Homework 3

706107402 Juwon Lee

Exercise 1

$$x=\frac{\Sigma^{-1}(\bar{R}-R_f1)}{(\bar{R}-R_f1)'\Sigma^{-1}(\bar{R}-R_f1)}=\frac{\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{21} & \sigma_2^2\end{pmatrix}^{-1}\left(\begin{pmatrix}\overline{R_1}\\ \overline{R_2}\end{pmatrix}-\begin{pmatrix}R_f\\ R_f\end{pmatrix}\right)}{\left(\begin{pmatrix}\overline{R_1}\\ \overline{R_2}\end{pmatrix}-\begin{pmatrix}R_f\\ R_f\end{pmatrix}\right)'\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}^{-1}\left(\begin{pmatrix}\overline{R_1}\\ \overline{R_2}\end{pmatrix}-\begin{pmatrix}R_f\\ R_f\end{pmatrix}\right)}=\frac{\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{21} & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}\overline{R_A}\\ \overline{R_B}\end{pmatrix}}{\left(\begin{pmatrix}\overline{R_A}\\ \overline{R_B}\end{pmatrix}\right)'\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{12} & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}\overline{R_A}\\ \overline{R_B}\end{pmatrix}}. \ (*)$$

$$\Sigma^{-1}\rightarrow\begin{bmatrix}1 & 0 & \sigma_1^2 & \sigma_{12}\\ 0 & 1 & \sigma_{12} & \sigma_2^2\end{bmatrix}\rightarrow\begin{bmatrix}\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & -\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & 1 & 0\\ -\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & \frac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & 0 & 1\end{bmatrix}.$$

$$(*)\ \frac{\begin{pmatrix}\frac{1}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}(\sigma_2^2\overline{R_A}-\sigma_{12}\overline{R_B})\\ \frac{1}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}(-\sigma_{12}\overline{R_A}+\sigma_1^2\overline{R_B})\end{pmatrix}}{(\overline{R_A}\quad\overline{R_B})\begin{pmatrix}\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & -\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\\ -\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & \frac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\end{pmatrix}\begin{pmatrix}\overline{R_A}\\ \overline{R_B}\end{pmatrix}}=\begin{pmatrix}\frac{\sigma_2^2\overline{R_A}-\sigma_{12}\overline{R_B}}{\sigma_2^2\overline{R_A}^2-2\sigma_{12}\overline{R_A}\overline{R_B}+\sigma_1^2\overline{R_B}^2}\\ \frac{-\sigma_{12}\overline{R_A}+\sigma_1^2\overline{R_B}^2}{\sigma_2^2\overline{R_A}-2\sigma_{12}\overline{R_A}\overline{R_B}+\sigma_1^2\overline{R_B}^2}\end{pmatrix}. \text{ QED}$$

Exercise 2

a. $\Sigma=\begin{pmatrix}0.2 & 0.1\\ 0.1 & 0.08\end{pmatrix}$, and $\Sigma^{-1}\rightarrow\begin{bmatrix}1 & 0 & 0.2 & 0.1\\ 0 & 1 & 0.1 & 0.08\end{bmatrix}\rightarrow\begin{pmatrix}\frac{40}{3} & -\frac{50}{3}\\ \frac{50}{3} & \frac{100}{3}\end{pmatrix}$.

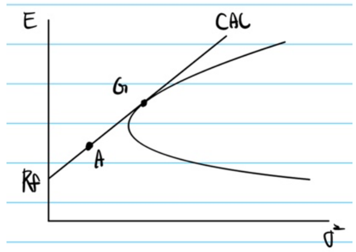
Then $x=\dfrac{\begin{pmatrix}\frac{40}{3} & -\frac{50}{3}\\ -\frac{50}{3} & \frac{100}{3}\end{pmatrix}\begin{pmatrix}0.12-0.04\\ \overline{R_B}-0.04\end{pmatrix}}{\begin{pmatrix}0.12-0.04\\ \overline{R_B}-0.04\end{pmatrix}'\begin{pmatrix}\frac{40}{3} & -\frac{50}{3}\\ -\frac{50}{3} & \frac{100}{3}\end{pmatrix}\begin{pmatrix}0.12-0.04\\ \overline{R_B}-0.04\end{pmatrix}}=\dfrac{\begin{pmatrix}\frac{1.2-50\overline{R_B}}{3}\\ \frac{100\overline{R_B}-8}{3}\end{pmatrix}}{\begin{pmatrix}0.08\\ \overline{R_B}-0.04\end{pmatrix}'\begin{pmatrix}\frac{40}{3} & -\frac{50}{3}\\ -\frac{50}{3} & \frac{100}{3}\end{pmatrix}\begin{pmatrix}0.08\\ \overline{R_B}-0.04\end{pmatrix}}$.

