Report on “Portfolio selection with a drawdown constraint”
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1. Introduction and Motivation

For those interested in portfolio optimization, unconstrained mean-variance optimization is the typical starting point (as it was for us). We chose this paper [1] as our project since we are interested in studying the results of a few optimizations, and the effects of adding a constraint to those optimizations.

The paper authors had two main concerns in performing this research. First, they were motivated by a paper by Chekhlov [2] that studied mean-drawdown optimization. They noted that this paper’s optimization did not allow for shorts, did not derive an analytical solution to determine constrained optimal portfolios and did not study the effects of maximum drawdown as a constraint. Second, they were interested in studying Merrill Lynch’s failed management of Unilever’s pension fund, which involved the application of a maximum drawdown constraint.

2. Data

We ran our analysis on the same data used in the paper. These data are 10 MSCI Country Indexes: Australia, Canada, France, Germany, Italy, Japan, Netherlands, Switzerland, UK and USA. Each index is composed of large and mid cap stocks. The time period is from 1970-2004. The data are still available on their website [3], but we found that the data have changed. The website provides data as yearly prices, and we converted the data to yearly returns. The risk free rate is set at 2.75%, which the paper notes is the 1-year Treasury security yield in December 2004. The benchmark portfolio is composed of the following indexes and weights: France (0.1), Germany (0.1), Japan (0.2), UK (0.2) and US (0.2). The remaining 5 indexes have weight 0.

2.1. Basic Statistics

Index Standard (Large+Mid) 1970-2004\										
	Australia	Canada	France	Germany	Italy	Japan	Netherlands	Switzerland	UK	USA
Mean %	7.67	8.57	11.21	11.	9.59	14.29	10.1	12.32	10.51	8.59
StdDev %	23.92	19.49	27.98	29.91	36.62	34.91	18.63	24.5	27.12	17.06

Correlation Matrix: Index Standard (Large+Mid) 1970-2004\										
	Australia	Canada	France	Germany	Italy	Japan	Netherlands	Switzerland	UK	USA
Australia	1.	0.68	0.63	0.36	0.5	0.48	0.62	0.38	0.59	0.52
Canada	0.68	1.	0.52	0.35	0.32	0.44	0.54	0.31	0.41	0.61
France	0.63	0.52	1.	0.75	0.75	0.5	0.81	0.7	0.5	0.56
Germany	0.36	0.35	0.75	1.	0.65	0.38	0.77	0.86	0.45	0.53
Italy	0.5	0.32	0.75	0.65	1.	0.43	0.62	0.61	0.33	0.43
Japan	0.48	0.44	0.5	0.38	0.43	1.	0.44	0.32	0.31	0.3
Netherlands	0.62	0.54	0.81	0.77	0.62	0.44	1.	0.82	0.67	0.78
Switzerland	0.38	0.31	0.7	0.86	0.61	0.32	0.82	1.	0.57	0.58
UK	0.59	0.41	0.5	0.45	0.33	0.31	0.67	0.57	1.	0.6
USA	0.52	0.61	0.56	0.53	0.43	0.3	0.78	0.58	0.6	1.

Tables 1: Means, Standard Deviations and Correlations

These tables provide the mean (%), standard deviation (%) and correlations for the 10 indexes. Comparing these to the tables provided in the paper, it is clear that the data changed, though the relations among the data are similar. For example, in both, Japan has the highest mean, and Italy has the highest standard deviation. The means of our data are lower than those in the paper. The changes only had minor effects on our results.

3. Mean-Variance Efficient Boundary Without a Risk Free Security

3.1. Unconstrained Efficient Boundary

Unconstrained mean-variance portfolio optimization for a given set of assets, without a risk free asset, is achieved through the following program:

$$
\begin{aligned}
&\min_w w^T \Sigma w \\
&s.t. \quad w^T 1 = 1, \\
&\qquad\quad w^T \mu = E,
\end{aligned}
$$

Program 1

The portfolio weights are given by w, the asset variance-covariance matrix is given by Σ, the asset means are given by μ, and the desired expected return is given by E. A closed form solution for w exists, and is given by the following:

$a = 1^T \Sigma^{-1} \mu$ | $b = \mu^T \Sigma^{-1} \mu$ | $c = 1^T \Sigma^{-1} 1$ | $w_\sigma = \frac{\Sigma^{-1} 1}{c}$ | $w_a = \frac{\Sigma^{-1} \mu}{a}$ | $\phi = \frac{E - b/a}{a/c - b/a}$ | $w_E = \phi w_\sigma + (1 - \phi) w_a$

Minimum-Variance portfolio: W_σ, has expected return $\frac{a}{c}$ and variance $\frac{1}{c}$

B/A Portfolio: W_a, has expected return $\frac{b}{a}$

Short sales are allowed, so a closed-form solution exists and is used to find the Mean-Variance efficient boundary: $\sigma(r_w) = \frac{1}{c} + \frac{(E[r_w] - a/c)^2}{d/c}$

Closed Form Solution 1

Closed form solution 1 indicates that the unconstrained mean-variance optimal portfolios in (E,σ)-space exhibit two fund separation. The first fund is the minimum variance portfolio, and the second fund is the B/A portfolio. The efficient frontier includes portfolios with $E[r_w] >= a/c$.

