

[4]By this we mean that the shape of $|\psi(x,t)|^2$ does not change: at different times $|\psi(x,t)|^2$ and $|\psi(x,t')|^2$ differ just by an overall displacement in x.

is much smaller than the typical momentum $m\omega x_0$, by just the same factor $\sim d/x_0$.

Problem. Prove that

$$\frac{\Delta p(t)}{\sqrt{\overline{\langle \hat{p}^2\rangle(t)}}} = \frac{\Delta x(t)}{\sqrt{\overline{\langle \hat{x}^2\rangle(t)}}} = \frac{1}{\sqrt{1+\frac{x_0^2}{d^2}}} \tag{4.46}$$

where the overlines on the expectation values denote time average.

4.4 Coherent states in the energy basis

We can get an interesting expression for the coherent state $|\tilde{x}_0\rangle$ by rewriting the momentum operator in terms of creation and annihilation operators. From (1.18) we have

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(\hat{a}^\dagger - \hat{a}) = i\,\frac{\hbar}{\sqrt{2}\,d}\,(\hat{a}^\dagger - \hat{a})\,. \tag{4.47}$$

The final form is also nice to see that units work. We now have that the coherent state (4.14) is given by

$$|\tilde{x}_0\rangle = \exp\Big(-\frac{i}{\hbar}\,\hat{p}\,x_0\Big)|0\rangle = \exp\Big(\frac{x_0}{\sqrt{2}d}\,(\hat{a}^\dagger - \hat{a})\Big)|0\rangle\,. \tag{4.48}$$

Since $\hat{a}|0\rangle = 0$ the above formula admits simplification: we should be able to get rid of all the $\hat{a}$'s! We could do this if we could split the exponential into two exponentials, one with the $\hat{a}^\dagger$'s to the *left* of another one with the $\hat{a}$'s. The exponential with the $\hat{a}$'s would stand near the vacuum and give no contribution, as we will see below. For this purpose we recall the commutator identity

$$e^{X+Y} = e^X\,e^Y\,e^{-\frac{1}{2}[X,Y]}\,, \quad \text{if } [X,Y] \text{ commutes with } X \text{ and with } Y. \tag{4.49}$$

Think of the term we are interested in as it appears in (4.48), and identify X and Y as

$$\exp\Big(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger - \frac{x_0}{\sqrt{2}d}\hat{a}\Big) \quad\to\quad X = \frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\,,\quad Y = -\frac{x_0}{\sqrt{2}d}\hat{a} \tag{4.50}$$

Then

$$[X,Y] = -\frac{x_0^2}{2d^2}\,[\,\hat{a}^\dagger\,,\hat{a}] = \frac{x_0^2}{2d^2} \tag{4.51}$$

and we find

$$\exp\Big(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger - \frac{x_0}{\sqrt{2}d}\hat{a}\Big) = \exp\Big(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\Big)\,\exp\Big(-\frac{x_0}{\sqrt{2}d}\hat{a}\Big)\,\exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big) \tag{4.52}$$

Since the last exponential is just a number, and $\exp(\gamma\hat{a})|0\rangle = |0\rangle$, for any γ, we have

$$\exp\Big(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger - \frac{x_0}{\sqrt{2}d}\hat{a}\Big)|0\rangle = \exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\exp\Big(\frac{x_0}{\sqrt{2}d}\hat{a}^\dagger\Big)|0\rangle\,. \tag{4.53}$$

As a result, our coherent state in (4.48) becomes

$$\boxed{|\tilde{x}_0\rangle = \exp\Big(-\frac{i}{\hbar}\,\hat{p}\,x_0\Big)|0\rangle = \exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\exp\Big(\frac{x_0}{\sqrt{2}d}\,\hat{a}^\dagger\Big)|0\rangle\,.} \tag{4.54}$$

