

NATIONAL UNIVERSITY OF HO CHI MINH CITY  
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FACULTY OF PHYSICS - ENGINEERING PHYSICS

UNDERGRADUATE THESIS

Thesis title:

Hofstadter butterfly in transition metal  
dichalcogenide monolayers

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Sudent: Tran Khoi Nguyen  
Supervisor: Dr. Huynh Thanh Duc

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## **ACKNOWLEDGMENTS**

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## LIST OF ABBREVIATIONS

<b>TMD</b> transition metal dichalcogenides . . . . .	iii
<b>TB</b> tight-binding . . . . .	4
<b>TBM</b> tight-binding model . . . . .	iii
<b>NN</b> nearest-neighbor . . . . .	3
<b>BZ</b> Brillouin zone . . . . .	7
<b>GGA</b> generalized-gradient approximation . . . . .	6
<b>SOC</b> spin orbit coupling . . . . .	15
<b>LLs</b> Landau levels . . . . .	17
<b>IQHE</b> integer Quantum Hall effect . . . . .	19
<b>EFA</b> Envelop Function Approximation . . . . .	30
<b>BCH</b> Baker-Campbell-Hausdorff . . . . .	31
<b>TKNN</b> Thouless-Kohmoto-Nightingale-Nijs . . . . .	22

## **Abstract**

**Keywords:**

# CHAPTER 1

## INTRODUCTION

## CHAPTER 2

### THEORY

#### 2.1 Three-band tight binding method without magnetic field

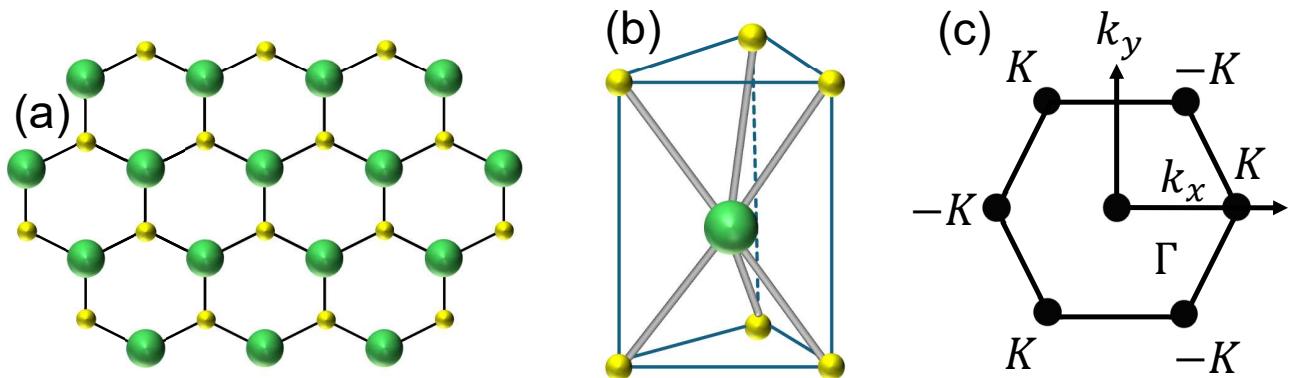


Figure 2.1: Top view of monolayer  $MX_2$ . The large sphere is  $M$  atom and the small sphere is  $X$ .

The time-independent Schrödinger equation for an electron in the crystal has the form

$$\left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] \psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \varepsilon_{\lambda}(\mathbf{k}) \psi_{\lambda, \mathbf{k}}(\mathbf{r}), \quad (2.1)$$

where  $U_0(\mathbf{r})$  is the periodic lattice potential,  $\psi_{\lambda, \mathbf{k}}(\mathbf{r})$  is the Bloch wavefunction of an electron in band  $\lambda$  with wave vector  $\mathbf{k}$  and  $\varepsilon_{\lambda}(\mathbf{k})$  is the band structure.

In the tight-binding model (TBM), the single-electron Bloch wavefunction can be expressed in terms of atomic orbitals as follows

$$\psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \sum_{j,i} C_{ji}^{\lambda}(\mathbf{k}) \sum_{\mathbf{R}} e^{i\mathbf{k} \cdot (\mathbf{R} + \mathbf{r}_i)} \phi_j(\mathbf{r} - \mathbf{R} - \mathbf{r}_i), \quad (2.2)$$

where  $\phi_j(\mathbf{r} - \mathbf{R} - \mathbf{r}_i)$  is the orbital  $j$  of an atom  $i$  localized on a lattice site  $\mathbf{R}$ , in which  $\mathbf{r}_i$  is the relativistic position of the atom  $i$  in the unit cell, and  $C_{ji}^\lambda(\mathbf{k})$  are the coefficients of linear expansion.

The unit cell of TMD involve one transition metal atom  $M$  and two chalcogenide atoms  $X$ . From the *ab initio* calculations, it is shown that the electron states near the band edges of  $MX_2$  are mainly contributed from the three  $d$  orbital of  $M$  atom, namely  $d_{z^2}, d_{xy}, d_{x^2-y^2}$  [1]. Since only the orbitals of atom  $M$  is included, we ignore the sum over the atom  $\mathbf{r}_i$  in the unit cell in Eq. (2.2). This model is called the three-band tight binding model. The three orbitals's wave function of  $M$  atom are denoted as

$$|\phi_1\rangle = |d_{z^2}\rangle; \quad |\phi_2\rangle = |d_{xy}\rangle; \quad |\phi_3\rangle = |d_{x^2-y^2}\rangle. \quad (2.3)$$

The Bloch wavefunction in this model has the form

$$\psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \sum_{j=1}^3 C_j^\lambda(\mathbf{k}) \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \phi_j(\mathbf{r} - \mathbf{R}). \quad (2.4)$$

The coefficients  $C_j^\lambda(\mathbf{k})$  are the solutions of the eigenvalue equation

$$\sum_{jj'}^3 \left[ H_{jj'}^{\text{TB}}(\mathbf{k}) - \varepsilon_\lambda(\mathbf{k}) S_{jj'}(\mathbf{k}) \right] C_j^\lambda(\mathbf{k}) = 0, \quad (2.5)$$

where

$$H_{jj'}^{\text{TB}}(\mathbf{k}) = \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle, \quad (2.6)$$

and

$$S_{jj'}(\mathbf{k}) = \sum_{\mathbf{R}} \langle \phi_j(\mathbf{r}) | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \approx \delta_{jj'}. \quad (2.7)$$

The three-band tight-binding model often called the nearest-neighbor (NN) since it only includes nearest-neighbor hopping does a decent job matching *ab initio* results near the band edges. However, it starts to break down in other parts of the band structure. This is because the model completely ignores the  $p$  orbitals from the  $X$  atoms, which still contributes to the conduction bands at  $\Gamma$  and the valence bands at  $M$ . The matrix

elements of the tight-binding (TB) Hamiltonian Eq. (2.6) are

$$H_{jj'}^{\text{NN}}(\mathbf{k}) = \mathcal{E}_{jj'}(\mathbf{0}) + e^{i\mathbf{k}\cdot\mathbf{R}_1} \mathcal{E}_{jj'}(\mathbf{R}_1) + e^{i\mathbf{k}\cdot\mathbf{R}_2} \mathcal{E}_{jj'}(\mathbf{R}_2) + e^{i\mathbf{k}\cdot\mathbf{R}_3} \mathcal{E}_{jj'}(\mathbf{R}_3) + e^{i\mathbf{k}\cdot\mathbf{R}_4} \mathcal{E}_{jj'}(\mathbf{R}_4) + e^{i\mathbf{k}\cdot\mathbf{R}_5} \mathcal{E}_{jj'}(\mathbf{R}_5) + e^{i\mathbf{k}\cdot\mathbf{R}_6} \mathcal{E}_{jj'}(\mathbf{R}_6), \quad (2.8)$$

where

$$\mathcal{E}_{jj'}(\mathbf{R}) = \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle, \quad (2.9)$$

and

$$\begin{aligned} \mathbf{R}_1 &= (a, 0), & \mathbf{R}_2 &= \left( \frac{a}{2}, -\frac{a\sqrt{3}}{2} \right), & \mathbf{R}_3 &= \left( -\frac{a}{2}, -\frac{a\sqrt{3}}{2} \right), \\ \mathbf{R}_4 &= (-a, 0), & \mathbf{R}_5 &= \left( -\frac{a}{2}, \frac{a\sqrt{3}}{2} \right), & \mathbf{R}_6 &= \left( \frac{a}{2}, \frac{a\sqrt{3}}{2} \right). \end{aligned} \quad (2.10)$$

Here,  $\mathbf{R}_{1-6}$  are the positions of the nearest neighbors  $M$  atoms, see Fig.

$g_n$	$x'$	$y'$	$z'$	$z'^2$	$x'y'$	$\frac{1}{2}(x'^2 - y'^2)$
$E$	$x$	$y$	$z$	$z^2$	$xy$	$\frac{1}{2}(x^2 - y^2)$
$C_3(\frac{-2\pi}{3})$	$-\frac{1}{2}x + \frac{\sqrt{3}}{2}y$	$-\frac{\sqrt{3}}{2}x - \frac{1}{2}y$	$z$	$z^2$	$-\frac{1}{2}xy + \frac{\sqrt{3}}{4}(x^2 + y^2)$	$-\frac{\sqrt{3}}{2}xy - \frac{1}{4}(x^2 - y^2)$
$C_3(\frac{-4\pi}{3})$	$-\frac{1}{2}x - \frac{\sqrt{3}}{2}y$	$\frac{\sqrt{3}}{2}x + \frac{1}{2}y$	$z$	$z^2$	$-\frac{1}{2}xy - \frac{\sqrt{3}}{4}(x^2 + y^2)$	$\frac{\sqrt{3}}{2}xy - \frac{1}{4}(x^2 - y^2)$
$\sigma_\nu$	$-x$	$y$	$z$	$z^2$	$-xy$	$\frac{1}{2}(x^2 - y^2)$
$\sigma'_\nu$	$\frac{1}{2}x - \frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}x - \frac{1}{2}y$	$z$	$z^2$	$\frac{1}{2}xy - \frac{\sqrt{3}}{4}(x^2 + y^2)$	$-\frac{\sqrt{3}}{2}xy - \frac{1}{4}(x^2 - y^2)$
$\sigma''_\nu$	$\frac{1}{2}x + \frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}x - \frac{1}{2}y$	$z$	$z^2$	$\frac{1}{2}xy + \frac{\sqrt{3}}{4}(x^2 + y^2)$	$\frac{\sqrt{3}}{2}xy - \frac{1}{4}(x^2 - y^2)$

Table 2.1: Some symmetry operators of the  $D_{3h}$  point group on basis functions taking  $(x, y, z)$  into  $(x', y', z')$ .  $C_3(\frac{-2\pi}{3})$  and  $C_3(\frac{-4\pi}{3})$  are the rotations by  $\frac{-2\pi}{3}$  and  $\frac{-4\pi}{3}$  around the  $z$  axis, respectively.  $\sigma_\nu$  is the reflection angular bisector of  $R_1$  and  $R_6$  in Fig. , and  $\sigma'_\nu, \sigma''_\nu$  are obtained through rotating  $\sigma_\nu$  around the  $z$  axis by  $2\pi/3$  and  $4\pi/3$ , respectively.

One parameterizes the matrices  $\mathcal{E}(\mathbf{0})$  and  $\mathcal{E}(\mathbf{R}_1)$  by

$$\begin{aligned} \mathcal{E}(\mathbf{0}) &= \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix}, \\ \mathcal{E}(\mathbf{R}_1) &= \begin{pmatrix} t_0 & t_1 & t_2 \\ -t_1 & t_{11} & t_{12} \\ t_2 & -t_{12} & t_{22} \end{pmatrix}. \end{aligned} \quad (2.11)$$

Given  $\mathcal{E}(\mathbf{R}_1)$ , the matrix  $\mathcal{E}(\mathbf{R}_{2-6})$  corresponding to all neighbor sites  $\mathbf{R}_{2-6}$  can be gen-

erated by

$$\mathcal{E}(g_n \mathbf{R}_1) = D(g_n) \mathcal{E}(\mathbf{R}_1) D^\dagger(g_n), \quad (2.12)$$

where  $D(g_n)$  is the matrix of the irreducible representation,  $g_n$  are symmetry operators of  $D_{3h}$  point groups,  $\{E, 2C_3, 3C_2, 2S_3, \sigma_h, 3\sigma_\nu\}$ . Particularly, we have  $\mathcal{E}(\mathbf{R}_2) = \mathcal{E}(\sigma'_\nu \mathbf{R}_1)$ ,  $\mathcal{E}(\mathbf{R}_3) = \mathcal{E}(C_3(-\frac{2\pi}{3})\mathbf{R}_1)$ ,  $\mathcal{E}(\mathbf{R}_4) = \mathcal{E}(\sigma_\nu \mathbf{R}_1)$ ,  $\mathcal{E}(\mathbf{R}_5) = \mathcal{E}(C_3(-\frac{4\pi}{3})\mathbf{R}_1)$ ,  $\mathcal{E}(\mathbf{R}_6) = \mathcal{E}(\sigma''_\nu \mathbf{R}_1)$ . Table 2.1 depicts the transformation of the basis functions under the action of symmetry operators. Also, from Table 2.1, we obtain irreducible matrices as follows

$$\begin{aligned} D(C_3(-\frac{2\pi}{3})) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, & D(C_3(-\frac{4\pi}{3})) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}, \\ D(\sigma_\nu) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & D(\sigma'_\nu) &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/2 & -\sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, \\ D(\sigma''_\nu) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{pmatrix}. \end{aligned} \quad (2.13)$$

