CARLETON UNIVERSITY

SCHOOL OF MATHEMATICS AND STATISTICS

HONOURS PROJECT

TITLE: MADELUNG'S FLUID: THE BRIDGE BETWEEN NONLINEAR SCHRÖDINGER AND KORTEWEG–DE VRIES EQUATIONS

AUTHOR: HOWL NHAN

SUPERVISOR: DR. DAVE AMUNDSEN

DATE: NOVEMBER 2024

Contents

1 Introduction

In this paper, we will explore the relationship between two significant mathematical models in theoretical physics: the nonlinear Schrödinger equation and the Korteweg-de Vries equation, framed within the context of the Madelung's fluid dynamics [14]. Under suitable physical hypothesis for the current velocity, this correspondence allows us to find vary envelope soliton solutions.

Our investigation includes analyzing cases where the current velocity is constant, as well as scenarios where perturbations introduce more complex. Notably, we extend our analysis beyond the conventional cubic nonlinear Schrödinger equation to encompass the cubic and quintic nonlinear Schrödinger equation, which supports the same types of soliton solutions under similar conditions. This broader perspective enables us to showcase the emergence of both bright and dark/gray envelope solitons linked to various families of the Korteweg-de Vries equation. These solutions highlight the dynamic range of wave structures possible within this theoretical framework, contributing to a deeper understanding of soliton behavior in nonlinear systems.

Furthermore, we provide a method for approaching solutions to the nonlinear Schrödinger equations by leveraging the soliton solutions of the associated Korteweg-de Vries equations [6, 7]. This method has offered deeper insights and underscores the utility of fluid-based language for interpreting complex wave phenomena. This approach is not only enriches the theoretical landscape of soliton research but also allows for a fluid-based interpretation of quantum phenomena, bridging classical and quantum mechanical descriptions. In summary, this paper purposes a novel method in solving the nonlinear Schrödinger equation starting from the Korteweg-de Vries equation give new insights and represents an alternative key of reading of the dark/gray envelope solitons based on the fluid language. For each derived solution, we will include corresponding graphs to aid in visualization and to demonstrate how various parameters influence the behavior of the solution. These visual aids will not only enhance understanding but also serve as a tool for validating the solutions under different conditions.

2 Background

2.1 Schrödinger Equation

The time-dependent Schrödinger equation is a fundamental partial differential equation in quantum mechanics that governs the evolution of the wave function $\Psi(x,t)$ for a non-relativistic quantum-mechanical system. Mathematically, it is expressed as:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t).$$

This is a standard linear Schrödinger equation in one dimension. It was published in 1926 by Erwin Schrödinger [17] and it contributed significantly to the development of quantum mechanics. However, its wave function $\Psi(x,t)$ was not easily visualizable in classical terms, leading to various interpretations of quantum mechanics. In the nonlinear case, we introduce a new term, namely U, which is a functional of $|\Psi(x,t)|^2$, to replace the potential $V(x,t)$. The term $U(|\Psi(x,t)|^2)$ represents a nonlinear potential that depends on the intensity of the wave function.

In a general framework, this wave function is viewed as a vector Ψ in a separable complex Hilbert space $\mathcal{H}$ and is postulated to be normalized under the Hilbert space's inner product $\langle\Psi,\Psi\rangle = 1$. This inner product is crucial in interpreting quantum mechanics, as it defines the probabilities associated with various measurable outcomes. Moreover, the wave solutions describe quantum objects as having dual particle-wave nature. The shape of Ψ is dictated by the potential term, which can lead to bound states (localized solutions), scattering states (extended solutions), or other complex behaviors depending on the potential.

2.2 Korteweg–de Vries Equation

The Korteweg–de Vries equation primarily models weakly nonlinear, long waves in systems where both nonlinearity and dispersion are significant. Also, it describes surface waves of long wavelength and small amplitude on shallow water. The standard one-dimensional Korteweg–De Vries equation is given by:

$$\frac{\partial}{\partial t}u + \varepsilon u \frac{\partial}{\partial x}u + \mu \frac{\partial^3}{\partial x^3}u = 0.$$

It was first introduced by Joseph Valentin Boussinesq in 1877 and rediscovered by Diederik Korteweg and Gustav de Vries in 1895 [13], who found the simplest solution, the one-soliton solution. Moreover, it was first derived in the context of shallow water waves but has since been applied to many physical systems, including plasma physics and elastic media. We will present a more detailed discussion of this equation in the Madelung's fluid framework as we delve into the problem.

2.3 Shared Characteristics Between Two Equations

The Schrödinger equation and the Korteweg–de Vries equation are two fundamental equations in physics, describing different types of systems. However, there are several interesting connections between them, particularly in the context of nonlinear wave theory and integrable systems. For example, they both belong to the family of integrable systems, meaning they exhibit special mathematical properties such as: almost conservation laws [3, 19], the ability to obtain soliton solutions, and exact solvability. With respect to the latter, the inverse scattering transform, originally developed for the Korteweg–de Vries equation [12], was later extended to solve the nonlinear Schrödinger equation [15], highlighting deep structural similarities between these equations. In this paper, we will show how these two equations connect or relate to each other in the specific context - solitary wave solutions.

3 History and Motivation

In the fall of 1926, Erwin Madelung, a German physicist, reformulated Schrödinger's quantum equation into a more classical and intuitive form, introducing what is now known as the hydrodynamic formulation of quantum mechanics or often referred to as Madelung's fluid [14]. This approach, introduced just months after Schrödinger's original publication, revises the Schrödinger equation which describes the behavior of quantum systems into a set of fluid-like equations. In this framework, the fluid density corresponds to the probability of finding a particle at a given position, while the fluid's velocity field is related to the gradient of the quantum phase.

Madelung's insight came from recognizing that the phase of the wave function could be connected to classical action, while the amplitude naturally describes a probability density, which he then interpreted as a fluid density. By expressing the wave function in polar form, he could make these connections explicit, leading to a fluid-like interpretation of quantum mechanics, where quantum effects appear as corrections to classical fluid equations.

3.1 Linear Schrödinger Equation in Madelung's Framework

Consider the modified linear Schrödinger equation (LSE) [21] in one dimension as follows:

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + mV(x,t)\right]\Psi(x,t), \tag{1}$$

where:

- $\Psi(x,t)$: the wave function.
- i: imaginary number $\mathrm{i} = \sqrt{-1}$.
- $\hbar$: Planck constant [energy time] = [mass length^2 time^{-1}].
- $\hat{H}$: Hamiltonian operator, which represents the total energy of the system included kinetic and potential energies.
- m: the mass of particle [mass].
- $V(x,t)$: the potential of surrounding environment in which the particle exists [length^2 time^{-2}].

Notice that the difference between this LSE and the standard LSE as mentioned above is the phase V is attached with the mass m.

3.2 Madelung's Fluid

Before Madelung applied his transformation, the equation for the wave function $\Psi(x,t)$ typically remained as its own Schrödinger equation. Madelung sought to bridge the gap between quantum mechanics and classical mechanics by providing a more intuitive interpretation - it is called *the first statistical description of quantum mechanics*. Since there is an imaginary part in the partial differential equation (PDE), Madelung represented the complex wave function Ψ in its polar form [16] corresponding to (1):

$$\Psi(x,t) = \sqrt{\rho(x,t)} \exp\left\{\frac{\mathrm{i}S(x,t)}{\hbar}\right\}, \tag{2}$$

where:

- $S(x,t)$ is the phase and velocity potential [length^3 time^{-1}] and is defined as:

$$v(x,t) = \frac{1}{m}\frac{\partial}{\partial x}S(x,t), \tag{3}$$

  such that $v(x,t)$: hydrodynamic velocity [length time^{-1}].

- $\rho(x,t)$ is the probability density function of the fluid:

$$\begin{aligned} |\Psi(x,t)|^2 &= \rho(x,t)\left[\cos\left(\frac{S(x,t)}{\hbar}\right)\right]^2 + \rho(x,t)\left[\sin\left(\frac{S(x,t)}{\hbar}\right)\right]^2 \\ &= \rho(x,t)\left(\left[\cos\left(\frac{S(x,t)}{\hbar}\right)\right]^2 + \left[\sin\left(\frac{S(x,t)}{\hbar}\right)\right]^2\right) \\ &= \rho(x,t) \\ \Rightarrow \rho(x,t) &= |\Psi(x,t)|^2, \end{aligned} \tag{4}$$

  and such that

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1.$$

  Notice that for simplicity, Ψ is considered dimensionless, while ρ has units of [length^{-1}].

The complete wave function $\Psi(x,t)$ is a complex-valued function where the phase $S(x,t)$ determines the oscillatory nature of the wave with $\sqrt{\rho(x,t)}$ insights the amplitude of the wave and is associated with the probability density $\rho(x,t)$ of finding a particle at position x and at time t.

For the convenience and simplicity of derivations and equations, the following notations will be interchanged:

$$\frac{\partial}{\partial x} f = \partial_x f,$$

and

$$\frac{\partial^2}{\partial x \partial y} f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) = \partial_x \partial_y f = \partial_{xy} f.$$

Madelung substituted (2) into (1). In order to do this, he first computed $\partial_t \Psi(x,t)$, and $\partial_{xx}\Psi(x,t)$:

$$\partial_t \Psi(x,t) = \left[\frac{1}{2\sqrt{\rho(x,t)}}\partial_t \rho(x,t) + \frac{\mathrm{i}}{\hbar}\sqrt{\rho(x,t)}\partial_t S(x,t)\right] \exp\left\{\frac{\mathrm{i}S(x,t)}{\hbar}\right\}, \tag{5}$$

$$\begin{aligned}\partial_{xx}\Psi(x,t) = \exp\left\{\frac{\mathrm{i}S(x,t)}{\hbar}\right\} &\left[\frac{\mathrm{i}}{\hbar}\left(\frac{1}{\sqrt{\rho(x,t)}}\partial_x \rho(x,t)\partial_x S(x,t) + \sqrt{\rho(x,t)}\partial_{xx} S(x,t)\right)\right. \\ &\left. -\frac{1}{4}\frac{(\partial_x \rho(x,t))^2}{(\rho(x,t))^{3/2}} + \frac{1}{2}\frac{\partial_{xx}\rho(x,t)}{\sqrt{\rho(x,t)}} - \frac{1}{\hbar^2}\sqrt{\rho(x,t)}\left(\partial_x S(x,t)\right)^2\right].\end{aligned} \tag{6}$$

From (5) and (6), with some arrangements, (1) changes to:

$$\begin{aligned}&\left[\frac{\mathrm{i}\hbar}{2\sqrt{\rho(x,t)}}\partial_t \rho(x,t) + \frac{\mathrm{i}\hbar}{2m}\left(\frac{1}{\sqrt{\rho(x,t)}}\partial_x \rho(x,t)\partial_x S(x,t) + \sqrt{\rho(x,t)}\partial_{xx} S(x,t)\right)\right. \\ &\quad -\sqrt{\rho(x,t)}\partial_t S(x,t) + \frac{\hbar^2}{2m}\left(\frac{\partial_{xx}\rho(x,t)}{2\sqrt{\rho(x,t)}} - \frac{(\partial_x \rho(x,t))^2}{4(\rho(x,t))^{3/2}}\right) \\ &\quad \left. -\frac{1}{2m}\sqrt{\rho(x,t)}(\partial_x S(x,t))^2 + mV(x,t)\sqrt{\rho(x,t)}\right] \exp\left\{\frac{\mathrm{i}S(x,t)}{\hbar}\right\} = 0.\end{aligned} \tag{7}$$

Since $\exp\left\{\frac{\mathrm{i}S(x,t)}{\hbar}\right\} \neq 0$, all terms in the squared brackets of (7) have to be equal to 0, he decomposed them into two equations by separating the real and imaginary parts. For the imaginary part, we have:

$$\frac{\hbar}{2\sqrt{\rho(x,t)}}\left[\partial_t \rho(x,t) + \frac{1}{m}\left(\partial_x \rho(x,t)\partial_x S(x,t) + \rho(x,t)\partial_{xx} S(x,t)\right)\right] = 0. \tag{8}$$

Since $\frac{\hbar}{2\sqrt{\rho(x,t)}} \neq 0$, and converts all $S(x,t)$ terms to $v(x,t)$ by (3), we obtain:

$$\begin{aligned}&\partial_t \rho(x,t) + (\partial_x \rho(x,t) v(x,t) + \rho(x,t)\partial_x v(x,t)) = 0 \\ \Rightarrow\ &\partial_t \rho(x,t) + \partial_x[\rho(x,t)v(x,t)] = 0.\end{aligned} \tag{9}$$

The imaginary part of (7) now becomes (9), it is a continuity equation for the Madelung fluid density.

For the real part, we have:

$$-\sqrt{\rho(x,t)}\left[\partial_t S(x,t) - \frac{\hbar^2}{2m}\left(\frac{\partial_{xx}\rho(x,t)}{2\sqrt{\rho(x,t)}} - \frac{(\partial_x \rho(x,t))^2}{4(\rho(x,t))^{3/2}}\right) + mV(x,t) + \frac{1}{2m}(\partial_x S(x,t))^2\right] = 0. \tag{10}$$

Since we are only interested in a non-trivial solution, implies that $\sqrt{\rho(x,t)} \neq 0$. With some arrangements and apply the spatial derivative, (10) becomes:

$$\partial_x \left[\partial_t S(x,t) + \frac{1}{2m}(\partial_x S(x,t))^2\right] = \partial_x \left[\frac{\hbar^2}{2m}\left(\frac{\partial_{xx}\rho(x,t)}{2\rho(x,t)} - \frac{(\partial_x \rho(x,t))^2}{4(\rho(x,t))^3}\right) - mV(x,t)\right]. \tag{11}$$

On the LHS of (11), we compute:

$$\partial_t v(x,t) = \frac{1}{m}\partial_{tx} S(x,t).$$

Since $v(x,t)$ is smooth, $S(x,t)$ is also smooth, by Schwarz's Theorem:

$$\partial_{tx} S(x,t) = \partial_{xt} S(x,t) = m\partial_t v(x,t).$$

Apply (3) on the LHS of (11), we get:

$$\begin{aligned}
\partial_{xt} S(x,t) + \frac{1}{2m}\partial_x(\partial_x S(x,t))^2 &= m\partial_t v(x,t) + \frac{1}{2m}\partial_x[m^2(v(x,t))^2] \\
&= m\partial_t v(x,t) + mv(x,t)\partial_x v(x,t) \\
&= m[\partial_t + v(x,t)\partial_x]v(x,t).
\end{aligned}$$

On the RHS of (11), we compute:

$$\begin{aligned}
\partial_{xx}\sqrt{\rho(x,t)} &= \frac{1}{2}\frac{\partial_{xx}\rho(x,t)}{\sqrt{\rho(x,t)}} - \frac{1}{4}\frac{(\partial_x \rho(x,t))^2}{(\rho(x,t))^{3/2}} \\
\Rightarrow \frac{1}{\sqrt{\rho(x,t)}}\partial_{xx}\sqrt{\rho(x,t)} &= \frac{1}{2}\frac{\partial_{xx}\rho(x,t)}{\rho(x,t)} - \frac{1}{4}\frac{(\partial_x \rho(x,t))^2}{(\rho(x,t))^3}.
\end{aligned} \tag{12}$$

Thus,

$$\frac{\hbar^2}{2m}\partial_x\left[\frac{\partial_{xx}\rho(x,t)}{2\rho(x,t)} - \frac{(\partial_x \rho(x,t))^2}{4(\rho(x,t))^3}\right] - m\partial_x V(x,t) = \frac{\hbar^2}{2m}\partial_x\left(\frac{1}{\sqrt{\rho(x,t)}}\partial_{xx}\sqrt{\rho(x,t)}\right) - m\partial_x V(x,t).$$

In conclusion, spatial derivative of (11) is:

$$[\partial_t + v(x,t)\partial_x]v(x,t) = \frac{\hbar^2}{2m^2}\partial_x\left(\frac{1}{\sqrt{\rho(x,t)}}\partial_{xx}\sqrt{\rho(x,t)}\right) - \partial_x V(x,t). \tag{13}$$

This equation represents the time-dependent Bernoulli equation for a barotropic flow, which can be interpreted as a hydrodynamic Hamilton-Jacobi equation derived from the real part of (1) with:

$$\frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho(x,t)}}\frac{\partial^2}{\partial x^2}\sqrt{\rho(x,t)}\right),$$

is usually called Bohm potential [2] in the literature which introduces quantum effects. Notice that as $\hbar \to 0$, the quantum potential term vanishes and the modified Hamilton-Jacobi equation reduces to the classical Hamilton-Jacobi equation, causing the system to behave like a classical particle. The unmodified Hamilton-Jacobi equation in this context is as follows:

$$\partial_t S(x,t) + \frac{1}{2m}(\partial_x S(x,t))^2 + V(x,t) = 0,$$

as derived above (for more details on derivation and connection, please refer to [10]). The action function $S(x,t)$ plays a dual role in classical and quantum mechanics:

- In quantum mechanics, it is the phase of the wave function.
- In classical mechanics, it corresponds to the action, which determines the particle's trajectory through the principle of least action.

