Vietnam National University, Ho Chi Minh City
University of Technology
Faculty of Computer Science and Engineering

MATHEMATICAL MODELING (CO2011)

Assignment (Semester: 221, Duration: 06 weeks)

"Dynamics of Love"

(Version 0.3)

Instructor(s): Nguyễn Tiến Thịnh
Nguyễn An Khương
Nguyễn Văn Minh Mẫn
Mai Xuân Toàn
Trần Hồng Tài

Student(s): Nguyễn Văn A – 221021 *(Group CC0x - Team 0y)*
Trần Văn B – 884714 *(Group CC0x - Team 0y)*
Lê Thị C – 368113 *(Group CC0x - Team 0y, **Leader**)*
Phạm Ngọc D – 975013 *(Group CC0x - Team 0y)*

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List of Symbols

i Unit imaginary number

$\mathbf{Re}\ \lambda$ The real part of the eigenvalue λ

$\mathbf{Im}\ \lambda$ The image part of the eigenvalue λ

t Time variable

t_0 The initial value of time t

h Time step

$\mathbf{u}$ A vector function dependent on t

$\dot{\mathbf{u}}$ The derivative of $\mathbf{u}$ with respect to t

$\mathbf{F}$ A vector function dependent on t, $\mathbf{u}$, and $\dot{\mathbf{u}}$

T The temperature of the surface of a certain object or the transpose operator

$\mathbb{N}$ Set of natural numbers

$\mathbb{R}$ Set of real numbers

$\mathbb{R}^+$ Set of positive real numbers

List of Acronyms

ODE (First-Order) Ordinary Differential Equation

IVP Initial-Value Problem

DL Deep Learning

ML Machine Learning

DS Dynamical System

Fig. Figure

Tab. Table

Sys. System of Equations

Eq. Equation

VF Vector Field

e.g. For Example

i.e. That Is

Ex. Exercise

List of Figures

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1 Introduction

This interesting assignment will show us how a real-life problem, such as a love affair between two people, can be solved with a straightforward system of (First-Order) Ordinary Differential Equations (ODEs). In general, a system of ODEs has the form

$$\mathbf{F}(t, \dot{\mathbf{u}}, \mathbf{u}) = 0, \tag{1}$$

where $\mathbf{F}$ is a vector function dependent on t, $\mathbf{u}$, and $\dot{\mathbf{u}}$, t is the time variable, $\mathbf{u}$ is a vector function dependent on t, and $\dot{\mathbf{u}}$ is the derivative of $\mathbf{u}$ with respect to t. Such a system describes the evolution of a quantity $\mathbf{u}$ in time. For example, if the heat transfers from the body of a certain object to the surrounding environment at the constant rate of 2 per second and the ambient temperature are 25 Kelvin degrees, the ODE

$$\dot{T} = -2(T - 25), \tag{2}$$

The Newton Law of Cooling describes the evolution of the temperature (denoted by T) of the object's surface in time. Moreover, by means of Eq. (2) and assume also that the initial value of T is 40 Kelvin degrees, we can easily see that $T(t) = 25 + 15e^{-2(t-t_0)}$ for $t \geq t_0 \geq 0$ (the initial time). This explicit formula of T allows us to predict the temperature of the object surface at any time $t \geq t_0 \geq 0$. The interesting thing is that $T(t) \to 25$ as $t \to +\infty$, i.e., the temperature of the object surface tends to convert to 25 Kelvin degrees for a considerable time. Unfortunately, not many ODEs can be solved explicitly. However, we will study a system of ODEs that describes the love affair between two people. Moreover, we always consider $t_0 = 0$ for simplicity.

