Annuity

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Recall: $\ddot{a}_{\overline{n}|}$ is the annuity-due certain, where payments are made at the beginning of each year. Also, the number of payments are certainly n (fixed) and: $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$ where d is the effective interest rate of discount ($d = \frac{i}{1+i}$).

1 Discrete annuity

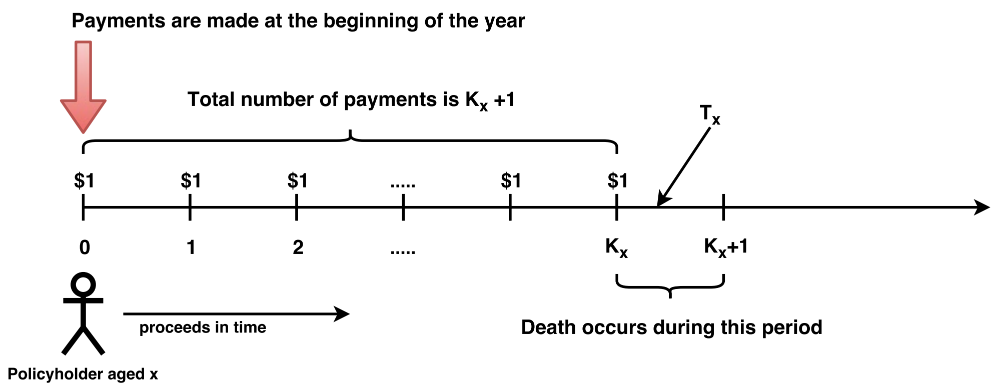
1.1 Different types of annuity-due

1.1.1 Whole life annuity-due

A whole life annuity-due is an on going series of payments made while someone remains alive. In other words, payments will be made at the beginning of each year as long as the policyholder is alive.
Let Y be the total present value of these payments. It is easy to realize that Y is a random variable as its value depends on the event of death (time of death). If we look at the diagram, if T_x (future lifetime) is in between K_x and $K_x + 1$ (where K_x is the curtate future lifetime, the integer parts of Tx), then:

$$Y = \ddot{a}_{\overline{K_x+1}|} = \frac{1 - v^{K_x+1}}{d} \tag{1}$$

As Y is the random variable of the present value of these payments, we are interest in calculating the average amount at time 0.



Notation: $E(Y) = \ddot{a}_x$.

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_tp_x = 1 + vp_x + v^2{}_2p_x + v^3{}_3p_x + ... \tag{2}$$

- If we have limiting age ω, replace ∞ by $\omega - x - 1$ in equation (2)
- Equation (2) is parallel to the continuous case $\bar{a}_x = \int_0^\infty v^t{}_tp_x dt$
- We certainly want to calculate the variance of Y ($Var(Y)$) by using the relationship between Y and Z where Z is the random variable present value of death benefit. In this case, we know from previous lecture that $Z = v^{K_x+1}$ if death occurs between K_x and $K_x + 1$ and death benefit is paid at the end of the year of death. Therefore, from (1):

$$Y = \frac{1-Z}{d} \to Var(Y) = \frac{1}{d^2}Var(1-Z) = \frac{1}{d^2}Var(Z) = \frac{{}^2A_x - A_x^2}{d^2}$$

where A_x is the APV (actuarial present value) of discrete whole life insurance issued to (x). **Note:** We can't calcuate $Var(Y)$ using ${}^2\ddot{a}_x - \ddot{a}_x^2$ because they are not equivalent. To understand why, read Example 2 in lecture notes (p.14-16)

1.1.2 n-year temporary life annuity-due

An n-year temporary life annuity-due makes payments at the beginning of each year as long as the policyholder is alive. The number of payments is limited to n years. In this case, we have two scenarios: (a) death occurs before time n ($K_x < n$) or (b) death occurs after time n ($K_x \geq n$).

$$Y = \begin{cases} \ddot{a}_{\overline{K_x+1}|} & \text{if } K_x < n \\ \ddot{a}_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$$

