

Hofstadter butterfly in transition metal dichalcogenide monolayers

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Overview

Group VI-B Transition Metal Dichalcogenides (TMD) are compound semiconductors of the type MX_2

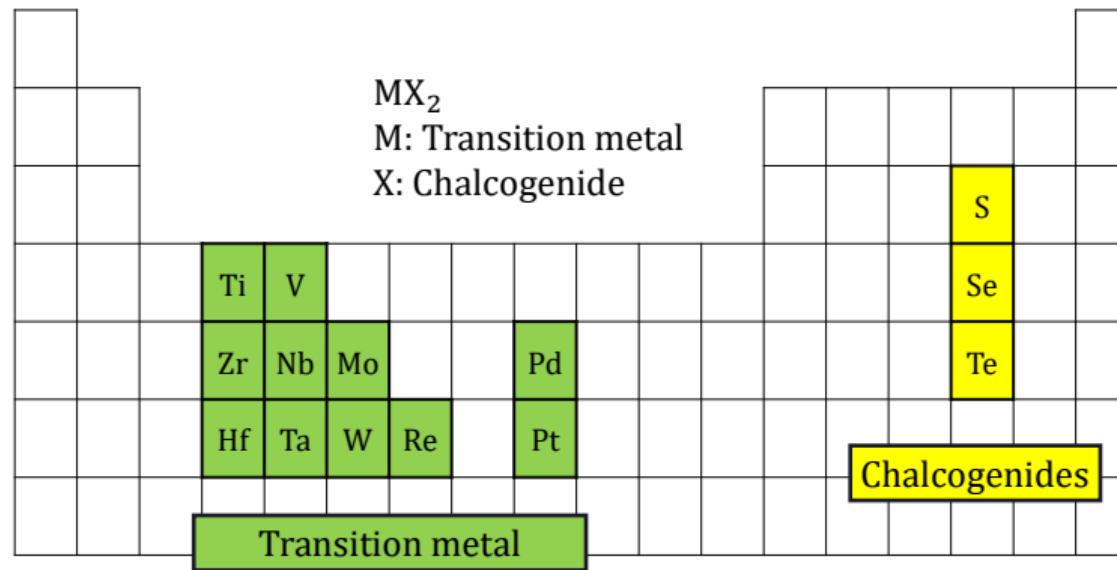


Figure: Transition metal dichalcogenides compound.

Transition Metal Dichalcogenides Monolayers

- One M layer sandwiched by two X layers as show in top view (a) and side view (b).
- Crystal structure has no central inversion symmetry.
- The symmetry of the lattice results in the hexagon Brillouin Zone (BZ).

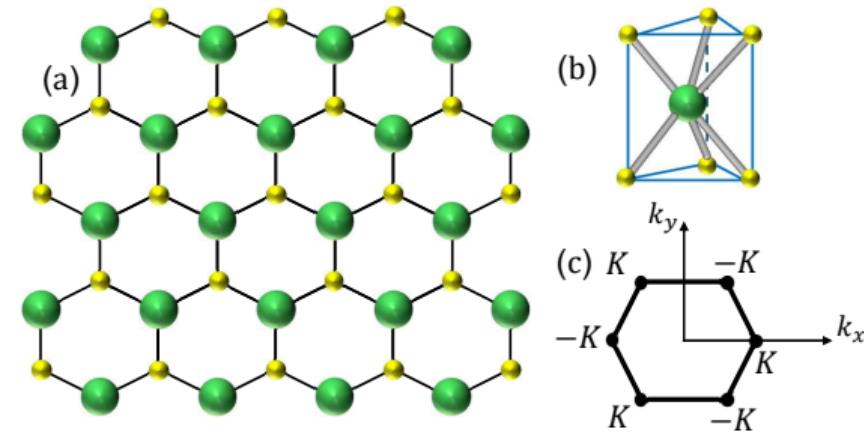


Figure: Structure and Brillouin Zone of Monolayer TMD^[1].

[1] Liu et al., "Three-band tight-binding model for monolayers of group-VIB transition metal dichalcogenides".

Transition Metal Dichalcogenides Monolayers

Properties

- Both mono-layer and few-layers remain stable at room temperature.
- TMD monolayer has the visible band gap in the band structure, which can be used in creating the transistor devices^[2].
- Strong spin-orbit coupling (SOC) in TMD monolayers lead to spin splitting of hundreds meV.

⇒ Promising material in electronic and optoelectronic applications.

[2] Radisavljevic et al., "Single-layer MoS₂ transistors"

Three-band tight-binding model without magnetic field

Time-independent Schrödinger equation for an electron in the crystal

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] |\psi_{\lambda, \mathbf{k}}(\mathbf{r})\rangle = \epsilon_{\lambda, \mathbf{k}} |\psi_{\lambda, \mathbf{k}}(\mathbf{r})\rangle. \quad (1)$$

Tight-binding (TB) wave function

$$|\psi_{\lambda, \mathbf{k}}(\mathbf{r})\rangle = \sum_j C_j^\lambda(\mathbf{k}) \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} |\phi_j(\mathbf{r} - \mathbf{R})\rangle. \quad (2)$$

The basis consists of three d -orbitals of the M atom:

$$|\phi_1\rangle = |d_{z^2}\rangle, |\phi_2\rangle = |d_{xy}\rangle, |\phi_3\rangle = |d_{x^2-y^2}\rangle. \quad (3)$$

The coefficients $C_j^\lambda(\mathbf{k})$ are the solutions of the eigenvalue equation

$$\sum_{jj'}^3 \left[H_{jj'}^{\text{TB}}(\mathbf{k}) - \epsilon_\lambda(\mathbf{k}) S_{jj'}(\mathbf{k}) \right] C_j^\lambda(\mathbf{k}) = 0. \quad (4)$$

Three-band tight-binding model without magnetic field

Overlap matrix elements

$$S_{jj'}(\mathbf{k}) = \sum_{\mathbf{R}} \langle \phi_j(\mathbf{r}) | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \approx \delta_{jj'}. \quad (5)$$