2 Love affairs and differential equations

This section is to study a system of ODEs that can describe the love affair between two people. For the sake of simplicity, let's consider the love of Romeo for Juliet and also the love of Juliet for Romeo. Romeo and Juliet are two famous characters in a tragedy written by William Shakespeare by the same name. Assume that the love of Romeo for Juliet can be measured as a differentiable function $R : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ and similarly differentiable $J : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ the love of Juliet for Romeo. The love affair between them can be described by the Initial-Value Problem (IVP) or also called the Dynamical System (DS)

$$\begin{cases} \dot{R} = aR + bJ, \\ \dot{J} = cR + dJ, \\ R(0) = R_0, \; J(0) = J_0. \end{cases} \tag{3}$$

Here R_0, $J_0 \in \mathbb{R}$ are the love of Romeo for Juliet and Juliet for Romeo at the initial time. The constant coefficients a, b, c and $d \in \mathbb{R}$ describe the interaction of the love of one to the other. Due to [Str88], we can determine the romantic style of the love of Romeo for Juliet and the romantic style of the love of Juliet for Romeo utilizing those coefficients. The author of [Str88] and his students came up with the following romantic styles of Romeo.

a	b	**Style**
$+$	$+$	Eager Beaver
$+$	$-$	Narcissistic Nerd
$-$	$+$	Cautious Lover
$-$	$-$	Hermit

Tab. 1: *Romantic styles of Romeo*

By definition, an "eager beaver" is encouraged by his/her feelings and by the love of the other for him/her, i.e., $a > 0$ and $b > 0$ while a "cautious lover" is retreated from his/her feelings but is spurred on the love of the other for him/her, I.E., $a < 0$ and $b > 0$. Indeed, it can be explained by looking at SYS. (3). The rate change of the love of Romeo for Juliet is nothing but a weighted sum of the love of Romeo for Juliet and similarly in the case of Juliet. For example, if $a = -b = -1$, then there is a change in the love of Romeo for Juliet, which is equal to $J - R$, the difference between the love of her for him and the love of him for her for all the time. If at a fixed time t, the love of Juliet for Romeo is more significant than his love for her, it seems to us that the love of Romeo for Juliet increases as well because $\dot{R}(t) > 0$. It means that Romeo tends to love Juliet if she loves him more than he loves her. On the contrary, if he loves her more than she loves him, that love will decrease at that time because of $\dot{R}(t) < 0$. In this case, Romeo is a very cautious person and, therefore, a "cautious lover" as described in [Str88].

Let's consider more concrete examples of the love between different types of lovers to understand more about the dynamics of love. The first example is the love between a cautious lover and a narcissistic nerd, which is given as the IVP

$$\begin{cases} \dot{R} = -3R + 3J, \\ \dot{J} = -2R + J, \\ R_0 = -4,\ J_0 = 2. \end{cases} \tag{4}$$

In this example, Romeo doesn't love Juliet; on the contrary, Juliet loves Romeo at the time we consider, I.E., $R_0 < 0$ and $J_0 > 0$. A narcissistic Juliet is a Juliet that is encouraged by her feelings and falls back in love with Romeo for her. However, if he hates her more, she will love him more. There are at least two ways to investigate the love between Romeo and Juliet. Because of the linear form of SYS. (4), the explicit solution can be found. To obtain the solution, we need to transform SYS. (4) into the vector form

$$\begin{cases} \dot{\mathbf{u}} = A\mathbf{u}, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{5}$$

where $A = \begin{pmatrix} -3 & 3 \\ -2 & 1 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} R & J \end{pmatrix}^T$, and $\mathbf{u}_0 = \begin{pmatrix} -4 & 2 \end{pmatrix}^T$. In this case, T denotes the transpose operator. The matrix A has two complex "eigenvalues" $\lambda_1 = -1 + \sqrt{2}i$ and $\lambda_2 = -1 - \sqrt{2}i$. Hence the solution to the SYS. (4) is

$$\begin{cases} R(t) = e^{-t}\left[7\sqrt{2}\sin\left(\sqrt{2}t\right) - 4\cos\left(\sqrt{2}t\right)\right], \\ J(t) = e^{-t}\left[2\cos\left(\sqrt{2}t\right) + 6\sqrt{2}\sin\left(\sqrt{2}t\right)\right]. \end{cases} \tag{6}$$