We see clearly that value of Y depends on when death occurs $\to Y$ is the random variable. Again, we want to calculate the average amount of these payments at

time 0 and the variance of Y.
Notation: $E(Y) = \ddot{a}_{x:\overline{n}|}$

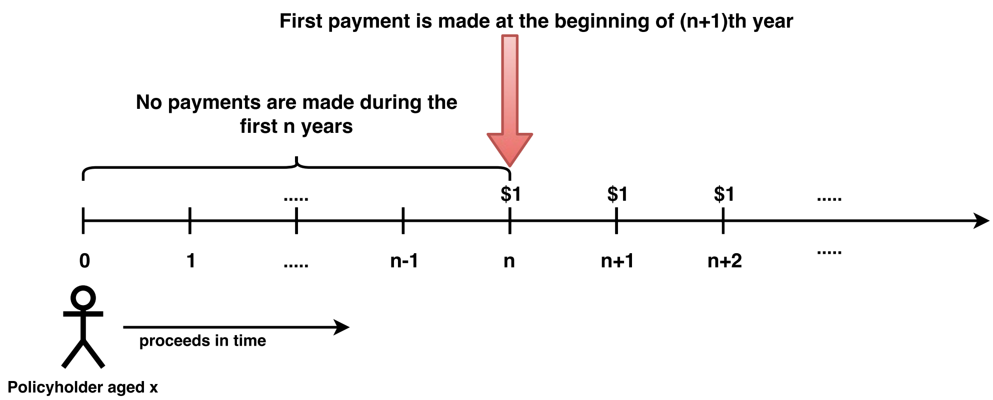
$$E(Y) = \ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = 1 + vp_x + v^2{}_2p_x + v^3{}_3p_x + ... + v^{n-1}{}_{n-1}p_x \quad (3)$$

To calculate the variance of Y, we can use the relationship between Y and Z where Z is the random variable present value of death benefit for discrete n-year endowment issued to (x).

$$Y = \frac{1-Z}{d} \rightarrow Var(Y) = \frac{Var(Z)}{d^2} = \frac{{}^2A_{x:\overline{n}|} - A^2_{x:\overline{n}|}}{d^2}$$

1.1.3 Deferred n-year life annuity

An n-year deferred life annuity is a life annuity in which payments start in n years if the policy holder is still alive at time n. Payments are made at the beginning of each year. If death occurs before time n, there will no payments.



Notation: $E(Y) = {}_{n|}\ddot{a}_x$

$$E(Y) = {}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k {}_kp_x \quad (4)$$

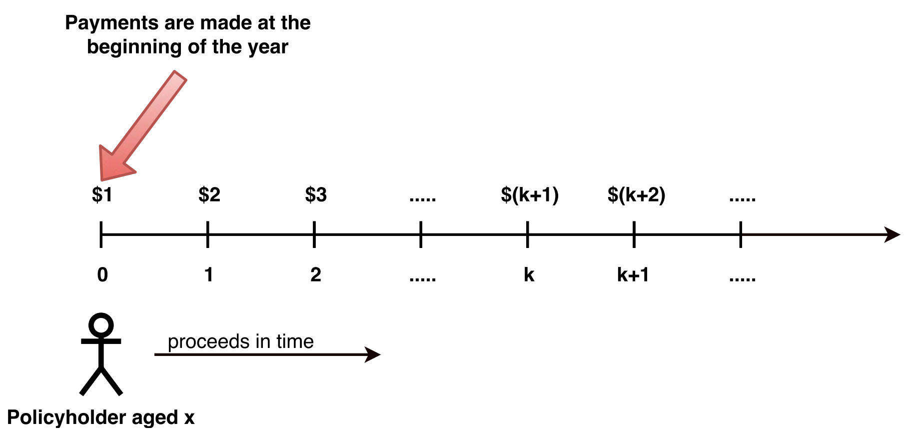
- Payments are deferred by a period of n years
- If there is limiting age ω, replace ∞ by $\omega - x - 1$
- There is no simple linear relationship between Y and Z, if we want to find $Var(Y)$, we need to use the first principle (e.g.: calculate $E(Y^2)$, then $Var(Y) = E[Y^2] - (E[Y])^2$)

1.1.4 Certain-and-life annuity (guaranteed annuity)

This type of annuity is the combination of an n-year annuity certain and an n-year deferred life annuity. In other words, there is a certain series of payments made for the next n years regardless the outcome of death (during these n years). After that, if the policy holder is still alive at time n, payments are made as long as policy holder is alive.
Notation: $E(Y) = \ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$