Therefore, we have

$$\begin{aligned} \mathcal{E}(\mathbf{R}_2) &= D(\sigma'_\nu) \mathcal{E}(\mathbf{R}_1) D^\dagger(\sigma'_\nu) \\ &= \begin{pmatrix} t_0 & \frac{1}{2}t_1 - \frac{\sqrt{3}}{2}t_2 & -\frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 \\ -\frac{1}{2}t_1 - \frac{\sqrt{3}}{2}t_2 & \frac{1}{4}t_{11} + \frac{3}{4}t_{22} & -\frac{\sqrt{3}}{4}t_{11} - t_{12} + \frac{\sqrt{3}}{4}t_{22} \\ \frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 & -\frac{\sqrt{3}}{4}t_{11} + t_{12} + \frac{\sqrt{3}}{4}t_{22} & \frac{3}{4}t_{11} + \frac{1}{4}t_{22} \end{pmatrix}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \mathcal{E}(\mathbf{R}_3) &= D(C_3(-\frac{2\pi}{3})) \mathcal{E}(\mathbf{R}_1) D^\dagger(C_3(-\frac{2\pi}{3})) \\ &= \begin{pmatrix} t_0 & -\frac{1}{2}t_1 + \frac{\sqrt{3}}{2}t_2 & -\frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 \\ \frac{1}{2}t_1 + \frac{\sqrt{3}}{2}t_2 & \frac{1}{4}t_{11} + \frac{3}{4}t_{22} & \frac{\sqrt{3}}{4}t_{11} + t_{12} - \frac{\sqrt{3}}{4}t_{22} \\ \frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 & \frac{\sqrt{3}}{4}t_{11} - t_{12} - \frac{\sqrt{3}}{4}t_{22} & \frac{3}{4}t_{11} + \frac{1}{4}t_{22} \end{pmatrix}, \end{aligned} \quad (2.15)$$

$$\mathcal{E}(\mathbf{R}_4) = D(\sigma_\nu) \mathcal{E}(\mathbf{R}_1) D^\dagger(\sigma_\nu) = \begin{pmatrix} t_0 & -t_1 & t_2 \\ t_1 & t_{11} & -t_{12} \\ t_2 & t_{12} & t_{22} \end{pmatrix}, \quad (2.16)$$

$$\begin{aligned}\mathcal{E}(\mathbf{R}_5) &= D(C(-\frac{4\pi}{3}))\mathcal{E}(\mathbf{R}_1)D^\dagger(C(-\frac{4\pi}{3})) \\ &= \begin{pmatrix} t_0 & -\frac{1}{2}t_1 - \frac{\sqrt{3}}{2}t_2 & \frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 \\ \frac{1}{2}t_1 - \frac{\sqrt{3}}{2}t_2 & \frac{1}{4}t_{11} + \frac{3}{4}t_{22} & -\frac{\sqrt{3}}{4}t_{11} + t_{12} + \frac{\sqrt{3}}{4}t_{22} \\ -\frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 & -\frac{\sqrt{3}}{4}t_{11} - t_{12} + \frac{\sqrt{3}}{4}t_{22} & \frac{3}{4}t_{11} + \frac{1}{4}t_{22} \end{pmatrix},\end{aligned}\quad (2.17)$$

$$\begin{aligned}\mathcal{E}(\mathbf{R}_6) &= D(\sigma_\nu'')\mathcal{E}(\mathbf{R}_1)D^\dagger(\sigma_\nu'') \\ &= \begin{pmatrix} t_0 & \frac{1}{2}t_1 + \frac{\sqrt{3}}{2}t_2 & \frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 \\ -\frac{1}{2}t_1 + \frac{\sqrt{3}}{2}t_2 & \frac{1}{4}t_{11} + \frac{3}{4}t_{22} & \frac{\sqrt{3}}{4}t_{11} - t_{12} - \frac{\sqrt{3}}{4}t_{22} \\ -\frac{\sqrt{3}}{2}t_1 - \frac{1}{2}t_2 & \frac{\sqrt{3}}{4}t_{11} - t_{12} - \frac{\sqrt{3}}{4}t_{22} & \frac{3}{4}t_{11} + \frac{1}{4}t_{22} \end{pmatrix},\end{aligned}\quad (2.18)$$

The nearest-neighbor tight-binding Hamiltonian now can be written as

$$H^{\text{NN}}(\mathbf{k}) = \begin{pmatrix} h_0 & h_1 & h_2 \\ h_1^* & h_{11} & h_{12} \\ h_2^* & h_{12}^* & h_{22} \end{pmatrix}\quad (2.19)$$

where

$$\begin{aligned}h_0 &= 2t_0(\cos 2\alpha + 2\cos \alpha \cos \beta) + \epsilon_1, \\ h_1 &= 2it_1(\sin 2\alpha + \sin \alpha \cos \beta) - 2\sqrt{3}t_2 \sin \alpha \sin \beta, \\ h_2 &= 2t_2(\cos 2\alpha - \cos \alpha \cos \beta) + 2i\sqrt{3}t_1 \cos \alpha \sin \beta, \\ h_{11} &= (t_{11} + 3t_{22}) \cos \alpha \cos \beta + 2t_{11} \cos 2\alpha + \epsilon_2, \\ h_{22} &= (3t_{11} + t_{22}) \cos \alpha \cos \beta + 2t_{22} \cos 2\alpha + \epsilon_2, \\ h_{12} &= \sqrt{3}(t_{22} - t_{11}) \sin \alpha \sin \beta + 4it_{12} \sin \alpha (\cos \alpha - \cos \beta),\end{aligned}\quad (2.20)$$

$$(\alpha, \beta) = \left( \frac{1}{2}k_x a, \frac{\sqrt{3}}{2}k_y a \right).\quad (2.21)$$

Eight additional parameters depicted in Table 2.2 are obtained by fitting the band with *ab initio* calculation results.

	$a(\text{\AA})$	$\epsilon_1$	$\epsilon_2$	$t_0$	$t_1$	$t_2$	$t_{11}$	$t_{12}$	$t_{22}$
MoS <sub>2</sub>	3.190	1.046	2.104	-0.184	0.401	0.507	0.218	0.338	0.057
WS <sub>2</sub>	3.191	1.130	2.275	-0.206	0.567	0.536	0.286	0.384	-0.061
MoSe <sub>2</sub>	3.326	0.919	2.065	-0.188	0.317	0.456	0.211	0.290	0.130
WSe <sub>2</sub>	3.325	0.943	2.179	-0.207	0.457	0.486	0.263	0.329	0.034
MoTe <sub>2</sub>	3.557	0.605	1.972	-0.169	0.228	0.390	0.207	0.239	0.252
WTe <sub>2</sub>	3.560	0.606	2.102	-0.175	0.342	0.410	0.233	0.270	0.190

Table 2.2: Fitted parameters in three-band NN TBM for generalized-gradient approximation (GGA) cases for MX<sub>2</sub> [1].

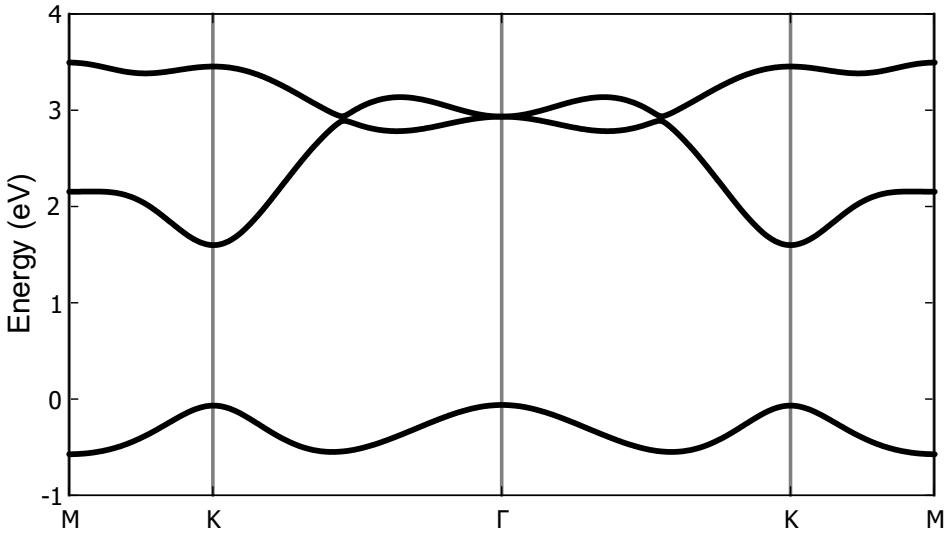


Figure 2.2: To obtain the band structure of monolayer MoS<sub>2</sub>, the eigenvalue of the Hamiltonian needs to be found at each  $k$  point across the entire the Brillouin zone (BZ). This figure illustrates the band structure along  $\Gamma$ -K direction using (GGA) fitting parameters.

## 2.2 Three-band tight binding method under a magnetic field

Under a uniform magnetic field given by a vector potential  $\mathbf{A}(\mathbf{r})$  the single electron Hamiltonian changes into

$$H = \frac{(-i\hbar\nabla + e\mathbf{A}(\mathbf{r}))^2}{2m} + U_0(\mathbf{r}) + g^* \mu_B \mathbf{B} \cdot \mathbf{L}, \quad (2.22)$$

where  $\mu_B = \frac{e\hbar}{2m}$  is Bohr magneton,  $g^*$  is an effective Landé g-factor,  $\mathbf{B} = \nabla \times \mathbf{A}$  is the uniform magnetic field, and  $\mathbf{L}$  is the angular momentum. It is possible to add a phase factor to the tight binding wavefunction

$$\psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \sum_{j=1}^3 C_j^\lambda(\mathbf{k}) \sum_{\mathbf{R}} e^{i\mathbf{k} \cdot \mathbf{R}} e^{\theta_{\mathbf{R}}(\mathbf{r})} \phi_j(\mathbf{r} - \mathbf{R}). \quad (2.23)$$

We now have

$$H_{jj'}(\mathbf{k}) = H_{jj'}^{\text{NN}}(\mathbf{k}) + H_{jj'}^Z(\mathbf{k}), \quad (2.24)$$

where

$$\begin{aligned}
H_{jj'}^{\text{NN}}(\mathbf{k}) &= \sum_{\mathbf{R}} \langle \phi_j(\mathbf{r}) | e^{-i\theta_0(\mathbf{r})} \left[ \frac{(-i\hbar\boldsymbol{\nabla} + e\mathbf{A}(\mathbf{r}))^2}{2m} + U_0(\mathbf{r}) \right] e^{i\mathbf{k}\cdot\mathbf{R}} e^{\theta_{\mathbf{R}}(\mathbf{r})} | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \\
&= \sum_{\mathbf{R}} \langle \phi_j(\mathbf{r}) | e^{i(\mathbf{k}\cdot\mathbf{R} + \theta_{\mathbf{R}} - \theta_0)} \left[ \frac{(-i\hbar\boldsymbol{\nabla} + e\mathbf{A} + \hbar\boldsymbol{\nabla}\theta_{\mathbf{R}})^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle,
\end{aligned} \tag{2.25}$$

and

$$H_{jj'}^Z(\mathbf{k}) = g^* \mu_B \mathbf{B} \cdot \sum_{\mathbf{R}} \langle \phi_j(\mathbf{r}) | e^{i(\mathbf{k}\cdot\mathbf{R} + \theta_{\mathbf{R}} - \theta_0)} \mathbf{L} | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle. \tag{2.26}$$

By choosing  $\theta_{\mathbf{R}} = -\frac{e}{\hbar} \int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'$  as Peierls substitution, the Hamiltonian in Eq. (2.25) now reads

$$\begin{aligned}
H_{jj'}^{\text{NN}}(\mathbf{k}) &= \sum_{\mathbf{R}} \langle \phi_j(\mathbf{r}) | e^{i\mathbf{k}\cdot\mathbf{R} - \frac{ie}{\hbar} \int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' + \frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \left[ -\frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \\
&= \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} e^{\frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \langle \phi_j(\mathbf{r}) | e^{-\frac{ie}{\hbar} \Phi_{\mathbf{R},\mathbf{r},\mathbf{0}}} \left[ -\frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle,
\end{aligned}$$

where  $\Phi_{\mathbf{R},\mathbf{r},\mathbf{0}} = \oint_{\mathbf{R},\mathbf{r},\mathbf{0}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'$  is the closed loop line intergral of  $\mathbf{A}$  along the triangle points  $\mathbf{R}, \mathbf{r}, \mathbf{0}$ , and  $\int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'$  is the path intergral along the two points  $\mathbf{R}, \mathbf{0}$ . Besides that, we have used the fact that

$$\int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' + \int_{\mathbf{r}}^{\mathbf{0}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' = \Phi_{\mathbf{R},\mathbf{r},\mathbf{0}} - \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'. \tag{2.27}$$

We can show that the flux term  $\Phi_{\mathbf{R},\mathbf{r},\mathbf{0}}$  is negligibly small [2] by two observations. When  $\mathbf{r}$  is far away from the lattice points  $\mathbf{R}$  and  $\mathbf{0}$ , the flux is large but since the atomic orbitals are highly localized at these two lattice points, the value of the hopping term is very small and the whole hopping term goes to zero. While  $\mathbf{r}$  is at or near any of these lattice points, the triangle formed is small, and assuming small magnetic field, the flux term  $\Phi_{\mathbf{R},\mathbf{r},\mathbf{0}}$  goes to zero, which giving us the Hamiltonian as

$$H_{jj'}^{\text{NN}}(\mathbf{k}) = \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} e^{\frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \boldsymbol{\nabla}^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle, \tag{2.28}$$

$$H_{jj'}^Z(\mathbf{k}) = g^* \mu_B \mathbf{B} \cdot \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} e^{\frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \langle \phi_j(\mathbf{r}) | \mathbf{L} | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle. \tag{2.29}$$

Considering only NN hopping, Eq (2.29) becomes

$$\begin{aligned}
H_{jj'}^{\text{NN}}(\mathbf{k}) &= \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}) \\
&= \mathcal{E}_{jj'}(\mathbf{0}) + e^{i\mathbf{k}\cdot\mathbf{R}_1} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}_1} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}_1) \\
&\quad + e^{i\mathbf{k}\cdot\mathbf{R}_2} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}_2} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}_2) + e^{i\mathbf{k}\cdot\mathbf{R}_3} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}_3} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}_3) \\
&\quad + e^{i\mathbf{k}\cdot\mathbf{R}_4} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}_4} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}_4) + e^{i\mathbf{k}\cdot\mathbf{R}_5} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}_5} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}_5) \\
&\quad + e^{i\mathbf{k}\cdot\mathbf{R}_6} e^{\frac{e}{\hbar} \int_0^{\mathbf{R}_6} A(\mathbf{r}') d\mathbf{r}'} \mathcal{E}_{jj'}(\mathbf{R}_6).
\end{aligned} \tag{2.30}$$