We don't want to explain why we obtain the solution SYS. (6) because it will be an exercise for the readers to study general explicit solutions to the IVP SYS. (5) for every initial value $\mathbf{u}_0$ and coefficients a, b, c, and d. In this chapter we focus only on the behavior of the solutions in some specific cases for large time to see how beautiful Mathematics is in the explanation of everything even the love between two people of different romantic styles or of the same romantic style. By looking at the solution SYS. (6), because $\sin$ and $\cos$ are bounded functions, both $R(t)$ and $J(t)$ quickly converge to zeros for t large enough as showed in FIG. 1. The convergence rate is explicit e^{-t}. Moreover, after a few simple calculation steps, from SYS. (6), we can see that both Romeo and Juliet have the same amount of love for the other, I.E., $R(t) = J(t)$ for $t = \dfrac{(\pi - 2\alpha)\sqrt{2}}{4} + k\pi$ for $k \in \mathbb{N}$, where $\alpha = \arctan\left(\dfrac{\sqrt{2}}{6}\right)$. Between those points, both Romeo and Juliet increase or decrease their love for the other with a small delay in time as we can see in FIG. 4. This is true in reality, for example, we observe the time interval $\left[0, \dfrac{(\pi - 2\alpha)\sqrt{2}}{4}\right]$, because Juliet in this case is a narcissistic nerd, the more Romeo hates her because $R_0 < 0$, the more she loves him. That is why starting from $t = 0$, Juliet increases her love for Romeo. On the other hand, Romeo is cautious, so because the love of Juliet for him increases, his love for Juliet therefore increases. However, when the love of Romeo for Juliet increases, the narcissistic Juliet at a certain point, she is going to be bored with his love for her and her love for him starts decreasing after then until both of their love for the other are equal to each other. Unfortunately, due to the damping term e^{-t}, their love for the other also decreases as time goes.

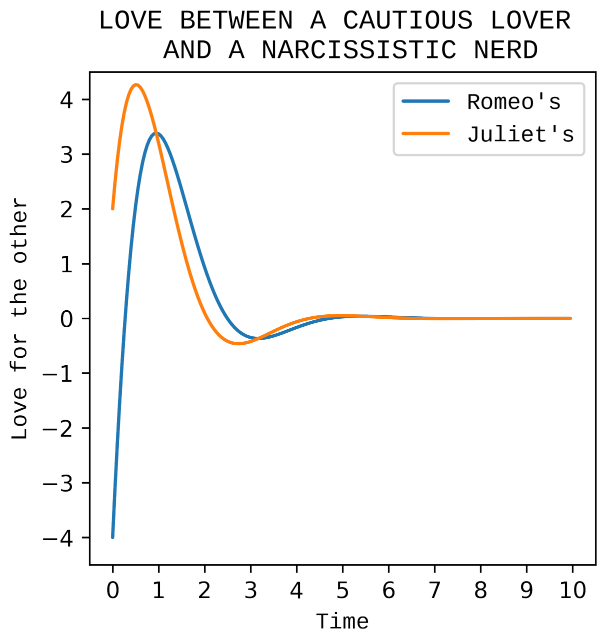

Fig. 1: *The love between a cautious lover and a narcissistic nerd*

Furthermore, we are also interested in the large-time behavior of solutions to IVP Sys. (5) without explicit formulae if they exist. That leads to the "qualitative study" of DSs. It can be done by observing the Vector Field (VF) caused by Sys. (5). The VF of Sys. (4) is sketched in Fig. 2. We can plot the VF by making a two-dimensional grid of R and J. Then at each point of the grid, we draw the vector $\left(\dot{R}, \dot{J}\right)$. The VFs helps us understand the orbit of an object appearing in them as well as the large-time behavior of the orbit. On the other hand, there are "steady points" in the fields, which are defined by $\mathbf{u}$ so that $\dot{\mathbf{u}} = \mathbf{0}$, which is equivalent to $A\mathbf{u} = \mathbf{0}$. They are exactly the "eigenvectors" associated with the eigenvalue 0 of the matrix A and the vector $\mathbf{0}$. The matrix A in Sys. (5) doesn't have any eigenvector associated with the eigenvalue 0. In this case, the Sys. (4) has only one steady state, which is $(0, 0)$. The steady states are interesting due to the fact that they do not change. A steady state can be either an "attractor" or a "repeller". If a steady state is a "local attractor", it attracts objects near it while a "global attractor" attracts every object appearing in the VF. Those things somehow describe the possibility of the existence of a solution to the IVP Sys. (5) for small initial value $\mathbf{u}_0$ or arbitrary initial value $\mathbf{u}_0$ and its large-time behavior. On the contrary, a "repeller" repels every object moving forward to it. In the case of the IVP Sys. (4), because $(0, 0)$ is a "global attractor", every trajectory in the VF is a "spiral" winding around and getting closer to the point $(0, 0)$ as time goes. The spiral pattern is indeed caused by "harmonic oscillations" of natural frequency $\sqrt{2}$ with "amplitudes" proportional to e^{-t} around $(0, 0)$ as we can see in the explicit formula Sys. (6) of the solution. The two terms are in fact corresponding to the "real parts" and the "imaginary parts" of the eigenvalues of the matrix A. Romeo and Juliet in this example no matter how wonderful or terrible their current love is, we just know that they neither love nor hate each other in the future.