While this form is quite nice to produce an expansion in energy eigenstates, the unit normalization of the state is not manifest anymore. Expanding the exponential with creation operators we get

$$\begin{aligned} |\tilde{x}_0\rangle &= \sum_{n=0}^{\infty}\exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\cdot\frac{1}{n!}\Big(\frac{x_0}{\sqrt{2}d}\Big)^n(\hat{a}^\dagger)^n|0\rangle \\ &= \sum_{n=0}^{\infty}\exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\cdot\frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}d}\Big)^n|n\rangle \end{aligned} \tag{4.55}$$

We thus have the desired expansion:

$$\boxed{|\tilde{x}_0\rangle = \sum_{n=0}^{\infty}c_n|n\rangle\,, \quad \text{with} \quad c_n = \exp\Big(-\frac{1}{4}\frac{x_0^2}{d^2}\Big)\cdot\frac{1}{\sqrt{n!}}\Big(\frac{x_0}{\sqrt{2}d}\Big)^n.} \tag{4.56}$$

Since the probability to find the energy E_n is equal to c_n^2, we note that

$$c_n^2 = \exp\Big(-\frac{x_0^2}{2d^2}\Big)\cdot\frac{1}{n!}\Big(\frac{x_0^2}{2d^2}\Big)^n \tag{4.57}$$

If we define the quantity $\lambda(x_0, d)$ as

$$\boxed{\lambda \equiv \frac{x_0^2}{2d^2}\,,} \tag{4.58}$$

we can then see that

$$\boxed{c_n^2 = \frac{\lambda^n}{n!}\,e^{-\lambda}\,.} \tag{4.59}$$

The probability to measure an energy $E_n = \hbar\omega(n+\frac{1}{2})$ in the coherent state is c_n^2, so the c_n^2's must define a probability distribution for $n \in \mathbb{Z}$, parameterized by λ. This is in fact the familiar *Poisson distribution*. It is straightforward to verify that

$$\sum_{n=0}^{\infty}c_n^2 = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} = e^{-\lambda}e^{\lambda} = 1\,, \tag{4.60}$$

as it should be. The physical interpretation of λ can be obtained by computing the expectation value of n:[5]

$$\langle n\rangle \;\equiv\; \sum_{n=0}^{\infty} n c_n^2 \;=\; e^{-\lambda}\sum_{n=0}^{\infty} n\,\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\sum_{n=0}^{\infty}\lambda\frac{d}{d\lambda}\frac{\lambda^n}{n!} \;=\; e^{-\lambda}\,\lambda\frac{d}{d\lambda}e^{\lambda} \;=\lambda\,. \tag{4.61}$$

Therefore λ equals the expected value $\langle n\rangle$. Note that $\langle n\rangle$ is just the expected value of the number operator $\hat{N}$ on the coherent state. Indeed,

$$\langle\tilde{x}_0|\hat{N}|\tilde{x}_0\rangle \;=\; \sum_{n,m} c_m c_n\langle m|\hat{N}|n\rangle \;=\; \sum_{n,m} c_m c_n\, n\delta_{m,n} \;=\; \sum_n n\, c_n^2 \;=\; \langle n\rangle\,. \tag{4.62}$$

It is also easy to verify (do it!) that

$$\langle n^2\rangle \;\equiv\; \sum_{n=0}^{\infty} n^2\, c_n^2 \;=\; \lambda^2+\lambda\,. \tag{4.63}$$

It then follows that

$$(\Delta n)^2 \;=\; \langle n^2\rangle - \langle n\rangle^2 \;=\; \lambda \qquad\to\qquad \Delta n \;=\; \sqrt{\lambda}\,. \tag{4.64}$$

In terms of energy we have $E=\hbar\omega(n+\frac{1}{2})$ so that

$$\langle E\rangle \;=\; \hbar\omega\Big(\langle n\rangle+\frac{1}{2}\Big) \;=\; \hbar\omega\Big(\lambda+\frac{1}{2}\Big)\,. \tag{4.65}$$

