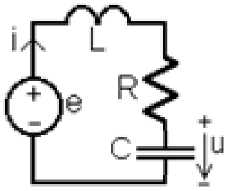
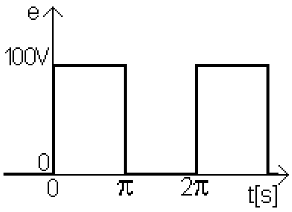
1) R = 1 Ω, L = 1 H, C = 1 F. Find u(t), i(t)





Step 1: Expand e(t) into Fourier series

$$e(t) = E_o + \sum_{k=1}^{\infty}\left(C_k\cos(k\omega_o t) + S_k\sin(k\omega_o t)\right)$$

$$\omega_o = \frac{2\pi}{T_o} = 1 \text{ rad/s}$$

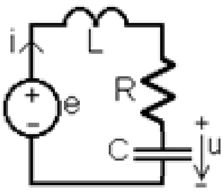
$$E_o = \frac{1}{T_o}\int_0^{T_o} e(t)dt = \frac{1}{2\pi}\int_0^{\pi} 100dt = 50 \text{ V}$$

$$C_k = \frac{2}{T_o}\int_0^{T_o} e(t)\cos(k\omega_o t)dt = \frac{1}{\pi}\int_0^{\pi} 100\cos(kt)dt = 0$$

$$S_k = \frac{2}{T_o}\int_0^{T_o} e(t)\sin(k\omega_o t)dt = \frac{1}{\pi}\int_0^{\pi} 100\sin(kt)dt = -\frac{100}{\pi k}\cos(kt)\Big|_0^{\pi} = -\frac{100}{\pi k}\left(\cos(k\pi) - 1\right)$$

$$= -\frac{100}{\pi k}\left((-1)^k - 1\right) = \begin{cases} 0 & \text{for k even} \\ \frac{200}{\pi k} & \text{for k odd} \end{cases}$$

$$e(t) = 50 + \sum_{k=1}^{\infty}\frac{100}{k\pi}\left(1-(-1)^k\right)\sin(kt) = 50 + \sum_{k=1,3,5,...}^{\infty}\frac{200}{k\pi}\sin(kt)\,[\text{V}]$$



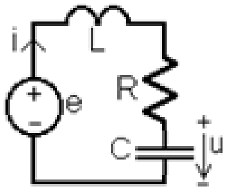
Step 2: Find the DC component

E_o = 50 V, ω = 0 [rad/s]

$$\Rightarrow \frac{1}{j\omega C} = \infty\ [\Omega] \Rightarrow \text{C is open circuit} \Rightarrow I_o = 0\ [\text{A}]$$

jωL = 0 [Ω] ⇒ L is short circuit

KVL ⇒ U_o = 50 [V]



$$e(t) = 50 + \sum_{k=1,3,5,...}^{\infty} \frac{200}{k\pi}\sin(kt)\,[V]$$

Step 3: Find the k^{th} harmonic

$$e_k(t) = \frac{200}{k\pi}\sin(kt)\ [V],\quad k \text{ odd}$$

$$\omega = k\ [rad/s] \Rightarrow Z = R + j\omega L + \frac{1}{j\omega C} = 1 + j(k - \frac{1}{k})\ [\Omega]$$

$$Z = \sqrt{1+\left(k-\frac{1}{k}\right)^2}\ \angle arctg(k-\frac{1}{k})\ \ [\Omega]$$

$$\Rightarrow\ i_k(t) = \frac{200}{k\pi\sqrt{1+\left(k-\frac{1}{k}\right)^2}}\sin(kt - arctg(k-\frac{1}{k}))\ [A]$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{jk} = \frac{1}{k}\angle\text{-}90°[\Omega]$$

$$\Rightarrow\ u_k(t) = \frac{200}{k^2\pi\sqrt{1+\left(k-\frac{1}{k}\right)^2}}\sin(kt - arctg(k-\frac{1}{k}) - 90°)\ \ [V]$$