To see a more general picture about the love dynamics, let's consider the love between an eager beaver and a hermit as follows.

$$\begin{cases} \dot{R} = 2R + 4J, \\ \dot{J} = -2R - 2J, \\ R_0 = \dfrac{5}{4},\ J_0 = \dfrac{5}{4}. \end{cases} \tag{7}$$

Because Juliet in this case is a hermit, she is always contrary to her feelings and the love of Romeo

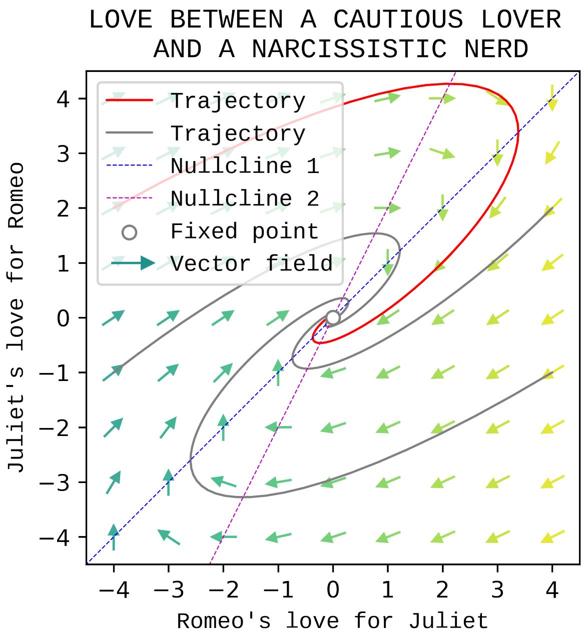

Fig. 2: *The phase portrait of the love between a cautious lover and a narcissistic nerd*

for her. The matrix A is $\begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$ and it has two complex eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. We thus obtain the exact solution

$$\begin{cases} R(t) = \dfrac{5}{4}\left[\cos(2t) + 3\sin(2t)\right], \\ J(t) = \dfrac{5}{4}\left[\cos(2t) - 2\sin(2t)\right] \end{cases} \tag{8}$$

for $t \geq 0$ to the IVP Sys. (7). Fig. 3 and Fig. 4 are respectively the plots of the exact solution in time and of the "phase portrait" of it. From the plots, we can see that the love is just harmonic oscillations around the origin. No damping effect occurs in this case. This is due to the fact that the real parts of the eigenvalues of the matrix A are actually zero. The love of those people is not steady, it always changes even sometimes they hate each other but after they increase their love. We thus call the origin a "center node". The loves are the same for all $t = \dfrac{k\pi}{2}$ for $k \in \mathbb{N}$.

We are going to consider a "strange" love between narcissistic nerds and the ending of the love is indeed dependent on how they start it. Let's begin with the IVP

$$\begin{cases} \dot{R} = R - 2J, \\ \dot{J} = -2R + J, \\ R_0,\ J_0. \end{cases} \tag{9}$$

If at the beginning the love of Romeo for Juliet is the same as her love for him, e.g., $R_0 = J_0 = 4$. The exact solution to the IVP Sys. (9) becomes

$$\begin{cases} R(t) = 4e^{-t}, \\ J(t) = 4e^{-t}. \end{cases} \tag{10}$$

It implies that $R(t) \to 0$ and $J(t) \to 0$ as $t \to +\infty$ (see Fig. 5). In this case, the origin is an attractor. In contradiction to the early example, the origin is not a global attractor. It is either an attractor or a repeller. Indeed, we consider the initial condition $R_0 = -J_0 = -\dfrac{1}{2}$ close to the origin to see what