In the presence of a perpendicular magnetic field  $\mathbf{B}\hat{z}$  applied to the plane of TMD, we choose the vector potential in the Landau gauge as  $\mathbf{A} = (0, Bx, 0)$ . For convenience, let us switch to a shorthand notation for these extra terms and define

$$\begin{aligned}
\theta_{m,n}^{m',n'} &= \frac{e}{\hbar} \int_{m,n}^{m',n'} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} \\
&= \frac{eB}{2\hbar} (x_{m'} + x_m)(y_{n'} - y_n),
\end{aligned} \tag{2.31}$$

in which  $x_m = \frac{ma}{2}$  ( $m = \pm 1, \pm 2$ ) and  $y_n = \frac{na\sqrt{3}}{2}$  ( $n = 0, \pm 1$ ) are the NN coordinates and  $a$  being the lattice constant, are shown in Fig.(2.3)

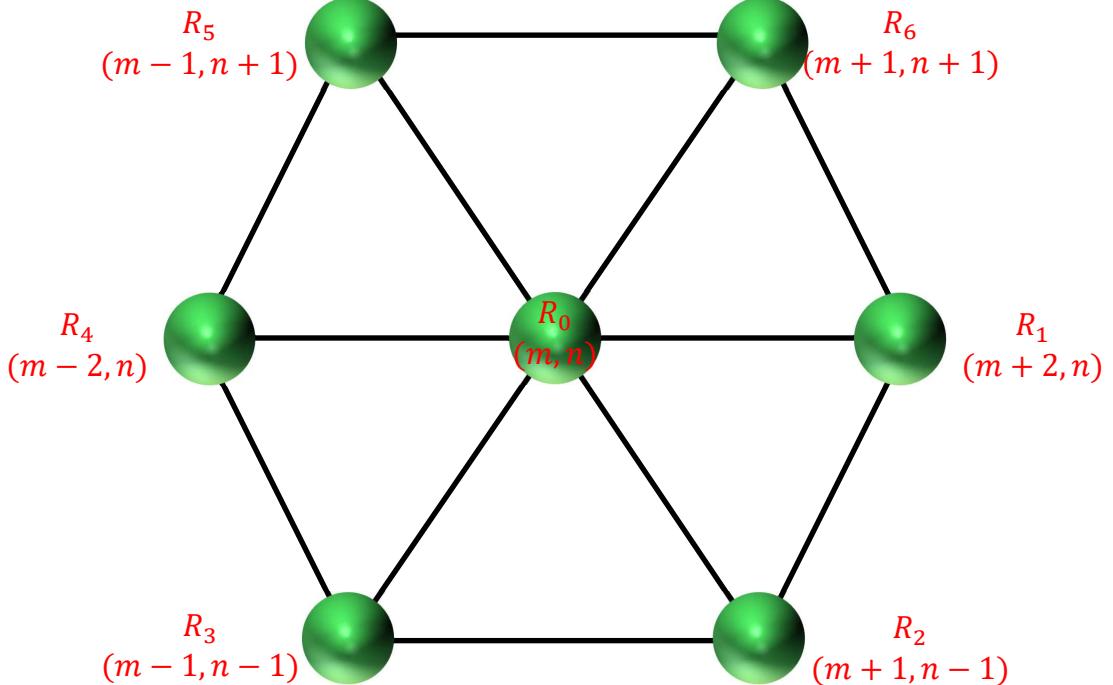


Figure 2.3: The TBM of TMD with six neighbors atom  $M$  rewrite with the site index.

Since  $dy = 0$  along the  $x$  direction,  $\theta_{m,n}^{m\pm 2,n} = 0$ , and using NN coordinates given for

lattice site, the  $\theta_{m,n}^{m',n'}$  can be written as

$$\theta_{m,n}^{m',n'} = \begin{cases} 0 & m' = m \pm 2, n' = n, \\ \pm \frac{e}{\hbar} \frac{Ba^2 \sqrt{3}}{4} (m + 1/2) & m' = m + 1, n' = n \pm 1, \\ \pm \frac{e}{\hbar} \frac{Ba^2 \sqrt{3}}{4} (m - 1/2) & m' = m - 1, n' = n \pm 1. \end{cases} \quad (2.32)$$

Identifying  $\frac{Ba^2 \sqrt{3}}{4}$  as the magnetic flux  $\Phi$  passing through per unit cell and  $\frac{h}{e}$  corresponds to the magnetic flux quantum  $\Phi_0$ , we obtain the following relation:

$$\begin{aligned} H_{jj'}(\mathbf{k}) = & E_{jj'}(\mathbf{0}) + e^{i\mathbf{k} \cdot \mathbf{R}_1} E_{jj'}(\mathbf{R}_1) + e^{-2i\pi(m+1/2)\Phi/\Phi_0} e^{i\mathbf{k} \cdot \mathbf{R}_2} E_{jj'}(\mathbf{R}_2) \\ & + e^{-2i\pi(m-1/2)\Phi/\Phi_0} e^{i\mathbf{k} \cdot \mathbf{R}_3} E_{jj'}(\mathbf{R}_3) + e^{i\mathbf{k} \cdot \mathbf{R}_4} E_{jj'}(\mathbf{R}_4) \\ & + e^{2i\pi(m-1/2)\Phi/\Phi_0} e^{i\mathbf{k} \cdot \mathbf{R}_5} E_{jj'}(\mathbf{R}_5) + e^{2i\pi(m+1/2)\Phi/\Phi_0} e^{i\mathbf{k} \cdot \mathbf{R}_6} E_{jj'}(\mathbf{R}_6). \end{aligned} \quad (2.33)$$

The Hamiltonian depends only on the site index  $m$  and does not invariant under the expansion of a lattice vector along the  $x$  axis. In order to restore this invariance, we can look at the case where the ratio of magnetic flux and flux quanta is a rational number  $\Phi/\Phi_0 = p/q$ . The crucial advantage of the Peierls phase approach is allowed the lattice periodicity can be restored provided a suitable “magnetic supercell” containing several original unit cells is constructed. One might ask what actually happens inside the magnetic unit cells? Do the NN still interact in the same way as they did in the tight-binding model? We will focus in this after we get the Hamiltonian for the magnetic unit cell.

The magnetic unit cell has lattice vectors  $q\mathbf{a}_1$  and  $\mathbf{a}_2$  is illustrated in Fig.

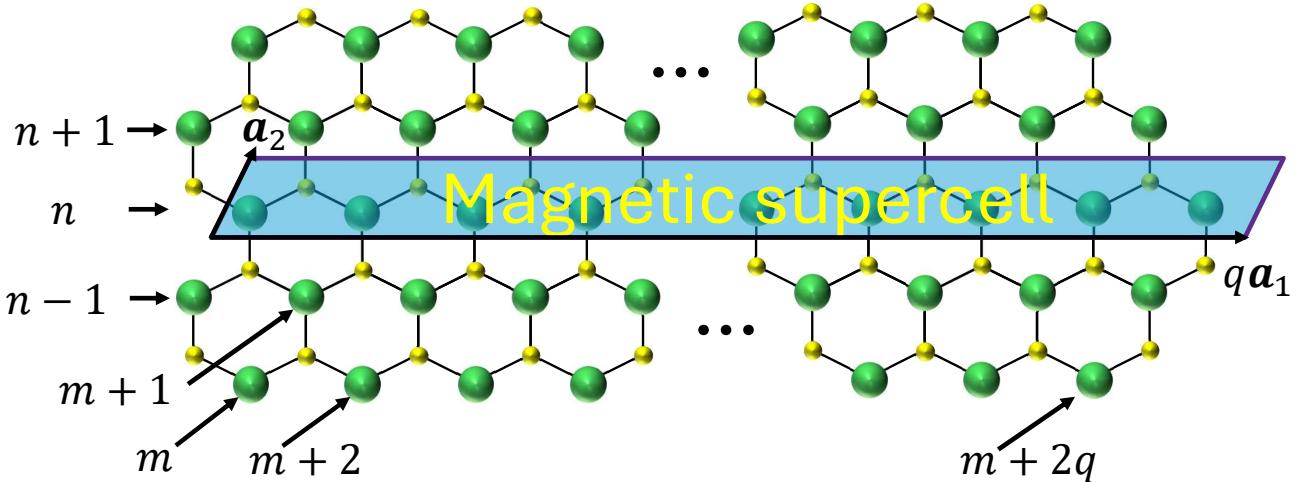


Figure 2.4: Magnetic unit cell for TMD monolayers.

In Section 2.1, we have ignore the sum over of relative positions  $\mathbf{r}_c$  in Eq (2.2) due to ignore the orbitals of  $X$  atoms. While, the magnetic unit cell consists  $2q$  atom  $M$ .

We, now, define a new basis set of  $6q$  atomic orbitals  $\{\phi_j(\mathbf{r} - \mathbf{r}_i)\}$  where  $j = 1, 2, 3$  and  $i = 1, 2, \dots, 2q$ . The new basis function now is

$$\psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \sum_{j,i} C_{ji}^{\lambda}(\mathbf{k}) \sum_{\alpha}^{N_{\text{UC}}} e^{i\mathbf{k} \cdot (\mathbf{R}_{\alpha} + \mathbf{r}_i)} \phi_j(\mathbf{r} - \mathbf{R}_{\alpha} - \mathbf{r}_i). \quad (2.34)$$

Here, we set  $\mathbf{r}_i$  refers to the position of an atom in a unit cell, while  $\mathbf{R}_{\alpha}$  denotes the position of different unit cells. The Hamiltonian matrix elements in the new basis is written as

$$H_{jj'}^{ii'}(\mathbf{k}) = \sum_{\alpha}^{N_{\text{UC}}} \sum_{\beta}^{N_{\text{UC}}} e^{i\mathbf{k} \cdot (\mathbf{R}_{\beta} - \mathbf{R}_{\alpha} + \mathbf{r}_{i'} - \mathbf{r}_i)} \langle \phi_j(\mathbf{r} - \mathbf{R}_{\alpha} - \mathbf{r}_i) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0 \right] | \phi_j(\mathbf{r} - \mathbf{R}_{\beta} - \mathbf{r}_{i'}) \rangle. \quad (2.35)$$

Now we center our system at  $\mathbf{r}' = \mathbf{r} - \mathbf{R}_{\alpha} - \mathbf{r}_i$  and define  $\mathbf{R}_{\gamma} = \mathbf{R}_{\alpha} - \mathbf{R}_{\beta}$ . This leads to

$$H_{jj'}^{ii'}(\mathbf{k}) = \sum_{\alpha}^{N_{\text{UC}}} \sum_{\gamma}^{N_{\text{UC}}} e^{-i\mathbf{k} \cdot (\mathbf{R}_{\gamma} + \mathbf{r}_i - \mathbf{r}_{i'})} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0 \right] | \phi_j(\mathbf{r} + \mathbf{R}_{\gamma} + \mathbf{r}_i - \mathbf{r}_{i'}) \rangle. \quad (2.36)$$

Taking the sum over  $\mathbf{R}$  and replacing  $\mathbf{r}_i = m\mathbf{a}_1 + n\mathbf{a}_2$ ,  $\mathbf{r}_{i'} = m'\mathbf{a}_1 + n'\mathbf{a}_2$ , notice that  $i = (m, n)$  with only considering the nearest-neighbors, we define our hopping terms in the new basis is given by

$$H_{ii'}^{jj'}(\mathbf{k}) = \sum_{\alpha}^{N_{\text{UC}}} \sum_{\gamma}^{N_{\text{UC}}} e^{-i\mathbf{k} \cdot \mathbf{R}_{\gamma}} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} + \mathbf{R}_{\gamma}) \rangle \delta_{i,i'}. \quad (2.37)$$

One can regconize that Eq. (2.37) resembles Eq. (2.6). Additionally, the equation not only describes the hopping between magnetic unit cells but also accounts for hopping between sites within magnetic unit cells. To address the previous question, it is important to note that we have enlarged the original unit cell into a magnetic unit cell, which now contains  $2q$  atoms  $M$ . The NN interactions are preserved inside the “supercell”, but they now involve neighboring magnetic unit cells due to the enlarged cell structure. The Hamiltonian has a discrete tranlastional invariance with a unit cell carrying  $2q$  unit cells along the  $x$  axis.

