1) a) $G_1(s) = \dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} = \dfrac{1}{(\frac{1}{w_n})^2 s^2 + 2\zeta \frac{1}{w_n} s + 1}$

$$\xrightarrow[u=\frac{w}{w_n}]{s=wj} \quad G_1 = \frac{1}{-u^2 + 2\zeta ju + 1} \implies |G_1| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}} \longrightarrow \frac{\partial |G_1|}{\partial u} = \frac{2u(1-u^2-2\zeta^2)}{((4\zeta^2-2)u^2 + u^4 + 1)^{\frac{3}{2}}} = 0 \implies u = \sqrt{1-2\zeta^2} = \frac{w}{w_n} \implies w = w_n\sqrt{1-2\zeta^2}$$

$\downarrow$

$$|G_1|_{max} = \left((2\zeta^2)^2 + (2\zeta\sqrt{1-2\zeta^2})^2\right)^{-\frac{1}{2}} = \left(4\zeta^4 + 4\zeta^2(1-2\zeta^2)\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{4\zeta^2 - 4\zeta^4}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

b) $G_2(s) = \dfrac{w_n(s+w_n)}{s^2 + 2\zeta w_n s + w_n^2} = \dfrac{\frac{1}{w_n}s + 1}{(\frac{1}{w_n})^2 s^2 + 2\zeta \frac{1}{w_n} s + 1}$

$$\xrightarrow[u=\frac{w}{w_n}]{s=wj} \quad G_2 = \frac{ju+1}{-u^2 + 2\zeta ju + 1} \implies |G_2| = \frac{\sqrt{1+u^2}}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}} \longrightarrow \frac{\partial |G_2|}{\partial u} = \frac{-u(u^4 + 2u^2 + 4\zeta^2 - 3)}{\sqrt{1+u^2}\,((4\zeta^2-2)u^2 + u^4 + 1)^{\frac{3}{2}}} = 0 \implies u^2 = -1 + \sqrt{1-(4\zeta^2-3)} = -1 + 2\sqrt{1-\zeta^2}$$

$$\implies u = \sqrt{2\sqrt{1-\zeta^2} - 1}$$

$$w = w_n\sqrt{2\sqrt{1-\zeta^2} - 1}$$

$\downarrow$

$$|G_2|_{max} = \frac{(1-\zeta^2)^{\frac{1}{4}}}{2\left((1-\zeta^2)(1-\sqrt{1-\zeta^2})\right)^{\frac{1}{2}}}$$

c) $$\frac{|G_2|_{max}}{|G_1|_{max}} = \frac{(1-\zeta^2)^{\frac{1}{4}}}{2\left((1-\zeta^2)(1-\sqrt{1-\zeta^2})\right)^{\frac{1}{2}}}\left(2\zeta\sqrt{1-\zeta^2}\right) = \frac{\zeta(1-\zeta^2)^{\frac{1}{4}}}{(1-\sqrt{1-\zeta^2})^{\frac{1}{2}}}$$

$$\lim_{\zeta \to 0} \frac{\zeta(1-\zeta^2)^{\frac{1}{4}}}{(1-\sqrt{1-\zeta^2})^{\frac{1}{2}}} \xrightarrow[\zeta = (1-u^4)^{\frac{1}{2}}]{u = (1-\zeta^2)^{\frac{1}{4}}} \lim_{u \to 1} \frac{(1-u^4)^{\frac{1}{2}} u}{(1-u^2)^{\frac{1}{2}}} = \lim_{u \to 1} \left(\frac{1-u^4}{1-u^2}\right)^{\frac{1}{2}} = \lim_{u \to 1} (1+u^2)^{\frac{1}{2}} = \sqrt{2} \implies \max|G_1| \approx \frac{\max|G_2|}{\sqrt{2}}$$

2 – a) $G_1(s) = \frac{k}{\tau s+1} \to \frac{k}{\tau wj+1} \quad \Rightarrow \quad |G_1(wj)| = \frac{k}{\sqrt{\tau^2 w^2+1}} = \frac{G_1(0)}{\sqrt{2}} = \frac{k}{\sqrt{2}} \quad \Rightarrow \quad \tau^2 w^2 + 1 = 2 \quad \Rightarrow \quad w_B = \frac{1}{\tau}$

DC gain : $G_1(0) = k$

b) $G_2(s) = \frac{k}{s^2 + 2\zeta w_n s + w_n^2} \to \frac{k}{2\zeta w_n wj + w_n^2 - w^2} \quad \Rightarrow \quad |G_2(wj)| = \frac{k}{\sqrt{(w_n^2 - w^2)^2 + (2\zeta w_n w)^2}} = \frac{k}{\sqrt{2}\, w_n^2}$

DC gain : $G_2(0) = \frac{k}{w_n^2}$

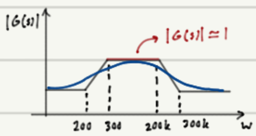
$$\Rightarrow 2w_n^4 = w_n^4 + w^4 - 2w_n^2 w^2 + 4\zeta^2 w_n^2 w^2 \quad \Rightarrow \quad w^4 + \left((4\zeta^2-2)\, w_n^2\right) w^2 - w_n^4 = 0 \quad \Rightarrow \quad w_B^2 = \frac{1}{2}\left(-(4\zeta^2-2)w_n^2 + \sqrt{(4\zeta^2-2)^2 w_n^4 + 4w_n^4}\right)$$

$$= 1 - 2\zeta^2 w_n^2 + \sqrt{(2\zeta^2-1)^2+1}\; w_n^2$$

$$= \left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right) w_n^2$$

$$\Rightarrow \quad w_B = \left(\sqrt{1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}\right) w_n$$

c) $G_3(s) = \frac{2}{3}\,\frac{(s+200)(s+300000)}{(s+300)(s+200000)}$



$$|G| = \sqrt{\frac{2}{3}\,\frac{(w^2+200^2)(w^2+300000^2)}{(w^2+300^2)(w^2+200000^2)}} = \frac{1}{\sqrt{2}} \longrightarrow w_B \simeq 100 \quad , \quad w_B \simeq 600000$$

جواب دقیق
از روی نمودار آبی

معادله خط تقریبی : $a = \frac{1-\frac{2}{3}}{300-200} = \frac{1}{300} \longrightarrow |G|_1 = \frac{1}{300} w = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad w_B = \frac{300}{\sqrt{2}}$

$$a = \frac{\frac{2}{3}-1}{300k-200k} = \frac{-1}{300k} \longrightarrow |G|_2 = \frac{-1}{300k} w + \frac{5}{3} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad w_B = \left(\frac{5}{3} - \frac{1}{\sqrt{2}}\right) 300k$$

