Homework 2

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1. Let $f : [a,b] \to \mathbb{R}$ be a function such that $\exists M > 0, \exists \alpha > 1$ such that

$$\forall x, y \in \mathbb{R}, \qquad |f(x) - f(y)| \leq M\,|x - y|^{\alpha}.$$

Prove that f is constant.

Proof. First we show that f must be continuous on $[a,b]$. Let $\varepsilon > 0$. Consider $\delta = \left(\frac{\varepsilon}{M}\right)^{\frac{1}{\alpha}}$. Then $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Now we show that f must also be differentiable. Let $x_0 \in [a,b]$ and let

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note that

$$\begin{aligned} 0 &\leq \left| \frac{f(x) - f(x_0)}{x - x_0} \right| \\ &= \frac{|f(x) - f(x_0)|}{x - x_0} \\ &\leq \frac{M\,|x - x_0|^{\alpha}}{x - x_0} \\ &= M\,|x - x_0|^{\alpha - 1} \end{aligned}$$

and as $\alpha > 1$, this clearly goes to 0 as $x \to x_0$. Thus by Squeeze Theorem $f' \equiv 0$.

Finally we appeal to the Mean Value Theorem; the equation $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$ must, for all $x_1, x_2 \in (a,b)$, hold for some $x \in (a,b)$. But we know that $\forall x \in (a,b), f'(x) = 0$, and hence that $\forall x_1, x_2 \in (a,b), f(x_2) = f(x_1)$. Therefore f is constant. $\square$

2. Let $n \in \mathbb{N}$, $n > 2$, $a, b \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = x^n + ax + b.$$

Prove that f vanishes in at most three points.

Proof. We have $f'(x) = nx^{n-1} + a$, and so at any critical point $f'(x) = 0$,

$$\begin{aligned} 0 &= nx^{n-1} + a \\ -\frac{a}{n} &= x^{n-1} \\ x &= \sqrt[n-1]{-\frac{a}{n}} \end{aligned}$$

Such an expression has at most 2 real solutions (1 when $n-1$ is odd or $a=0$. For $n-1$ even, it depends on the sign of a: 2 solutions for $a<0$ and 0 for $a>0$.)

The Mean Value Theorem tells us that each vanishing point of a function after the first requires a corresponding point where $f'(x)=0$ somewhere before, in order to satisfy $f(x_2)-f(x_1)=0=(x_2-x_1)f'(x)$.

Therefore f vanishes in at most three points. $\square$

3. Show that $(\sin(x))' = \cos(x)$.

Proof. The proof relies upon a few trigonometric identities and the facts that: $\lim_{x\to 0}\frac{\sin x}{x}=1$ (which can be proven by considering the areas of the two right triangles with base along the x-axis and height equal to $|\sin x|$ and $|\tan x|$, respectively, and then employing the Squeeze Theorem); and $\lim_{x\to 0}\frac{1-\cos x}{x}=0$ (which can be proven using the identity $1-\cos^2 x=\sin^2 x$ and some clever algebra).

$$\begin{aligned}
(\sin x)' &\equiv \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{(\cos x)(\sin h)+(\sin x)(\cos h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{(\cos x)(\sin h)}{h}+\frac{(\sin x)(\cos h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{(\cos x)(\sin h)}{h}+\lim_{h\to 0}\frac{(\sin x)(\cos h)-\sin x}{h}\\
&= \lim_{h\to 0}(\cos x)\frac{\sin h}{h}+\lim_{h\to 0}\frac{(\sin x)(\cos h-1)}{h}\\
&= \cos x\cdot\left[\lim_{h\to 0}\frac{\sin h}{h}\right]-\sin x\cdot\left[\lim_{h\to 0}\frac{1-\cos h}{h}\right]\\
&= \cos x\cdot 1 - sinx\cdot 0\\
&= \cos x
\end{aligned}$$

$\square$

4. For $a\geq 0$ define $f_a:\mathbb{R}\to\mathbb{R}$ by

$$f_a(x)=\begin{cases} x^a\sin\left(\frac{1}{x}\right), & x>0\\ 0, & x\leq 0\end{cases}$$

For which values of a:

1. ...is f_a continuous (on $\mathbb{R}$)?

   Proof. As the composition of continuous functions, f_a is continuous on $\mathbb{R}-\{0\}$ for all a. The only possible problem is at $x=0$. Consider when $a=0$. The $\lim_{x\to 0}x^0=1$ and so $\lim_{x\to 0}f_a(x)=\sin\left(\frac{1}{x}\right)$, which is not even defined; that function has crazy oscillations as $x\to 0$. But for any $a>0$, $\lim_{x\to 0}x^a=0$, and since the sin term is bounded, the whole thing reaches 0 at 0. As a consequence, for any $\epsilon>0,\exists\delta>0,|f_{a>0}(\delta)|=\epsilon$, and hence f_a is continuous on all of $\mathbb{R}$. $\square$

2. ...does $f'_a(0)$ exist?

Proof. By definition,

$$\begin{aligned} f_a'(0) &= \lim_{x\to 0} \frac{f_a(x) - f_a(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{f_a(x) - 0}{x} \\ &= \lim_{x\to 0} x^{a-1} \sin\left(\frac{1}{x}\right) \end{aligned}$$

$f_a'(0)$ exists when this limit converges, and by a similar argument to above, this limit only converges when $a > 1$, so that $\lim_{x\to 0} x^{a-1} = 0$. □

3. ...is f_a' continuous at $x = 0$?

