Lecture 7

Plane Wave Maths

Nazmul Haque Turja
Research and Development Assistant, BUET

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$.

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$+x \to$

$E_{y/z} = E_0 e^{j(\omega t - kx)}$

$$E_y = E_0\, e^{j(\omega t - k_z z)}$$

$\frac{E}{H} = Z_F$

$H_x = \frac{E_y}{Z_F}$

$$Z_F = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_0 = \sqrt{\mu_0 / \varepsilon_0}$$

a) $\vec{E}(\omega t) = E_0 \cos(\omega t - kz)\,\vec{e}_x$; $E_0 = 100 \frac{V}{m}$

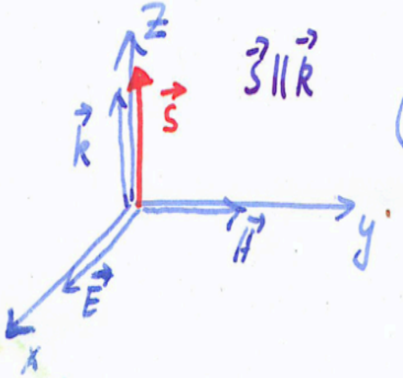
(or $\vec{E}(\omega t) = E_0 \cos(\omega t - kz)\,\vec{e}_y$)

(or a mixture of both)

$\hat{e}_z \times \hat{e}_x$

$\hat{e}_y$

with $E_0 = 100 \frac{V}{m}$; $\omega = 2\pi f = 2\pi \cdot 300 \cdot 10^6 \frac{1}{s}$



$\vec{S} \parallel \vec{k}$

$(\vec{E} \times \vec{H}) \parallel \vec{k}$; $\vec{H} \parallel (\vec{k} \times \vec{E})$; $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}$ here:

$\vec{H} = \vec{e}_z \times \vec{E} \cdot \frac{1}{Z}$

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}} = Z_0 \cdot \sqrt{\frac{1}{\varepsilon_r}} = \frac{1}{3} Z_0 \quad \text{with } Z_0 \approx 120\pi\,\Omega$$

$$\vec{H} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{Z} = \begin{pmatrix} 0 \\ \frac{E_x}{Z} \\ 0 \end{pmatrix} = \frac{E_0}{Z} \cos(\omega t - kz)\,\vec{e}_y$$

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$.

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$$\vec{S} = \vec{E} \times \vec{H} = \frac{1}{Z_F} E_y^2 \, \vec{e}_z$$

$$\begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ E_x & 0 & 0 \\ 0 & H_y & 0 \end{vmatrix} = \vec{e}_z \,(E_x H_y)$$

$E_x \to H_y$

b) $$\vec{S} = \vec{E} \times \vec{H} = \begin{pmatrix} E_0 \cos(\omega t - kz) \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \frac{E_0}{Z} \cos(\omega t - kz) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{E_0^2}{Z} \cos^2(\omega t - kz) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{E_0^2}{2Z}\left(1 + \cos(2\omega t - 2kz)\right) \end{pmatrix}$$

Average power density

$$\overline{\vec{S}} = \frac{1}{T} \int_0^T \vec{S}(\omega t)\,dt = \frac{E_0^2}{2Z} \vec{e}_z = 39.79 \frac{\mathrm{W}}{\mathrm{m}^2} \vec{e}_z$$

$E_0 = 100\,\mathrm{V/m}$, $Z = 40\pi$

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$. lossy

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$$K = \omega\sqrt{\mu\underline{\varepsilon}} = \frac{2\pi}{\lambda}$$

$$Z_F = \sqrt{\frac{\mu}{\underline{\varepsilon}}} \qquad \underline{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\varepsilon\,\omega}\right)$$

$$\tan\delta = \frac{\sigma}{\omega\,\varepsilon} \qquad \text{with} \quad \varepsilon = \varepsilon_0\,\varepsilon_r$$

4.2c) $\sigma = 10\,\frac{S}{m}$ (very lossy!) (salt water) 16

impedance:

$\underline{Z} = \sqrt{\frac{\mu}{\underline{\varepsilon}}}$ with $\mu = \mu_0$ and

$$\underline{\varepsilon} = \varepsilon_0\varepsilon_r(1 - j\tan\delta)$$

$$\tan\delta = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_0\varepsilon_r}$$

$$= \frac{10\,\frac{S}{m}}{2\pi\cdot 300\cdot 10^6\,\frac{1}{s}\cdot 8.854\cdot 10^{-12}\,\frac{As}{Vm}\cdot 9}$$

$$= 66.57 \quad \text{(no unit!)}$$

$$\underline{Z} = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r(1-j\tan\delta)}} = \frac{Z_0}{\sqrt{\varepsilon_r(1-j\tan\delta)}}$$

