1)

1. $G_1(s) = \dfrac{5s+5}{s^4+3s^3+3s^2+3s+2}$

$$\begin{array}{c|ccc}
s^4 & 1 & 3 & 2 \\
s^3 & 3 & 3 & 0 \\
s^2 & 2 & 2 & 0 \\
s^1 & \cancel{0}\ 4 & \cancel{0}\ 0 & \cancel{0}\ 0 \\
s^0 & 2 & 0 & 0
\end{array}
\quad \longrightarrow a(s) = 2s^2+2 \qquad \frac{da(s)}{ds} = 4s$$

marginally stable

2. $G_2(s) = \dfrac{8s+1}{s^4+3s^3+3s^2+9s+1}$

$$\begin{array}{c|ccc}
s^4 & 1 & 3 & 1 \\
s^3 & 3 & 9 & 0 \\
s^2 & \epsilon & 1 & 0 \\
s^1 & C & 0 & 0 \\
s^0 & 1 & 0 & 0
\end{array}
\qquad C = \frac{9\epsilon-3}{\epsilon} \qquad \lim_{\epsilon \to 0^+} C = -\infty$$

unstable (two unstable poles)

3. $G_3(s) = \dfrac{4}{s^4+4s^3+3s^2-4s-4}$

$$\begin{array}{c|ccc}
s^4 & 1 & 3 & -4 \\
s^3 & 4 & -4 & 0 \\
s^2 & 4 & -4 & 0 \\
s^1 & \cancel{0}\ 8 & \cancel{0}\ 0 & \cancel{0}\ 0 \\
s^0 & -4 & 0 & 0
\end{array}
\quad \longrightarrow a(s) = 4s^2-4 \qquad \frac{da(s)}{ds} = 8s$$

unstable (one unstable pole)

4. $G_4(s) = \dfrac{2s-1}{s^5+2s^4+2s^3+4s^2+11s+10}$

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \epsilon & 6 & 0 \\
s^2 & C & 10 & 0 \\
s^1 & d & 0 & 0 \\
s^0 & 10 & 0 & 0
\end{array}
\qquad C = \frac{4\epsilon-12}{\epsilon} \qquad \lim_{\epsilon \to 0^+} C = -\infty \qquad d = \frac{6C-10\epsilon}{C} \qquad \lim_{C \to -\infty} d = 6$$

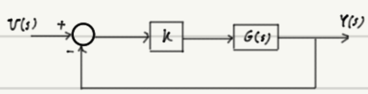
unstable (two unstable poles)

5. $G_5(s) = \dfrac{2}{s^6+7s^5+20s^4+30s^3+25s^2+11s+2}$

$$\begin{array}{c|cccc}
s^6 & 1 & 20 & 25 & 2 \\
s^5 & 7 & 30 & 11 & 0 \\
s^4 & \frac{110}{7} & \frac{164}{7} & 2 & 0 \\
s^3 & 19.56 & 10.1 & 0 & 0 \\
s^2 & 15.3 & 2 & 0 & 0 \\
s^1 & 7.55 & 0 & 0 & 0 \\
s^0 & 2 & 0 & 0 & 0
\end{array}$$

stable

2) $G(s) = \dfrac{2s^3+s^2+s-1}{s^3-5s^2+2s+8}$



$$\frac{Y(s)}{U(s)} = \frac{KG(s)}{1+KG(s)} \quad \Rightarrow 1+KG(s) = 0 \quad \Rightarrow \frac{(2k+1)s^3+(k-5)s^2+(k+2)s-k+8}{s^3-5s^2+2s+8} = 0$$

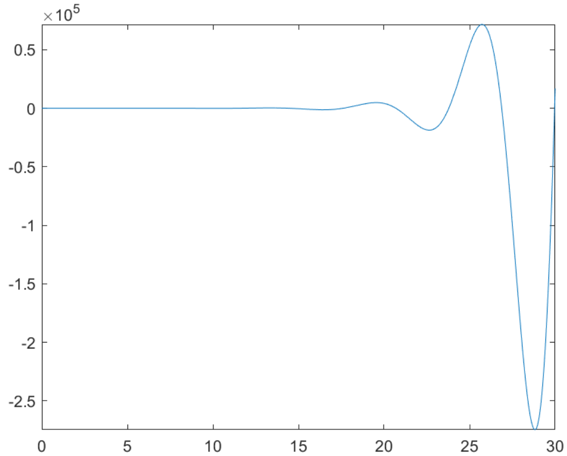
$$\begin{array}{c|cc}
s^3 & 2k+1 & k+2 \\
s^2 & k-5 & -k+8 \\
s^1 & C & 0 \\
s^0 & -k+8 & 0
\end{array}$$

$$C = \frac{(2k+1)(-k+8)-(k+2)(k-5)}{-(k-5)} = \frac{3(k^2-6k-6)}{k-5}$$

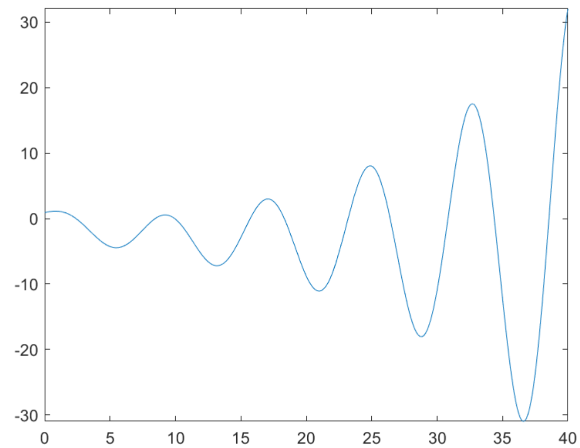
$$\left.\begin{array}{l}
2k+1>0 \longrightarrow k > \frac{-1}{2} \\
k-5>0 \longrightarrow k>5 \\
C>0 \longrightarrow k>3+\sqrt{15} \ \cup \ 3-\sqrt{15}<k<5 \\
-k+8>0 \longrightarrow k<8
\end{array}\right\} \longrightarrow \underbrace{3+\sqrt{15}}_{\approx 6.87} < k < 8$$

