Homework 1

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Exercise 1

a. Assume that the variance of the returns of security A is 0.16 and the variance of security B is 0.25. The variance of a portpolio of 50% A and 50% B is 0.0525. What is the covariance between securities A and B?

   Answer) $V(A) = 0.16,\ V(B) = 0.25,\ V(0.5A + 0.5B) = 0.0525.$
   $V(0.5A + 0.5B) = 0.5^2V(A) + 0.5^2V(B) + 2Cov(A,B) = 0.25V(A) + 0.25V(B) + 2Cov(A,B).$
   $\rightarrow 0.25 * 0.16 + 0.25 * 0.25 + 2Cov(A,B) = 0.0525$, so that
   $\rightarrow 0.04 + 0.0625 + 2Cov(A,B) = 0.0525,$
   $\rightarrow 2Cov(A,B) = -0.05.$
   $\therefore\ Cov(A,B) = -0.025.$ QED

b. Suppose you are constructing two portfolios using the same n stocks. What is the expression of the covariance between these two portfolios in summations form and in matrix/vector form.

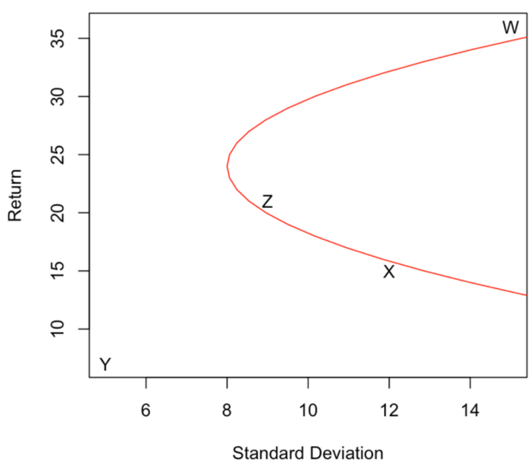
   Answer)
   Portpolio A: $x_1^A R_1^A + x_2^A R_2^A + \cdots + x_n^A R_n^A = A = x^{A\prime} R^A$
   Portpolio B: $x_1^B R_1^B + x_2^B R_2^B + \cdots + x_n^B R_n^B = B = x^{B\prime} R^B$.

   $Cov(A,B) = E(AB) - E(A)E(B) = \frac{\sum_{i=1}^n (AB)_i}{n} - \frac{\sum_{i=1}^n A_i}{n} * \frac{\sum_{i=1}^n B_i}{n}.$
   Or equivalently,
   $Cov(A,B) = E(AB) - E(A)E(B) = E(x^A R^A x^B R^B) - E(x^A R^A)E(x^B R^B).$

Exercise 2

1. Which one of the following portfolios cannot lie on the efficient frontier as described by Markowitz? Please explain your answer.

Portpolio	Expected Return (%)	Standard Deviation (%)
W	15	36
X	12	15
Y	5	7
Z	9	21



Thus, Y doesn't lie on the efficient frontier.

2. Suppose all stocks have $E(R) = 15\%, \sigma = 60\%$, and common correlation coefficient $\rho = 0.5$. What are the expected return and standard deviation of an equally weighted portfolio of $n = 25$ stocks?

Answer) $E\left(\frac{1}{25}R_1 + \cdots + \frac{1}{25}R_{25}\right) = \frac{1}{25}E(R_1 + \cdots + R_{25}) = \frac{1}{25} * 25 * 0.15 = 0.15$,
$V\left(\frac{1}{25}R_1 + \cdots + \frac{1}{25}R_{25}\right) = \frac{1}{25^2}[0.6^2 + \cdots + 0.6^2 + 2 * 0.5 * 0.6^2 + \cdots + 2 * 0.5 * 0.6^2]$,
$= \frac{1}{25^2}\left[0.36 * 25 + 0.36 * \binom{25}{2}\right] = \frac{0.36}{25^2}\left[25 + \frac{24*25}{2}\right] = \frac{0.36}{25}[1 + 12] = 0.1872.$

3. Refer to question (2). What is the smallest number of stocks necessarily to generate a portfolio with standard deviation of at most 43%?

Answer) $V\left(\frac{1}{n}R_1 + \cdots + \frac{1}{n}R_n\right) = \frac{1}{n^2}V(R_1 + \cdots + R_n)$
$= \frac{1}{n^2}[0.6^2 + \cdots + 0.6^2 + 2 * 0.5 * 0.6^2 + \cdots + 2 * 0.5 * 0.6^2] = 0.43^2$,
$\to \frac{1}{n^2}\left[n * 0.36 + \frac{n(n-1)}{2}0.36\right] = \frac{0.36}{n}\left[1 + \frac{n-1}{2}\right] = 0.1849$,
$\to 1.947 * \frac{n+1}{2} = n$, $0.053n = 1.947, n = 36.736$.
Thus, we should take more than $n = 37$.

