Homework 5

Nathaniel Hamovitz
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1. Let (X, d) be a metric space. Prove that if $\{A_\alpha : \alpha \in I\}$ is a family of connected subsets of X such that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in I} A_\alpha$ is also connected.

Proof. $\square$

2. Rudin, Chapter 3, Exercise 14, parts (a) - (d). If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

Proof. By definition of limit, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, n > N \Rightarrow d(s_n, s) < \varepsilon$. □

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

Proof. □

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

Proof. □

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $n a_n \to 0$.]

3. Define $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$, $n \in \mathbb{N}$. Prove that the sequence $(a_n)_{n=1}^{\infty}$ converges and find its limit.

Proof. Note that

$$
\begin{aligned}
a_1 &= \sqrt{2} & &= \sqrt{2} = 2^{\frac{1}{2}} & & \\
a_2 &= \sqrt{2\sqrt{2}} & &= \sqrt{2}\sqrt{\sqrt{2}} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} & &= 2^{\frac{1}{2}+\frac{1}{4}} \\
a_3 &= \sqrt{2\sqrt{2\sqrt{2}}} & &= \sqrt{2}\sqrt{\sqrt{2}}\sqrt{\sqrt{\sqrt{2}}} = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} & &= 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}
\end{aligned}
$$

$$a_n = 2^{\sum_{j=1}^{n}\left(\frac{1}{2}\right)^j}$$

That's a geometric series which sums to 1, and thus the limit is $2^1 = 2$. I know we haven't learned series yet in this course, but this was my first instinct and I think it's a cool way of doing it. $\square$

4. Let $S = (a_n)_{n=1}^{\infty}$ be a bounded real sequence. Let

$$L_S = \left\{ x \in \mathbb{R} : \text{ exists a subsequence of } S : (a_{n_k})_{k=1}^{\infty} \text{ such that } \lim_{k \to \infty} a_{n_k} = x \right\}.$$

Prove:

(a) $L_S \neq \emptyset$ is closed and bounded.

Proof. First, that $L_S \neq \emptyset$.

Let $C = \{a_n : n \in \mathbb{N}\}$. We use cases. Case 1: C is finite; that is, $C = \{p_1, p_2, \ldots, p_k\}$. Note that C is the image of the naturals under the function $n \mapsto a_n$, and so by pigeonhole principle, $\exists i_0 \in \{1, \ldots, k\}$ such that $a_n = p_{i_0}$ for infinitely many n. This defines a convergent subsequence, and hence $p_{i_0} \in L_S$; therefore $L_S \neq \emptyset$.

Case 2: C is infinite. Note that C must be bounded, because S is bounded. That is, $\exists R > 0, C \subseteq B_R(0)$. That ball is compact (by Heine-Borel) and thus $\exists x_0 \in C'$. By definition, $\forall r > 0, B_r(x_0) \cap C - \{x_0\} \neq \emptyset$. That is equivalent to $\forall k \in \mathbb{N}, B_{\frac{1}{k}}(x_0) \cap C - \{x_0\} \neq \emptyset$. (For the forward direction, $\frac{1}{k}$ is a radius greater than 0; for the backwards, note that $\forall r > 0, \exists k \in \mathbb{N}, r > \frac{1}{k}$.) Now define a subsequence (a_{n_k}) such that for each k, a_{n_k} is the point in C guaranteed to exists within the ball of radius $\frac{1}{k}$ around x_0 by the above. Then $\lim a_{n_k} = x_0$ and hence $x_0 \in L_S$.

Step 2: L_S is bounded.

Since C is bounded, $\exists k > 0, \forall n \in \mathbb{N}, |a_n| \leq k$. Take some $x \in L_S$. Then $x = \lim a_{n_j}$ for some subsequence of S. Taking the absolute value, $|x| = \left|\lim a_{n_j}\right| = \lim \left|a_{n_j}\right| \leq k$. Thus $x \in \overline{B_k(0)}$ and so $L_S \subseteq \overline{B_k(0)}$, and therefore L_S is bounded.

Step 3: L_S is closed. If $L_S' = \emptyset$, then L_S is closed. Thus let $x \in L_S'$. Then $\forall r > 0, B_r$

□

(b) L_S has only one point ($L_S = \{x_0\}$) if and only if the sequence $\{a_n\}_{n=1}^{\infty}$ converges ($\lim_{n\to\infty} a_n = x_0$).

Proof. ($\Longleftarrow$). Suppose that the sequence converges. Suppose for a contradiction that L_S has more than one element. Clearly $x_0 \in L_S$; consider the subsequence formed by taking every point in the original sequence. Name one of the other elements $y \in L_S$. Let $d = |x_0 - y|$ and let $\varepsilon = \frac{d}{4}$. By the definition of convergence, $\exists N \in \mathbb{N}$ such that $n_j > N$ implies $|a_{n_j} - y| < \varepsilon$. Then all those a_{n_j} are not within ε of x_0, which means that $\nexists N, n > N \Rightarrow |a_n - x_0| < \varepsilon$. Therefore x_0 is not the limit of (a_n), and we have a contradiction. □

(c) If $\alpha \in L_S$, then

$$\limsup_{n\to\infty} a_n = M \geq \alpha.$$

Proof. □

(d) Prove that $M \in L_S$. Hence, $M = \sup L_S$.

Proof. From above, we know the set is closed. Therefore it must contain its supremum. (This is a result from above). □