In summary, (9) and (13) together form a system:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}[\rho(x,t)v(x,t)] = 0,$$
$$\left[\frac{\partial}{\partial t} + v(x,t)\frac{\partial}{\partial x}\right] v(x,t) = \frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho(x,t)}}\frac{\partial^2}{\partial x^2}\sqrt{\rho(x,t)}\right) - \frac{\partial}{\partial x}V(x,t).$$

It is called Madelung's fluid, and each equation in the system describes different characteristics of a quantum fluid, as mentioned in the introduction of **Sec. 3**. The system is formally analogous to classical fluid dynamics but with a quantum correction.

4 Preliminary

4.1 Nonlinear Schröodinger Equation

Now consider the following nonlinear Schröodinger equation (NLSE):

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + mU(|\Psi(x,t)|^2)\right]\Psi(x,t), \tag{14}$$

where U is function of $|\Psi(x,t)|^2$. If $U(|\Psi(x,t)|^2) \propto |\Psi(x,t)|^2$ [21], this becomes the well-known completely integrable NLS [1, 9, 20]. This equation is used to describe many physical systems, including wave propagation in optical fibersand Bose-Einstein condensates. Moreover, when the potential is given by:

$$U(|\Psi(x,t)|^2) = q_0|\Psi(x,t)|^{2\beta} + q_1|\Psi(x,t)|^{4\beta},$$

where q_0, q_1 are constants, and β is a parameter that controls the type of nonlinearity. For $\beta = 1$, the equation corresponds to cubic and quintic nonlinearity. This form of nonlinearity can arise in certain physical systems, such as nonlinear opticsand plasma physics, where higher-order effects need to be considered in addition to the standard cubic nonlinearity.

4.2 From NLSE to Generalized KdVE

Although in the NLSE, $V(x,t)$ is replaced by the nonlinear term $U(|\Psi(x,t)|^2)$, the set of coupled fluid dynamics equations, according to (14), are derived like the linear case in **Sec. 3.2**:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}[\rho(x,t)v(x,t)] = 0, \tag{15}$$
$$\left[\frac{\partial}{\partial t} + v(x,t)\frac{\partial}{\partial x}\right] v(x,t) = \frac{\hbar^2}{2m^2}\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{\rho(x,t)}}\frac{\partial^2}{\partial x^2}\sqrt{\rho(x,t)}\right) - \frac{\partial}{\partial x}U(|\Psi(x,t)|^2). \tag{16}$$

From the Madelung's fluid (15) and (16), we can derive an evolution equation for the density $\rho(x,t)$ with an arbitrary velocity function $v(x,t)$ [6]. For simplicity in deriving equations, all functions will simplify to the function name (i.e. $f(x,t) = f$).

First, by multiplying v with (15):

$$\begin{aligned} & v\partial_t\rho + v\partial_x[\rho v] = 0 \\ \Rightarrow\ & v\partial_t\rho + v^2\partial_x\rho + \rho v\partial_x v = 0 \\ \Rightarrow\ & \rho v\partial_x v = -v\partial_t\rho - v^2\partial_x\rho. \end{aligned}$$

Then, by multiplying ρ with the LHS of (16) and combine with the above computation:

$$\begin{aligned}\rho(\partial_t + v\partial_x)v &= \rho\partial_t v + \rho v\partial_x v \\ &= \rho\partial_t v - v\partial_t\rho - v^2\partial_x\rho. \end{aligned} \tag{17}$$

On the other hand, we will integrate (16) with respect to x to obtain:

$$\begin{aligned} &\int \partial_t v + v\partial_x v \ dx = \int \frac{\hbar^2}{2m^2}\partial_x\left(\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho}\right) - \partial_x U \ dx \\ \Rightarrow &\int \partial_t v \ dx + \frac{1}{2}v^2 = \frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho} - U + c_0(t) \\ \Rightarrow & -\frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho} = -\int \partial_t v \ dx - \frac{1}{2}v^2 - U + c_0(t) \\ \Rightarrow & -\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho} = -2\int \partial_t v \ dx - v^2 - 2U + 2c_0(t). \end{aligned} \tag{18}$$

Now multiplying the result above by $\partial_x\rho$. Note that $\partial_x\rho = \partial_x(\sqrt{\rho})^2 = 2\sqrt{\rho}\partial_x\sqrt{\rho}$, we have:

$$\begin{aligned} & -\left(\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho}\right)\partial_x\rho = -2\partial_x\rho\int \partial_t v \ dx - v^2\partial_x\rho - 2U\partial_x\rho + 2c_0(t)\partial_x\rho \\ \Rightarrow & -\frac{2\hbar^2}{m^2}\partial_x\rho^{1/2}\partial_{xx}\rho^{1/2} = -2\partial_x\rho\int \partial_t v \ dx - v^2\partial_x\rho - 2U\partial_x\rho + 2c_0(t)\partial_x\rho. \end{aligned} \tag{19}$$

Notice that:

$$\begin{aligned} \partial_x\rho^{1/2} &= \frac{1}{2}\rho^{-1/2}\partial_x\rho \\ \Rightarrow \partial_{xx}\rho^{1/2} &= \frac{1}{2}\rho^{-1/2}\partial_{xx}\rho - \frac{1}{4}\rho^{-3/2}(\partial_x\rho)^2 \\ \Rightarrow \partial_{xxx}\rho^{1/2} &= \frac{1}{2}\rho^{-1/2}\partial_{xxx}\rho - \frac{3}{4}\rho^{-3/2}\partial_x\rho\partial_{xx}\rho + \frac{3}{8}\rho^{-5/2}(\partial_x\rho)^3. \end{aligned}$$

And

$$\begin{aligned} \partial_x\rho^{1/2}\partial_{xx}\rho^{1/2} &= \frac{1}{2}\rho^{-1/2}\partial_x\rho\left(\frac{1}{2}\rho^{-1/2}\partial_{xx}\rho - \frac{1}{4}\rho^{-3/2}(\partial_x\rho)^2\right) \\ &= \frac{1}{4}\rho^{-1}\partial_x\rho\partial_{xx}\rho - \frac{1}{8}\rho^{-2}(\partial_x\rho)^3 \\ \Rightarrow \partial_x\rho\partial_{xx}\rho &= 4\rho\partial_x\rho^{1/2}\partial_{xx}\rho^{1/2} + \frac{1}{2}\rho^{-2}(\partial_x\rho)^3. \end{aligned}$$

Thus, we can derive:

$$\begin{aligned} \partial_x\left(\frac{\partial_{xx}\rho^{1/2}}{\rho^{1/2}}\right) &= \frac{1}{\rho}(\rho^{1/2}\partial_{xxx}\rho^{1/2} - \partial_x\rho^{1/2}\partial_{xx}\rho^{1/2}) \\ &= \frac{1}{\rho}\left[\frac{1}{2}\partial_{xxx}\rho - \frac{3}{4}\rho^{-1}\partial_x\rho\partial_{xx}\rho + \frac{3}{8}\rho^{-2}(\partial_x\rho)^3 - \partial_x\rho^{1/2}\partial_{xx}\rho^{1/2}\right] \\ &= \frac{1}{\rho}\left[\frac{1}{2}\partial_{xxx}\rho - 3\partial_x\rho^{1/2}\partial_{xx}\rho^{1/2} - \frac{3}{8}\rho^{-2}(\partial_x\rho)^3 + \frac{3}{8}\rho^{-2}(\partial_x\rho)^3 - \partial_x\rho^{1/2}\partial_{xx}\rho^{1/2}\right] \\ &= \frac{1}{\rho}\left(\frac{1}{2}\partial_{xxx}\rho - 4\partial_x\rho^{1/2}\partial_{xx}\rho^{1/2}\right). \end{aligned} \tag{20}$$

Now multiplying (16) by ρ. On the LHS, applying (17); and on the RHS, applying (20), we obtain:

$$\rho\frac{\partial}{\partial t}v - v\frac{\partial}{\partial t}\rho - v^2\frac{\partial}{\partial x}\rho = \frac{\hbar^2}{4m^2}\frac{\partial^3}{\partial x^3}\rho - \frac{2\hbar^2}{m^2}\frac{\partial}{\partial x}\rho^{1/2}\frac{\partial^2}{\partial x^2}\rho^{1/2} - \rho\frac{\partial}{\partial x}U. \tag{21}$$

Combining (19) and (21), we get:

$$-2\frac{\partial}{\partial x}\rho\int\frac{\partial}{\partial t}v\ dx - v^2\frac{\partial}{\partial x}\rho - 2U\frac{\partial}{\partial x}\rho + 2c_0(t)\frac{\partial}{\partial x}\rho = \rho\frac{\partial}{\partial t}v - v\frac{\partial}{\partial t}\rho - v^2\frac{\partial}{\partial x}\rho - \frac{\hbar^2}{4m^2}\frac{\partial^3}{\partial x^3}\rho + \rho\frac{\partial}{\partial x}U$$

$$\Downarrow$$

$$\begin{aligned}
&-\rho(x,t)\frac{\partial}{\partial t}v(x,t) + v(x,t)\frac{\partial}{\partial t}\rho(x,t) + 2\left[c_0(t) - \int\frac{\partial}{\partial t}v(x,t)\ dx\right]\frac{\partial}{\partial x}\rho(x,t)\\
&-\left(\rho(x,t)\frac{\partial}{\partial x}U(|\Psi(x,t)|^2) + 2U(|\Psi(x,t)|^2)\frac{\partial}{\partial x}\rho(x,t)\right) + \frac{\hbar^2}{4m^2}\frac{\partial^3}{\partial x^3}\rho(x,t) = 0.
\end{aligned} \tag{22}$$

Equation (22) will be used to establish the connection between NSLE (13) and a wide class of Korteweg-de Vries equation (KdVE) for the density ρ. This connection will be demonstrated by exploring two following special cases: (i) $v(x,t) = v_0$, an arbitrary constant; and (ii) $v(x,t) = v(\xi)$, where $\xi = x - u_0 t$ is the combined variable with u_0 is an arbitrary real constant. Each case highlighting how different physical regimes can be captured within the framework of the KdVE for the density, thereby illustrating the versatility of this formalism.

5 Motion With Constant Current Velocity

5.1 Stationary Madelung's Fluid

As mentioned above, we will consider the first case where the current velocity is an arbitrary non-zero constant:

$$v(x,t) = v_0. \tag{23}$$

Since the velocity v is fixed in time t, consequently, we will assume $c_0(t) = c_0$, a constant. Moreover, the Madelung fluid density equation (15) transforms into:

$$\frac{\partial}{\partial t}\rho(x,t) + v_0\frac{\partial}{\partial x}\rho(x,t) = 0. \tag{24}$$

This is a linear advection equation. Thus the solution ρ is:

$$\rho(x,t) = \rho(\tau) = \rho(x - v_0 t), \tag{25}$$

where $\tau = x - v_0 t$, a combined variable.

Moreover, from the assumption (23) and the change of variables of (25), (18) implies:

$$\begin{aligned}
&-\frac{\hbar^2}{m^2}\frac{d^2}{d\tau^2}\rho^{1/2} = -v_0^2\rho^{1/2} - 2U\rho^{1/2} + 2c_0\rho^{1/2}\\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{d^2}{d\tau^2}\rho^{1/2} + 2U\rho^{1/2} = (2c_0 - v_0^2)\rho^{1/2}\\
\Rightarrow\ &-\frac{\hbar^2}{2m^2}\frac{d^2}{d\tau^2}\rho^{1/2} + U\rho^{1/2} = E\rho^{1/2},
\end{aligned} \tag{26}$$

where,

$$E = c_0 - \frac{v_0^2}{2} = \text{constant}, \tag{27}$$

represents the total energy of the system. Since E is a constant, this implies that the system is in a stationary state. Also notice that from the physical assumption (23), the anti-derivative of (3) with respect to x is:

$$\begin{aligned} v_0 &= \frac{1}{m}\partial_x S(x,t) \\ \Rightarrow \partial_x S(x,t) &= m v_0. \end{aligned} \tag{28}$$

Recall that from (10), since we work in the NLSE, we replace the linear term V with the nonlinear term U. Next, simplifying the equation with (12) and changing the parameter of the derivative term of ρ into ξ, we obtain:

$$\begin{aligned} & -\rho^{1/2}\left[\partial_t S - \frac{\hbar^2}{2m}\left(\frac{\partial_{xx}\rho}{2\rho^{1/2}} - \frac{(\partial_x \rho)^2}{4\rho^{3/2}}\right) + mU + \frac{1}{2m}(\partial_x S)^2\right] = 0 \\ \Rightarrow & -\rho^{1/2}\partial_t S - \frac{1}{2m}\rho^{1/2}(\partial_x S)^2 + \frac{\hbar^2}{2m}\partial_{xx}\rho^{1/2} - mU\rho^{1/2} = 0 \\ \Rightarrow & -\rho^{1/2}\partial_t S - \frac{1}{2m}\rho^{1/2}(\partial_x S)^2 = -\frac{\hbar^2}{2m}\partial_{xx}\rho^{1/2} + mU\rho^{1/2} \\ \Rightarrow & -\frac{1}{m}\rho^{1/2}\partial_t S - \frac{1}{2m^2}\rho^{1/2}(\partial_x S)^2 = -\frac{\hbar^2}{2m^2}\partial_{xx}\rho^{1/2} + U\rho^{1/2} \\ \Rightarrow & -\frac{1}{m}\rho^{1/2}\partial_t S - \frac{1}{2m^2}\rho^{1/2}(\partial_x S)^2 = -\frac{\hbar^2}{2m^2}\frac{d^2}{d\tau^2}\rho^{1/2} + U\rho^{1/2} \\ \Rightarrow & -\frac{1}{m}\rho^{1/2}\partial_t S - \rho^{1/2}\frac{v_0^2}{2} = E\rho^{1/2} \\ \Rightarrow & -\frac{1}{m}\rho^{1/2}\partial_t S = c_0\rho^{1/2} \\ \Rightarrow & \ \partial_t S = -mc_0. \end{aligned} \tag{29}$$

Therefore, from (28) and (29), we can conclude that

$$S(x,t) = mv_0 x - mc_0 t + c_1,$$

where c_1 is an arbitrary constant. Specifically, $S(x,t)$ governs the phase evolution of the wave function and is a key part of the Wentzel-Kramers-Brillouin (WKB) [8] or eikonal approximation [18] used in semi-classical or classical limits of quantum mechanics. As a result, we get:

$$\Psi(x,t) = C\sqrt{\rho(x - v_0 t)}\exp\{\mathrm{i}(kx - \omega t)\}.$$

where $C = \exp\{\mathrm{i}c_1\}, k = \dfrac{v_0 m}{\hbar}$ represents the wavevector and $\omega = \dfrac{c_0 m}{\hbar}$ is the angular frequency. However, we can absorb the constant C into $\sqrt{\rho(x - v_0 t)}$ because Ψ represents the state at time t, it must be normalized:

$$\int |\Psi(x,t)|^2\, dx = \int \left(\sqrt{\rho(x - v_0 t)}\right)^2\, dx = \int \rho(x - v_0 t)\, dx = 1.$$

Thus $\rho(x - v_0 t)$ must, in fact, be a normalized function. Implies that:

$$\Psi(x,t) = \sqrt{\rho(x - v_0 t)}\exp\{\mathrm{i}(kx - \omega t)\}. \tag{30}$$

It is indispensable to highlight that even though (26) is generally a nonlinear stationary Schrödinger equation, due to an arbitrary function $U(\rho) = U(|\Psi(x,t)|^2)$ in the equation. Nevertheless, it still describes a type of stationary state within the configurational τ-space, representing a nonlinear eigenvalue problem.