4. Refer to question (3). As n gets larger, is it true that $\sigma_p = \sigma\sqrt{\rho}$? Please explain your answer.

Answer) Claim $\sigma_p^2 = \sigma^2\rho \to \sigma_p^2 = 0.6^2 * 0.5 = 0.18$.
$V\left(\frac{1}{n}R_1 + \cdots + \frac{1}{n}R_n\right) = \frac{0.36}{n} + \frac{0.18(n-1)}{n}$, so that
$\lim_{n\to\infty} \frac{0.36}{n} + \frac{0.18(n-1)}{n} = 0.18$. QED

Exercise 3

The mean returns and variance covariance matrix of the returns of three stocks are given below:

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Mean returns:
        C         XOM        AAPL       ^GSPC
    0.005174   0.010617   0.016947   0.010846

Variance-covariance matrix:
            C      XOM     AAPL    ^GSPC
C     0.010025 0.000000 0.000000 0.000000
XOM   0.000000 0.002123 0.000000 0.000000
AAPL  0.000000 0.000000 0.005775 0.000000
^GSPC 0.000000 0.000000 0.000000 0.001217
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Assume short sales are allowed. Compute the composition of the minimum risk portfolio using only the three stocks.

Answer) Let C = C, XOM = X, AAPL = A.
Then $\min_{x_C,x_X,x_A} V(x_C R_C + x_X R_X + x_A R_A) = \min_{x_C,x_X,x_A} [x_C^2 V(R_C) + x_X^2 V(R_X) + x_A^2 V(R_C)]$
subject to $x_C + x_X + x_A = 1$. (This is because they don't have any correlation.)

$= \min_{x_C,x_X,x_A} [x_C^2 V(R_C) + x_X^2 V(R_X) + (1 - x_C - x_X)^2 V(R_A)]$.
$\frac{\partial \mathcal{L}}{\partial x_C} = 2x_C V(R_C) - 2(1 - x_C - x_X)V(R_C) = 0$,
$\frac{\partial \mathcal{L}}{\partial x_X} = 2x_X V(R_X) - 2(1 - x_C - x_X)\ \ V(R_C) = 0$, so that $x_C V(R_C) = x_X V(R_X)$.
$\to x_C = 0.212x_X$.

$= \min_{x_C,x_X,x_A} [(1 - x_X - x_A)^2 V(R_C) + x_X^2 V(R_X) + x_A^2 V(R_A)]$.
$\to x_X V(R_X) = x_A V(R_A)$, so that $x_A = 0.367619x_X$.

Therefore, $0.212x_X + x_X + 0.367619x_X = 1$, so that $x_X = 0.633064, x_C = 0.13421, x_A = 0.232726$.

Exercise 4

Assume that the average variance of the return for an individual security is 50 and that the average covariance is 10. What is the variance of an equally weighted portfolio of 5, 10, 20, 50, and 100 securities?

Answer) $V(R_i) = 50,\ Cov(R_i, R_j) = 10.$
$V\left(\frac{1}{5}R_1 + \cdots + \frac{1}{5}R_5\right) = \frac{1}{25}[50 + \cdots + 50 + 2 * 10 + \cdots + 2 * 10] = \frac{1}{25}\left[250 + 20 * \frac{5*4}{2}\right] = 18.$
$V\left(\frac{1}{10}R_1 + \cdots + \frac{1}{10}R_{10}\right) = \frac{1}{100}[50 + \cdots + 50 + 2 * 10 + \cdots + 2 * 10] = \frac{1}{100}\left[500 + 20 * \frac{10*9}{2}\right] = 14.$
$V\left(\frac{1}{20}R_1 + \cdots + \frac{1}{20}R_{20}\right) = \frac{1}{400}\left[50 * 20 + 20 * \binom{20}{2}\right] = \frac{1}{400}\left[1000 + 20 * \frac{20*19}{2}\right] = 2.5 + 9.5 = 13.$
$V\left(\frac{1}{50}R_1 + \cdots + \frac{1}{50}R_{50}\right) = \frac{1}{2500}\left[50 * 50 + 20 * \binom{50}{2}\right] = \frac{1}{2500}\left[2500 + 20 * \frac{50*49}{2}\right] = 1 + 9.8 = 10.8.$
$V\left(\frac{1}{100}R_1 + \cdots + \frac{1}{100}R_{100}\right) = \frac{1}{10000}\left[50 * 100 + 20 * \binom{100}{2}\right] = \frac{1}{10000}\left[5000 + 20 * \frac{100*99}{2}\right] = 0.5 + 9.9 = 10.4.$

Exercise 5

What is the composition of the minimum risk portfolio using n risky stocks? Show the entire derivation using matrix/vector notation. Using this composition give the expression for the expected return and variance of the minimum risk portfolio?

$V\left(\frac{1}{n}R_1 + \cdots + \frac{1}{n}R_n\right) = \frac{1}{n^2}\left[50 * n + 20 * \binom{n}{2}\right] = \frac{1}{n^2}\left[50n + 20 * \frac{n(n-1)}{2}\right] = \frac{50n + 10n(n-1)}{n^2}.$
Thus, as $n \to \infty$, then $V\left(\frac{1}{n}R_1 + \cdots + \frac{1}{n}R_n\right) \to 10.$