3.2. Maximum Drawdown Constrained Efficient Boundary

Maximum drawdown constrained mean-variance portfolio optimization for a given set of assets, without a risk free asset, is achieved through the following program:

$$\begin{aligned} &\min_w w^T \Sigma w \\ &s.t. \quad w^T 1 = 1, \\ &\qquad w^T \mu = E, \\ &\qquad -w^T r_s \leq D \ \forall s \in \{1, \ldots, S\} \end{aligned}$$

Program 2

In addition to the previously listed variables, the maximum drawdown constraint given by D, the number of states is given by S and the return of the assets at each state is given by r_s. A closed form solution for w does not exist. We used Mathematica's NMinimize function and the Nelder-Mead numerical method with S+2 linear constraints to solve for w. Using the programmatically determined optimal portfolios, the appendix to the paper [4] demonstrates that a closed form solution exists for w:

$e_{s_k} = 1^T \Sigma^{-1} r_{s_k}$ | $w_{s_k} = \frac{\Sigma^{-1} r_{s_k}}{e_{s_k}}$ | $A_{E,D} = [1 \ E \ -D \ \ldots \ -D]$ | $M = [1 \ \mu \ r_{s_3} \ \ldots \ r_{s_{K+2}}]$

$N = M^T \Sigma^{-1} M$ | $[\lambda_1 \ \lambda_2 \ \lambda_3 \ldots \lambda_{K+2}] = N^{-1} A_{E,D}$

$[\phi_1 \ \phi_2 \ \phi_3 \ldots \phi_{K+2}] = [\lambda_1 c \ \ \lambda_2 a \ \ \lambda_3 e_{s_3} \ldots \lambda_{K+2} e_{s_{K+2}}]$ | $w_{E,D} = \phi_1 w_\sigma + \phi_2 w_a + \sum_{k=3}^{K+2} \phi_k w_{s_k}$

Closed Form Solution 2

Closed form solution 2 indicates that the maximum drawdown constrained mean-variance optimal portfolios in (E,σ)-space exhibit K+2 fund separation, where K is the number of states bound by the maximum drawdown constraint. The first fund is the minimum variance portfolio, and the second fund is the B/A portfolio. The remaining K funds are unconstrained mean-variance inefficient funds given by w_{sk}. The efficient frontier includes portfolios with $E[r_w] >= a/c$.

3.3. Analysis of Results

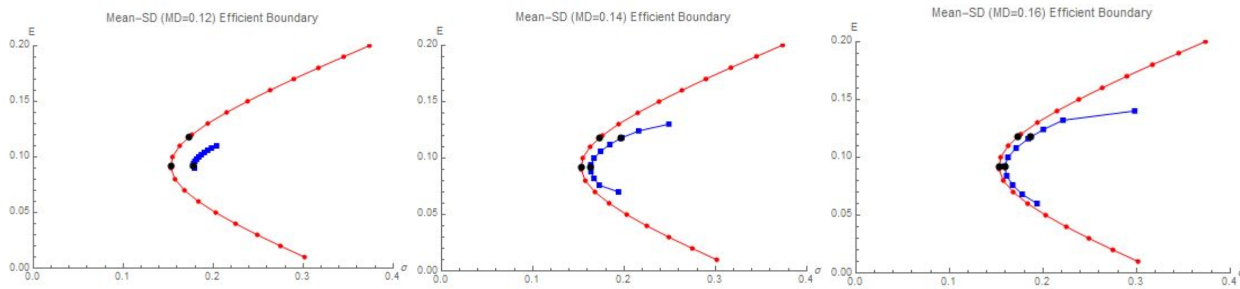

Figure 1: Constrained and Unconstrained Mean-Variance Efficient Boundaries

In Figure 1, the same unconstrained mean-variance efficient boundary (red) is plotted in each of the graphs. From left to right, the maximum drawdown constrained mean-variance efficient boundaries (blue) are plotted for maximum drawdown constraints: 0.12, 0.14 and 0.16. The constrained efficient boundary is unconstrained inefficient, in each of these cases. Setting the constraint too low (e.g. to 0.1) yields no feasible portfolios. As the constraint increases, the range of feasible expected returns increases, and the constrained efficient boundary shifts closer to the unconstrained efficient boundary. Setting the constraint to infinity is equivalent to solving the unconstrained optimization. Feasible portfolios closer to the edges of the feasible expected return range require relatively higher standard deviations to meet the constraint. Interestingly, the efficient portfolio with expected return a/c is the minimum variance portfolio in all unconstrained and constrained cases.

