AMS 513 Spring 2019, Stony Brook University – Instructor: Pinezich

Computer Project 2

This project has two parts. The first part is analytical; in the second part you will write a Mathematica notebook related to Part 1. The following are acceptable formats to write up your Part 1 solution: (1) Within the Mathematica notebook of Part 2; (2) A latex document output in PDF; (3) a Word document (or Word converted to PDF); (4) Neatly printed by hand and scanned into electronic format. All material should be submitted on the BlackBoard.

Part 1

A nickel, a dime and a quarter are tossed (with values 5, 10, 25, resp.). Let X be the random variable given by the sum of the coin values that land with a head. Denoting a three-toss outcome as $\omega = (\omega_N \omega_D \omega_Q)$, X can be written

$$X(\omega) = 5 \times \mathbb{I}_{\{\omega_N = H\}} + 10 \times \mathbb{I}_{\{\omega_D = H\}} + 25 \times \mathbb{I}_{\{\omega_Q = H\}}.$$

Let A be the event that exactly two heads occur in the outcome.

(a) Given that the coins are all fair, what is the expected value of X given that A has occurred, that is, $\mathbb{E}(X|A)$?

(b) Suppose the probability of a head is given by $p_N = \frac{4}{5}$, $p_D = \frac{1}{2}$, $p_Q = \frac{1}{4}$, what is $\mathbb{E}(X|A)$?

(c) With the probabilities from (b), what is the generalized conditional expectation $\mathbb{E}(X|\sigma(A))$?

Part 2

Create a Mathematica notebook that estimates the conditional expectations in Part 1 (a), (b), (c). For (c) you need to estimate the conditional expectations on the atoms of $\sigma(A)$. Here are some hints:

- Construct a list of three tosses using the `RandomVariate` function with the appropriate distribution.
- Define a function `ifTwoHeads` that returns the value $X(\omega)$ of a three-toss sequence $\omega = \omega_N \omega_D \omega_Q$ if there are exactly two heads in it. Use the `Function` command to do this.
- Make a list `lotsOfTrials` of three-toss sequences using the `Table` function, for some number N of trials.
- Use the `Map` function to apply `ifTwoHeads` to `lotsOfTrials`.

You will then have use a few more steps to derive the numerical answers for (a), (b), and (c). Write your code in a way that makes it easy to change the probability values, and the conditions, as needed.

Part 1 Solution: $\mathbb{E}(X|A)$ is the conditional expectation with respect to an outcome, given by

$$\mathbb{E}[X|A] = \frac{1}{\mathbb{P}(A)} \int_A X(\omega) d\mathbb{P}(\omega).$$

The event A is given by $A = \{HHT, HTH, THH\}$.

(a) Since $\mathbb{P}(\{\omega\}) = \frac{1}{8}$ for each possible outcome ω, $\mathbb{P}(A) = \frac{3}{8}$. Since the outcome space is discrete, the integral becomes a summation over the outcome values weighted by the outcome probabilities:

$$\begin{aligned} \int_A X(\omega) d\mathbb{P}(\omega) &= (5+10) \times \mathbb{P}(HHT) + (5+25) \times \mathbb{P}(HTH) + (10+25) \times \mathbb{P}(THH) \\ &= (15+30+35) \times \frac{1}{8} \;=\; 10. \end{aligned}$$

Therefore:

$$\mathbb{E}(X|A) = \frac{80}{3} = 26\frac{2}{3}.$$

Note, in this case it is easier to just average the three possible outcomes, since the likelihoods of each are equal.

(b) The outcome probabilities are

$$\begin{aligned} \mathbb{P}(HHT) &= \frac{4}{5} \times \frac{1}{2} \times \frac{3}{4} = \frac{12}{40} \\ \mathbb{P}(HTH) &= \frac{4}{5} \times \frac{1}{2} \times \frac{1}{4} = \frac{4}{40} \\ \mathbb{P}(THH) &= \frac{1}{5} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{40} \\ \mathbb{P}(A) &= \frac{17}{40}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_A X(\omega) d\mathbb{P}(\omega) &= (5+10) \times \frac{12}{40} + (5+25) \times \frac{4}{40} + (10+25) \times \frac{1}{40} \\ &= 15 \times \frac{12}{40} + 30 \times \frac{4}{40} + 35 \times \frac{1}{40} \;=\; \frac{67}{8}, \end{aligned}$$

and

$$\mathbb{E}(X|A) = 19\frac{12}{17}.$$

(c) To compute the conditional expectation with respect to a σ-algebra (also called the *generalized conditional expectation*), the value of $\mathbb{E}(X|\sigma(A))(\omega)$ must be found for every $\omega \in \Omega$. In this simple example, $\sigma(A) = \{\emptyset, A, A^c, \Omega\}$, and so A and A^c are the atoms of $\sigma(A)$ (see study problem on atoms). A random variable that is measurable with respect to a σ-algebra $\mathcal{G}$ is constant on the atoms of $\mathcal{G}$. Since random variable $\mathbb{E}(X|\sigma(A))$ is measurable with respect to $\sigma(A)$, find the values of $\mathbb{E}(X|\sigma(A))(\omega)$ for $\omega \in A$ and for $\omega \in A^c$. For $\omega \in A^c = \{HHH, TTT, HTT, THT, TTH\}$:

$$\begin{aligned}
\mathbb{P}(HHH) &= \frac{4}{5} \times \frac{1}{2} \times \frac{1}{4} = \frac{4}{40} \\
\mathbb{P}(TTT) &= \frac{1}{5} \times \frac{1}{2} \times \frac{3}{4} = \frac{3}{40} \\
\mathbb{P}(HTT) &= \frac{4}{5} \times \frac{1}{2} \times \frac{3}{4} = \frac{12}{40} \\
\mathbb{P}(THT) &= \frac{1}{5} \times \frac{1}{2} \times \frac{3}{4} = \frac{3}{40} \\
\mathbb{P}(TTH) &= \frac{1}{5} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{40} \\
\mathbb{P}(A^c) &= \frac{23}{40}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{A^c} X(\omega) d\mathbb{P}(\omega) &= (5 + 10 + 25) \times \frac{4}{40} + 0 \times \frac{3}{40} + 5 \times \frac{12}{40} + 10 \times \frac{3}{40} + 25 \times \frac{1}{40} \\
&= \frac{40 \times 4}{40} + \frac{5 \times 12}{40} + \frac{10 \times 3}{40} + \frac{25}{40} = \frac{55}{8},
\end{aligned}$$

and

$$\mathbb{E}(X|A^c)(\omega) = \frac{55/8}{23/40} = 11\frac{22}{23}.$$

Therefore

$$\mathbb{E}(X|\sigma(A))(\omega) = \begin{cases} 19\frac{12}{17} & \text{if} \quad \omega \in A \\ \\ 11\frac{22}{23} & \text{if} \quad \omega \in A^c. \end{cases}$$