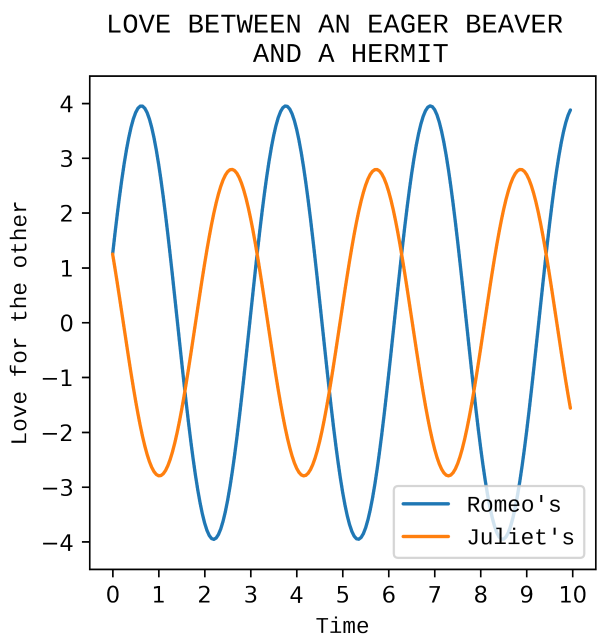

Fig. 3: *The love between an eager beaver and a hermit*

happens. The solution is

$$\begin{cases} R(t) = -\dfrac{e^{3t}}{2}, \\ J(t) = \dfrac{e^{3t}}{2}. \end{cases} \tag{11}$$

We can see that $R(t) \to -\infty$ and $J(t) \to +\infty$ as $t \to +\infty$ (see). This would be the same for every initial condition satisfying $R_0 = -J_0$; in this case, the origin behaves as a repeller. Due to the attractor-repeller property, trajectories in the VF of the SYS. (9) are branches of "hyperbolas" except for the ones lying on the two "asymptotes" $R = J$ and $R = -J$. We can verify that by looking at FIG. 8 and by observing the solution corresponding to the initial condition $R_0 = \dfrac{7}{3}$ and $J_0 = 3$. Moreover, the IVP SYS. (9) then has the solution

$$\begin{cases} R(t) = -\dfrac{e^{3t}}{3} + \dfrac{8e^{-t}}{3}, \\ J(t) = \dfrac{e^{3t}}{3} + \dfrac{8e^{-t}}{3}. \end{cases} \tag{12}$$

As $t \to +\infty$, $R(t) \to \infty$ and $J(t) \to -\infty$. Their love of Romeo for Juliet just slightly decreases at the beginning. Still, it increases quickly after that while the love of Juliet for him never increases but decreases "exponentially" in time (poor Romeo!). In this case, we call the origin a "saddle-node".

Let's see what conclusions we can make from the previous examples. Let λ_1 and λ_2 be the two eigenvalues of A. We have the following table.

Re λ_1	Re λ_2	$\lvert\text{Im } \lambda_1\rvert$	$\lvert\text{Im } \lambda_2\rvert$	**Type**
$-$	$-$	$+$	$+$	Spiral-In
0	0	$+$	$+$	Center
$+$	$-$	0	0	Saddle

Tab. 2: *(Incomplete) Phase-portrait classification*

For a good understanding of the subject, see [Lue79, HS74, Arn92].

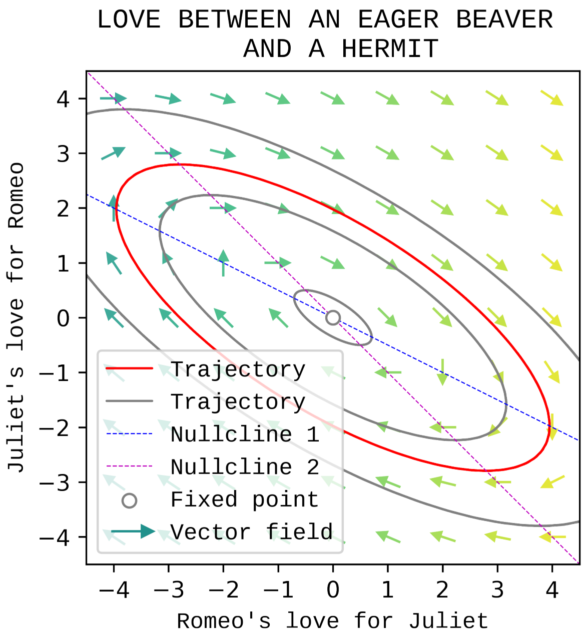

Fig. 4: *The phase portrait of the love between an eager beaver and a hermit*

3 Exercises

Students prepare a report consisting of the following exercises.