Thus, $\frac{1.2-50\overline{R_B}}{3}=\frac{100\overline{R_B}-8}{3}$, so that $150\overline{R_B}=9.2$, $\overline{R_B}=\frac{9.2}{150}=0.0613333$ …. QED

b. So that $\frac{100\overline{R_B}-8}{3}=0$, $\overline{R_B}=0.8$. QED

Exercise 3

(1) $\sum_{i=1}^{n}\left(\overline{R_i}-R_f\right)x_i+R_f=E.$



For G, $\sum_{i=1}^{n}\left(\overline{R_i^G}-R_f\right)x_i^G+R_f=E.$

Let $\Omega=\{(\mathbf{R},\mathbf{x})\}$ subject to $\sum_{i=1}^{n}\left(\overline{R_i}-R_f\right)x_i+R_f=E.$

If we know $(\mathbf{R_G},\mathbf{x_G})\in\Omega$,

Then we can find any $(\mathbf{R_A},\mathbf{x_A})\in\Omega$ such that $x_1^A+x_2^A+x_3^A=1.$

$\therefore\ \forall(\mathbf{R_A},\mathbf{x_A})\equiv CAL.$

(2) $\sum_{i=1}^{n}\left(\overline{R_i}-R_f\right)x_i+R_f=E.$

Let $\left(\overline{R_i}-R_f\right)=\mathbf{P}$, then $\mathbf{P}'\cdot\mathbf{x}+R_f=E.$

Then we can find as above by using $\mathbf{x}$.

Exercise 4

a.

Table	$\overline{R}$	σ
Portfolio (P)	0.67%	2.37%
ABC Company (A)	1.25%	2.95%

1. $0.9 * 0.0067 + 0.1 * 0.0125 = 0.00728.$
2. $\rho=Corr(P,A)=\frac{Cov(P,A)}{sd(P)sd(A)}.$

   ➔ $Cov(P,A)=\rho*sd(P)*sd(A)=0.4*0.0237*0.0295=0.00027966.$
3. $V(0.9P+0.1A)=0.9^2V(P)+0.1^2V(A)+2*0.9*0.1*Cov(P,A)=0.0005140102,$

   $sd(0.9P+0.1A)=\sqrt{V(0.9P+0.1A)}=0.02267179.$

b. $\overline{R_g}=0.0042,\ \sigma_g=0.$

1. $0.9 * 0.0067 + 0.1 * 0.0042 = 0.00645.$
2. $Cov(P,G)=0,$

   $V(0.9P+0.1G)=0.9^2V(P)+0.1^2V(G)+2*0.9*0.1*Cov(P,G)$

   $=0.81*0.0237^2=0.0004549689.$

   Thus, $sd(0.9P+0.1G)=0.02133.$

Exercise 5

a. $\min \frac{1}{2}\mathbf{x}'\Sigma\mathbf{x}$ subject to $\overline{\mathbf{R}'\mathbf{x}} = E,\ \mathbf{1}'\mathbf{x} = 1.$

Show that $\mathbf{x} = g + hE.$

$\min Q = \min \frac{1}{2}\mathbf{x}'\Sigma\mathbf{x} + \lambda_1[E - \overline{\mathbf{R}'\mathbf{x}}] + \lambda_2[1 - \mathbf{1}'\mathbf{x}].$

$\frac{\partial Q}{\partial \mathbf{x}}: \frac{1}{2}\mathbf{x}'\Sigma - \lambda_1\overline{\mathbf{R}'} - \lambda_2\mathbf{1}' = 0,\ \frac{\partial Q}{\partial \lambda_1}: E - \overline{\mathbf{R}'}\mathbf{x} = 0,\ \frac{\partial Q}{\partial \lambda_2}: 1 - \mathbf{1}'\mathbf{x} = 0.$

$\frac{1}{2}\mathbf{x}'\Sigma = \lambda_1\overline{\mathbf{R}'} + \lambda_2\mathbf{1}',$

$\mathbf{x}'\Sigma = \gamma_1\overline{\mathbf{R}'} + \gamma_2\mathbf{1}',$

$\mathbf{x}' = \gamma_1\overline{\mathbf{R}'}\Sigma^{-1} + \gamma_2\mathbf{1}'\Sigma^{-1}.$

Because $\gamma_1 = \frac{CE-A}{D},\ \gamma_2 = \frac{B-AE}{D},$

$\mathbf{x}' = \frac{CE-A}{D}\overline{\mathbf{R}'}\Sigma^{-1} + \frac{B-AE}{D}\mathbf{1}'\Sigma^{-1} = \frac{E\Sigma^{-1}(CE-A)' + 1\Sigma^{-1}(B-AE)'}{D}$

$= \frac{1}{D}\left[B\Sigma^{-1}\mathbf{1} - A\Sigma^{-1}\overline{\mathbf{R}}\right] + \frac{1}{D}\left[C\Sigma^{-1}\overline{\mathbf{R}} - A\Sigma^{-1}\mathbf{1}\right]E = g + hE.$ QED

b. Claim $Cov(R_a, R_b) = \frac{C}{D}\left(E_aE_b - \frac{A}{C}E_a - \frac{A}{C}E_b + \frac{A^2}{C^2}\right) + \frac{1}{C}$

$= \frac{C}{D}E_aE_b - \frac{A}{D}(E_a + E_b) + \frac{A^2+D}{DC} = \frac{C}{D}E_aE_b - \frac{A}{D}(E_a + E_b) + \frac{A^2+BC-A^2}{DC}$

$= \frac{C}{D}E_aE_b - \frac{A}{D}(E_a + E_b) + \frac{B}{D} = \frac{CE_aE_b - A(E_a+E_b) + B}{D}.$

$Cov(R_a, R_b) = \sigma_{AB} = \frac{C*E_aE_b - AE_a - AE_b + B}{D}.$ QED