Homework 3

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1. Rudin Ch. 2, Exc. 17: Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Solution. E is not countable; you can draw a bijection between E and the set of functions from $\mathbb{N}$ to $[0,1]$ (if the ith digit is 4, then $f(i) \equiv 0$; if the ith digit is 7, then $f(i) \equiv 1$), a set which we showed in class is uncountable, by Cantor's Diagonal Argument.

E is not dense in $[0,1]$. Because the decimal expansions of the elements of E contain only 4 and 7, $E \subset [0.4, 0.5) \cup [0.7, 0.8)$. Consider $0.6 \in [0,1]$. 0.6 is not in either of those intervals, and so could not possibly be a limit point of E. (To see this concretely, consider $B_{0.05}(0.6)$, which contains no elements of E.) Therefore E is not dense in $[0,1]$.

E is compact. We are in $\mathbb{R}$, and in $\mathbb{R}^n$, any set that is closed and bounded is compact. E is clearly bounded (for example, by 0 below and 1 above). It remains to show that E is closed.

We use the contrapositive. Suppose that $x \notin E$. Then $x = 0.a_1a_2\dots$ and $\exists j \in \mathbb{N}, a_j \notin \{4,7\}$. Consider $\varepsilon = 10^{-(j+2)}$. Then $B_\varepsilon(x) = (0.a_1a_2\dots a_ja_{j+1}[a_{j+2}-1]a_{j+3}a_{j+4}\dots, 0.a_1a_2\dots a_ja_{j+1}[a_{j+2}+1]a_{j+3}a_{j+4}\dots)$. The jth digit of every digit in that ball is $a_j \notin \{4,7\}$, and so $B_\varepsilon(x) - \{x\} \cap E = \emptyset$; thus x is not a limit point of E. (This is a little oversimplified. To restate the idea in words: if $x \notin E$, then there is some digit of x which is not 4 or 7, and you can choose some radius around x such that the digits only vary in places past that non-4-or-7 digit.) No point not in E is a limit point, and so E is closed.

E is perfect. We saw above that E is closed. It remains to show that every point in E is a limit point of E. Let $x \in E$; then $x = 0.a_1a_2\dots$ where $a_i \in \{4,7\}$. Let $\varepsilon > 0$. Then $\exists n \in \mathbb{N}, 3 \cdot 10^{-n} < \varepsilon$. We construct the number $y = b_1b_2\dots$ where all $b_i = a_i$ except for $i = n$, where if $a_n = 4$ then $b_n = 7$, and if $a_n = 7$ then $b_n = 4$. Thus $y \in E$ and $|x - y| = 3 \cdot 10^{-n} < \varepsilon$; that is, a ball around x of any radius will contain another point in E.

□

2. Let (X, d) be a metric space. For $A \subset X$ we define its boundary ∂A and $\partial A = \overline{A} \cap \overline{A^c}$. Prove:

(a) $\partial A = \partial(A^c)$

Proof. $\partial(A^c) = \overline{A^c} \cap \overline{(A^c)^c}$ by definition, which is $\overline{A^c} \cap \overline{A}$, which is the definition of ∂A. □

(b) $\partial A = \overline{A} - A^o$

Proof. First let $x \in \partial A$. Then $x \in \overline{A}$ and $x \in \overline{A^c}$. By a lemma on the previous homework, we know that $\overline{A^c} = (A^o)^c$. Hence, since $x \in \overline{A^c}$, $x \notin A^o$. Therefore $x \in \overline{A}$ and $x \notin A^o$, and so $x \in \overline{A} - A^o$. Therefore $\partial A \subseteq \overline{A} - A^o$.

Now let $x \in \overline{A} - A^o$. Then $x \in \overline{A}$ and $x \notin A^o$. Hence $x \in (A^o)^c$, and by a lemma on the previous homework, $(A^o)^c = \overline{A^c}$. Thus $x \in \overline{A^c}$, and so $x \in \overline{A} \cap \overline{A^c}$. That's the definition of boundary, and so $x \in \partial A$, and therefore $\overline{A} - A^o \subseteq \partial A$.

Therefore $\partial A = \overline{A} - A^o$ □

(c) $\overline{A} = \partial A \cup A^o$

Proof. First let $x \in \overline{A}$. Because $\partial A = \overline{A} - A^o$, we have that $\partial A \cup A^o = (\overline{A} - A^o) \cup A^o$. That is, all the points in the closure except those in the interior, and also all those in the interior. Hence $x \in \partial A \cup A^o$, and so $\overline{A} \subseteq \partial A \cup A^o$.

Now let $x \in \partial A \cup A^o$. Again, this is all the points in the closure except those in the interior, and also all those in the interior. If $x \in \overline{A}$, we are done; and $x \in A^o - \overline{A}$ is a contradiction. (If x is an interior point, a neighborhood around is is a subset of A. That neighborhood contains x, and so $x \in A \subseteq \overline{A}$.) Hence $x \in \overline{A}$, and so $\partial A \cup A^o \subseteq \overline{A}$.

Therefore $\overline{A} = \partial A \cup A^o$. □

(d) $X = A^o \cup \partial A \cup (A^c)^o$

Proof. We first do a manipulation of the RHS to make it easier.

$$
\begin{aligned}
&A^o \cup \partial A \cup (A^c)^o \\
&A^o \cup (\overline{A} - A^o) \cup (A^c)^o \\
&A^o \cup \overline{A} \cup (A^c)^o \\
&A^o \cup A \cup A' \cup (A^c)^o \\
&A \cup A' \cup (A^c)^o
\end{aligned}
$$

First let $x \in X$. We use cases. Case 1: $x \in (A^c)^o$. Case 2: $x \notin (A^c)^o$. Then $\forall r > 0, \exists y \in B_r(x), y \in (A^c)^c$. $(A^c)^c = A$ and so $y \in A$. That's the definition of a limit point, and so then $x \in A'$. Thus in either case, $x \in A' \cup (A^c)^o$, and hence $x \in A \cup A' \cup (A^c)^o$. Therefore $X \subseteq A \cup A' \cup (A^c)^o$.