- To calculate $Var(Y)$, need to use the first principle

1.1.5 Increasing annuity (arithmetically)



Increasing whole life annuity
Notation: $E(Y) = (I\ddot{a})_x$

$$E(Y) = (I\ddot{a})_x = \sum_{k=0}^{\infty}(k+1)v^k {}_kp_x = 1 + 2vp_x + 3v^2 {}_2p_x + 4v^3 {}_3p_x + ... \qquad (5)$$

- If there is limitting age ω, replace ∞ by $\omega - x - 1$
- To calculate $Var(Y)$, use the first principle

Increasing n-year temporary life annuity
Notation: $E(Y) = (I\ddot{a})_{x:\overline{n}|}$

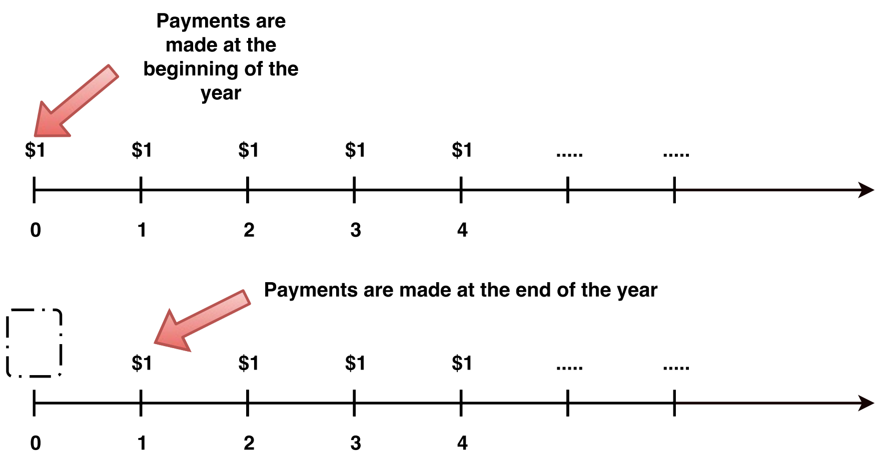
$$E(Y) = (I\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1}(k+1)v^k {}_kp_x \qquad (6)$$

Note: Besides arithmetically increasing annuity, we also have geometrically increasing annuity. Please read the lecture notes on page 31 for more details.

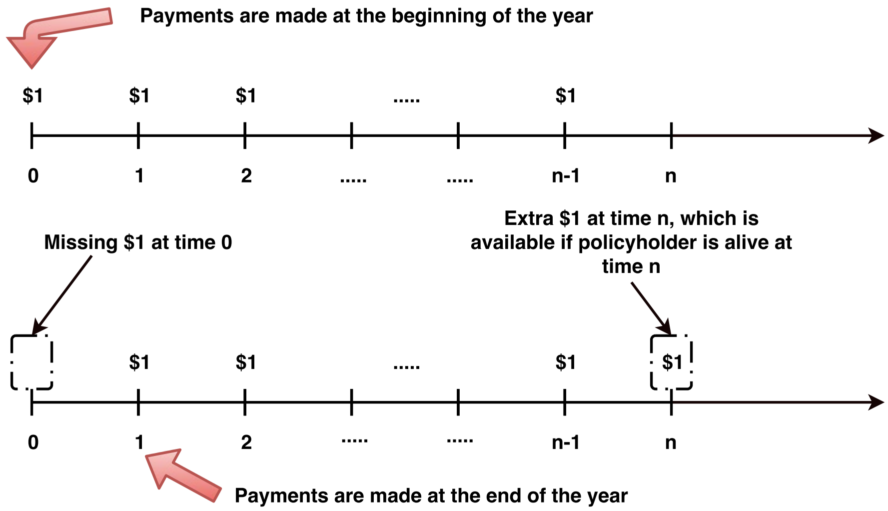
1.2 Relationship between life annuity-due and annuity-immediate

Recall: Certain annuity-immediate is when series of payments are made at the end of each year and we are certain about the total number of payments.
Life annuity-immediate is similar as payments are made at the end of the year and we must consider the randomness in the number of payments as it depends on the event of death. At the same time, we can derive the relationship between life annuity-due and life annuity-immediate. The easiest way is to draw out diagrams to compare the cash flows.
We have several equations to describe relationship between annuity-due and annuity-immediate:

- Whole life annuity: $a_x = \ddot{a}_x - 1$



- n-year temporary annuity: $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + {}_nE_x$