3 – a) $G_1(s) = \frac{1}{\tau s+1} \to \frac{1}{\tau wj+1} \quad \Rightarrow \quad |G_1(wj)| = \frac{1}{\sqrt{(\tau w)^2+1}} = 1 \quad \Rightarrow \quad \tau^2 w^2 + 1 = 1 \quad \Rightarrow \quad w_c = 0$

b) $G_2(s) = \frac{w_n^2}{s(s+2\zeta w_n)} \longrightarrow \frac{w_n^2}{wj(wj+2\zeta w_n)} = \frac{w_n^2}{2\zeta w_n wj - w^2} \quad \Rightarrow \quad |G_2(wj)| = \frac{w_n^2}{\sqrt{(2\zeta w_n w)^2 + w^4}} = 1 \quad \Rightarrow \quad w_n^4 = 4\zeta^2 w_n^2 w^2 + w^4$

$$\Rightarrow \quad w^2 = \frac{1}{2}\left(-4\zeta^2 w_n^2 + \sqrt{(4\zeta^2 w_n^2)^2 + 4w_n^4}\right) = -2\zeta^2 w_n^2 + \sqrt{4\zeta^2+1}\; w_n^2 = \left(-2\zeta^2 + \sqrt{4\zeta^2+1}\right) w_n^2 \quad \Rightarrow \quad w_c = \sqrt{\sqrt{4\zeta^2+1} - 2\zeta^2}\; w_n$$

c) $G_3(s) = \frac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} \longrightarrow \frac{w_n^2}{w_n^2 - w^2 + 2\zeta w_n wj} \quad \Rightarrow \quad |G_3(wj)| = \frac{w_n^2}{\sqrt{(w_n^2-w^2)^2 + (2\zeta w_n w)^2}} = 1 \quad \Rightarrow \quad w_n^4 = w_n^4 + w^4 - 2w_n^2 w^2 + 4\zeta^2 w_n^2 w^2$

$$\Rightarrow \quad w^2\left(w^2 + w_n^2(4\zeta^2 - 2)\right) = 0 \quad \begin{cases} w_c = 0 \\ w_c = w_n\sqrt{2 - 4\zeta^2} \end{cases}$$

4_ a) $y(t) = u(t) * g(t) = \int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau \leq \|u\|_2 \int_{-\infty}^{\infty} g(\tau)\, d\tau$

$$\longrightarrow \|y\|_2 \leq \sqrt{\int_0^{\infty} \left(\|u\|_2 \int_{-\infty}^{\infty} g(\tau)\, d\tau \right)^2 dt} = \|u\|_2 \, \|g\|_2$$

$$\Rightarrow \|y\|_2 \leq \|u\|_2 \max |G(j\omega)|$$

$$\checkmark \quad \|g\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)\, G(-j\omega)\, d\omega} \leq \sqrt{\frac{(\max |G(j\omega)|)^2}{2\pi}} \leq \max |G(j\omega)|$$

b) $\|S\| = \sup \frac{\|y\|_2}{\|u\|_2}$

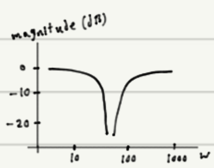
$$\Rightarrow \|S\| = \max |G(j\omega)|$$

$$\|y\|_2 \leq \|u\|_2 \max |G(j\omega)| \Rightarrow \frac{\|y\|_2}{\|u\|_2} \leq \max |G(j\omega)|$$

5_ notch filter : $G_n(s) = \frac{s^2 + 2\zeta w_n s + w_n^2}{w_n^2} \times \frac{a w_n}{s + a w_n} \times \frac{\frac{w_n}{a}}{s + \frac{w_n}{a}}$ $\qquad w_n = 50 \;,\; a = \frac{51}{49}$

برای طراحی این فیلتر نیازمندیم که بازهٔ مشخص شده را حذف کند اما به سایر محتواهای فرکانسی آسیبی نزند. پس نیازمند قطب های هستیم تا پاسخ فرکانسی را به این نتیجه برساند. همچنین برای حالت ایده‌آن می‌توان $\zeta = 0$ درنظر گرفت.

$$G_n(s) = \frac{s^2 + w_n^2}{(s + a w_n)(s + \frac{w_n}{a})} = \frac{s^2 + 50^2}{(s + \frac{50 \times 51}{49})(s + \frac{50 \times 49}{51})}$$



6- $G(s) = \dfrac{w_n^2}{s(s+2\zeta w_n)}$ $\qquad T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{w_n^2}{s(s+2\zeta w_n)+w_n^2}$

a) $\xrightarrow[u=\frac{w}{w_n}]{s=wj} T = \dfrac{1}{ju(ju+2\zeta)+1} = \dfrac{1}{2\zeta uj + 1 - u^2} \Rightarrow |T| = \dfrac{1}{\left((1-u^2)^2+(2\zeta u)^2\right)^{\frac{1}{2}}} = 1 \Rightarrow u^4 + (4\zeta^2-2)u^2 = 0$

$\Rightarrow u^2 = 2-4\zeta^2 \Rightarrow u = \sqrt{2-4\zeta^2}$

$\Rightarrow w_{cc} = w_n\sqrt{2-4\zeta^2}$

$\angle G(jw_{-180}) = -180^\circ \longrightarrow w = \infty \quad \Rightarrow \quad G.M. = \infty$

$|G(w_c)| = 1 \longrightarrow w \simeq 9 \ \frac{rad}{s} \quad \Rightarrow \quad P.M. = 180 + \angle G(w_c) = 180 - 140 = 40^\circ$

b) greater G.M. $\Rightarrow$ better stability for closed-loop system

$\Rightarrow T(s) = \dfrac{kG(s)}{1+kG(s)} \quad (k \geqslant 1)$

$\rightarrow$ increasing K won't effect stability $\Rightarrow k \geqslant 1$

c) $e^{-\tau s}G(s) \longleftrightarrow g(t-\tau)$: delay

$D.M. = \dfrac{P.M.}{w_c} = \dfrac{40}{9} \times \left(\dfrac{\pi}{180}\right) \simeq 0.077\,s \quad \Rightarrow \quad 0 \leqslant \tau \leqslant 0.077$

d) $e_{ss} = \lim\limits_{s\to 0} \dfrac{s}{1+G(s)}\dfrac{1}{s} = \lim\limits_{s\to 0} \dfrac{s(s+2\zeta w_n)}{s(s+2\zeta w_n)+w_n^2} = \dfrac{0}{w_n^2} = 0$

