Word Embedding

Group 13

Instructor: Dr. Nguyễn An Khương

Team Members

Name	Student ID
Ngô Hoài Tú	2570536
Nguyễn Hoàng Kiên	2570435
Huỳnh Đức Nhâm	2570154
Trần Quốc Việt	2570276

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Introduction to Word Embeddings

Problem Statement

In the field of Natural Language Processing (NLP), the fundamental challenge lies in translating human language—which consists of discrete, symbolic, and unstructured data—into a format that machine learning algorithms can process.

Computers operate on numerical data, specifically vectors and matrices. Therefore, we need a robust mapping function to transform the vocabulary of a language into a vector space.

Introduction to Word Embeddings

The Semantic Gap

The core problem is not merely assigning numbers to words, but preserving semantic meaning.

- **Traditional approaches** treated words as atomic symbols with no inherent relationship.
- **The Result:** The system could not understand that "laptop" and "notebook" are related.
- **The Goal:** Word Embedding bridges this gap by creating a continuous vector space where geometric proximity reflects semantic similarity.

Introduction to Word Embeddings

Applications

- Semantic Search.
- Machine Translation
- Recommendation Systems
- Sentiment Analysis

Solving CS Problems

- The Curse of Dimensionality
- Unsupervised Feature Extraction:
- Analogical Reasoning

The One-Hot Encoding Approach

The One-Hot Encoding Approach

Definition and Mechanism

- Before distributed representations, the standard method was **One-Hot Encoding.**
- A One-Hot vector represents a word as a vector $v \in R^{|V|}$ where only one element is 1 and all others are 0. For a vocabulary V=[apple,banana,cat] :

$$w_{\text{apple}} = [1, 0, 0]$$
$$w_{\text{banana}} = [0, 1, 0]$$
$$w_{\text{cat}} = [0, 0, 1]$$

The One-Hot Encoding Approach

Why is One-Hot a bad choice?

While simple, One-Hot encoding suffers from critical flaws:

- **High Dimensionality**: For 100,000 words, every vector length is 100,000. This is computationally wasteful.
- **Data Sparsity**: Vectors are mostly zeros, making gradient updates inefficient in neural networks.
- **Lack of Semantic Similarity (Orthogonality)**: In One-Hot space, every word is orthogonal (90°) to every other word.The dot product of any two distinct words is always zero.

The model learns zero information about the relationship between synonyms

The One-Hot Encoding Approach

Transition to Probability-Based Representations

To overcome One-Hot limitations, we shift from counting to predicting.

We utilize Self-Supervised Learning based on the Distributional Hypothesis:

> *"A word is characterized by the company it keeps."*

By analyzing the probability of a word appearing within a specific context, Word2Vec (CBOW/Skip-gram) learns dense vectors where similar words have similar probability distributions.

Prerequisite Mathematical Concepts

[Reference] Mikolov, T., Sutskever, I., Chen, K., Corrado, G. S., & Dean, J. (2013). Distributed representations of words and phrases and their compositionality.

Prerequisite Mathematical Concepts

Dot Product

- Definition:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- Geometrically, it relates to the angle θ :

$$a \cdot b = ||a|| \, ||b|| \cos(\theta)$$

Prerequisite Mathematical Concepts

Cosine Similarity

- To overcome the magnitude dependency of the dot product, we use Cosine Similarity. This is the gold standard in NLP for determining semantic similarity. It measures the cosine of the angle θ between two vectors.

$$\text{Cosine Similarity} = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||\,||\mathbf{b}||}$$

Prerequisite Mathematical Concepts

What is Conditional Probability?

Key Question: What is $P(A)$ given that B already happened?

Real-world examples:

- $P(\text{rain} \mid \text{cloudy sky})$
- $P(\text{pass exam} \mid \text{studied 10 hours})$

Word2Vec Questions:

Model	Question
Skip-Gram	$P(\text{"cat"} \mid \text{"pet"}) = ?$
CBOW	$P(\text{"coffee"} \mid \text{"I"}, \text{"drink"}, \text{"every"}, \text{"morning"}) = ?$

Prerequisite Mathematical Concepts

The Formula of Conditional Probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Where:

- $P(A \mid B)$ = probability of A **given** B
- $P(A \cap B)$ = probability that **both** A and B happen
- $P(B)$ = probability that B happens

Key Insight: *Conditioning = **narrowing down** the sample space*

Prerequisite Mathematical Concepts

Conditional Probability (connection to Word2Vec)

Model	Direction	Formula
Skip-Gram	center → context	$P(w_{\text{context}} \mid w_{\text{center}})$
CBOW	context → center	$P(w_{\text{center}} \mid w_{\text{context}_1}, w_{\text{context}_2}, \ldots)$

Example: "The cat sat on the mat"

- **Skip-Gram** asks: Given "cat", predict:

$$P(\text{"The"} \mid \text{"cat"}), P(\text{"sat"} \mid \text{"cat"}), P(\text{"on"} \mid \text{"cat"})$$

- **CBOW asks**: Given ["The","sat","on"], predict:

$$P(\text{"cat"} \mid \text{"The"}, \text{"sat"}, \text{"on"})$$

Prerequisite Mathematical Concepts

What is Maximum Likelihood Estimation (MLE)?

Core Idea: Find parameters that make observed data **MOST PROBABLE**

$$\hat{\theta}_{\text{MLE}} = \arg\max_{\theta} P(\text{Data} \mid \theta)$$

Analogy: *"Detective work" - which best explains what we observed?*

Prerequisite Mathematical Concepts

Likelihood vs Probability

Concept	Fixed	Variable	Question
Probability	θ	Data	"Given $\theta = 0.5$, what's $P(7 \text{ heads})$?"
Likelihood	Data	θ	"Given 7 heads, is $\theta = 0.5$ or $\theta = 0.7$ more likely?"

Key Insight: *Same number, different interpretation!*

- Probability: Variable is fixed, Data varies
- Likelihood: Data is fixed, Variable varies

Prerequisite Mathematical Concepts

Why Log-Likelihood?

Instead of $\mathcal{L}(\theta)$, we use $\ell(\theta) = \log \mathcal{L}(\theta)$

Three Reasons:

Reason	Problem	Solution
Numerical Stability	$0.01^{1000} = 10^{-2000}$ (underflow!)	Log keeps numbers manageable
Products → Sums	Hard to differentiate products	$\log(a \times b) = \log a + \log b$
Same Maximum	Need equivalent optimization	$\log$ is monotonically increasing

Prerequisite Mathematical Concepts

From MLE to Loss Function

$$\text{Maximize } \mathcal{L}(\theta) \iff \text{Maximize } \log \mathcal{L}(\theta) \iff \text{Minimize } -\log \mathcal{L}(\theta)$$

Negative Log-Likelihood (NLL) = Word2Vec Loss!

$$\mathcal{L}_{\text{loss}} = -\sum_{t=1}^{T} \sum_{j} \log P(w^{(t+j)} \mid w^{(t)}; \theta)$$

Loss	Likelihood	Model Quality
Lower	Higher	Better
Higher	Lower	Worse

Prerequisite Mathematical Concepts

What is Softmax?

