Homework 2

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Axler 2e, Exercise 5.18: Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

Solution. Consider

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is its own inverse. □

Axler 2e, Exercise 5.19: Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

Solution. Consider

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

T is not invertible because it is not injective; note that (for example) $T(2,0)^t = T(1,1)^t = (2,2)^t$. $\square$

Axler 2e, Exercise 6.1: Prove that if x, y are nonzero vectors in $\mathbb{R}^2$, then

$$\langle x, y\rangle = \|x\| \, \|y\| \cos\theta,$$

where θ is the angle between x and y (thinking of x and y as arrows with initial point at the origin).

Proof. Consider the traingle formed by the vectors x, y, and $x - y$. From the Law of Cosines (remember high school geometry?) we then have that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\,\|x\|\,\|y\|\cos\theta.$$

Applying some properties of the norm and the inner product, we see

$$\begin{aligned}
\|x - y\|^2 &= \|x\|^2 + \|y\|^2 - 2\,\|x\|\,\|y\|\cos\theta \\
\langle x - y, x - y\rangle &= \\
\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle &= \\
\|x\|^2 + \|y\|^2 - 2\,\langle x, y\rangle &= \|x\|^2 + \|y\|^2 - 2\,\|x\|\,\|y\|\cos\theta \\
-2\,\langle x, y\rangle &= -2\,\|x\|\,\|y\|\cos\theta \\
\langle x, y\rangle &= \|x\|\,\|y\|\cos\theta
\end{aligned}$$

□

Axler 2e, Exercise 6.10: On $\mathcal{P}_2(\mathbb{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx.$$

Apply the Gram-Schmidt prodedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$.

Solution. Let $e_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{\langle 1,1 \rangle}} = \frac{1}{\sqrt{1}} = 1.$

Let

$$\begin{aligned}
e_2 &= \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} \\
&= \frac{x - \langle x, 1 \rangle \cdot 1}{\|x - \langle x, 1 \rangle \cdot 1\|} \\
&= \frac{x - \left(\int_0^1 x\,dx\right)(1)}{\left\|x - \left(\int_0^1 x\,dx\right)(1)\right\|} \\
&= \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} \\
&= \left(x - \frac{1}{2}\right)\left(\left\langle x - \frac{1}{2}, x - \frac{1}{2} \right\rangle\right)^{-1} \\
&= \left(x - \frac{1}{2}\right)\left(\int_0^1 \left(x - \frac{1}{2}\right)^2\right)^{-1} \\
&= 12\left(x - \frac{1}{2}\right) \\
&= 12x - 6
\end{aligned}$$

□

Axler 2e, Exercise 6.11: What happens if the Gram-Schmidt procedure is applied to a list of vectors that is not linearly independent?

Solution. □

Axler 2e, Exercise 6.14: Find an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$ (with inner product as in Exercise 10) such that the differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbb{R})$ has an upper-triangular matrix with respect to this basis.

Solution. $\square$

Axler 2e, Exercise 6.15: Suppose U is a subspace of V. Prove that

$$\dim U^{\perp} = \dim V - \dim U.$$

Proof. $\square$

Axler 2e, Exercise 6.18: Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and

$$\|Pv\| \leq \|v\|$$

for every $v \in V$, then P is an orthogonal projection.

Proof. Let $P \in \mathcal{L}(V)$. Suppose that $P^2 = P$ and $\forall v \in V, \|Pv\| \leq \|v\|$. $\square$