Apply changing variable $\tau = x - v_0 t$:

$$\frac{\partial}{\partial t}\rho(x,t) = \left(\frac{d}{dt}[x - v_0 t]\right)\left(\frac{d}{d\tau}\rho(x - v_0 t)\right) = -v_0 \frac{d}{d\tau}\rho(\tau),$$
$$\frac{\partial}{\partial x}\rho(x,t) = \left(\frac{d}{dx}[x - v_0 t]\right)\left(\frac{d}{d\tau}\rho(x - v_0 t)\right) = \frac{d}{d\tau}\rho(\tau).$$

With the assumptions and derivations of (23) and (25), we can decompose (22) term-by-term:

$$-\rho(x,t)\frac{\partial}{\partial t}v(x,t) = -\rho(x,t)\frac{\partial}{\partial t}v_0 = 0,$$
$$v(x,t)\frac{\partial}{\partial t}\rho(x,t) = v_0\left(-v_0\frac{d}{d\tau}\rho(\tau)\right) = -v_0^2\frac{d}{d\tau}\rho(\tau),$$
$$2\left[c_0(t) - \int \frac{\partial}{\partial t}v(x,t)\ dx\right]\frac{\partial}{\partial x}\rho(x,t) = 2\left[c_0 - \int \frac{\partial}{\partial t}v_0\ dx\right]\frac{d}{d\tau}\rho(\tau) = 2c_0\frac{d}{d\tau}\rho(\tau),$$
$$\rho(x,t)\frac{\partial}{\partial x}U(|\Psi(x,t)|^2) + 2U(|\Psi(x,t)|^2)\frac{\partial}{\partial x}\rho(x,t) = \rho(\tau)\frac{d}{d\tau}U(|\Psi(\tau)|^2) + 2U(|\Psi(\tau)|^2)\frac{d}{d\tau}\rho(\tau),$$
$$\frac{\partial^3}{\partial x^3}\rho(x,t) = \frac{d^3}{d\tau^3}\rho(\tau).$$

Groups the first two non-zero terms together and applies (27), (22) changes to:

$$2E\frac{d}{d\tau}\rho - \left(2U\frac{d}{d\tau}\rho + \rho\frac{d}{d\tau}U\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0. \tag{31}$$

5.2 Quadratic Nonlinear Potential

If we assume function U as follows:

$$U(|\Psi|^2) = U(\rho) = q_0\rho^\beta, \tag{32}$$

with $q_0, \beta \in \mathbb{R}$; (31) becomes a stationary modified KdVE:

$$\begin{aligned}
&2E\frac{d}{d\tau}\rho - \left(2U\frac{d}{d\tau}\rho + \rho\frac{d}{d\tau}U\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E\frac{d}{d\tau}\rho - \left(2q_0\rho^\beta\frac{d}{d\tau}\rho + \rho\frac{d}{d\tau}\left(q_0\rho^\beta\right)\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E\frac{d}{d\tau}\rho - \left(2q_0\rho^\beta\frac{d}{d\tau}\rho + \beta q_0\rho\rho^{\beta-1}\frac{d}{d\tau}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E\frac{d}{d\tau}\rho - (\beta+2)q_0\rho^\beta\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0.
\end{aligned} \tag{33}$$

When taking $\beta = 1$, (33) simplifies to a cubic nonlinearity:

$$2E\frac{d}{d\tau}\rho - 3q_0\rho\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0, \tag{34}$$

which is analogous to the standard KdVE:

$$\frac{\partial}{\partial t}u + \varepsilon u\frac{\partial}{\partial x}u + \mu\frac{\partial^3}{\partial x^3}u = 0, \tag{35}$$

where ε and μ are some arbitrary constants. More specifically, (34) has the form of the KdVE for wave solution with stationary profile. This implies that the potential term $U(\rho)$, defined in (32), introduces a nonlinearity of arbitrary order β to the system. Thus $-(\beta+2)q_0\rho^\beta\frac{d}{d\tau}\rho$ generalizes the classical cubic nonlinearity of the KdVE to higher-order nonlinearities, depending on β, which describes the steepening or narrowing of the wave. Additionally, the term $\frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho$ governs the wave's dispersion, where different wave components travel at vary velocities, causing the wave to spread over time. Since the last two terms of (34) correspond to the same characteristics as the last two terms of (35), we can conclude that the remaining term $2E\frac{d}{d\tau}\rho$ represents the time evolution of the wave propagating in one direction.

Furthermore, (34) can, under certain conditions, produce both periodic (cnoidal waves) [22] and localized [12, 23] solutions. In order to obtain these types of solutions, the nonlinear and dispersive effects are perfectly balanced, meaning that the rate of wave steepening due to nonlinearity is exactly countered by the spreading effect of dispersion. This balance allows the wave to propagate without changing shape.

5.2.1 Periodic Solution (Cnoidal Waves)

Cnoidal waves are characterized by their oscillatory, periodic nature and are often expressed in terms of Jacobi elliptic functions $(\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k))$. The general form of a cnoidal wave solution is:

$$\rho(\tau) = A[\mathrm{cn}(B\tau, k)]^2,$$

where A, B are some constants. The parameter k is the elliptic modulus (or modulus parameter) of the Jacobi elliptic functions, $0 \leq k \leq 1$:

- $k = 0$, the cnoidal wave becomes purely sinusoidal (a simple periodic wave).
- $k \to 0$, the solution becomes a sinusoidal wave (a simple harmonic wave).
- $k \to 1$, the solution becomes more "sharply peaked" and resembles a train of solitons with flat regions in between.
- $k = 1$, the cnoidal wave reduces to a solitary wave (soliton).

Cnoidal wave solutions from (34) arise when the system has periodic boundary conditions. Nevertheless, we are not interested in this case so we will not discover more about it. If you want to discover more about this type of solution and its application, please refer to [22].

5.2.2 Localized Solution

The localized solutions of this equation correspond to solitons, which are stable, solitary waves with a fixed shape. The general well-known soliton solution for the standard KdVE is of the form:

$$\rho(\tau) = A[\mathrm{sech}(B\tau)]^2,$$

where A, B are some constants.

Localized solution arises when the boundary conditions require the wave to decay to zero. These solutions are localized pulses that maintain their shape during propagation and do not repeat.

5.2.3 Conclusion

Since $\rho = |\Psi|^2$, implies that ρ is a non-negative function. Notably, if ρ is a localized solution of (34), then $\rho^{1/2}$ also serves as a localized solution of (26).

Now substitute (32) with $\beta = 1$, into the NLSE (14) to obtain the cubic NSLE:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + mq_0|\Psi|^2\Psi. \tag{36}$$

With the same physical assumption (23), hence $|\Psi|^2$ is a soliton solution of the following KdVE deriving from (34):

$$\begin{aligned}
&2E\frac{d}{d\tau}\rho - 3q_0\rho\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2|E|\frac{d}{d\tau}\rho - 3|q_0|\rho\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &\frac{2|E|}{|q_0|}\frac{d}{d\tau}\rho - 3\rho\frac{d}{d\tau}\rho + \frac{\hbar^2}{4|q_0|m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &-\frac{2|E|}{|q_0|v_0}\frac{\partial}{\partial t}\rho - 3\rho\frac{\partial}{\partial x}\rho + \frac{\hbar^2}{4|q_0|m^2}\frac{\partial^3}{\partial x^3}\rho = 0,
\end{aligned} \tag{37}$$

with $v_0 \neq 0$. For E and q_0, we take the absolute value here because for different solutions will acquire different conditions for E and q_0 (they can be negative). However, in the context of the equation, only their magnitudes are physically meaningful, so we need the absolute values on them.

The reason for this transformation is that we aim to rewrite the equation in a form resembling the KdVE by balancing nonlinear term and dispersive term. This balance is central to the behavior of solitons and wave-like solutions, which the KdVE is known to describe. Thus, rewriting in terms of t and x makes it possible to recognize and analyze the wave solution, unlike (34) just describes a wave in a co-moving frame in general.

We can find the localized solutions of (34) by applying the *Inverse Scattering Transform* (IST) method (presented in [12]). In the next subsections, we will consider two different boundary conditions to acquire distinguished solitary wave solutions.

5.3 Bright Solitary Wave Solution

A bright soliton occurs when the solution represents a localized wave with a peak or amplitude higher than the surrounding medium (i.e., a "bump" or "pulse"). These solitons are typically found in media where the nonlinearity is attractive, leading to the self-focusing of the wave.

A bright soliton is localized and confined, meaning the wave amplitude tends to zero far from the soliton's center. Thus ρ satisfies the following boundary conditions in the τ-space:

$$\lim_{\tau\to\pm\infty}\rho(\tau) = 0. \tag{38}$$

This ensures the soliton is localized, with no influence at infinity. Provided that $q_0 < 0$ and $E < 0$. The solution of (34) has the following form:

$$\rho(x - v_0 t) = \frac{2|E|}{|q_0|}\left[\operatorname{sech}\left(\frac{m\sqrt{2|E|}}{\hbar}(x - v_0 t)\right)\right]^2. \tag{39}$$

Correspondingly, by virtue of (30), the envelope solition of the cubic NLSE (36) is as follows:

$$\Psi(x,t) = \left(\frac{2|E|}{|q_0|}\right)^{1/2} \operatorname{sech}\left(\frac{m\sqrt{2|E|}}{\hbar}(x-v_0t)\right) \exp\left\{\frac{im}{\hbar}\left(v_0x - \left(E+\frac{v_0^2}{2}\right)t\right)\right\}. \tag{40}$$

Solutions (39) and (40) are interconnected, which describe standard KdVE soliton and NLSE envelop soliton respectively. Note that:

$$\sqrt{\rho(x-v_0t)} = \left(\frac{2|E|}{|q_0|}\right)^{1/2} \operatorname{sech}\left(\frac{m\sqrt{2|E|}}{\hbar}(x-v_0t)\right), \tag{41}$$

is a solution of (26), where (26) is the nonlinear-stationary-state equation for functional U of ρ, namely $U(\rho) = q_0\rho$. This shows how the soliton solution of the KdVE directly translates to the envelope of the soliton in the NLSE. Moreover, let $\Delta = \dfrac{\hbar}{m\sqrt{2|E|}}$, $\rho_m = \dfrac{2|E|}{|q_0|}$ and they satisfy the following property:

$$\Delta^2\rho_m = \frac{\hbar^2}{m^2|q_0|} = \text{ constant.} \tag{42}$$

This reflects the scaling properties of the soliton:

- Δ represents the characteristic length scale of the soliton, related to how quickly the soliton decays away from its center. A smaller value of Δ corresponds to a narrower, more localized soliton.

- ρ_m is the maximum density of the soliton. Higher ρ_m corresponds to a soliton with a larger peak density.

For $v_0 = 0$, (40) becomes a nonlinear localized stationary state of the cubic NLSE (36) with $q_0 < 0$ and $E < 0$. This solution describes a static bright soliton.

Furthermore, for $0 < \beta < \infty$, the general solutions of (33) and (36) respectively are:

$$\rho(x-v_0t) = \frac{(1+\beta)|E|}{|q_0|}\left[\operatorname{sech}\left(\frac{m\sqrt{2|E|}}{\hbar}(x-v_0t)\right)\right]^{2/\beta}, \tag{43}$$

$$\Psi(x,t) = \left(\frac{(1+\beta)|E|}{|q_0|}\right)^{1/(2\beta)} \left[\operatorname{sech}\left(\frac{\beta m\sqrt{2|E|}}{\hbar}(x-v_0t)\right)\right]^{1/\beta} \exp\left\{\frac{im}{\hbar}\left[v_0x - \left(E+\frac{v_0^2}{2}\right)t\right]\right\}. \tag{44}$$

To explore the bright soliton-like solutions for various values of β in the context of the modified NLSE and the modified KdVE please refer to Tables I ($\beta \leq 1$) and II ($\beta > 1$) in reference [7]. Some of these values correspond to typical cases relevant for the several scientific and technological applications.

5.4 Visualization of Bright Soliton

We will provide the graphs about bright soliton in terms of real part of Ψ and ρ respectively.

(i) Figure 1, 2: $v_0 = 50, E = -10, q_0 = -5, m = 1$ and $\hbar = 5$:

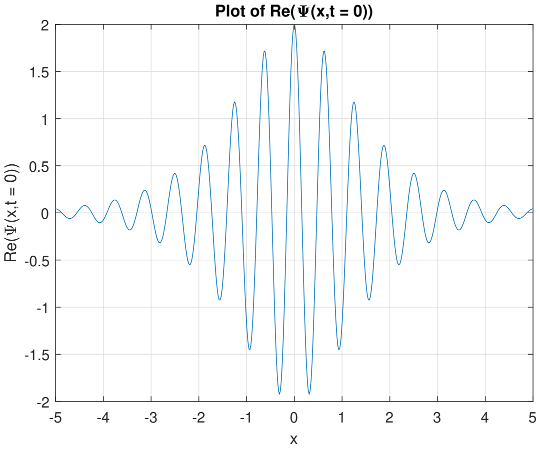

Figure 1: The real part of Ψ.

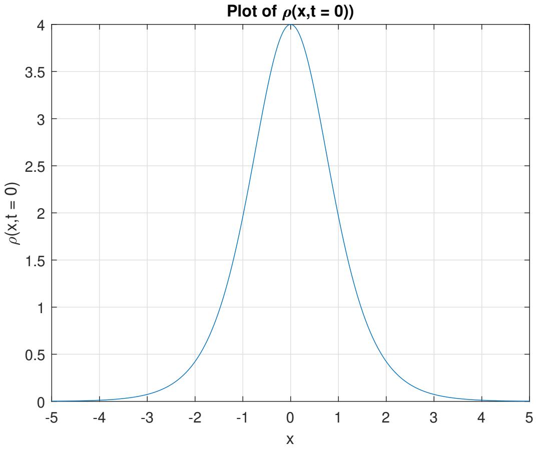

Figure 2: The envelope ρ.

(ii) Figure 3,4: $v_0 = 20, E = -10, q_0 = -5, m = 1$ and $\hbar = 5$. If we decrease v_0, we expect that the oscillation of the real part of Ψ decreases:

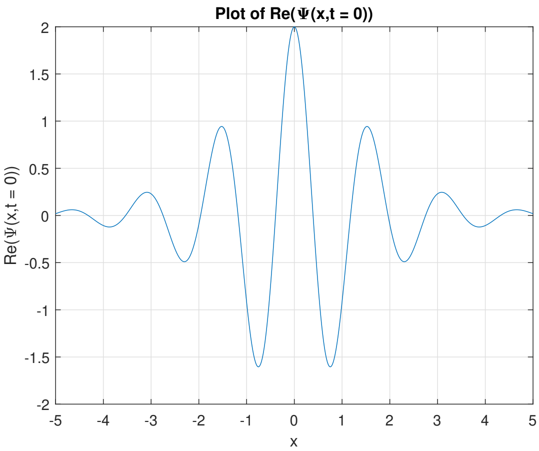

Figure 3: The real part of Ψ with .

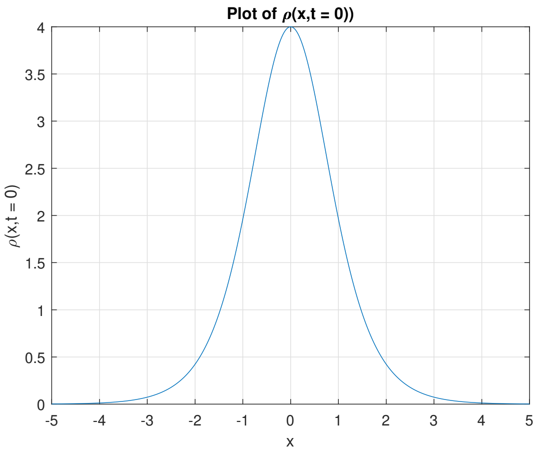

Figure 4: The envelope ρ.

