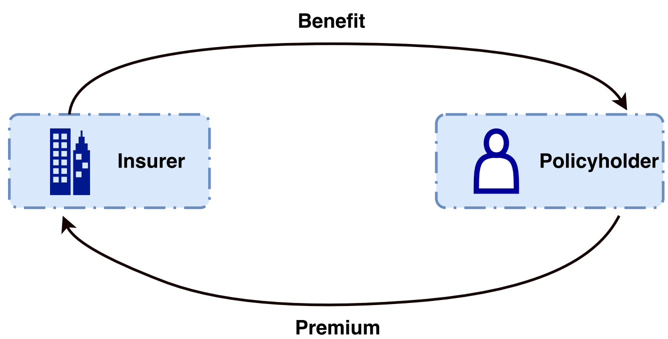
Net Premium

Nhan Huynh

February 2019

1 Benefit premium and equivalence principle

1.1 How does insurance work?



1.2 Equivalence principle

We want to determine the value of premium so that the actuarial present value of the premiums at time 0 is equivalent to the actuarial present value of future benefit at time 0.
Equivalence principle:

$$\boxed{\text{APV of all premiums (t=0)} = \text{APV of all benefits (t=0)}}$$

Premiums determined from this principle are called net premiums or benefit premiums.
Q: What is the net premium for a discrete whole life insurance?
A: Let P be the amount of net premium charged for a discrete whole life insurance. As long as the policyholder is alive, he/she must pay the amount of P to the insurer at the beginning of each year. Thus, the APV of all premiums at time 0 is $P\ddot{a}_x$. If we assume death benefit is \$1 and it is payable at the end of year of death, the APV of the benefit at time 0 is A_x. Using equivalence principle, set $P\ddot{a}_x = A_x$. Thus, $P = \frac{A_x}{\ddot{a}_x}$. We have an actuarial notation for this type of net premium as: $P(A_x)$.

1.3 Summary

Plan	Premium formula (Continuous)	Premium formula (Discrete)
Whole life	$\begin{aligned}\bar{P}(\bar{A}_x) &= \frac{\bar{A}_x}{\bar{a}_x} \\ &= \frac{\delta \bar{A}_x}{1-\bar{A}_x} \\ &= \frac{1}{\bar{a}_x} - \delta\end{aligned}$	$\begin{aligned}P(A_x) &= \frac{A_x}{\ddot{a}_x} \\ &= \frac{dA_x}{1-A_x} \\ &= \frac{1}{\ddot{a}_x} - d\end{aligned}$
n-year endowment	$\begin{aligned}\bar{P}(\bar{A}_{x:\overline{n}\rvert}) &= \frac{\bar{A}_{x:\overline{n}\rvert}}{\bar{a}_{x:\overline{n}\rvert}} \\ &= \frac{\delta \bar{A}_{x:\overline{n}\rvert}}{1-\bar{A}_{x:\overline{n}\rvert}} \\ &= \frac{1}{\bar{a}_{x:\overline{n}\rvert}} - \delta\end{aligned}$	$\begin{aligned}P(A_{x:\overline{n}\rvert}) &= \frac{A_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}} \\ &= \frac{dA_{x:\overline{n}\rvert}}{1-A_{x:\overline{n}\rvert}} \\ &= \frac{1}{\ddot{a}_{x:\overline{n}\rvert}} - d\end{aligned}$
n-year term	$\bar{P}(\bar{A}^{1}_{x:\overline{n}\rvert}) = \frac{\bar{A}^{1}_{x:\overline{n}\rvert}}{\bar{a}_{x:\overline{n}\rvert}}$	$P(A^{1}_{x:\overline{n}\rvert}) = \frac{A^{1}_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{n}\rvert}}$
n-year pure endowment	$\bar{P}(A_{x:\overset{1}{\overline{n}\rvert}}) = \frac{A_{x:\overset{1}{\overline{n}\rvert}}}{\bar{a}_{x:\overline{n}\rvert}}$	$P(A_{x:\overset{1}{\overline{n}\rvert}}) = \frac{A_{x:\overset{1}{\overline{n}\rvert}}}{\ddot{a}_{x:\overline{n}\rvert}}$
m-payment whole life	$\begin{aligned}{}_m\bar{P}(\bar{A}_x) &= \frac{\bar{A}_x}{\bar{a}_{x:\overline{m}\rvert}} \\ &= \frac{\delta \bar{A}_x}{1-\bar{A}_{x:\overline{m}\rvert}} \\ &= \frac{1-\delta \bar{a}_x}{\bar{a}_{x:\overline{m}\rvert}}\end{aligned}$	$\begin{aligned}{}_mP(A_x) &= \frac{A_x}{\ddot{a}_{x:\overline{m}\rvert}} \\ &= \frac{dA_x}{A_{x:\overline{m}\rvert}} \\ &= \frac{1-d\ddot{a}_x}{\ddot{a}_{x:\overline{m}\rvert}}\end{aligned}$
m-payment n-year term	${}_m\bar{P}(\bar{A}^{1}_{x:\overline{n}\rvert}) = \frac{\bar{A}^{1}_{x:\overline{n}\rvert}}{\bar{a}_{x:\overline{m}\rvert}}$	${}_mP(A^{1}_{x:\overline{n}\rvert}) = \frac{A^{1}_{x:\overline{n}\rvert}}{\ddot{a}_{x:\overline{m}\rvert}}$
m-payment n-year endowment	${}_m\bar{P}(A_{x:\overset{1}{\overline{n}\rvert}}) = \frac{\bar{A}_{x:\overset{1}{\overline{n}\rvert}}}{\bar{a}_{x:\overline{m}\rvert}}$	${}_mP(A_{x:\overset{1}{\overline{n}\rvert}}) = \frac{A_{x:\overset{1}{\overline{n}\rvert}}}{\ddot{a}_{x:\overline{m}\rvert}}$
Special Relationship	$P(A_{x:\overline{n}\rvert}) = P(A^{1}_{x:\overline{n}\rvert}) + P(A_{x:\overset{1}{\overline{n}\rvert}})$	