TB Hamiltonian matrix elements

$$H_{jj'}^{\text{TB}}(\mathbf{k}) = \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \langle \phi_j(\mathbf{r}) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle. \quad (6)$$

Three-band tight-binding model without magnetic field

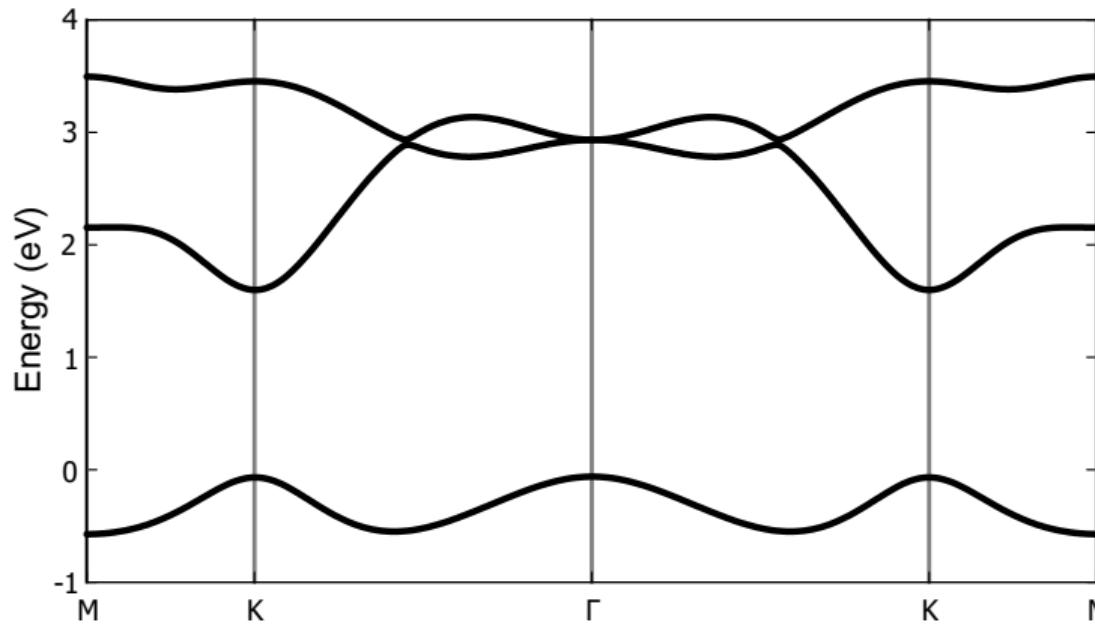


Figure: Band structure of MoS_2 monolayer^[3].

[3] Liu et al., "Three-band tight-binding model for monolayers of group-VIB transition metal dichalcogenides".

Three-band tight-binding model under a magnetic field

TB Hamiltonian matrix elements change to

$$H = \frac{(-i\hbar \nabla + e\mathbf{A}(\mathbf{r}))^2}{2m} + U_0(\mathbf{r}) + g^* \mu_B \mathbf{B} \cdot \mathbf{L}, \quad (7)$$

It is possible to add a phase factor to the basis functions

$$\psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \sum_{j=1}^3 C_j^\lambda(\mathbf{k}) \sum_{\mathbf{R}} e^{i\mathbf{k} \cdot \mathbf{R}} e^{i\theta_{\mathbf{R}}(\mathbf{r})} \phi_j(\mathbf{r} - \mathbf{R}). \quad (8)$$

By choosing $\theta_{\mathbf{R}} = -\frac{e}{\hbar} \int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'$ as Peierls substitution, the Hamiltonian matrix elements as the form

$$\begin{aligned} H_{jj'}^{\text{TB}}(\mathbf{k}) = & \sum_{\mathbf{R}} e^{i\mathbf{k} \cdot \mathbf{R}} e^{\frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \langle \phi_j(\mathbf{r}) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \\ & + g^* \mu_B \mathbf{B} \cdot \sum_{\mathbf{R}} e^{i\mathbf{k} \cdot \mathbf{R} + \frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} \langle \phi_j(\mathbf{r}) | \mathbf{L} | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle. \end{aligned} \quad (9)$$

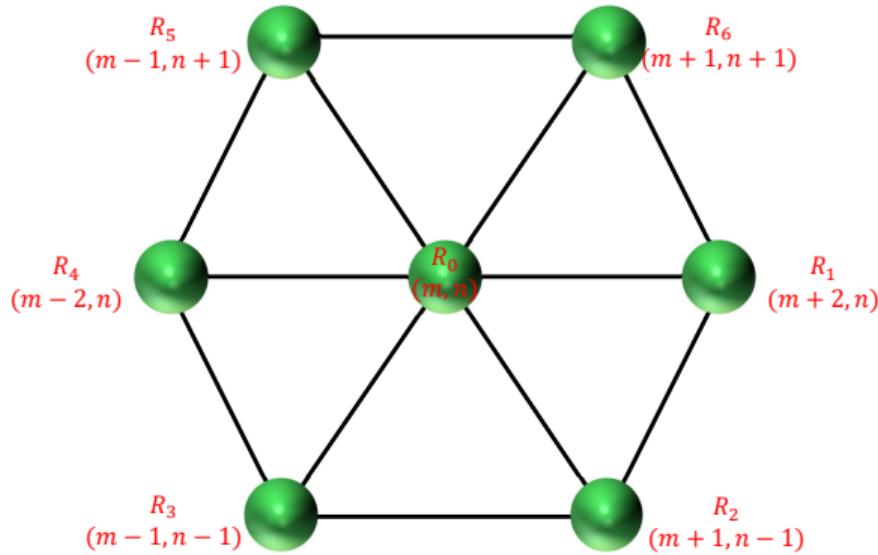


Figure: The TB model of TMDC with six neighbors atom M .