2 Range of a Gain for Closed-Loop Stability

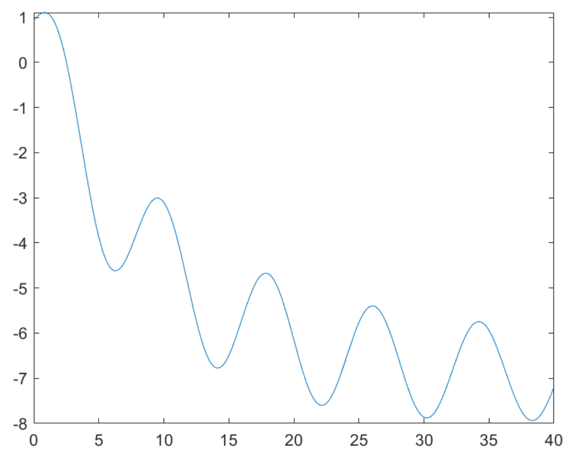
For different values of k we have these step responces:



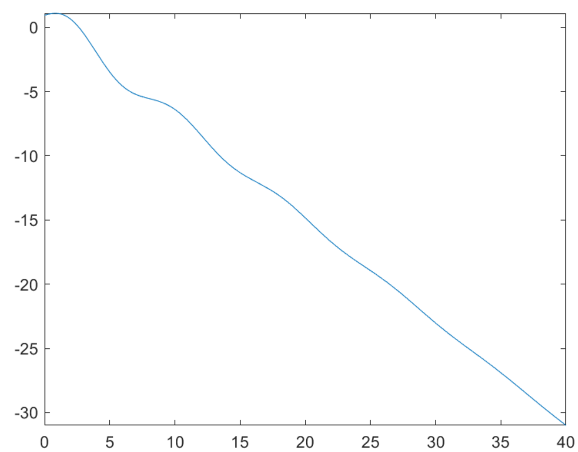
k = 3



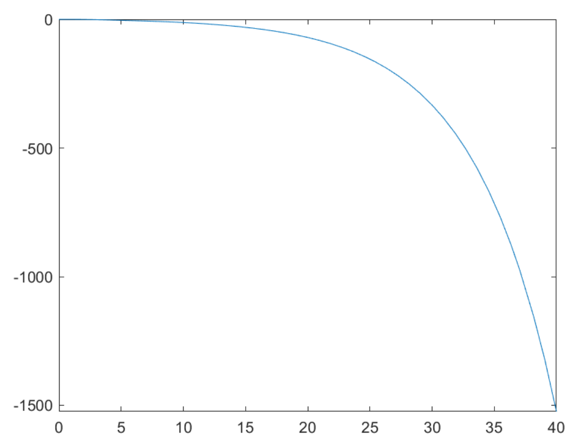
k = 6



k = 7



k = 8



k = 10

3_ a) $y(t) = \int_{-\infty}^{t} u(t-\tau)\, h(\tau)\, d\tau$

$$\xrightarrow{t=T_s} \|y\|_\infty = y(T_s) = \int_0^{T_s} u(T_s-\tau)\, h(\tau)\, d\tau \leqslant \int_0^{T_s} |u(T_s-\tau)|\,|h(\tau)|\, d\tau \leqslant \int_0^{\infty} |u(T_s-\tau)|\,|h(\tau)|\, d\tau \leqslant \int_0^{\infty} \underset{\downarrow}{\|u\|_\infty}\, |h(\tau)|\, d\tau \leqslant \|u\|_\infty \int_0^{\infty} |h(\tau)|\, d\tau$$

از همه مقادیر $|u|$ بزرگتر است

b) $u = a\, \mathrm{sgn}(h(t))\, u(t)$ $\quad (a>0)$

$$\Rightarrow \|y\|_\infty = \|u\|_\infty \int_0^\infty |h(\tau)|\, d\tau \quad \Rightarrow \quad \|S\| = \int_0^\infty |h(\tau)|\, d\tau$$

c) 1) $S_1 : G_1(s) = \frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} e^{-t} \longrightarrow \|S\| = \int_0^\infty |e^{-\tau}|\, d\tau = \int_0^\infty e^{-\tau}\, d\tau = -e^{-\tau}\Big|_0^\infty = 1 \longrightarrow$ stable

2) $S_2 : G_2(s) = \frac{1}{s-1} \xrightarrow{\mathcal{L}^{-1}} e^{t} \longrightarrow \|S\| = \int_0^\infty |e^{\tau}|\, d\tau = \int_0^\infty e^{\tau}\, d\tau = e^{\tau}\Big|_0^\infty = \infty \longrightarrow$ unstable

3) $S_3 : G_3(s) = \frac{1}{s^2+1} \xrightarrow{\mathcal{L}^{-1}} \sin(t) \longrightarrow \|S\| = \int_0^\infty |\sin(\tau)|\, d\tau = \frac{\sin(\tau)}{|\sin(\tau)|} \int_0^\infty \sin(\tau)\, d\tau = \frac{-\cos(\tau)\,\sin(\tau)}{|\sin(\tau)|}\Big|_0^\infty = \ ? \longrightarrow$ unstable

4) $S_4 : G_4(s) = \frac{1}{(s+1)^2} \xrightarrow{\mathcal{L}^{-1}} te^{-t} \longrightarrow \|S\| = \int_0^\infty |\tau e^{-\tau}|\, d\tau = \int_0^\infty e^{-\tau}|\tau|\, d\tau = -e^{-\tau}|\tau|\Big|_0^\infty + \int_0^\infty \frac{\tau e^{-\tau}}{|\tau|}\, d\tau$

$= -e^{-\tau}|\tau|\Big|_0^\infty - \frac{\tau e^{-\tau}}{|\tau|}\Big|_0^\infty = \frac{-(\tau^2+\tau)\,e^{-\tau}}{|\tau|}\Big|_0^\infty = 1 \longrightarrow$ stable

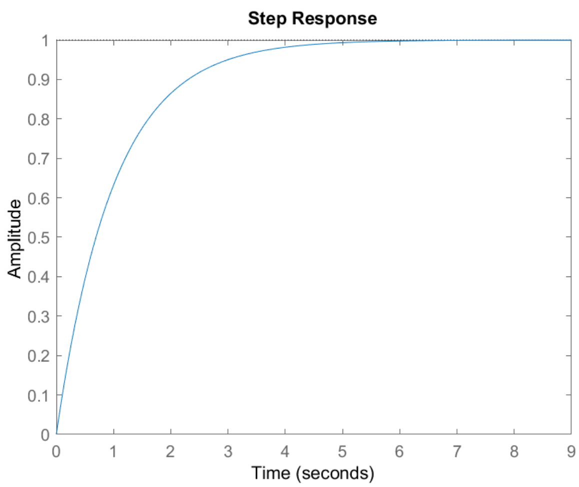
5) $S_5 : h_5(t) = \frac{1}{t+1} \longrightarrow \|S\| = \int_0^\infty \left|\frac{1}{\tau+1}\right| d\tau = \frac{\tau+1}{|\tau+1|} \int_0^\infty \frac{1}{\tau+1}\, d\tau = \frac{\tau+1}{|\tau+1|} \ln(\tau+1)\Big|_0^\infty = \infty \longrightarrow$ unstable