Similarly,

$$\langle E^2\rangle \;=\; \hbar^2\omega^2\Big\langle\Big(n+\frac{1}{2}\Big)^2\Big\rangle \;=\; \hbar^2\omega^2\Big\langle n^2+n+\frac{1}{4}\Big\rangle \;=\; \hbar^2\omega^2\Big(\lambda^2+\lambda+\lambda+\frac{1}{4}\Big)\,, \tag{4.66}$$

so that

$$\langle E^2\rangle \;=\; \hbar^2\omega^2\Big(\lambda^2+2\lambda+\frac{1}{4}\Big) \tag{4.67}$$

The energy uncertainty is thus obtained as

$$(\Delta E)^2 \;=\; \langle E^2\rangle - \langle E\rangle^2 \;=\; \hbar^2\omega^2\Big(\lambda^2+2\lambda+\frac{1}{4}-\Big(\lambda+\frac{1}{2}\Big)^2\Big) \;=\; \hbar^2\omega^2\,\lambda\,, \tag{4.68}$$

so that

$$\Delta E \;=\; \hbar\omega\sqrt{\lambda} \;=\; \hbar\omega\,\frac{x_0}{\sqrt{2}d}\,. \tag{4.69}$$

Note now the fundamental inequality, holding for $x_0/d \gg 1$,

$$\hbar\omega \quad\ll\quad \Delta E=\hbar\omega\,\frac{x_0}{\sqrt{2}d} \quad\ll\quad \langle\, E\,\rangle=\hbar\omega\Big(\frac{x_0}{\sqrt{2}d}\Big)^2+\frac{1}{2}\hbar\omega\,. \tag{4.70}$$

[5]Here we are thinking of n as a random variable of the probability distribution. In the quantum viewpoint $\langle n\rangle$ is simply the expectation value of the number operator.

We see that the uncertainty ΔE is big enough to contain about $\frac{x_0}{\sqrt{2}d}$ levels of the harmonic oscillator –a lot of levels. But even then, ΔE is about a factor $\frac{x_0}{\sqrt{2}d}$ smaller than the expected value $E\rangle$ of the energy. So, alternatively,

$$\frac{\Delta E}{\hbar\omega} \simeq \frac{x_0}{\sqrt{2}d} \simeq \frac{\langle E\rangle}{\Delta E}\,. \tag{4.71}$$

This is part of the semi-classical nature of coherent states.

Example of Poisson distribution. Consider a sample of radioactive material with $N_0 \gg 1$ atoms at $t = 0$. Assume that the half-lifetime of the material is τ_0, which means that the number $N(t)$ of atoms that have not yet decayed after time $t > 0$ is given by

$$N(t) = N_0 \exp(-t/\tau_0) \quad \rightarrow \quad \frac{dN}{dt}(t=0) = \frac{N_0}{\tau_0}\,.$$

It follows that in the time interval $t \in [0, \Delta t]$, with $\Delta t \ll \tau_0$ we expect a number of decays

$$\frac{N_0 \Delta t}{\tau_0} \equiv \lambda\,.$$

One can then show that the probability p_n to observe n decays during that same time interval Δt is (approximately) given by the Poisson distribution: $p_n = \frac{\lambda^n}{n!}e^{-\lambda}$.