Note: Review the relationship between insurance and annuity. For example, if we have continuous case: $\bar{a}_x = \frac{1-\bar{A}_x}{\delta} \rightarrow \bar{A}_x = 1 - \delta \bar{a}_x$

2 The loss-at-issue random variable

We want to determine the difference between the benefits (expenses) and the series of net premium (revenue) at the issue time (time 0). Let ${}_0L$ be the loss-at-issue random variable (RV because both the PV of benefits and PV of premiums depend on the timing of death).

Ex1: Let's consider the continuous case of whole life insurance where death benefit is payable (by the insurer to policyholder) at the moment of death, and premiums must be paid (to the insurer by the policyholder) as long as policyholder is alive.

Notation: Let ${}_0L$ be the loss-at-issue random variable. We also have T_x as the random variable for the future lifetime.

1. Since death benefit is payable at the moment of death, the present value of death benefit at time 0 is: Bv^{T_x}, where B is the amount of death benefit.

2. If π is the level of premium, the present value of premiums at time 0 is: $\pi\bar{a}_{\overline{T_x}|}$.

3. Therefore, the loss-at-issue is: ${}_0L = Bv^{T_x} - \pi\bar{a}_{\overline{T_x}|}$. Since death benefit is fixed and paid at the moment of death, the larger value of T_x, the smaller value of PV of death benefit. Similarly, as long as the policyholder is alive, the number of premium payments increases $\rightarrow$ the PV of premiums increases. Thus, ${}_0L$ is the decreasing function of T_x.

4. What is the probability that ${}_0L > 0$?
   We have: ${}_0L = Bv^{T_x} - \pi(\frac{1-v^{T_x}}{\delta}) = (B + \frac{\pi}{\delta})v^{T_x} - \frac{\pi}{\delta}$. Also: $v = e^{-\delta}$.

   Therefore:
   $$ {}_0L > 0 \iff T_x < -\frac{1}{\delta} ln\Big(\frac{\pi}{B\delta + \pi}\Big) $$
   Since T_x is the random variable future lifetime, let F be the distribution function of T_x:
   $$ Pr({}_0L > 0) = Pr\bigg(T_x < -\frac{1}{\delta} ln\Big(\frac{\pi}{B\delta + \pi}\Big)\bigg) = F\bigg(-\frac{1}{\delta} ln\Big(\frac{\pi}{B\delta + \pi}\Big)\bigg) $$
   If we are given the form of survival function and other details such as δ, π, and B, we can calculate $Pr({}_0L > 0)$.