6) $S_6 : h_6(t) = \frac{1}{t^2+1} \longrightarrow \|S\| = \int_0^\infty \left|\frac{1}{\tau^2+1}\right| d\tau = \int_0^\infty \frac{1}{\tau^2+1}\, d\tau = \arctan(\tau)\Big|_0^\infty = \frac{\pi}{2} \longrightarrow$ stable

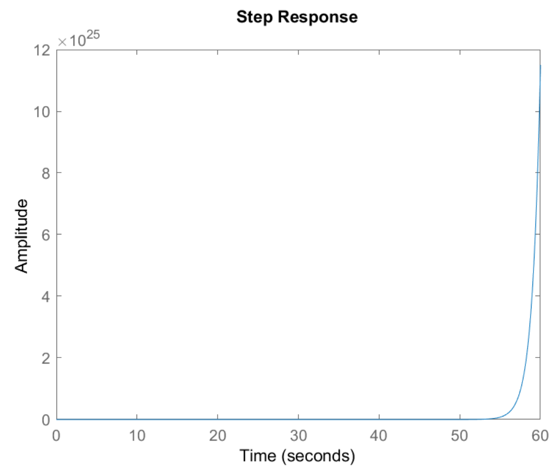
3 Evaluating Stability Condition Using Impulse Response

c)

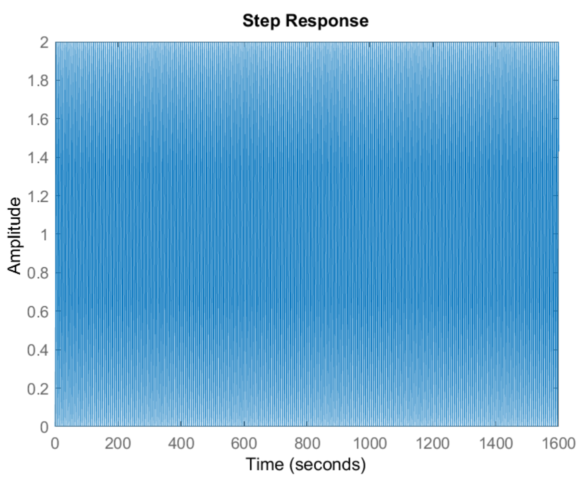
1)



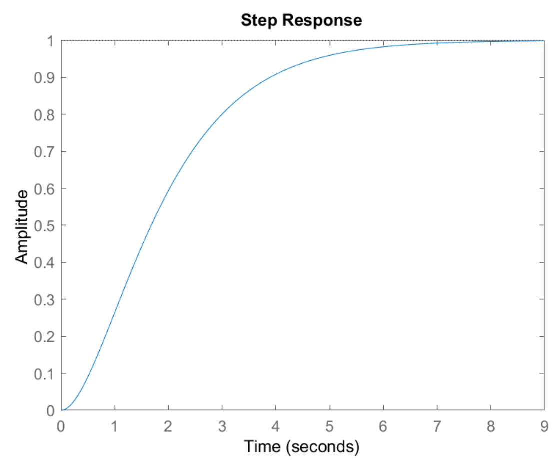
2)



3)



4)



5)

Input

$$\int_0^\infty \frac{1}{1+(t-T)}\,dT$$

Result

(integral does not converge)

($u(t)*h_5(t)$)

6)

Definite integral

$$\int_0^\infty \frac{1}{1+(t-T)^2}\,dT = \tan^{-1}(t) + \frac{\pi}{2}$$

($u(t)*h_6(t)$)

4) a) $\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} < 0.02 \Rightarrow T_s = \frac{\ln(0.02\sqrt{1-\zeta^2})}{-\zeta w_n}$

$0<\zeta<1 \rightarrow \zeta^2 \ll 1$

$\rightarrow T_s = \frac{4}{\zeta w_n}$

$\zeta = 1 \Rightarrow Y(s) = \frac{w_n^2}{s(s^2+2w_n s+w_n^2)} = \frac{w_n^2}{s(s+w_n)^2} = \frac{1}{s} - \frac{1}{s+w_n} - \frac{w_n}{(s+w_n)^2} \longrightarrow y(t) = 1 - e^{-w_n t} - w_n t e^{-w_n t}$

$\Rightarrow e^{-w_n t} + w_n t e^{-w_n t} < 0.02 \Rightarrow \ln(1+w_n T_s) - w_n T_s = 4 \Rightarrow w_n T_s \simeq 6 \Rightarrow T_s = \frac{6}{w_n}$

b) $M_p = \max(y(t)) - 1 = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

$Y(s) = \frac{w_n^2}{s(s^2+2w_n s+w_n^2)} \Rightarrow sY(s) = \frac{w_n^2}{s^2+2w_n s+w_n^2}$

$\Rightarrow \dot{y}(t) = \mathcal{L}^{-1}\{sY(s)\} = \frac{w_n}{\sqrt{1-\zeta^2}} e^{-\zeta w_n t} \sin(w_n\sqrt{1-\zeta^2}\, t) = 0 \Rightarrow t = \frac{n\pi}{w_n\sqrt{1-\zeta^2}} \Rightarrow T_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}}$

$y(T_p) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \sin(\pi + \cos^{-1}\zeta)$

$\zeta \to 0 \Rightarrow \begin{cases} \cos^{-1}\zeta \to \frac{\pi}{2} \\ \sin(\pi+\cos^{-1}\zeta) \longrightarrow -1 \\ \frac{1}{\sqrt{1-\zeta^2}} \longrightarrow 1 \end{cases}$

$\rightarrow \max(y(t)) = 1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \Rightarrow M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$

for not having overshoot the poles must be real-valued

for fastest response we should make dominant pole as fast as possible

so we want two real-valued poles at the same location $\Rightarrow \zeta = 1 \longrightarrow s_{1,2} = -W_n$

$\zeta = 1 \longrightarrow G(s) = \frac{w_n^2}{(s+w_n^2)}$

MATLAB Assignments

5 Curve Fitting Toolbox

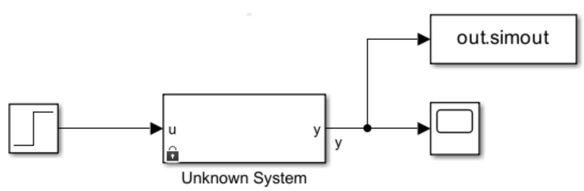
a)

