Homework 1

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Math 104A, Atzberger, F22

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1 Ch 1.1

2 Show that the following equations have at least one solution in the given intervals.

a. $\sqrt{x} - \cos x = 0$ on $[0, 1]$

Solution. We use the Intermediate Value Theorem, which is allowed because the LHS is a continuous function of x. $\sqrt{(0)} - \cos(0) = -1 < 0$ and $\sqrt{(1)} - \cos(1) \approx 0.460 > 0$. Thus by IVT, $\exists c \in [0, 1], \sqrt{c} - \cos c = 0$. $\square$

b. $e^x - x^2 + 3x - 2 = 0$ on $[0, 1]$.

Solution. The process is similar. $e^0 - 0^2 + 3(0) - 2 = -1 < 0$ and $e^1 - 1^2 + 3(1) - 2 = e > 0$. $\square$

c. $-3\tan(2x) + x = 0$ on $[0, 1]$.

Solution. Here, testing the endpoints finds us the solution directly. $-3\tan(2(0)) + 0 = 0$ and $0 \in [0, 1]$. $\square$

3 Find intervals containing solutions to the following equations.

a. $x - 2^{-x} = 0$

Solution. $[0, 1]$ works; $0 - 2^0 = -1 < 0$ and $1 - 2^{-1} = 1 - \frac{1}{2} = \frac{1}{2} > 0$. $\square$

c. $3x - e^x = 0$

Solution. $[0, 1]$ works again; $3(0) - e^0 = -1 < 0$ and $3(1) - e^1 = 3 - e \approx 0.3 > 0$. $\square$

8 Suppose $f \in C[a, b]$ and $f'(x)$ exists on (a, b). Show that if $f'(x) \neq 0$ for all $x \in (a, b)$, then there can exists at most one number $p \in [a, b]$ with $f(p) = 0$

Proof. The derivative is everywhere nonzero, so there are no critical points. Thus the function can't possibly "turn around," because doing so would create a local extrema, which would be a critical point, which is impossible. Hence there might or might not be one crossing of the x-axis in the interval but there can't be more than one. $\square$

9 Let $f(x) = x^3$.

a. Find the second Taylor polynomial $P_2(x)$ about $x_0 = 0$

Solution.

$$P_2(x) = 0 + 0 + 0 = 0$$

$\square$

b. Find $R_2(0.5)$ and the actual error in using $P_2(0.5)$ to approximate $f(0.5)$.

Solution.

$$R_2(x) = 1(x)^3$$

Because the $\xi(x)$ term falls out in the third derivative, R_2 is the exact error. Thus $R_2(0.5) = 0.125$ is the exact error in using $P_2(0.5)$ to approximate 0.5^3. $\square$

c. Now about $x_0 = 1$:

Solution.

$$P_2(x) = 1 + 3(x-1) + \frac{6}{2}(x-1)^2$$

$\square$

d. And the associated remainder term again:

Solution. Using the formula for the remainder term at $x_0 = 1$,

$$R_2(x) = 1(x-1)^3$$

Then $R_2(0.5) = -0.125$ and $P_2(0.5) = 1 + 3(-0.5) + 3(-0.5)^2 = 1 - 1.5 + 0.75 = 0.25$, giving an exact error of -0.125. This makes sense because the $\xi(x)$ term fell out of the derivative again. $\square$

11 Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.
We'll need three derivatives:

$$\begin{aligned}
f(x) &= e^x \cos x \\
f'(x) &= e^x (\cos x - \sin x) \\
f''(x) &= -2e^x \sin x \\
f^{(3)} &= -2e^x(\sin x + \cos x)
\end{aligned}$$

a. Thus by Taylor, $P_2(x) = 1\left[1 + 1 \cdot x + 0 \cdot x^2\right] = 1 + x$ and so $P_2(0.5) = 1.5$. Likewise $R_2(x) = -\frac{1}{3}e^{\xi(x)}\left(\sin \xi(x) + \cos \xi(x)\right)(x^3)$. sin and cos are both bounded in magnitude by 1, so a reasonable upper bound for absolute error would be $R_2(0.5) \leq \frac{1}{3}e^0.5(2)(0.5^3) \approx 1.37393 \cdot 10^{-1}$.

