Homework 1

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Problem 1. Find a 4×4 matrix with no real eigenvalues, and prove it. Find a 4×4 matrix with exactly one eigenvalue λ, such that the dimension of the subspace formed by the collection of all eigenvectors for that eigenvalue is 2-dimensional, and prove it.

Solution. One such matrix is

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which has a characteristic polynomial of $\lambda^4 + 1$, which has no real roots.

One such matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly $\lambda = 2$ is an eigenvalue, with corresponding eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$\operatorname{span}\{v_1, v_2\}$ is a 2-dimensional subspace. □

Problem 2. Let V be a vector space of dimension n over a field k, $T : V \mapsto V$ a linear transformation such that $T^n = 0$ and $T^{n-1} \neq 0$. Assume $v \in V$ is not contained in the kernel of T^{n-1}.

Prove $B = \{v, \dots, T^{n-1}v\}$ is a basis for V. Compute the matrix for T with respect to B. Let $c \in k$ and define $S : V \mapsto V$ such that $S(u) = cu + T(v)$. Compute the matrix of S with respect to B.

Proof. B is clearly the correct size to be a basis for V, to it is sufficient to show B is linearly independent. Suppose $a_1 v + a_2 Tv + \cdots + a_n T^{n-1} v = 0$. Then transform both sides by T; now we have $a_1 Tv + a_2 T^2 v + \cdots + a_n T^n v = 0$. Because $T^n v = 0$, this simplifies to $a_1 Tv + a_2 T^2 v + \cdots a_{n-1} T^{n-1} v = 0$. Continue applying T and simplifying, $n-1$ times until only the a_1 term remains. Now the equation is $a_1 T^{n-1} v = 0$. Because $v \notin \ker T^{n-1}$, $T^{n-1} v \neq 0$, and thus we have that $a_1 = 0$.

Thus, returning to our hypothesis, the a_1 term drops out and we have $a_2 Tv + \cdots + a_n T^{n-1} v = 0$. Repeat the process, this time multiplying by T^{n-2}; at the end we have $a_2 T^{n-1} v = 0$ and thus $a_2 = 0$. This process, of multiplying by T^{n-k} until the equation simplifies to $a_k T^{n-1} v = 0$ and hence $a_k = 0$, and then dropping the a_k term, can be repeated all the way until we are left with $a_n T^{n-1} v = 0$. Then $a_n = 0$ and we have found that all $a_i = 0$.

Therefore B is linearly independent, and hence a basis for V.

The columns of a matrix with respect to some basis are equivalent to the product of that matrix with the standard basic vectors. Let b_i be the ith basis vector in b. Then $b_2 = Tb_1$, $b_3 = Tb_2$, and so on until $b_n = Tb_{n-1}$ and $0 = Tb_n$. Then, because the ith basic vector with respect to it's own basis is just the ith column of I, we have that

$$T_B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & & \ddots & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

We examine S by a similar process and find that

$$S_T = \begin{pmatrix} c & 0 & 0 & \cdots & 0 \\ 1 & c & & \ddots & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & c & \vdots \\ 0 & \cdots & 0 & 1 & c \end{pmatrix}$$

□

Axler 2nd edition, Exercise 5-3 Prove or give a counterexample: if U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

Proof. We prove the contrapositive: that if a subspace U of V is not $\{0\}$ or V, then there exists some operator on V which U is not invariant under. Suppose there is such a U. Then exists $v_1 \in U$ and $v_2 \notin U$ such that $v_1 \neq v_2 \neq 0$. Since v_1 and v_2 are linearly independent, $\text{span}\{v_1, v_2\}$ is a subspace of v. Now we draw upon Theorem 2.34 in Axler 3rd edition, which tell us that there exists some other subspace of V such that $\text{span}\{v_1, v_2\} \oplus W = V$.

To define a linear operator on V, it suffices to specify it's behavior on v_1, v_2, and all $w \in W$. We construct T such that $T(v_1) = v_2$, $T(v_2) = v_1$, and $T(w) = w$. U is not invariant under T, and therefore such a U is not invariant under every operator on V. □

Axler 2nd edition, Excercise 5-10 Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda \in \mathbb{F} \setminus \{0\}$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

Proof. ($\Longrightarrow$) Suppose λ is an eigenvalue of T. Then $\exists \vec{v} \in V$ such that

$$\begin{aligned} T\vec{v} &= \lambda\vec{v} \\ T^{-1}T\vec{v} &= T^{-1}\lambda\vec{v} \\ \vec{v} &= T^{-1}\lambda\vec{v} \\ \vec{v}\frac{1}{\lambda} &= T^{-1}\vec{v} \end{aligned}$$

That is, $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

($\Longleftarrow$) The process is extremely similar. Supose $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}. Then $\exists \vec{v} \in V$ such that

$$\begin{aligned} T^{-1}\vec{v} &= \frac{1}{\lambda}\vec{v} \\ \vec{v} &= T\frac{1}{\lambda}\vec{v} \\ \lambda\vec{v} &= T\vec{v} \end{aligned}$$

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Therefore λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}. □

Axler 2nd edition, Excercise 5-15 (In $\mathbb{C}$.) Suppose $T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbb{C})$, and $a \in \mathbb{C}$. Prove that a is an eigenvalue of $p(T)$ if and only iff $a = p(\lambda)$ for some eigenvalue λ of T.

Proof. ($\Longrightarrow$) Suppose that a is an eigenvalue of of $p(T)$. Then $\exists \vec{v}$ such that $p(T)v = a\vec{v}$ or equivalently that

$$(p(T) - aI)\,\vec{v} = \vec{0}.$$

We can multiply out the polynomial on the LHS to get

$$[c_n T^n + \cdots + c_1 T + (c_0 - a)I]\,\vec{v} = \vec{0}.$$

By the Fundamental Theorem of Algebra, this polynomial can be written as the product of linear factors

$$[c(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)]\,\vec{v} = \vec{0}.$$

But for that to be true, $\vec{v}$ must be in the null space of $(T - \lambda_j I)$ for some j between 1 and n. Then $T\vec{v} = \lambda_j \vec{v}$ and so λ_j is an eigenvalue corresponding to eigenvector $\vec{v}$. Additionally, λj is a root of $p(z) - a$, and so $p(\lambda_j) - a = 0$ or $p(\lambda_j) = a$.

($\Longleftarrow$) Suppose $a = p(\lambda)$ for some eigenvalue λ of T. Then λ is a root of $p(\cdot) - a$.

I can't quite figure out the rest of this proof and it's very late. □

Axler 2nd edition, Excercise 5-16 Show that the result in the previous exercise does not hold if $\mathbb{C}$ is replaced with $\mathbb{R}$.

Proof. The proof above relies on the Fundamental Theorem of Algebra, which is only valid under the complex field; it has many counterexamples among $p(\mathbb{F})$. □