Exercise 1 (2.5 points)**.** Write on the report a very detailed introduction to the IVPs Sys. (3) and the formulae of its exact solutions for general a, b, c, and d and initial condition R_0 and J_0. Then complete Tab. 2 for all possible cases of eigenvalues of general 2×2 matrix A.

Exercise 2 (2.5 points)**.** For each combination of romantic styles in Tab. 1, give two concrete examples of IVPs Sys. (3). Apply the formulae in Ex. 1 to find the exact solutions. Plot all the solutions and the phase portraits.

Exercise 3 (2.5 points)**.** Let's assume that the love between Romeo and Juliet is perturbed by outer conditions, e.g. their families and social prejudices. In this case, the love is modeled by the IVP

$$\begin{cases} \dot{R} = aR + bJ + f(t), \\ \dot{J} = cR + dJ + g(t), \\ R(0) = R_0,\ J(0) = J_0. \end{cases} \tag{13}$$

Here f and g are two real functions dependent on t, e.g., $f(t) = t - 1$ and $g(t) = t^2$. Could we find the exact solution to the general IVP Sys. (13)? If we could, give the formula of the solution and five specific examples with their exact solutions. Otherwise, find general conditions on f and g so that the IVP Sys. (13) has a solution and also give five specific examples of such IVPs without finding the exact solutions.

A more general and also complicated love between Romeo and Juliet is the IVP

$$\begin{cases} \dot{R} = f(t, R, J), \\ \dot{J} = g(t, R, J), \\ R(0) = R_0,\ J(0) = J_0, \end{cases} \tag{14}$$

where f and g are two real functions dependent on t, R, and J. Similarly, find conditions on f and g so that the IVP Sys. (14) has a solution. For example, a solution exists for IVP Sys. (14) where $f(R, J) = R(1 - J)$ and $g(R, J) = J(R - 1)$ (this is also known as the Lotka–Volterra equations in

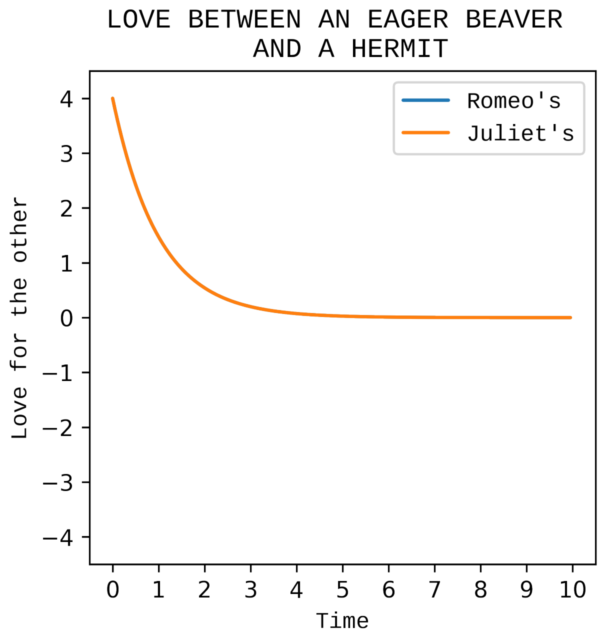

Fig. 5: *The love between two narcissistic nerds corresponding to* $R_0 = J_0 = 4$

Biology to model the interaction of two species). Give five specific examples of such IVPs without finding the exact solutions.