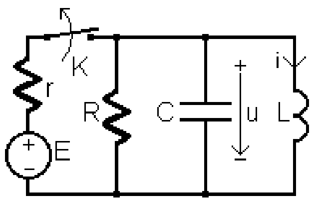
Step 4: Apply the superposition principle

$$i(t) = \sum_{k=1,3,5...}^{\infty}\frac{200}{k\pi\sqrt{1+\left(k-\frac{1}{k}\right)^2}}\sin(kt - arctg(k-\frac{1}{k}))\ \ [A]$$

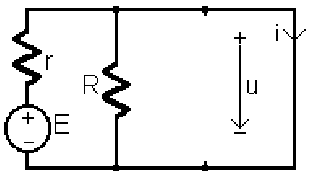
$$u(t) = 50 + \sum_{k=1,3,5}^{+\infty}\frac{200}{k^2\pi\sqrt{1+\left(k-\frac{1}{k}\right)^2}}\sin(kt - arctg(k-\frac{1}{k}) - 90°)\ \ [V]$$

---

2) E = 100 V(DC), r = R = 100 Ω, L = 100 mH, C = 10 μF. t < 0: K closed. t > 0: K open. Find u(t),i(t)

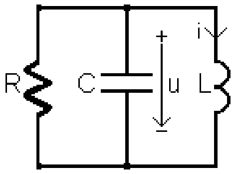


t < 0: K is closed for very long time $\Rightarrow$ the circuit is in DC steady-state (since E is a DC voltage source) $\Rightarrow$ C acts as an open circuit and L acts as a short circuit $\Rightarrow$ u = 0 V, i = E/r = 1 A.



t > 0: K is open.

KCL $\Rightarrow \frac{u}{R} + C\frac{du}{dt} + i = 0$



Natural response: Taking derivative and applying the fact that $u = L\frac{di}{dt}$, one gets

$$\frac{L}{R}\frac{di}{dt} + CL\frac{d^2i}{dt^2} + i = 0$$

$$\frac{L}{R}p + CLp^2 + 1 = 0$$

$$\frac{1}{RC}p + p^2 + \frac{1}{LC} = 0$$

$$\Rightarrow \frac{1}{R}\frac{du}{dt} + C\frac{d^2u}{dt^2} + \frac{u}{L} = 0 \Rightarrow \frac{d^2u}{dt^2} + \frac{1}{RC}\frac{du}{dt} + \frac{1}{LC}u = 0$$

The characteristic equation: $p^2 + \frac{1}{RC}p + \frac{1}{LC} = 0$

$$\Rightarrow p = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -500 \pm j500\sqrt{3}$$

$\Rightarrow i_n(t) = A_1e^{-500t}\cos(500\sqrt{3}\,t + A_2)$ [A]

Forced response: $i_f(t) = 0$ [A]

Complete response: $i(t) = i_f(t) + i_n(t) = A_1e^{-500t}\cos(500\sqrt{3}\,t + A_2)$ [A]

$u(t) = L\frac{di}{dt} = -50A_1e^{-500t}\cos(500\sqrt{3}\,t + A_2) - 50\sqrt{3}A_1e^{-500t}\sin(500\sqrt{3}\,t + A_2)$ [V]

$i(0^+) = i(0^-) = 1$ (current through inductor is continuous) $\Rightarrow A_1\cos(A_2) = 1$

$u(0^+) = u(0^-) = 0$ (voltage across capacitor is continuous)

$\Rightarrow A_1\cos(A_2) + \sqrt{3}A_1\sin(A_2) = 0 \Rightarrow \sqrt{3}A_1\sin(A_2) = -1 \Rightarrow A_1\sin(A_2) = -\frac{1}{\sqrt{3}}$

