Homework 4

Nathaniel Hamovitz
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1. Consider $\mathbb{R}$ with the usual distance. Using only the definition, prove

(a) $A = [0, 1)$ is not compact.

Proof. Consider the open cover $\{\sigma_n\}_{n\in\mathbb{N}}$, where $\sigma_n = \left(-\frac{1}{2}, 1 - \frac{1}{n}\right)$. As the infimum of each interval remains constant and the supremum is increasing we have that $\forall n \in \mathbb{N}, \sigma_n \subsetneq \sigma_{n+1}$.

Given any finite subcover $\{\sigma_{n_i}\}$, we can consider $j = \max\{n_i\}$. From above, we see that $\cup_{n_i}\sigma_{n_i} \subseteq \sigma_j$. But there must exist $x \in [0, 1)$ such that $x \notin \sigma_j$; consider $x = 1 - \frac{1}{2j}$.

Hence there is at least one open cover of A from which it is not possible to extract a finite subcover, and so A is not compact. $\square$

(b) $D = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\} \cup \{1\}$ is compact.

Proof. Let $\{\sigma_\alpha\}_{\alpha\in I}$ be an open covering of D. Then there must be some open set σ_0 such that $1 \in \sigma_0$.

σ_0 contains all but finitely many elements of D; each remaining element must be contained in some set σ_n. There are finitely many, so considering those σ_n along with σ_0 gives us a finite subcovering of D. $\square$

2. Prove that if $A_1, \ldots, A_n, \ldots$ is a countable collection of countable sets, then

$$\bigcup_{n=1}^{\infty} A_n \qquad \text{is also countable.}$$

Proof. Let $U = \bigcup_{n=1}^{\infty} A_n$.

Write each A_n in a row, like so (here the exponent is index within each set):

$$\begin{aligned} A_1 &= \left\{a_1^1, a_1^2, a_1^3, \ldots, a_1^n, \ldots\right\} \\ A_2 &= \left\{a_2^1, a_2^2, a_2^3, \ldots, a_2^n, \ldots\right\} \\ A_3 &= \left\{a_3^1, a_3^2, a_3^3, \ldots, a_3^n, \ldots\right\} \end{aligned}$$

Now consider the function/sequence $f : \mathbb{N} \mapsto U$ constructed by drawing arrows from top right to bottom left, so that (for example)

$$\begin{aligned} f(1) &= a_1^1 \\ f(2) &= a_1^2 \\ f(3) &= a_2^1 \\ f(4) &= a_1^3 \\ f(5) &= a_2^2 \\ f(6) &= a_3^1 \end{aligned}$$

and so on. Then f puts U in bijection with the naturals, and so U is countable. (Note that for it to be a true bijection, you must skip any a_i^j you have previously encountered when constructing the sequence.) $\square$

3. Consider $\mathbb{R}^n$ with the usual distance. Prove that if $A \subset \mathbb{R}^n$ is uncountable, then $A' \neq \emptyset$ (i.e. the set of limit points of A is nonempty).

Proof. First we place an open ball of radius 1 around every rational. Clearly their union is the whole space; that is,

$$\mathbb{R}^n = \bigcup_{n=1}^{\infty} B_1(q_n)$$

where $\mathbb{Q}^n = \bigcup_{n=1}^{\infty} \{q_n\}$.

Suppose for a contradiction that $A \cap B_1(q_n)$ is finite for all $n \in N$. Note that

$$\begin{aligned} A &= A \bigcap \mathbb{R}^n \\ &= A \bigcap \left[\bigcup_{n=1}^{\infty} B_1(q_n) \right] \\ &= \bigcup_{n=1}^{\infty} \left[A \bigcap B_1(q_n) \right], \end{aligned}$$

which is the countable union of finite sets. By the previous problem, that implies A is finite or countable. However, we know A is uncountable, and so we have a contradiction. Hence there must exist $n_0 \in \mathbb{N}$ such that $A \cap B_1(q_n)$ is infinite (in fact, that it is uncountable).

Note that

$$A \cap B_1(q_{n_0}) \subseteq B_1(q_{n_0}) \subsetneq \overline{B_1(q_{n_0})}.$$

Then by Bolzano-Weierstrass $(A \cap B_1(q_{n_0}))' \neq \emptyset$.