- n-year deferred annuity: ${}_{n|}a_x = {}_{n|}\ddot{a}_x - {}_nE_x$

1.3 The expected accumulated annuity value

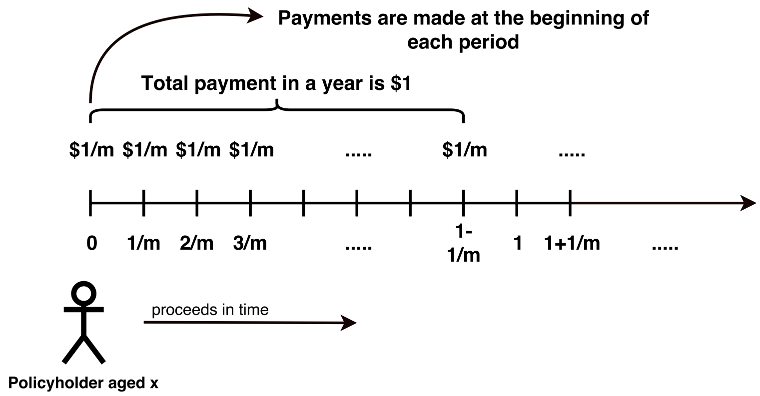
Let's take a look at the present value of the annuity-due certain, $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$. Then, the future value of this annuity-due certain after n years is $(1+i)^n\ddot{a}_{\overline{n}|}$. If we recall the notation for this future value (accumulated value), it is $\ddot{s}_{\overline{n}|}$. Or, $\ddot{s}_{\overline{n}|} = (1+i)^n\ddot{a}_{\overline{n}|} \rightarrow \ddot{a}_{\overline{n}|} = v^n\ddot{s}_{\overline{n}|}$.
Similarly to the certain annuity, to calculate the expected accumulated annuity value, which is denoted as $\ddot{s}_{x:\overline{n}|}$, we can write out a similar equation such that: $\ddot{a}_{x:\overline{n}|} = v^n{}_np_x\ddot{s}_{x:\overline{n}|} = {}_nE_x\ddot{s}_{x:\overline{n}|}$. In this case, ${}_nE_x$ can be thought as the "discount factor" in life annuity. We see much more of this in next quarter when studying reserves.

1.4 mthly life annuity

1.4.1 mthly life annuity-due

It is not necessary to have annual payment for life annuity. It can also be monthly or weekly. For example, if we have mthly whole life annuity-due:

- Payments are made at the beginning of each $1/m$ of a year
- The amount of each payment is $\$1/m$
- Total payment over a period of one year is $1



Notation: $E(Y) = \ddot{a}_x^{(m)}$

$$E(Y) = \ddot{a}_x^{(m)} = \frac{1}{m} \sum_{k=0}^{\infty} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x \tag{7}$$

Similarly, we have formulas to calculate mthly n-year temporary life annuity-due and mthly n-year deferred life annuity-due.

- mthly n-year temporary annuity: $\ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1}{m} \sum_{k=0}^{mn-1} v^{\frac{k}{m}} {}_{\frac{k}{m}}p_x$
- mthly n-year deferred annuity: ${}_{n|}\ddot{a}_x = v^n {}_np_x \ddot{a}_{x+n}^{(m)}$

1.4.2 Relationship between mthly annuity-due and annuity-immediate

- Whole life: $a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$
- Temporary life: $a_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} - \frac{1}{m} + \frac{1}{m} {}_nE_x$
- Deferred life: ${}_{n|}a_x^{(m)} = {}_{n|}\ddot{a}_x^{(m)} - \frac{{}_nE_x}{m}$

1.4.3 Approximating mthly life annuity under UDD assumption

Policy	Formula
Whole life	$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$
n-year temporary	$\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)(1 - {}_nE_x)$
n-year deferred	${}_{n|}\ddot{a}_x = \alpha(m){}_{n|}\ddot{a}_x - \beta(m){}_nE_x$

In the above table, $\alpha(m) = \frac{id}{i^{(m)}d^{(m)}}$ and $\beta(m) = \frac{i - i^{(m)}}{i^{(m)}d^{(m)}}$