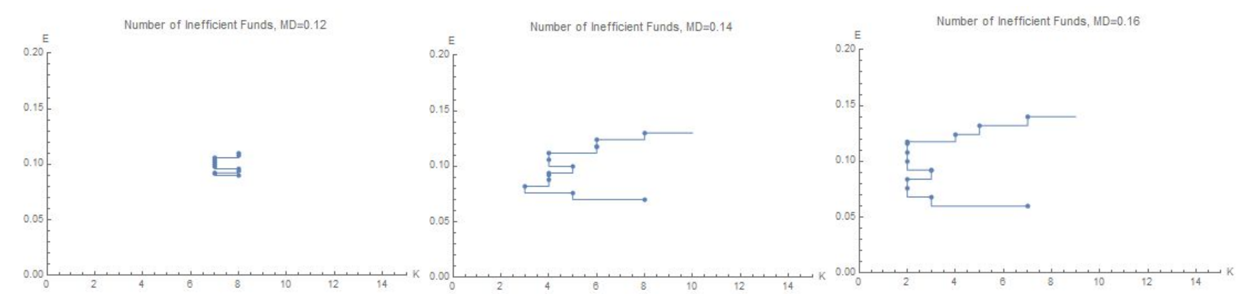

Figure 2: Number of Inefficient Funds in Constrained Mean-Variance Optimization

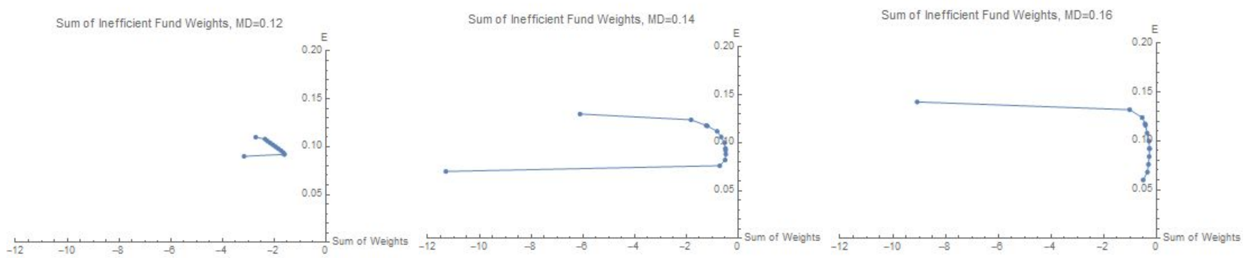

Figure 3: Sum of Inefficient Fund Weights in Constrained Mean-Variance Optimization

Figures 2 and 3 characterize the K+2 fund separation of the constrained mean-variance optimal portfolios for constraint values 0.12, 0.14 and 0.16. As noted previously, these graphs also show that as the constraint increases, the range of feasible expected returns increases. Feasible portfolios closer to the edges of the feasible expected return range exhibit higher fund separation: they are combinations of more inefficient funds with a larger negative sum of weights. As the constraint increases, the fund separation decreases for a given expected return. In other words, the portfolio with that expected return is a combination of fewer inefficient funds with a smaller negative sum of weights. The sum of inefficient weights is minimized near the minimum variance portfolio, and starting from this portfolio, the sum increases as the expected return increases and decreases. This is not the case for the number of inefficient funds, where the number increases, decreases, then increases again. In other words, a fund separation with a smaller number of funds does not guarantee a small sum of inefficient weights. There are never more than 8 inefficient funds, though the sum of their weights can differ greatly.