We can choose  $\mathbf{R}_{\beta} = \mathbf{0}$ , this will not affect the Hamiltonian because the sum over  $\mathbf{R}$  is

only considering the NN atoms, Hamiltonian with the Peierls phase now is

$$\begin{aligned}
H_{jj'}^{ii'}(\mathbf{k}) &= e^{i\theta_{m,n}^{m,n}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{0})} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r}) \rangle \delta_{m',m}^{n',n} \\
&+ e^{i\theta_{m,n}^{m+2,n}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{R}_1)} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}_1) \rangle \delta_{m,m+2}^{n',n} \\
&+ e^{i\theta_{m,n}^{m-2,n}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{R}_4)} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}_4) \rangle \delta_{m',m-2}^{n',n} \\
&+ e^{i\theta_{m,n}^{m+1,n-1}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{R}_2)} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}_2) \rangle \delta_{m',m+1}^{n',n-1} \\
&+ e^{i\theta_{m,n}^{m-1,n-1}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{R}_3)} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}_3) \rangle \delta_{m',m-1}^{n',n-1} \\
&+ e^{i\theta_{m,n}^{m-1,n+1}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{R}_5)} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}_5) \rangle \delta_{m',m-1}^{n',n+1} \\
&+ e^{i\theta_{m,n}^{m+1,n+1}} e^{i\mathbf{k}\cdot(\mathbf{0}-\mathbf{R}_6)} \langle \phi_j(\mathbf{r}) | \left[ -\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}_6) \rangle \delta_{m',m+1}^{n',n+1}.
\end{aligned} \tag{2.38}$$

Simplizing Eq. (2.38), we get the Hamiltonian for the magnetic unit cell

$$\begin{aligned}
H_{ii'}^{jj'}(\mathbf{k}) &= \mathcal{E}_{jj'}(\mathbf{0}) \delta_{m',m}^{n',n} + e^{-i\mathbf{k}\cdot\mathbf{R}_1} \mathcal{E}_{jj'}(\mathbf{R}_1) \delta_{m',m+2}^{n',n} + e^{-i\mathbf{k}\cdot\mathbf{R}_4} \mathcal{E}_{jj'}(\mathbf{R}_4) \delta_{m',m-2}^{n',n} \\
&+ e^{-2i\pi(m+1/2)p/q} e^{-i\mathbf{k}\cdot\mathbf{R}_2} \mathcal{E}_{jj'}(\mathbf{R}_2) \delta_{m',m+1}^{n',n-1} + e^{-2i\pi(m-1/2)p/q} e^{-i\mathbf{k}\cdot\mathbf{R}_3} \mathcal{E}_{jj'}(\mathbf{R}_3) \delta_{m',m-1}^{n',n-1} \\
&+ e^{2i\pi(m-1/2)p/q} e^{-i\mathbf{k}\cdot\mathbf{R}_5} \mathcal{E}_{jj'}(\mathbf{R}_5) \delta_{m',m-1}^{n',n+1} + e^{2i\pi(m+1/2)p/q} e^{-i\mathbf{k}\cdot\mathbf{R}_6} \mathcal{E}_{jj'}(\mathbf{R}_6) \delta_{m',m+1}^{n',n+1}.
\end{aligned} \tag{2.39}$$

Now, for given flux ratio  $p/q$ , only the  $q$  determines the periodicity of the magnetic cell assuming  $p$  and  $q$  are mutually prime numbers. When we plot the band energies while varying the  $p$ , we obtain the famous Hofstadter butterfly [3], a complex fractal structure as seen in Fig.(2.6). This structure is generated at the  $K$ -point. This fractal spectrum is a result of two competing effects, lattice periodicity and magnetic unit cell periodicity enforced by the presence of the magnetic field. Eq. (2.39) give the following matrix which must be diagonalized to obtain the energy eigenvalues

$$H = \begin{pmatrix} h_0 & h_1 & h_2 \\ h_1^* & h_{11} & h_{12} \\ h_2^* & h_{12}^* & h_{22} \end{pmatrix}, \tag{2.40}$$

with

$$H_{jj'} = \begin{pmatrix} \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(1)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(1)} \\ B_{jj'}^{(2)} & \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(2)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) \\ \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(3)} & \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(3)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(q-2)} & \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(q-2)} \\ A_{jj'}^{(q-1)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & \cdots & 0 & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(q-1)} & \mathcal{E}_{jj'}(\mathbf{0}) \end{pmatrix}, \quad (2.41)$$

where  $A_{jj'}^{(m)} = \mathcal{E}_{jj'}(\mathbf{R}_2)e^{-2i\pi(m+1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_2} + \mathcal{E}_{jj'}(\mathbf{R}_6)e^{2i\pi(m+1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_6}$ , and  $B_{jj'}^{(m)} = \mathcal{E}_{jj'}(\mathbf{R}_3)e^{-2i\pi(m-1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_3} + \mathcal{E}_{jj'}(\mathbf{R}_5)e^{2i\pi(m-1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_5}$  and  $h_0, h_1, h_2, h_{11}, h_{12}, h_{22}$  are submatrices of size  $3q \times 3q$ . (A visualization is shown in Fig. (2.5))

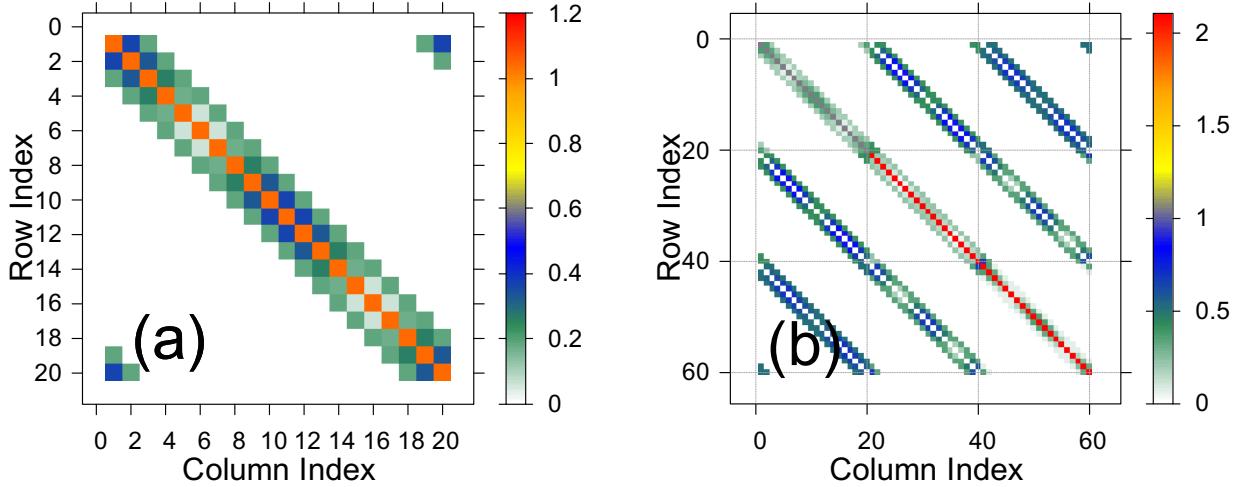


Figure 2.5: A simple and intuitive visualization of sub-matrix  $h_0$  for single-band(a) and matrix  $H$  for three band(b) using standard plotter with  $q = 20$ . (a): orange squares, blue squares and green squares correspond to  $\epsilon_1, 2t_0 \cos \zeta_1, t_0$ , respectively.

The magnetic field enters the TB Hamiltonian only through the fraction  $p/q$ , which is the magnetic flux through the primitive unit cell of the lattice. In general, as the lattice geometry evolves, the area of the primitive unit cell changes  $m$  times.

Another observation is that the lattice constant  $a$  and the magnetic field  $B$  always appears together in an expression with the magnetic field ( $\frac{Ba^2\sqrt{3}}{4}$ ). This quantity reflects the flux per plaquette in the super magnetic unit cell, which is relevant in the context of Aharonov-Bohm effect [4]. Since the expression involves the product  $Ba^2$ , this implies that increasing  $B$  by a certain amount is mathematically equivalent to increasing  $a$ . In other words, for energy calculations, increasing the strength of the magnetic field is physically equivalent to increasing the lattice constant, as both affect the system in the same way through the flux per unit cell.

The spectrum exhibits several notable symmetries. First, it depends only on the

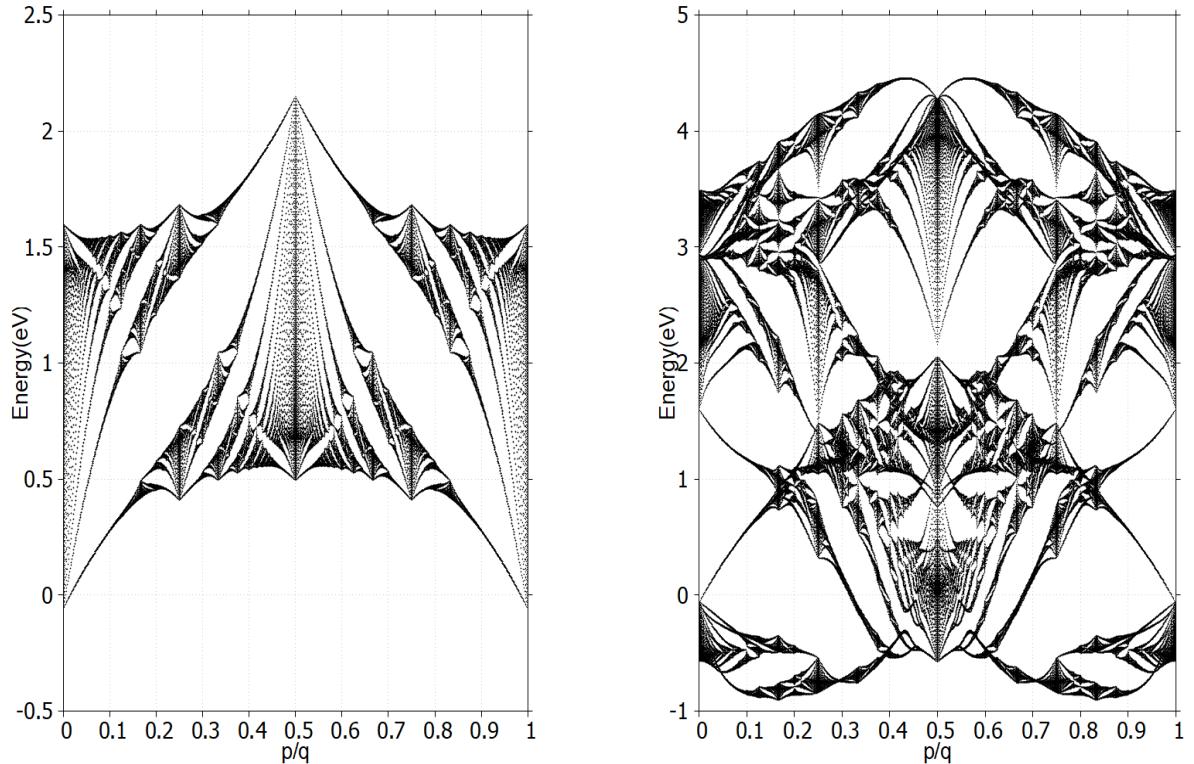


Figure 2.6: Hofstadter butterfly for single-band  $|dz\rangle \equiv |\phi_1^1(x, y)\rangle$  (left) and all band (right) with  $q = 797$  and vary  $p$  from 1 to  $q$  with field strength  $B_0 = 4.6928 \times 10^4$  T. Here on  $x$ -axis represents the flux in units of quantum flux enclosed by the unit cell and  $y$ -axis represents the Energy.

flux ratio  $p/q$ , meaning that shifting  $p/q$  by an integer  $c$  (i.e.,  $p/q \rightarrow p/q + c$ ) leaves the spectrum unchanged. Additionally, the spectrum remains invariant under the transformation  $p/q \rightarrow -p/q$ , since if  $\psi$  is an eigenstate with energy  $E$  for flux  $p/q$ , its complex conjugate  $\psi^*$  is an eigenstate with the same energy for flux  $-p/q$ . These two symmetries are general and not specific to the  $MX_2$  case. The third symmetry involves changing  $p/q$  to  $p/q + 1/2$ , which is equivalent to flipping the sign of the hopping energies  $t_i$  (i.e.,  $t_i \rightarrow -t_i$ ), resulting in an inversion of the spectrum.

The role of the eight hopping constants  $t$  is just to set an energy scale. Change the hopping constants amounts to stretching the butterfly spectrum vertically, which is an overall scaling to the energy levels. Thus it does not give rise to any interesting physical phenomenon.

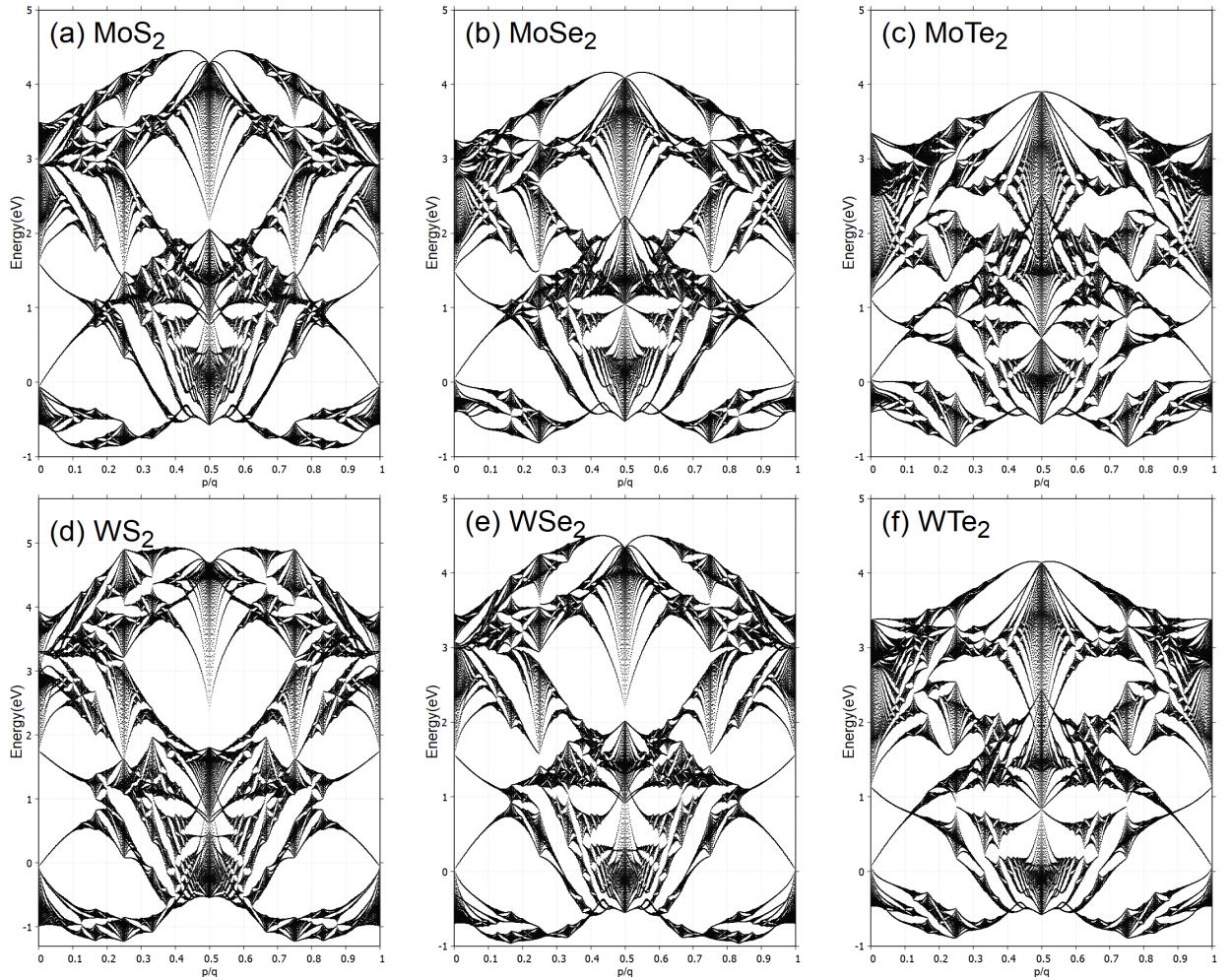


Figure 2.7: The Hofstadter's butterflies of  $MX_2$  monolayers using GGA parameters from Table 1.