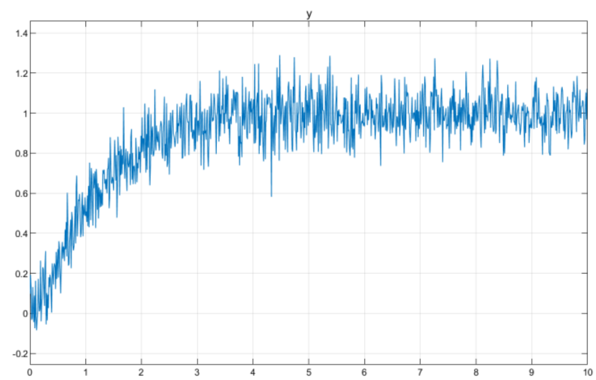
$$G(s) = \frac{K}{\tau s+1} \quad , \quad U(s) = \frac{1}{s}$$

```
%% 5_a

syms K T s
step_res1 = ilaplace(K/(T*s+1) * (1/s));
```

```
val =

K - K*exp(-t/T)
```

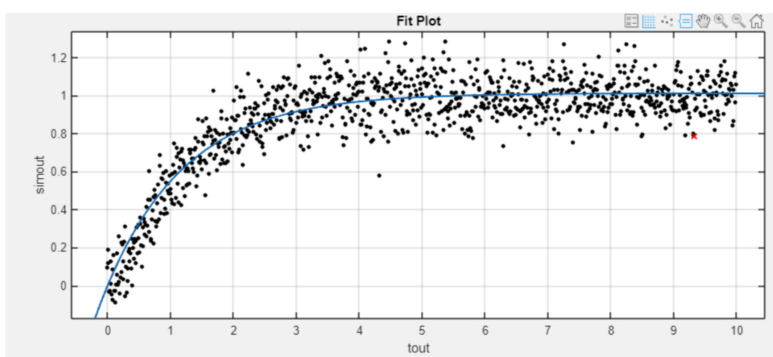
b)





c)

```
General model:
    f(x) = a*(1-exp(-b*x))
Coefficients (with 95% confidence bounds):
      a =       1.012  (1.003, 1.021)
      b =      0.7834  (0.7488, 0.818)
```



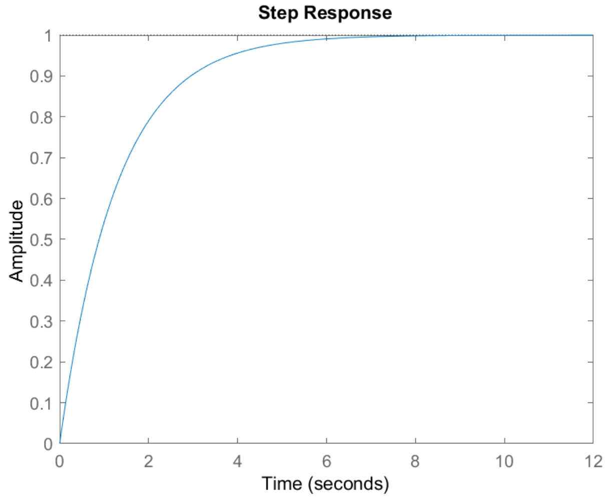
From data we have:

$f(x) = a\,(1- e^{-bx}) = 1.012\,(1- e^{-0.7834x}) \approx 1\,(1- e^{-0.78x})$

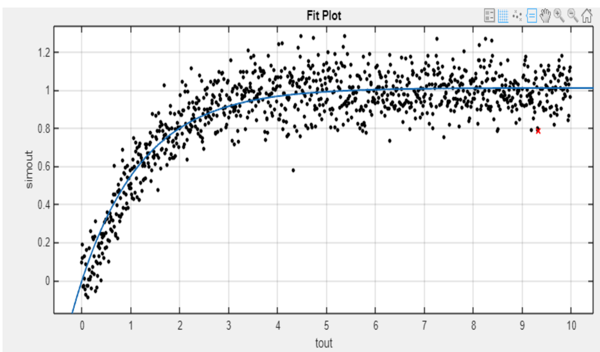
compare to result in part a:

$$K = 1 \quad , \quad \tau = \frac{1}{0.78} = 1.28$$

d)



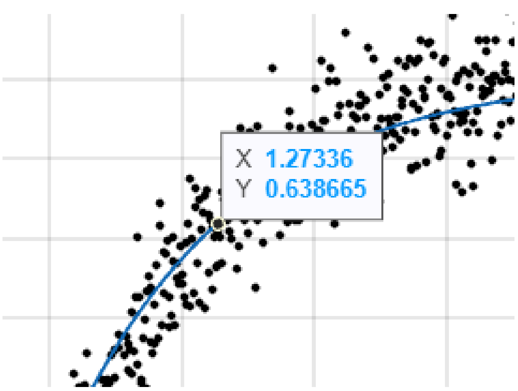

```
%% 5_d

K2 = 1;
T2 = 1.28;

step( tf([K2],[T2,1]) )
```



As we see the result of step response for our system is similar to the output from the unknown system.

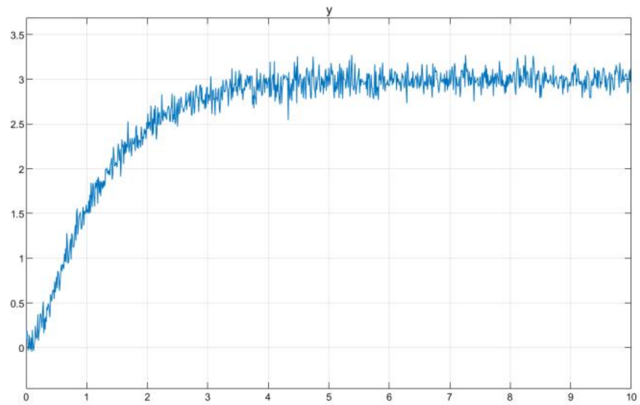
e) For the DC gain we should consider the value of response as the time goes to infinity. We can say that DC gain could be a little more than one (because of a). so 1 is a proper estimation for K (DC gain)

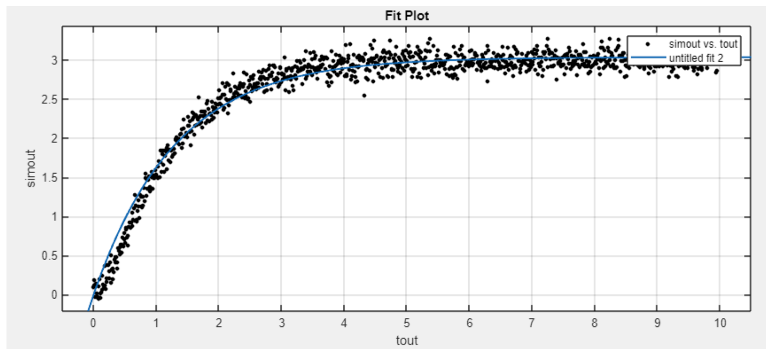
We know: $1\text{-}e^{-1} = 0.632$, so the time constant could be estimated to be the time when the curve reaches to 63.2% of its maximum value.