4. Mean-Variance Efficient Boundary With a Risk Free Security

4.1. Unconstrained Efficient Boundary

Unconstrained mean-variance portfolio optimization for a given set of assets, with a risk free asset, is achieved through the following program:

$$\begin{aligned} &\min_{w\in\mathbb{R}^J} w^T \Sigma w \\ &s.t. \quad w^T \mu - \left(1 - w^T 1\right) r_f = E, \end{aligned}$$

Program 3

In addition to the previously listed variables,, the risk free rate is given by r_f.. A closed form solution for w exists, and is given by the following:

J = # risky assets | r_f = risk free rate | $w_f = [0\ 0\ 0\ \ldots\ 1]$ | $w_t = [\ \frac{\Sigma^{-1}(\mu - 1\, r_f)^T}{a - c\, r_f}\ \ldots\ 0\ ]$

$\theta = \frac{E - E[r_{w_t}]}{r_f - E[r_{w_t}]}$ | $w_E = \theta w_f + (1 - \theta)\, w_t$

Risk Free portfolio: w_f, has expected return r_f and variance 0

Tangent Portfolio: w_t, portfolio on the mean-variance boundary without risk free security whose tangent intersects $(0, r_f)$ in $(E[r_w], \sigma[r_w])$ space

A closed-form solution is used to find the Mean-Variance efficient boundary: $\sigma(r_w) = \text{Abs}\left(\frac{E[r_w] - r_f}{\sqrt{h}}\right)$ | $h = (\mu - 1\, r_f)^T\, \Sigma^{-1} (\mu - 1\, r_f)$

Closed Form Solution 3

Closed form solution 3 indicates that the unconstrained mean-variance optimal portfolios in (E,σ)-space exhibit two fund separation. The first fund is the risk free portfolio, and the second fund is the tangent portfolio. The efficient frontier includes portfolios with $E[r_w] >= r_f$.

4.2. Maximum Drawdown Constrained Efficient Boundary

Maximum drawdown constrained mean-variance portfolio optimization for a given set of assets, with a risk free asset, is achieved through the following program:

$$\begin{aligned} &\min_{w \in \mathbb{R}^J}\ w^T \Sigma w \\ &s.t.\quad w^T \mu - (1 - w^T 1)\, r_f = E, \\ &\qquad -w^T r_s - (1 - w^T 1)\, r_f \le D\ \ \forall\, s \in \{1,\ \ldots,\ S\} \end{aligned}$$

Program 4

A closed form solution for this program for w does not exist. We used Mathematica's NMinimize function and the Nelder-Mead numerical method with S+1 linear constraints to solve for w. Using the programmatically determined optimal portfolios, the appendix to the paper [4] demonstrates that a closed form solution exists for w:

$w_{s_k} = \left[\frac{\Sigma^{-1}(r_{s_k} - 1\, r_f)^T}{e_{s_k} - c\, r_f}\ \ldots 0 \right]$ | $B_{E,D}^{(K+1)\text{x}1} = [\text{E-}r_f\ \ \text{-D-}r_f\ \ \ldots\ \ \text{-D-}r_f]$ | $R = \left[\mu - 1\, r_f\ \ r_{s_3} - 1\, r_f\ \ldots\ r_{s_{K+2}} - 1\, r_f\right]$

$T = R^T\, \Sigma^{-1}\, R$ | $[\gamma_1\ \gamma_3\ \ldots\ \gamma_{K+2}] = T^{-1}\, B_{E,D}$

$[\theta_1\, \theta_2\, \theta_3 \ldots\, \theta_{K+2}] = [1 - \sum_{k=2}^{K+2} \theta_k\ \ \gamma_1(a - c r_f)\ \ \gamma_3(e_{s_3} - \text{c}r_f)\ \ldots\ \gamma_{K+2}(e_{s_{K+2}} - \text{c}r_f)]$ | $w_{E,D} = \theta_1\, w_f + \theta_2\, w_t + \sum_{k=3}^{K+2} \theta_k\, w_{s_k}$

Closed Form Solution 4

Closed form solution 4 indicates that the maximum drawdown constrained mean-variance optimal portfolios in (E,σ)-space exhibit K+2 fund separation, where K is the number of states bound by the maximum drawdown constraint. The first fund is the risk free portfolio, and the second fund is the tangent portfolio. The remaining K funds are unconstrained mean-variance inefficient funds given by w_{sk}. The efficient frontier includes portfolios with $E[r_w] >= r_f$.