Three-band tight-binding model under a magnetic field

The Eq (9) is consist only 1-atom M in the unit cell, and the Hamiltonian does not invariant under the expansion of lattice vector along the x axis. In order to restore this invariance, we expanded the origin unit cell into a magnetic unit cell, which now contains 2-atoms M .

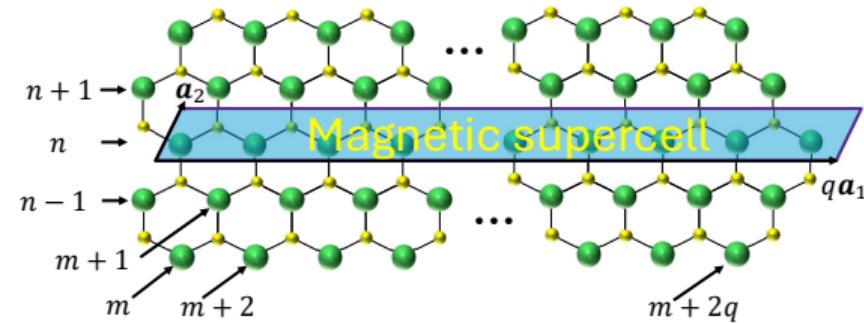


Figure: Magnetic unit cell for TMD monolayers.

Hamiltonian of magnetic unit cell

The new basis set of $3q$ atomic orbitals is defined as

$$\psi_{\lambda, \mathbf{k}}(\mathbf{r}) = \sum_{j,i} C_{ji}^{\lambda}(\mathbf{k}) \sum_{\mathbf{R}_{\alpha}}^{N_{UC}} e^{i\mathbf{k} \cdot (\mathbf{R}_{\alpha} + \mathbf{r}_i)} \phi_j(\mathbf{r} - \mathbf{R}_{\alpha} - \mathbf{r}_i), \quad (10)$$

in which \mathbf{r}_i is the position of an atom in a unit cell, while \mathbf{R}_{α} is the position of different unit cells. The Hamiltonian matrix elements in the new basis is written as

$$H_{ii'}^{jj'}(\mathbf{k}) = \sum_{\mathbf{R}_{\alpha}}^{N_{UC}} \sum_{\mathbf{R}_{\beta}}^{N_{UC}} e^{i\mathbf{k} \cdot (\mathbf{R}_{\beta} - \mathbf{R}_{\alpha} + \mathbf{r}_{i'} - \mathbf{r}_i)} \langle \phi_j(\mathbf{r} - \mathbf{R}_{\alpha} - \mathbf{r}_i) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0 \right] | \phi_j(\mathbf{r} - \mathbf{R}_{\beta} - \mathbf{r}_{i'}) \rangle. \quad (11)$$

Now we center our system at $\mathbf{r}' = \mathbf{r} - \mathbf{R}_{\alpha} - \mathbf{r}_i$ and define $\mathbf{R}_{\gamma} = \mathbf{R}_{\alpha} - \mathbf{R}_{\beta}$. This lead us to

$$H_{ii'}^{jj'}(\mathbf{k}) = \sum_{\alpha}^{N_{UC}} \sum_{\gamma}^{N_{UC}} e^{-i\mathbf{k} \cdot (\mathbf{R}_{\gamma} + \mathbf{r}_i - \mathbf{r}_{i'})} \langle \phi_j(\mathbf{r}) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0 \right] | \phi_j(\mathbf{r} + \mathbf{R}_{\gamma} + \mathbf{r}_i - \mathbf{r}_{i'}) \rangle. \quad (12)$$

Hamiltonian of magnetic unit cell

Only considering the nearest-neighbors, we define our hopping terms in the new basis

$$H_{jj'}^{ii'}(\mathbf{k}) = \sum_{\alpha}^{N_{\text{UC}}} \sum_{\gamma}^{N_{\text{UC}}} e^{-i\mathbf{k} \cdot \mathbf{R}_{\gamma}} \langle \phi_j(\mathbf{r}) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} + \mathbf{R}_{\gamma}) \rangle \delta_{i,i'}. \quad (13)$$

Note that $i = (m, n)$, taking the sum over \mathbf{R} and plugging the Peierls phase into Eq. (13), we get the Hamiltonian under a magnetic field

$$H_{jj'}^{ii'}(\mathbf{k}) = e^{\frac{ie}{\hbar} \int_{\mathbf{0}}^{\mathbf{R}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} e^{i\mathbf{k} \cdot (\mathbf{0} - \mathbf{R})} \langle \phi_j(\mathbf{r}) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \delta_{i,i'}. \quad (14)$$

Hamiltonian of magnetic unit cell

Given the NN coordinates, we rewrite the Hamiltonian (14) as

$$\begin{aligned}
 H_{jj'}^{ii'}(\mathbf{k}) &= e^{\frac{ie}{\hbar} \int_{m,n}^{m',n'} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}'} e^{-i\mathbf{k} \cdot \mathbf{R}} \langle \phi_j(\mathbf{r}) | \left[-\frac{\hbar^2 \nabla^2}{2m} + U_0(\mathbf{r}) \right] | \phi_{j'}(\mathbf{r} - \mathbf{R}) \rangle \delta_{m,m'}^{n,n'} \\
 &= E_{jj'}(\mathbf{0}) \delta_{m,m}^{n,n} + e^{-i\mathbf{k} \cdot \mathbf{R}_1} E_{jj'}(\mathbf{R}_1) \delta_{m,m+2}^{n,n} + e^{-i\mathbf{k} \cdot \mathbf{R}_4} E_{jj'}(\mathbf{R}_4) \delta_{m-2,m'}^{n,n'} \\
 &\quad + e^{-2i\pi(m+1/2)p/q} e^{-i\mathbf{k} \cdot \mathbf{R}_2} E_{jj'}(\mathbf{R}_2) \delta_{m,m+1}^{n,n-1} + e^{-2i\pi(m-1/2)p/q} e^{-i\mathbf{k} \cdot \mathbf{R}_3} E_{jj'}(\mathbf{R}_3) \delta_{m,m-1}^{n,n-1} \\
 &\quad + e^{2i\pi(m-1/2)p/q} e^{-i\mathbf{k} \cdot \mathbf{R}_5} E_{jj'}(\mathbf{R}_5) \delta_{m,m-1}^{n,n+1} + e^{2i\pi(m+1/2)p/q} e^{-i\mathbf{k} \cdot \mathbf{R}_6} E_{jj'}(\mathbf{R}_6) \delta_{m,m+1}^{n,n+1}.
 \end{aligned} \tag{15}$$

The Hamiltonian now invariant under the expansion of a lattice vector along the x axis and depend on the site index m .