As we see the value of time constant that we've estimated (1.28) could be a proper value for it.

f) We set the input amplitude on 3.



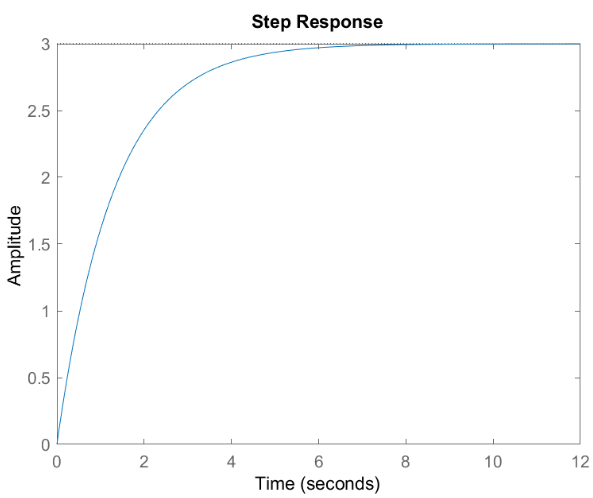


General model:
    f(x) = a*(1-exp(-b*x))
Coefficients (with 95% confidence bounds):
    a = 3.034 (3.023, 3.044)
    b = 0.769 (0.7552, 0.7827)

$f(x) = a\ (1- e^{bx}\ )= 3.034\ (1- e^{-0.769x}\ ) \approx 3\ (1- e^{-0.77x})$

compare to results in part a:

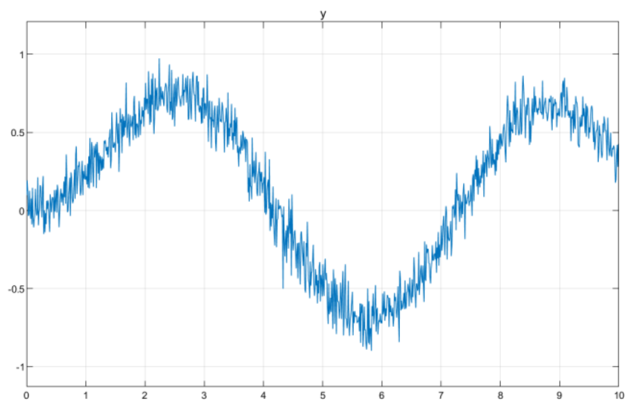
$K = 3 \quad , \quad \tau = \frac{1}{0.77} = 1.3$




```
%% 5_f

K3 = 3;
T3 = 1.3;

step( tf([K3],[T3,1]) )
```

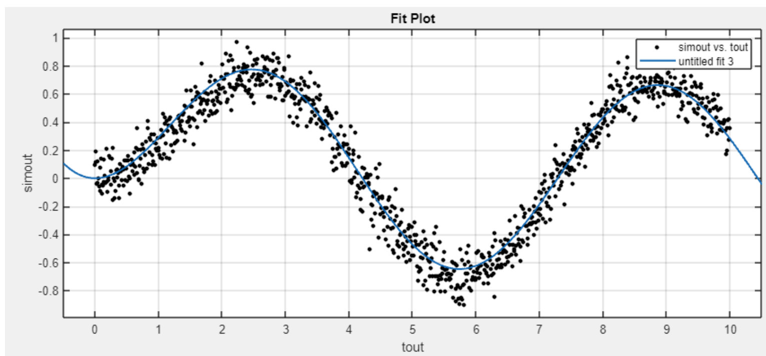
As we see the result of step response for our system is similar to the output from the unknown system.

g) input : sin(t)




```
val =

(K*sin(t) - K*T*cos(t))/(T^2 + 1) + (K*T*exp(-t/T))/(T^2 + 1)
```

L{ sin(t) } = $\frac{1}{s^2+1}$ => response:

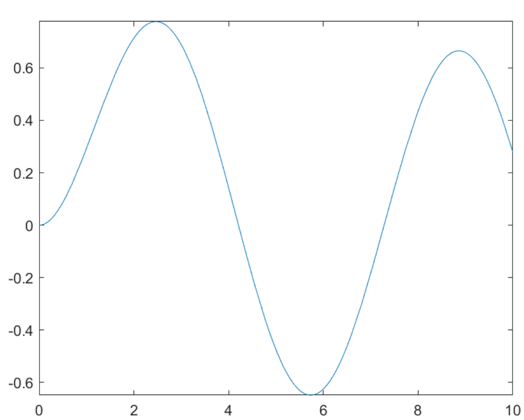


General model:
f(x) = (K*sin(x)-K*T*cos(x)+K*T*exp(-x/T))/(T^2+1)
Coefficients (with 95% confidence bounds):
K = 1.238 (1.21, 1.266)
T = 1.582 (1.535, 1.63)

$$f(x) = \frac{1}{T^2+1}\left(K\sin(t) - KT\cos(t) + KT\, e^{-\frac{t}{T}}\right)$$

$$= \frac{1}{(1.582)^2+1}\left(1.238\sin(t) - 1.238 \times 1.582\cos(t) + 1.238 \times 1.582\, e^{-\frac{t}{1.582}}\right)$$

$$\approx 0.35\sin(t) - 0.56\cos(x) + 0.56\, e^{-0.63x} \qquad (K = 1.24 \;,\; T = 1.58)$$