4.3. Analysis of Results

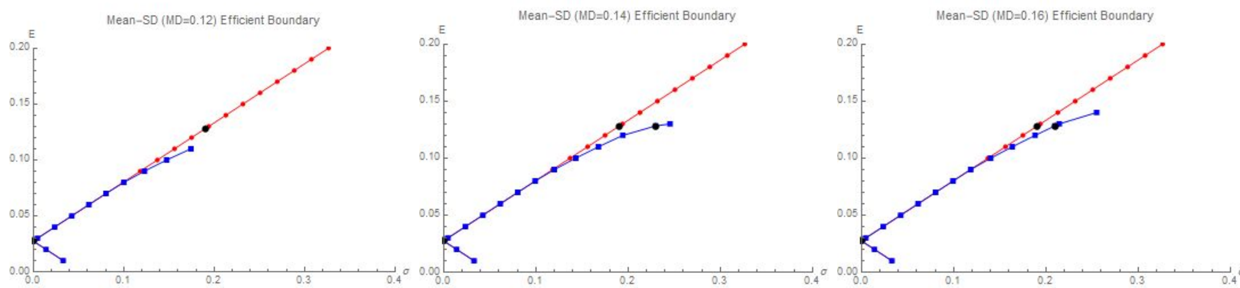

Figure 4: Constrained and Unconstrained Mean-Variance Efficient Boundaries

In Figure 4, the same unconstrained mean-variance efficient boundary (red) is plotted in each of the graphs. From left to right, the maximum drawdown constrained mean-variance efficient boundaries (blue) are plotted for maximum drawdown constraints: 0.12, 0.14 and 0.16. The constrained efficient boundary is unconstrained inefficient for higher expected returns, in each of these cases. However, the shift is less pronounced than in the constrained mean-variance optimization without the risk free security. Due to the risk free asset, the maximum drawdown constraint has no (or minimal) effect on the efficient boundary for lower expected returns, and feasible portfolios exist for all constraint values greater than or equal to 0. As the constraint increases, the range of feasible expected returns increases, though this only apparent for higher expected returns since the risk free rate is low. Also, as the constraint increases, the constrained efficient boundary shifts closer to the unconstrained efficient boundary. Setting the constraint to infinity is equivalent to solving the unconstrained optimization. Feasible portfolios closer to the max of the feasible expected return range require relatively higher standard deviations to meet the constraint.

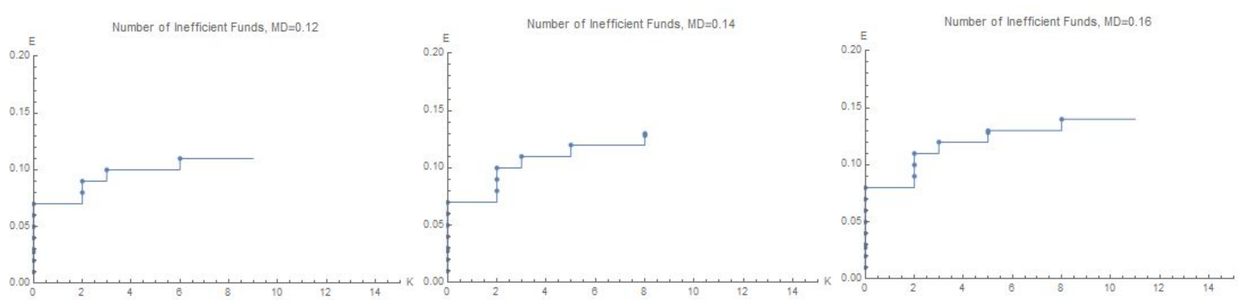

Figure 5: Number of Inefficient Funds in Constrained Mean-Variance Optimization

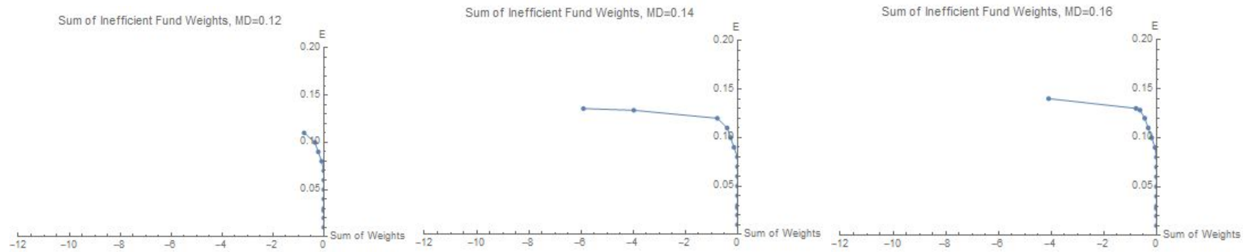

Figure 6: Sum of Inefficient Fund Weights in Constrained Mean-Variance Optimization

Figures 5 and 6 characterize the K+2 fund separation of the constrained mean-variance optimal portfolios for constraint values 0.12, 0.14 and 0.16. As noted previously, these graphs also show that as the constraint increases, the max feasible expected return increases. Feasible portfolios close to the max feasible expected return exhibit higher fund separation. Feasible portfolios with lower expected returns are characterized almost completely by the risk free and tangent portfolios, as shown by their low fund separation. As the constraint increases, the fund separation decreases for a given expected return. The number of inefficient funds and sum of inefficient weights is 0 for the risk free portfolio, and starting from this portfolio, both the number and sum never decrease as the expected return increases, unlike in the optimization without the risk free security. There are never more than 8 inefficient funds, though the sum of their weights can differ greatly.