Hamiltonian of magnetic unit cell

$$H_{jj'} = \begin{pmatrix} \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(1)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(1)} \\ B_{jj'}^{(2)} & \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(2)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) \\ \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(3)} & \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(3)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{E}_{jj'}(\mathbf{R}_1) & 0 & \cdots & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(q-2)} & \mathcal{E}_{jj'}(\mathbf{0}) & A_{jj'}^{(q-2)} \\ A_{jj'}^{(q-1)} & \mathcal{E}_{jj'}(\mathbf{R}_1) & \cdots & 0 & 0 & \mathcal{E}_{jj'}(\mathbf{R}_4) & B_{jj'}^{(q-1)} & \mathcal{E}_{jj'}(\mathbf{0}) \end{pmatrix}, \quad (16)$$

where $A_{jj'}^{(m)} = \mathcal{E}_{jj'}(\mathbf{R}_2)e^{-2i\pi(m+1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_2} + \mathcal{E}_{jj'}(\mathbf{R}_6)e^{2i\pi(m+1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_6}$, and
 $B_{jj'}^{(m)} = \mathcal{E}_{jj'}(\mathbf{R}_3)e^{-2i\pi(m-1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_3} + \mathcal{E}_{jj'}(\mathbf{R}_5)e^{2i\pi(m-1/2)p/q}e^{-i\mathbf{k}\cdot\mathbf{R}_5}$

Hamiltonian of magnetic unit cell

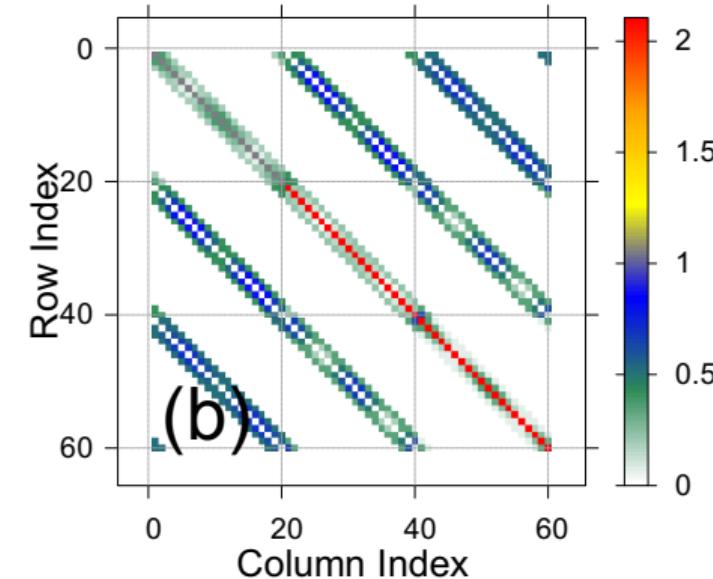
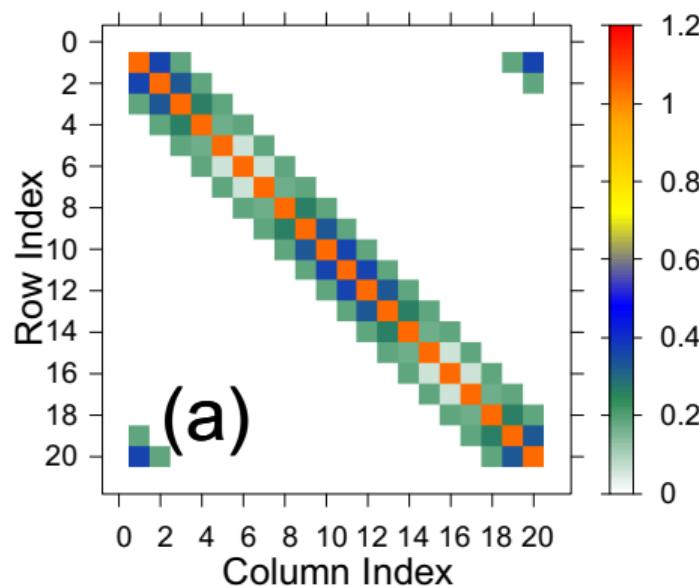


Figure: An visualization of sub-matrix h_0 for single-band(a) and matrix H for three band(b) using standard plotter with $q = 20$.

Hofstadter butterfly^[4] - Energy band structure

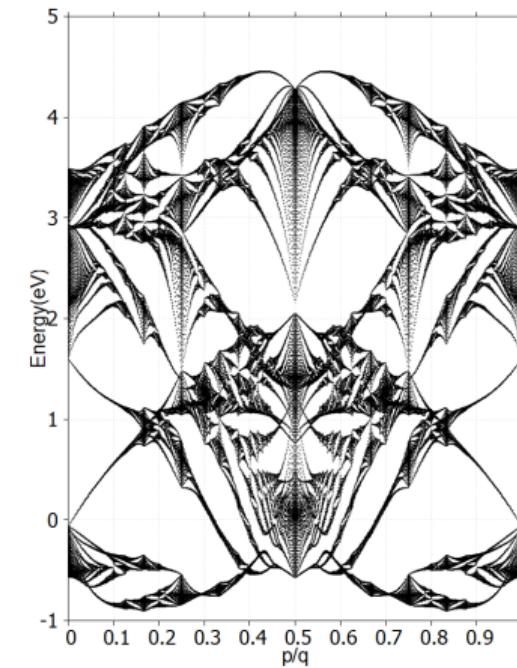
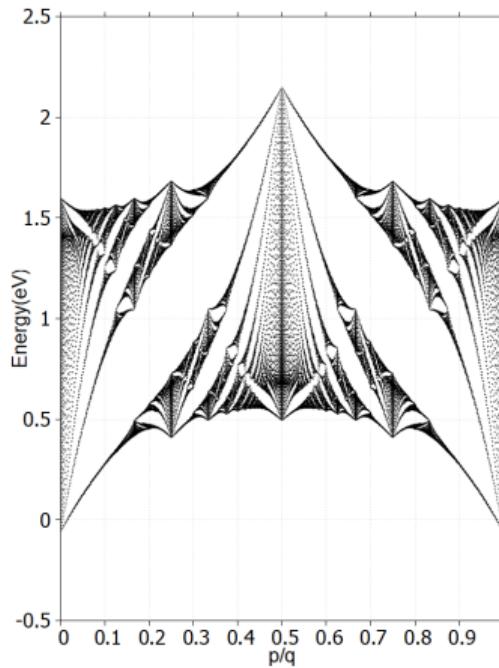


