Problem 10.20

Using Eq. 10.88 and integration by parts:

$$\begin{aligned}
f(\theta) &= -\frac{2m}{\hbar^2\kappa}\int_0^\infty rAe^{-\mu r^2}\sin(\kappa r)\,dr = -\frac{2mA}{\hbar^2\kappa}\int_0^\infty \frac{d}{dr}\left(-\frac{1}{2\mu}e^{-\mu r^2}\right)\sin(\kappa r)\,dr \\
&= \frac{2mA}{2\mu\hbar^2\kappa}\left\{e^{-\mu r^2}\sin(\kappa r)\Big|_0^\infty - \int_0^\infty e^{-\mu r^2}\frac{d}{dr}\left[\sin(\kappa r)\right]\,dr\right\} \\
&= \frac{mA}{\mu\hbar^2\kappa}\left\{0-\kappa\int_0^\infty e^{-\mu r^2}\cos(\kappa r)\,dr\right\} = -\frac{mA}{\mu\hbar^2}\left(\frac{\sqrt{\pi}}{2\sqrt{\mu}}e^{-\kappa^2/4\mu}\right) \\
&= -\frac{mA\sqrt{\pi}}{2\hbar^2\mu^{3/2}}e^{-\kappa^2/4\mu}, \quad \text{where} \quad \kappa = 2k\sin(\theta/2) \quad \text{(Eq. 10.89)}.
\end{aligned}$$

From Eq. 10.14, then,

$$\frac{d\sigma}{d\Omega} = \frac{\pi m^2A^2}{4\hbar^4\mu^3}e^{-\kappa^2/2\mu},$$

and hence

$$\begin{aligned}
\sigma &= \int \frac{d\sigma}{d\Omega}\,d\Omega = \frac{\pi m^2A^2}{4\hbar^4\mu^3}\int e^{-4k^2\sin^2(\theta/2)/2\mu}\sin\theta\,d\theta\,d\phi \\
&= \frac{\pi^2m^2A^2}{2\hbar^4\mu^3}\int_0^\pi e^{-2k^2\sin^2(\theta/2)/\mu}\sin\theta\,d\theta; \quad \text{write } \sin\theta = 2\sin(\theta/2)\cos(\theta/2) \quad \text{and let } x\equiv\sin(\theta/2) \\
&= \frac{\pi^2m^2A^2}{2\hbar^4\mu^3}\int_0^1 e^{-2k^2x^2/\mu}2x\,2\,dx = \frac{2\pi^2m^2A^2}{\hbar^4\mu^3}\int_0^1 xe^{-2k^2x^2/\mu}\,dx \\
&= \frac{2\pi^2m^2A^2}{\hbar^4\mu^3}\left[-\frac{\mu}{4k^2}e^{-2k^2x^2/\mu}\right]\Big|_0^1 = -\frac{\pi^2m^2A^2}{2\hbar^4\mu^2k^2}\left(e^{-2k^2/\mu}-1\right) \\
&= \boxed{\frac{\pi^2m^2A^2}{2\hbar^4\mu^2k^2}\left(1-e^{-2k^2/\mu}\right).}
\end{aligned}$$

Problem 10.21

(a) In the first Born approximation (Equation 10.79):

$$\begin{aligned}
f(\theta,\phi) &\approx -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0}V(\mathbf{r}_0)\,d^3\mathbf{r}_0 = -\frac{m}{2\pi\hbar^2}\frac{2\pi\hbar^2 b}{m}\sum_i\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0}\delta^3(\mathbf{r}_0-\mathbf{r}_i)\,d^3\mathbf{r}_0 \\
&= -b\sum_i e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_i} = -b\sum_i e^{-i\mathbf{q}\cdot\mathbf{r}_i}; \\
\frac{d\sigma}{d\Omega} &= |f|^2 = b^2\left|\sum_i e^{-i\mathbf{q}\cdot\mathbf{r}_i}\right|^2. \quad \checkmark
\end{aligned}$$