(iii) Figure 5, 6: $v_0 = 50, E = -10, q_0 = -5, m = 1$ and $\hbar = 2$. If we adjust $\hbar$, we can control the width of the soliton $\dfrac{1}{\Delta}$ without effecting the peak of the soliton ρ_m as we showed above. Since the width of ρ is more narrow as $\hbar$ decreases, as the consequence, the real part of the wave Ψ is being compressed:

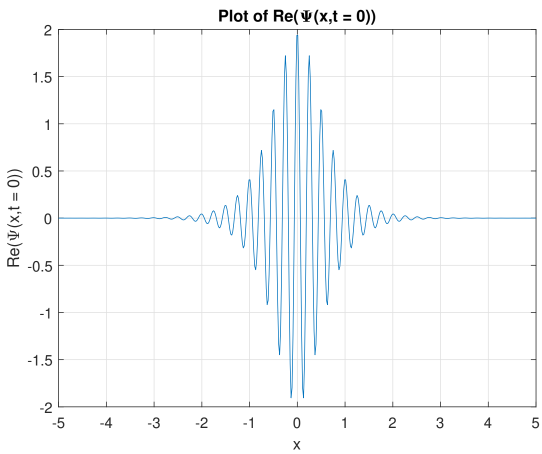

Figure 5: The real part of Ψ.

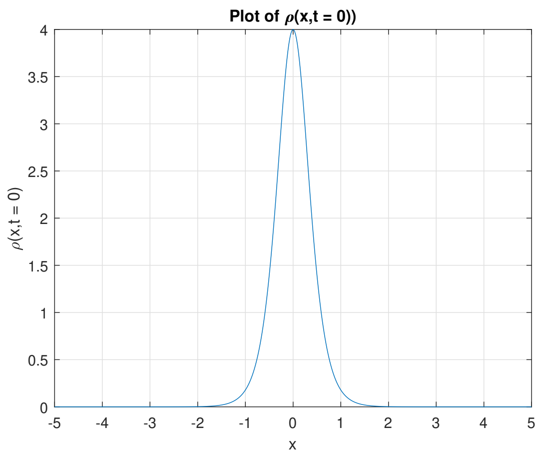

Figure 6: The envelope ρ.

5.5 Dark Solitary Wave Solution

A dark soliton arises in systems with repulsive (defocusing) interactions and typically correspond to phase shifts in the wave, where the amplitude goes to zero at some point but remains nonzero elsewhere.

We will consider the cases for which

$$\lim_{\tau \pm \infty} \rho(\tau) = \rho_0 > 0,$$

and we express ρ as:

$$\rho(\tau) = \rho_0 + \rho_1(\tau). \tag{45}$$

Hence, (34) changes to:

$$\begin{aligned} & 2E\frac{d}{d\tau}\rho - 3q_0\rho\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0 \\ \Rightarrow\ & 2E\frac{d}{d\tau}\rho_1 - 3q_0(\rho_0 + \rho_1)\frac{d}{d\tau}\rho_1 + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho_1 = 0 \\ \Rightarrow\ & (2E - 3q_0\rho_0)\frac{d}{d\tau}\rho_1 - 3q_0\rho_1\frac{d}{d\tau}\rho_1 + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho_1 = 0 \\ \Rightarrow\ & 2E_1\frac{d}{d\tau}\rho_1 - 3q_0\rho_1\frac{d}{d\tau}\rho_1 + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho_1 = 0, \end{aligned} \tag{46}$$

where $E_1 = E - \dfrac{3}{2}q_0\rho_0$. Now we assume the following boundary conditions for ρ_1:

$$\lim_{\tau \to \pm\infty} \rho_1(\tau) = 0. \tag{47}$$

We will compute the derivatives of $\sqrt{\rho(\tau)}$ based on the assumption (45):

$$\begin{aligned} \frac{d}{d\tau}\sqrt{\rho(\tau)} &= \frac{1}{2}(\rho(\tau))^{-1/2}\frac{d}{d\tau}\rho_1(\tau) \\ \frac{d^2}{d\tau^2}\sqrt{\rho(\tau)} &= -\frac{1}{4}(\rho(\tau))^{-3/2}\left(\frac{d}{d\tau}\rho_1(\tau)\right)^2 + \frac{1}{2}(\rho(\tau))^{-1/2}\frac{d^2}{d\tau^2}\rho_1(\tau). \end{aligned} \tag{48}$$

For the convenience and simplicity of derivations and equations, the following notation will be interchanged:

$$\frac{d}{dx}f(x) = f'(x).$$

Thus, substitute (32) with $\beta = 1$, (45) and (48) into (26), we can derive:

$$\begin{aligned}
& -\frac{\hbar^2}{2m^2}\frac{d^2}{d\tau^2}\rho^{1/2} + U(\rho)\rho^{1/2} = E\rho^{1/2} \\
\Rightarrow\ & -\frac{\hbar^2}{2m^2}\rho^{-1/2}\frac{d^2}{d\tau^2}\rho^{1/2} + q_0(\rho_0+\rho_1) = E \\
\Rightarrow\ & -\frac{\hbar^2}{2m^2}\left[-\frac{1}{4}\left(\frac{\rho_1'}{\rho}\right)^2 + \frac{1}{2}\frac{\rho_1''}{\rho}\right] + q_0(\rho_0+\rho_1) = E.
\end{aligned}$$

Now apply the boundaries assumption (47) and E is simplified as:

$$E = q_0\rho_0, \tag{49}$$

and the result from that is:

$$E_1 = -\frac{1}{2}q_0\rho_0. \tag{50}$$

If we look for solutions corresponding to $\rho_1(\tau) > 0\ \forall\tau$, there is no soliton solutions can be found. However, soliton solutions exist when

$$q_0 > 0 \Rightarrow E_1 < 0, \tag{51}$$

which corresponds to $\rho_1(\tau) < 0\ \forall\tau$. Notice that, in fact (49), (48) and (51) are completely consistent. Furthermore, we want $\rho(x)$ to be non-negative which provided that

$$\rho_0 \geq |\rho_1(\tau)|\ \forall\tau.$$

Thus the dark solitons are possible to obtain. Consequently, solving (46) for ρ_1 with (45) implies the following soliton solution for ρ:

$$\rho(x - v_0t) = \rho_0\left[\tanh\left(\frac{m\sqrt{q_0\rho_0}}{\hbar}(x - v_0t)\right)\right]^2. \tag{52}$$

Correspondingly, according to (30), the envelop soliton of the cubic NLSE (36) is:

$$\Psi(x,t) = \sqrt{\rho_0}\left|\tanh\left(\frac{m\sqrt{q_0\rho_0}}{\hbar}(x - v_0t)\right)\right|\exp\left\{\frac{im}{\hbar}\left[v_0x - \left(q_0\rho_0 + \frac{v_0^2}{2}\right)t\right]\right\}. \tag{53}$$

Note that in this case, $\Delta = \dfrac{\hbar}{m\sqrt{q_0\rho_0}}$ and $\rho_m = -\rho_0$ are soliton's width and soliton's minimum amplitude respectively. They still satisfy the property such that is similar to (42):

$$\Delta^2\,|\rho_m| = \frac{\hbar^2}{m^2|q_0|} = \text{ constant}. \tag{54}$$

For $v_0 = 0$, (40) becomes a nonlinear localized stationary state of the cubic NLSE (36) with $q_0 > 0$ and $E_1 = q_0\rho_0 > 0$.

We would like to emphasize that all the solitary waves found, in this section, for the cubic NLSE (36) have an amplitude ρ_m is independent of the soliton velocity v_0.

5.6 Visualization of Dark Soliton

As similar for the bright soliton, the dark soliton also is visualized in terms of real part of Ψ and ρ respectively.

(i) Figure 7, 8: $v_0 = 50, q_0 = 10, \rho_0 = 2, m = 1$ and $\hbar = 5$:

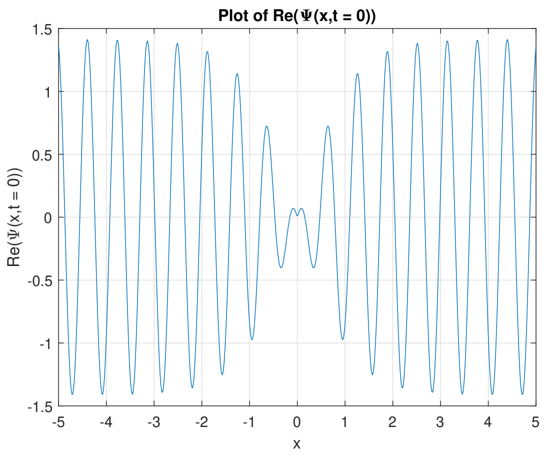

Figure 7: The real part of Ψ.

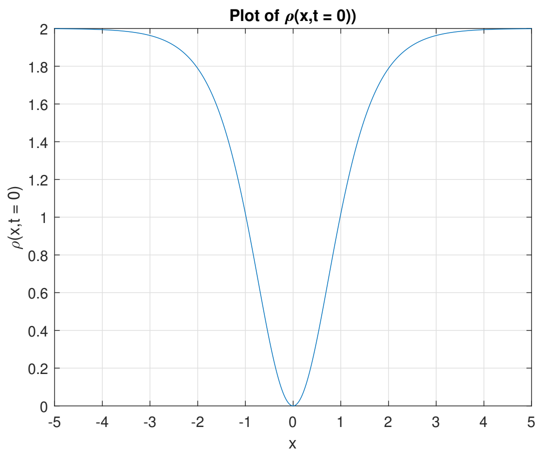

Figure 8: The envelope ρ.

(ii) Figure 9, 10: $v_0 = 20, q_0 = 10, \rho_0 = 2, m = 1$ and $\hbar = 5$. If we decrease v_0, the real part of Ψ oscillates slower:

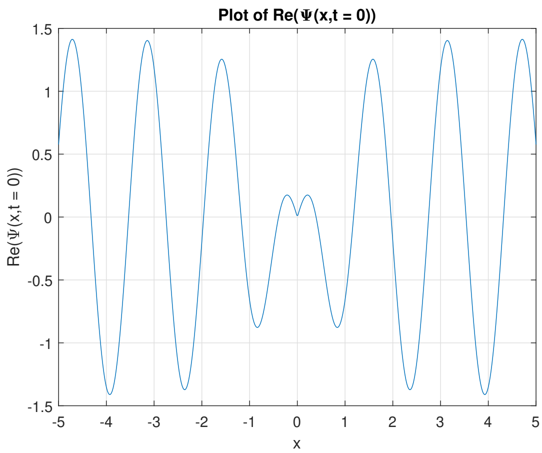

Figure 9: The real part of Ψ.

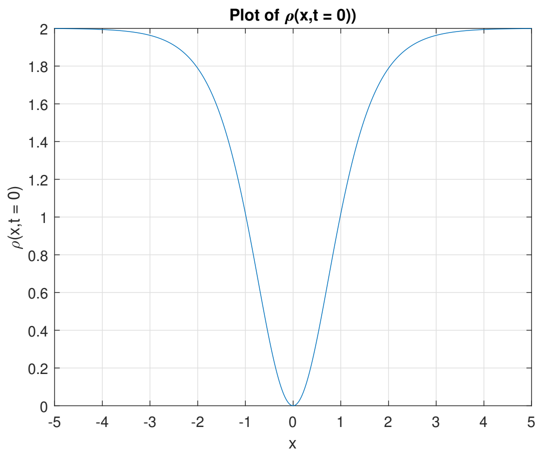

Figure 10: The envelope ρ.

(iii) Figure 11, 12: $v_0 = 50, q_0 = 50, \rho_0 = 2, m = 1$ and $\hbar = 5$. We want to narrow down the width of the envelope soliton without varying the amplitude, in order to do that, we either increase q_0 or decrease $\hbar$. Here we choose q_0 to adjust:

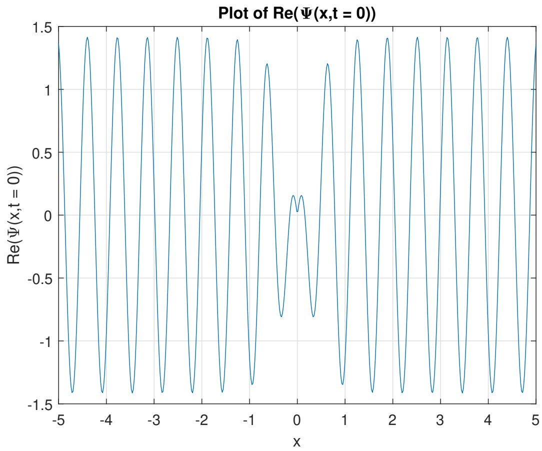

Figure 11: The real part of Ψ.

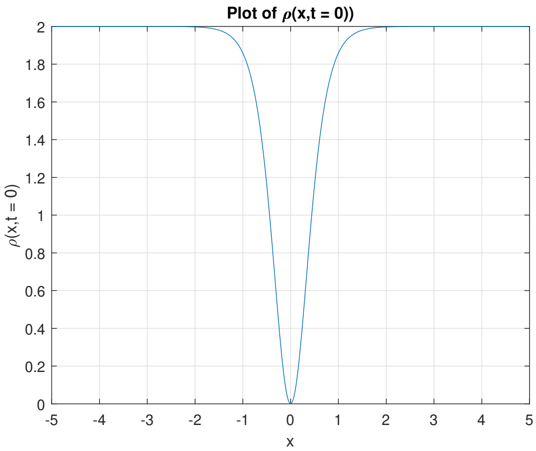

Figure 12: The envelope ρ.

5.7 Quartic Nonlinear Potential

Now we assume function U as follows:

$$U(|\Psi|^2) = U(\rho) = q_0\rho + q_1\rho^2, \tag{55}$$

where $\beta = 1$ with $q_0, q_1 \in \mathbb{R}$. Now substitute (55) into the NLSE (14) to obtain the cubic+quintic NSLE:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + m(q_0|\Psi|^2 + q_1|\Psi|^4)\Psi. \tag{56}$$

Moreover, (31) changes to:

$$\begin{aligned}
&2E\frac{d}{d\tau}\rho - \left(2U\frac{d}{d\tau}\rho + \rho\frac{d}{d\tau}U\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E\frac{d}{d\tau}\rho - \left(2(q_0\rho + q_1\rho^2)\frac{d}{d\tau}\rho + \rho\frac{d}{d\tau}\left(q_0\rho + q_1\rho^2\right)\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E\frac{d}{d\tau}\rho - \left(2q_0\rho\frac{d}{d\tau}\rho + 2q_1\rho^2\frac{d}{d\tau}\rho + q_0\rho\frac{d}{d\tau}\rho + 2q_1\rho^2\frac{d}{d\tau}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E\frac{d}{d\tau}\rho + \frac{9}{16}\frac{q_0^2}{q_1}\frac{d}{d\tau}\rho - \frac{9}{16}\frac{q_0^2}{q_1}\frac{d}{d\tau}\rho - \left(3q_0\rho\frac{d}{d\tau}\rho + 4q_1\rho^2\frac{d}{d\tau}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2\left(E + \frac{9}{32}\frac{q_0^2}{q_1}\right)\frac{d}{d\tau}\rho - \left(4q_1\rho^2 + 3q_0\rho + \frac{9}{16}\frac{q_0^2}{q_1}\right)\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2\left(E + \frac{9}{32}\frac{q_0^2}{q_1}\right)\frac{d}{d\tau}\rho - 4q_1\left(\rho^2 + \frac{3}{4}\frac{q_0}{q_1}\rho + \frac{9}{64}\frac{q_0^2}{q_1^2}\right)\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0\\
\Rightarrow\ &2E_2\frac{d}{d\tau}\rho - 4q_1\left(\rho + \frac{3}{8}\frac{q_0}{q_1}\right)^2\frac{d}{d\tau}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\tau^3}\rho = 0,
\end{aligned} \tag{57}$$

where $E_2 = E + \dfrac{9}{32}\dfrac{q_0^2}{q_1}$.