e) $\zeta = \dfrac{P.M.}{100} = \dfrac{40}{100} = 0.4$

$P.O. = 100\, e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 25.38\ \%$

$w_c = w_n\sqrt{2-4\zeta^2} = 9 \Rightarrow w_n = 7.17$

$T_r = \dfrac{2.16\zeta + 0.6}{w_n} = 0.2$

$T_{s\,(2\%)} = \dfrac{4}{\zeta w_n} = 1.39$

f) $M_p = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.36$

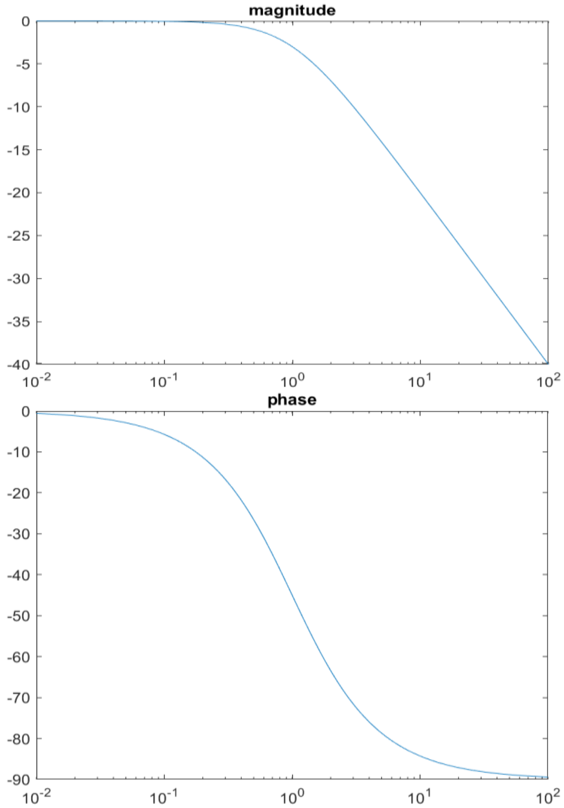
$W_r = w_n\sqrt{1-2\zeta^2} = 5.9$

$W_B = \left(\sqrt{1-2\zeta^2+\sqrt{4\zeta^4-4\zeta^2+2}}\right) w_n = 9.85$

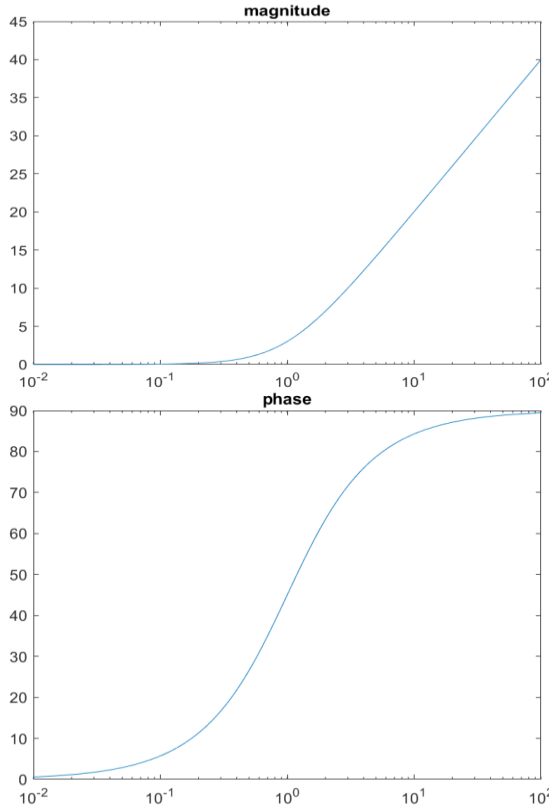
MATLAB Assignments

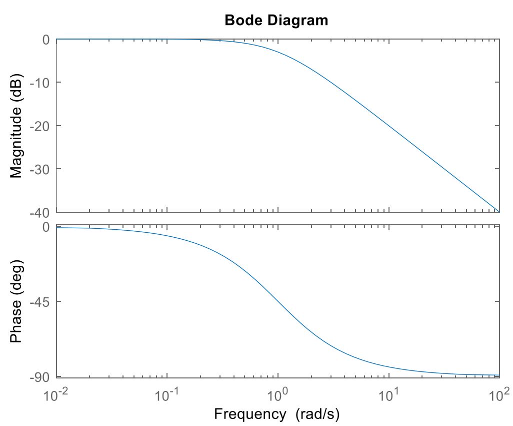
7 Bode Diagram Plot

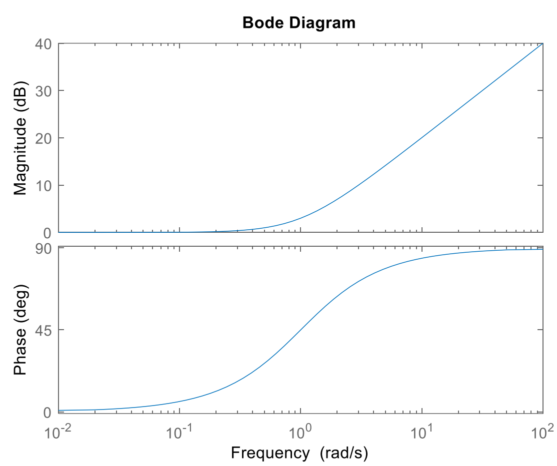
a) $G_1(s)$



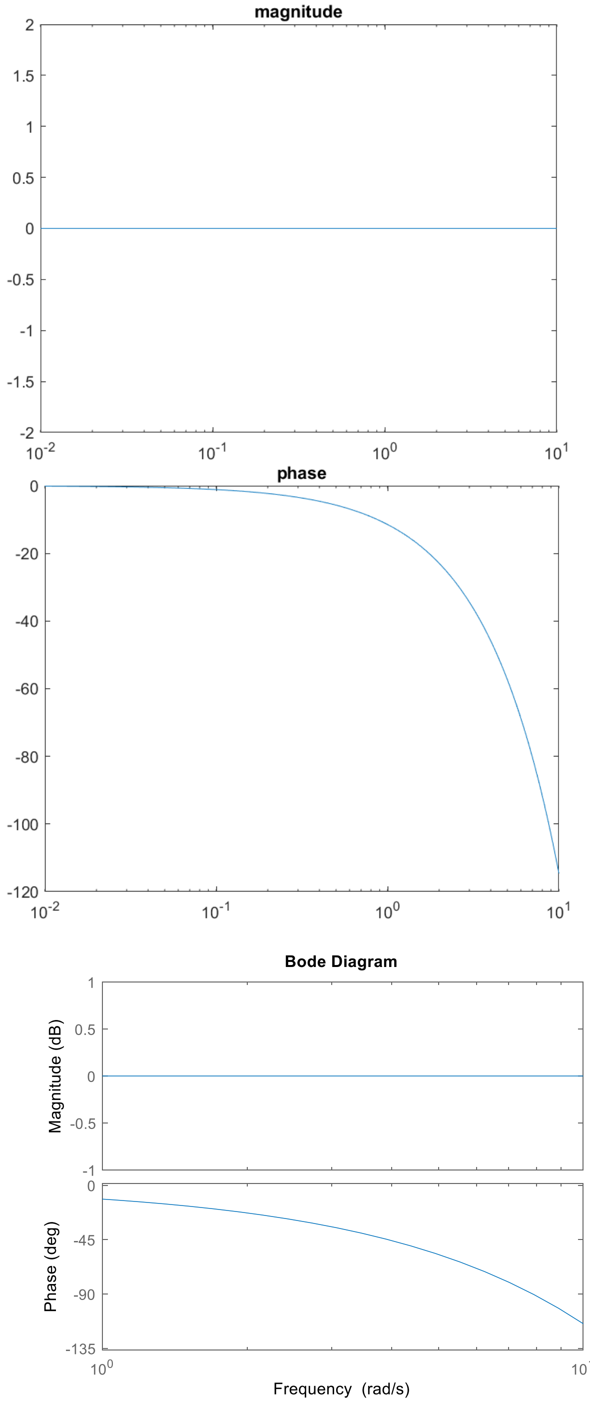
b) $G_2(s)$



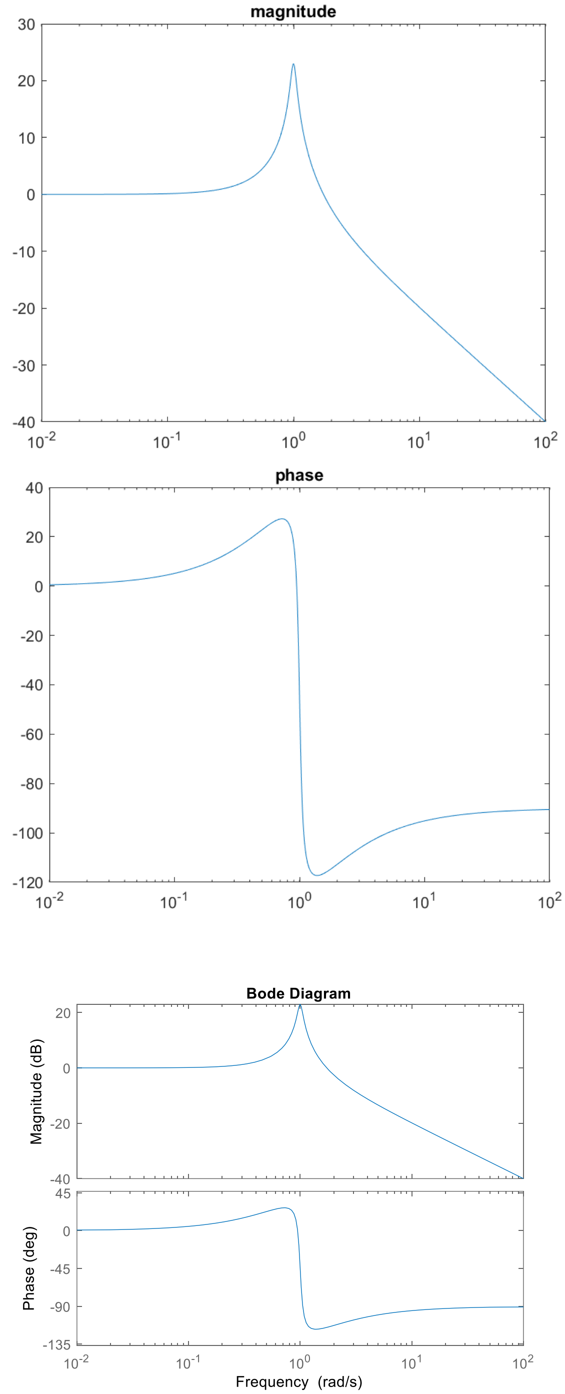




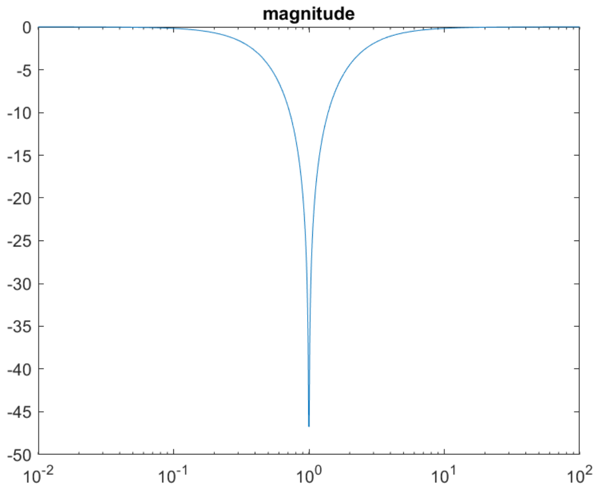
c) $G_3(s)$

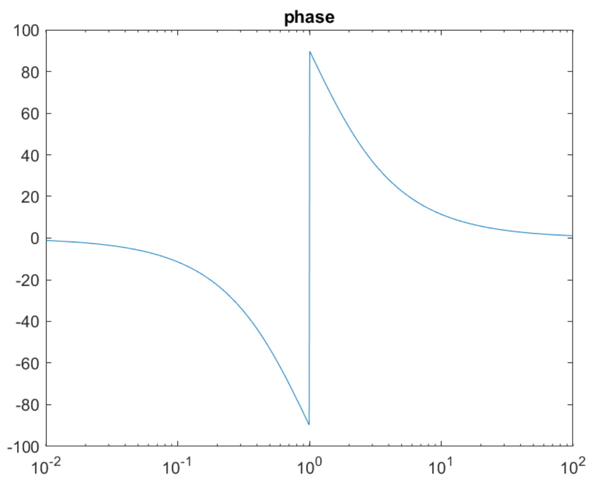