b. As above, $R_2(x) = -\frac{1}{3}e^{\xi(x)}\left(\sin \xi(x) + \cos \xi(x)\right)(x^3)$. We take the absolute value and note that the sum of the trigonometric functions must always be ≤ 2, and the other parts of the function are monotonically increasing. Thus on $[0,1]$, $|R_2| \leq \frac{1}{3} \cdot e^1 \cdot 2 \cdot 1^3 = \frac{2}{3}e$.

c. $\int_0^1 P_2(x)dx = \int_0^1 1 + x\, dx = \left[x + \frac{1}{2}x^2\right]_0^1 = 1.5$

d. $\int_0^1 |R_2(x)|\, dx = \frac{1}{3}\int_0^1 e^{\xi(x)}\left[\sin \xi(x) + \cos \xi(x)\right] x^3\, dx \leq \frac{2}{3}\int_0^1 e^x x^3\, dx \approx 0.356$

14 Let $f(x) = 2x\cos(2x) - (x-2)^2$ and $x_0 = 0$
We'll need five derivatives:

$$\begin{aligned}
f &= 2x\cos(2x) - (x-2)^2 \\
&= 2x\cos(2x) - x^2 + 4x - 4 \\
f' &= 2\cos(2x) + 2x\,(-\sin(2x)) \cdot 2 - 2x + 4 \\
&= 2\left[\cos(2x) - 2x\sin(2x) - x + 2\right] \\
f^{(2)} &= 2\left[-\sin(2x) \cdot 2 - 2\sin(2x) - 2x\cos(2x) \cdot 2 - 1\right] \\
&= 2\left[-4\sin(2x) - 4x\cos(2x) - 1\right] \\
&= -8\,(\sin(2x) + x\cos(2x)) - 2 \text{(used derivative-caluclator for the remaining three)} \\
f^{(3)} &= 16x\sin(2x) - 24\cos(2x) \\
f^{(4)} &= 64\sin(2x) + 32x\cos(2x) \\
f^{(5)} &= 160\cos(2x) - 64x\sin(2x)
\end{aligned}$$

a. Now we plug in $x_0 = 0$, giving

$$P_3(x) = -4 + 6x - x^2 - 4x^3$$

and thus $P_3(0.4) = -2.016$

b. By formula,

$$R_3(x) = \frac{64\sin(2\xi(x)) + 32\xi(x)\cos(2\xi(x))}{24}(x^4).$$

A reasonable upper bound for the error at $x = 0.4$ would be $R_3(0.4) \leq \frac{64\cdot 1 + 32\cdot 0.4\cdot 1}{24}(0.4^4) = 8.192\cdot 10^{-2}$.
The absolute error is $e_{\text{abs}} = |P_3(0.4) - f(0.4)| \approx 1.336536748 \cdot 10^{-2}$.

c. The fourth derivative at 0 is itself 0, so in fact $P_4(x) = P_3(x) = -4 + 6x - x^2 - 4x^3$, and again $P_4(0.4) = -2.016$.

d. However, moving up a derivative allows us the bound the error much tighter; now $R_4(x) = \frac{160\cos(2\xi(x)) - 64\xi(x)\sin(2\xi(x))}{120}(x^5$ and applying the same heuristics as before for the bound gives us $R_4(0.4) \leq \frac{160-64(0)}{120}(0.4^5) = 1.365\bar{3} \cdot 10^{-2}$, an error bound about 6 times smaller. Finally, because $P_4(x) = P_3(x)$, the absolute error remains the same at $\approx 1.336536748 \cdot 10^{-2}$.