Purpose: Convert arbitrary scores → **probability distribution**

Properties:

1. All outputs are positive (between 0 and 1)
2. All outputs sum to exactly 1

Perfect for: *Multi-class classification where we need probability of each class*

Prerequisite Mathematical Concepts

The Softmax Formula

$$\text{softmax}(z_i) = \frac{e^{z_i}}{\sum_{j=1}^{K} e^{z_j}}$$

Components:

- **Numerator**: e^{z_i} - exponential of the score
- **Denominator**: $\sum_{j=1}^{K} e^{z_j}$ - normalization constant

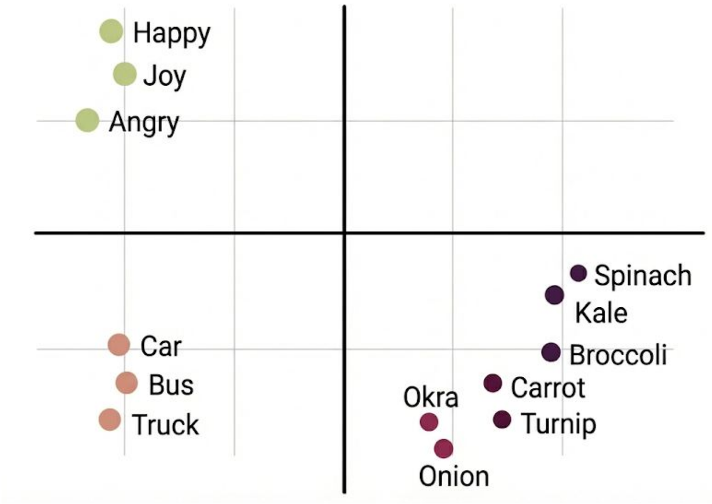
For the entire vector:

$$\text{softmax}(\mathbf{z}) = \left(\frac{e^{z_1}}{\sum_j e^{z_j}}, \frac{e^{z_2}}{\sum_j e^{z_j}}, \ldots, \frac{e^{z_K}}{\sum_j e^{z_j}} \right)$$

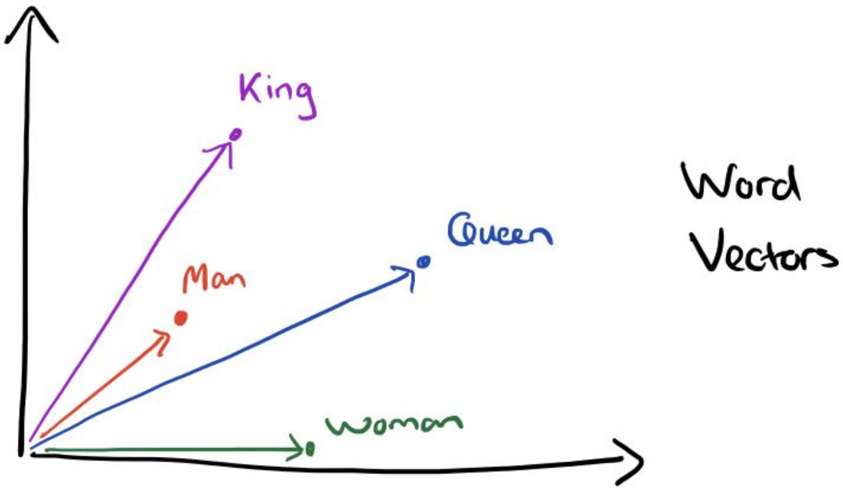
Self-Supervised Word to Vector Method

[Reference] Mikolov, T., Sutskever, I., Chen, K., Corrado, G. S., & Dean, J. (2013). Distributed representations of words and phrases and their compositionality.

Goal



Similarity in vector space



Optimize word offset technique

$$vector(''\text{King}'') - vector(''\text{Man}'') + vector(''\text{Woman}'') \sim vector(''\text{Queen}'')$$

Missions

Preserve the linear regularities among words

Maximize accuracy of simple algebraic vector operations

Minimizing computational complexity by using simple model

$$\dots\ \underbrace{w_{-2}\ w_{-1}}_{\text{context}}\ \overset{\text{center}}{w_0}\ \underbrace{w_1\ w_2}_{\text{context}}\ \dots$$

Continuous Bag of Words Model (CBOW)

The Dual-Vector of a Word

We have a dictionary of words calling **V**. Suppose, each word at index *i* is represented by a vector $\mathbf{v}_i$ (R^N) - This is also our desired outcome.

A model **M** (not CBOW model) help us predict a blank in a sentence.
Ex: **"we are ___ friends forever"**,

M("we", "are", "friends", "forever") = "good"

Continuous Bag of Words Model (CBOW)

The Dual-Vector of a Word

$$M(v_{we},\ v_{are},\ v_{friends},\ v_{forever}) = \text{''good''} \sim M(\bar{v}) = \text{''good''} \sim \max p(\text{''good''} \mid \bar{v}\ )$$

$$\bar{v} = \frac{v_{we} + v_{are} + v_{friends} + v_{forever}}{4}$$

Continuous Bag of Words Model (CBOW)

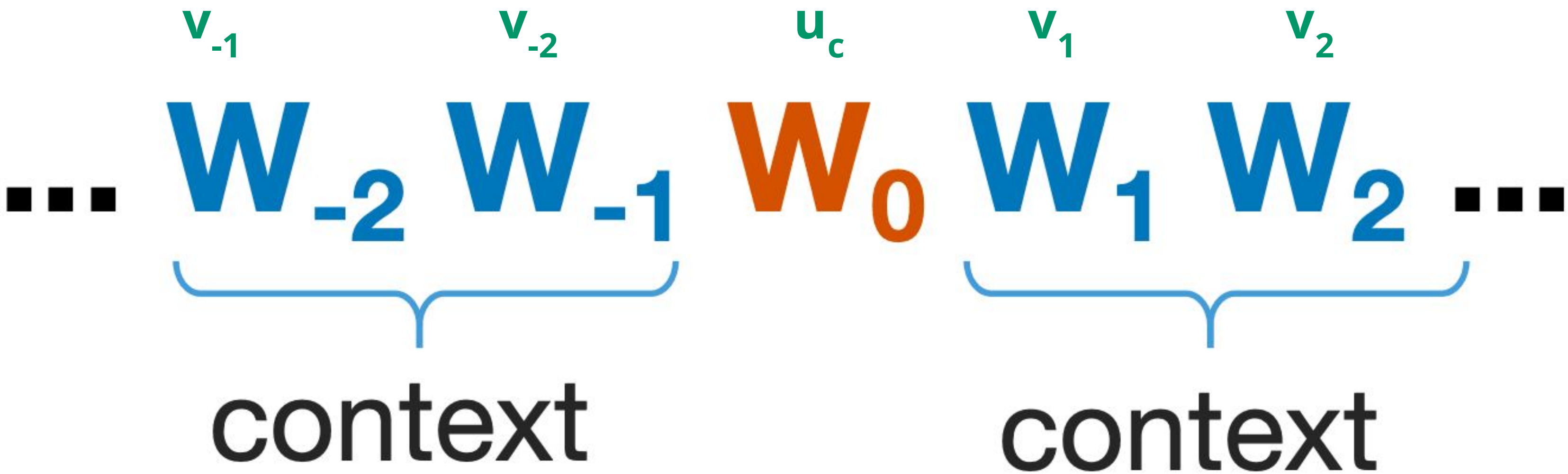
The Dual-Vector of a Word

We define $\mathbf{u}_{good}$ ($\boldsymbol{R}^{N}$) to be the label of class “good”