magnitude
phase
Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

d) $G_4(s)$

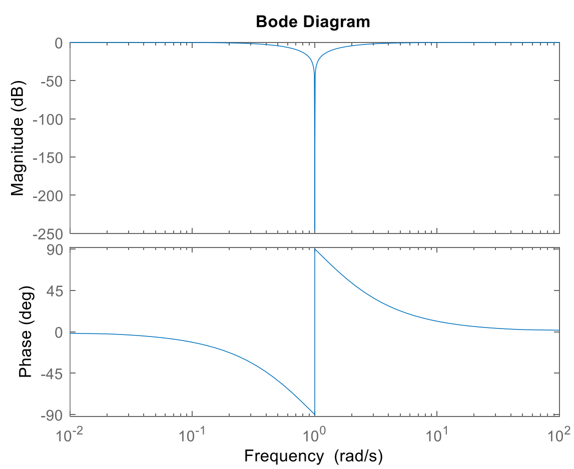

magnitude
phase
Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

e) $G_5(s)$





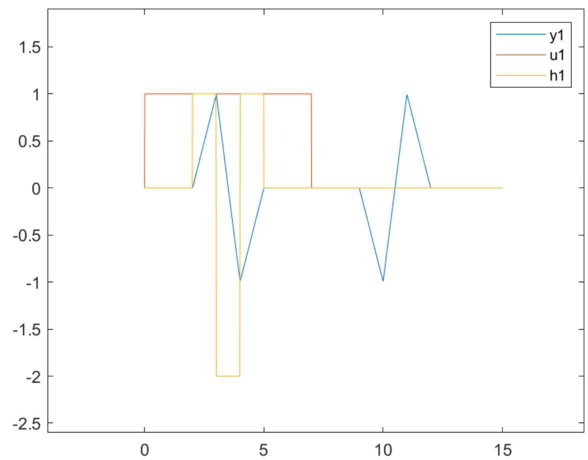

```
s = tf('s');
w = logspace(-2,2,1000);

G1 = 1/(s+1);
G2 = (s+1);
G3 = exp(-0.2*s);
G4 = (s+1)/(s^2+0.1*s+1);
G5 = (s^2+1)/(s+1)^2;


f = freqresp(G5,w);
magnitude = abs(f(1,:));
phase = angle(f(1,:));
figure
semilogx(w,20*log10(magnitude))
%ylim([-2,2])
title('magnitude')
figure
semilogx(w,(phase)*180/pi)
title('phase')

figure
bode(G5)
```



8 Convolution

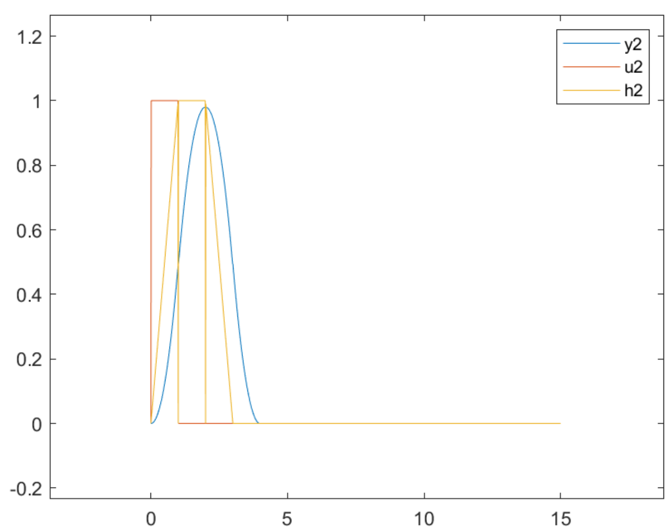
a)

$$h_1(t) = \begin{cases} 1 & 2 \le t < 3 \\ -2 & 3 \le t < 4 \\ 1 & 4 \le t < 5 \\ 0 & \text{O.W.} \end{cases} \qquad u_1(t) = \begin{cases} 1 & 0 \le t < 7 \\ 0 & \text{O.W.} \end{cases}$$