(b)

$$\sum_i e^{-i\mathbf{q}\cdot\mathbf{r}_i} = \sum_{l,m,n}e^{-i(q_xla+q_yma+q_zna)} = \sum_{l=0}^{N-1}e^{-iq_xla}\sum_{m=0}^{N-1}e^{-iq_yma}\sum_{n=0}^{N-1}e^{-iq_zna}.$$

Now, letting $u \equiv \exp(-iq_x a)$,

$$\sum_{l=0}^{N-1} e^{-iq_x la} = \sum_{l=0}^{N-1} u^l = \frac{1-u^N}{1-u} = \frac{1-e^{-iq_x aN}}{1-e^{-iq_x a}} = \frac{e^{-iq_x aN/2}\left(e^{iq_x aN/2} - e^{-iq_x aN/2}\right)}{e^{-iq_x a/2}\left(e^{iq_x a/2} - e^{-iq_x a/2}\right)}$$
$$= e^{-iq_x a(N-1)/2}\,\frac{\sin(q_x aN/2)}{\sin(q_x a/2)},$$

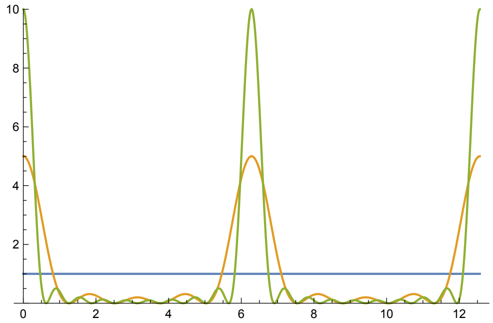
so

$$\frac{d\sigma}{d\Omega} = b^2 \frac{\sin^2(q_x aN/2)}{\sin^2(q_x a/2)} \frac{\sin^2(q_y aN/2)}{\sin^2(q_y a/2)} \frac{\sin^2(q_z aN/2)}{\sin^2(q_z a/2)}. \quad \checkmark$$

(c) Here's the graph:

```
f[N_, x_] := (1/N) (Sin[N x/2]/Sin[x/2])^2

Plot[{f[1, x], f[5, x], f[10, x]}, {x, 0, 4 π}, PlotRange → {0, 10}]
```



(d) The vectors $\mathbf{k}$, $\mathbf{k}'$ and $\mathbf{G}$ form an isosceles triangle (Figure 10.11), and

$$\sin(\theta/2) = \frac{G/2}{k} = \frac{\pi}{ka}\sqrt{l^2+m^2+n^2},$$

so, in terms of the wavelength $\lambda = 2\pi/k$,

$$\theta = 2\arcsin\left(\frac{\lambda}{2a}\sqrt{l^2+m^2+n^2}\right).$$

The smallest (nonzero) angles occur when $(l^2+m^2+n^2) = 1,\ 2,$ or 3, corresponding to angles

$$\boxed{\pi/3, \pi/2, \text{and } 2\pi/3.}$$

Problem 10.22

(a) $-\dfrac{\hbar^2}{2m}\left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right)\psi(r,\theta) + V(r)\psi(r,\theta) = E\psi(r,\theta)$. Let $\psi(r,\theta) = R(r)\,\Theta(\theta)$:

$-\dfrac{\hbar^2}{2m}\left\{\Theta\left[\dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr}\right] + \dfrac{R}{r^2}\dfrac{d^2\Theta}{d\theta^2}\right\} + V(r)\,R\Theta = E\,R\Theta.$ Multiply by r^2 and divide by $R\Theta$:

$$-\frac{\hbar^2}{2m}\left[\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}\right]-\frac{\hbar^2}{2m}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+r^2V(r)=r^2E.$$ The Θ term must be constant:

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=-j^2 \quad\Rightarrow\quad \frac{d^2\Theta}{d\theta^2}=-j^2\,\Theta \quad\Rightarrow\quad \Theta(\theta)=Ae^{ij\theta},$$

where A is a constant (which we might as well absorb into R), and j could be positive or negative (it has to be an integer, since $\Theta(\theta+2\pi)=\Theta(\theta)$). This leaves the radial equation,

$$-\frac{\hbar^2}{2m}\left[r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}\right]+\frac{\hbar^2}{2m}j^2\,R+r^2V(r)R=r^2E\,R.$$