$$p(\text{"good"} \mid \bar{v}\ ) = p(u_{good} \mid \bar{v}\ )$$

The blank word is called **center word**, the surrounding words are called **context words**

Continuous Bag of Words Model (CBOW)



Continuous Bag of Words Model (CBOW)

The Dual-Vector of a Word

Set context window size to **m**, we have:

$$p(output = u_c \mid input = v_1, v_2, \ldots, v_{2m})$$

A word at index ***i*** in dictionary **V** has two vector ($\mathbf{R}^N$)

$\boldsymbol{v}_i$ is the feature vector (outcome) of a word - representing its context role

$\boldsymbol{u}_i$ is the auxiliary vector of a word - representing its center role

Continuous Bag of Words Model (CBOW)

The Dual-Vector of a Word

We define new vector concept calling **h** $\in \mathbb{R}^N$ to represent the context (list of words, **not only a single** word) in vector space

h

$\boxed{\text{“we” “are” “friend” “forever”}}$

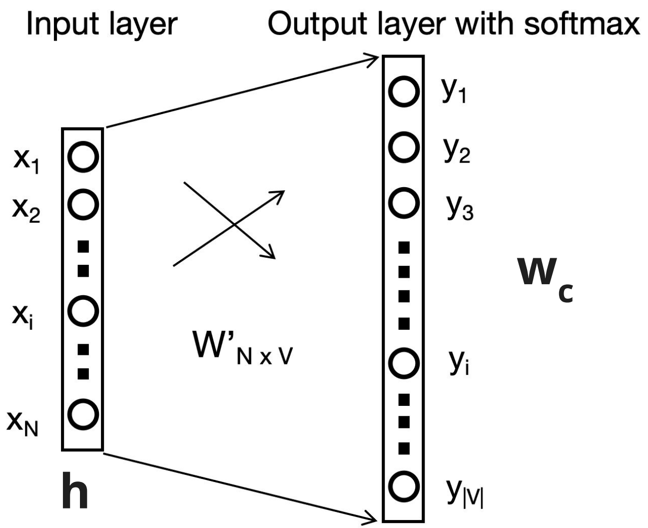
Context to Center Word

We define a new simple model M' containing only two layers:

- Input layer is context h

- Output layer is One-hot vector $y \in \mathbb{R}^{|V|}$ of the center word w_c (c is index in Dict).

The output layer is handled with the softmax function



Context to Center Word

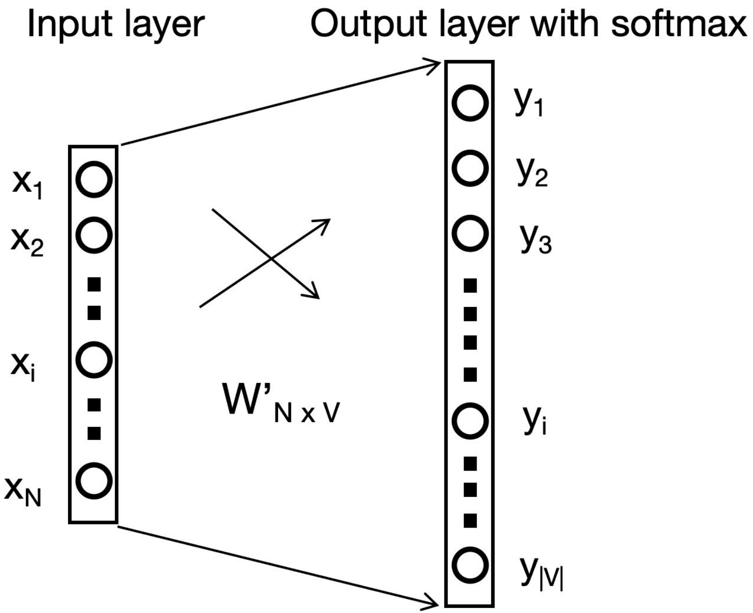
$$y = W'^T . x$$

Following the softmax of this simple neural net, we have:

$$p(word_i \mid x = h) = y_i = \frac{exp(W'^T_{(\cdot,\, i)} \cdot x)}{\Sigma_{j=1}^{|V|} exp(W'^T_{(\cdot,\, j)} \cdot x)}$$

where y_i is the output of the *i-th* unit in the output,

$W'_{(\cdot,\, i)} \in \mathbb{R}^{N x 1}$ is the *i-th* column of the matrix **W '**



Context to Center Word

$$y = W'^{T} . x$$

Following the softmax of this simple neural net, we have:

$$p(word_i \mid x = h) = y_i = \frac{exp(W'^{T}_{(\cdot,\, i)} \cdot x)}{\Sigma_{j=1}^{|V|} exp(W'^{T}_{(\cdot,\, j)} \cdot x)}$$

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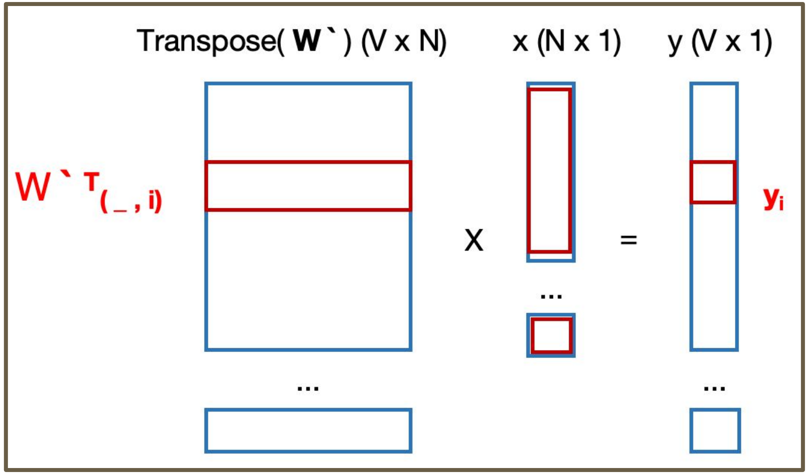


