Homework 8

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Exercise 1

	Flows at $t=0$	Flows at $t=1$	
Action		$S_1 = uS_0$	$S_1 = dS_0$
Write 1 call	C	$-C_u$	$-C_d$
Buy α shares of stock	$-\alpha S_0$	$\alpha u S_0$	$\alpha d S_0$

This is hedged portfolio, so $-C_u + \alpha u S_0 = -C_d + \alpha d S_0$.
$\rightarrow \alpha S_0(u-d) = C_u - C_d$
$\therefore \alpha = \frac{C_u - C_d}{S_0(u-d)}$.

And because of time period, $(\alpha S_0 - C)e^{rt} = \alpha d S_0 - C_d$
$\rightarrow \alpha S_0 - C = (\alpha d S_0 - C_d)e^{-rt}$
$\rightarrow C = \alpha S_0 - (\alpha d S_0 - C_d)e^{-rt} = (\alpha S_0 e^{rt} - \alpha d S_0 + C_d)e^{-rt}$
$\rightarrow C = \left[\frac{C_u - C_d}{S_0(u-d)} S_0 e^{rt} - \frac{C_u - C_d}{S_0(u-d)} d S_0 + C_d\right] e^{-rt}$
$\rightarrow C = \left[\frac{C_u - C_d}{u-d} e^{rt} - \frac{C_u - C_d}{u-d} d + C_d\right] e^{-rt}$
$\rightarrow C = \left[\frac{e^{rt} - d}{u-d} C_u + \frac{u - e^{rt}}{u-d} C_d\right] e^{-rt}$

Let $p = \frac{e^{rt}-d}{u-d}, 1-p = \frac{u-e^{rt}}{u-d}$, then
$\therefore\ C = (pC_u + (1-p)C_d)e^{-rt}$.

EX) $S_0 = 50,\ E = 50,\ u = 1.05,\ d = 0.95,\ r = 0.01,\ t = 1$.
$p = \frac{e^{rt}-d}{u-d} = \frac{e^{0.01*1} - 0.95}{1.05 - 0.95} = 0.6005$, so that $1 - p\ =\ 0.3995$.

1. $S_1^u = uS_0 = 55,\ C_u = \max(55-50,0) = 5$
2. $S_1^d = dS_0 = 45,\ C_d = \max(45-50,0) = 0$

Thus, $C = (pC_u + (1-p)C_d)e^{-rt} = (0.6005 * 5 + 0.3995 * 0)e^{-0.01*1} = 2.973$.

Exercise 2

$S_0 = 50, t = 3$ months, $u\ =\ 1.06, d\ =\ 0.95, r\ =\ 0.05,\ E\ =\ 51$.

		$C_{u^2}(56.18)$
	$C_u(53)$	
$C(50)$		$C_{ud}(50)$
	$C_{\mathrm{d}}(47.5)$	
		$C_{d^2}(45.125)$

$r_p = (1.05)^{\frac{6}{12}*\frac{1}{2}} - 1 = (1.05)^{\frac{3}{12}} - 1 = 0.01227223$,
$p = \frac{e^{0.01227223} - 0.95}{1.06 - 0.95} = 0.5668$, so that $1 - p\ =\ 0.4332$.

$C_u = \max(53 - 51,0) = 2,\ C_u = \max(47.5 - 50,0) = 0$, then,
$C = (pC_u + (1-p)C_d)e^{-rt} = (0.5668 * 2 + 0.4332 * 0)e^{-0.01227223} = 1.120$.

At the second node,
$C_{u^2} = \max(56.18 - 51,0) = 5.18,\ C_{ud} = \max(50 - 51,0) = 0,\ C_{d^2} = \max(45.125 - 51,0) = 0$.
$C|C_u = (pC_{u^2} + (1-p)C_{ud})e^{-rt} = (0.5668 * 5.18 + 0.4332 * 0)e^{-0.01227223}\ =\ 2.900$.
$C|C_d = (pC_{ud} + (1-p)C_{d^2})e^{-rt} = (0.5668 * 0 + 0.4332 * 0)e^{-0.1227223}\ =\ 0$.

Exercise 3

$S_0 = 50, t = 3$ months, $u = 1.06, d = 0.95, r = 0.05, E = 51.$

$$\begin{array}{ccc} & & P_{u^2}(56.18) \\ & P_u(53) & \\ P(50) & & P_{ud}(50) \\ & P_d(47.5) & \\ & & P_{d^2}(45.125) \end{array}$$

$P_u = \max(51-53,0) = 0,\ P_d = \max(51-47.5,0) = 3.5$
$P_{u^2} = \max(51-56.18,0) = 0,\ P_{ud} = \max(51-50,0) = 1,\ P_{d^2} = \max(51-45.125,0) = 5.875.$

$r_p = (1.05)^{\frac{6}{12}*\frac{1}{2}} - 1 = (1.05)^{\frac{3}{12}} - 1 = 0.01227223,$
$p = \frac{e^{0.01227223} - 0.95}{1.06-0.95} = 0.5668,$ so that $1-p = 0.4332.$

$P = (pP_u + (1-p)P_d)e^{-rt} = (0.5668*0 + 0.4332*3.5)e^{-0.01227223} = 1.498.$
$P|P_u = (pP_{u^2} + (1-p)P_{ud})e^{-rt} = (0.5668*0 + 0.4332*1)e^{-0.01227223} = 0.988.$
$P|P_d = (pP_{ud} + (1-p)P_{d^2})e^{-rt} = (0.5668*1 + 0.4332*5.875)e^{-0.01227223} = 3.074.$

Put-call Parity: $P + S_0 = C + Ee^{-rt}$

1. $P + S_0 = 1.498 + 50 = 51.498$
2. $C + Ee^{-rt} = 1.120 + 51 * e^{-0.01227223} = 51.498.$

Thus, put-call parity holds.