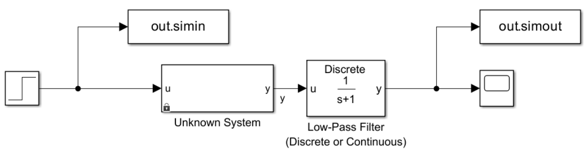


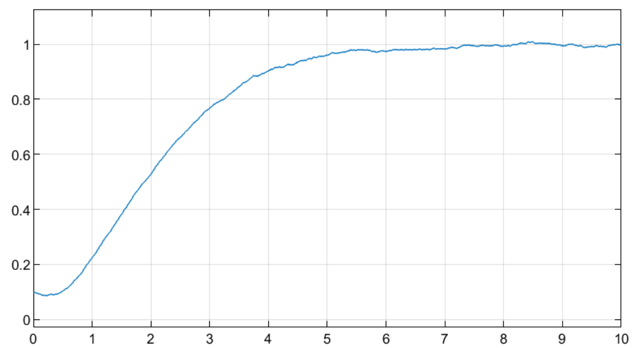
```
%% 5_g_2

K4 = 1.24;
T4 = 1.58;

fplot( ilaplace( (K4/(T4*s+1))*(1/(s^2+1))
) )
xlim([0 10])
```

As we see the result of step response for our system is similar to the output from the unknown system.

h) Increasing data length means to have more data to calculate the proper equation for the output curve. This also helps us to understand the steady state behavior of the response better. So we can say that increasing data length will increase the accuracy of estimating output equation and coefficients.

i)





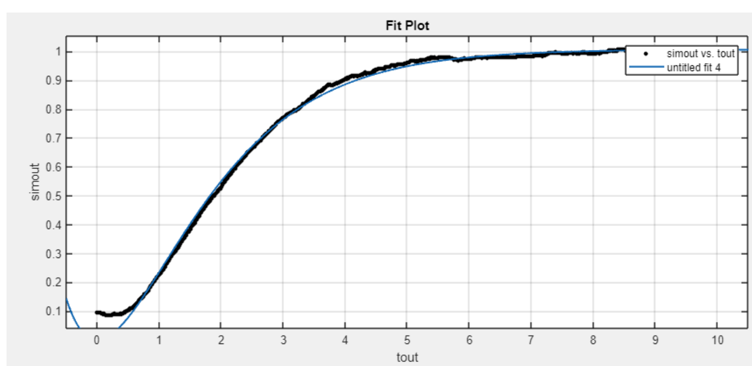
Time constant: a = 1 => $f_C = \frac{1}{2\pi a} = 0.16\ Hz$

G(s)F(s) = $\frac{k}{\tau s+1}\frac{1}{as+1}$

=> step response:

```
val =

K - (K*T*exp(-t/T))/(T - a) + (K*a*exp(-t/a))/(T - a)
```

General model:
f(x) = K+(-K*T*exp(-x/T)+K*exp(-x))/(T-1)
Coefficients (with 95% confidence bounds):
K = 1.006 (1.004, 1.008)
T = 1.186 (1.174, 1.198)



K ≈ 1 , τ ≈ 1.18 => G(s) = $\frac{1}{1.18s+1}$

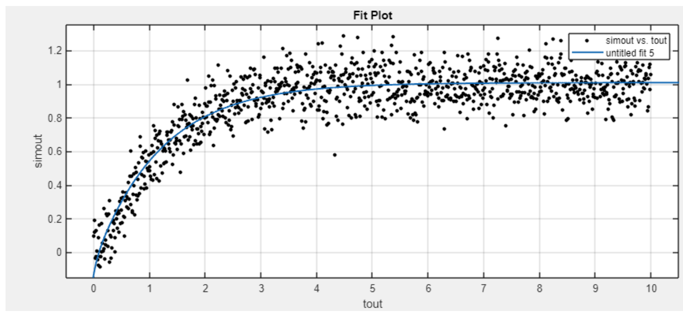
j)

Answers from different inputs have near values for K and τ. Although the values we got from sinusoidal input were a little different from the others.

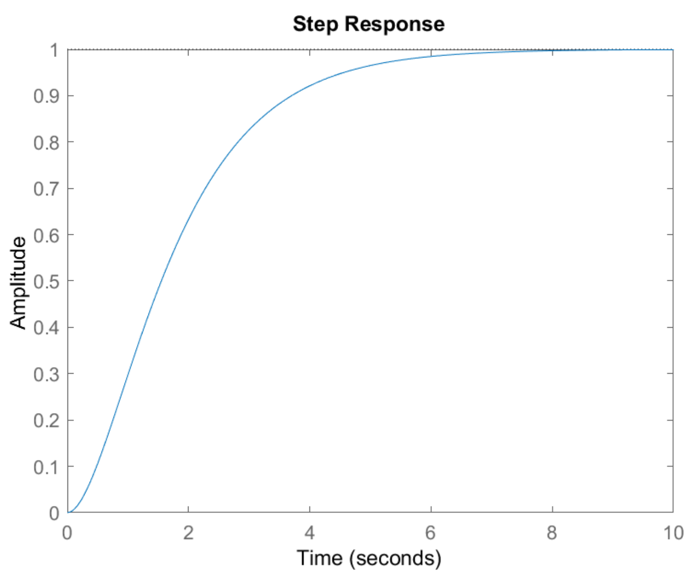
k)

```
val =

K - (K*T1*exp(-t/T1))/(T1 - T2) + (K*T2*exp(-t/T2))/(T1 - T2)
```

General model:
f(x) = K-(K*T1*exp(-x/T1)+K*T2*exp(-x/T2))/(T1-T2)
Coefficients (with 95% confidence bounds):
K = 1.009 (0.9999, 1.017)
T1 = 1.197 (1.135, 1.258)
T2 = 0.06852 (0.04278, 0.09426)

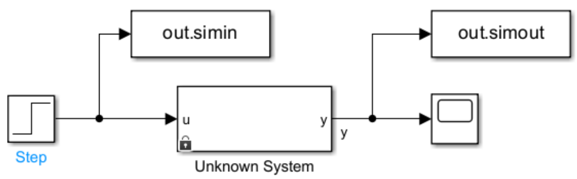


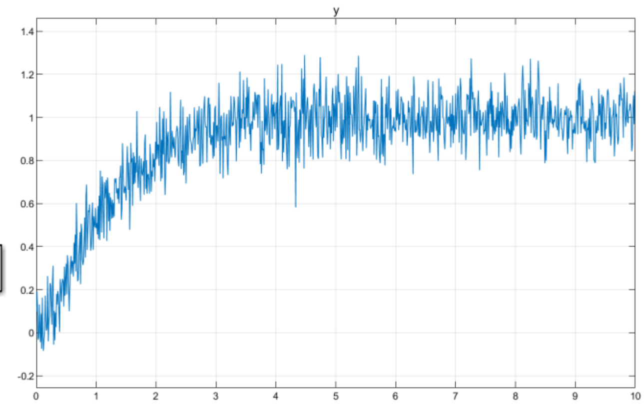
$K \approx 1$, $\tau_1 \approx 1.2$, $\tau_2 \approx 0.68$