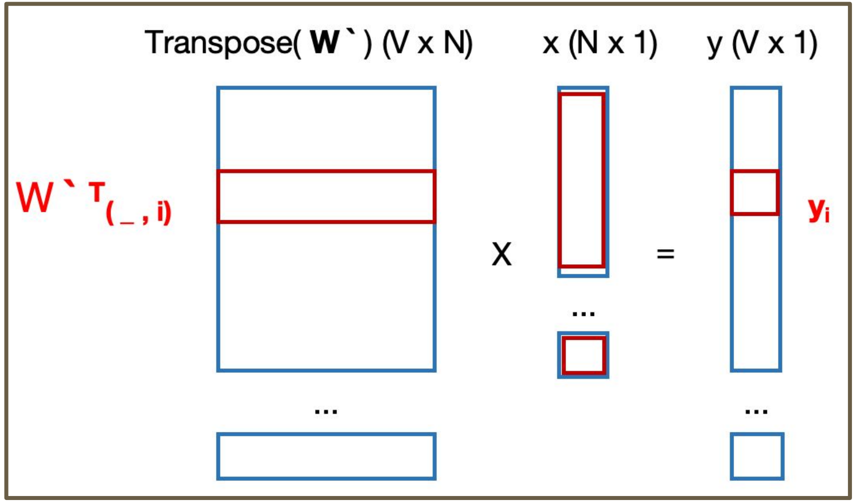
Context to Center Word

Call c is the true index of the center word, the target of training is maximizing the probability:

$$p(word_c \mid x = h)$$

The maximum $\boldsymbol{p(word_c \mid x = h)}$ is related with the weights column $\boldsymbol{W'(\cdot, c)}$ so we assign:

$$u_c := W'_{(\cdot,\, c)} \text{ or } u_i := W'_{(\cdot,\, i)}$$



Therefore:

$$p(word_c \mid x = h) = y_c = \frac{exp(u_c^T \cdot h)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot h)}$$

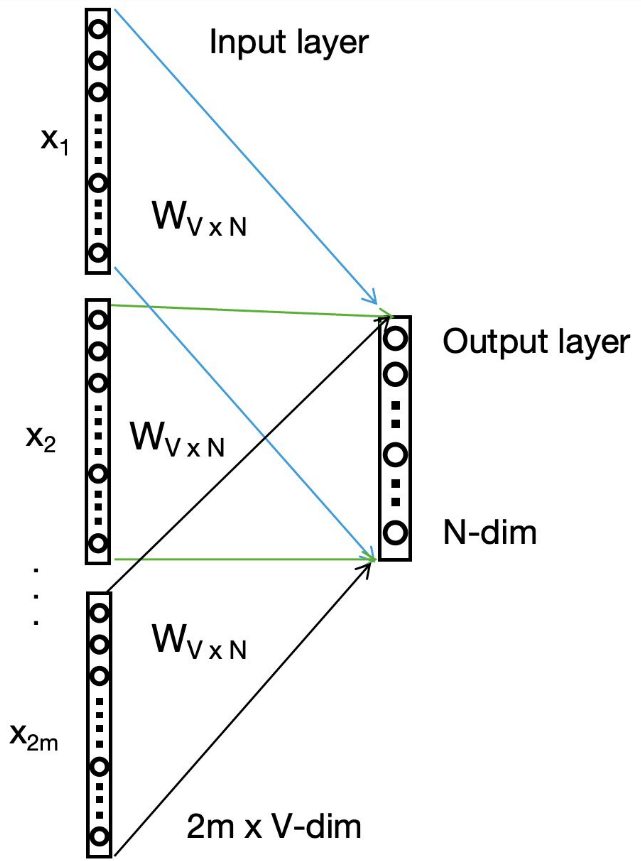
Surrounding words to Context vector

We define a new simple model $\boldsymbol{M}$ `` containing only two layers:

- Input layer is multi-inputs which are One-hot vectors $x_i \in \mathbb{R}^{|V|}$ for context $w_1, w_2, \ldots, w_{2m}$

- Output layer is $y \in \mathbb{R}^N$

*Target of **M ' '** : the output vector is the representation of **the context (list of words)** in vector space (vector h).*



Surrounding words to Context vector

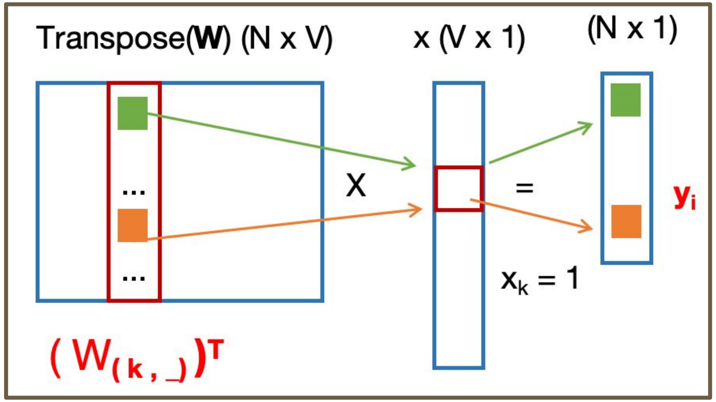
We take the average of the vectors of the input context words, and use the product of the input:

$$y = \frac{1}{2m}(W^T x_1 + W^T x_2 + \cdots + W^T x_{2m}) = \frac{1}{2m} W^T \cdot (x_1 + x_2 + \cdots + x_{2m})$$

W is the weights of the model M'' shared for all inputs.

Given a context, assuming $x^k_i = 1$ and $x^{k'}_i \neq 0$ for $k \neq k'$, we set:

$$W^T \cdot x_i = W^T_{(k,\cdot)} := v^T_{w_k}$$



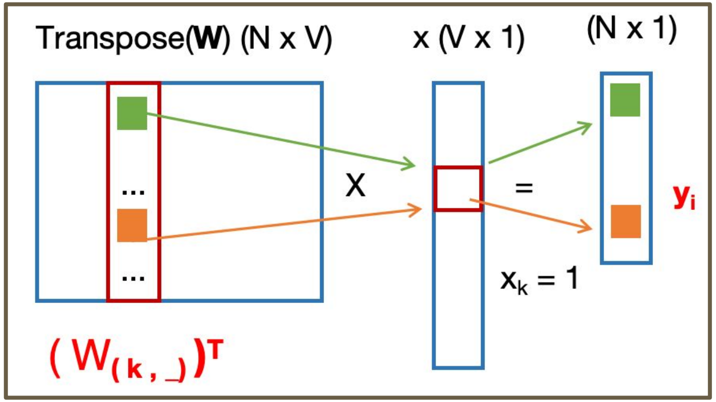
We can assign $\boldsymbol{W}_{(k,\,\cdot\,)} := \boldsymbol{v}_{w\,k}$, because the output vector is contributed by $\boldsymbol{word}_k$ at $\boldsymbol{x}_i$ ~ $\boldsymbol{W}_{(k,\cdot)}$