5.8 Bright Solitary Wave Solution

Notice that the boundaries condition for the bright solution is:

$$\lim_{\tau\to\pm\infty}\rho(\tau) = 0.$$

However, this physical assumption is unsuitable in order to obtain the solution from (57). To be more concrete, bright soliton requires a precise balance between dispersive spreading and focusing nonlinearity. In a system with quadratic nonlinearity, this balance can create a localized wave packet with finite energy. Higher-order (quartic) nonlinearities do not generally offer the same balance conditions necessary for bright solitons. Instead, they often create instability or a lack of localization in the absence of quadratic nonlinearity. In general, bright soliton needs a strong focusing force to rise above a zero background, which only quadratic nonlinearity can provide. Unlike dark and gray solitons, they represent dips (or phase shifts) in a non-zero background. Quartic nonlinearity can support these dips since it allows for a defocusing effect on a non-zero background, compatible with dark and gray solitons (will be shown later). In conclusion, it is impossible to have bright soliton with the quartic nonlinearity potential.

5.9 Dark Solitary Wave Solution

Since we want to achieve the dark solution, the boundary conditions remain the same as (45) and (47):

$$\lim_{\tau\to\pm\infty}\rho(\tau)=\rho_0,$$

where $\rho_0=-\dfrac{3}{8}\dfrac{q_0}{q_1}>0$. Notice that from (26) and apply the same boundaries assumption for dark solution, we can likewise obtain:

$$\begin{aligned}
&-\frac{\hbar^2}{2m^2}\frac{d^2}{d\tau^2}\rho^{1/2}+U(\rho)\rho^{1/2}=E\rho^{1/2}\\
\Rightarrow\;&-\frac{\hbar^2}{2m^2}\rho^{-1/2}\frac{d^2}{d\tau^2}\rho^{1/2}+q_0(\rho_0+\rho_1)+q_1(\rho_0+\rho_1)^2=E\\
\Rightarrow\;&-\frac{\hbar^2}{2m^2}\left[-\frac{1}{4}\left(\frac{\rho_1'}{\rho}\right)^2+\frac{1}{2}\frac{\rho_1''}{\rho}\right]+q_0(\rho_0+\rho_1)+q_1(\rho_0+\rho_1)^2=E\\
\Rightarrow\;&E=q_0\rho_0+q_1\rho_0^2,
\end{aligned}\tag{58}$$

as we did in **Sec. 4.5**. Moreover, giving $E_2<0$ and $q_1<0$, the solution of (57) is:

$$\rho(x-v_0t)=\frac{3q_0}{8|q_1|}\left[1-\operatorname{sech}\left(\sqrt{\frac{3m^2}{8\hbar^2}\frac{q_0^2}{|q_1|}}(x-v_0t)\right)\right],\tag{59}$$

and, consequently, we also obtain the solution for (56):

$$\Psi(x,t)=\sqrt{\frac{3q_0}{8|q_1|}}\left[1-\operatorname{sech}\left(\sqrt{\frac{3m^2}{8\hbar^2}\frac{q_0^2}{|q_1|}}(x-v_0t)\right)\right]^{1/2}\exp\left\{\frac{im}{\hbar}\left[v_0x-\left(q_0\rho_0+q_1\rho_0^2+\frac{v_0^2}{2}\right)t\right]\right\}.\tag{60}$$

For $\Delta=\sqrt{\dfrac{8\hbar^2}{3m^2}\dfrac{|q_1|}{q_0^2}}$, $\rho_m=\dfrac{3q_0}{8|q_1|}$ and they satisfy the similar property:

$$\Delta^2|\rho_m|=\frac{\hbar^2}{m^2|q_0|}=\text{ constant.}\tag{61}$$

5.10 Visualization of Dark Soliton

Since this is still a dark soliton, we will provide the graphs for the real part of the wave and the envelope soliton:

(i) Figure 13, 14: $v_0=50, q_0=20, q_1=-5, m=1$ and $\hbar=5$:

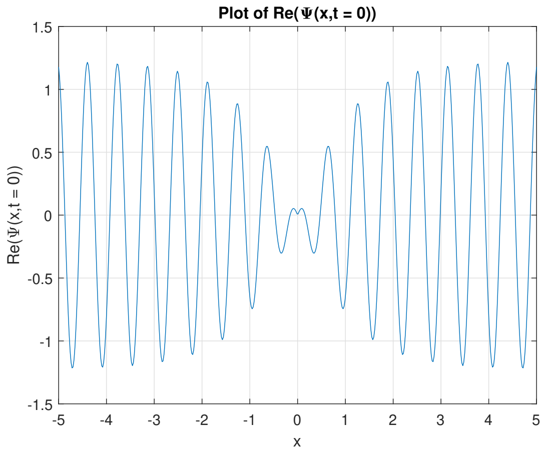

Figure 13: The real part of Ψ.

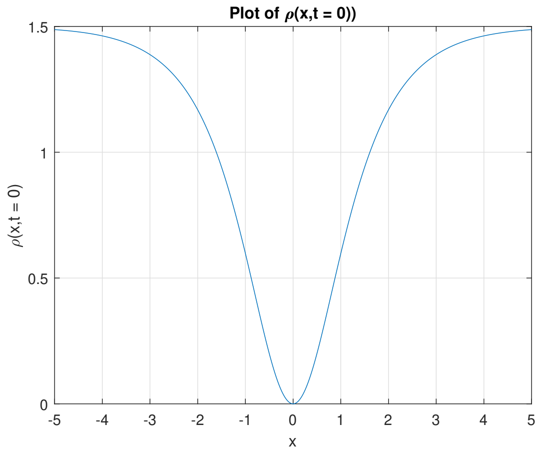

Figure 14: The envelope ρ.

(ii) Figure 15, 16: $v_0 = 20, q_0 = 20, q_1 = -5, m = 1$ and $\hbar = 5$. The wave propagates slower as v_0 decreases:

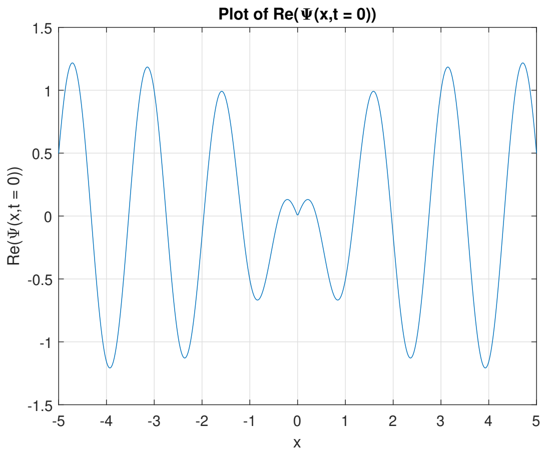

Figure 15: The real part of Ψ.

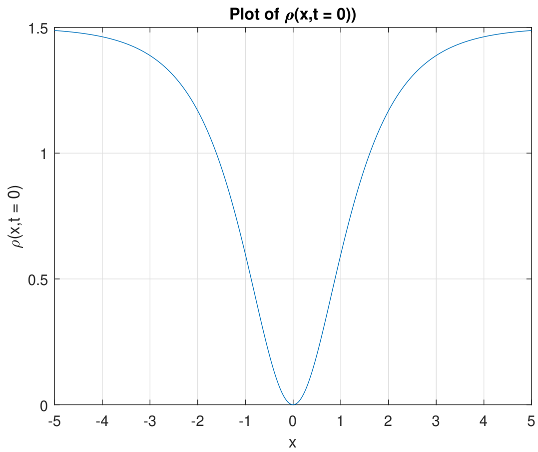

Figure 16: The envelope ρ.

(iii) Figure 17, 18: $v_0 = 50, q_0 = 20, q_1 = -5, m = 1$ and $\hbar = 2$. Now making the soliton's width narrow by deducing $\hbar$:

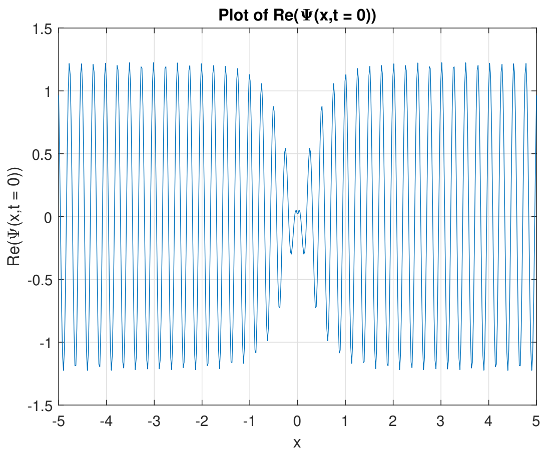

Figure 17: The real part of Ψ.

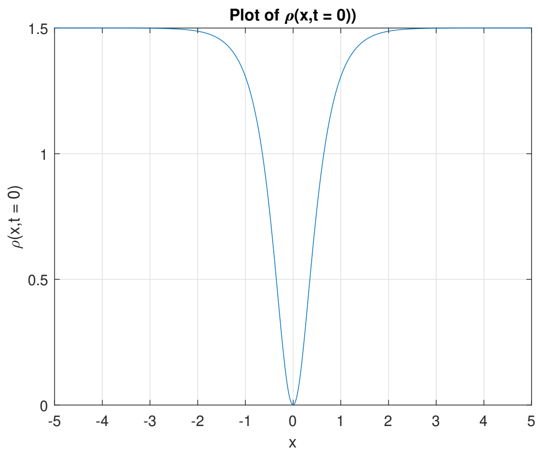

Figure 18: The envelope ρ.

6 Motion With Stationary-Profile Current Velocity

6.1 Formulation of The Problem

In this section, we will assume both ρ and v are functions of the combined variable $\xi = x - u_0 t$ where $u_0 \in \mathbb{R}$:

$$\rho(x,t) = \rho(\xi) = \rho_0 + \rho_1(\xi), \tag{62}$$

$$v(x,t) = v(\xi) = v_0 + v_1(\xi), \tag{63}$$

where v_0 represents an arbitrary constant current velocity associated with the Madelung's fluid background motion, as explained in **Sec. 5**, while $v_1(\xi)$ is a large perturbation in the current velocity.

With those assumptions, (15) changes to:

$$\frac{d}{d\xi}[\rho(\xi)v(\xi)] = u_0 \frac{d}{d\xi}\rho(\xi). \tag{64}$$

The integrand of (64) is easily to be evaluated:

$$\begin{aligned} \int \frac{d}{d\xi}[\rho(\xi)v(\xi)]\ d\xi &= \int u_0 \frac{d}{d\xi}\rho(\xi)\ d\xi \\ \Rightarrow \rho(\xi)v(\xi) &= u_0\rho(\xi) + A_0 \\ \Rightarrow v(\xi) &= v_0 + v_1(\xi) = u_0 + \frac{A_0}{\rho(\xi)}, \end{aligned} \tag{65}$$

where A_0 is an arbitrary constant. Notice that we still assume the arbitrary function $c_0(t)$ appearing in (19) (due to the integration of (16) with respect to x) to be a constant c_0 and it contributes to the energy conservation.

Apply the new assumptions of ρ and v on (22), first we apply change of variable into $\xi = x - u_0 t$:

$$\begin{aligned} &-\rho\partial_t v + v\partial_t\rho + 2\left[c_0 - \int \partial_t v\ dx\right]\partial_x\rho - (\rho\partial_x U + 2U\partial_x\rho) + \frac{\hbar^2}{4m^2}\partial_{xxx}\rho = 0 \\ \Rightarrow\ &\rho\left(u_0\frac{d}{d\xi}v\right) - v\left(u_0\frac{d}{d\xi}\rho\right) + 2\left[c_0 + \int u_0\frac{d}{d\xi}v\ d\xi\right]\frac{d}{d\xi}\rho - \left(\rho\frac{d}{d\xi}U + 2U\frac{d}{d\xi}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\ \Rightarrow\ &u_0\rho\frac{d}{d\xi}v - u_0 v\frac{d}{d\xi}\rho + 2\left[c_0 + u_0 v\right]\frac{d}{d\xi}\rho - \left(\rho\frac{d}{d\xi}U + 2U\frac{d}{d\xi}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\ \Rightarrow\ &u_0\rho\frac{d}{d\xi}v + 2c_0\frac{d}{d\xi}\rho + u_0 v\frac{d}{d\xi}\rho - \left(\rho\frac{d}{d\xi}U + 2U\frac{d}{d\xi}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0. \end{aligned}$$

Now for the first term of the above expression, we apply (64) and we get:

$$\begin{aligned} u_0\rho\frac{d}{d\xi}v &= u_0\rho\frac{d}{d\xi}\left(u_0 + \frac{A_0}{\rho}\right) \\ &= A_0 u_0\rho\frac{d}{d\xi}\rho^{-1} \\ &= -u_0\frac{A_0}{\rho}\frac{d}{d\xi}\rho \\ &= -u_0(v-u_0)\frac{d}{d\xi}\rho \\ &= -u_0 v\frac{d}{d\xi}\rho + u_0^2\frac{d}{d\xi}\rho, \end{aligned}$$

substitute it back in, we obtain:

$$
\begin{aligned}
&-u_0 v\frac{d}{d\xi}\rho+u_0^2\frac{d}{d\xi}\rho+2c_0\frac{d}{d\xi}\rho+u_0 v\frac{d}{d\xi}\rho-\left(\rho\frac{d}{d\xi}U+2U\frac{d}{d\xi}\rho\right)+\frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho=0\\
&\Rightarrow (2c_0+u_0^2)\frac{d}{d\xi}\rho-\left(2U\frac{d}{d\xi}\rho+\rho\frac{d}{d\xi}U\right)+\frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho=0, \qquad (66)
\end{aligned}
$$

which is very similar to (31).

6.2 Quadratic Nonlinear Potential

Suppose U has the form (32) with $\beta=1$, (65) becomes:

$$
\begin{aligned}
&(2c_0+u_0^2)\frac{d}{d\xi}\rho-\left(2U\frac{d}{d\xi}\rho+\rho\frac{d}{d\xi}U\right)+\frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho=0\\
\Rightarrow\ &(2c_0+u_0^2)\frac{d}{d\xi}\rho-\left(2q_0\rho\frac{d}{d\xi}\rho+\rho\frac{d}{d\xi}(q_0\rho)\right)+\frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho=0\\
\Rightarrow\ &(2c_0+u_0^2)\frac{d}{d\xi}\rho-\left(2q_0\rho\frac{d}{d\xi}\rho+q_0\rho\frac{d}{d\xi}\rho\right)+\frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho=0\\
\Rightarrow\ &(2c_0+u_0^2)\frac{d}{d\xi}\rho-3q_0\rho\frac{d}{d\xi}\rho+\frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho=0, \qquad (67)
\end{aligned}
$$

which has the same form as the modified KdVE (34).

6.3 Gray Solitary Wave Solution

Gray solitons can form in systems with defocusing nonlinearity, where the wave's dispersion balances against repulsive nonlinear effects. They occur when the wave propagates on a nonzero background field, allowing for a localized dip in amplitude without the wave dropping to zero. Gray solitons require a non-constant velocity, with the depth of the dip decreasing as the velocity increases.