5. What is $\mathbb{E}({}_0L)$?
   Since ${}_0L = Bv^{T_x} - \pi\bar{a}_{\overline{T_x}|} \rightarrow \mathbb{E}({}_0L) = B\mathbb{E}(v^{T_x}) - \pi\mathbb{E}(\bar{a}_{\overline{T_x}|}) = B\bar{A}_x - \pi\bar{a}_x$.

6. What is $Var({}_0L)$?
   $$ Var({}_0L) = Var\bigg[\Big(B+\frac{\pi}{\delta}\Big)v^{T_x} - \frac{\pi}{\delta}\bigg] = \Big(B+\frac{\pi}{\delta}\Big)^2 Var(v^{T_x}) = \Big(B+\frac{\pi}{\delta}\Big)^2 ({}^2\bar{A}_x - \bar{A}_x^2) $$

Ex2: Construct the ${}_0L$ for a discrete n-year endowment insurance.
Recall that if Z is the present value random variable for a discrete n-year endowment insurance with both death benefit and endowment benefit equal to B (note: it is possible to have different values between death benefit and endowment), then:

$$Z = \begin{cases} Bv^{K_x+1} & \text{if } K_x < n \\ Bv^n & \text{if } K_x \geq n, \end{cases}$$

and we know that $\mathbb{E}(Z) = A_{x:\overline{n}|}$. In this case:

$${}_0L = \begin{cases} Bv^{K_x+1} - \pi \ddot{a}_{\overline{K_x+1}|} = \big(B + \frac{\pi}{d}\big)v^{K_x+1} - \frac{\pi}{d} & \text{if } K_x < n \\ Bv^n - \pi\ddot{a}_{\overline{n}|} = \big(B + \frac{\pi}{d}\big)v^n - \frac{\pi}{d} & \text{if } K_x \geq n \end{cases}$$

Therefore, $\mathbb{E}({}_0L) = BA_{x:\overline{n}|} - \pi\ddot{a}_{x:\overline{n}|}$, and $Var({}_0L) = \big(B + \frac{\pi}{d}\big)^2({}^2A_{x:\overline{n}|} - A^2_{x:\overline{n}|})$.

We can derive similar formulas for loss-at-issue random variable if we have other insurance products such as: continuous (discrete) n-year endowment, continuous (discrete) n-year term, and continuous n-year pure endowment.

3 Percentile Premium

Goal: We want to find the minimum value of net premium P such that the insurance company has probability of at most α of a positive financial loss, or $P({}_0L > 0) \leq \alpha$. Such a problem is called percentile premium problem. Let's consider a fully continuous whole life insurance of B on (x). Let π be the level of net premium ($P = \pi$). We know:

$$Pr({}_0L > 0) = F_x\bigg[-\frac{1}{\delta}ln\bigg(\frac{\pi}{B\delta + \pi}\bigg)\bigg]$$

F_x is the cdf function of T_x, or F_x is the mortality function of T_x. Suppose we have t_α such that: $F_x(t) = \alpha$, or t_α is the $100\alpha^{th}$ percentile of T_x. Then:

$$-\frac{1}{\delta}ln\bigg(\frac{\pi}{B\delta + \pi}\bigg) \leq t_\alpha \implies \pi \geq \frac{B\delta}{e^{\delta t_\alpha} - 1} = \frac{B}{\frac{e^{\delta t_\alpha} - 1}{\delta}}$$

Note that if we have a certain continuous annuity (PSTAT 171) with present value: $\bar{a}_{\overline{n}|} = \frac{1-e^{-\delta n}}{\delta}$, then the accumulated value of this continuous annuity is: $\bar{s}_{\overline{n}|} = \bar{a}_{\overline{n}|}e^{\delta n} = \frac{e^{\delta n}-1}{\delta}$. Therefore, the smallest premium must be:

$$\pi = \frac{B}{\frac{e^{\delta t_\alpha} - 1}{\delta}} = \frac{B}{\bar{s}_{\overline{t_\alpha}|}}$$

* Use the same method to determine the percentile premiums for whole life (discrete), n-year term (discrete/continuous), and n-year endowment (discrete/continuous).