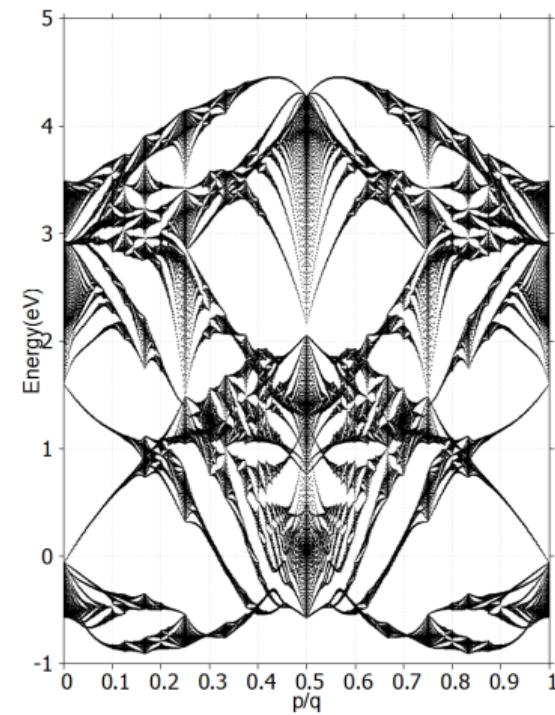
Figure: Hofstadter butterfly for single-band $|dz\rangle \equiv |\phi_1^1(x, y)\rangle$ (left) and three-band (right).

[4] Hofstadter, "Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields".

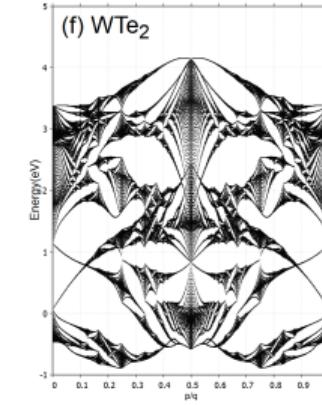
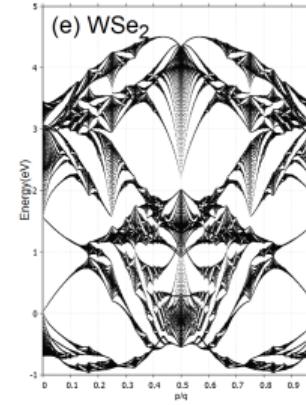
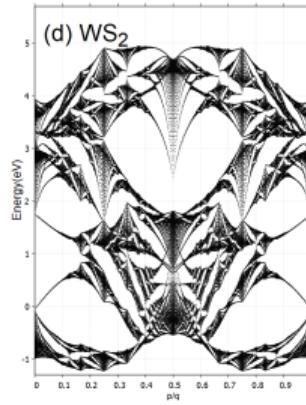
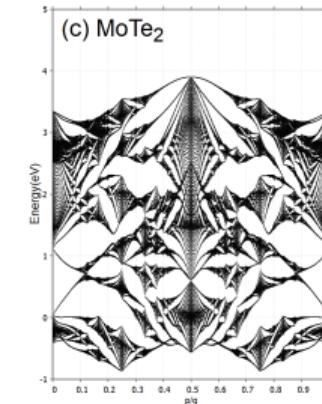
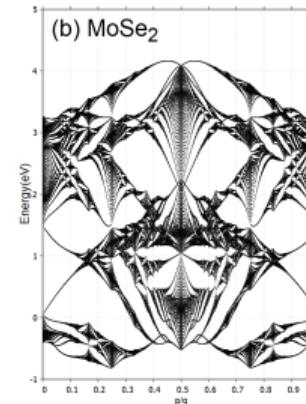
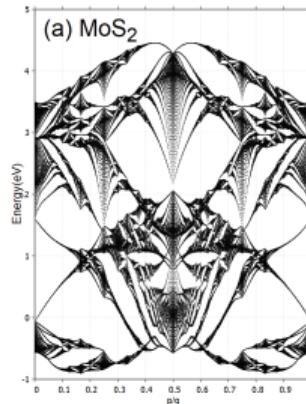
Hofstadter butterfly

Properties

- The spectrum depends only on the flux ratio p/q .
- The spectrum is also invariant under reversal of the magnetic field $\frac{p}{q} \rightarrow -\frac{p}{q}$.
- It inverts when the flux is higher than $p/q + 1/2$ (i.e., $1 - p/q$).
- At weak magnetic field, Landau levels can clearly be seen from the Hofstadter spectrum.