Observe that boundary conditions similar to those in (38) cannot be applied because of the restrictions imposed by (64). In fact, under that assumption, v diverges when $\xi\to\pm\infty$. Therefore, in order to obtain this specific solution, we will impose the following boundary conditions:

$$\lim_{\xi\to\pm\infty}\rho(\xi)=\rho_0, \qquad (68)$$

where ρ_0 is a positive constant and

$$\lim_{\xi\to\pm\infty}v(\xi)=v_0. \qquad (69)$$

Consequently, (65) is continuous with the boundary conditions, we obtain:

$$A_0=-\rho_0(u_0-v_0). \qquad (70)$$

Since the assumption for ρ_1 remains the same, the derivation for $E=q_0\rho_0$ is exactly the same as (49). Moreover, with the expression of ρ as:

$$\rho(\xi)=\rho_0+\rho_1(\xi), \qquad (71)$$

replace it into (67), we get:

$$\begin{aligned}
&(2c_0+u_0^2)\frac{d}{d\xi}\rho - 3q_0\rho\frac{d}{d\xi}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0\\
\Rightarrow\ &(2c_0+u_0^2)\frac{d}{d\xi}\rho_1 - 3q_0(\rho_0+\rho_1)\frac{d}{d\xi}\rho_1 + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho_1 = 0\\
\Rightarrow\ &(2c_0+u_0^2-3q_0\rho_0)\frac{d}{d\xi}\rho_1 - 3q_0\rho_1\frac{d}{d\xi}\rho_1 + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho_1 = 0\\
\Rightarrow\ &2E_3\frac{d}{d\xi}\rho_1 - 3q_0\rho_1\frac{d}{d\xi}\rho_1 + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho_1 = 0,
\end{aligned} \tag{72}$$

where $E_3 = c_0 + \frac{1}{2}u_0^2 - \frac{3}{2}q_0\rho_0$. Now we will derive the formula for c_0, starting with (18):

$$\begin{aligned}
&-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho} = -2\int \partial_t v\ dx - v^2 - 2U + 2c_0\\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0\int \frac{d}{d\xi}v\ d\xi - v^2 - 2U + 2c_0\\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0 v - v^2 - 2U + 2c_0.
\end{aligned}$$

Next, we substitute (65) in and apply the boundary conditions (68) and (69):

$$\begin{aligned}
&-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0\left(u_0+\frac{A_0}{\rho}\right) - \left(u_0+\frac{A_0}{\rho}\right)^2 - 2U + 2c_0\\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0^2 + \frac{2A_0u_0}{\rho} - u_0^2 - \frac{2A_0u_0}{\rho} - \frac{A_0^2}{\rho^2} - 2U + 2c_0\\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0^2 - u_0^2 - \frac{A_0^2}{\rho^2} - 2U + 2c_0\\
\Rightarrow\ &0 = u_0^2 - (u_0-v_0)^2 - 2q_0\rho_0 + 2c_0\\
\Rightarrow\ &-2c_0 = u_0^2 - u_0^2 + 2u_0v_0 - v_0^2 - 2q_0\rho_0\\
\Rightarrow\ &c_0 = q_0\rho_0 + \frac{1}{2}v_0^2 - u_0v_0.
\end{aligned} \tag{73}$$

Additionally, we can rewrite E_3 as follows:

$$\begin{aligned}
2E_3 &= 2c_0 + u_0^2 - 3q_0\rho_0\\
&= 2q_0\rho_0 + v_0^2 - 2u_0v_0 + u_0^2 - 3q_0\rho_0\\
&= (u_0-v_0)^2 - q_0\rho_0.
\end{aligned} \tag{74}$$

We still want the boundary conditions for ρ_1 is the same as for the dark solution:

$$\lim_{\xi\to\pm\infty}\rho_1(\xi) = 0, \tag{75}$$

provided from (68) and (71). In order to find the gray solution, we will consider two cases $\rho_1 > 0$ and $\rho_1 < 0$ like we did for the dark solution.

For $\rho_1 > 0$, (72) would have a solution for

$$q_0 < 0, \tag{76}$$
$$E_3 < 0. \tag{77}$$

However, with (74) we get:

$$2E_3 = (u_0 - v_0)^2 + |q_0|\rho_0 > 0,$$

which contradicts (77). Thus, there is no solution for the case $\rho_1 > 0$.

Now consider $\rho_1 < 0$, the solution for (72) will require:

$$q_0 > 0, \tag{78}$$

$$E_3 < 0, \tag{79}$$

provided that the condition $|\rho_1(\xi)| \leq \rho_0 \ \forall \xi$ is satisfied. Consequently,

$$(u_0 - v_0)^2 \leq q_0\rho_0, \tag{80}$$

which is equivalent with:

$$v_0 - \sqrt{q_0\rho_0} \leq u_0 \leq v_0 + \sqrt{q_0\rho_0}. \tag{81}$$

With all assumptions, we can obtain the solution of (71):

$$\rho_1(\xi) = -\frac{q_0\rho_0 - (u_0 - v_0)^2}{q_0} \left[\operatorname{sech} \left(\frac{m\sqrt{q_0\rho_0 - (u_0 - v_0)^2}}{\hbar} \xi \right) \right]^2, \tag{82}$$

and therefore, the solution ρ for (26) is:

$$\begin{aligned} \rho(x - u_0 t) &= \rho_0 \left[1 - A^2 \left[\operatorname{sech} \left(\frac{m\sqrt{A^2 q_0\rho_0}}{\hbar} (x - u_0 t) \right) \right]^2 \right] \\ &= \rho_0 \left[1 - A^2 \left[\operatorname{sech} \left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar} (x - u_0 t) \right) \right]^2 \right], \end{aligned} \tag{83}$$

where

$$A^2 = \frac{q_0\rho_0 - (u_0 - v_0)^2}{q_0\rho_0} \geq 0. \tag{84}$$

Notice that if $A^2 \leq 1$ implies that $-1 \leq A \leq 1$. Moreover, with the assumption $|\rho_1| \leq \rho_0$ leads to (80). Thus, there is no contradictions to (84), which proves the consistency of those above conditions. Furthermore, we have $\Delta = \dfrac{\hbar}{m|A|\sqrt{q_0\rho_0}}$ and $\rho_m = \rho_0 A^2$, the soliton's property still holds:

$$\Delta^2 |\rho_m| = \frac{\hbar^2}{m^2 |q_0|} = \text{ constant}. \tag{85}$$

With the form of (65) and the results of (70) and (82), we can determine the velocity function

v_1 as follows:

$$
\begin{aligned}
v_0 + v_1(\xi) &= u_0 + \frac{A_0}{\rho(\xi)} \\
\Rightarrow v_1(\xi) &= u_0 - v_0 + \frac{A_0}{\rho_0\left[1 - A^2\left[\operatorname{sech}\left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar}\xi\right)\right]^2\right]} \\
\Rightarrow v_1(\xi) &= u_0 - v_0 + \frac{-\rho_0(u_0 - v_0)}{\rho_0\left[1 - A^2\left[\operatorname{sech}\left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar}\xi\right)\right]^2\right]} \\
\Rightarrow v_1(\xi) &= (u_0 - v_0)\left[1 - \frac{1}{1 - A^2\left[\operatorname{sech}\left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar}\xi\right)\right]^2}\right] \\
\Rightarrow v_1(x - u_0t) &= -\frac{(u_0 - v_0)A^2\left[\operatorname{sech}\left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar}(x - u_0t)\right)\right]^2}{1 - A^2\left[\operatorname{sech}\left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar}(x - u_0t)\right)\right]^2}.
\end{aligned} \tag{86}
$$

We will now construct the solution for Ψ in the form of (2), along with the relation between v and S as defined in (3). This can be expressed as:

$$
v_0 + v_1(\xi) = \frac{1}{m}\left(\frac{\partial}{\partial x}S_0(x,t) + \frac{d}{d\xi}S_1(\xi)\right), \tag{87}
$$

where $S(x,t)$ has been split into two components:

$$
S(x,t) = S_0(x,t) + S_1(\xi), \tag{88}
$$

where $S_0(x,t)$ is the "background" phase and $S_1(\xi)$ is the "perturbated" phase. From (87), it is clear to see that $v_0 = \frac{1}{m}\frac{\partial}{\partial x}S_0$ and $v_1 = \frac{1}{m}\frac{d}{d\xi}S_1$, we can integrate them with respect to x and ξ respectively, therefore:

$$
S_0(x,t) = mv_0x - m\left(q_0\rho_0 + \frac{v_0^2}{2}\right)t, \tag{89}
$$

where $E = q_0\rho_0$ since the boundary conditions for ρ are the same as the dark soliton, and

$$
S_1(x - u_0t) = C - \frac{A\hbar(u_0 - v_0)}{|A|\sqrt{(1 - A^2)q_0\rho_0}}\arctan\left[\frac{A}{\sqrt{1 - A^2}}\tanh\left(\frac{m|A|\sqrt{q_0\rho_0}}{\hbar}(x - u_0t)\right)\right], \tag{90}
$$

where C is an arbitrary constant.

In conclusion, the solution for the cubic NLSE is:

$$
\Psi(x,t) = \sqrt{\rho(x - u_0t)}\exp\left\{\frac{\mathrm{i}}{\hbar}S(x,t)\right\}, \tag{91}
$$

where $\rho(x - u_0t)$ is (83) and $S(x,t)$ is the sum of (89) and (90).

Note that for $u_0 - v_0 \neq 0$, this solution represents the gray envelope soliton (*i.e.* the minimum amplitude does not reach zero at both end sides) with $-1 \leq A \leq 1$. However, when $u_0 - v_0 = 0$ implies that $A = \pm 1$ and $A_0 = 0$. Consequently, it can be observed from (65), $v(\xi)$ is identical to v_0 and $v_1(\xi)$ vanishes. Therefore, this will become the exact dark solution that we have shown in **Sec. 5.5**. Thus, basically, gray soliton is the extended version of dark soliton by means of the non-homogeneity of the current velocity $v(\xi)$.

6.4 Visualization of Gray Soliton

Besides the real part of Ψ and ρ are being provided; due to the complexity of v and S by adding a non-homogeneous term, there will be two more additional graphs, namely v_1 and S_1 to demonstrate gray soliton.

(i) Figure 19, 20, 21, 22: $u_0 = 7, v_0 = 10, q_0 = 20, \rho_0 = 5, m = 1$ and $\hbar = 5$:

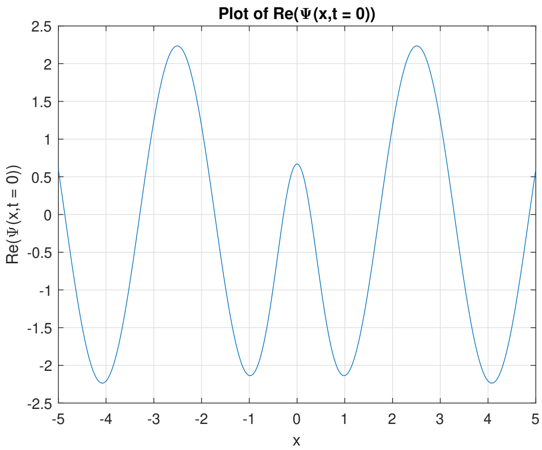

Figure 19: The real part of Ψ.

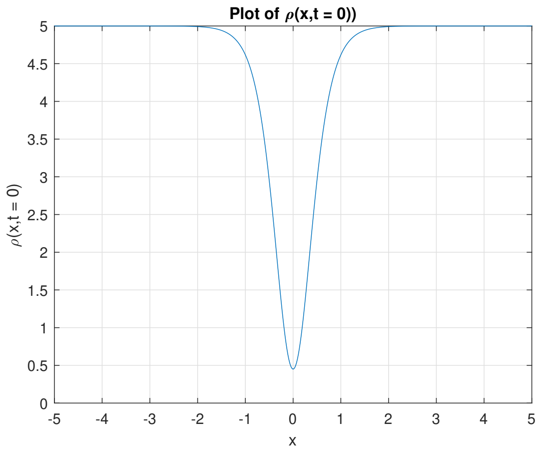

Figure 20: The envelope ρ.

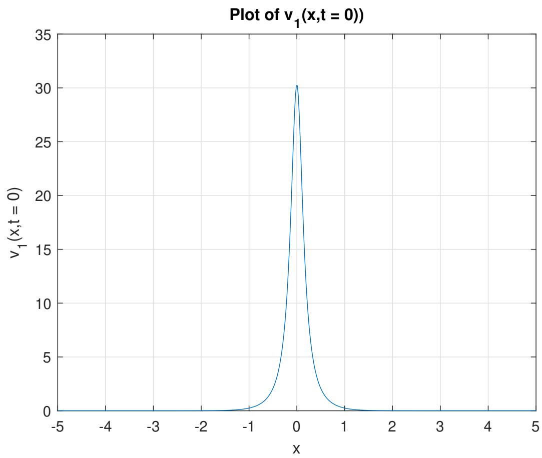

Figure 21: The velocity v_1.

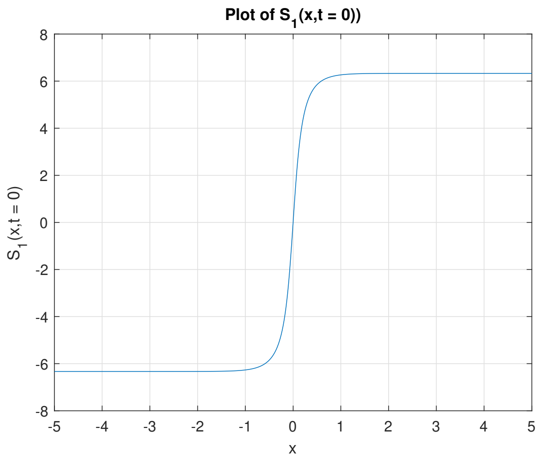

Figure 22: The phase S_1.

(ii) Figure 23, 24, 25, 26: $u_0 = 15, v_0 = 10, q_0 = 20, \rho_0 = 5, m = 1$ and $\hbar = 5$. Now consider $u_0 > v_0$, which means the soliton travels faster than the phase propagates:

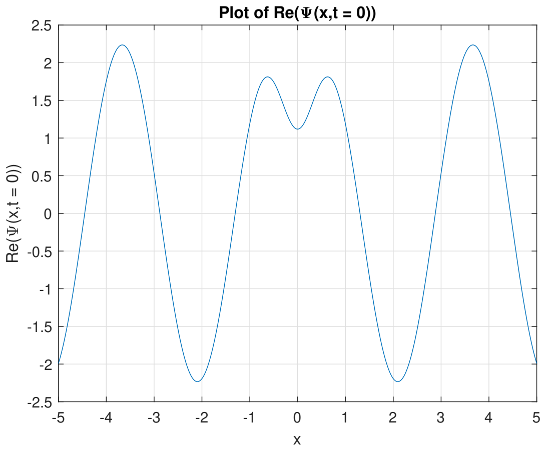

Figure 23: The real part of Ψ.

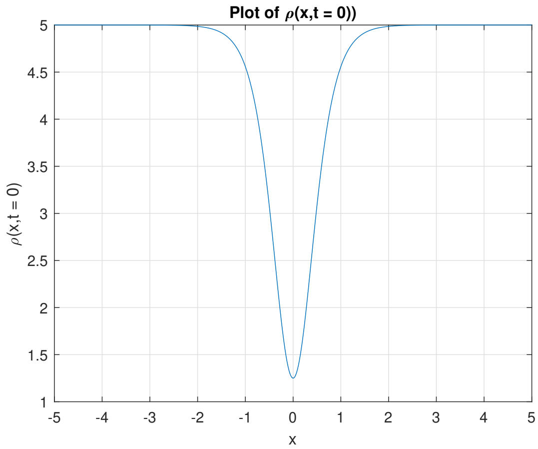

Figure 24: The envelope ρ.

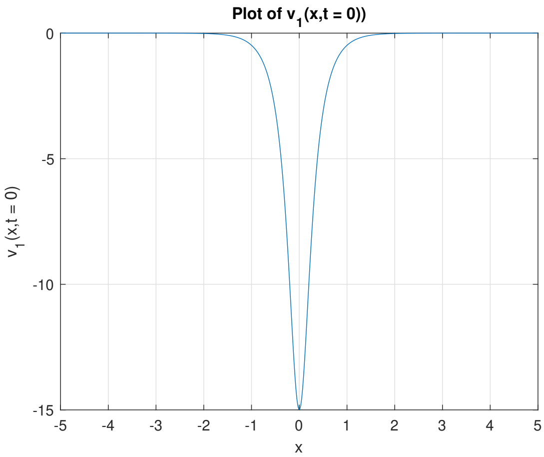

Figure 25: The velocity v_1.

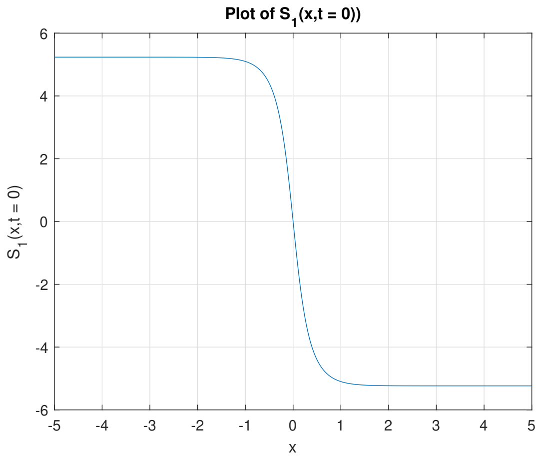

Figure 26: The phase S_1.