5. Mean-TEV Efficient Boundary Without a Risk Free Security

5.1. Unconstrained Efficient Boundary

Unconstrained mean-tev portfolio optimization for a given set of assets, without a risk free asset, is achieved through the following program:

$$\begin{aligned} &\min_x x^T \Sigma x \\ &s.t. \quad x^T 1 = 0, \\ &\qquad\quad x^T \mu = E - E\left[r_{w_b}\right], \end{aligned}$$

Program 5

In addition to the previously listed variables, the benchmark portfolio weights are given by w_b, portfolio weights are given by $w = x + w_b$ and the benchmark portfolio expected return is given by $E[r_{wb}]$. A closed form solution for w exists, and is given by the following:

w_b = benchmark portfolio weights | $\pi = \frac{E - E[r_{w_b}]}{b/a - a/c}$ | $x_E = \pi(w_a - w_\sigma)$ | $w_E^\epsilon = w_b + x_E$

Tracking Error Volatility (TEV): $\sigma^\epsilon[r_w] = \sqrt{(w - w_b)^T\, \Sigma(w - w_b)}$

Short sales are allowed, so a closed-form solution exists to find weights for the portfolios on the Mean-TEV efficient boundary. The equations are given above.

Closed Form Solution 5

Closed form solution 5 indicates that the unconstrained mean-tev optimal portfolios in (E,TEV)-space exhibit three fund separation. The first fund is the benchmark portfolio, the second fund is the minimum variance portfolio, and the third fund is the B/A portfolio. The benchmark portfolio is mean-tev efficient but not necessarily mean-variance efficient. The paper notes that in practice, the benchmark portfolio is mean-variance inefficient. The minimum variance and B/A portfolios are mean-variance efficient. The efficient frontier includes portfolios with $E[r_w] >= E[r_{wb}]$.

5.2. Maximum Drawdown Constrained Efficient Boundary

Maximum drawdown constrained mean-tev portfolio optimization for a given set of assets, without a risk free asset, is achieved through the following program:

$$\begin{aligned} &\min_x x^T\, \Sigma x \\ &s.t. \quad x^T\, 1 = 0, \\ &\qquad\; x^T\, \mu = E - E\left[r_{w_b}\right], \\ &\qquad\; x^T\, r_s \le D^\epsilon \;\; \forall\, s \in \{1,\ \ldots,\ S\} \end{aligned}$$

Program 6

A closed form solution for w does not exist. We used Mathematica's NMinimize function and the Nelder-Mead numerical method with S+2 linear constraints to solve for w. Using the programmatically determined optimal portfolios, the appendix to the paper [4] demonstrates that a closed form solution exists for w:

$C_{E,D^\epsilon}^{(K+2)\mathrm{x}1} = [0\;\; E - E\left[r_{w_b}\right]\;\; -D^\epsilon\;\; \ldots\;\; -D^\epsilon]$ | $[\delta_1\, \delta_2\, \delta_3 \ldots\, \delta_{K+2}] = N^{-1}\, C_{E,D^\epsilon}$

$[\pi_1\, \pi_2\, \pi_3 \ldots\, \pi_{K+2}] = [\delta_1\, c\;\; \delta_2\, a\;\; \delta_3\, e_{s_3} \ldots\, \delta_{K+2}\, e_{s_{K+2}}]$ | $x_{E,D^\epsilon} = \pi_1\, w_\sigma + \pi_2\, w_a + \sum_{k=3}^{K+2} \pi_k\, w_{s_k}$ | $w_{E,D^\epsilon}^\epsilon = w_b + x_{E,D^\epsilon}$

Closed Form Solution 6

Closed form solution 6 indicates that the maximum drawdown constrained mean-tev optimal portfolios in both (E,TEV)-space and (E,σ)-space exhibit K+3 fund separation, where K is the number of states bound by the maximum drawdown constraint. The first fund is the benchmark portfolio, the second fund is the minimum variance portfolio, and the third fund is the

B/A portfolio. The remaining K funds are unconstrained mean-tev inefficient funds given by w_{sk}. The efficient frontier includes portfolios with $E[r_w] >= E[r_{wb}]$.