Exercise 4 (2.5 points)**.** This exercise shows us how to solve the IVPs Sys. (13) and Sys. (14) "numerically" when the existence of solutions is guaranteed. The most straightforward numerical scheme is the explicit Euler method.

```
def ExplicitEuler(f, g, t0, R0, J0, h):
        R1 = R0 + f(t0, R0, J0) * h
        J1 = J0 + g(t0, R0, J0) * h
        return R1, J1
```

This function receives the values R_0 and J_0 of $R(t)$ and $J(t)$ at time t_0 and returns the approximate values R_1 and J_1 at $t_1 = t_0 + h$, where h is the time step. The "local truncation error" at t_1 is thus defined by

$$\mathcal{E}(t_1) := \sqrt{[R(t_1) - R_1]^2 + [J(t_1) - J_1]^2}. \tag{15}$$

Prove that $\mathcal{E}(t_1)$ is proportional to h^2.

One of the advantages of this numerical scheme is the fast running time. However, this scheme is not "stable" for large-time step h. For some problems, it requires $h < 1$. In this case, we must consider the "implicit" Euler method. Study and implement the implicit Euler method for five specific examples of IVPs Sys. (14). Plot the solutions. Prove that the implicit Euler method also has $\mathcal{E}(t_1)$ proportional to h^2. What are the cons of the implicit Euler method?

Exercise 5 (Optional + 2.5 points)**.** Consider 1000 data of Romeo's and Juliet's love for the other in the file `https://tinyurl.com/2cypybcw`. Those data are generated from the exact solution to the IVP Sys. (3) with initial condition $R_0 = -2$ and $J_0 = 3$ and time step $h = 0.001$. Some "noises" are also added to the data. Could you use those data to estimate the coefficients a, b, c, and d? What are they? *Hints: you might want to use Machine Learning (ML)/Deep Learning (DL) techniques, for example, by constructing a simple neural network to learn from the data the coefficients* $a, b, c,$ *and* d *based on the idea in* [NVN+22]*. You might also follow a more general framework in* [Duf21].

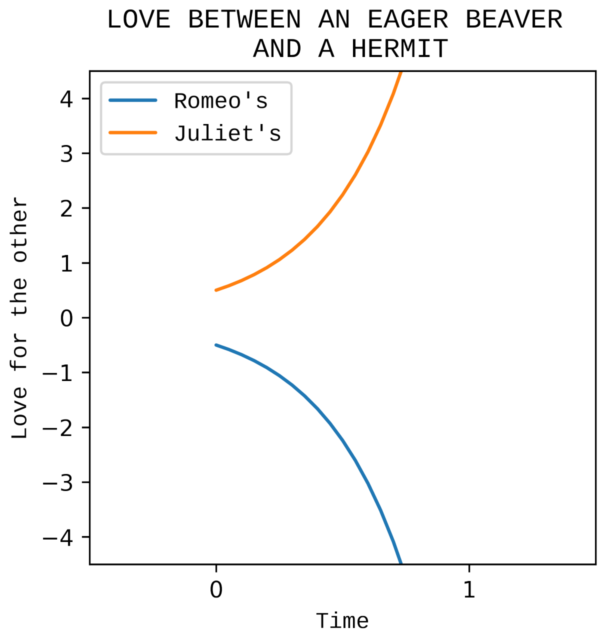

Fig. 6: *The love between two narcissistic nerds corresponding to* $R_0 = -J_0 = -\frac{1}{2}$

4 Instructions and requirements

Students have to follow the instructions and comply with the requirements below. *Lecturers do not solve the cases arising because students do not follow the instructions or do not comply with the requirements.*

4.1 Instructions

Students must work closely with the other members of their team. Fill in and check your team at https://tinyurl.com/2p9e8mzw (for CC0x students) or https://tinyurl.com/25dt87vb (for L0x students).

All aspects related to this assignment will be quizzed (about 10 - 12 of about 25 multiple-choice questions) in the subject's final exam. Therefore, team members must work together so that all of you understand all aspects of the assignment. The team leader should organize the work team to meet this requirement.

During the work, if you have any questions about the assignment, **please post that question on the forum** https://tinyurl.com/29h2w5tu **(for CC0x students) and** https://tinyurl.com/3jb5h723 **(for L0x students)**.

Regarding the background knowledge related to the topic, students should refer to all the references. However, it would be best if you put all of them in the reference section of your report.