4.5 General coherent states and time evolution

We wrote earlier coherent states using creation and annihilation operators:

$$|\tilde{x}_0\rangle = \exp\Big(-\frac{i}{\hbar}\,\hat{p}\,x_0\Big)|0\rangle = \exp\Big(\frac{x_0}{\sqrt{2}d}\,(\hat{a}^\dagger - \hat{a})\Big)|0\rangle\,. \tag{4.72}$$

Such coherent states can be written as

$$|\alpha\rangle \equiv e^{\alpha\,(\hat{a}^\dagger - \hat{a})}|0\rangle\,, \quad \text{with} \quad \alpha = \frac{x_0}{\sqrt{2}d}\,. \tag{4.73}$$

This notation is not free of ambiguity: the label α in the coherent state above is now the coefficient of the factor $\hat{a}^\dagger - \hat{a}$ in the exponential. An obvious generalization is to take α to be a complex number: $\alpha \in \mathbb{C}$. This must be done with a little care, since the key property of the operator in the exponential (4.73) is that it is antihermitian (thus the exponential is unitary, as desired). We thus define

$$\boxed{|\alpha\rangle \equiv D(\alpha)|0\rangle \equiv \exp\big(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\big)|0\rangle\,, \quad \text{with} \quad \alpha \in \mathbb{C}\,.} \tag{4.74}$$

In this definition we introduced the unitary 'displacement' operator

$$D(\alpha) \equiv \exp\Big(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\Big)\,. \tag{4.75}$$

Since $D(\alpha)$ is unitary it is clear that $\langle\alpha|\alpha\rangle = 1$.

The action of the annihilation operator on the states $|\alpha\rangle$ is quite interesting,

$$\begin{aligned} \hat{a}|\alpha\rangle \;&=\; \hat{a}\, e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; \big[\,\hat{a}\,,\, e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}\big]|0\rangle \\ &=\; \big[\,\hat{a}\,,\, \alpha\hat{a}^\dagger - \alpha^*\hat{a}\,\big]e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle \;=\; \alpha e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}|0\rangle\,, \end{aligned} \tag{4.76}$$

so that we conclude that

$$\boxed{\hat{a}\,|\alpha\rangle \;=\; \alpha\,|\alpha\rangle\,.} \tag{4.77}$$

This result is kind of shocking: we have found eigenstates of the *non-hermitian* operator $\hat{a}$. Because $\hat{a}$ is not hermitian, our theorems about eigenstates and eigenvectors of hermitian operators do not apply. Thus, for example, the eigenvalues need not be real (they are not, in general $\alpha \in \mathbb{C}$), eigenvectors of different eigenvalue need not be orthogonal (they are not!) and the set of eigenvectors need not form a complete basis (coherent states actually give an overcomplete basis!).

Ordering the exponential in the state $|\alpha\rangle$ in (4.74) we find

$$|\alpha\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2} e^{\alpha\hat{a}^\dagger}|0\rangle\,. \tag{4.78}$$

Exercise. Show that

$$\langle\beta|\alpha\rangle \;=\; \exp\Big(-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \beta^*\alpha\Big)\,. \tag{4.79}$$

Hint: You may find it helpful to evaluate $e^{\beta^*\hat{a} + \alpha\hat{a}^\dagger}$ in two different ways using (4.49).

To find the physical interpretation of the complex number α we first note that when real, as in (4.73), α encodes the initial position x_0 of the coherent state (more precisely, it encodes the expectation value of $\hat{x}$ in the state at $t = 0$). For complex α, its real part is still related to the initial position:

$$\langle\alpha|\hat{x}|\alpha\rangle \;=\; \frac{d}{\sqrt{2}}\,\langle\alpha|(\hat{a} + \hat{a}^\dagger)|\alpha\rangle \;=\; \frac{d}{\sqrt{2}}(\alpha + \alpha^*) \;=\; d\sqrt{2}\;\mathrm{Re}(\alpha)\,, \tag{4.80}$$

where we used (1.18) and (4.77) both on bras and on kets. We have thus learned that

$$\mathrm{Re}(\alpha) \;=\; \frac{\langle\hat{x}\rangle}{\sqrt{2}\,d}\,. \tag{4.81}$$