(iii) Figure 27, 28, 29, 30: $u_0 = 15, v_0 = 10, q_0 = 20, \rho_0 = 5, m = 1$ and $\hbar = 2$. With the same parameters like in (ii), now we compress the wave solution by making $\hbar$ smaller:

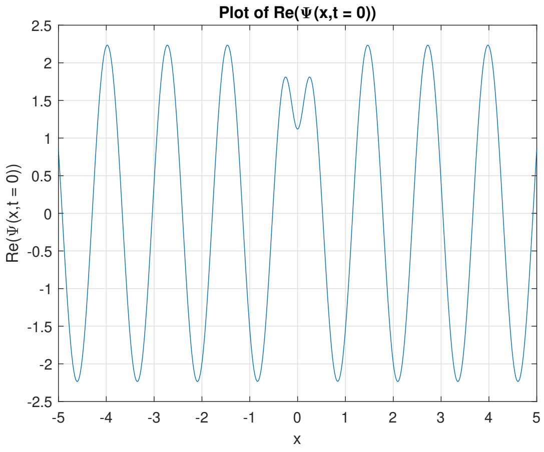

Figure 27: The real part of Ψ.

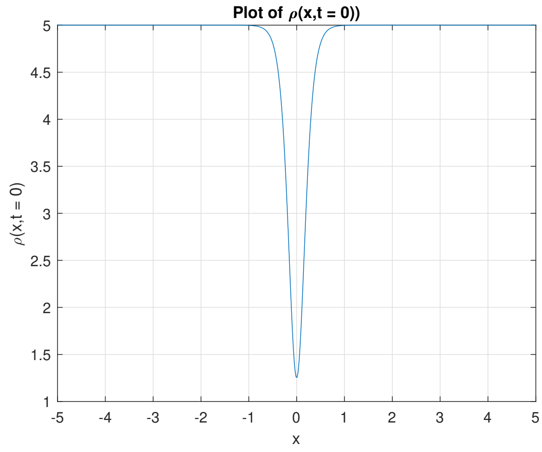

Figure 28: The envelope ρ.

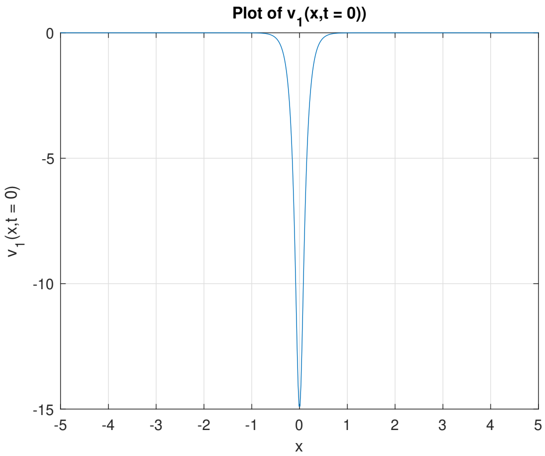

Figure 29: The velocity v_1.

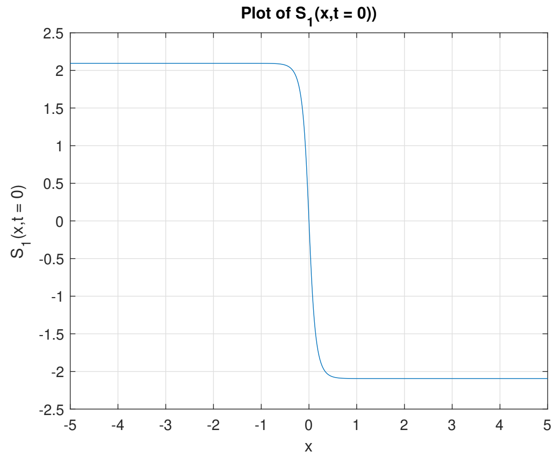

Figure 30: The phase S_1.

6.5 Quartic Nonlinear Potential

Now consider U with exact same form as (55) and substitute it into (66):

$$
\begin{aligned}
&(2c_0 + u_0^2)\frac{d}{d\xi}\rho - \left(2U\frac{d}{d\xi}\rho + \rho\frac{d}{d\xi}U\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\
\Rightarrow\ &(2c_0 + u_0^2)\frac{d}{d\xi}\rho - \left(2(q_0\rho + q_1\rho^2)\frac{d}{d\xi}\rho + \rho\frac{d}{d\xi}(q_0\rho + q_1\rho^2)\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\
\Rightarrow\ &(2c_0 + u_0^2)\frac{d}{d\xi}\rho - \left(2q_0\rho\frac{d}{d\xi}\rho + 2q_1\rho^2\frac{d}{d\xi}\rho + q_0\rho\frac{d}{d\xi}\rho + q_1\rho\frac{d}{d\xi}\rho^2\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\
\Rightarrow\ &(2c_0 + u_0^2)\frac{d}{d\xi}\rho - \left(3q_0\rho\frac{d}{d\xi}\rho + 4q_1\rho^2\frac{d}{d\xi}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\
\Rightarrow\ &(2c_0 + u_0^2)\frac{d}{d\xi}\rho + \frac{9}{16}\frac{q_0^2}{q_1}\frac{d}{d\xi}\rho - \frac{9}{16}\frac{q_0^2}{q_1}\frac{d}{d\xi}\rho - \left(3q_0\rho\frac{d}{d\xi}\rho + 4q_1\rho^2\frac{d}{d\xi}\rho\right) + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\
\Rightarrow\ &\left(2c_0 + u_0^2 + \frac{9}{16}\frac{q_0^2}{q_1}\right)\frac{d}{d\xi}\rho - \left(4q_1\rho^2 + 3q_0\rho + \frac{9}{16}\frac{q_0^2}{q_1}\right)\frac{d}{d\xi}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0 \\
\Rightarrow\ &2E_4\frac{d}{d\xi}\rho - 4\left(\rho + \frac{3}{8}\frac{q_0}{q_1}\right)\frac{d}{d\xi}\rho + \frac{\hbar^2}{4m^2}\frac{d^3}{d\xi^3}\rho = 0.
\end{aligned} \tag{92}
$$

where $E_4 = c_0 + \dfrac{1}{2}u_0^2 + \dfrac{9}{32}\dfrac{q_0^2}{q_1}$ and $\rho_0 = -\dfrac{3}{8}\dfrac{q_0}{q_1}$ - which akin to the dark soliton's quartic nonlinear potential scenario. Now we will derive the formula for c_0, starting with (18) and (65) as we did in **Sec. 6.3**:

$$
\begin{aligned}
&-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho} = -2\int \partial_t v\ dx - v^2 - 2U + 2c_0 \\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0\int \frac{d}{d\xi}v\ d\xi - v^2 - 2U + 2c_0 \\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0 v - v^2 - 2U + 2c_0 \\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0\left(u_0 + \frac{A_0}{\rho}\right) - \left(u_0 + \frac{A_0}{\rho}\right)^2 - 2U + 2c_0 \\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = 2u_0^2 + \frac{2A_0u_0}{\rho} - u_0^2 - \frac{2A_0u_0}{\rho} - \frac{A_0^2}{\rho^2} - 2U + 2c_0 \\
\Rightarrow\ &-\frac{\hbar^2}{m^2}\frac{1}{\sqrt{\rho}}\frac{d^2}{d\xi^2}\sqrt{\rho} = u_0^2 - \frac{A_0^2}{\rho^2} - 2U + 2c_0.
\end{aligned}
$$

Apply the boundary conditions (68) and (69), we continue deriving:

$$
\begin{aligned}
&0 = u_0^2 - (u_0 - v_0)^2 - 2q_0\rho_0 - 2q_1\rho_0^2 + 2c_0 \\
\Rightarrow\ &-2c_0 = u_0^2 - u_0^2 + 2u_0v_0 - v_0^2 - 2q_0\rho_0 - 2q_1\rho_0^2 \\
\Rightarrow\ &c_0 = \frac{1}{2}v_0^2 - u_0v_0 + q_0\rho_0 + q_1\rho_0^2 \\
\Rightarrow\ &c_0 = \frac{1}{2}v_0^2 - u_0v_0 - \frac{15}{64}\frac{q_0^2}{q_1}.
\end{aligned} \tag{93}
$$

Additionally, we can rewrite E_4 as follows:

$$
\begin{aligned}
2E_4 &= 2c_0 + u_0^2 + \frac{9}{16}\frac{q_0^2}{q_1} \\
&= v_0^2 - 2u_0v_0 - \frac{15}{32}\frac{q_0^2}{q_1} + u_0^2 + \frac{9}{16}\frac{q_0^2}{q_1} \\
&= (u_0 - v_0)^2 + \frac{3}{32}\frac{q_0^2}{q_1}.
\end{aligned} \tag{94}
$$

This is consistent with [5] where they derived this formula using a different strategy and solved for the solution.

6.6 Gray Solitary Wave Solution

As mentioned before, in order to obtain the gray solution, we need $\rho_1 < 0$, the boundary conditions must be unchanged as in (68) and (69). Moreover, for (68), we have:

$$\lim_{\xi\to\pm\infty}\rho(\xi) = \rho_0,$$

where $\rho_0 = -\dfrac{3}{8}\dfrac{q_0}{q_1} > 0$, provided that:

$$
\begin{aligned}
q_0 &> 0, \\
q_1 &< 0.
\end{aligned}
$$

Since the assumption for ρ_1 remains the same, the derivation for $E = q_0\rho_0 + q_1\rho_0^2$ is exactly the same as (58). Again assume $E_4 < 0$ implies that $|\rho_1(\xi)| \leq \rho_0 \; \forall \xi$. As the result:

$$\begin{aligned}
&\frac{1}{2}(u_0 - v_0)^2 \leq \frac{3}{64}\frac{q_0^2}{|q_1|} \\
&\Rightarrow (u_0 - v_0)^2 \leq \frac{3}{32}\frac{q_0^2}{|q_1|} \\
&\Rightarrow v_0 - \sqrt{\frac{3}{32}\frac{q_0^2}{|q_1|}} \leq u_0 \leq v_0 + \sqrt{\frac{3}{32}\frac{q_0^2}{|q_1|}} \\
&\Rightarrow v_0 - \frac{q_0}{4}\sqrt{\frac{3}{2|q_1|}} \leq u_0 \leq v_0 + \frac{q_0}{4}\sqrt{\frac{3}{2|q_1|}},
\end{aligned} \tag{95}$$

must be satisfied to obtain the solution. Thus the soliton solution for (92) is:

$$\rho(x - u_0 t) = \frac{3}{8}\frac{q_0}{|q_1|} - \sqrt{\frac{9}{64}\frac{q_0^2}{q_1^2} - \frac{3}{2}\frac{(u_0 - v_0)^2}{|q_1|}}\,\mathrm{sech}\left(\left(\frac{2m}{\hbar}\sqrt{\frac{3q_0^2}{32|q_1|} - (u_0 - v_0)^2}\right)(x - u_0 t)\right). \tag{96}$$

From (65) and (70), we can obtain the solution v:

$$\begin{aligned}
&v_0 + v_1(\xi) = u_0 + \frac{A_0}{\rho(\xi)} \\
&\Rightarrow v_1(\xi) = u_0 - v_0 + \frac{A_0}{\frac{3}{8}\frac{q_0}{|q_1|} - \sqrt{\frac{9}{64}\frac{q_0^2}{q_1^2} - \frac{3}{2}\frac{(u_0 - v_0)^2}{|q_1|}}\,\mathrm{sech}\left(\left(\frac{2m}{\hbar}\sqrt{\frac{3q_0^2}{32|q_1|} - (u_0 - v_0)^2}\right)(x - u_0 t)\right)} \\
&\Rightarrow v_1(\xi) = (u_0 - v_0)\left[1 + \frac{\frac{3}{8}\frac{q_0}{q_1}}{\frac{3}{8}\frac{q_0}{|q_1|} - \sqrt{\frac{9}{64}\frac{q_0^2}{q_1^2} - \frac{3}{2}\frac{(u_0 - v_0)^2}{|q_1|}}\,\mathrm{sech}\left(\left(\frac{2m}{\hbar}\sqrt{\frac{3q_0^2}{32|q_1|} - (u_0 - v_0)^2}\right)(x - u_0 t)\right)}\right] \\
&\Rightarrow v_1(\xi) = -\frac{(u_0 - v_0)\sqrt{\frac{9}{64}\frac{q_0^2}{q_1^2} - \frac{3}{2}\frac{(u_0 - v_0)^2}{|q_1|}}\,\mathrm{sech}\left(\left(\frac{2m}{\hbar}\sqrt{\frac{3q_0^2}{32|q_1|} - (u_0 - v_0)^2}\right)(x - u_0 t)\right)}{\frac{3}{8}\frac{q_0}{|q_1|} - \sqrt{\frac{9}{64}\frac{q_0^2}{q_1^2} - \frac{3}{2}\frac{(u_0 - v_0)^2}{|q_1|}}\,\mathrm{sech}\left(\left(\frac{2m}{\hbar}\sqrt{\frac{3q_0^2}{32|q_1|} - (u_0 - v_0)^2}\right)(x - u_0 t)\right)}.
\end{aligned} \tag{97}$$

Therefore, the background and perturbated phases, after taking the anti-derivative of v_0 and v_1 accordingly, are:

$$S_0(x, t) = m v_0 x - m\left(q_0\rho_0 + q_1\rho_0^2 + \frac{v_0^2}{2}\right)t, \tag{98}$$

where $E = q_0\rho_0 + q_1\rho_0^2$ since the boundary conditions for ρ are the same as the dark soliton, and

$$S_1(x - u_0 t) = C - \mathrm{sign}(u_0 - v_0) B \arctan\left[D \tanh\left(\left(\frac{m}{\hbar}\sqrt{\frac{3q_0^2}{32|q_1|} - (u_0 - v_0)^2}\right)(x - u_0 t)\right)\right], \tag{99}$$

where C is an arbitrary constant and

$$B = 2m\sqrt{\frac{3}{32}\frac{q_0^2}{|q_1|} + (u_0 - v_0)^2}, \tag{100}$$

and

$$D = \frac{\frac{3}{8}\frac{q_0}{|q_1|} - \sqrt{\frac{9}{64}\frac{q_0^2}{q_1^2} - \frac{3}{2}\frac{(u_0 - v_0)^2}{|q_1|}}}{|u_0 - v_0|\sqrt{\frac{3}{2|q_1|}}}. \tag{101}$$

In conclusion, the solution for the cubic+quintic NLSE is:

$$\Psi(x,t) = \sqrt{\rho(x - u_0 t)}\exp\left\{\frac{\mathrm{i}}{\hbar}S(x,t)\right\}, \tag{102}$$

where $\rho(x - u_0 t)$ is (96) and $S(x,t)$ is the sum of (98) and (98). Once again, when $u_0 \neq v_0$, it represents the gray soliton. On the other hand, when $u_0 = v_0$, the solution simplifies to dark soliton as we expected. For more various interesting types of soliton such as up-shifted soliton, upper-shifted soliton, etc. please refer to [5].

6.7 Visualization of Gray Soliton

The four main graphs $\mathrm{Re}(\Psi), \rho, v_1$ and S_1 are provided to demonstrate the gray soliton.

(i) Figure 31, 32, 33, 34: $u_0 = 8, v_0 = 10, q_0 = 40, q_1 = -3, m = 1$ and $\hbar = 5$:

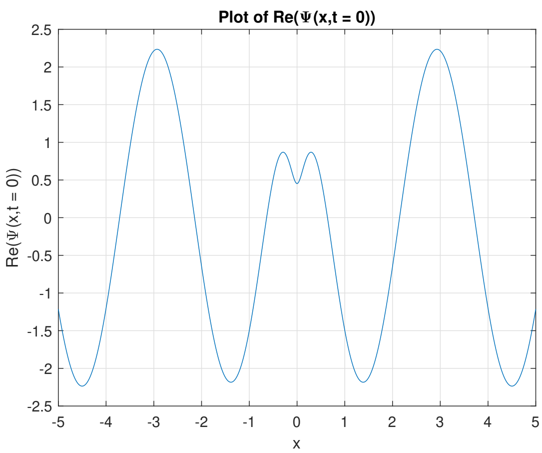

Figure 31: The real part of Ψ.

