Homework 3

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Axler 2e, Exercise 6.19: Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T iff $P_U T P_U = T P_U$.

Proof. ($\Longrightarrow$) Suppose that U is invariant under T. Then $\forall u \in U, Tu \in U$. Let $v \in V$. Then $v = u + u'$ where $u \in U$ and $u' \in U^\perp$. By the definition of the orthogonal projection, $P_U v = u$. Because U is invariant under T, $Tu = u$. And because $u \in U$, P_U is the identity. Thus the action of $P_U T P_U$ and $T P_U$ is the same for all $v \in V$.

($\Longleftarrow$) Suppose that $P_U T P_U = T P_U$. Let $u \in U$. Then $P_U u = u \in U$ and so $P_U(Tu) = Tu$. Hence $Tu \in U$, and therefore U is invariant under T. $\square$

Axler 2e, Exercise 6.25: Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ such that

$$\int_0^1 p(x)\cos(\pi x)\,dx = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

Proof. Because of the properties of the integral, $\phi : \mathcal{P}_2(\mathbb{R}) \mapsto \mathbb{R}$ defined by $\phi(p) = \int_0^1 p(x)\cos(\pi x)\,dx$ is a linear functional on $\mathcal{P}_2(\mathbb{R})$. Therefore, by the Riesz Representation Theorem, there must exist some vector (polynomial) $q \in \mathcal{P}_2(\mathbb{R})$ such that $\phi(p) = \langle p, q\rangle$. We can define the inner product as the integral from 0 to 1 of the product of the two polynomials; now the question asks us to find the q which Riesz says must exist.

Luckily, Riesz also tells us a formula for such a q! Starting from an orthonormal basis $e_1, \ldots, e_n$, we have $q = \overline{\phi(e_1)}e_1 + \cdots + \overline{\phi(e_n)}e_n$; as we are dealing only with the reals here, we simplify to

$$q = \phi(e_1)e_1 + \cdots + \phi(e_n)e_n.$$

We found an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ in the previous homework:

$$\begin{aligned} e_1 &= 1 \\ e_2 &= \sqrt{3}(2x-1) \\ e_2 &= \sqrt{5}(6x^2 - 6x + 1) \end{aligned}$$

The final answer is

$$q(x) = \frac{12 - 24x}{\pi^2}$$

$\square$

Axler 2e, Exercise 6.26: Fix a vector $v \in V$ and define $T \in \mathcal{L}(V, \mathbb{F})$ by $Tu = \langle u, v\rangle$. For $a \in \mathbb{F}$, find a formula for $T^* a$.

Proof. $T^* \in \mathcal{L}(\mathbb{F}, V)$ is defined as the operator such that $\langle Tu, a\rangle = u, T^*a$. We manipulate:

$$\begin{aligned}
\langle Tu, a\rangle &= \langle u, T^*a\rangle \\
\langle \langle u, v\rangle, a\rangle &= \\
\langle u, v\rangle \overline{a} &= \\
\langle \overline{a}u, v\rangle &= \\
\overline{a}\langle u, v\rangle &= \\
\langle u, av\rangle &= \langle u, T^*a\rangle
\end{aligned}$$

and so $T^*a = av$. □

Axler 2e, Exercise 6.30: Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective iff T^* is surjective.

Proof. T is injective iff $\operatorname{null} T = \{0\}$ (by Axler 3e Thm 3.16). But $\operatorname{null} T = (\operatorname{range} T^*)^\perp$, so we have $(\operatorname{range} T^*)^\perp = \{0\}$. That is the case iff $\operatorname{range} T^* = W$, which is equivalent to T^* being surjective. Therefore T^* is surjective. □

(b) T is surjective iff $T*$ is injective.

Proof. Replace T above with T^*. □

Axler 2e, Exercise 6.31: For every $T \in \mathcal{L}(V, W)$, prove that

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$

Proof. Note that by the Fundamental Theorem of Linear Maps (Axler 3e Thm 3.22), $\dim \operatorname{null} T* = \dim W - \dim \operatorname{range} T^*$. Thus it remains to show that $\dim \operatorname{range} T^* = \dim V - \dim \operatorname{null} T$; to do so, notice that $\operatorname{range} T^* = (\operatorname{null} T)^\perp$ and clearly $\dim(\operatorname{null} T)^\perp = \dim V - \dim \operatorname{null} T$. □

and

$$\dim \operatorname{range} T^* = \dim \operatorname{range} T.$$

Proof.

$$\begin{aligned}
\dim V - \dim \operatorname{null} T &= \dim \operatorname{range} T \\
\dim(\operatorname{null} T)^\perp &= \dim \operatorname{range} T \\
\dim \operatorname{range} T^* &= \dim \operatorname{range} T
\end{aligned}$$

□