15) 4th Taylor,

$f(x) = xe^{x^2}$

$f' = e^{x^2} + x \cdot 2xe^{x^2} = (2x^2+1)e^{x^2}$

$f'' = 2xe^{x^2} + 2\left(2xe^{x^2} + x^2 \cdot 2xe^{x^2}\right) = 2xe^{x^2} + 4\left[xe^{x^2} + x^3e^{x^2}\right]$

$= \cancel{x}e^{x^2}\left[2 + 4 + 4x^2\right] = (4x^3 + 6x)e^{x^2}$

<deriv-calc>:

$f^{(3)} = (8x^4 + 24x^2 + 6)e^{x^2}$

$f^{(4)} = (16x^5 + 80x^3 + 60x)e^{x^2}$

$f^{(5)} = (32x^6 + 240x^4 + 360x^2 + 60)e^{x^2}$

$P_4(x) = 0(\sim) + (2 \cdot 0 + 1)e^0(x) + 0 \cdot x^2 + \frac{6e^0}{6}x^3 + \frac{0}{24} \cdot x^4$

$= 1x + 1x^3 = \cancel{ex(1+x^2)}\ x + x^3$

$R_4(x) = \left(\frac{32\xi^6(x) + 240\xi^4(x) + 360\xi^2(x) + 60}{120}\right)x^5$

(a) R_4 is monotically inc on $[0, \infty)$ so upper bdd for error of P_4 on $[0,$

$\Rightarrow R_4(0.4) \underset{\xi(0.4)=0.4}{\leq} 0.0105706772 81$

(b) $\int_0^{0.4} f(x)\,dx = \int_0^{0.4} P_4(x)\,dx + \int_0^{0.4} R_4(x)\,dx$

$= \int_0^{0.4} ex + \cancel{e}x^3\,dx = \left[\frac{e}{2}x^2 + \frac{e}{4}x^4\right]_0^{0.4} = \cancel{0.23495955}\ 0.0864$

(c) Upper bound for error = bound $\int_0^{0.4} R_4(x)\,dx$

$\hookrightarrow 0.4 \cdot \left\{R_4(0.4) \leq R_{4,\ \xi(x)=0.4}(0.4) \text{ at (a)}\right\} = 0.00422826994$

(d) $P_4' = \cancel{e + 3ex^2}\ 1 + 3x^2$

$P_4'(0.2) = \cancel{3.0444475648}\ 1.2$

$err = \left|f'(0.2) - P_4'(0.2)\right| \approx \cancel{1.920400012}$

$0.0040756361277 8$

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a. The nice property of the Taylor polynomial is that the first n derivatives match the original function at x_0. It's a quite good approximation locally, but then becomes very bad.

b. From the clues given, we now that $f(1) = -1$, $f'(1) = 4$, and $f''(1) = 6$. Thus at $x_0 = 1$, $P_2(x) = -1 + 4(x-1) + 3(x-1)^2$.

2 Ch 1.2

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c. $p = e; p^* = 2.718$. $e_{abs} = |e - 2.718| \approx 0.282 \cdot 10^{-3}$; $e_{rel} = \frac{|e-2.718|}{e} \approx 0.104 \cdot 10^{-3}$.

d. $p = \sqrt{2}; p^* = 1.414$. $e_{abs} = \left|\sqrt{2} - 1.414\right| \approx 0.214 \cdot 10^{-3}$; $e_{rel} = \frac{|e-2.718|}{\sqrt{2}} \approx 0.151 \cdot 10^{-3}$.

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a. $p = e^{10}; p^* = 22000$. $e_{abs} = \left|e^{10} - 22000\right| \approx 0.265 \cdot 10^2$; $e_{rel} = \frac{\left|e^{10}-22000\right|}{e^{10}} \approx 0.120 \cdot 10^{-2}$.

b. $p = 10^\pi; p^* = 1400$. $e_{abs} = |10^\pi - 1400| \approx 0.145 \cdot 10^2$; $e_{rel} = \frac{|10^\pi - 1400|}{10^\pi} \approx 0.105 \cdot 10^{-1}$.