Due to the significant mass of the transition-metal atom  $M$ , its spin orbit coupling (SOC) can be large. For simplicity, we consider only the on-site contribution, which corresponds to the  $\mathbf{L} \cdot \mathbf{S}$  term originating from the  $M$  atoms. Using the basis set  $\{|d_{z^2}, \uparrow\rangle, |d_{xy}, \uparrow\rangle, |d_{x^2-y^2}, \uparrow\rangle, |d_{z^2}, \downarrow\rangle, |d_{xy}, \downarrow\rangle, |d_{x^2-y^2}, \downarrow\rangle\}$ , we derive the SOC term in the Hamiltonian as

$$H' = \lambda \mathbf{L} \cdot \mathbf{S} = \frac{\lambda}{2} \begin{pmatrix} L_z & L_x - iL_y \\ L_x + iL_y & -L_z \end{pmatrix}, \quad (2.42)$$

in which

$$L_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix}, \quad (2.43)$$

is the matrix of  $\hat{L}_z$  ( $z$  component of the orbital angular momentum) in bases of  $d_{z^2}, d_{xy}, d_{x^2-y^2}$  and  $\lambda$  is characterized the strength of the SOC. Noting that, under the three bases, the

matrix elements of  $\hat{L}_x$  and  $\hat{L}_y$  are all zeros. Therefore the full TB Hamiltonian for the magnetic unit cell with the SOC as follows

$$\begin{aligned} H_{\text{SOC}}(\mathbf{k}) &= \mathbf{I}_2 \otimes H(\mathbf{k}) + \mathbf{I}_q \otimes H' \\ &= \begin{pmatrix} H_{3q \times 3q}(\mathbf{k}) + \frac{\lambda}{2} L_z & 0 \\ 0 & H_{3q \times 3q}(\mathbf{k}) - \frac{\lambda}{2} L_z \end{pmatrix}, \end{aligned} \quad (2.44)$$

in which  $\mathbf{I}_2$  is the  $2 \times 2$  identity matrix.

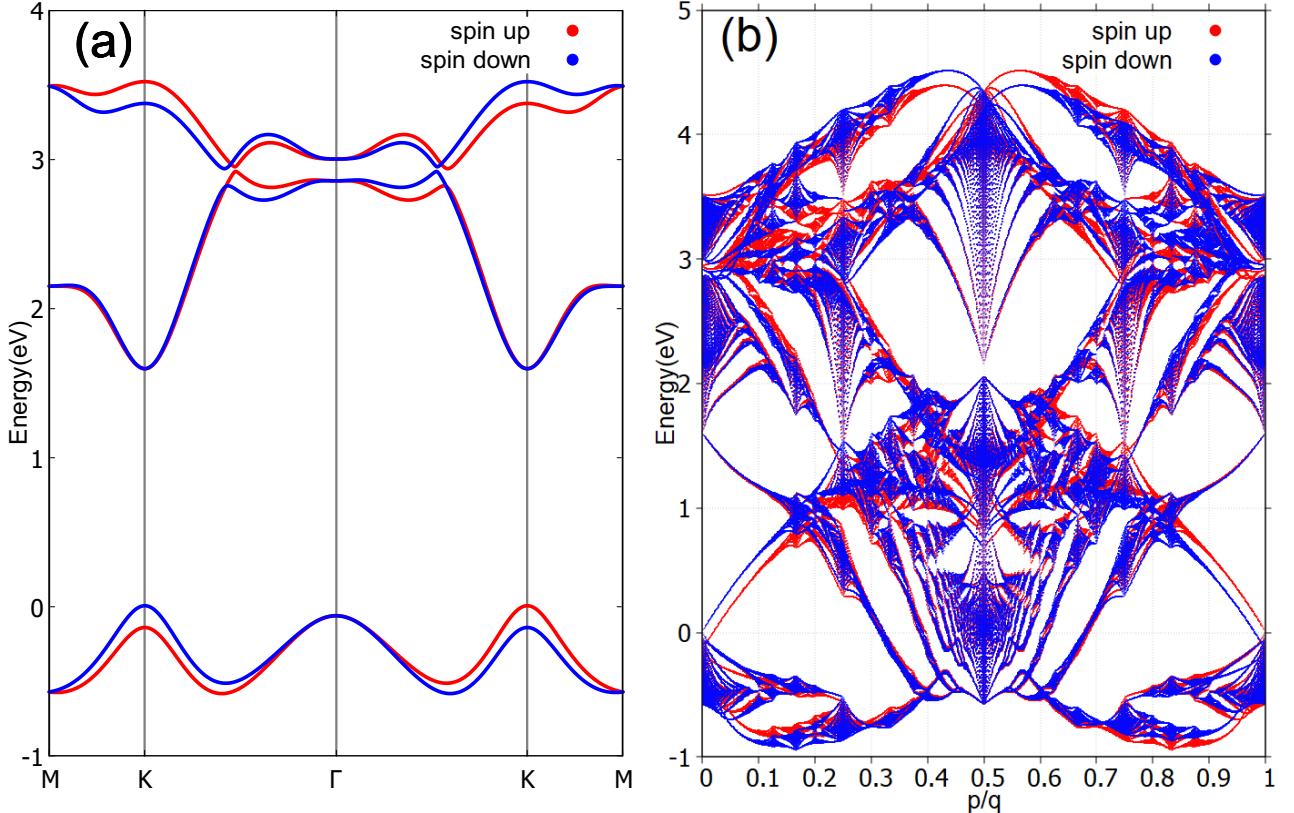


Figure 2.8: Band structure of monolayer  $MoS_2$  along  $\Gamma$ -K direction, SOC causes huge spin splittings in band-structure at  $K$  and  $-K$  points.

An alternative approach to the derivation of the Hamiltonian under an uniform magnetic field is given in Appendix B.

## 2.3 Landau levels

In solid-state physics, the behavior of electrons in magnetic fields is usually introduced by using the Hamiltonian

$$H = \frac{\mathbf{p} + e\mathbf{A}(\mathbf{r})^2}{2m}, \quad (2.45)$$

and the energy eigenfunctions are known as Landau levels (LLs)

$$E_n = (n + 1/2) \hbar\omega_c. \quad (2.46)$$

This treatment is for free electrons, but near the bottom of the two-dimensional tight-binding band of TMD we must find a regime in which the electron behaves as a nearly one (At least with a nearly free dispersion relation).

Recalling the result obtained for the dispersion relation of an electron within the TBM

$$h_0 = 2t_0(\cos 2\alpha + 2 \cos \alpha \cos \beta) + \epsilon_1, \quad (2.47)$$

The dispersion energy is approximately free-electron-like by Taylor expansion to second order of  $\mathbf{k}$

$$\begin{aligned} h_0(\mathbf{k}) &\approx 2t_0 \left[ 1 - \frac{a^2 k_x^2}{2} + 2 \left( 1 - \frac{a^2 k_x^2}{8} \right) \left( 1 - \frac{3a^2 k_y^2}{8} \right) \right] \\ &= t_0 \frac{3}{16} (32 + a^4 k_x^2 k_y^2) - t_0 \frac{3}{2} a^2 (k_x^2 + k_y^2) + \epsilon_1, \end{aligned} \quad (2.48)$$

the term  $a^4$  is negligibly small, then we have

$$h_0(\mathbf{k}) \approx 6t_0 - \frac{3}{2} t_0 a^2 (k_x^2 + k_y^2) + \epsilon_1. \quad (2.49)$$

One of the ways derivation of effective mass  $m^*$  is substitution  $\hbar\mathbf{k} \rightarrow \mathbf{\Pi} + e\mathbf{A}$ , with Landau gauge  $\mathbf{A} = (0, Bx, 0)$

$$\begin{aligned} h_0(\mathbf{\Pi}) &\approx 6t_0 - \frac{3}{2} t_0 \frac{a^2}{\hbar^2} \left[ \Pi_x^2 + (\Pi_y + eBx)^2 \right] + \epsilon_1 \\ &\approx 6t_0 - \frac{3}{2} t_0 \frac{a^2}{\hbar^2} \Pi_x^2 - \frac{3}{2} t_0 \frac{a^2}{\hbar^2} (eB)^2 \left[ x - \left( -\frac{\hbar k_y}{eB} \right) \right]^2 + \epsilon_1. \end{aligned} \quad (2.50)$$

The Eq (2.47) can be rewrite in the form as

$$E(\mathbf{\Pi}) = 6t_0 - \left[ \frac{1}{2m^*} \Pi_x^2 + \frac{1}{2} m^* \omega_c^2 (x - x_0)^2 \right] + \epsilon_1, \quad (2.51)$$

where  $m^* = \frac{\hbar^2}{3t_0 a^2}$  is the effective mass and  $x_0 = \frac{\hbar k_y}{eB}$ . Subsequently, the cyclotron frequency is

$$\omega_c = \frac{eB}{m^*} = \frac{8\pi\sqrt{3}t_0 p}{\hbar q}, \quad (2.52)$$

and therefore the Landau levels near the bottom of the band structure can be written

as

$$\begin{aligned} E_n &= 6t_0 - \hbar\omega_c(n + 1/2) + \epsilon_1 \\ &= t_0 \left( 6 - 8\pi\sqrt{3} \frac{p}{q}(n + 1/2) \right) + \epsilon_1, \end{aligned} \quad (2.53)$$

in linear order of an uniform-flux, where  $n$  is Landau index. These levels give rise to what is called “the Landau fan”, being very important in the de Haas-van Alphen and Shubnikov-de Haas effects [5] which predicts oscillations of the magnetic moment of a metal depending on an applied magnetic field.

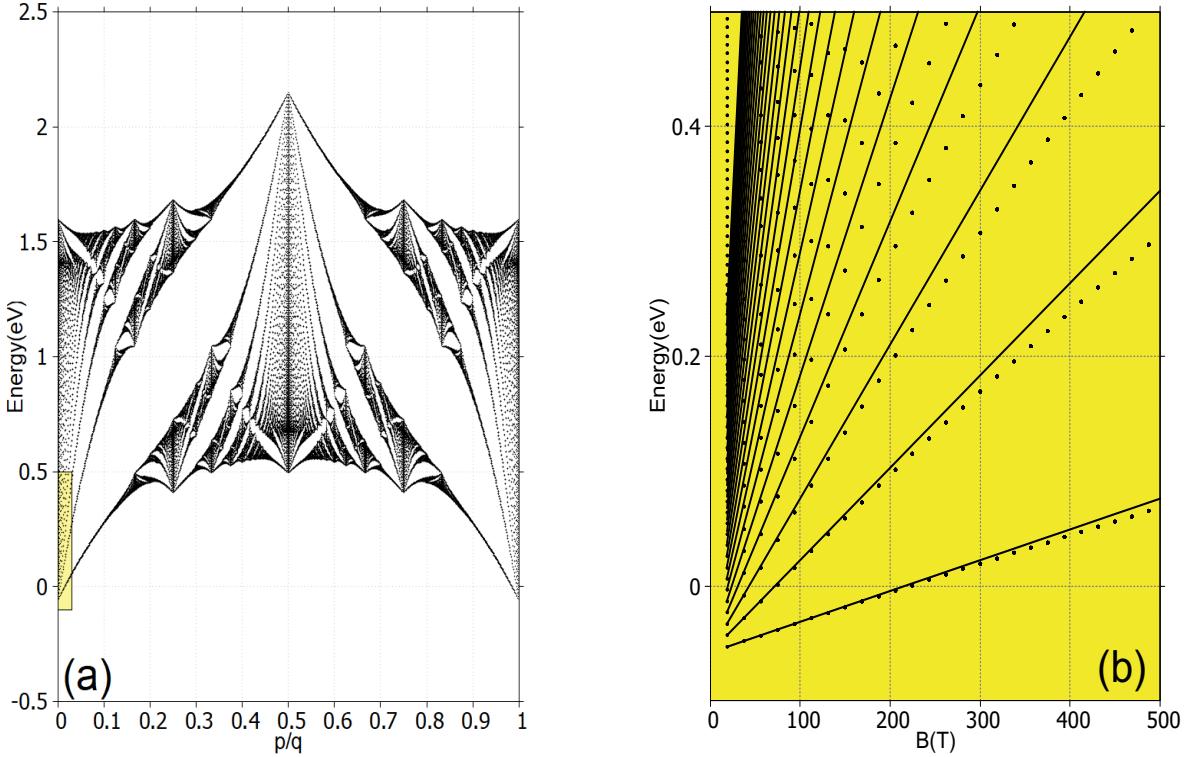


Figure 2.9: (a) Same plot as Fig (2.6) but considering a small area and (b) shows superposition of the Landau fan diagram and the Hofstadter butterfly. Display the first  $n = 30$  levels near the bottom of the conduction band for a magnetic field up to  $B = 500$  T.

In Fig 2.9 we compare the spectrum of a small section of single-band with  $p/q = 1/797$ , which is equivalent to small magnetic field, the spectrum of  $\text{MoS}_2$ , with the energy of Landau levels given by Eq.(2.50) show standard equally spaced LLs [6–9] near the bottom of the bands, as plotted in Fig 2.9(b). The fan of LLs can be clearly seen emerging from the pattern in Fig 2.9(a).

In Fig 2.9(a), there is just single-band in case zero field, with the effective mass  $m^* = \frac{\hbar}{3t_0 a^2}$ . The numerical result for this portion of the spectrum are shown in Fig 2.9 for  $p/q \geq 1/797$ . The first few LLs are clearly seen, and the asymptotic slopes  $p/q$  at large  $q$  given by Eq. (2.50) are shown for comparison for the first five Landau levels at  $B \leq 100$  T. At the values of  $B$  the fit is not ideal, but it does seem to be improving

with the decreasing  $p/q$ .

Figure 2.9 displays a blowup of the low uniform magnetic region and the LLs as a function of  $\Phi/\Phi_0$  [10]. The Landau levels are all close to being linear in  $B$ , resulting from the magnetic quantization of parabolic bands at  $B = 0$  T i.e. increasing values of  $B$ , these LLs are sequentially depleted; for  $B = 200$  T the levels are completely filled up to the level  $n = 10$ ; for  $B = 500$  T it happens the same, only this time are filled up to the level  $n = 4$  and so on.