Showcase Hofstadter butterflies



Hofstadter butterfly with SOC

Three-band TB Hamiltonian under magnetic field with SOC has the form

$$H_{\text{SOC}}(\mathbf{k}) = \begin{pmatrix} H_{3q \times 3q}(\mathbf{k}) + \frac{\lambda}{2} \mathbf{I}_q \otimes L_z & 0 \\ 0 & H_{3q \times 3q}(\mathbf{k}) - \frac{\lambda}{2} \mathbf{I}_q \otimes L_z \end{pmatrix}, \quad L_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} \quad (17)$$

Hofstadter butterfly with SOC

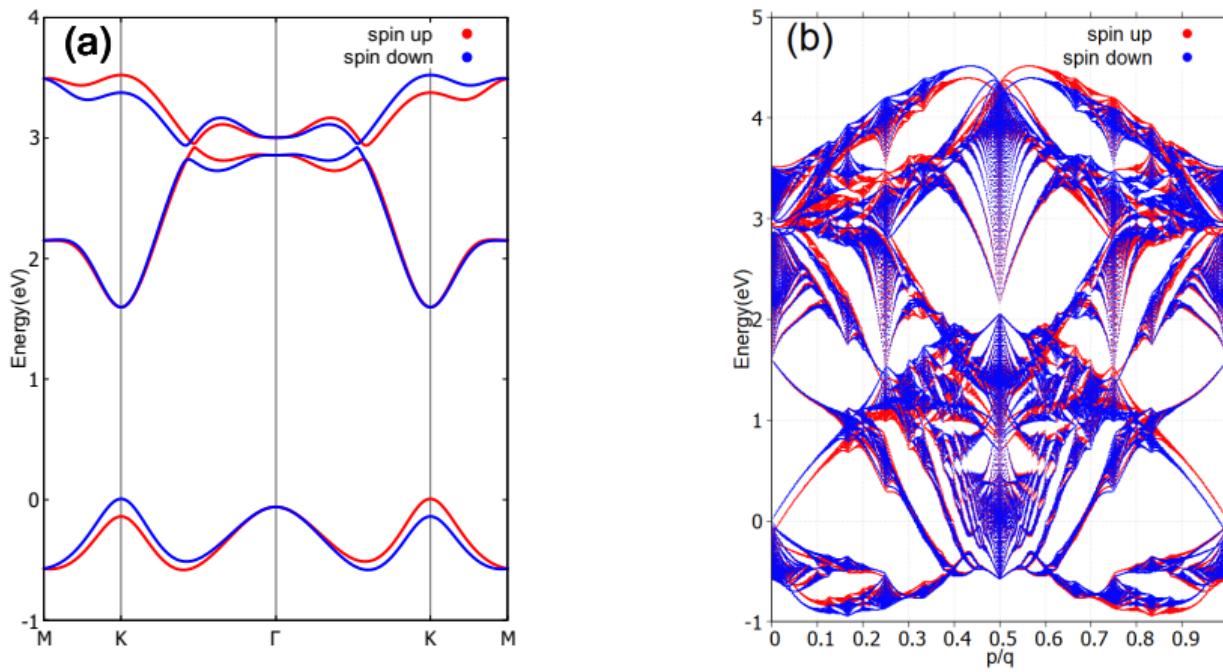


Figure: Band structure of monolayer MoS_2 along Γ -K direction with SOC.

Landau levels

The Hamiltonian for free-electron

$$H = \frac{\mathbf{p} + e\mathbf{A}(\mathbf{r})^2}{2m}, \quad (18)$$

and the energy eigenfunctions are known as Landau levels

$$E_n = (n + 1/2) \hbar \omega_c, \quad (19)$$

where ω_c is cyclotron frequency and n is Landau level index.

Envelope Function Approximation

The dispersion relation of an electron within TBM

$$h_0 \approx 6t_0 - \frac{3}{2}t_0 a^2 (k_x^2 + k_y^2) + \epsilon_1. \quad (20)$$

Using Envelop Function Approximation (EFA) substitution $\hbar\mathbf{k} \rightarrow \mathbf{\Pi} + e\mathbf{A}$ with Landau gauge $\mathbf{A} = (0, Bx, 0)$

$$h_0(\mathbf{\Pi}) \approx 6t_0 - \frac{3}{2}t_0 \frac{a^2}{\hbar^2} \left[\Pi_x^2 + (\Pi_y + eBx)^2 \right] + \epsilon_1. \quad (21)$$

Eq. (21) can be rewrite in the form as

$$E(\mathbf{\Pi}) = 6t_0 - \left[\frac{1}{2m^*} \Pi_x^2 + \frac{1}{2} m^* \omega_c^2 (x - x_0)^2 \right] + \epsilon_1, \quad (22)$$

where $m^* = \frac{\hbar^2}{3t_0 a^2}$ is the effective mass and $x_0 = \frac{\hbar k_y}{eB}$. Subsequently, the cyclotron frequency is

$$\omega_c = \frac{eB}{m^*} = \frac{8\pi\sqrt{3}t_0}{\hbar} \frac{p}{q}, \quad (23)$$

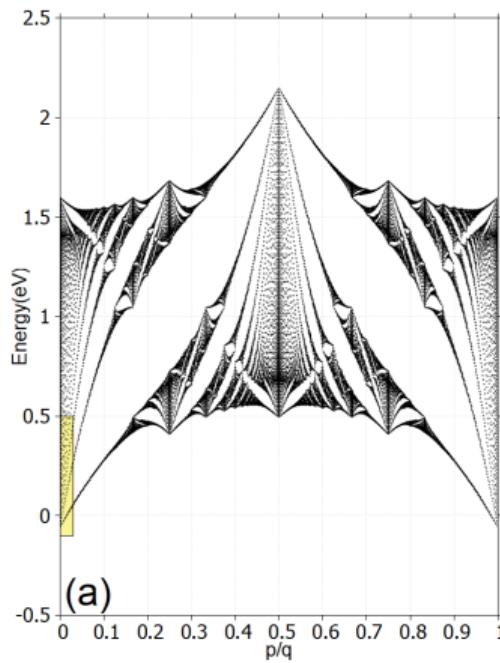
Envelope Function Approximation

and therefore the Landau levels near the bottom of the band structure can be written as

