Homework 2

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1. Prove that there exists $r \in (0,1)$ such that

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\} \cap \{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$$

is empty.

Proof. Denote the set on the left by C_r for each r and notice that the set on the right is $\mathbb{Q} \times \mathbb{Q}$. $\mathbb{Q} \times \mathbb{Q}$ is countable, because a finite product of countable sets is countable.

But the set $\{C_r : r \in (0,1)\}$ is uncountable, by bijection with $(0,1)$.

Let $A_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\} \cap \{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$. Suppose for a contradiction that $\forall r \in (0,1), \exists p_r \in A_r$. Then there is a map $\psi : (0,1) \mapsto \mathbb{Q} \times \mathbb{Q}$ defined by $r \mapsto p_r$. Note that ψ is an injection. (Proof: Let $r, s \in (0,1)$. Suppose $r \neq s$. $r = \|p_r\|$ and $s = \|p_s\|$, so $\|p_r\| \neq \|p_s\|$. If p_r and p_s have different norms, then clearly $p_r \neq s_r$.) But an injection from an uncountable set to a countable one is an impossibility, and so we have a contradiction. □

2. Use the notation introduced in class and in the book (see Exercise 9 and Definition 2.26 Chapter 2). Let A, B be subsets of a metric space (X, d). Prove of give a counter-example of the following statements:

(i) $(A \cup B)^o = A^o \cup B^o$

Counterexample. Consider $A = (0, 1)$ and $B = [1, 2]$. Then $(A \cup B) = (0, 2]$ and so $(A \cup B)^o = (0, 2)$. But $A^o = (0, 1)$ and $B^o = (1, 2)$, so their intersection is empty. □

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. First let $x \in \overline{A \cup B}$. Then $x \in A \cup B$ or $x \in (A \cup B)'$. In the former case, $x \in A$ or $x \in B$; in the latter, $x \in A'$ or $x \in B'$. In any case, x is in one of A, B, A', and B', which are precisely the four sets which $\overline{A} \cup \overline{B}$ is the union of. Thus $x \in overline A \cup \overline{B}$.

Now let $y \in overline A \cup \overline{B}$. Then $y \in A \cup A' \cup B \cup B'$; and re-ordering the union gives $\overline{A \cup B}$. □

(iii) $(A \cap B)^o = A^o \cap B^o$

Proof. First let $x \in (A \cap B)^o$. x is an interior point of $A \cap B$, so there is some neighborhood N of x such that $N \subset A \cap B$. Thus $N \subset A$ and $N \subset B$. Hence $x \in A^o$ and $x \in B^o$, respectively; and so overall $x \in A^o \cap B^o$.

Now let $y \in A^o \cap B^o$. Then y is an interior point of both A and B, so there are some neighborhoods $B_{r_1}(y)$ and $B_{r_2}(y)$ such that $B_{r_1}(y) \in A$ and $B_{r_2}(y) \in B$. Without loss of generality, suppose $r_1 \leq r_2$; then $B_{r_1}(y) \subseteq B_{r_2}(y)$ and so $B_{r_1}(y) \in (A \cap B)$. Therefore y is an interior point of $A \cap B$, or in other words $y \in A \cap B$. □

(iv) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

Counterexample. Consider $A = (0, 1)$ and $B = (1, 2)$. Then $(A \cap B) = \emptyset$ and so $\overline{(A \cap B)} = \emptyset$. But $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$, so their intersection is $\{1\}$.

□

3. Prove that if $A \subset \mathbb{R}$ is bounded above, then

$$\sup A \in \overline{A} = A \cup A'.$$

Proof. Let $m = \sup A$. If $m \in A$ then clearly $m \in \overline{A}$. Suppose $m \notin A$. By the definition of sup, $\forall \varepsilon > 0, \exists a \in A, m - \varepsilon < a < m$. This is exactly the definition of a limit point, and hence $m \in \overline{A}$. $\square$

4. Consider $\mathbb{R}^n$ with the usual metric. Prove that $A \subseteq \mathbb{R}^n$ is open and closed iff $A = \mathbb{R}^n$ or $A = \emptyset$.

Proof. ($\Longrightarrow$) Let $A \subseteq \mathbb{R}^n$ such that A is both open and closed. Suppose for a contradiction that A is nonempty but $A \subsetneq \mathbb{R}^n$. Then $\exists x \in A$ and $\exists y \in A^c$.

Consider the path from y to x defined by $\gamma(t) = (1-t)y + tx$, $t \in [0,1]$. Let $s_0 = \inf\{s : \gamma(s) \in A\}$.

Then $\forall \varepsilon > 0, B_\varepsilon(\gamma(s_0)) \cap A^c \neq \emptyset$. (If that neighborhood were entirely in A^c, then s_0 would not be the infimum.) Then every neighborhood of $\gamma(s_0)$ contains points both in A and in A^c. This is a contradiction.

($\Longleftarrow$) $\mathbb{R}^n$ is both open (because every neighborhood of every point in $\mathbb{R}^n$ is a subset of R^n) and closed (because every point of $\mathbb{R}^n$ is a limit point of $\mathbb{R}^n$). Now we refer to Rudin Them 2.23, which says that a set is open iff it's complement is closed, and that a set is closed iff it's complement is open. $(\mathbb{R}^n)^c = \emptyset$, and so $\emptyset$ is both open and closed as well. $\square$

5. Prove that in any metric space (X, d), if $A \subseteq X$, then $\overline{(A^c)} = (A^o)^c$.

Proof. First let $x \in \overline{A^c}$. Then $x \in A^c$ or $x \in (A^c)'$. Every interior point of a set is in that set, so if $x \in A^c$ then $x \in (A^o)^c$. If $x \in (A^c)'$, then every neighborhood of x contains a point not in A. But then no neighborhood of x is contained in A, and so $x \in (A^o)^c$.

Now let $y \in (A^o)^c$. Then y is not an interior point of A, so there is no neighborhood of y contained entirely within A. That is, every neighborhood of y contains a point from A^c. Thus $y \in (A^c)'$ and so $y \in \overline{(A^c)}$.

Therefore $\overline{(A^c)} = (A^o)^c$.

$\square$