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a.
(i) $\frac{4}{5} + \frac{1}{3} = \frac{12+5}{15} = \frac{17}{15} = 0.11\bar{3} \cdot 10^1$

(ii) 3-digit chopping: $\frac{4}{5} + \frac{1}{3} = 0.800 + 0.333 = 1.133$ becomes $0.113 \cdot 10^1$, for a relative error of $\approx 0.294 \cdot 10^{-2}$.

(iii) 3-digit rounding: $\frac{4}{5} + \frac{1}{3} = 0.800 + 0.333 = 1.133$ becomes $0.113 \cdot 10^1$, for the same relative error of $\approx 0.294 \cdot 10^{-2}$.

b.
(a) $\frac{4}{5} \cdot \frac{1}{3} = \frac{4}{15} = 0.26\bar{6} \cdot 10^0$

(b) 3-digit chopping: $\frac{4}{5} \cdot \frac{1}{3} = 0.800 \cdot 0.333 = 0.2664$ becomes $0.266 \cdot 10^0$, for a relative error of $0.25 \cdot 10^{-2}$.

(c) 3-digit rounding: $\frac{4}{5} \cdot \frac{1}{3} = 0.800 \cdot 0.333 = 0.2664$ becomes $0.266 \cdot 10^0$, for the same relative error of $0.25 \cdot 10^{-2}$.

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a. $\dfrac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$ becomes $\frac{0.9285 - 0.8571}{2(2.718) - 5.400} = \frac{0.0714}{0.036} = 1.98\bar{3}$ becomes $0.1983 \cdot 10^{-1}$, for a relative error of $\approx 0.153 \cdot 10^{-1}$

b. $-10\pi + 6e - \dfrac{3}{62}$ becomes $-10(3.141) - 6(2.718) - 0.04838 = 31.41 - fl(16.308) - 0.04838 = 31.41 - 16.30 - 0.04838 = fl(15.11) - 0.04838 = fl(15.06162) = 0.1506 \cdot 10^2$

c. $\dfrac{2}{9} \cdot \dfrac{9}{7}$ becomes $fl\left(fl\left(\frac{2}{9}\right) \cdot fl\left(\frac{9}{7}\right)\right) = 0.2855 \cdot 10^0$

d. $\dfrac{\sqrt{13} + \sqrt{11}}{\sqrt{13} - \sqrt{11}}$ becomes $fl\left(\frac{fl\left(fl\left(\sqrt{13}\right) + fl\left(\sqrt{11}\right)\right)}{fl\left(fl\left(\sqrt{13}\right) - fl\left(\sqrt{11}\right)\right)}\right) = 0.2394 \cdot 10^2$

15 For problem 15a, the classic quadratic formula gives $x_1 = 92.26$ (with errors $e_{abs} = 0.015420372687700024$ and $e_{rel} = 0.00016716833390104444$) and $x_2 = 0E + 2$ (with errors $e_{abs} = 0.005420372687694908$ and $e_{rel} = 1.0$). The rationalized quadratic formula gives $x_1 = -Infinity$ (with errors $e_{abs} = 0$ and $e_{rel} = 0.0$) and $x_2 = 0.005421$ (with errors $e_{abs} = 6.273123050919496e - 07$ and $e_{rel} = 0.00011573231975654487$).

For problem 15b, the classic quadratic formula gives $x_1 = 0E + 2$ (with errors $e_{abs} = 0.005419735788228408$ and $e_{rel} = 1.0$) and $x_2 = -92.26$ (with errors $e_{abs} = 0.004580264211782037$ and $e_{rel} = 4.964764373626535e-05$). The rationalized quadratic formula gives $x_1 = 0.005421$ (with errors $e_{abs} = 1.264211771591535e-06$ and $e_{rel} = 0.0002332607752461634$) and $x_2 = Infinity$ (with errors $e_{abs} = 0$ and $e_{rel} = 0.0$).

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- **a.** The classic quadratic formula gives $x_1 = 1.903$ (with errors $e_{abs} = 0.0006535189493659388$ and $e_{rel} = 0.00034353308184164813$) and $x_2 = 0.743$ (with errors $e_{abs} = 0.0004048300139565253$ and $e_{rel} = 0.0005445619904688071$). The rationalized quadratic formula gives $x_1 = 1.903$ (with errors $e_{abs} = 0.0006535189493659388$ and $e_{rel} = 0.00034353308184164813$) and $x_2 = 0.7430$ (with errors $e_{abs} = 0.0004048300139565253$ and $e_{rel} = 0.0005445619904688071$).