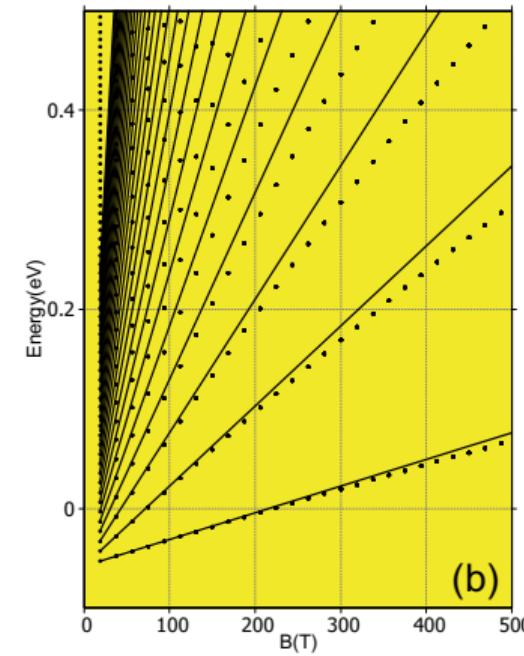
$$\begin{aligned} E_n &= 6t_0 - \hbar\omega_c(n + 1/2) + \epsilon_1 \\ &= t_0 \left(6 - 8\pi\sqrt{3} \frac{p}{q} (n + 1/2) \right) + \epsilon_1. \end{aligned} \tag{24}$$

Plot Eq. (25) while varying p from $1 \rightarrow q$ which Landau level index n we get

Landau levels



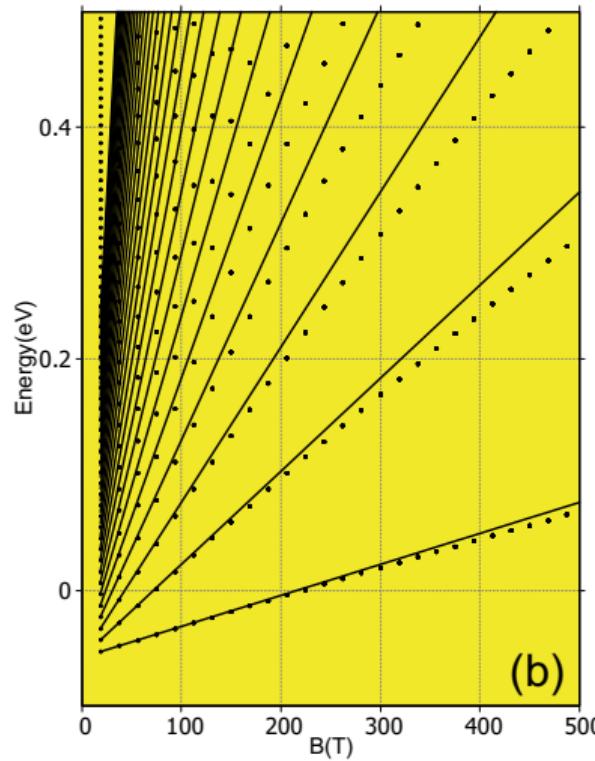
(a)



(b)

Figure: Landau levels appear in weak magnetic field.

Landau levels



Properties

- Increase linearly with respect to B .
- At weak magnetic field, the Landau levels are well-described.
- Increasing $B \rightarrow$ fewer filled levels.
- At extremely strong field \rightarrow only one level is filled.

The Classical Hall effect

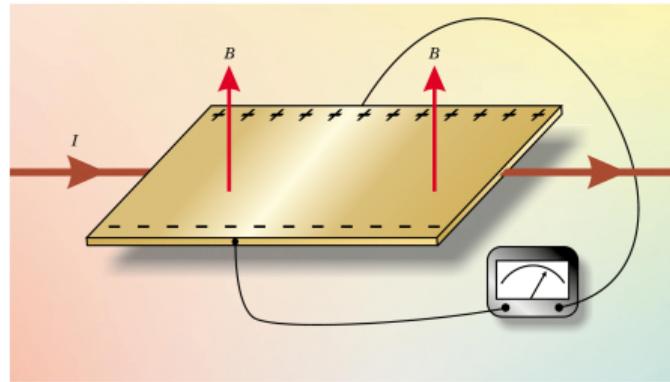
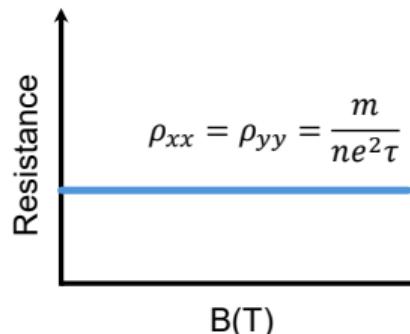


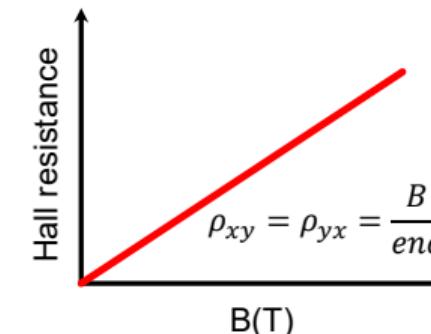
Figure: Classical Hall effect



The Hall effect can be described through Ohm's law

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad (25)$$

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}. \quad (26)$$



The Quantum Hall effect

It was first discovered in early 1980^[5].

The resistance in a MOSFET under a strong magnetic field depicts interesting phenomena.