It is natural to conjecture that the imaginary part of α is related to the momentum expectation value on the initial state. So we explore

$$\langle\alpha|\hat{p}|\alpha\rangle \;=\; \frac{i\hbar}{\sqrt{2}d}\,\langle\alpha|(\hat{a}^\dagger - \hat{a})|\alpha\rangle \;=\; -\frac{i\hbar}{\sqrt{2}d}(\alpha - \alpha^*) \;=\; -\frac{i\hbar}{\sqrt{2}d}(2i\mathrm{Im}(\alpha)) \;=\; \frac{\hbar\sqrt{2}}{d}\;\mathrm{Im}(\alpha)\,, \tag{4.82}$$

and learn that

$$\mathrm{Im}(\alpha) \;=\; \frac{\langle \hat{p}\rangle\, d}{\sqrt{2}\,\hbar}\,. \tag{4.83}$$

The identification of α in terms of expectation values of $\hat{x}$ and $\hat{p}$ is now complete:

$$\boxed{\alpha \;=\; \frac{\langle \hat{x}\rangle}{\sqrt{2}\,d} \;+\; i\,\frac{\langle \hat{p}\rangle\, d}{\sqrt{2}\,\hbar}\,.} \tag{4.84}$$

A calculation in the problem set shows that

$$\alpha\,\hat{a}^\dagger - \alpha^*\hat{a} \;=\; -\frac{i}{\hbar}\big(\hat{p}\,\langle x\rangle - \langle \hat{p}\rangle\,\hat{x}\big)\,, \tag{4.85}$$

affording yet another rewriting of the general coherent state (4.74), valid when α is defined as in (4.84):

$$\boxed{|\alpha\rangle \;=\; \exp\Big(-\frac{i\hat{p}\,\langle x\rangle}{\hbar} + \frac{i\langle \hat{p}\rangle\,\hat{x}}{\hbar}\Big)|0\rangle\,.} \tag{4.86}$$

In order to find the time evolution of the coherent state we can use a trick from the Heisenberg picture. We have using (4.74)

$$|\alpha, t\rangle \;\equiv\; e^{-i\frac{Ht}{\hbar}}|\alpha\rangle \;=\; e^{-i\frac{Ht}{\hbar}}e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}|0\rangle \;=\; \Big(e^{-i\frac{Ht}{\hbar}}e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}e^{i\frac{Ht}{\hbar}}\Big)e^{-i\frac{Ht}{\hbar}}|0\rangle \tag{4.87}$$

For a time independent Hamiltonian (as that of the SHO) and a Schrödinger operator $\mathcal{O}$, we have

$$\mathcal{O}_H(t) \;=\; e^{iHt/\hbar}\,\mathcal{O}\,e^{-iHt/\hbar} \tag{4.88}$$

and therefore with the opposite signs for the exponentials we get

$$e^{-iHt/\hbar}\,\mathcal{O}\,e^{iHt/\hbar} \;=\; \mathcal{O}_H(-t)\,. \tag{4.89}$$

Such a relation is also valid for any function of an operator:

$$e^{-iHt/\hbar}\,F(\mathcal{O})\,e^{iHt/\hbar} \;=\; F(\mathcal{O}_H(-t))\,. \tag{4.90}$$

as you can convince yourself is the case whenever $F(x)$ has a good Taylor expansion in powers of x. It then follows that back in (4.87) we have

$$|\alpha, t\rangle \;=\; \exp\Big(\alpha\hat{a}^\dagger(-t) - \alpha^*\hat{a}(-t)\Big)e^{-i\omega t/2}|0\rangle\,. \tag{4.91}$$

Recalling ((3.53)) that $\hat{a}(t) \;=\; e^{-i\omega t}\,\hat{a}$, and thus $\hat{a}^\dagger(t) = e^{i\omega t}\,\hat{a}^\dagger$, we find

$$|\alpha, t\rangle \;=\; e^{-i\omega t/2}\exp\Big(\alpha e^{-i\omega t}\hat{a}^\dagger - \alpha^* e^{i\omega t}\hat{a}\Big)|0\rangle\,. \tag{4.92}$$