Surrounding words to Context vector

Given a context, assuming $x^k_i = 1$ and $x^{k'}_i \neq 0$ for $k \neq k'$, we set:

$$W^T \cdot x_i = W^T_{(k,\cdot)} := v_{w_k}$$

We can assign $\boldsymbol{W}_{(k,\,\cdot\,)} := \boldsymbol{v}_{w\,k}$, because the output vector is contributed by $\boldsymbol{word}_k$ at $\boldsymbol{x}_i$ ~ $\boldsymbol{W}_{(k,\cdot)}$



Therefore:

$$y = \frac{1}{2m}\left(v_{w_1} + v_{w_2} + \ldots + v_{w_{2m}}\right) = \frac{1}{2m}\left(v_1 + v_2 + \ldots + v_{2m}\right)$$

*Then, we assign **y := h** (context vector)*

Full CBOW Model

From M `

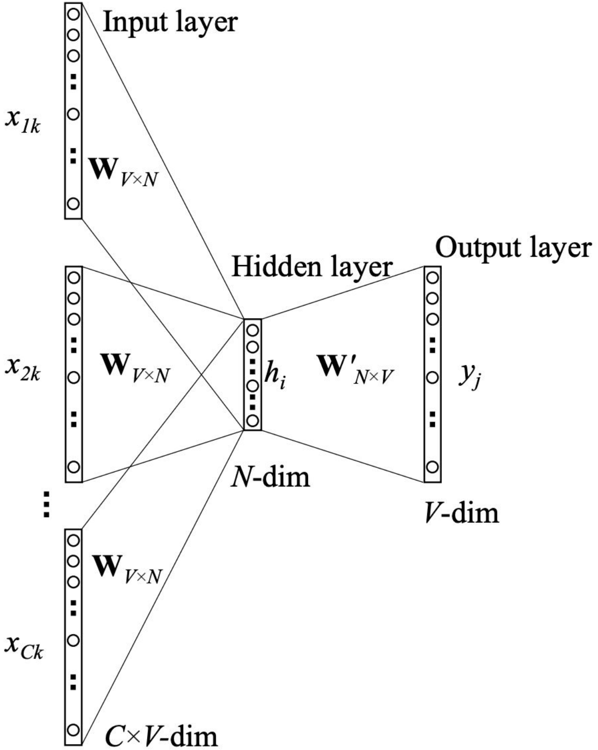
$$p(word_c \mid x = h) = y_c = \frac{exp(u_c^T \cdot h)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot h)}$$

From M ``

$$h = \frac{1}{2m}(v_1 + v_2 + \ldots + v_{2m})$$

Combine

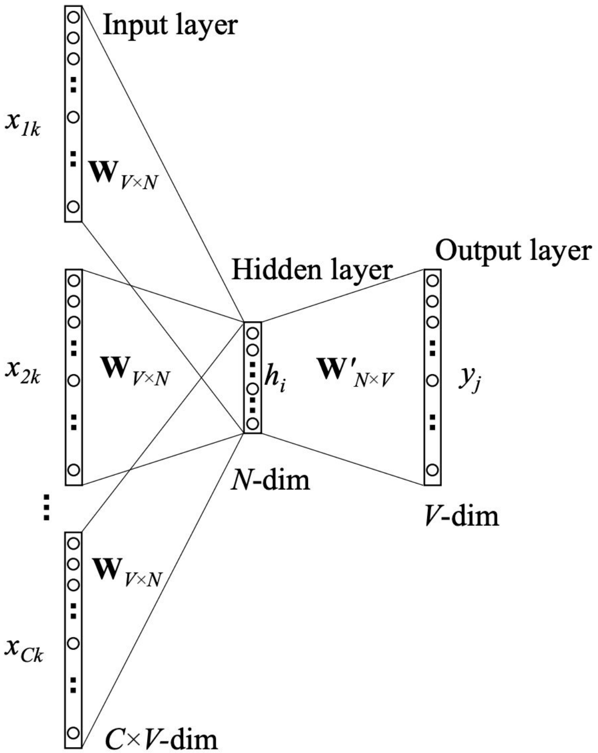
$$p(w_c \mid \mathscr{W}_o) = \frac{exp(\frac{1}{2m} u_c^T (v_{o_1} + v_{o_2} + \cdots + v_{o_{2m}}))}{\Sigma_{i=1}^{|V|} exp(\frac{1}{2m} u_i^T (v_{o_1} + v_{o_2} + \cdots + v_{o_{2m}}))} = \frac{exp(u_c^T \cdot \bar{v_o})}{\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \bar{v_o})}$$



Full CBOW Model

Given a text sequence **T** for training, Following the Maximum likelihood estimation (MLE), we will train the full CBOW model by maximizing this likelihood function:

$$\prod_{t=1}^{T} p\left(w^{(t)} \mid w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)}\right)$$



CBOW model Training

$$max\, p(w_c \mid \mathscr{W}_o) \Leftrightarrow max\, y_i$$

$$\Leftrightarrow max\; log\, y_i = log(exp(u_c^T \cdot \bar{v}_o)) - log(\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \bar{v}_o)) = u_c^T \cdot \bar{v}_o - log(\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \bar{v}_o)) \; := -L$$

with y_i is the **i-th** element of the *y* vector - **i** is index of **word c** in dictionary

y_i

***V*-dim**

Thus: $L = log(\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \bar{v}_o)) - u_c^T \cdot \bar{v}_o$

Loss of sequence T:

$$Ł = -\Sigma_{t=1}^{T} log\, p\left(w^{(t)} \mid w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)}\right) = \Sigma_{t=1}^{T} L_t$$

CBOW model Training

Through differentiation, we can obtain its gradient with respect to any context word vector $\boldsymbol{v}_{\mathbf{o}_i}$ (i=1,...,2m) as:

$$\frac{\partial L}{\partial v_{o_i}} = \frac{\partial \log p(w_c \mid \mathscr{W}_o)}{\partial v_{o_i}}$$

CBOW model Training

Through differentiation, we can obtain its gradient with respect to any context word vector v_{o_i} (i=1,...,2m) as:

$$\frac{\partial L}{\partial v_{o_i}} = \frac{\partial \log p(w_c \mid \mathscr{W}_o)}{\partial v_{o_i}} = \frac{\partial \left(log(\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))) - u_c^T \cdot \frac{1}{2m}(v_1 + .. + v_{o_i} + .. + v_{2m})\right)}{\partial v_{o_i}}$$

$$\boxed{= \frac{(\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m})))'}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))} - \frac{1}{2m} u_c}$$