5.3. Analysis of Results

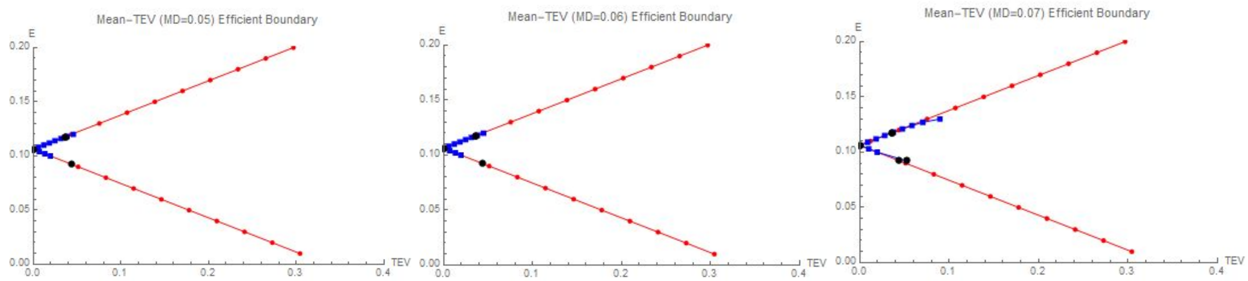

Figure 7: Constrained and Unconstrained Mean-TEV Efficient Boundaries in (E,TEV)-Space

In Figure 7, the same unconstrained mean-tev efficient boundary (red) is plotted in each of the graphs. From left to right, the maximum drawdown constrained mean-tev efficient boundaries (blue) are plotted for maximum drawdown constraints: 0.05, 0.06 and 0.07. To be clear to the reader, mean-tev graphs do not appear in the paper. We decided to analyze these for our own benefit. Unlike the mean-variance efficient boundary, the mean-tev efficient boundary is 2-piece linear. The constrained efficient boundary is unconstrained inefficient for feasible portfolios close to the edge of the range of feasible expected returns, in each of these cases. However, the shift is less pronounced than in the constrained mean-variance optimization. Due to the benchmark portfolio, the maximum drawdown constraint has no (or minimal) effect on the efficient boundary for expected returns closer to the benchmark portfolio expected return. Feasible portfolios exist for all constraint values greater than or equal to 0, but decreasing the constraint too much (e.g. to 0.01) yields practically no other feasible portfolios. As the constraint increases, the range of feasible expected returns increases. Setting the constraint to infinity is equivalent to solving the unconstrained optimization. Feasible portfolios closer to the edges of the feasible expected return range require relatively higher tevs to meet the constraint.

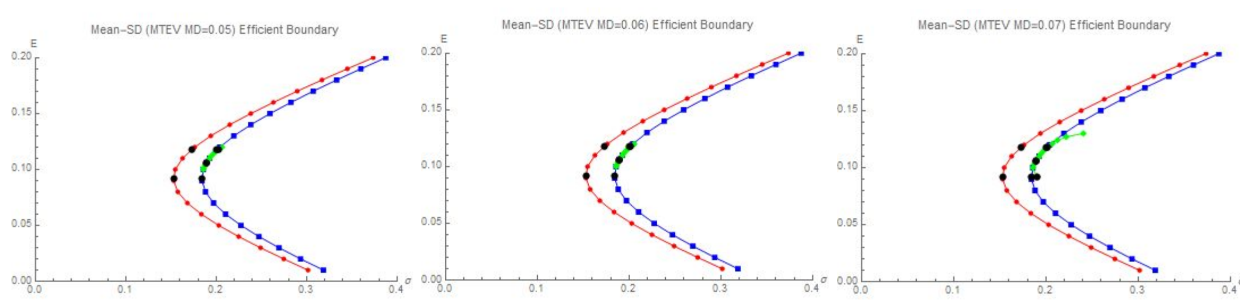

Figure 8: Unconstrained Mean-Variance, Unconstrained Mean-TEV and Constrained Mean-TEV Efficient Boundaries in (E,σ)-Space

In *Figure* 8, the same unconstrained mean-variance (red) and mean-tev (blue) efficient boundaries are plotted in each of the graphs. From left to right, the maximum drawdown constrained mean-tev efficient boundaries (green) are plotted for maximum drawdown constraints: 0.05, 0.06 and 0.07. Much of the analysis performed on Figure 7 is also applicable here. Additionally, Figure 8 also demonstrates that both unconstrained and constrained mean-tev optimizations yield efficient boundaries that are mean-variance inefficient.