Looking at the exponential we see that it is in fact the displacement operator with $\alpha \to \alpha e^{-i\omega t}$. As a result we have shown that

$$\boxed{\quad |\alpha, t\rangle \;=\; e^{-i\omega t/2} |e^{-i\omega t}\alpha\rangle\,. \quad} \tag{4.93}$$

This is how a coherent state $|\alpha\rangle$ evolves in time: up to an irrelevant phase, the state remains a coherent state with a time-varying parameter $e^{-i\omega t}\alpha$. In the complex α plane the state is represented by a vector that rotates in the clockwise direction with angular velocity ω. The α plane can be viewed as having a real axis that gives $\langle x\rangle$ (up to a proportionality constant) and an imaginary axis that gives $\langle \hat{p}\rangle$ (up to a proportionality constant). It is a phase space and the evolution of any state is represented by a circle. This is illustrated in Figure 3.

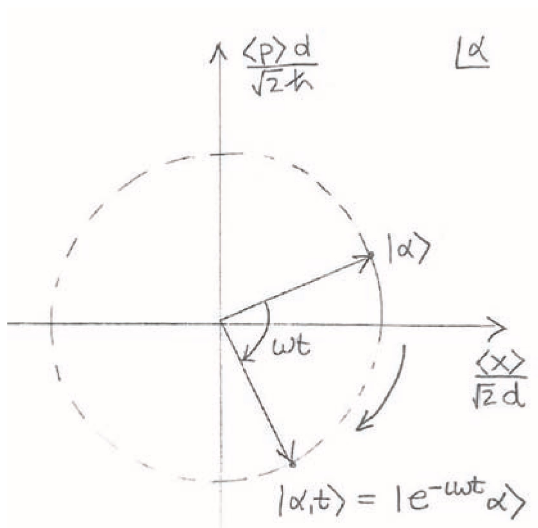

Figure 3: Time evolution of the coherent state $|\alpha\rangle$. The real and imaginary parts of α determine the expectation values $\langle x\rangle$ and $\langle p\rangle$ respectively. As time goes by the α parameter of the coherent state rotates clockwise with angular velocity ω.

An alternative, conventional, calculation of the time evolution begins by expanding the exponential in (4.78) to find:

$$|\alpha\rangle \;=\; e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, \alpha^n |n\rangle\,. \tag{4.94}$$

The time-evolved state is then given by the action of $\exp(-iHt/\hbar)$:

$$
\begin{aligned}
|\alpha, t\rangle \;\equiv\; e^{-i\frac{Ht}{\hbar}}|\alpha\rangle \;&=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,\alpha^n e^{-i\hbar\omega(n+\frac{1}{2})t/\hbar}|n\rangle \\
&=\; e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,\alpha^n e^{-i\omega t n}e^{-i\omega t/2}|n\rangle \\
&=\; e^{-i\omega t/2}\;\underbrace{e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,(e^{-i\omega t}\alpha)^n|n\rangle}\;.
\end{aligned}
\tag{4.95}
$$

Using (4.94) and noting that $|e^{-i\omega t}\alpha|^2 = |\alpha|^2$, we identify the terms under the brace as a coherent state $|\alpha e^{-i\omega t}\rangle$. This gives the earlier result (4.93).

In the coherent state $|\alpha\rangle$ the expectation value of $\hat{N}$ is easily calculated

$$
\langle\hat{N}\rangle_\alpha \;=\; \langle\alpha|\hat{N}|\alpha\rangle \;=\; \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; \langle\alpha|\alpha^*\alpha|\alpha\rangle = |\alpha|^2\,. \tag{4.96}
$$

