Homework 3

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1. Find

(a) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^7}{n^8}$

Solution. □

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin \frac{k}{n}}{n}$

Solution. □

2. Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \frac{1}{q}, & x = \frac{p}{q} \text{ irreducible,} \\ 0, & x \text{ irrational.} \end{cases}$$

Prove that f is Riemann integrable and calculate $\int_0^1 f(x)\,dx$.

Proof. □

3. Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Prove if f is Riemann integrable, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx.$$

Proof. □

4. Let $\{q_1, q_2, \dots\} = \mathbb{Q} \cap [0,1]$. Define $g_n : [0,1] \to \mathbb{R}$ as

$$g_n(x) = \begin{cases} 1, & x = q_n, \\ 0, & x \neq q_n. \end{cases}$$

Is

$$G(x) = \sum_{n=1}^{\infty} \frac{g_n(x)}{n^2}$$

Riemann integrable?

Proof. □

5. Let $f : [0,1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0, \\ 0, & x = 0. \end{cases}$$

Prove that f is Riemann integrable.

Proof. $\square$