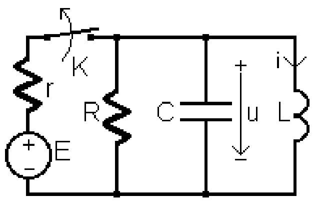
$\Rightarrow \tan(A_2) = -\frac{1}{\sqrt{3}} \Rightarrow A_2 = -30° \Rightarrow A_1 = \frac{2}{\sqrt{3}}$

$$i(t) = \frac{2}{\sqrt{3}}e^{-500t}\cos(500\sqrt{3}\,t - 30°) \text{ [A]}$$

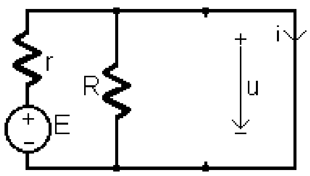
$$u(t) = -\frac{100}{\sqrt{3}}e^{-500t}\cos(500\sqrt{3}\,t - 30°) - 100e^{-500t}\sin(500\sqrt{3}\,t - 30°) \text{ [V]}$$

---

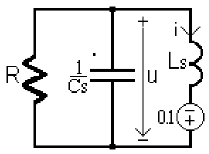
2) E = 100 V(DC), r = R = 100 Ω, L = 100 mH, C = 10 μF. t < 0: K closed. t > 0: K open. Find u(t),i(t)

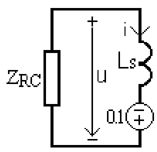


<u>t < 0</u>: K is closed for very long time ⇒ the circuit is in DC steady-state (since E is a DC voltage source) ⇒ C acts as an open circuit and L acts as a short circuit ⇒ u = 0 V, i = E/r = 1 A.



<u>t > 0</u>: K is open.





$$Z_{RC} = \frac{R\frac{1}{Cs}}{R+\frac{1}{Cs}} = \frac{R}{RCs+1} = \frac{10^5}{s+1000}$$

KVL: $-0.1 + (Z_{RC}+Ls)I(s) = 0$

$$\Rightarrow I(s) = \frac{0.1}{\frac{10^5}{s+1000}+0.1s} = \frac{0.1(s+1000)}{0.1s^2+100s+10^5} = \frac{s+1000}{s^2+10^3s+10^6} = \frac{s+500+500}{(s+500)^2+(500\sqrt{3})^2}$$

$$\Rightarrow i(t) = e^{-500t}\cos(500\sqrt{3}\,t) + \frac{1}{\sqrt{3}}e^{-500t}\sin(500\sqrt{3}\,t) \text{ [A]}$$

$$i(t) = \frac{2}{\sqrt{3}}e^{-500t}\cos(500\sqrt{3}\,t - 30°) \text{ [A]}$$

$$= \frac{2}{\sqrt{3}} e^{-500t} [\cos(30°)\cos(500\sqrt{3}\,t) + \sin(30°)\sin(500\sqrt{3}\,t)$$

$$= \frac{2}{\sqrt{3}} e^{-500t} [\frac{\sqrt{3}}{2}\cos(500\sqrt{3}\,t) + \frac{1}{2}\sin(500\sqrt{3}\,t)]$$

$$= e^{-500t} [\cos(500\sqrt{3}\,t) + \frac{1}{\sqrt{3}}\sin(500\sqrt{3}\,t)]$$

KVL $\Rightarrow$ LsI(s) – 0.1 – U(s) = 0

$$\Rightarrow U(s) = LsI(s) - 0.1 = \frac{0.1s(s+1000)}{s^2+10^3s+10^6} - 0.1 = \frac{0.1s(s+1000) - 0.1(s^2+10^3s+10^6)}{s^2+10^3s+10^6}$$

$$= \frac{-10^5}{s^2+10^3s+10^6} = \frac{-10^5}{(s+500)^2+(500\sqrt{3})^2}$$

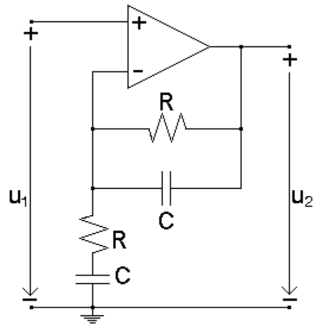
$$\Rightarrow u(t) = -\frac{200}{\sqrt{3}} e^{-500t}\sin(500\sqrt{3}\,t)\ [V]$$