   Proof. Clearly $f_a' = 0$ on $x < 0$, we saw above that $f_a'(0) = 0$ if $a > 1$, and for $x > 0$ the function is well-behaved and we can use our usual differentiation rules:

   $$\begin{aligned} f_a'(x) &= ax^{a-1} \sin\left(\frac{1}{x}\right) + x^a \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), \quad x > 0 \\ &= ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right) \end{aligned}$$

   Perfectly continuous on $x > 0$; to ensure continuity at $x = 0$, this expression must go to 0 as $x \to 0$. Our smallest exponent of x now is $a - 2$, and so we need $a > 2$. □

4. ...does $f_a''(0)$ exist?

   Proof. Once again we appeal to the limit definition:

   $$\begin{aligned} f_a''(0) &= \lim_{x\to 0} \frac{f_a'(x) - f_a'(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x\to 0} ax^{a-2} \sin\left(\frac{1}{x}\right) - x^{a-3} \cos\left(\frac{1}{x}\right) \end{aligned}$$

   The derivative exists iff this limit converges; our smallest exponent is now $a - 3$ and so $f_a''(0)$ will exist when $a > 3$.

   □

5. ...is f_a'' continuous at $x = 0$?

   Proof. And now we must again consider the expression for f_a'' on $x > 0$:

   $$f_a''(x) = (a)(a-1)x^{a-2} \sin\left(\frac{1}{x}\right) + ax^{a-1} \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - (a-2)x^{a-3} \cos\left(\frac{1}{x}\right) - x^{a-2} \sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

   And the smallest power of x, coming from the last term, will be $a - 4$; therefore f_a'' is continuous at $x = 0$ when $a > 4$. □

Rudin 5-2. (pt 1) Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b).

Proof. This is a simple application of the Mean Value Theorem. Let $x_1, x_2 \in (a,b)$ such that $x_2 > x_1$, the equation $f(x_1) - f(x_2) = (x_2 - x_1)f'(x)$ for *some* $x \in (x_1, x_2)$. The RHS is positive, and so we have $f(x_1) > f(x_2)$. $\square$

Rudin 5-2. (pt 2) Now let g be the inverse function of f. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad x \in (a,b).$$

Proof. We first show that g is differentiable on (a,b). Let $x_0 \in (a,b)$. As $g = f^{-1}$, $x - x_0 = g(f(x)) - g(f(x_0))$. Hence

$$\begin{aligned}
1 &= \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\
1 &= \frac{g(f(x)) - g(f(x_0))}{x - x_0} \cdot 1 \\
1 &= \frac{g(f(x)) - g(f(x_0))}{x - x_0} \cdot \frac{f(x) - f(x_0)}{f(x) - f(x_0)} \\
1 &= \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}
\end{aligned}$$

Now take the limit on both sides as $x \to x_0$:

$$\begin{aligned}
\lim_{x \to x_0} 1 &= \lim_{x \to x_0} \left[\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0} \right] \\
1 &= \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}
\end{aligned}$$

Finally note that since f' exists, f is continuous, and thus as $x \to x_0$, $|f(x) - f(x_0)| \to 0$. Hence, while the LHS remains 1, the RHS is in the form of derivative formulas, and we can write $1 = g'(f(x)) \cdot f'(x)$. Simply divide by $f'(x)$ (recalling that f' is always positive) and we have

$$g'(f(x)) = \frac{1}{f'(x)}.$$

$\square$

Rudin 5-3. Suppose g is a real function on $\mathbb{R}$, with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough.

Proof. Let $x_1, x_2 \in \mathbb{R}$ (WLOG, suppose $x_1 \leq x_2$) and suppose that $f(x_1) = f(x_2)$. Then $f(x_2) - f(x_1) = 0$ and so by the Mean Value Theorem either $x_2 - x_1 = 0$ or $f'(x) = 0$ for some $x \in (x_1, x_2)$.

Suppose for a contradiction that $f'(x) = 0$. We have $f'(x) = 1 + \varepsilon g'(x)$. Hence

$$\begin{aligned}
0 &= 1 + \varepsilon g'(x) \\
g'(x) &= -\frac{1}{\varepsilon} \\
|g'(x)| &= \frac{1}{\varepsilon}
\end{aligned}$$

Now consider when $\varepsilon \leq \dfrac{1}{M+1}$. Then $|g'(x)| = M + 1 > M$, which is a contradiction. Thus $x_2 = x_1$, and therefore f is injective. $\square$

Rudin 5-15. Suppose $a \in \mathbb{R}$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leq 4M_0M_2.$$

Proof. It can be shown from the hypothesis that $\forall h > 0$,

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

(This is given as a hint in Rudin; deriving it involves a clever reformulation of the Taylor Polynomial Theorem.) From there by a property of the supremum we have that $M_1 \leq hM_2 + \frac{M_0}{h}$. All these values are nonnegative so we can square and find $M_1^2 \leq h^2M_2^2 + \frac{M_0^2}{h^2} + 2M_2M_0$. Call the RHS $g(h)$; as $M_1^2 \leq g(h)$ holds for all positive h, it suffices to show there $\exists h > 0, g(h) = 4M_0M_2$. Consider $h = \sqrt{\frac{M_0}{M_2}}$ (can be deduced by solving for h directly or by considering the critical points of g); then

$$\begin{aligned} g(h) &= \frac{M_0}{M_2}M_2^2 + \frac{M_0^2}{\frac{M_0}{M_2}} + 2M_2M_0 \\ &= M_0M_2 + M_0M_2 + 2M_0M_2 \\ &= 4M_0M_2 \end{aligned}$$

As the above holds for $h > 0$, we must also consider when $M_0 = 0$; in that case, $f \equiv 0$ and so $f = f' = f'' = 0$ and indeed $0^2 \leq 4 \cdot 0 \cdot 0$.

$\square$