4.2 Requirements

- Deadline for submission: **December 04, 2022**. Students have to answer each question clearly and coherently.
- Write a report by using LaTeX in accordance with **the layout as in the template file** (you can find it on https://tinyurl.com/mt29ftrd).
- Each team when submitting its report **need to submit also a log file (diary)** in which clearly state: **weekly work progress for all 06 week(s)**, tasks, content of opinions exchanged of the members, . . .

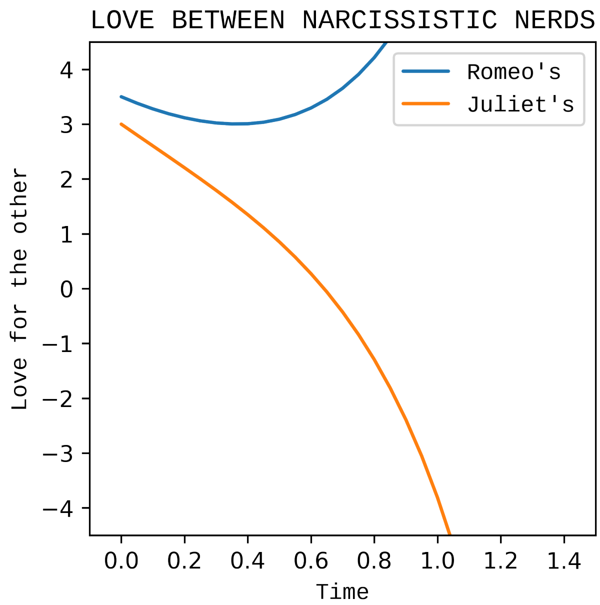

Fig. 7: *The love between two narcissistic nerds corresponding to* $R_0 = \frac{7}{3}$ *and* $J_0 = 3$

- Programming languages: C++/Python/Java/...

4.3 Submission

- Students must submit their team report via the BK-eLearning system (to be opened in the coming weeks): compress all necessary files (.pdf file, .tex file, coding file, ...) into a single file named "*Assignment-CO2011-CSE221-Team_name.zip*" and submit it on `https://tinyurl.com/56xkvy4r` (for CC0x students) or `https://tinyurl.com/3knhtwd8` (for L0x students).
- Noting that for each team, **only the leader will submit the report of the team.**

5 Evaluation and cheating treatment

5.1 Evaluation

Each assignment will be evaluated as follows.

Content	Score (%)
- Analyze, answer coherently, systematically, focus on the goals of the questions and requests	30%
- The programs are neatly written and executable	30%
- Correct, clear, and intuitive graphs & diagrams	20%
- Background section is well written, correct, and appropriate	15%
- Well written report and correct	5%

Tab. 3: *Evaluation*

5.2 Cheating treatment

The assignment has been done by each group separately. Students in a group will be considered as cheating if:

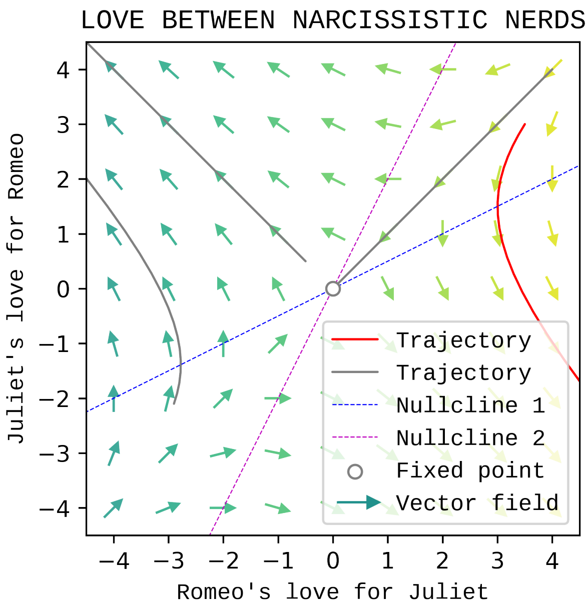

Fig. 8: *The phase portrait of the love between two narcissistic nerds*

- There is an unusual similarity among the reports (especially in the background section). In this case, ALL submissions that are similar are considered cheating. Therefore, the students of a group must protect the work of their group.
- They do not understand the works written by themselves. You can consult from any source, but make sure that you know the meaning of everything you will have written.

If the article is found as cheating, students will be judged according to the university's regulations.

References

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