To find the uncertainty ΔN we also compute

$$
\begin{aligned}
\langle\hat{N}^2\rangle_\alpha \;&=\; \langle\alpha|\,\hat{a}^\dagger\hat{a}\;\hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; |\alpha|^2\langle\alpha|\,\hat{a}\;\hat{a}^\dagger|\alpha\rangle \\
&=\; |\alpha|^2\langle\alpha|\,[\hat{a}\,,\hat{a}^\dagger] + \hat{a}^\dagger\hat{a}|\alpha\rangle \;=\; |\alpha|^2\big(1+|\alpha|^2\big)\,.
\end{aligned}
\tag{4.97}
$$

From these results we get

$$
(\Delta N)^2 \;=\; \langle\hat{N}^2\rangle_\alpha - \langle\hat{N}\rangle_\alpha^2 \;=\; |\alpha|^2 + |\alpha|^4 - |\alpha|^4 = |\alpha|^2 \tag{4.98}
$$

so that

$$
\boxed{\Delta N \;=\; |\alpha|\,.} \tag{4.99}
$$

in Figure 3 the magnitude of the rotating phasor is ΔN and the square of the magnitude is the expectation value $\langle\hat{N}\rangle_\alpha$.

We will soon discuss electromagnetic fields and waves as coherent states of photons. For such waves a number/phase uncertainty exists. A rough argument goes as follows. For a wave with N photons with frequency ω, the energy is $E = N\hbar\omega$ and the phase ϕ of the wave goes like $\phi = \omega t$. It follows that $\Delta E \sim \Delta N\hbar\omega$ and $\Delta\phi = \omega\Delta t$ (with the admittedly ambiguous meaning of Δt). Therefore

$$
\Delta E\Delta t \geq \frac{\hbar}{2} \quad\rightarrow\quad \Delta N\hbar\omega\,\frac{\Delta\phi}{\omega} \geq \frac{\hbar}{2} \quad\rightarrow\quad \Delta N\Delta\phi \geq \frac{1}{2}\,. \tag{4.100}
$$

A better intuition for this result follows from our coherent state $|\alpha\rangle$ for which we know that $\Delta N = |\alpha|$. The position and momentum uncertainties are the same as for the ground state:

$$
\Delta x \;=\; \frac{d}{\sqrt{2}}\,,\quad \Delta p \;=\; \frac{\hbar}{d\sqrt{2}} \tag{4.101}
$$

When we measure x on the state $|\alpha\rangle$ we expect to get a good fraction of values in a range Δx about the expected value $\langle x\rangle$ of x. This is, of course, just a rough estimate.

$$\text{Representative range for measured } x \;=\; [\langle x\rangle - \tfrac{1}{2}\Delta x\,,\, \langle x\rangle + \tfrac{1}{2}\Delta x] \tag{4.102}$$

Dividing by $\sqrt{2}d$ we have

$$\text{Representative range for measured } \frac{x}{\sqrt{2}d} \;=\; \Big[\frac{\langle x\rangle}{\sqrt{2}d} - \tfrac{1}{2}\,,\, \frac{\langle x\rangle}{\sqrt{2}d} + \tfrac{1}{2}\Big] \tag{4.103}$$

It follows that the position measurements, indicated on the horizontal axis of Figure 3, spread over a representative range of width one. Similarly, for momentum we have

$$\text{Representative range for measured } p \;=\; [\langle p\rangle - \tfrac{1}{2}\Delta p\,,\, \langle p\rangle + \tfrac{1}{2}\Delta p] \tag{4.104}$$

Multiplying by $d/(\sqrt{2}\hbar)$, we have

$$\text{Representative range for measured } \frac{p\,d}{\sqrt{2}\hbar} \;=\; \Big[\frac{\langle p\rangle\, d}{\sqrt{2}\hbar} - \tfrac{1}{2}\,,\, \frac{\langle p\rangle\, d}{\sqrt{2}\hbar} + \tfrac{1}{2}\Big] \tag{4.105}$$

It follows that the momentum measurements, indicated on the vertical axis of Figure 3, spread