Properties

- On Hall plateaux

$$\rho_{xx} = \rho_{yy} = 0, \quad \rho_{xy} = \rho_{yx} = \text{const.}$$

$$\sigma_{xx} = \sigma_{yy} = 0, \quad \sigma_{xy} = \sigma_{yx} = 1/\text{const.}$$

- Between Hall plateaux

$$\rho_{xx}, \rho_{yy} \neq 0 \Rightarrow \sigma_{xx}, \sigma_{yy} \neq 0,$$

$$\rho_{xy} = \frac{R_K}{\nu}, \quad \nu = 1, 2, 3, \dots$$

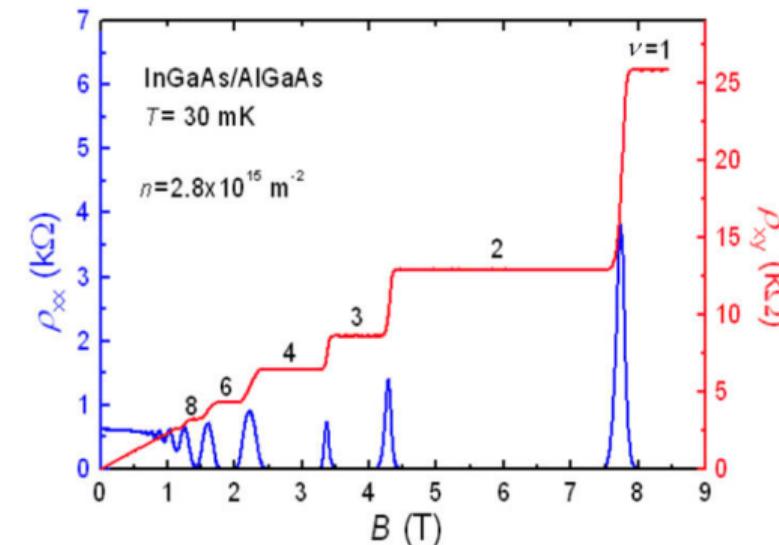


Figure: The Quantum Hall effect by Klitzing, Dorda, and Pepper (1980).

The Quantum Hall effect

The contribution to the Hall conductance is given by

$$\sigma_{xy} = \frac{e^2}{h} \sum_n^{\text{occ.}} \frac{1}{2\pi} \iint_{\text{BZ}} dk_x dk_y \Omega_n^z(\mathbf{k}), \quad (27)$$

since the integral over the Brillouin zone is an integer, we arrived at the TKNN formula^[6]

$$\sigma_{xy} = \frac{e^2}{h} \nu, \quad \nu = 1, 2, \dots \quad (28)$$

and combine with Streda formula^[7]

$$\sigma_{xy}(B, E_F) = e \frac{\partial N(E, B)}{\partial B} \bigg|_{E=E_F}, \quad (29)$$

we have the Diophantine equation

$$r = q \times s_r + p \times \nu_r. \quad (30)$$

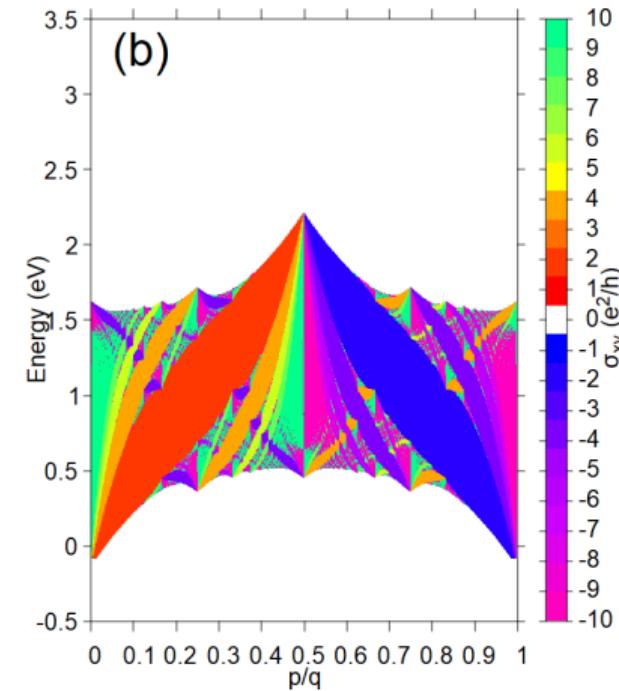
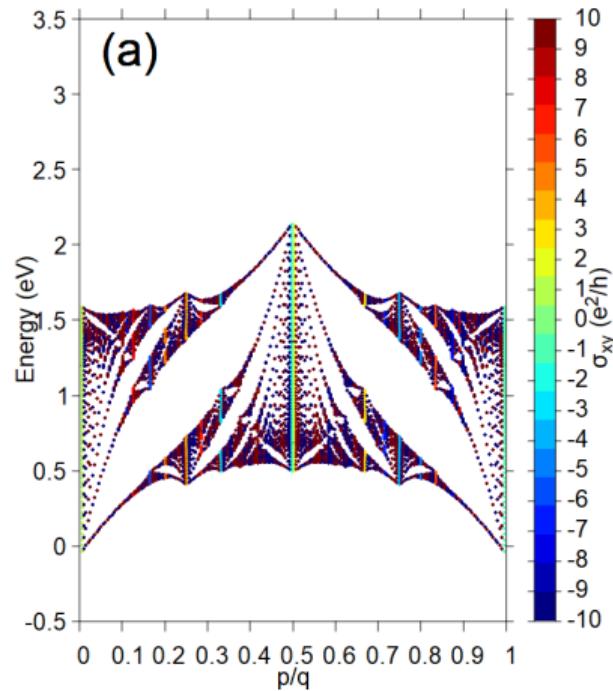
[6] Thouless et al., "Quantized Hall Conductance in a Two-Dimensional Periodic Potential".

[7] Streda, "Theory of quantised Hall conductivity in two dimensions".

The Diophantine equation

- p, q are the flux number Φ and flux quanta Φ_0 .
$$r = q \times s_r + p \times \nu_r.$$
- The Chern number ν determines the topological and the contribution to the Hall conductance.
- Identifying s as an index of energy gap in the butterfly.

Color the butterfly



$$r = q \times s_r + p \times \nu_r$$

Summary:

- We confirm the Hofstadter butterfly in this model corrected compared to previous study.
- From three-band TB + magnetic field → QHE.

Further research:

- Fractional Quantum Hall effect
- High Harmonic Generation
- High-order Side-band Generation
- Photovoltaic effect

Thank you for your listening.

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