Let $u(r)\equiv\sqrt{r}\,R(r)$, so $R=\dfrac{u}{\sqrt{r}}$, $\dfrac{dR}{dr}=\dfrac{1}{\sqrt{r}}\dfrac{du}{dr}-\dfrac{1}{2}\dfrac{u}{r^{3/2}}$, $\dfrac{d^2R}{dr^2}=\dfrac{1}{\sqrt{r}}\dfrac{d^2u}{dr^2}-\dfrac{1}{r^{3/2}}\dfrac{du}{dr}+\dfrac{3}{4}\dfrac{u}{r^{5/2}}$. Then

$$-\frac{\hbar^2}{2m}\left[\left(\frac{1}{\sqrt{r}}\frac{d^2u}{dr^2}-\frac{1}{r^{3/2}}\frac{du}{dr}+\frac{3}{4}\frac{u}{r^{5/2}}\right)+\frac{1}{r}\left(\frac{1}{\sqrt{r}}\frac{du}{dr}-\frac{1}{2}\frac{u}{r^{3/2}}\right)\right]+\frac{\hbar^2}{2m}j^2\frac{1}{r^2}\frac{u}{\sqrt{r}}+V(r)\frac{u}{\sqrt{r}}=E\frac{u}{\sqrt{r}};$$

$$\boxed{\psi(r,\theta)=\frac{u}{\sqrt{r}}e^{ij\theta},\qquad -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left[V(r)+\frac{\hbar^2}{2m}\frac{(j^2-1/4)}{r^2}\right]u=Eu.}$$

(b) $-\dfrac{\hbar^2}{2m}\dfrac{d^2u}{dr^2}=E\,u \;\Rightarrow\; \dfrac{d^2u}{dr^2}=-k^2u$ where $k\equiv\frac{\sqrt{2mE}}{\hbar}$. The general solution is $u(r)=Ae^{ikr}+Be^{-ikr}$, but for an *outgoing* wave we want only the first term: $R(r)=A\dfrac{e^{ikr}}{\sqrt{r}}$. Notice that this asymptotic form is the same for all j, so when we form the general linear combination of separable solutions (from Part (a)), at large r the radial term will factor out, leaving some function of θ: $\psi\approx Af(\theta)\dfrac{e^{ikr}}{\sqrt{r}}$. ✓

$$-\frac{\hbar^2}{2m}\nabla^2\psi=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)Ae^{ikx}=-\frac{\hbar^2}{2m}(-k^2)Ae^{ikx}=E\,\psi.\quad ✓$$

(c) $-\dfrac{\hbar^2}{2m}\dfrac{d^2u}{dr^2}+\dfrac{\hbar^2}{2m}\dfrac{(j^2-1/4)}{r^2}u=Eu \quad\Rightarrow\quad \dfrac{d^2u}{dr^2}-\dfrac{j^2-1/4)}{r^2}u=-k^2u.$

This is a form of Bessel's equation; in Boas' notation (her Equation 16.1), $a=1/2,\ b=k,\ c=1,\ p=j$, and the solution is $u(r)=\sqrt{r}\,Z_j(kr)$, where Z is a Bessel/Neumann/Hankel function of order j. The outgoing one is $H^{(1)}$, so $u(r)=\sqrt{r}H_j^{(1)}(kr)$, the separable solution is $\psi(r,\theta)=H_j^{(1)}(kr)e^{ij\theta}$, and the general solution is the linear combination

$$\psi_{\text{out}}(r,\theta)=A\sum_{-\infty}^{\infty}c_j\,H_j^{(1)}(kr)\,e^{ij\theta}.$$

Combining this with the incident wave (Ae^{ikx}) we have

$$\boxed{\psi(r,\theta)=A\left(e^{ikx}+\sum_{-\infty}^{\infty}c_j\,H_j^{(1)}(kr)\,e^{ij\theta}\right).}$$