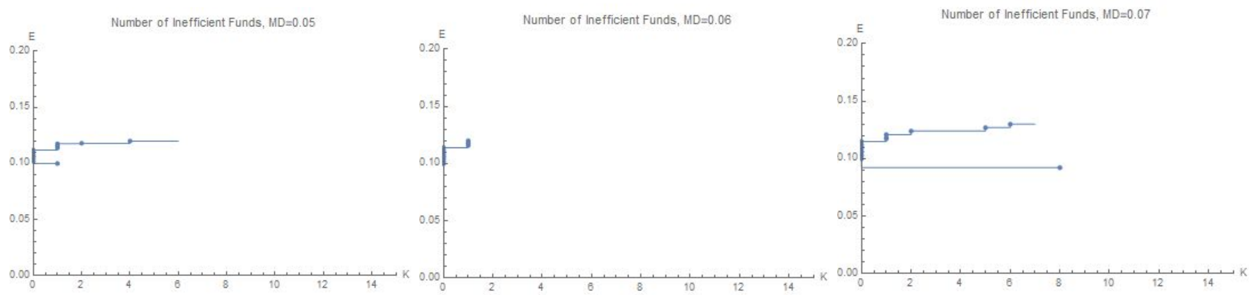

Figure 9: Number of Inefficient Funds in Constrained Mean-TEV Optimization

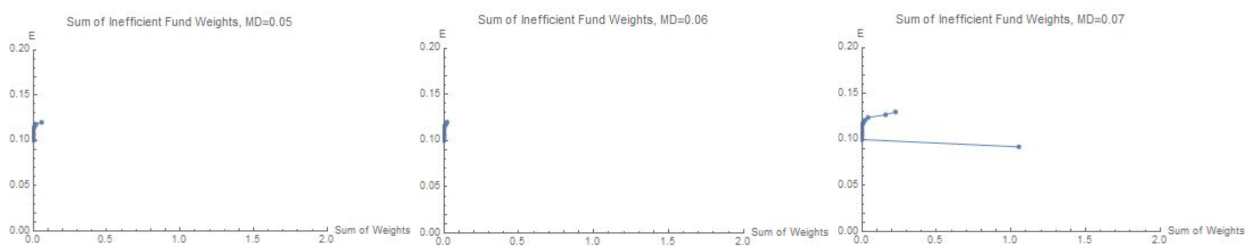

Figure 10: Sum of Inefficient Fund Weights in Constrained Mean-TEV Optimization

Figures 9 and 10 characterize the K+3 fund separation of the constrained mean-tev optimal portfolios for constraint values 0.05, 0.06 and 0.07. As noted previously, these graphs also show that as the constraint increases, the range of feasible expected returns increases. Feasible portfolios closer to the edges of the feasible expected return range exhibit higher fund separation. As the constraint increases, the fund separation decreases for a given expected return. The number of inefficient funds and sum of inefficient weights is 0 for the benchmark portfolio, and starting from this portfolio, both the number and sum never decrease as the expected return increases, unlike in the mean-variance optimization. There are never more than 8 inefficient funds, though the sum of their weights can differ greatly. Comparing the fund separation here to the previous fund separations, there are 2 interesting differences. First, notice that the sum of weights is always non-negative here, whereas they were always non-positive in the previous constrained optimizations. Second, the increase in the constraint from 0.05 to 0.06 is the only case where the feasible expected return range did not noticeably increase. This is most likely a quirk of the data or program implementation.

6. Conclusions

Although our data came from the same source with the same indexes and time period as the data in the paper, they are different, as explained in the Data section. However, the difference only had minor effects on our results, when compared to the results in the paper. All efficient boundaries were roughly equivalent in shape and length of the feasible expected return range. All constrained efficient boundaries were located similarly, with respect to the unconstrained efficient boundaries. We had to increase the smallest maximum drawdown coefficient from 0.10 to 0.12 for both constrained mean-variance optimizations.

Our fund separation graphs also showed similar characteristics to those in the paper. Naturally, there were differences, but major features were the same. The number of inefficient funds is locally constant. Also, the number was never greater than 8, though this will depend on the data. Lower maximum drawdown constraints yielded optimal portfolios with higher fund separation. For all constraint values, optimal portfolios closer to the edges of the feasible expected return ranges had higher fund separation, relative to those closer to the center. For the constrained mean-variance fund separations, the sum of inefficient weights was negative, while it was positive for the constrained mean-tev optimization.

6.1. Issues and Other Concerns

We covered much of the paper in our project but did not include the utility function $U(E, \sigma) = E - \frac{\rho}{2}\sigma^2$. Its implementation is not explained in the paper (another paper is referenced), but the results of its implementation are discussed. The data change issue, previously mentioned, was ultimately only a minor concern. Matrix inversion is applied in several places, and Mathematica provided warnings about badly conditioned matrices, though it was not apparent to us that this caused any issues. Initially, we used a minimum inequality for the maximum drawdown constraint optimizations. This caused our constrained mean-tev optimization to take approximately 10 minutes. Replacing this with S inequalities, as described that section, reduced the runtime to a few seconds.

References

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