with $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 120\pi\,\Omega$

$$1 - j\tan\delta = 1 - j\,66.57 \approx 66.58\,e^{-j\,89.14^\circ}$$

complex

$$\frac{1}{\sqrt{1-j\tan\delta}} = \frac{1}{\sqrt{66.58\,e^{-j\,89.14^\circ}}} \approx \frac{1}{8.16}\,e^{+j\,44.57^\circ}$$

$$\left(\frac{1}{\sqrt{e^{-j\varphi}}} = e^{+j\varphi/2}\right)$$

$$\Rightarrow \underline{Z} = \frac{Z_0}{\sqrt{\varepsilon_r}}\cdot\frac{1}{8.16}\,e^{j\,44.57^\circ} \approx 10.97\,\Omega + j\,10.81\,\Omega$$

wave number:

$$\underline{k} = \omega\sqrt{\mu\underline{\varepsilon}} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}\sqrt{1-j\tan\delta}$$

$$= \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_r}\cdot 8.16\,e^{-j\,44.57^\circ}$$

$$\approx (109.6 - j\,108)\,\frac{1}{m}$$

$$= k' - j k''$$

complex

4.2 Problem 2

A plane wave travels in the $+z$ direction in a dielectric lossless medium with a relative permittivity of $\varepsilon_r = 9$, at a frequency of $300\,\mathrm{MHz}$ and with an electric field amplitude of $100\,\mathrm{V/m}$.

a) Write the complete time-domain expressions for the vector fields $\vec{E}$ and $\vec{H}$.

b) Calculate the average power density of the wave.

Now, assume the dielectric material has a conductivity $\sigma = 10\,\mathrm{S\,m^{-1}}$.

c) What are the wave impedance and wave number of the wave?

d) Determine the average power density of the wave.

$$\vec{E}_x = E_0 \cos(\omega t - kx)$$

$$H_y = H_0 \cos(\omega t - kx + \phi)$$

$$\vec{\bar{S}} = \frac{1}{T}\int_0^T \vec{S}(t)\,dt = \frac{1}{2}\Re\{\vec{E}\times\vec{H}^*\}$$

$$\underline{Z} = \frac{Z_0}{\sqrt{\varepsilon_r}}\cdot\frac{1}{8.16}e^{j44.57^\circ} \approx 10.97\,\Omega + j\,10.81\,\Omega$$

4.2 d)

$$\vec{E}(\omega t) = Re\{\vec{\underline{E}}_0\, e^{j\omega t}\}$$

$$\text{with } \vec{\underline{E}}_0 = 100\tfrac{V}{m}\cdot e^{-j\underline{k}z}\cdot\vec{e}_x$$

$$= 100\tfrac{V}{m}\, e^{-j(k'-jk'')z}\,\vec{e}_x$$

$$= 100\tfrac{V}{m}\cdot \underbrace{e^{-k''z}}_{\text{damping}}\cdot\underbrace{e^{-jk'z}}_{\text{phasor}}\,\vec{e}_x$$

a+ib

a−ib

$$\vec{H}(\omega t) = Re\{\vec{\underline{H}}_0\, e^{j\omega t}\}$$

$$\text{with } \vec{\underline{H}}_0 = \frac{E_0}{\underline{Z}}e^{-k''z}\cdot e^{-jk'z}\,\vec{e}_y$$

$$\Rightarrow \vec{\bar{S}} = \tfrac{1}{2}Re\{\vec{\underline{E}}\times\vec{\underline{H}}^*\} = Re\{\vec{\underline{S}}\}$$ (mean active power flow)

$$\underline{Z} = |\underline{Z}|e^{j\varphi_z} \quad (= 10.97\,\Omega + j10.81\,\Omega)$$

$$\Rightarrow \vec{\underline{H}}_0 = \frac{E_0}{|\underline{Z}|}\cdot e^{-k''z}\cdot e^{-j(k'z+\varphi_z)}\cdot\vec{e}_y$$ (phase)

$$\vec{\underline{S}} = \tfrac{1}{2}\vec{\underline{E}}_0\times\vec{\underline{H}}_0^* = \frac{E_0^2}{2|\underline{Z}|}\cdot e^{-2k''z}\cdot e^{+j\varphi_z}\cdot\vec{e}_z$$

$$\Rightarrow \vec{\bar{S}} = \tfrac{1}{2}Re\{\vec{\underline{E}}\times\vec{\underline{H}}^*\} = \frac{E_0^2}{2|\underline{Z}|}e^{-2k''z}\cos(\varphi_z)\cdot\vec{e}_z$$

$$= \frac{(100\frac{V}{m})^2}{2\sqrt{10.97^2+10.81^2}\,\Omega}\cdot\cos(44.57^\circ)\,e^{-2k''z}\cdot\vec{e}_z$$

$$= 231.3\, e^{-2k''z}\,\frac{W}{m^2}\cdot\vec{e}_z$$