6 System Identification Toolbox

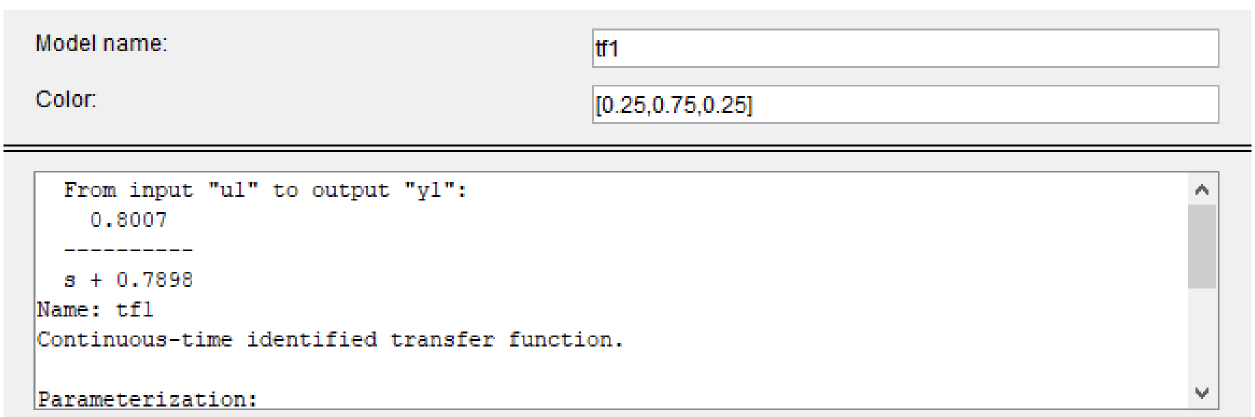
a)





b)

Result from System Identification toolbox :



u(t)

K = 0.8007   a = 0.7898

c)

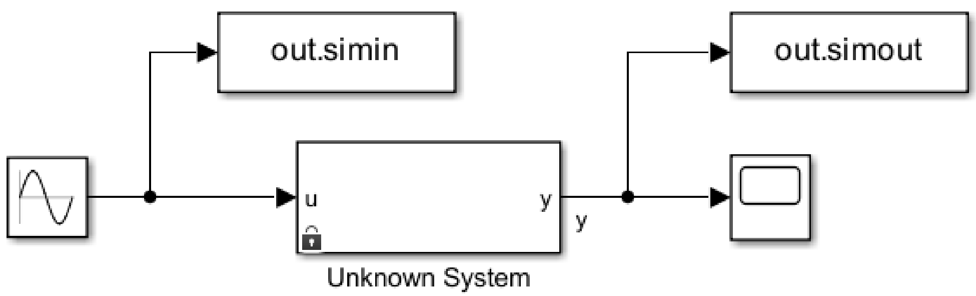
Model name:	tf2
Color:	[0.25,0.75,0.25]

```
  From input "u1" to output "y1":
    0.7865
  ----------
  s + 0.7765
Name: tf2
Continuous-time identified transfer function.

Parameterization:
```

3u(t)

K = 0.7865 a = 0.7765

d)



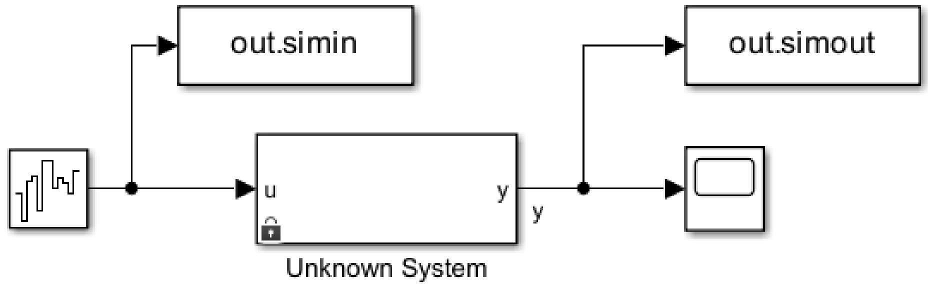
Model name:	tf3
Color:	[0.25,0.75,0.25]

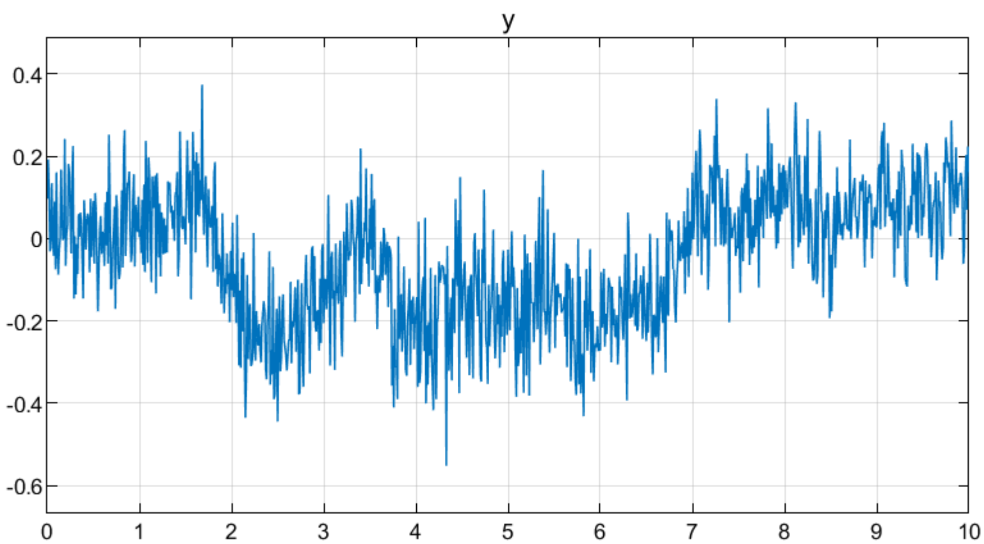
```
  From input "u1" to output "y1":
    0.7859
  ----------
  s + 0.6404
Name: tf3
Continuous-time identified transfer function.

Parameterization:
```

Sin(t)