## 2.4 Hall effects

### 2.4.1 The classical Hall effect

In this section, we will discuss the integer Quantum Hall effect (IQHE) where the flux number and flux quanta is an integer. This phenomenon can be understood without taking into account the Coulomb interaction between electrons. This means we will continue using the single electron Hamiltonian that we described in Section 2. Starting with an electric field  $\mathbf{E}$  established in the solid results in a current density  $\mathbf{J}$  linearly related to the field through Ohm's law

$$\mathbf{J} = \boldsymbol{\sigma} \mathbf{E}, \quad (2.54)$$

$$\mathbf{E} = \sigma^{-1} \mathbf{J} = \rho \mathbf{J}, \quad (2.55)$$

where  $\boldsymbol{\sigma}$  is the conductivity tensor and the resistivity  $\rho$  is defined as the inverse of the conductivity. This remains true when both are tensors

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad (2.56)$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}, \quad (2.57)$$

$$(2.58)$$

The off-diagonal components of resistivity tensor is  $\rho_{xy} = \rho_{yx} = \frac{B}{en}$ . Usually we measure the resistance  $R$ , which differs from the resistivity  $\rho$  by geometric factors. However, for  $\rho_{xy}$ , this thing coincide. To see this, consider a sample of material of length  $L$  in the  $y$ -direction. We drop a voltage  $V_y$  in the  $y$ -direction and measure the resulting current

$I_x$  in the  $x$ -direction. The transverse resistance is

$$R_{xy} = \frac{V_y}{I_x} = \frac{LE_y}{LJ_x} = \frac{E_y}{J_x} = -\rho_{xy}. \quad (2.59)$$

For a current  $I_x$  flowing in the  $x$ -direction, and the corresponded electric field  $E_y$  in the  $y$ -direction, the Hall coefficient is defined by

$$R_H = \frac{\rho_{xy}}{B} = \frac{1}{en}. \quad (2.60)$$

We see that the Hall coefficient depends only on microscopic information about the material: the charge and density of conduction particles. The Hall coefficient does not depend on the scattering time  $\tau$ ; it remains unaffected by the specific frictional mechanism present in the material.

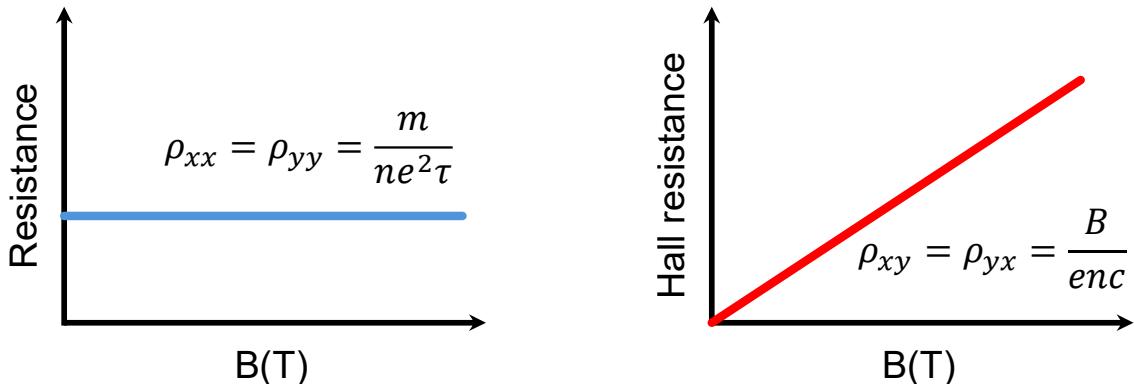


Figure 2.10: The longitudinal resistance on the right figure and the Hall resistance in the left figure. The graph shows both the longitudinal resistance and the Hall resistance is linear to the increasing magnetic field.

### 2.4.2 The Quantum Hall effect

In two dimensional, there is a crucial relationship between the conductivity tensor  $\sigma$  and the resistivity tensor  $\rho$  is given by

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix}^{-1} = \frac{1}{\rho_{xx}\rho_{yy} - \rho_{xy}\rho_{yx}} \begin{bmatrix} \rho_{yy} & -\rho_{xy} \\ -\rho_{yx} & \rho_{xx} \end{bmatrix}. \quad (2.61)$$

In the previous study, we had arrived at the classical Hall resistance. And this remains stable as expect whenever we still trust classical mechanics. But the world is also consist quantum mechanics. This becomes important at low temperatures and strong magnetic fields.

There are two related phenomena which are associated to two different quantum Hall effects. These are called the integer and fractional quantum Hall effects. In this

work, we first discovered and subsequently understood theoretically the integer quantum Hall effect in the Hofstadter butterfly.

Let's take a look at the experimental data for the quantum Hall effect were performed in 1980 by von Klitzing *et al.* [11]

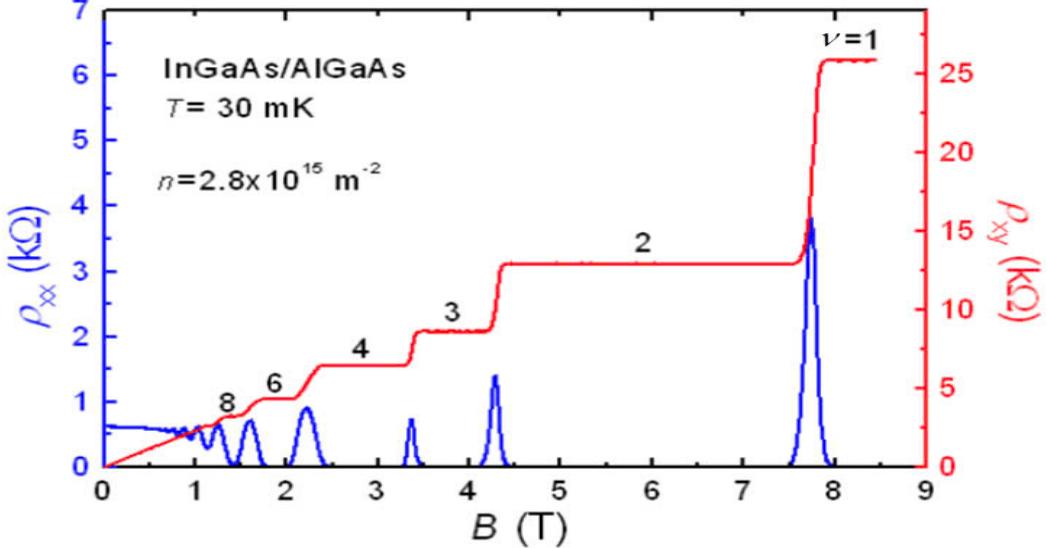


Figure 2.11: This is the integer quantum Hall effect. For this von Klitzing was awarded the 1985 Nobel prize.

Both the Hall resistivity  $\rho_{xy}$  and the longitudinal resistivity  $\rho_{xx}$  depict interesting behaviour. Perhaps the most striking feature in the figure is the fact that the Hall resistivity  $\rho_{xy}$  sits on a plateau for a range of magnetic field, before jumping dramatically to the next plateau. On these plateau, the Hall resistance now defined

$$\rho_{xy} = \frac{R_K}{\nu}, \quad \nu = 1, 2, \dots, \quad (2.62)$$

and  $R_K$  is Klitzing's resistance constant.

The contribution to the Hall conductance from a single subband is given by [12–15]

$$\sigma_{xy} = \frac{e^2}{h} \sum_n^{\text{occ.}} \frac{1}{2\pi} \iint_{Bz} dk_x dk_y \Omega_n^z(\mathbf{k}), \quad (2.63)$$

In general, the Berry curvature intergrated over a closed manifold is quantized in the units of  $e^2/h$  and equals to the net number of monopoles inside. This number is called the Chern number and is responsible for a number of quantization effects. Therefore the Hall conductivity is quantized for a two dimensional band insulator of noninteracting electrons.

### 2.4.3 Colored the spectrum

With the cyclotron frequency in Section 2.4, the electron energy is quantized to the Landau levels.

We calculate the quantum Hall conductivity by the Streda formula [16]

$$\sigma_{xy}(B, E_F) = e \frac{\partial \rho(E_F, B)}{\partial B}, \quad (2.64)$$

where  $\rho(E_F, B)$  is the cumulative energy density of state at Fermi-energy  $E_F$ . As shown in Fig, the Hall conductivity is quantized at colored points. Since the integral for the whole Brillouin zone respectively Berry curvature, we arrived at the Thouless-Kohmoto-Nightingale-Nijs (TKNN)'s formula

$$\sigma_{xy} = \frac{e^2}{h} \nu, \quad \nu = 1, 2, \dots \quad (2.65)$$

$\nu$  is guaranteed to be an integer given by the Chern number. Combining Eq(2.39) and Eq(2.40), we have

$$\frac{\partial \rho}{\partial B} = \frac{e}{h} \nu. \quad (2.66)$$

Assuming that  $B$  has slight variation

$$\rho = c + \frac{e}{h} B \nu, \quad c \text{ is constant.} \quad (2.67)$$

Before this, we have defined  $\frac{p}{q} = \frac{eB a^2 \sqrt{3}}{4h}$ , with  $S = \frac{\sqrt{3}a^2}{4}$  is the area of the unit cell. Multiply  $S$  with Eq(2.42), we have

$$\rho \times S = c + \frac{p}{q} \nu, \quad (2.68)$$

and the density of electron in a single band is given by  $\frac{1}{Sq}$ , thus when there are  $r$  bands below the Fermi energy level, the density of electron for  $r$ -th band is

$$\rho = \frac{r}{Sq}. \quad (2.69)$$

Then, the Eq. (2.68), is written as,

$$r = c \times q + p \times \nu_r, \quad (2.70)$$

in this equation  $r, q, p, \nu_r$  are integers, thus,  $c \times q$  must be an integer. On the one hand, since  $c$  is independent of  $q$ , and  $q$  can change when the magnetic field is varied without making a point of contact, then  $c$  itself must be an integer, namely  $s_r$ . Thus we have

$$r = q \times s_r + p \times \nu_r, \quad (2.71)$$

which is the Diophantine equation. In order to compute the Hall conductivity of the lattice model for an electron in a background magnetic field. We can only do this for rational fluxes  $\frac{\Phi}{\Phi_0} = \frac{p}{q}$ . In this case, we can use the TKNN formula, but with the Chern number, which used to be defined by integrating over the Brillouin zone, now arising by integrating over the magnetic Brillouin zone. Others derivation is in [17],[18].

The Hofstadter butterfly may be colored in various ways. For instance, we may color the points of the butterfly by their Chern number, as illustrated in the Fig2.11(a). The disadvantage of this, is that there will be many points in the butterfly and so the fine details of the coloring may be obscured.

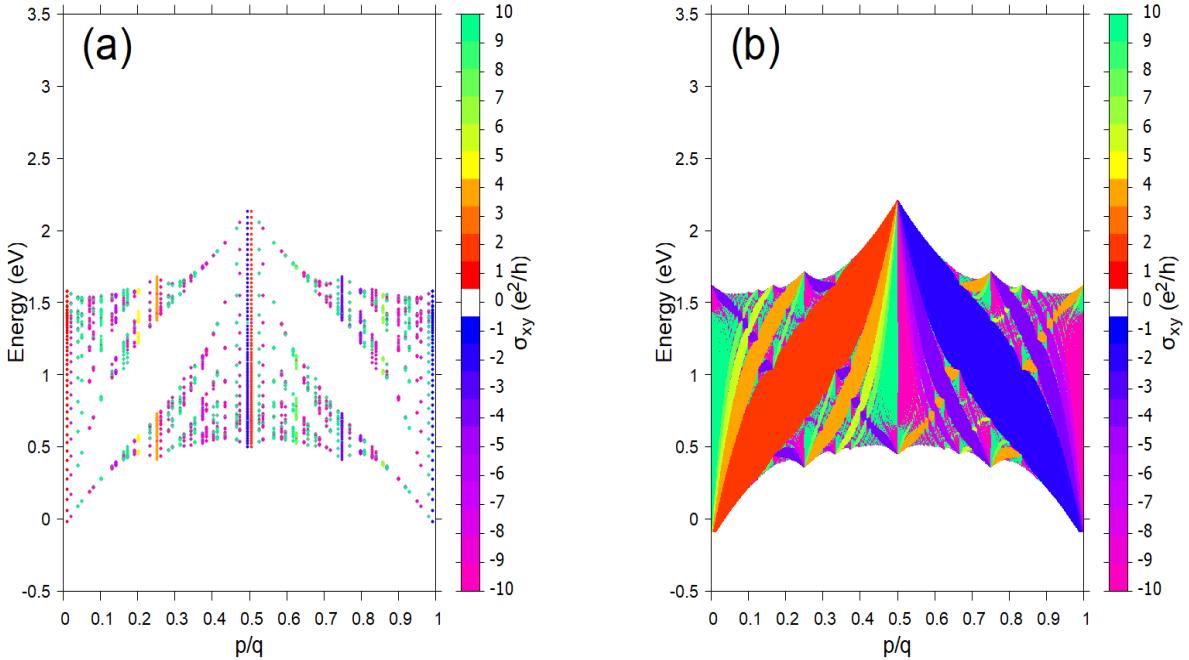


Figure 2.12:  $q = 99$  và  $q = 797$

Fig 2.11(b) shows the Hofstadter butterfly, color-coded according to the Hall conductance. Moreover, the Chern number magnitude range will increase with  $q$  and so it is difficult to define a fixed scale. In this work, we use a scale for Chern numbers with magnitude up to 10, as well as the color palette made famous by Avron *et al.* [19].

Any Chern number out of range  $-10 \leq \nu \leq 10$  is equal to zero. Regions with zero Hall conductance and the corresponding spectrum are left blank. The two largest gaps near the center of the figure are associated with small intergers where the color coding is faithful. The colored picture emphasizes the gaps while the standard Hofstadter butterfly emphasizes the spectrum. The colored figure is prettier and displays the regular aspects of the diagram: gaps exhibit better-defined behavior than spectra.

The transverse conductance in each energy gap, depicted in figure, is given by the sum of the Chern numbers of the occupied bands. At first discussed by Wannier in reference [20], even a small change in the flux modifies radically the underlying band structure, thereby altering the integrated density of states.

This results consistent with the Quantum Hall effect and the Shubnikov-de Haas effects.