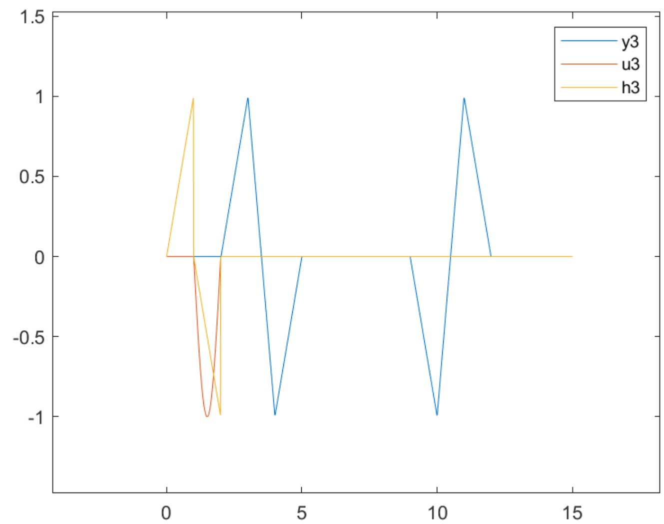
b)

$$h_2(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \\ 0 & \text{O.W.} \end{cases} \qquad u_2(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{O.W.} \end{cases}$$



c)

$$h_3(t) = \begin{cases} t & 0 \leq t < 1 \\ 1-t & 1 \leq t < 2 \\ 0 & \text{O.W.} \end{cases} \qquad u_3(t) = \begin{cases} \sin(\pi t) & 1 \leq t < 2 \\ 0 & \text{O.W.} \end{cases}$$




```
%% 8

clear; clc;

Ts = 0.01;
t = 0:Ts:15;
sz = size(t);

h1 = (2<t&t<3)*1 + (3<t&t<4)*-2 + (4<t&t<5)*1;
u1 = (0<t&t<7)*1;
y1 = conv(u1,h1)*Ts;
y1 = y1(1:sz(2));
plot(t,y1,t,u1,t,h1)
legend('y1','u1','h1')

h2 = (0<t&t<1).*t + (1<t&t<2)*1 + (2<t&t<3).*(3-t);
u2 = (0<t&t<1)*1;
y2 = conv(u2,h2)*Ts;
y2 = y2(1:sz(2));
%plot(t,y2,t,u2,t,h2)
%legend('y2','u2','h2')

h3 = (0<t&t<1).*t + (1<t&t<2).*(1-t);
u3 = (1<t&t<2).*sin(pi*t);
y3 = conv(u1,h1)*Ts;
y3 = y3(1:sz(2));
%plot(t,y3,t,u3,t,h3)
%legend('y3','u3','h3')
```

9 Impulse Response Truncation

First we find the transfer function of modified system and them we compare impulse response, step response and bode diagram of original system with modified one at 95 percentage for settling time.

9 – a) $T_1(s) = \frac{1}{2s+1}$

$$t_1(t)\,(u(t) - u(t-T_s)) = \frac{e^{-\frac{t}{2}}}{2}\,(u(t) - u(t-T_s))$$

$$u(t) \longleftrightarrow \frac{1}{s}$$

$$\frac{e^{-\frac{t}{2}}}{2}\,u(t) \longleftrightarrow \frac{1}{2}\,\frac{1}{(s+\frac{1}{2})}$$

$$u(t-T_s) \longleftrightarrow \frac{-1}{s}\,e^{-sT_s}$$

$$\frac{e^{-\frac{t}{2}}}{2}\,u(t-T_s) \longleftrightarrow \frac{1}{2}\,\frac{-1}{(s+\frac{1}{2})}\,e^{-(s+\frac{1}{2})T_s}$$

$$\Rightarrow \quad M_1(s) = \frac{1}{2s+1}\,(1 - e^{-sT_s}e^{-\frac{1}{2}T_s})$$

b) $T_2(s) = \frac{1}{s^2+s+1}$

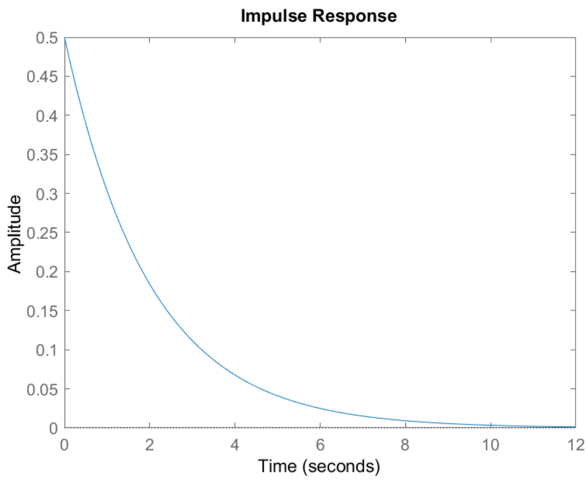
$$t_2(t)\,(u(t) - u(t-T_s)) = \frac{2}{\sqrt{3}}\,e^{-\frac{t}{2}}\,\sin(\frac{\sqrt{3}}{2}t)\,(u(t) - u(t-T_s)) = \frac{j}{\sqrt{3}}\,(e^{\frac{-1-j\sqrt{3}}{2}t} - e^{\frac{-1+j\sqrt{3}}{2}t})\,(u(t) - u(t-T_s))$$

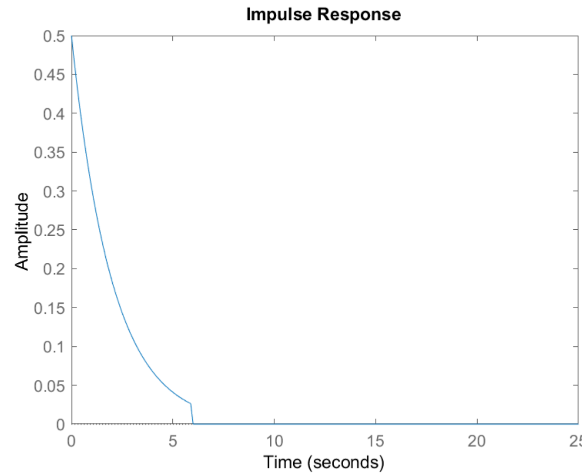
$$a = \frac{-1-j\sqrt{3}}{2}, \qquad b = \frac{-1+j\sqrt{3}}{2} \quad \Rightarrow \quad \frac{j}{\sqrt{3}}\,(e^{at} - e^{bt})\,(u(t) - u(t-T_s))$$