- **b.** The classic quadratic formula gives $x_1 = -0.07798$ (with errors $e_{abs} = 0.0004287938339695291$ and $e_{rel} = 0.005468695703666852$) and $x_2 = -4.060$ (with errors $e_{abs} = 0.00038027344468982704$ and $e_{rel} = 9.367218367827243e-05$). The rationalized quadratic formula gives $x_1 = -0.07840$ (with errors $e_{abs} = 8.793833969525378e-06$ and $e_{rel} = 0.00011215366975477655$) and $x_2 = -4.082$ (with errors $e_{abs} = 0.02238027344469007$ and $e_{rel} = 0.005512898978762303$).

- **c.** The classic quadratic formula gives $x_1 = 1.223$ (with errors $e_{abs} = 0.00012977027895555437$ and $e_{rel} = 0.00010611941954393391$) and $x_2 = -2.223$ (with errors $e_{abs} = 0.00012977027895555437$ and $e_{rel} = 5.837960184110239e-05$). The rationalized quadratic formula gives $x_1 = 1.223$ (with errors $e_{abs} = 0.00012977027895555437$ and $e_{rel} = 0.00010611941954393391$) and $x_2 = -2.223$ (with errors $e_{abs} = 0.00012977027895555437$ and $e_{rel} = 5.837960184110239e-05$).

- **d.** The classic quadratic formula gives $x_1 = 6.235$ (with errors $e_{abs} = 0.001759153700807481$ and $e_{rel} = 0.0002820621507828506$) and $x_2 = -0.3205$ (with errors $e_{abs} = 0.00017937060119210813$ and $e_{rel} = 0.0005593456194448491$). The rationalized quadratic formula gives $x_1 = 6.240$ (with errors $e_{abs} = 0.0032408462991924125$ and $e_{rel} = 0.0005196362757201132$) and $x_2 = -0.3208$ (with errors $e_{abs} = 0.00012062939880785883$ and $e_{rel} = 0.0003761682536101697$).

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- **a.** For problem 17a, the classic quadratic formula gives $x_1 = 92.24$ (with errors $e_{abs} = 0.004579627312310208$ and $e_{rel} = 4.964657360695745e-05$) and $x_2 = 0.01500$ (with errors $e_{abs} = 0.009579627312305092$ and $e_{rel} = 1.7673373888205772$). The rationalized quadratic formula gives $x_1 = 33.32$ (with errors $e_{abs} = 58.924579627312305$ and $e_{rel} = 0.6387863640918536$) and $x_2 = 0.005418$ (with errors $e_{abs} = 2.372687694907581e-06$ and $e_{rel} = 0.0004377351580074840$).

- **b.** For problem 17b, the classic quadratic formula gives $x_1 = 0E+2$ (with errors $e_{abs} = 0.005419735788228408$ and $e_{rel} = 1.0$) and $x_2 = -92.25$ (with errors $e_{abs} = 0.005419735788223079$ and $e_{rel} = 5.8747072028339875e-05$). The rationalized quadratic formula gives $x_1 = 0.005417$ (with errors $e_{abs} = 2.7357882284081286e-06$ and $e_{rel} = 0.0005047825826399551$) and $x_2 = Infinity$ (with errors $e_{abs} = 0$ and $e_{rel} = 0.0$).

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- **a.** With the nesting technique, the polynomial becomes

$$f(x) = (((1.01e^x - 4.6)e^x - 3.11)e^x + 12.2)e^x - 1.99$$

- **b.** Normal evaluation of $f(1.53)$: -6.09 (absolute error 1.52, relative error 0.200)

- **c.** Nested evaluation of $f(1.53)$: -7.07 (absolute error 0.540, relative error 0.0710)

- **d.** see above