(d) At large r:

$$\psi(r,\theta)\sim A\left\{e^{ikx}+\sqrt{\frac{2}{\pi}}\,e^{-i\pi/4}\left(\sum_{-\infty}^{\infty}c_j(-i)^j e^{ij\theta}\right)\frac{e^{ikr}}{\sqrt{kr}}\right\}.$$

Comparing Equation (10.109), we read off

$$f(\theta) = \sqrt{\frac{2}{\pi k}}\, e^{-i\pi/4} \sum_{-\infty}^{\infty} c_j (-i)^j e^{ij\theta}. \quad \checkmark$$

(e) Following the argument leading to Equation (10.14), the probability that an incident particle (traveling at speed v) passes through an infinitesimal length (differential impact parameter) db is

$$dP = |\psi_{\text{incident}}|^2\, da = |A|^2 (v\, dt) db = |\psi_{\text{scattered}}|^2\, da = \frac{|A|^2 |f|^2}{r} (v\, dt) r\, d\theta \Rightarrow db = |f|^2\, d\theta \Rightarrow D(\theta) = |f(\theta)|^2. \quad \checkmark$$

$$B = \int_0^{2\pi} D(\theta)\, d\theta = \int_0^{2\pi} |f(\theta)|^2\, d\theta = \int_0^{2\pi} \frac{2}{\pi k} \left(\sum_j (-i)^j c_j e^{ij\theta} \right) \left(\sum_{j'} (i)^{j'} c_{j'}^* e^{-ij'\theta} \right) d\theta$$

$$= \frac{2}{\pi k} \sum_{j,j'} (i)^{j'-j} c_j c_{j'}^* \int_0^{2\pi} e^{i(j-j')\theta}\, d\theta = \frac{2}{\pi k} \sum_{j,j'} (i)^{j'-j} c_j c_{j'}^* (2\pi \delta_{j,j'}) = \frac{4}{k} \sum_{j=-\infty}^{\infty} |c_j|^2. \quad \checkmark$$

(f)

$$\psi(a,\theta) = A \left\{ \sum_{j=-\infty}^{\infty} (i)^j J_j(ka) e^{ij\theta} + \sum_{j=-\infty}^{\infty} c_j H_j^{(1)}(ka) e^{ij\theta} \right\} = A \sum_{j=-\infty}^{\infty} \left(i^j J_j + c_j H_j^{(1)} \right) e^{ij\theta} = 0.$$

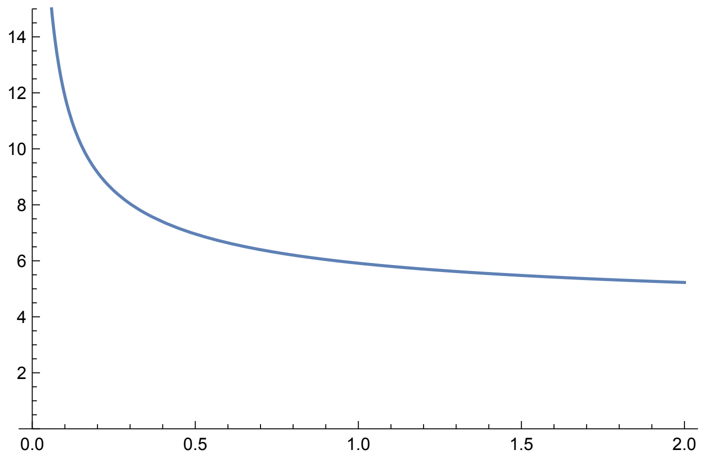
Each coefficient must vanish (prove it, if you like, by multiplying by $e^{-ij'\theta}$ and integrating over θ from 0 to 2π):

$$i^j J_j(ka) + c_j H_j^{(1)}(ka) = 0 \quad \Rightarrow \quad c_j = -\frac{i^j J_j(ka)}{H_j^{(1)}(ka)} \quad \Rightarrow \quad \boxed{B = \frac{4}{k} \sum_{j=-\infty}^{\infty} \left| \frac{J_j(ka)}{H_j^{(1)}(ka)} \right|^2.}$$