Clearly $A \cup A' \cup (A^c)^o \subseteq X$.

Therefore $X = A \cup A' \cup (A^c)^o$. □

3. If $X = \mathbb{R}^2$ with the usual distance, find:

(1) $\partial B_1(0)$

Solution. $\partial B_1(0) = \overline{B_1(0)} - (B_1(0))^o$. The closure is the set $\{(x, y) : x^2 + y^2 \leq 1\}$ and the interior is the ball itself, because balls are open. Thus the boundary is the sphere of radius 1, $\{(x, y) : x^2 + y^2 = 1\}$. $\square$

(2) $\partial(\mathbb{Q} \times \mathbb{Q})$

Solution. Because $\mathbb{Q}$ is dense in the reals, $\overline{(\mathbb{Q}^2)} = \mathbb{R}^2$. Because $\mathbb{Q}$ is dense in the reals, no point $q \in \mathbb{Q}$ is an interior point, and so $(\mathbb{Q}^2)^o = \emptyset$. Thus $\partial\mathbb{Q}^2 = \overline{\mathbb{Q}^2} - (\mathbb{Q})^o = \mathbb{R}^2$.

$\square$

(3) $\partial(\mathbb{Z} \times \mathbb{Z})$

Solution. Because $\mathbb{Z}^2$ is a discrete set (every point is isolated), its closure is itself and it has no interior points. Thus $\partial\mathbb{Z}^2 = \overline{\mathbb{Z}^2} - (\mathbb{Z}^2)^o = \mathbb{Z}^2 - \emptyset = \mathbb{Z}^2$.

$\square$

(4) $\partial\{(x, x) : x \in \mathbb{R}\}$

Solution. Let $A_4 = \{(x, x) : x \in \mathbb{R}\}$. Because A_4 is a line embedded in 2D space, its closure is itself, and it has no interior points. Thus $\partial A_4 = \overline{A_4} - A_4^o = A_4 - \emptyset = A_4$.

$\square$

4. Let

$$A = \bigcup_{k \in \mathbb{N}} B_{\frac{1}{2^k}}(q_k)$$

where q_i are the elements of $\mathbb{Q}$.

Find ∂A.

Solution. Because A is the union of a collection of open sets, A is open (by Theorem 2.24(a) in Rudin). Thus $A = A^o$. Therefore $\partial A = \overline{A} - A^o = \overline{A} - A$.

I claim that $\overline{A} = \mathbb{R}$. Clearly $\overline{A} \subseteq \mathbb{R}$.

For the other direction, first we show a lemma: for any $D, E \subseteq X$, $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$. Let $x \in \overline{X}$. We use cases. If $x \in X$, then $x \in Y$ and we are done. If $x \in X'$, then $\forall r > 0, B_r(x) - \{x\} \int X \neq \emptyset$. That is $\exists x_2 \in X, x_2 \neq x, x_2 \in B_r(x)$. Then $x_2 \in Y$ as well, satisfying all the same conditions otherwise, and so $x \in Y'$.

We know $\mathbb{Q} \subseteq A$ because A is the union of balls around the rationals. By the lemma above, $\overline{\mathbb{Q}} \subseteq \overline{A}$. But $\overline{\mathbb{Q}} = \mathbb{R}$, and thus $\mathbb{R} \subseteq \overline{A}$. Therefore $\overline{A} = \mathbb{R}$.

Hence

$$\partial A = \mathbb{R} - A = \mathbb{R} - \bigcup_{k \in \mathbb{N}} B_{\frac{1}{2^k}}(q_k) = \bigcap_{k \in \mathbb{N}} (\mathbb{R} - B_{\frac{1}{2^k}}(q_k))$$

$\square$

5. Consider $\mathbb{R}^n$ with the usual metric. Let $A \subseteq \mathbb{R}^n$. Prove that $\partial A = \emptyset$ iff ($A = \mathbb{R}^n$ or $A = \emptyset$).

Proof. ($\Longrightarrow$) Suppose $\partial A = \emptyset$. We have from 2.c above that $\overline{A} = \partial A \cup A^o$. That becomes $A \cup A' = \emptyset \cup A^o = A^o$; that is, the set and its limit points are all interior points.

Suppose for a contradiction that $A \subsetneq \mathbb{R}^n$ but $A \neq \emptyset$. Then $\exists a \in A$ and $\exists b \in A^c$. There must be a path $\gamma : [0,1] \mapsto \mathbb{R}^n$ such that $\gamma(0) = a$ and $\gamma(1) = b$.

Consider $t_c = \sup \{t : \gamma(t) \in A\}$. Then clearly $c = \gamma(t_c)$ is in some sense "on the boundary" of A. $c \in A'$ but $c \notin A^o$, which is a contradiction.

Therefore $A = \mathbb{R}^n$ or $A = \emptyset$.

($\Longleftarrow$) Suppose $A = \mathbb{R}^n$ or $A = \emptyset$. We use cases. If $A = \emptyset$, we have $\overline{\emptyset} = \emptyset^o = \emptyset$, so $\partial\emptyset = \overline{\emptyset} - \emptyset^o = \emptyset$. If $A = \mathbb{R}^n$, we have $\overline{\mathbb{R}^n} = (\mathbb{R}^n)^o = \mathbb{R}^n$, so $\partial\mathbb{R}^n = \overline{\mathbb{R}^n} - (\mathbb{R}^n)^o = \mathbb{R}^n$. $\square$