We now prove a lemma: for any sets $A, B \subseteq X$, $A \subseteq B$ implies $A' \subseteq B'$. Let $x \in A'$. Then $\forall r > 0, B_r(x) \cap A - \{x\} \neq \emptyset$. Because $A \subseteq B$, $B_r(x) \cap A \subseteq B_r(x) \cap B$ and so we have $\forall r > 0, B_r(x) \cap B - \{x\} \neq \emptyset$. Thus $x \in B'$ and $A' \subseteq B'$.

Now note that $A \cap B_1(q_{n_0}) \subseteq A$, so by the above lemma $(A \cap B_1(q_{n_0}))' \subseteq A'$. The smaller set is nonempty and so the larger must be as well; $A' \neq \emptyset$. $\square$

4. Consider $\mathbb{R}^2$ with the usual distance, and define

$$E = \{(\sin(n), \cos(m)) : n, m \in \mathbb{Z}\}$$

Prove that $E' \neq \emptyset$.

Proof. Let $B = [-1, 1] \times [-1, 1]$. Clearly $E \subsetneq B$. B is closed and bounded in $\mathbb{R}^2$ and so by Heine-Borel compact.

Consider the function $f : \mathbb{N} \mapsto \mathbb{R}^2$ where $f(n) = (\sin n, \cos 0) = (\sin n, 1)$. f is an injective map because $\sin a = \sin b$ iff $a - b = 2\pi k$ for some $k \in \mathbb{Z}$, and no natural numbers are a multiple of 2π apart (if two were, then π could be written as a rational number). Therefore $f[\mathbb{N}]$ is infinite. Now notice that $f[\mathbb{N}] \subsetneq E$, and hence E is infinite.

We now apply the Bolzano-Weierstrass theorem. E is an infinite set and a subset of a compact set, and therefore $E' \neq \emptyset$. $\square$

5. Let $K_1, \ldots, K_n, \ldots$ be a family of compact subsets of $\mathbb{R}^n$ considered with the usual distance.

(a) Is $K_1 \cup K_2$ compact?

Yes.

Proof. Let $\{\sigma_\alpha\}_{\alpha \in I}$ be an open covering of $K_1 \cup K_2$. Because K_1 and K_2 are both compact, it is possible to extract finite subcoverings $\{\sigma_{\alpha_i}\}_{i=1}^{N}$ and $\left\{\sigma_{\alpha_j}\right\}_{j=1}^{M}$ such that $K_1 \subseteq \cup_{i=1}^{N} \sigma_{\alpha_i}$ and $K_2 \subseteq \cup_{j=1}^{M} \sigma_{\alpha_j}$. Thus

$$K_1 \cup K_2 \subseteq \left[\cup_{i=1}^{N} \sigma_{\alpha_i}\right] \cup \left[\cup_{j=1}^{M} \sigma_{\alpha_j}\right]$$

and as both those collections are finite, so is their union. □

(b) Is $\bigcup_{n=1}^{\infty} K_n$ compact?

No.

Proof. Consider $K_n = \overline{B_1(q_n)}$, where $\mathbb{Q}^n = q_1, q_2, \ldots$. Each K_n is closed and bounded, and thus compact, but their union is $\mathbb{R}^n$, which is open and not bounded and so not compact.

□

(c) Is $\bigcap_{n=1}^{\infty} K_n$ compact?

Yes.

Proof. (A more general proof using the definition of compactness is possible, but if we restrict ourself to $\mathbb{R}^n$, the proof becomes simple.)

Each K_n is closed and bounded, because it is compact.

By definition of boundedness, $\exists x \in \mathbb{R}^n, \exists M \in \mathbb{R}, K_1 \subseteq B_M(x)$. By definition of intersection, $\cap_{n=1}^{\infty} K_n \subseteq K_1 \subseteq B_M(x)$ and hence the intersection is bounded.

The intersection, including the infinite intersection, of closed sets is closed; hence $\cap_{n=1}^{\infty} K_n$ is closed.

Therefore by Heine-Borel $\cap_{n=1}^{\infty} K_n$ is compact. □