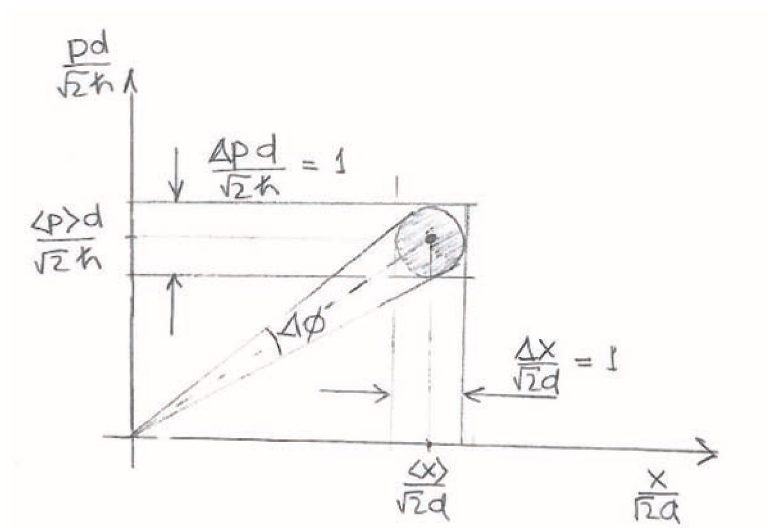

Figure 4: When doing measurements on $|\alpha\rangle$ the uncertainties on the value of α can be represented by a blob of unit diameter centered at α. The projections of this blob on the axes are, up to scale, the uncertainties Δx and Δp.

over a representative range of width one. We can thus reconsider the plot, this time indicating the ranges of values expected on the horizontal and vertical axes. Those ranges can be viewed as some kind of uncertainty in the value of α that we could find by measurements on the state

$|\alpha\rangle$. We draw a blob of unit diameter centered at α whose projections down along the axes reproduce the uncertainty ranges. This is shown in Figure 4. In the spirit of the discussion on time dependence, this blob must be imagined rotating with angular frequency ω. In such picture we have a phase ambiguity $\Delta\phi$, represented in the picture as the angle subtended by the uncertainty blob. Since the blob has diameter one and is centered at α, which is a distance $|\alpha|$ from the origin, we have

$$\Delta\phi \simeq \frac{1}{|\alpha|} \tag{4.106}$$

Recalling that $\Delta N = |\alpha|$ we finally obtain that for our coherent state

$$\Delta N \Delta\phi \simeq 1\,. \tag{4.107}$$

This is a familiar relation for coherent states of light. It then relates the uncertainty in the number of photons to the uncertainty in the phase of the wave.

5 Squeezed states

Squeezed states of the harmonic oscillator are states that are obtained by acting on the ground state with an exponential that includes terms quadratic in creation operators. They are the most general states for which $\Delta x \Delta p = \hbar/2$, thus achieving saturation of the uncertainty bound.

5.1 Squeezed vacuum states

One useful way to motivate the introduction of squeezed states is to consider the ground state of a harmonic oscillator Hamiltonian with mass and frequency parameters m_1 and ω_1, respectively:

$$H_1 = \frac{p^2}{2m_1} + \frac{1}{2}\, m_1 \omega_1^2 x^2\,. \tag{5.1}$$

Such ground state has uncertaintites Δx and Δp that follow from (1.58) :

$$\begin{aligned} \Delta x &= \sqrt{\frac{\hbar}{2m_1\omega_1}}\,, \\ \Delta p &= \sqrt{\frac{\hbar m_1 \omega_1}{2}}\,. \end{aligned} \tag{5.2}$$

Note that the product of uncertainties saturates the lower bound:

$$\Delta x \Delta p = \frac{\hbar}{2}\,. \tag{5.3}$$

Now we consider the following situation: suppose at time $t = 0^-$ the wavefunction is indeed that of the ground state of the oscillator. At $t = 0$, however, the oscillator parameters *change*