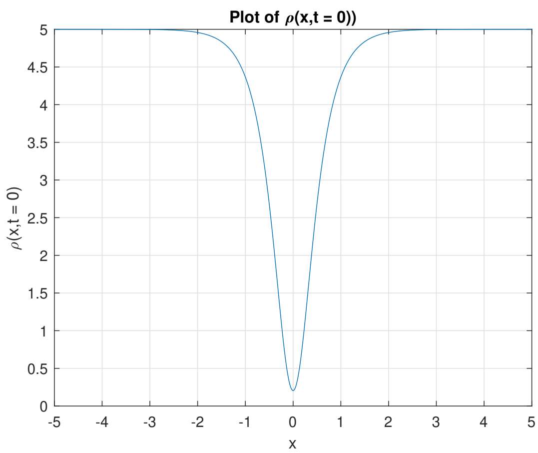

Figure 32: The envelope ρ.

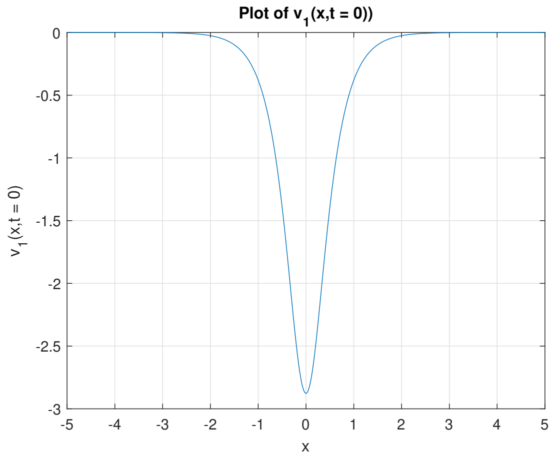

Figure 33: The velocity v_1.

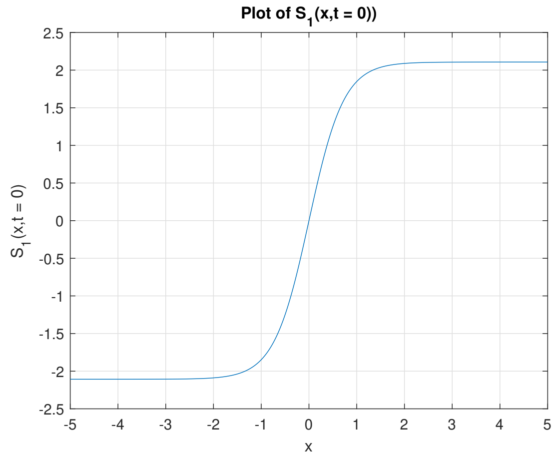

Figure 34: The phase S_1.

(ii) Figure 35, 36, 37, 38: $u_0 = 14, v_0 = 10, q_0 = 40, q_1 = -3, m = 1$ and $\hbar = 5$. This time we let the soliton wave propagate faster the phase changes:

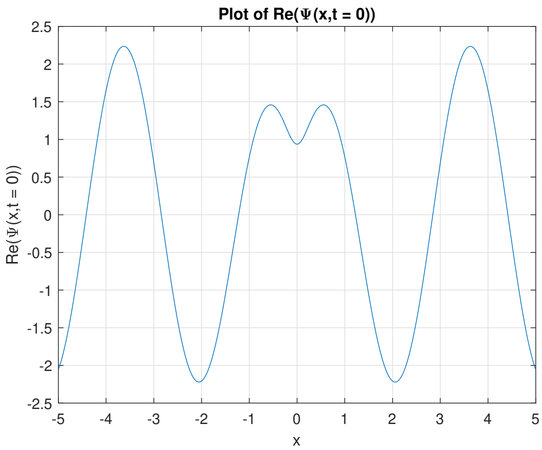

Figure 35: The real part of Ψ.

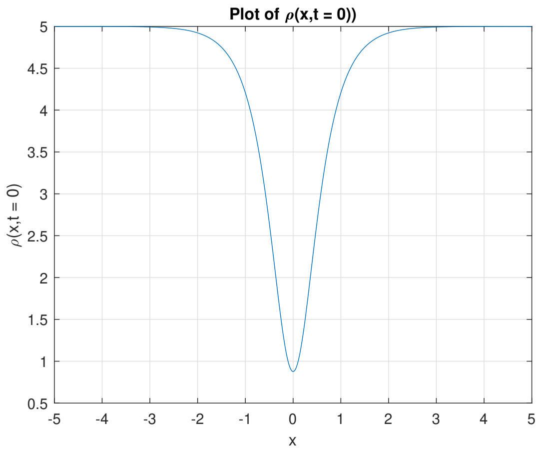

Figure 36: The envelope ρ.

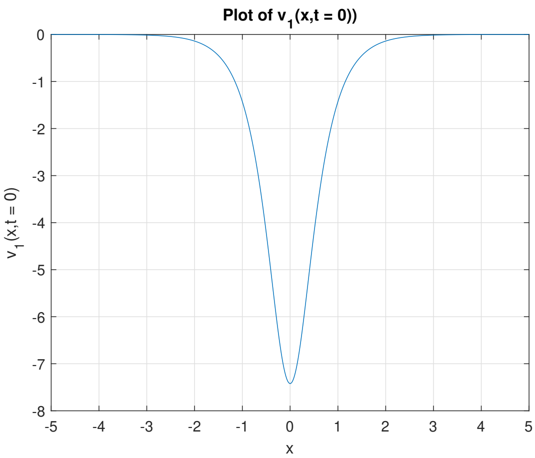

Figure 37: The velocity v_1.

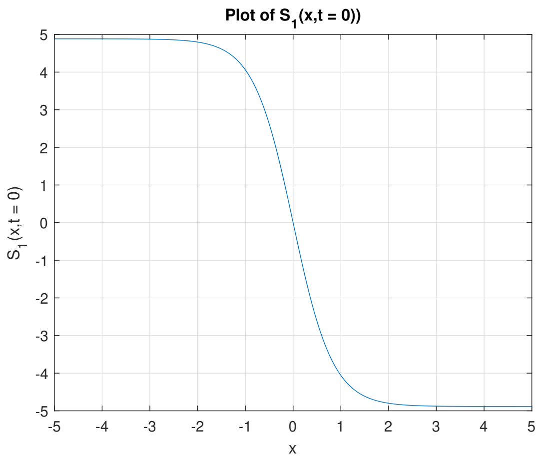

Figure 38: The phase S_1.

(iii) Figure 39, 40, 41, 42: $u_0 = 14, v_0 = 10, q_0 = 40, q_1 = -3, m = 1$ and $\hbar = 2$. Now squeezing the width of soliton to compress the wave by making $\hbar$ smaller:

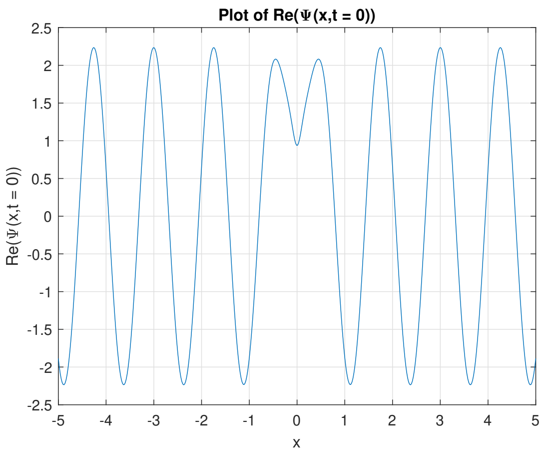

Figure 39: The real part of Ψ.

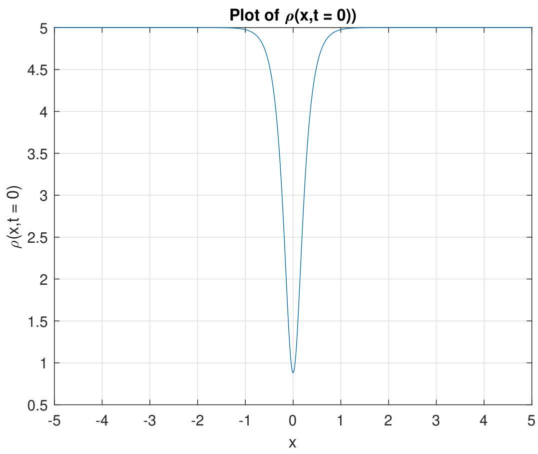

Figure 40: The envelope ρ.

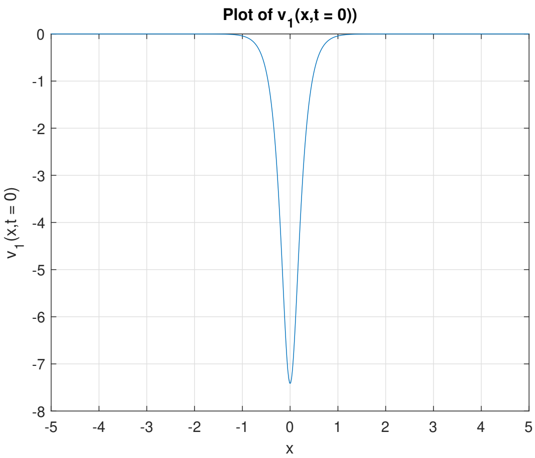

Figure 41: The velocity v_1.

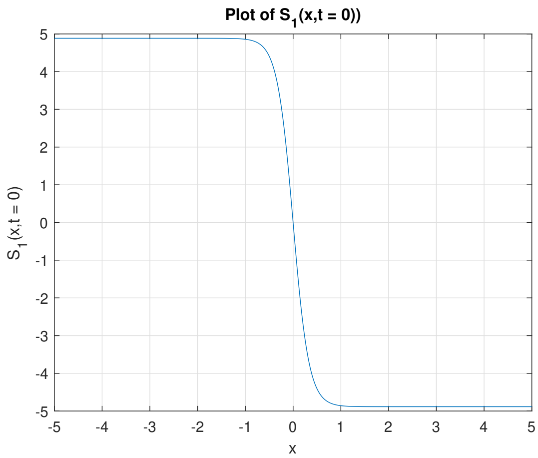

Figure 42: The phase S_1.

7 Conclusion

This paper has demonstrated a correspondence between soliton-like solutions and envelope soliton-like solutions within broad families of the modified KdVE and modified NLSE, respectively. This connection is established through the framework of Madelung's fluid, which serves as the fluid dynamic counterpart to the NLSE. We have examined both the case of a constant current velocity ($v(\xi) = v_0$), and the case involving a perturbation with an arbitrarily large amplitude and a stationary profile ($v(\xi) = v_0 + v_1(\xi)$). Under appropriate constraints, this framework yields specific solutions (bright and dark/gray envelop solitary solutions) for a wide family of NLSE, derived from the known soliton solutions of the corresponding KdVE. In both scenarios, the coupled motion and continuity equations (Madelung's fluid system, which is equivalent to NLSE) can be transformed into a suitable KdVE for stationary-profile waves. Additionally, regarding the visualization of envelope solitons, it is important to note that if we account m and fix the Planck's constant $\hbar$, adjusting the soliton's width without affecting its amplitude becomes nearly impossible.

While we have derived a range of solutions under specific assumptions, it is important to acknowledge that many of these assumptions are necessary to achieve these "ideal" solutions. One of natural extensions to this work would be to explore cases where the current velocity $v(x,t)$ changes over time t independently with space x. Introducing this physical assumption would significantly enhance the framework of the Madelung's fluid picture, allowing it to capture the dynamic, time-evolving behavior of coherent nonlinear structures more accurately. It is possible to investigate of the existence of such solutions. Alternatively, what if we consider non-stationary-profile solutions (multi-solitary solutions) in the standard theory of KdVE (*e.g* by the inverse scattering method or by the Hirota's method [4]). By allowance of a more general dependence between the density ρ and the current velocity v (while still satisfying the Madelung's fluid system) and by defining the explicit form of the nonlinear potential functional $U(\rho)$, it is conceivable that multi-soliton solutions could also be found for the Madelung's fluid.

These analysis has implications for for the understanding of quantum fluids, nonlinear wave propagation, and soliton dynamics across various physical contexts. This study helps explain how solitary waves, which play an essential role in fields like fluid dynamics and plasma physics, can emerge from quantum mechanical descriptions and connect to classical wave phenomena. Additionally, the Madelung's fluid framework offers a unique perspective for further exploration, such as investigating the stability of solutions [21], and examining de Broglie matter waves in their normal modes [11]. These avenues not only deepen the theoretical understanding of soliton behavior but also bond the connection between quantum mechanics and classical wave theory, enhancing our ability to model and predict complex wave dynamics in practical applications.

8 References

[1] M. J. Ablowitz, A. Ramani, and H. Segur. A connection between nonlinear evolution equations and ordinary differential equations of p-type. ii. *Bulg. J. Phys*, 10:13–26, May 1983.

[2] David Bohm. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I. *Phys. Rev.*, 85:166–179, Jan 1952.

[3] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Almost conservation laws and global rough solutions to a Nonlinear Schrödinger equation, 2002.

[4] P. G. Drazin and R. S. Johnson. *Solitons: An Introduction.* Cambridge University Press, Cambridge, 1989.

[5] Renato Fedele. Envelope Solitons versus Solitons. *Physica Scripta*, 65:502, 02 2006.

[6] Renato Fedele and Hans Schamel. Solitary waves in the Madelung's fluid: Connection between the nonlinear Schrödinger equation and the Korteweg-de Vries equation. *The European Physical Journal B - Condensed Matter*, 27:313–320, Jun 2002.

[7] Renato Fedele, Hans Schamel, and Padma K Shukla. Solitons in the madelung's fluid. *Physica Scripta*, 2002(T98):18, Jan 2002.

[8] Michael Fowler. *Quantum Mechanics.* LibreTexts Chemistry, Charlottesville.

[9] V. Gerdzhikov and M. Ivanov. The quadratic bundle of a general form and the nonlinear evolution equations. Pt. 1. *Bulg. J. Phys*, 10:13–26, 1983.

[10] Eyal Heifetz and Eliahu Cohen. Toward a Thermo-hydrodynamic Like Description of Schrödinger Equation via the Madelung Formulation and Fisher Information. *Foundations of Physics*, 45(11):1514–1525, Jul 2015.

[11] Eyal Heifetz, Anirban Guha, and Leo Maas. de Broglie Normal Modes in the Madelung Fluid. *Foundations of Physics*, 53, Mar 2023.

[12] Vladimir Iosifovich. Karpman. *Non-linear waves in dispersive media, by V. I. Karpman. Translated by F. F. Cap. Translation edited by S. M. Hamberger.* International series of monographs in natural philosophy v. 71. Pergamon Press, Oxford, [1st ed.] edition, 1974 - 1975.

[13] D. J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philosophical Magazine*, 39(240):422–443, 1895.

[14] Erwin Madelung. Quantentheorie in hydrodynamischer form. *Z. Phys.*, 60:322–326, 1926.

[15] Alan C. Newell. The Inverse Scattering Transform, Nonlinear Waves, Singular Perturbations And Synchronized Solitons. *The Rocky Mountain Journal of Mathematics*, 8(1/2):25–52, 1978.

[16] Yoshio Nishiyama. The Relation between Classical Mechanics and Wave Mechanics. *Progress of Theoretical Physics*, 30(5):657–690, Nov 1963.

[17] Erwin Schrö"dinger. An Undulatory Theory of the Mechanics of Atoms and Molecules. *Physical Review*, 28:1049–1070, Dec 1926.

[18] K. V Shajesh. *Eikonal Approximation.* Norman.

[19] Gigliola Staffilani. Kdv and Almost Conservation Laws, 2002.

[20] Takayuki Tsuchida. Integrable discretizations of derivative nonlinear schrödinger equations. *Journal of Physics A: Mathematical and General*, 35(36):7827, Aug 2002.

[21] Anca Visinescu, Denisa Grecu, Renato Fedele, and Sergio De Nicola. Madelung fluid description of the generalized derivative nonlinear Schrödinger equation: Special solutions and their stability. *Theoretical and Mathematical Physics*, 160:1066–1074, Jul 2009.

[22] G. B. Whitham. On the periodic solutions of the korteweg-de vries equation. *Proceedings of the Royal Society of London*, 299(1452):6–21, 1967.

[23] G. B. Whitham. *Linear and Nonlinear Waves.* John Wiley and Sons, New York, 1974.