Homework 2

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Exercise 1

Claim $BC - A^2 > 0$.

$\rightarrow \left(\bar{R}'\Sigma^{-1}\bar{R}\right) * \left(1'\Sigma^{-1}1\right) - \left(1'\Sigma^{-1}\bar{R}\right)\left(1'\Sigma^{-1}\bar{R}\right) > 0.$

$(A\bar{R} - B1)'\Sigma^{-1}(A\bar{R} - B1) > 0,$
$= (A\bar{R})'\Sigma^{-1}(A\bar{R}) - (A\bar{R})'\Sigma^{-1}(B1) - (B1)'\Sigma^{-1}(A\bar{R}) + (B1)'\Sigma^{-1}(B1) > 0,$
$= A^2 * B - A^2 * B - B * A^2 + B^2 * C > 0,$
$= B^C - A^B = B(BC - A^2) > 0.$ ($\because B > 0$) QED

Exercise 2

$min\{\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij} + \lambda_1[E - \sum_{i=1}^{m} x_i E_i] + \lambda_2[1 - \sum_{i=1}^{m} x_i]\}.$
(1) $\frac{\partial\mathcal{L}}{\partial x_i}: \frac{1}{2}\sum_{j=1}^{m} x_j \sigma_{ij} - \lambda_1 E_i - \lambda_2 = 0,$
(2) $\frac{\partial\mathcal{L}}{\partial \lambda_1}: E - \sum_{i=1}^{m} x_i E_i = 0,$
(3) $\frac{\partial\mathcal{L}}{\partial \lambda_2}: 1 - \sum_{i=1}^{m} x_i = 0.$

(1) $\mathrm{x_k} = \lambda_1 \sum_{j=1}^{m} v_{kj} E_j + \lambda_2 \sum_{j=1}^{m} v_{kj}$, $k = 1, \ldots, m.$
(2) $\sum_{k=1}^{m} x_k E_k = \lambda_1 \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj} E_j E_k + \lambda_2 \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj} E_k.$
(3) $\sum_{k=1}^{m} x_k = \lambda_1 \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj} E_j + \lambda_2 \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj}.$

Let $A = \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj} E_j$, $B = \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj} E_j E_k$, $C = \sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj}$.
Then
(2) $E = \sum_{i=1}^{m} x_i E_i = \sum_{k=1}^{m} x_k E_k = B\lambda_1 + A\lambda_2.$
(3) $1 = \sum_{i=1}^{m} x_i = \sum_{k=1}^{m} x_k = A\lambda_1 + C\lambda_2.$

Thus, $\begin{pmatrix} B & A \\ A & C \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} E \\ 1 \end{pmatrix}$, so that $\lambda_1 = \frac{CE-A}{BC-A^2} = \frac{CE-A}{D}$, $\lambda_2 = \frac{B-AE}{BC-A^2} = \frac{B-AE}{D}$, where $D = BC - A^2$.

Therefore,
(1) $x_k = \lambda_1 \sum_{j=1}^{m} v_{kj} E_j + \lambda_2 \sum_{j=1}^{m} v_{kj}$, $k = 1, \ldots, m,$
$= \frac{(CE-A)\sum_{j=1}^{m} v_{kj}E_j + (B-AE)\sum_{j=1}^{m} v_{kj}}{D} = \frac{E\sum_{j=1}^{m} v_{kj}(CE_j - A) + \sum_{j=1}^{m} v_{kj}(B - AE_j)}{D}$, $(\because E = E_j)$.

And if $\bar{E} = \frac{A}{C}$, then

$x_k = \frac{\frac{A}{C}\sum_{j=1}^{m}\left(C * \frac{A}{C} - A\right) + \sum_{j=1}^{m} v_{kj}\left(B - A * \frac{A}{C}\right)}{D} = \frac{\left(B - \frac{A^2}{C}\right)\sum_{j=1}^{m} v_{kj}}{BC - A^2} = \frac{\sum_{j=1}^{m} v_{kj}}{C}$. QED

Exercise 3

$\rho = \frac{Cov(R_A, R_B)}{\sigma_A \sigma_B} = \frac{E(R_A R_B) - E(R_A)E(R_B)}{\sigma_A \sigma_B}.$

Exercise 4

a. $\Omega^{-1} = [v_{ij}] = \begin{pmatrix} 166.21139 & -22.40241 \\ -22.40241 & 220.41076 \end{pmatrix}$.
$C = \sum_{i=1}^{2}\sum_{j=1}^{2} v_{ij} = 341.8173$.

$\overline{x_1} = \frac{\sum_{j=1}^{2} v_{1j}}{C} = \frac{166.21139-22.40241}{341.8173} = 0.4207188$.
$\overline{x_2} = \frac{\sum_{j=1}^{2} v_{2j}}{C} = \frac{-22.40241+220.41076}{341.8173} = 0.5792812$.
Thus, $x = (0.4207188, 0.5792812)$.

b. $R = x * \overline{R_G} + (1-x) * R_f = x * 0.01315856 + (1-x) * 0.011$
$= 0.011 + (0.01315856 - 0.011)x = 0.011 + 0.00215856x$.

Thus, $0.01219724 = 0.011 + 0.00215856\,x$,
$\to x = \frac{0.01219724-0.011}{0.00215856} = 0.5546475$.

c. $\sigma_B = 0.03$.
No. Given $\sigma_B = 0.03$, we cannot make efficient portfolio.

Exercise 5

$R = 0.011 + 0.00215856x$.
Actually, I can't understand what is the exact meaning of 'perfect correlation', i.e. $|Cov(A,B)| = 1$
But because they have linear relationship, so that having correlation in terms of those.