1.4.4 Approximating mthly life annuity using Woolhouse's formula

Policy	Approximation formula
Whole life	$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$
n-year temporary	$\ddot{a}_{x:\overline{n}\rvert}^{(m)} = \ddot{a}_{x:\overline{n}\rvert} - \frac{m-1}{2m}(1 - v^n)_n p_x - \frac{m^2-1}{12m^2}(\delta + \mu_x - v^n {}_n p_x(\delta + \mu_{x+n}))$
n-year deferred	${}_{n\rvert}\ddot{a}_x^{(m)} = {}_{n\rvert}\ddot{a}_x - \frac{m-1}{2m} v^n {}_n p_x - \frac{m^2-1}{12m^2} v^n {}_n p_x(\delta + \mu_{x+n})$

1.5 Relating different policies

- Whole life annuity-due is combination between the n-year temporary life annuity-due and n-year deferred life annuity-due.

  $$\ddot{a}_x = \ddot{a}_{x:\overline{n}\rvert} + {}_{n\rvert}\ddot{a}_x.$$

  Note that it works the same way if we have continous whole life annuity: $\bar{a}_x = \bar{a}_{x:\overline{n}\rvert} + {}_{n\rvert}\bar{a}_x$

- Relationship between n-year deferred annuity-due and whole life annuity.

  $${}_{n\rvert}\ddot{a}_x = v^n {}_n p_x \ddot{a}_{x+n} = {}_n E_x \ddot{a}_{x+n}$$

  Note that it works the same way if we have continuous n-year deferred annuity. Also, this formula is similar to what we have learned in insurance chapter: ${}_{n\rvert}A_x = {}_n E_x A_{x+n}$

1.6 Recursions

- Whole life annuity-due: $\ddot{a}_x = 1 + v p_x \ddot{a}_{x+1}$. If we think about it, it makes perfect sense. At the issued time (age x or time 0), the first payment is made because at time 0, the probability of being alive is 1. After year 1, there are two possibilities: (1) if death occurs with probability q_x, no more payment will be made; (2) if the policyholder is alive with probability p_x, continue to provide whole life annuity, which has value of $\ddot{a}_{x+1}$ at time 1.
- n-year temporary life annuity: $\ddot{a}_{x:\overline{n}\rvert} = 1 + v p_x \ddot{a}_{x+1:\overline{n-1}\rvert}$
- mthly whole life annuity-due: $\ddot{a}_x^{(m)} = \frac{1}{m} + v^{\frac{1}{m}} {}_{\frac{1}{m}}p_x \ddot{a}_{x+\frac{1}{m}}^{(m)}$

2 Continuous annuity

Policy	Notation	Formula
Wholfe life	$\bar{a}_x$	$\bar{a}_x = \int_0^\infty v^t {}_tp_x dt = \frac{1-\bar{A}_x}{\delta}$
n-year temporary	$\bar{a}_{x:\overline{n}\rvert}$	$\bar{a}_{x:\overline{n}\rvert} = \int_0^n v^t {}_tp_x dt = \frac{1-\bar{A}_{x:\overline{n}\rvert}}{\delta}$
n-year deferred	${}_{n\rvert}\bar{a}_x$	${}_{n\rvert}\bar{a}_x = \int_n^\infty v^t {}_tp_x dt$
n-year certain and life	$\bar{a}_{\overline{x:\overline{n}\rvert}}$	$\bar{a}_{\overline{x:\overline{n}\rvert}} = \bar{a}_{\overline{n}\rvert} + {}_{n\rvert}\bar{a}_x$
Continuous increasing whole life	$(I\bar{a})_x$	$(I\bar{a})_x = \int_0^\infty tv^t {}_tp_x dt$

Note: If we have constant force of mortality:
(a) Whole life

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1-\frac{\mu}{\mu+\delta}}{\delta} = \frac{\delta}{\mu+\delta}\left(\frac{1}{\delta}\right) = \frac{1}{\mu+\delta}$$

(b) n-year deferred

$${}_{n\rvert}\bar{a}_x = v^n {}_np_x \bar{a}_{x+n} = e^{-(\mu+\delta)n}\frac{1}{\mu+\delta} = \frac{e^{-(\mu+\delta)n}}{\mu+\delta}$$

(c) n-year temporary

$$\bar{a}_{x:\overline{n}\rvert} = \bar{a}_x - {}_{n\rvert}\bar{a}_x = \frac{1}{\mu+\delta} - e^{-(\mu+\delta)n}\frac{1}{\mu+\delta} = \frac{1-e^{-(\mu+\delta)n}}{\mu+\delta}$$