CBOW model Training

Through differentiation, we can obtain its gradient with respect to any context word vector $\boldsymbol{v}_{\mathbf{O}\boldsymbol{i}}$ (i=1,...,2m) as:

$$= \frac{(\Sigma_{i=1}^{|V|} exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m})))'}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))} - \frac{1}{2m} u_c$$

$$\boxed{= \frac{\Sigma_{i=1}^{|V|} \left(exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m})) \cdot (u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))'\right)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))} - \frac{1}{2m} u_c}$$

CBOW model Training

Through differentiation, we can obtain its gradient with respect to any context word vector v_{oi} (i=1,...,2m) as:

$$= \frac{\Sigma_{i=1}^{|V|} \left(exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m})) \cdot (u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))'\right)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))} - \frac{1}{2m} u_c$$

$$\boxed{= \frac{\Sigma_{i=1}^{|V|} \left(exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m})) \cdot (u_i \cdot \frac{1}{2m})\right)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))} - \frac{1}{2m} u_c}$$

CBOW model Training

Through differentiation, we can obtain its gradient with respect to any context word vector $\boldsymbol{v}_{\mathbf{o}_i}$ (i=1,...,2m) as:

$$= \frac{\Sigma_{i=1}^{|V|}\ \left(exp(u_i^T \cdot \frac{1}{2m}(v_1+\ldots+v_{2m})) \cdot (u_i \cdot \frac{1}{2m})\right)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1+\ldots+v_{2m}))} - \frac{1}{2m}u_c$$

$$\boxed{\begin{aligned} &= \frac{1}{2m}\left(\frac{\Sigma_{i=1}^{|V|}\ \left(exp(u_i^T \cdot \frac{1}{2m}(v_1+\ldots+v_{2m})) \cdot u_i\right)}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1+\ldots+v_{2m}))} - u_c\right) \\ &= \frac{1}{2m}\left(\Sigma_{i=1}^{|V|}\ \frac{exp(u_i^T \cdot \frac{1}{2m}(v_1+\ldots+v_{2m})) \cdot u_i}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1+\ldots+v_{2m}))} - u_c\right) \end{aligned}}$$

CBOW model Training

Through differentiation, we can obtain its gradient with respect to any context word vector v_{oi} (i=1,...,2m) as:

$$= \frac{1}{2m}\left(\Sigma_{i=1}^{|V|} \frac{exp(u_i^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}) \cdot u_i}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \frac{1}{2m}(v_1 + \ldots + v_{2m}))} - u_c\right)$$

$$\boxed{= \frac{1}{2m}\left(\Sigma_{i=1}^{|V|} \frac{exp(u_i^T \cdot \bar{v_o}) \cdot u_i}{\Sigma_{j=1}^{|V|} exp(u_j^T \cdot \bar{v_o})} - u_c\right) = \frac{1}{2m}\left(\Sigma_{i=1}^{|V|} p(w_i \mid \mathscr{W}_o) . u_i \;- u_c\right)}$$

Since $2m \ll |V|$ in language tasks: $O(|V|)$

Skip-Gram Model

Reversely to model M:

$$p(\text{"we"}, \text{"are"}, \text{"friends"}, \text{"forever"} \mid \text{"good"}) \sim p(u_{\text{we}}, u_{\text{are}}, u_{\text{friends}}, u_{\text{forever}} \mid v_{\text{good}}) \text{ on model } M$$

Therefore, for any word with index $\boldsymbol{i}$ in the dictionary, we call vector $\boldsymbol{v_i} \in \mathbb{R}^N$ as its representation in center role and vector $\boldsymbol{u_i} \in \mathbb{R}^N$ as its representation in context role.

Why CBOW and Skip-Gram are consistent:

Following the conditional probability on model M, we have:

$$P(A|B).P(B) = P(B|A).P(A)$$

or:

$$P(center = C \mid context = O).P(context = O) = P(context = O \mid center = C).P(center = C)$$

Architecture

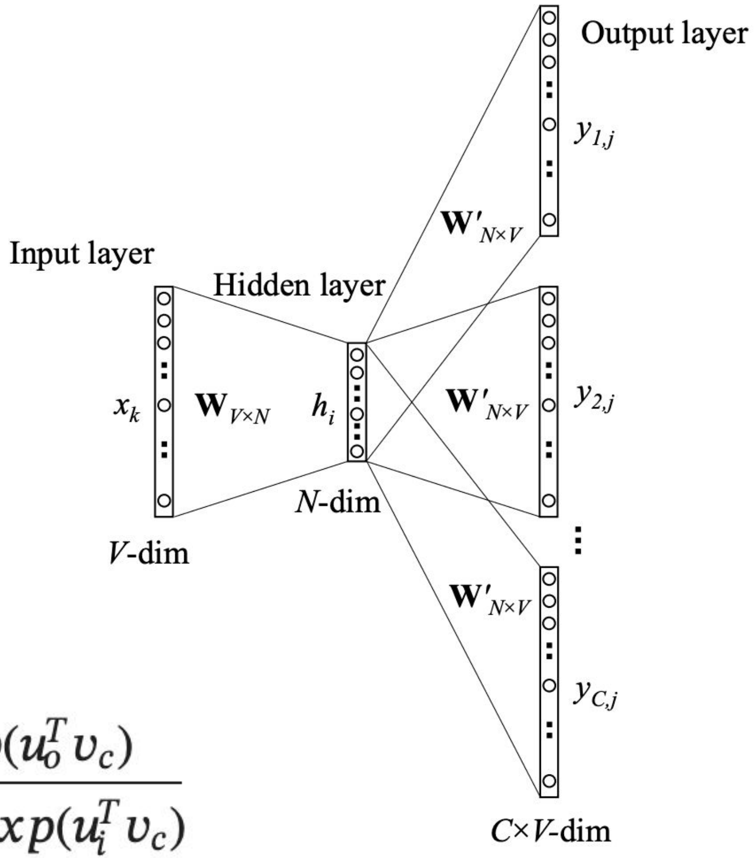
$W_{V\,x\,N}$ is the weights matrix whose each row at index $\boldsymbol{i}$ is the $\boldsymbol{v_i}$ vector of word $\boldsymbol{i}$ in dictionary.

$W'_{N\,x\,V}$ is the weights matrix whose each column at index $\boldsymbol{j}$ is the $\boldsymbol{u_j}$ vector of word $\boldsymbol{j}$ in dictionary.

