Homework 1

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Problem 1. Prove that if $n \in \mathbb{N}$, then $\sqrt{n} \in \mathbb{N} \cup (\mathbb{Q})^c$. Hint: Review and use the unique-prime-factorization theorem.

Proof. We know by the unique prime factorization that $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, where p_i are distinct primes and $k_i \in \mathbb{N}$. If all k_i are even, then $\sqrt{n} = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$, where $m_i = \frac{k_i}{2} \in \mathbb{N}$, and it is evident that $\sqrt{n} \in \mathbb{N}$.

If at least one k_i is odd, then $\sqrt{n}$ must be irrational. Without loss of generality (multiplication commutes), assume k_1 is odd. It suffices to show that $\sqrt{p_1}$ is irrational, because the product of an irrational number is irrational.

Assume for a contradiction that $\sqrt{p_1} \in \mathbb{Q}$. Then we can write $\sqrt{n} = \sqrt{p_1} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$. Without loss of generality, we assume a and b are coprime. Now we have $\left(\frac{a}{b}\right)^2 = p_1$. Then $a^2 = b^2 p_1$, which implies $p_1 \mid a^2$. Recall that for any prime p and square number t^2, $p \mid t^2 \Rightarrow p \mid t$. Thus we have $p_1 \mid a$, and thus $p_1^2 \mid a^2$, and thus (referring to the original equation) $p_1^2 \mid b^2 p_1$, which implies $p_1 \mid b^2$, and therefore $p_1 \mid b$. This is a contradiction, as we assuemd a and b were coprime. Hence $\sqrt{p_1} \in (\mathbb{Q})^c$, and so $\sqrt{n} = \sqrt{p_1}\sqrt{p_1^{k_1-1} p_2^{k_2} \cdots p_m^{k_m}} \in (\mathbb{Q})^c$.

Therefore $\sqrt{n} \in \mathbb{N} \cup (\mathbb{Q})^c$. □

Problem 2. Find an example of a set $A \subset \mathbb{R}$ such that $\lambda = \sup A$ is well defined but we may not be able to find it explicitly.

Solution. One example is the set $A = \{\cos(n) : n \in \mathbb{N}\}$. A is bounded, and so by the least upper bound property must have a supremum, but it is unclear what the supremum is. □

Problem 3. Rudin, Exercise 1-16: Suppose $k \geq 3$, $\hat{x}$, $\hat{y} \in \mathbb{R}^k$, $|\hat{x} - \hat{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\hat{z} \in \mathbb{R}^k$ such that

$$|\hat{z} - \hat{x}| = |\hat{z} - \hat{y}| = r$$

Proof. Without loss of generality (all the relevant operations are invariant under translation and rotation), take $\hat{x}$ and $\hat{y}$ to lay along the x-axis, equidistant from the origin. That is, $\hat{x} = \left(-\frac{d}{2}, 0, \ldots, 0\right)$ and $\hat{y} = \left(\frac{d}{2}, 0, \ldots, 0\right)$.

I claim that the intersection is a sphere in dimension $k-1$, the radius of which is given by $A = \sqrt{r^2 - \left(\frac{d}{2}\right)^2}$. The value of A is derived from the Pythagorean theorem in 3 dimensions, starting from $\left(\frac{d}{2}\right)^2 + A^2 = r^2$.

The curve $C = \{(0, A\cos\theta, A\sin\theta, 0, \ldots, 0) : \theta \in [0, 2\pi)\}$ is clearly infinite, and lies along the intersection mentioned above. Thus it suffices to show that $C \subseteq \{z \in \mathbb{R}^k : |\hat{x} - \hat{z}| = |\hat{y} - \hat{z}| = r\}$.