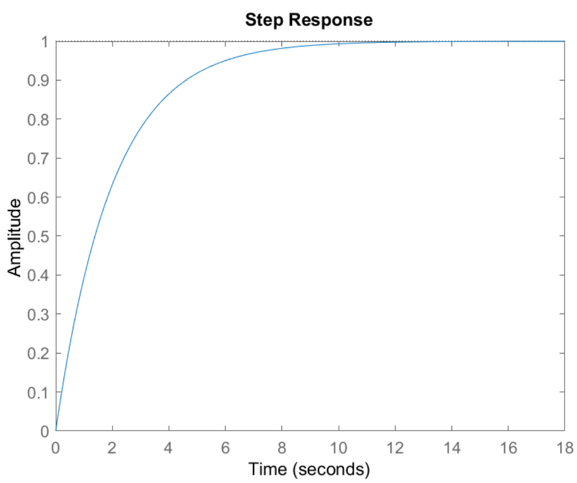
$$M_2(s) = \frac{j}{\sqrt{3}}\left[\frac{1}{s-a} - \frac{1}{s-b} - \frac{1}{(s-a)}\,e^{-(s-a)T_s} + \frac{1}{(s-b)}\,e^{-(s-b)T_s}\right]$$

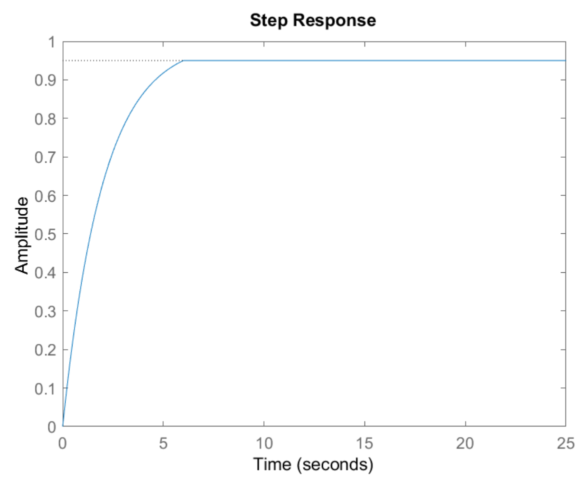
➔ $G_1(s)$

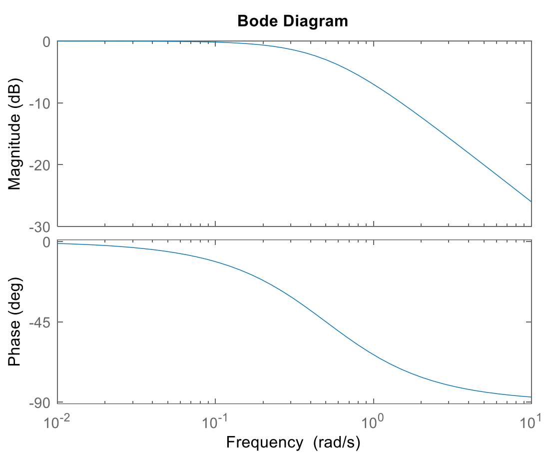
(original) (modified)