Similar to CBOW, at output layer with softmax, we have $\boldsymbol{p(w_o \mid w_c)}$ formula as below, where $\boldsymbol{w_o}$ is the word at index $\boldsymbol{o}$ and $\boldsymbol{w_c}$ is the word at index $\boldsymbol{c}$ in the dictionary.

$$h = W^T_{(c,\cdot)} = v_c$$

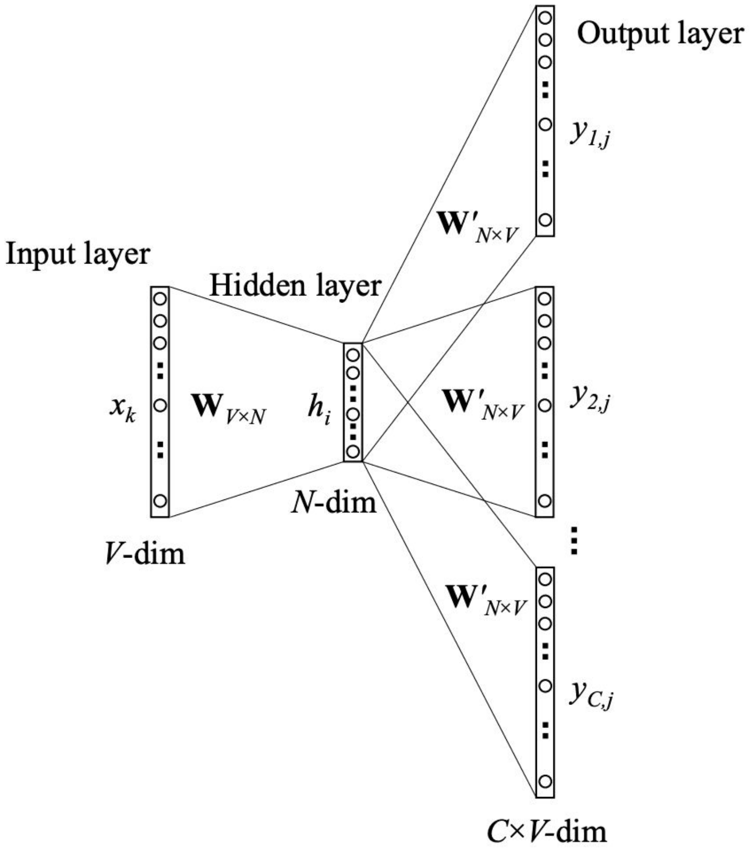
$$p(w_o \mid w_c) = y^o_{ij} = softmax(W'^T \cdot h)^o = \frac{exp(u_o^T v_c)}{\sum_{i=1}^{|V|} exp(u_i^T v_c)}$$



Training

Maximum likelihood estimation

$$\prod_{t=1}^{T} \prod_{-m \leq j \leq m, j \neq 0} p(w^{t+j} \mid w^{(t)})$$



Training

$$p(w_o \mid w_c) = \frac{\exp(u_o^T v_c)}{\sum_{i=1}^{|V|} \exp(u_i^T v_c)}$$

Maximizing this probability is equivalent to maximizing its logarithm:

$$\max\ p(w_o \mid w_c) \iff \max\ \log p(w_o \mid w_c)$$

Then:

$$\log p(w_o \mid w_c) = u_o^T v_c - \log\left(\sum_{i=1}^{|V|} \exp(u_i^T v_c)\right) := -L$$

Thus, the loss for one center–context pair is:

$$L = \log\left(\sum_{i=1}^{|V|} \exp(u_i^T v_c)\right) - u_o^T v_c$$

Training

Differentiating with respect to v_c:

$$\frac{\partial L}{\partial v_c} = \frac{\left(\sum_{i=1}^{|V|} \exp(u_i^T v_c)\right)'}{\sum_{j=1}^{|V|} \exp(u_j^T v_c)} - u_o = \frac{\sum_{i=1}^{|V|} \exp(u_i^T v_c) \cdot (u_i^T v_c)'}{\sum_{j=1}^{|V|} \exp(u_j^T v_c)} - u_o = \frac{\sum_{i=1}^{|V|} \exp(u_i^T v_c) \cdot u_i}{\sum_{j=1}^{|V|} \exp(u_j^T v_c)} - u_o$$

Recognizing the softmax probability:

$$p(w_i \mid w_c) = \frac{\exp(u_i^T v_c)}{\sum_{j=1}^{|V|} \exp(u_j^T v_c)}$$

we obtain:

$$\frac{\partial L}{\partial v_c} = \sum_{i=1}^{|V|} p(w_i \mid w_c)\, u_i \; - \; u_o$$

Complexity:

$$\boxed{O(|V|)}$$

Approximate Training

Why approximate training?

Fully Softmax

$$O(|\mathcal{V}|)$$

For a typical Word2Vec with a vocabulary size of $|V| = 3{,}000{,}000$:

- $\approx 3{,}000{,}000$ dot products per update
- $\approx 3{,}000{,}000$ gradient updates

Negative Sampling

$O(k)$ where typically $k = 5$ to 10 15.

Hierarchical Softmax

$O(\log_2 |V|)$ $O(\log_2 |3,000,000|) \approx 22$

Negative Sampling in Skip-Gram Model

The probability model will be:

$$P(D = 1 \mid w_c, w_o) = \sigma(\mathbf{u}_o^\top \mathbf{v}_c)$$

The joint probability will be:

$$\prod_{t=1}^{T} \prod_{-m \leq j \leq m,\ j \neq 0} P(D = 1 \mid w^{(t)}, w^{(t+j)})$$

Negative sampling rewrites the conditional probability:

$$\begin{aligned} P(w^{(t+j)} \mid w^{(t)}) \\ = P_S \prod_K P_k \end{aligned}$$

$$= P(D = 1 \mid w^{(t)}, w^{(t+j)}) \prod_{k=1,\ w_k \sim P(w)}^{K} P(D = 0 \mid w^{(t)}, w_k)$$

Negative Sampling in Skip-Gram Model

The logarithmic loss

$$
\begin{aligned}
&-\log P(w^{(t+j)} \mid w^{(t)}) \\
&= -\log P(D=1 \mid w^{(t)}, w^{(t+j)}) - \sum_{k=1,\, w_k \sim P(w)}^{K} \log P(D=0 \mid w^{(t)}, w_k)
\end{aligned}
$$

because of classification binary so $P(D=0 \mid w^{(t)}, w_k) = 1 - P(D=1 \mid w^{(t)}, w_k)$, we have:

$$
= -\log \sigma\left(\mathbf{u}_{i_{t+j}}^\top \mathbf{v}_{i_t}\right) - \sum_{k=1,\, w_k \sim P(w)}^{K} \log\left(1 - \sigma\left(\mathbf{u}_{h_k}^\top \mathbf{v}_{i_t}\right)\right)
$$

with $\sigma(x) + \sigma(-x) = 1$, We can infer:

$$
= -\log \sigma\left(\mathbf{u}_{i_{t+j}}^\top \mathbf{v}_{i_t}\right) - \sum_{k=1,\, w_k \sim P(w)}^{K} \log \sigma\left(-\mathbf{u}_{h_k}^\top \mathbf{v}_{i_t}\right)
$$