Let $\hat{z} \in C$. Then $\exists \theta \in [0, 2\pi)$ such that $\hat{z} = (0, A\cos\theta, A\sin\theta, 0, \ldots, 0)$. We have:

$$|\hat{z} - \hat{x}|^2 = \left| \begin{pmatrix} 0 \\ A\cos\theta \\ A\sin\theta \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{d}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \frac{d}{2} \\ A\cos\theta \\ A\sin\theta \\ 0 \\ 0 \end{pmatrix} \right|^2$$

$$= \left(\frac{d}{2}\right)^2 + A^2\cos^2\theta + A^2\sin^2\theta + 0^2 + \cdots + 0^2$$

$$= \left(\frac{d}{2}\right)^2 + A^2(1) + 0 = \left(\frac{d}{2}\right)^2 + A^2 = r^2$$

$$|\hat{z} - \hat{x}| = r$$

The calculation is similar for $|\hat{z} - \hat{y}|$; the only difference is the sign of $\frac{d}{2}$, and it's squared after so it doesn't matter. $\square$

(b) If $2r = d$, there is exactly one such $\hat{z}$.

Proof. We assume $\hat{x}$ and $\hat{y}$ are placed on the x-axis as before.

Suppose the point $\hat{p} = (p_1, p_2, \ldots, p_n)$ satisfies the conditions. Then $|x - p| = |y - p| = r$, or $|x - p|^2 = |y - p|^2 = r^2$.

Using the generalized formula for the norm, we then have $(p_1 + \frac{d}{2})^2 + p_2^2 + \cdots + p_k^2 = r^2$ and $(p_1 - \frac{d}{2})^2 + p_2^2 + \cdots + p_k^2 = r^2$, and subtracting these two equations gives us that

$$\left(p_1 + \frac{d}{2}^2\right) - \left(p_1 - \frac{d}{2}^2\right) = 0$$

$$p_1^2 + 2p_1\frac{d}{2} + \left(\frac{d}{2}\right)^2 - p_1^2 + 2p_1\frac{d}{2} - \left(\frac{d}{2}\right)^2 = 0$$

$$4p_1\frac{d}{2} = 0$$

$$p_1 d = 0$$

and as $d > 0$, we find $p_1 = 0$.

Hence $\hat{p} = (0, p_2, \ldots, p_k)$. Now we return to our assumption, that $|\hat{x} - \hat{p}|^2 = r^2$.

$$|\hat{x} - \hat{p}|^2 = r^2$$

$$\left| \begin{pmatrix} -\frac{d}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ p_2 \\ \vdots \\ p_k \end{pmatrix} \right|^2 = r^2$$

$$\left(-\frac{d}{2}\right)^2 + p_2^2 + \cdots + p_k^2 = r^2 = \left(\frac{d}{2}\right)^2$$

$$p_2^2 + \cdots + p_k^2 = 0$$

but all $p_i^2 \geq 0$, so their sum can only be 0 if all $p_i^2 = 0$, and thus all $p_i = 0$.

Again, the calculation is similar for $\hat{y}$; the only difference is the sign of $\frac{d}{2}$, and it is squared afterward.

Therefore $\hat{p} = (0, \ldots, 0)$ satisfies $|\hat{z} - \hat{x}| = |\hat{z} - \hat{y}| = r$; and it is the only point which could do so.

$\square$

(c) If $2r < d$, there is no such $\hat{z}$.

Proof. Suppose for a contradiction that such a $\hat{z}$ did exist. Then we have, by hypothesis, $|\hat{x} - \hat{y}| = d > 2r = |\hat{x} - \hat{z}| + |\hat{z} - \hat{y}| = 2r$. But the Triangle Inequality tells us that $|\hat{x} - \hat{y}| \leq |\hat{x} - \hat{z}| + |\hat{z} - \hat{y}|$, and we have a contradiction. $\square$

(d) How must these statements be modified if k is 2 or 1?

Solution. If $k = 2$, then there are two such $\hat{z}$ when the circles overlap; the solutions for (b) and (c) remain the same.

If $k = 1$, then when the "spheres" overlap there are no such solutions. When $2r = d$, there is again one solution, at $\frac{x+y}{2}$. And the solution for (c) is the same. $\square$