#### 2.4.4 Solving the diophantine equation

We have defined that the magnetic flux through a unit cell is  $\frac{\Phi}{\Phi_0} = \frac{p}{q}$ . For  $p$  and  $q$  are mutually prime numbers, we defined the pairs  $(\nu_r, s_r) = (m, n)$  as the solutions

$$pm + qn = \gcd(p, q). \quad (2.72)$$

Fortunately,  $p$  and  $q$  are co-prime, Eq (2.47) is now

$$pm + qn = 1, \quad (2.73)$$

By deviding  $p$  by  $q$ , we get a quotient  $a$  and a remainder  $b$ . They satisfy

$$q = pa + b. \quad (2.74)$$

By using the Euclidean Algorithm, we can easily find  $(m, n)$ . For instance, the rational magnetic flux is  $\frac{p}{q} = \frac{30}{47}$ , then the diophantine equation now is  $30m + 47n = 1$ , and

$$\begin{aligned} 47 &= 30 \times 1 + 17, \\ 30 &= 17 \times 1 + 13, \\ 17 &= 13 \times 1 + 4, \\ 13 &= 4 \times 3 + 1. \end{aligned} \quad (2.75)$$

At this point we stop, because we arrived at the greatest common divisor, so the algorithm is over. The next step is solve for the remainders

$$\begin{aligned}
47 &= 30 \times 1 + 17 \Rightarrow 17 = 47 \times 1 + 30 \times (-1), \\
30 &= 17 \times 1 + 13 \Rightarrow 13 = 30 \times 1 + 17 \times (-1), \\
17 &= 13 \times 1 + 4 \Rightarrow 4 = 17 \times 1 + 13 \times (-1), \\
13 &= 4 \times 3 + 1 \Rightarrow 1 = 13 \times 1 + 4 \times (-3).
\end{aligned} \tag{2.76}$$

We are going to take this last remainder equation, and do backwards substitute until we get the very first remainder

$$\begin{aligned}
1 &= 13 \times 1 + 4 \times (-3) \\
&= 13 \times 1 + [17 \times 1 + 13 \times (-1)] \times (-3) \\
&= 13 \times 4 + 17 \times (-3) \\
&= [30 \times 1 + 17 \times (-1)] \times 4 + 17 \times (-3) \\
&= 30 \times 4 + 17 \times (-7) \\
&= 30 \times 4 + [47 \times 1 + 30 \times (-1)] \times (-7) \\
&= 30 \times (11) + 47 \times (-7),
\end{aligned} \tag{2.77}$$

we find the solution for  $(m, n)$  is  $(11, -7)$ .

### 2.4.5 Wannier diagram

To further explore the intricate fractal nature of the Hofstadter spectrum, we shall now achieve a simplified replica of Fig, a powerful tool for visualizing the relationship between magnetic flux and electron filling in the system. Wannier's diagram provides a graphical representation of the allowed energy gaps in the Hofstadter spectrum as a function of the magnetic flux per unit cell and the electron filling factor. This diagram is particularly insightful because it captures the topological properties of the system through the distribution and behavior of these energy gaps.

The fundamental of Wannier's diagram lies in the Diophantine gap equation

$$\frac{n}{n_0} = \nu \frac{\Phi}{\Phi_0} + s, \tag{2.78}$$

where  $\frac{n}{n_0}$  is the electron filling factor, representing the ratio of the number of electrons per unit cell,  $\nu$  is the Chern number associated with the quantized Hall conductance, and  $s$  is another integer that corresponds to the gap index, effectively indicating the electron filling within the spectrum.

The Diophantine equation is crucial in understanding the quantization of Hall conductance in the Hofstadter butterfly. The Chern number  $t$  determines the topological nature of the bands and their contribution to the Hall conductance, while the integer  $s$  identifies specific energy gaps in the spectrum. These gaps are directly linked to incompressible quantum Hall states, which are of significant interest in both theoretical and experimental condensed matter physics.

The calculated Wannier's diagram, the energy density of state as a function of filling factor  $\frac{n}{n_0}$  and magnetic flux  $\Phi$  as illustrated in Fig, reveal the presence of these energy gaps across the spectrum. Each colored line in the diagram corresponds to an energy gap where the system exhibits an imcompressible quantum Hall state, characterized by a quantized Hall conductance. The diagram not only provides a clear visualization of the gap structure but also offers insights into the topological phases that arise in monolayer TMD systems under varying magnetic flux conditions.

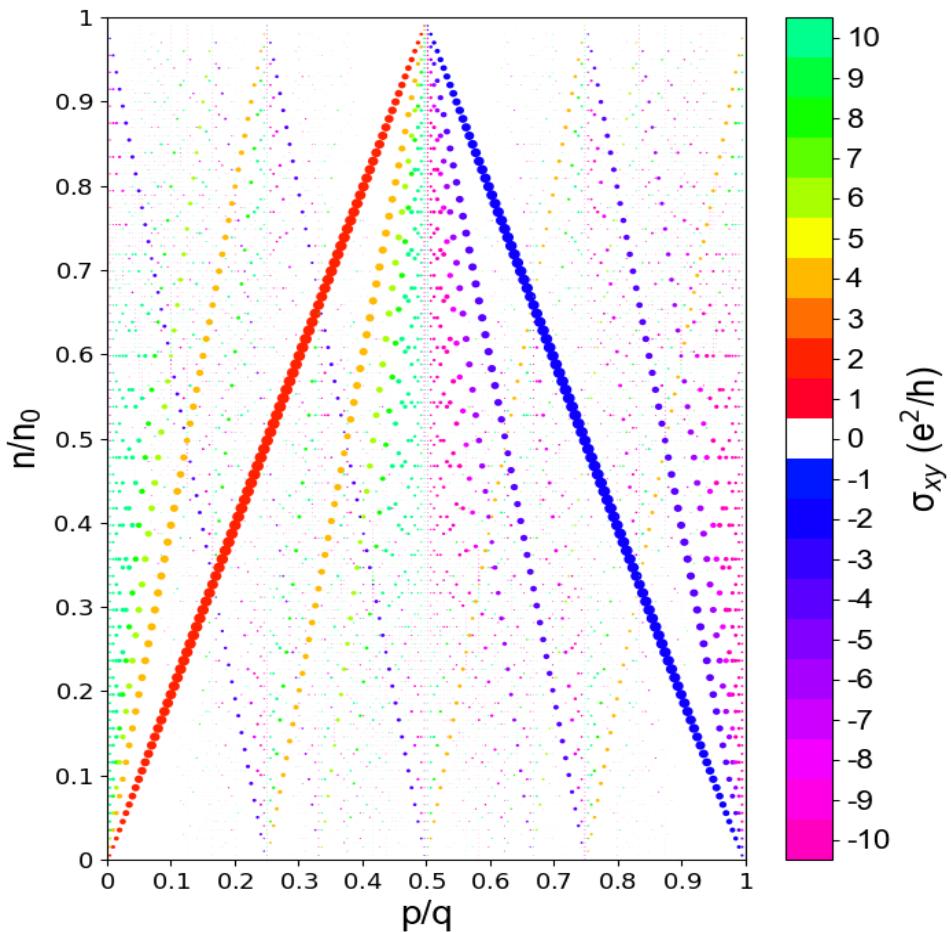


Figure 2.13: Wannier diagram

## **CHAPTER 3**

### **RESULT AND DISCUSSION**

## CHAPTER 4

# CONCLUSION AND FUTURE WORK

### 4.1 Conclusion

In our research, we have calculated the Hofstadter butterfly of monolayer MoS<sub>2</sub> and others transition metal dichalcogenide types by using a tight-binding three-band model. In addition, we have explored the rich and complex physics of monolayer MoS<sub>2</sub>, such as Landau levels and integer quantum Hall effect (IQHE), in the presence of external magnetic fields. The research conducted within these pages has demonstrated the unique interplay between the superlattice and magnetic fields, which leads to the emergence of fascinating quantum phenomena.

In section 2.1, we have studied the tight-binding three-band model for monolayers of MX<sub>2</sub> using only the M- $d_{z^2}$ ,  $d_{xy}$  and  $d_{x^2-y^2}$  orbitals. When only NN M-M hoppings are included, we calculated the hopping energies using the symmetry of the  $D_{3h}$  point group we derived eight hopping parameters from Ref [1].

In section 2.2, we focused on the Hofstadter physics in monolayer TMD, where the lattice gives rise to a rich Hofstadter spectrum when subjected to a magnetic field. The detailed analysis revealed key features of the spectrum, including the SOC and the emergence of topological quantum Hall states. In addition, the study also demonstrated that there are many ways to derive the Hofstadter spectrum two of those is using the Peierls substitution or Envelope Function Approximation.

In section 2.3 and section 2.4, extended the investigation into the realm of Hall effects, introducing the Landau levels, the integer quantum Hall effect and applying it to monolayer TMD systems. We also shown that how the Hofstadter butterfly can be colored in various ways using the Chern number.

Overall, while this study provides valuable insights, we acknowledge several limitations. Firstly, in section 2.3 and 2.4, our calculation was restricted to the single-

band approximation due to the computational complexity of multi-band interactions. Specifically, incorporating three-band model would require significantly more resources, particularly in calculating Chern numbers, which are numerically intensive for larger Hamiltonian matrices. Secondly, for tractability, we adopted the convection from Ref [11] of fixing the Fermi energy at mid-spectrum. While this simplification is justified in certain regimes, it may not hold under strong magnetic field, where the density of electrons may be changed corresponding the magnetic field.

## 4.2 Future work

For further research, we can utilize thi

## APPENDIX A

### Harper's equation

We now consider the case of hexagonal lattice with one band as a basis under an uniform magnetic field given by the Landau gauge  $\mathbf{A} = (0, Bx, 0)$ . Given

$$\begin{aligned}
h_0 &= 2t_0 (\cos 2\alpha + 2 \cos \alpha \cos \beta) + \epsilon_1 \\
&= 2t_0 \left[ \cos(k_x a) + 2 \cos\left(\frac{k_x a}{2}\right) \cos\left(\frac{\sqrt{3}k_y a}{2}\right) \right] + \epsilon_1 \\
&= 2t_0 \left\{ \cos(k_x a) + \cos\left[\left(k_x + \sqrt{3}k_y\right) \frac{a}{2}\right] + \cos\left[\left(k_x - \sqrt{3}k_y\right) \frac{a}{2}\right] \right\} + \epsilon_1 \\
&= 2t_0 \left\{ \cos\left(\Pi_x \frac{a}{\hbar}\right) + \cos\left[\left(\Pi_x + \sqrt{3}eBx + \sqrt{3}\Pi_y\right) \frac{a}{2\hbar}\right] \right. \\
&\quad \left. + \cos\left[\left(\Pi_x - \sqrt{3}eBx - \sqrt{3}\Pi_y\right) \frac{a}{2\hbar}\right] \right\} + \epsilon_1 \tag{A.1} \\
&= t_0 \left[ e^{i\Pi_x \frac{a}{\hbar}} + e^{-i\Pi_x \frac{a}{\hbar}} + e^{i(\Pi_x + \sqrt{3}eBx + \sqrt{3}\Pi_y)a/2\hbar} + e^{-i(\Pi_x + \sqrt{3}eBx + \sqrt{3}\Pi_y)a/2\hbar} \right. \\
&\quad \left. + e^{i(\Pi_x - \sqrt{3}eBx - \sqrt{3}\Pi_y)a/2\hbar} + e^{-i(\Pi_x - \sqrt{3}eBx - \sqrt{3}\Pi_y)a/2\hbar} \right] + \epsilon_1.
\end{aligned}$$

We replaced  $\hbar\mathbf{k}$  in the above function by the operators  $\mathbf{\Pi} + e\mathbf{A}/c$  in order to create an operator out of  $h_0$ . However, the quantity  $\hbar\mathbf{k}$  is represents the crystal momentum, it is more precise interpretation is to regard  $\mathbf{k}$  as a quantum number which describes a Bloch state. This method can be achived by using Envelop Function Approximation (EFA). When this substitution is made, the Hamiltonian element is seen to contain translation operators  $\exp[a\Pi_x/\hbar]$ ,  $\exp[a\sqrt{3}\Pi_y/(2\hbar)]$ . The Landau gauge  $\mathbf{A} = (0, Bx, 0)$  was chosen, then only the translation along  $y$  are multiplied by phases. [3]However, we must be very careful regarding how the operators act on the wave functions, since  $[x, \Pi_x] \neq 0$ . In their article, Gumbs and Fekete [21] incorrectly applied the modified translation operators, leading to completely incorrect results. In this work, we treat the

operators more correctly by applying the Baker-Campbell-Hausdorff (BCH) formula and taking into account the commutation relation  $[x, \Pi_x] = i\hbar$

$$\begin{aligned} e^{\pm i(\Pi_x + \sqrt{3}eBx)a/2\hbar} &= e^{\pm i\Pi_x a/2\hbar} e^{\pm i\sqrt{3}eBxa/2\hbar} e^{-\frac{1}{2}[\pm i\Pi_x, \pm i\sqrt{3}eBx]a^2/2\hbar^2} \\ &= e^{\pm i\Pi_x a/2\hbar} e^{\pm i\sqrt{3}eBxa/2\hbar} e^{\mp i\sqrt{3}eBa^2/8\hbar}. \end{aligned} \quad (\text{A.2})$$

Substituting  $x = \frac{ma}{2}$  into (B.2), this leads to

$$e^{\pm i(\Pi_x + \sqrt{3}eBx)a/2\hbar} = e^{\pm i\Pi_x a/2\hbar} e^{\pm i\sqrt{3}eB(m+1/2)a^2/4\hbar}. \quad (\text{A.3})$$

And

$$\begin{aligned} e^{\pm i(\Pi_x - \sqrt{3}eBx)a/2\hbar} &= e^{\pm i\Pi_x a/2\hbar} e^{\mp i\sqrt{3}eBxa/2\hbar} e^{-\frac{1}{2}[\pm i\Pi_x, \mp i\sqrt{3}eBx]a^2/2\hbar^2} \\ &= e^{\pm i\Pi_x a/2\hbar} e^{\mp i\sqrt{3}eBxa/2\hbar} e^{\mp i\sqrt{3}eBa^2/8\hbar}, \end{aligned} \quad (\text{A.4})$$

substituting  $x = \frac{ma}{2}$  into (B.4), this leads to

$$e^{\pm i(\Pi_x - \sqrt{3}eBx)a/2\hbar} = e^{\pm i\Pi_x a/2\hbar} e^{\mp i\sqrt{3}eB(m-1/2)a^2/4\hbar}. \quad (\text{A.5})$$

The operators  $e^{\pm i\Pi_x a/2\hbar}, e^{\pm i\Pi_y \sqrt{3}a/2\hbar}$  can be recognized as translational operators, we can rewrite (B.3) as

$$\begin{aligned} &t_0\varphi_0(x+a, y) + t_0\varphi_0(x-a, y) + t_0\varphi_0(x + \frac{a}{2}, y + \frac{a\sqrt{3}}{2}) e^{\frac{ie}{\hbar}B(m+1/2)\frac{a^2\sqrt{3}}{4}} \\ &+ t_0\varphi_0(x + \frac{a}{2}, y - \frac{a\sqrt{3}}{2}) e^{-\frac{ie}{\hbar}B(m+1/2)\frac{a^2\sqrt{3}}{4}} + t_0\varphi_0(x - \frac{a}{2}, y + \frac{a\sqrt{3}}{2}) e^{\frac{ie}{\hbar}B(m+1/2)\frac{a^2\sqrt{3}}{4}} \\ &+ t_0\varphi_0(x - \frac{a}{2}, y - \frac{a\sqrt{3}}{2}) e^{-\frac{ie}{\hbar}B(m-1/2)\frac{a^2\sqrt{3}}{4}} + \epsilon_1\varphi_0(x, y) = E_1\varphi_0(x, y), \end{aligned} \quad (\text{A.6})$$

for the sake of simplicity we have defined  $\varphi_0 \equiv |d_{z^2}\rangle$ .