Exercise 4

$P_{d^2} > P_d > P_{ud} > P_u = P_{u^2}.$

Thus, suppose that a person has Expected Utility,

1. If there is u in first node, then hold is the strategic dominance.
2. If there is d in first node,
   a. To exercise now, then utility is 3.5.
   b. To hold, then the expected utility is $(0.5668*1 + 0.4332*5.875)e^{-0.01227223} = 3.074.$
   Then exercise is the strategic dominance.

Exercise 5

a. $k = \frac{\log\left(\frac{E}{d^n S_0}\right)}{\log\left(\frac{u}{d}\right)} = \frac{\log\left(\frac{60}{0.833^{10} 50}\right)}{\log\left(\frac{1.2}{0.833}\right)} = 5.5 \rightarrow 6.$

b. $$\begin{pmatrix} C_{u^{10}} \\ C_{u^9 d} \\ C_{u^8 d^2} \\ C_{u^7 d^3} \\ C_{u^6 d^4} \\ C_{u^5 d^5} \\ C_{u^4 d^6} \\ C_{u^3 d^7} \\ C_{u^2 d^8} \\ C_{u d^9} \\ C_{d^{10}} \end{pmatrix} = \begin{pmatrix} 309.5868 \\ 214.9908 \\ 149.2992 \\ 103.68 \\ 72 \\ 50 \\ 34.72222 \\ 24.11265 \\ 16.7449 \\ 11.6284 \\ 8.075279 \end{pmatrix}$$

c. $$\begin{pmatrix} Intrinsic_1 \\ Intrinsic_2 \\ Intrinsic_3 \\ Intrinsic_4 \\ Intrinsic_5 \\ Intrinsic_6 \\ \vdots \\ Intrinsic_{10} \end{pmatrix} = \begin{pmatrix} 249.58682 \\ 154.99085 \\ 89.29920 \\ 43.68000 \\ 12 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

d. $p' = \frac{pu}{1+r} = 0.808773.$

1. $C = 50P(X \geq k) - 60e^{-1}P(Y \geq k),$
   Where $X \sim B(10, p')$ and $Y \sim B(10, p)$
   $= 28.51598.$
2. $\left[\binom{10}{10}C_{u^{10}}p^{10} + \binom{10}{9}C_{u^9 d}p^9(1-p) + \binom{10}{8}C_{u^8 d^2}p^8(1-p)^2 + \binom{10}{7}C_{u^7 d^3}p^7(1-p)^3 + \binom{10}{6}C_{u^6 d^4}p^6(1-p)^4\right]e^{-1} = 28.40256.$

Exercise 6

	Flows at $t = 1$	Flows at $t = 2$	
Action	(given C_u)	$S_2 = u^2 S_0$	$S_2 = \mathrm{u}dS_0$
Write 1 call	C_u	$-C_{u^2}$	$-C_{\mathrm{ud}}$
Buy α shares of stock	$-\alpha u S_0$	$\alpha u^2 S_0$	$\alpha \mathrm{u}dS_0$

	Flows at $t = 1$	Flows at $t = 2$	
Action	(given C_d)	$S_2 = d^2 S_0$	$S_2 = duS_0$
Write 1 call	C_d	$-C_{d^2}$	$-C_{\mathrm{du}}$
Buy α shares of stock	$-\alpha d S_0$	$\alpha \mathrm{d}^2 S_0$	$\alpha d\mathrm{u}S_0$

Because they are hedged portfolio,
$$-C_{u^2} + \alpha u^2 S_0 = -C_{ud} + \alpha udS_0$$
$$-C_{d^2} + \alpha d^2 S_0 = -C_{du} + \alpha duS_0$$

Then,
$$\alpha = \frac{C_{u^2} - C_{ud}}{uS_0(u-d)}$$
$$\alpha = \frac{C_{d^2} - C_{du}}{dS_0(d-u)}.$$

Then because of time period,
$$(-C_u + \alpha uS_0)e^{rt} = -C_{ud} + \alpha udS_0$$
$$(-C_d + \alpha dS_0)e^{rt} = -C_{du} + \alpha duS_0$$

Then,
$$C_u = \alpha uS_0 - (\alpha udS_0 - C_{ud})e^{-rt} = (\alpha uS_0 e^{rt} - \alpha udS_0 + C_{ud})e^{-rt}$$
$$C_d = \alpha dS_0 - (\alpha duS_0 - C_{du})e^{-rt} = (\alpha dS_0 e^{rt} - \alpha duS_0 + C_{du})e^{-rt}$$

Thus,
$$C_u = [\alpha uS_0(e^{rt} - d) + C_{ud}]e^{-rt}$$
$$C_d = [\alpha dS_0(e^{rt} - u) + C_{du}]e^{-rt}$$

When we take α,
$$C_u = \left[\frac{C_{u^2} - C_{ud}}{uS_0(u-d)} uS_0(e^{rt} - d) + C_{ud}\right] e^{-rt}$$
$$C_d = \left[\frac{C_{d^2} - C_{du}}{dS_0(d-u)} dS_0(e^{rt} - u) + C_{du}\right] e^{-rt}$$

Then,
$$C_u = \left[\frac{e^{rt} - d}{u-d} C_{u^2} + \frac{u - e^{rt}}{u-d} C_{ud}\right] e^{-rt}$$
$$C_d = \left[\frac{e^{rt} - u}{d-u} C_{d^2} + \frac{d - e^{rt}}{d-u} C_{du}\right] e^{-rt}$$

Therefore, we can conclude that

When one-step binomial tree makes upward movement, we will take $p = \frac{e^{rt} - d}{u-d}$ for the second period.

QED