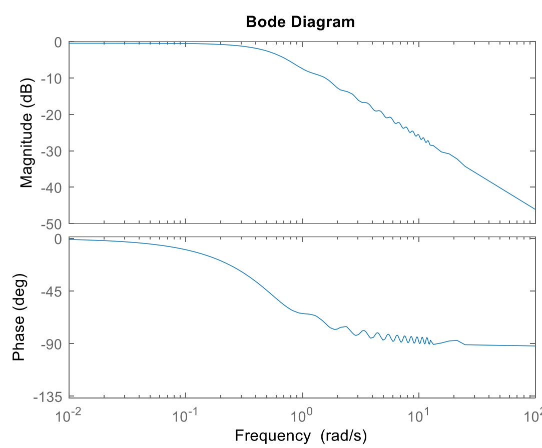






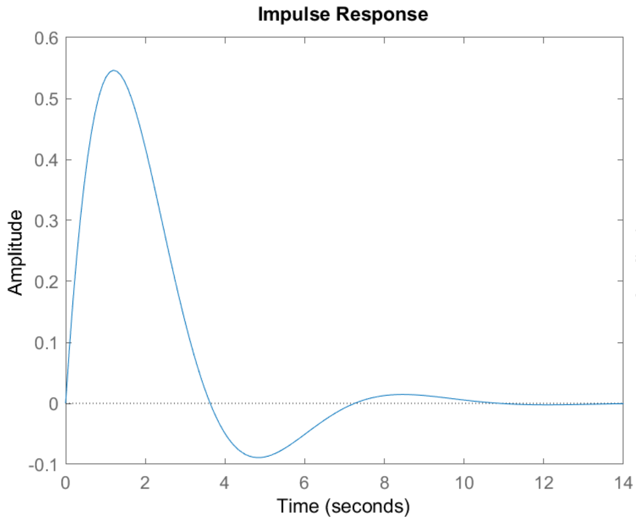


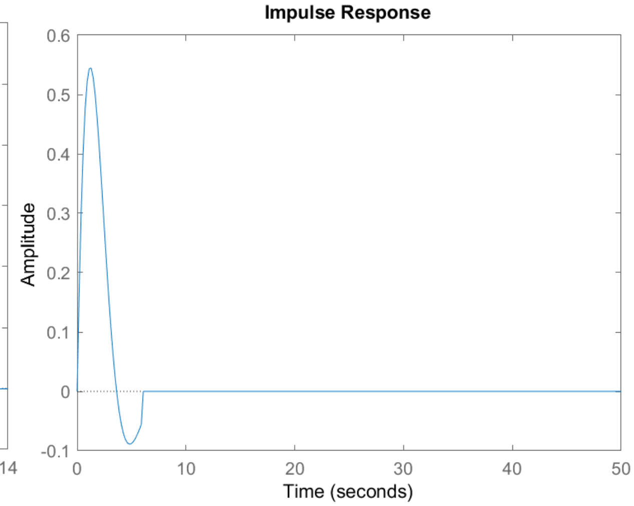


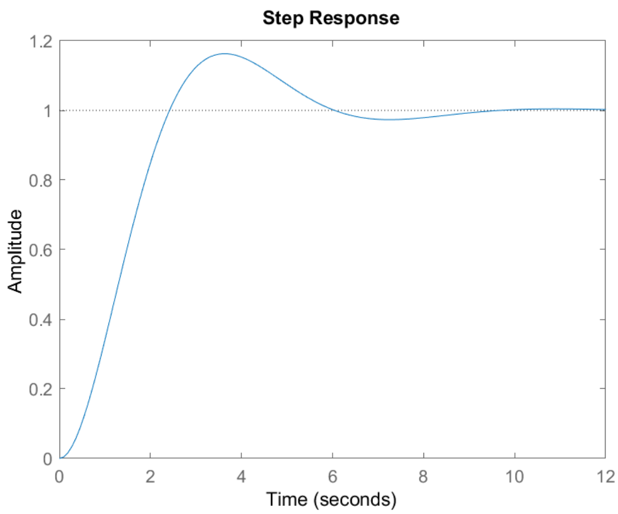


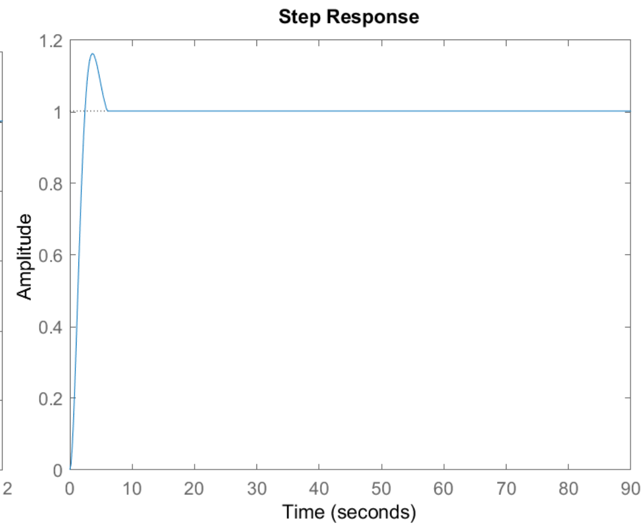
➔ $G_2(s)$

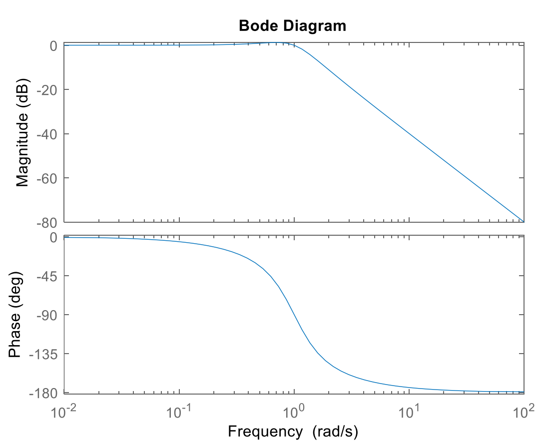
(original) (modified)

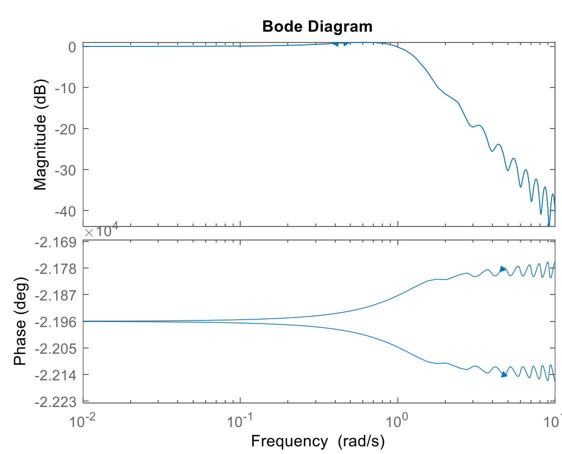








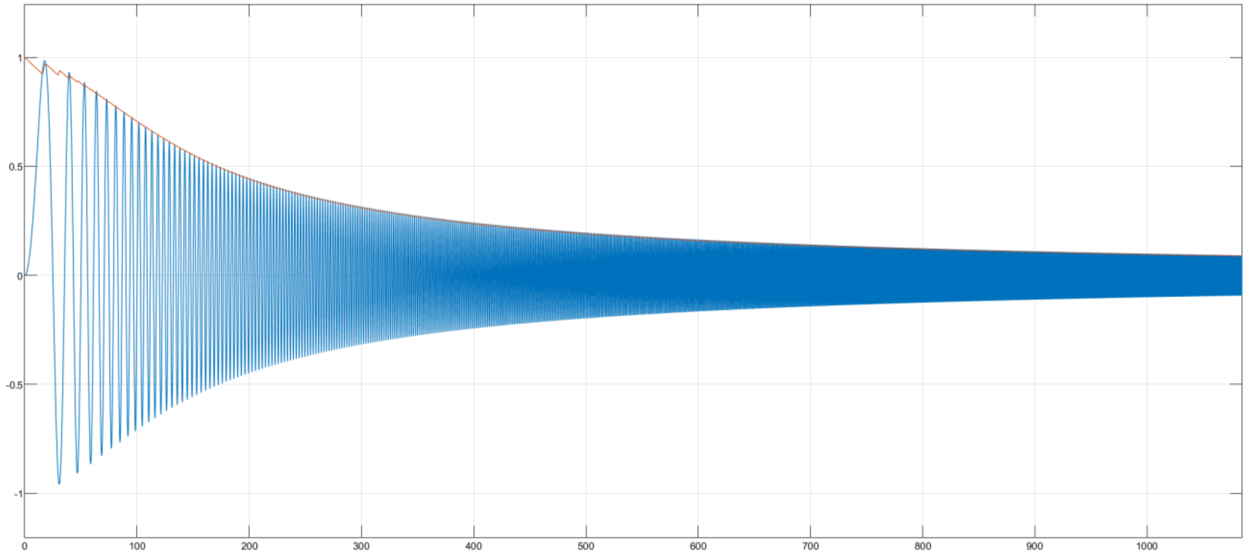


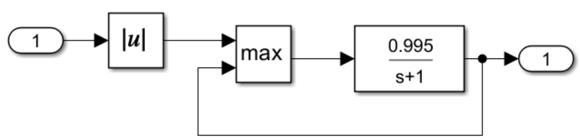


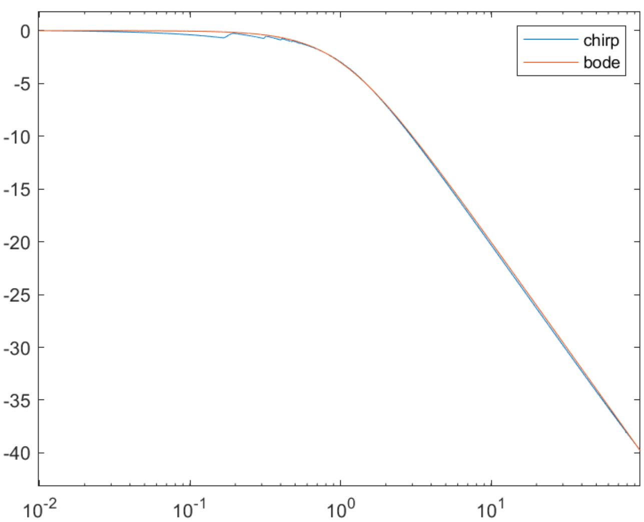
```
%% 9

clear;  clc;

s = tf('s');
tau1 = 3*2;
tau2 = 3/(0.5*1);

T1 = 1/(2*s+1);
T2 = 1/(s^2+s+1);

G1 = 1/(2*s+1) * (1-exp( -tau1*s )*exp( -tau1*0.5 ));
a = (-1-sqrt(3)*1i)/2;
b = (-1+sqrt(3)*1i)/2;
G2 = 1i/sqrt(3) * ( 1/(s-a) - 1/(s-b) - 1/(s-a)*exp(a*tau2)*exp(-tau2*s) +
1/(s-b)*exp(b*tau2)*exp(-tau2*s) );

figure
impulse(T1)
figure
step(T1)
figure
bode(T1)
```