To create the plot, note that $H_j^{(1)} = J_j + iN_j$, while J_j and N_j (the Neumann function) are *real*, so $\left| \frac{J_j}{H_j^{(1)}} \right|^2 = \left[1 + (N_j/J_j)^2\right]^{-1}$. Then

```
b[x_] := 4/x Sum[(1 + (BesselY[j, x]/BesselJ[j, x])^2)^-1, {j, -100, 100}]

Plot[b[x], {x, 0, 2}, PlotRange → {0, 15}]
```



Problem 10.23

(a) The center of mass and relative coordinates are $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. Therefore interchanging the two particles leaves $\mathbf{R}$ untouched, but switches the sign of $\mathbf{r}$. Since the wave function must be symmetric under interchange, this means $\psi_r(-\mathbf{r}) = \psi_r(\mathbf{r})$. ✓

(b) Because the potential is spherically symmetric, if $\psi_r(\mathbf{r})$ is a solution to Equation 10.118 with energy E_r, so too is $\psi_r(-\mathbf{r})$. We can therefore construct a symmetric solution from the sum: $\psi_r(\mathbf{r}) + \psi_r(-\mathbf{r})$. Applying this to Equation 10.12 gives

$$\psi(r,\theta) = A\left\{e^{ikz} + e^{-ikz} + [f(\theta) + f(\pi-\theta)]\,\frac{e^{ikr}}{r}\right\},$$

using the fact that in polar coordinates $\mathbf{r} \to -\mathbf{r}$ is achieved by $(r,\,\theta,\,\phi) \to (r,\,\pi-\theta,\,\phi+\pi)$. We can then read off the scattering amplitude: $\boxed{f_B(\theta) = f(\theta) + f(\pi-\theta).}$ ✓

(c) From Equation 10.25,

$$\begin{aligned} f_B(\theta) &= \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell[P_\ell(\cos\theta) + P_\ell(\cos(\pi-\theta))] \\ &= \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell[P_\ell(\cos\theta) + P_\ell(-\cos\theta)] = \sum_{\ell\,\text{even}} 2(2\ell+1)a_\ell\, P_\ell(\cos\theta), \end{aligned}$$

since $P_\ell(-x) = (-1)^\ell P_\ell(x)$ (Equation 4.28).

(d) In this case the wave function must be *odd*: $\psi_r(-\mathbf{r}) = -\psi_r(\mathbf{r})$. We construct *anti*symmetric solutions, $\psi_r(\mathbf{r}) - \psi_r(-\mathbf{r})$, and by the same reasoning we get $\boxed{f_F(\theta) = f(\theta) - f(\pi-\theta).}$ This time the partial waves include only *odd* values of ℓ.

(e) $f_F(\pi/2) = f(\pi/2) - f(\pi-\pi/2) = f(\pi/2) - f(\pi/2) = 0.$ ✓

(f) $f(\theta) = \dfrac{A}{\sin^2(\theta/2)}$; $f_{B/F} = A\left[\dfrac{1}{\sin^2(\theta/2)} \pm \dfrac{1}{\sin^2((\theta-\pi/2)/2)}\right]$. But $\sin^2(\theta/2) = \dfrac{1}{2}[1-\cos\theta]$, and hence $\sin^2((\theta-\pi/2)/2) = \dfrac{1}{2}[1-\cos(\theta-\pi/2)] = \dfrac{1}{2}[1+\cos\theta]$, so $f_{B/F} = 2A\left(\dfrac{1}{1-\cos\theta} \pm \dfrac{1}{1+\cos\theta}\right) = 2A\dfrac{(1+\cos\theta)\pm(1-\cos\theta)}{1-\cos^2\theta}$. $f_B = 4A\left(\dfrac{1}{\sin^2\theta}\right)$, $f_F = 4A\left(\dfrac{\cos\theta}{\sin^2\theta}\right)$.

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_B = (4A)^2\left(\frac{1}{\sin^4\theta}\right); \quad \left(\frac{d\sigma}{d\Omega}\right)_F = (4A)^2\left(\frac{\cos^2\theta}{\sin^4\theta}\right).}$$

In the plot below the dashed line is $(d\sigma/d\Omega)_B$, and the solid line is $(d\sigma/d\Omega)_F$.