Negative Sampling in Continuous Bag-of-words Model

The probability model will be:

$$P(D = 1 \mid w_c, w_{o_1}, \ldots, w_{o_{2m}}) = \sigma(\mathbf{u}_o^\top \bar{\mathbf{v}}_o)$$

The joint probability will be:

$$\prod_{t=1}^{T} P(D = 1 \mid w^{(t)}, w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)})$$

$$P(w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)}) \mid w^{(t)})$$
$$= P_S \prod_K P_k$$

$$= P(D = 1 \mid w^{(t)}, w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)}) \prod_{k=1,\, w_k \sim P(w)}^{K} P(D = 0 \mid w_k, w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)})$$

Negative Sampling in Continuous Bag-of-words Model

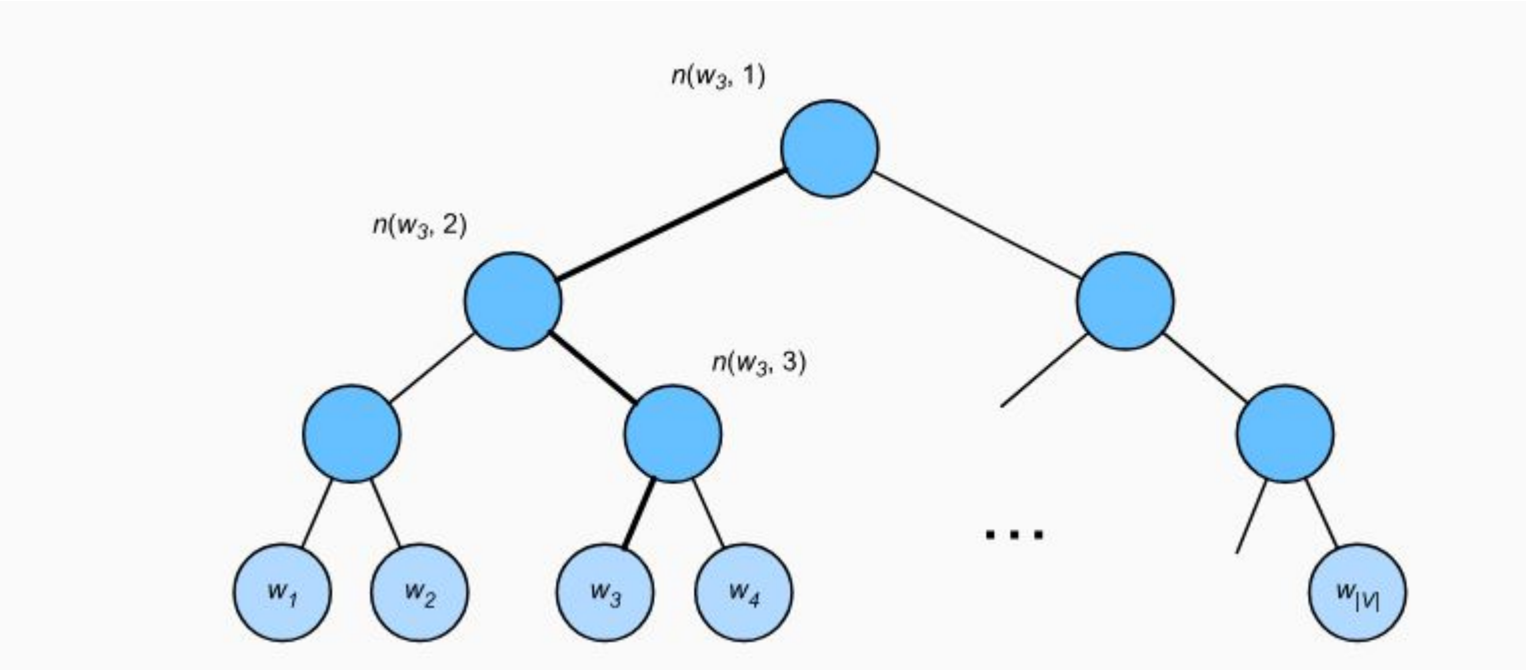
The logarithmic loss:

$$
\begin{aligned}
&- \log P(w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)} \mid w^{(t)}) \\
&= -\log P(D=1 \mid w^{(t)}, w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)}) - \sum_{k=1,\, w_k \sim P(w)}^{K} \log \\
&\quad P(D=0 \mid w_k, w^{(t-m)}, \ldots, w^{(t-1)}, w^{(t+1)}, \ldots, w^{(t+m)}) \\
&= -\log \sigma\left(\mathbf{u}_{i_{t+j}}^{\top} \mathbf{v}_{i_t}\right) - \sum_{k=1,\, w_k \sim P(w)}^{K} \log\left(1 - \sigma\left(\mathbf{u}_{h_k}^{\top} \mathbf{v}_{i_t}\right)\right)
\end{aligned}
$$

with $\sigma(x) + \sigma(-x) = 1$, We can infer (todo: cleaning):

$$
= -\log \sigma\left(\mathbf{u}_{i_{t+j}}^{\top} \bar{\mathbf{v}}_{i_t}\right) - \sum_{k=1,\, w_k \sim P(w)}^{K} \log \sigma\left(-\mathbf{u}_{h_k}^{\top} \bar{\mathbf{v}_{i_t}}\right)
$$

Hierarchical Softmax in Skip Gram Model



Hierarchical Softmax in Skip Gram Model

Given:

- $L(w)$: number of nodes on the path from the root node to the leaf node representing word w in the binary tree
- $n(w, j)$ is the j^{th} node on the path, with context word vector $u_{n(w,j)}$

$$P(w_o \mid w_c) = \prod_{j=1}^{L(w_o)-1} \sigma\left(\llbracket n(w_o, j+1) = \text{leftChild}(n(w_o, j)) \rrbracket \cdot \mathbf{u}_{n(w_o,j)}^\top \mathbf{v}_c\right)$$

- $\text{leftChild}(n)$ is the left child node of node n: if x is true, $\llbracket x \rrbracket = 1$; otherwise $\llbracket x \rrbracket = -1$

Hierarchical Softmax in Continuous Bag-of-words Model

Given:

- $L(w)$: number of nodes on the path from the root node to the leaf node representing word w in the binary tree
- $n(w,j)$ is the j^{th} node on the path, with center word vector $\theta_{n(w,j)}$
- The context vector as the average:

$$\bar{\mathbf{v}} = \frac{1}{2m} \sum_{j=1}^{2m} \mathbf{v}_{o_j}$$

so the conditional probability:

$$P(w_o \mid w_c) = \prod_{j=1}^{L(w_o)-1} \sigma\left([\![n(w_o, j+1) = \text{leftChild}(n(w_o, j))]\!] \cdot \theta_{n(w,j)}^\top \bar{\mathbf{v}} \right)$$

- $\text{leftChild}(n)$ is the left child node of node n: if x is true, $[\![x]\!] = 1$; otherwise $[\![x]\!] = -1$

Thanks for watching and listening!