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Axiom of Choice and Equivalent Examples

Bressoud defines the Axiom of Choice (AoC) as follows: "given a set S, there is a mapping that assigns to each non-empty subset of S one of the elements of that subset". To mere mortals, this seems blatantly obvious. However, to some, it isn't as clear cut that the AoC is a valid axiom to include in the foundation of Mathematics. Hausdorff, Banach and Tarski, whose combined work resulted in "the pea and the sun theorem" (Banach-Tarski Paradox), all said it wasn't. Godel showed that the AoC was consistent with the Zermelo-Fraenkel (ZF) axioms, which form the foundation of Set Theory. Thus, he accepted it. Cohen showed that the real number system could be constructed with or without the AoC. To him, either inclusion or exclusion of the AoC was acceptable.

The notion of a choice function helps clarify the AoC. Given a collection of sets C, let X be an element of C and x be an element of X. If there exists a choice function, f, such that $f(X) = x$ for all X, then it is unnecessary to invoke the AoC. The ZF axioms are sufficient to choose an element from X. However, if there is no such choice function f, then it is necessary to invoke the AoC in order to choose an element from X. For example, consider an infinite collection of pairs of shoes. The choice function "pick the left shoe" allows us to choose a shoe from each pair. Suppose instead of shoes, we have indistinguishable socks. "pick the left sock" is no longer a valid choice function, since socks fit on either foot. In fact, no choice function exists, because there is no way to distinguish between the socks within each pair in the collection. Therefore, the AoC is necessary to choose a sock.

Well-Ordering is equivalent to the AoC. A Well-Ordered set is one in which every subset of that set has a least element. Trichotomy is also equivalent to the AoC. From Linear (and Abstract) Algebra, every vector space has a basis. Here, it is necessary to choose one basis from among a possibly infinite number of indistinguishable bases. The Hausdorff Maximal Principle states that in any partially order set, every totally ordered subset is contained in a maximal totally ordered subset (of which, there can be many which are indistinguishable).

To me, it seems that the use of the word "arbitrary", in regard to sets, is also equivalent to the AoC. As a student of Real Analysis, the AoC is important because it is a necessary condition for the existence of non-measurable sets. If we do not accept the AoC as valid, then all sets must be measurable. Bressoud notes that it is our choice to make when accepting or rejecting the AoC. I choose to accept it. That way, when I have trouble analyzing a difficult integral over a complex set, I can just chalk it up to the set being non-measurable and move on!