10 Chirp Signal and Frequency Response

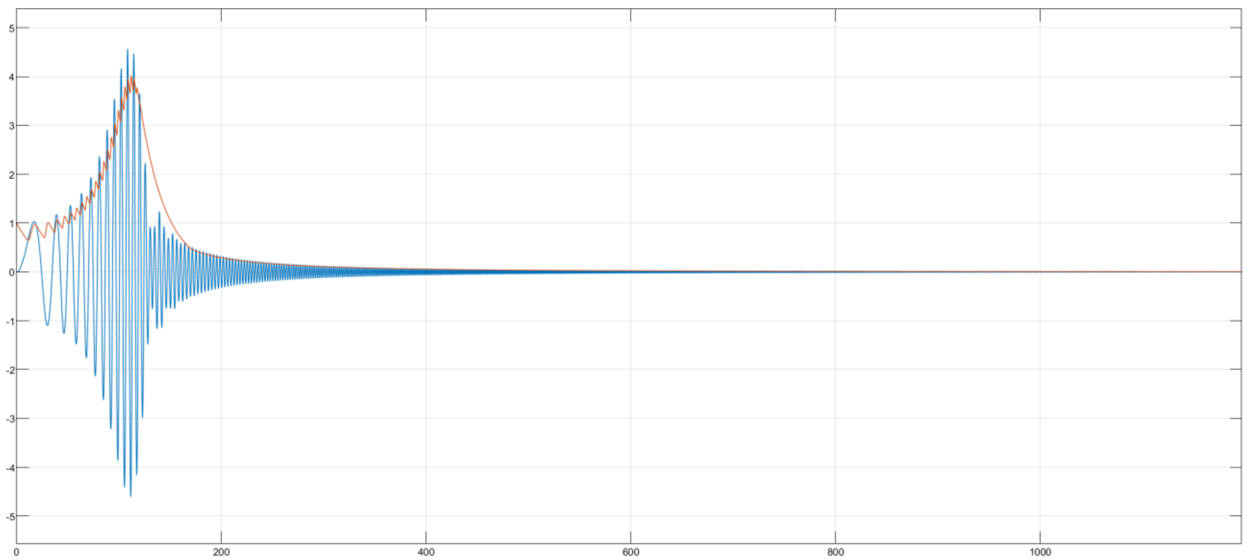
a)



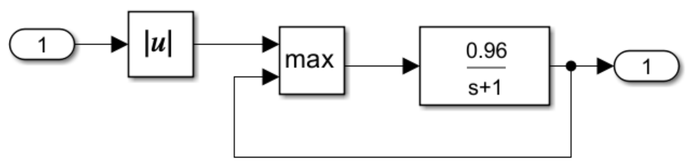




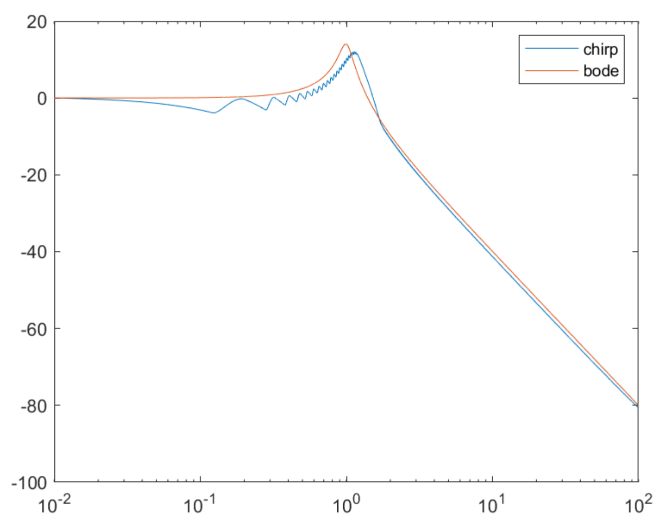
b)



5
4
3
2
1
0
-1
-2
-3
-4
-5
0
200
400
600
800
1000

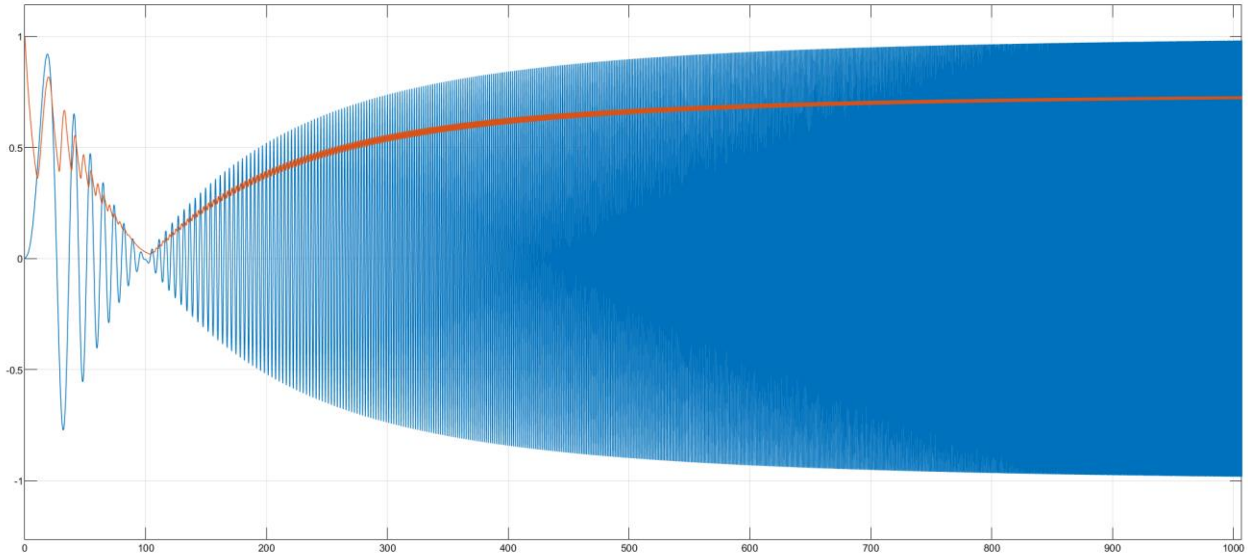


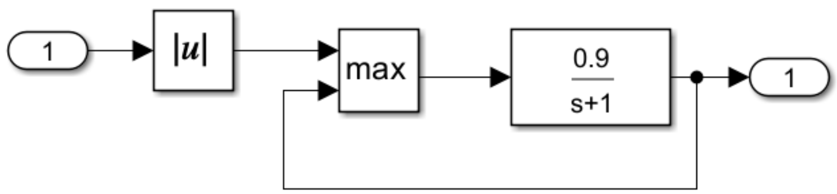
1
|u|
max
0.96
s+1
1

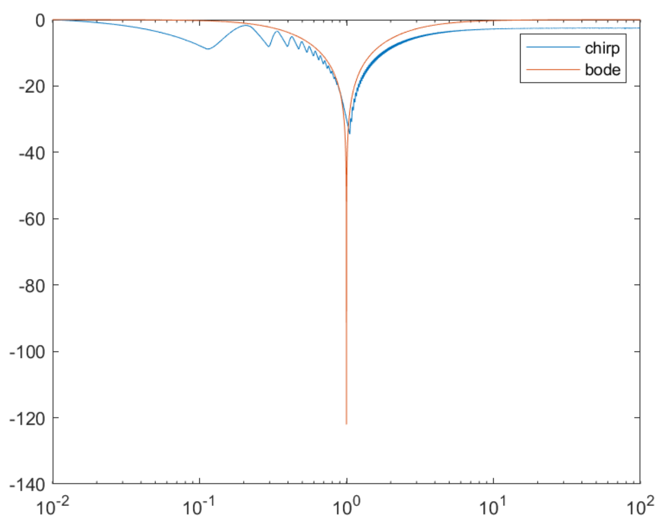


chirp
bode
20
0
-20
-40
-60
-80
-100
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

c)







In this part we have a notch filter that for a better result we have a trade off in increasing or decreasing the capacitor value.

By decreasing transfer function numerator in detector we will have better elimination of our certain frequency otherwise by increasing it we will see smoother plot specially in other frequencies. So maybe another detector will work better.