K = 0.7859 a = 0.6404

e)





output

Model name: tf4

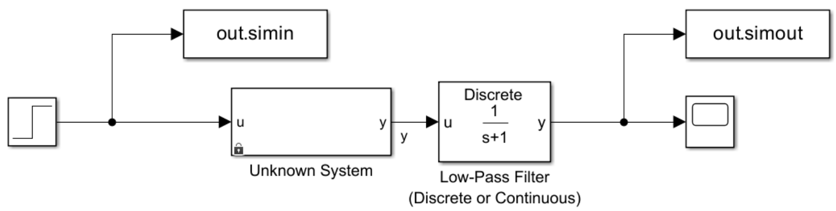
Color: [0.75,0.75,0.2]

```
  From input "u1" to output "y1":
    0.7051
  ----------
  s + 0.8805
Name: tf4
Continuous-time identified transfer function.

Parameterization:
```

White noise

K = 0.7051 a = 0.8805

f)



Model name: tf5

Color: [0.75,0.75,0.75]

```
  From input "u1" to output "y1":
    0.4161
  ----------
  s + 0.3852
Name: tf5
Continuous-time identified transfer function.

Parameterization:
```

u(t) with low-pass filter

$K = 0.4161 \qquad a = 0.3852$

g)

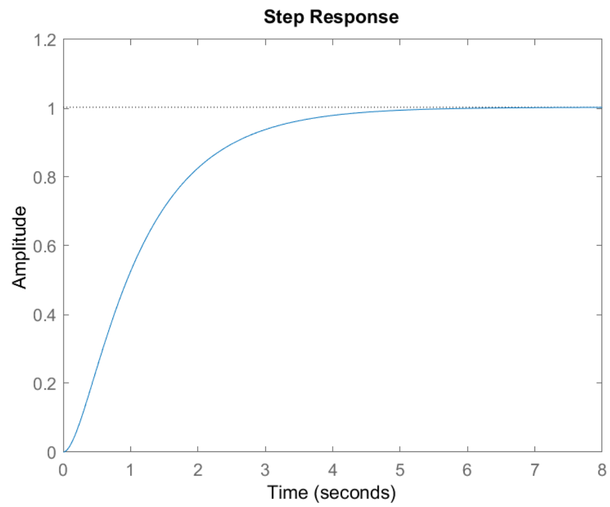
Answers from different inputs have near values for K and a. Although the values we got from white noise and sinusoidal input and filtered system were a little different from the others.

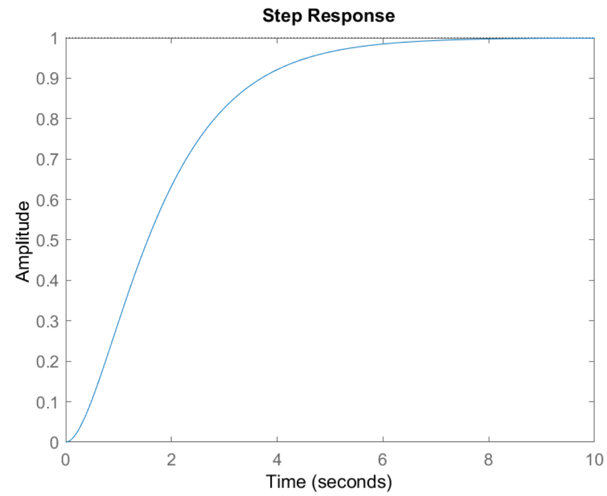
h) Result from System Identification toolbox :

Model name: tf6

Color: [0,0,1]

```
  From input "u1" to output "y1":
           4.25
  ---------------------
  s^2 + 5.228 s + 4.238
Name: tf6
Continuous-time identified transfer function.

Parameterization:
```



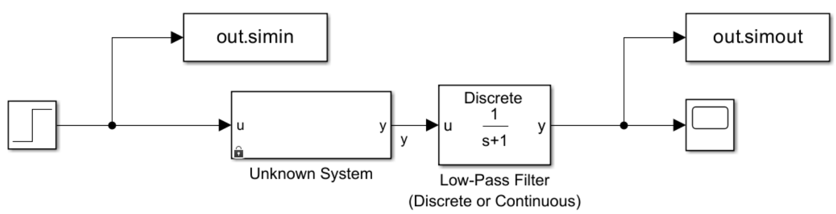


Step response of result (part h) **Step response of Curve Fitting**

$$\frac{4.25}{s^2 + 5.228\, s + 4.238} = \frac{4.25}{(s+1.0031)(s+4.2249)} \qquad \frac{1}{(s+1.2)(s+0.68)} = \frac{1}{s^2 + 1.88\, s + 0.816}$$

Result from Curve Fitting toolbox is similar to our system and had better performance compare to System Identification toolbox.

*) suggested method:



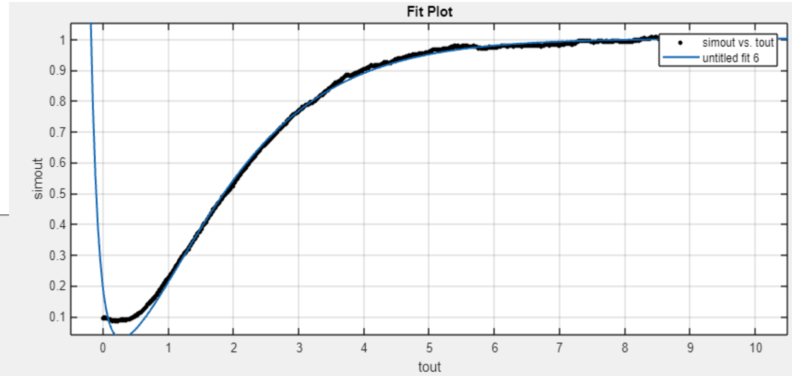
Step response:

```
val =

K - (K*a^2*exp(-t/a))/((T1 - a)*(T2 - a)) - (K*T1^2*exp(-t/T1))/((T1 - T2)*(T1 - a)) + (K*T2^2*exp(-t/T2))/((T1 - T2)*(T2 - a))
```

=

```
K - (K*exp(-t))/((T1 - 1)*(T2 - 1)) - (K*T1^2*exp(-t/T1))/((T1 - T2)*(T1 - 1)) + (K*T2^2*exp(-t/T2))/((T1 - T2)*(T2 - 1))
```

General model:
f(x) = K-K*(exp(-x)*(T1-T2)+(T1^2)*exp(-x/T1)*(T2-1)-(T2^2)*exp(-x/T2)*(T2-1))/((T1-T2)*(T1-1)*(T2-1))
Coefficients (with 95% confidence bounds):
K = 1.003 (1.002, 1.004)
T1 = 1.074 (1.068, 1.079)
T2 = 0.109 (0.1058, 0.1122)



$$K \approx 1 \quad , \quad \tau_1 \approx 1.07 \quad , \quad \tau_2 \approx 0.11$$