It is reasonable to assume planewave behavior in the  $y$  direction, since the coefficients in the above equation only involve  $x$ . Therefore, we can assume the partial solution for  $y$  to be in the form

$$\varphi(\frac{ma}{2}, \frac{na\sqrt{3}}{2}) = e^{ik_y n \frac{a\sqrt{3}}{2}} \varphi(m), \quad (\text{A.7})$$

which reduces (B.6) to

$$\begin{aligned} &t_0\varphi_0(m+2) + t_0\varphi_0(m-2) + t_0\varphi_0(m+1)e^{2i\pi(m+1/2)p/q} e^{ik_y a\sqrt{3}/2} \\ &+ t_0\varphi_0(m+1)e^{-2i\pi(m+1/2)p/q} e^{-ik_y a\sqrt{3}/2} + t_0\varphi_0(m-1)e^{2i\pi(m-1/2)p/q} e^{ik_y a\sqrt{3}/2} \\ &+ t_0\varphi_0(m-1)e^{-2i\pi(m-1/2)p/q} e^{-ik_y a\sqrt{3}/2} + \epsilon_1\varphi_0(m) = E_1\varphi_0(m), \end{aligned} \quad (\text{A.8})$$

this is equivalent to Eq. 2.16 we have mentioned in Section 2.2. Equation B.8 is sometimes called “Harper’s equation”. [22] Since different  $m$  values give different equations, one reaches a unique set of equations when  $\Phi/\Phi_0$  is a rational number  $p/q$  and  $m$  goes through  $q$  different values, essentially resulting in the Hamiltonian matrix written for a magnetic unit cell enlarged in  $x$  direction  $q$  times.

In the case of TMD presented in [1], the contribution of the  $X$  atom has been neglected, leading to the transformation of the hexagonal crystal structure of TMD into a regular triangular lattice. From there, we can map the triangular lattice to the case of the square lattice. In the triangular lattice, it has been established that the translation operators must satisfy the Baker-Campbell-Hausdorff formula.

## APPENDIX B

### Cyclotron frequency for all band

Starting with using Taylor expansion to second order of  $\mathbf{k}$  for Hamiltonian elements in Eq()

$$\begin{aligned}
h_0 &= t_0(6 - \frac{3}{2}a^2(k_x^2 + k_y^2)) + \epsilon_1, \\
h_1 &= it_1 3ak_x - \frac{3}{2}t_2 a^2 k_x k_y, \\
h_2 &= 2t_2(\frac{3}{8}a^2 k_x^2 + \frac{3}{8}a^2 k_y^2) + i3t_1 a k_y, \\
h_{11} &= (t_{11} + 3t_{22})(1 - \frac{1}{8}a^2 k_x^2 - \frac{3}{8}a^2 k_y^2) + 2t_{11}(1 - \frac{1}{2}a^2 k_x^2) + \epsilon_2, \\
h_{22} &= (3t_{11} + t_{22})(1 - \frac{1}{8}a^2 k_x^2 - \frac{3}{8}a^2 k_y^2) + 2t_{22}(1 - \frac{1}{2}a^2 k_x^2) + \epsilon_2, \\
h_{12} &= \sqrt{3}(t_{22} - t_{11})a^2 k_x k_y.
\end{aligned} \tag{B.1}$$

In this (C.1), we have neglected coefficents of terms  $a^3, a^4$  by the small of large limit  $a$  this leads to and using substitution  $\hbar\mathbf{k} \rightarrow (\mathbf{\Pi} + e\mathbf{A})$

$$\begin{aligned}
h_0 &= t_0(6 - \frac{3}{2\hbar^2}a^2(\Pi_x^2 + (\Pi_y + eBx)^2)) + \epsilon_1, \\
h_1 &= it_1 \frac{3}{\hbar}a\Pi_x - \frac{3}{2\hbar^2}t_2 a^2 \Pi_x (\Pi_y + eBx), \\
h_2 &= 2t_2(\frac{3}{8\hbar^2}a^2 \Pi_x^2 + \frac{3}{8\hbar^2}a^2(\Pi_y + eBx)^2) + i\frac{3}{\hbar}t_1 a(\Pi_y + eBx), \\
h_{11} &= (t_{11} + 3t_{22})(1 - \frac{1}{8\hbar^2}a^2 \Pi_x^2 - \frac{3}{8\hbar^2}a^2(\Pi_y + eBx)^2) + 2t_{11}(1 - \frac{1}{2\hbar^2}a^2 \Pi_x^2) + \epsilon_2, \\
h_{22} &= (3t_{11} + t_{22})(1 - \frac{1}{8\hbar^2}a^2 \Pi_x^2 - \frac{3}{8\hbar^2}a^2(\Pi_y + eBx)^2) + 2t_{22}(1 - \frac{1}{2\hbar^2}a^2 \Pi_x^2) + \epsilon_2, \\
h_{12} &= \frac{\sqrt{3}(t_{22} - t_{11})a^2}{\hbar^2} \Pi_x (\Pi_y + eBx).
\end{aligned} \tag{B.2}$$

Instead of doing as we have done in Section 2, there is an alternative way to determine the energy spectrum. The Hamiltonian can be simplified by a suitably chosen canonical transformation, or ladder (creation and annihilation) operators can be used instead of

position and momentum operators, but the description of the motion in the  $xy$ -plane requires two commuting sets of operators now. Since  $x$  and  $\Pi_y$  appear together in the combination  $x + \frac{1}{eB}\Pi_x$ , the appropriate choice in this case is [23, 24]

$$\begin{aligned} a &= \sqrt{\frac{eB}{2\hbar}} \left( x + \frac{1}{eB}\Pi_y + \frac{i}{eB}\Pi_x \right), \\ a^\dagger &= \sqrt{\frac{eB}{2\hbar}} \left( x + \frac{1}{eB}\Pi_y - \frac{i}{eB}\Pi_x \right), \\ b &= \sqrt{\frac{eB}{2\hbar}} \left( y + \frac{1}{eB}\Pi_x + \frac{i}{eB}\Pi_y \right), \\ b^\dagger &= \sqrt{\frac{eB}{2\hbar}} \left( y + \frac{1}{eB}\Pi_x - \frac{i}{eB}\Pi_y \right). \end{aligned} \tag{B.3}$$

The inverse transformation is then

$$\begin{aligned} x + \frac{1}{eB}\Pi_y &= \sqrt{\frac{\hbar}{2eB}} (a + a^\dagger), \\ \Pi_x &= i\sqrt{\frac{\hbar eB}{2}} (a^\dagger - a), \\ y + \frac{1}{eB}\Pi_y &= \sqrt{\frac{\hbar}{2eB}} (b + b^\dagger), \\ \Pi_y &= i\sqrt{\frac{\hbar eB}{2}} (b^\dagger - b). \end{aligned} \tag{B.4}$$

It follows from the canonical commutation relations of the position and momentum operators that the ladder operators satisfy bosonic commutation relations

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \tag{B.5}$$

and

$$[a, a] = [a^\dagger, a^\dagger] = [b, b] = [b^\dagger, b^\dagger] = 0, \tag{B.6}$$

moreover the operators  $a(a^\dagger)$  and  $b(b^\dagger)$  commute with each other, too. As in the usual one-dimensional harmonic oscillator

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{B.7}$$

where  $|n\rangle$  is an eigenstate of the usual number operators  $a^\dagger a|n\rangle = n|n\rangle$ , with  $n \geq 0$  an

interger. In terms of them, the Hamiltonian (C.2) can be cast in form

$$\begin{aligned}
h_0 &= -6t_0 \frac{a^2 e B}{2\hbar} (a^\dagger a^- + \frac{1}{2}) + 6t_0 + \epsilon_1, \\
h_1 &= 3t_1 \sqrt{\frac{a^2 e B}{2\hbar}} (a^- - a^\dagger) - \frac{3i t_2 a^2 e B}{4\hbar^2} (a^\dagger a^\dagger - a^- a^- - 1), \\
h_2 &= 3i t_1 \sqrt{\frac{a^2 e B}{2\hbar}} (a^\dagger + a^-) + \frac{3t_2 a^2 e B}{8\hbar^2} (a^\dagger a^\dagger + a^- a^-), \\
h_{11} &= 3(t_{11} + t_{22}) + \frac{3t_{11} a^2 e B}{8\hbar^2} (a^\dagger a^\dagger + a^- a^-) - 3(t_{11} + t_{22}) \frac{a^2 e B}{2\hbar} (a^\dagger a^- + \frac{1}{2}) + \epsilon_2, \\
h_{22} &= 3(t_{11} + t_{22}) + \frac{3t_{22} a^2 e B}{8\hbar^2} (a^\dagger a^\dagger + a^- a^-) - 3(t_{11} + t_{22}) \frac{a^2 e B}{2\hbar} (a^\dagger a^- + \frac{1}{2}) + \epsilon_2, \\
h_{12} &= \sqrt{3}i(t_{22} - t_{11}) \frac{a^2 e B}{2\hbar} (a^\dagger a^\dagger - a^- a^- - 1).
\end{aligned} \tag{B.8}$$

Note that there are still linear-in- $\mathbf{k}$  matrix elements between the  $|d_z\rangle$ ,  $|d_{xy}\rangle$ ,  $|d_{x^2-y^2}\rangle$ . In the higher order of  $\mathbf{k}$ , these bands do couple, but for the sake of simplicity, in this work, will be neglected. We can reduce the Hamiltonian in the form

$$\begin{pmatrix} h_0 & 0 & 0 \\ 0 & h_{11} & h_{12} \\ 0 & h_{12}^* & h_{22} \end{pmatrix} \tag{B.9}$$

By diagonalizing the Hamiltonian

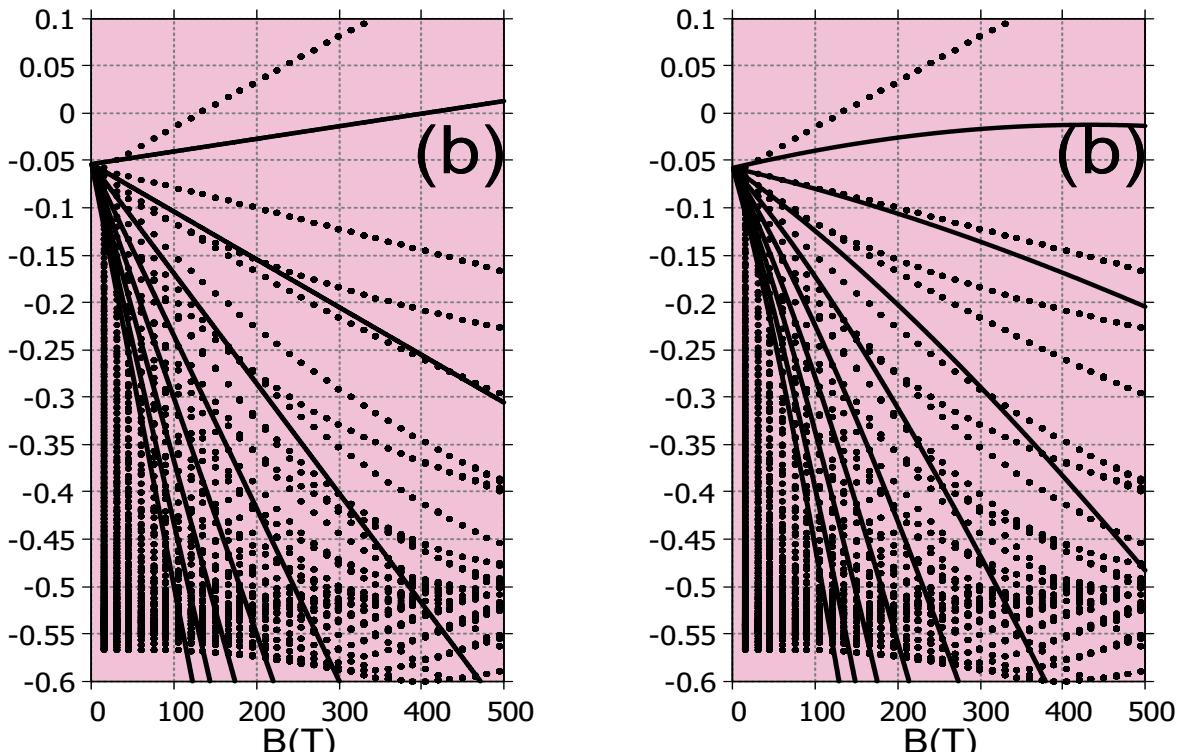


Figure B.1: A comparision bewteen two methods, figure on the right take data from maple, both depicts Landau levels.

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