----

3) Given the circuit in Fig.1 with R = 10 kΩ, C = 1 μF.

a) Find the voltage transfer function $K_u(j\omega) = \frac{\dot{U}_2}{\dot{U}_1}$

b) Find $u_2(t)$ if $u_1(t)$ = 100 + 100sin(500t) + 100sin(1000t) mV

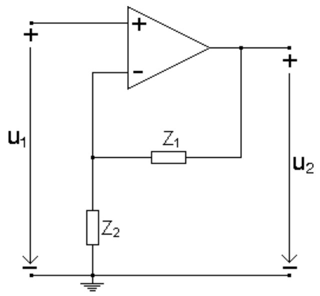


Solution

a) Find the voltage transfer function $K_u(j\omega) = \frac{\dot{U}_2}{\dot{U}_1}$

$$Z_1 = \frac{R\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} = \frac{R}{1+j\omega CR}$$

$$Z_2 = R + \frac{1}{j\omega C} = \frac{1+j\omega CR}{j\omega C}$$



$$\dot{U}_- = \frac{Z_2}{Z_1+Z_2}\dot{U}_2 = \dot{U}_+ = \dot{U}_1$$

$$\Rightarrow K_u(j\omega) = \frac{\dot{U}_2}{\dot{U}_1} = \frac{Z_1+Z_2}{Z_2} = 1 + \frac{Z_1}{Z_2} = 1 + \frac{j\omega CR}{(1+j\omega CR)^2}$$

b) Find $u_2(t)$ if $u_1(t)$ = 100 + 100sin(500t) + 100sin(1000t) mV.

DC component: u_{10} = 100 mV, ω = 0 rad/s

$K_U(0) = \frac{\dot{U}_{20}}{\dot{U}_{10}} = 1 \Rightarrow u_{20} = 100$ V

AC component with ω = 500 rad/s: $u_{11} = 100\sin(500t)$ mV

$$K_U(j500) = \frac{\dot{U}_{21}}{\dot{U}_{11}} = 1 + \frac{j500.10^{-6}.10^4}{(1+j500.10^{-6}.10^4)^2} = 1 + \frac{j5}{(1+j5)^2} = \frac{1-5+j10+j5}{(1+j5)^2}$$

$$= \frac{-4+j15}{(1+j5)^2} = \frac{-4+j15}{(1+j5)^2} = \frac{15.5\angle 105°}{26\angle 157.4°} = 0.6\angle\text{-}52.4°$$

$\Rightarrow u_{21}(t) = 60\sin(500t-52.4°)$ mV

AC component with ω = 1000 rad/s: $u_{12} = 100\sin(1000t)$ mV

$$K_U(j1000) = \frac{\dot{U}_{22}}{\dot{U}_{12}} = 1 + \frac{j10}{(1+j10)^2} = \frac{1-100+j20+j10}{(1+j10)^2} = \frac{-99+j30}{(1+j10)^2}$$

$$= \frac{103.4\angle 163°}{10\angle 168.6°} = 10.34\angle\text{-}5.6°$$

$\Rightarrow u_{22}(t) = 1034\sin(1000t-5.6°)$ mV

Superposition:

$\Rightarrow u_o(t) = 100 + 60\sin(500t-52.4°) + 1034\sin(1000t-5.6°)$ mV

---

4) The circuit in Fig.2A has R = 1 Ω, C = 1 F. e is a periodic voltage source with period T_o = 2π s (Fig.2B). Expand e(t) into Fourier series. Find an expression of u(t)

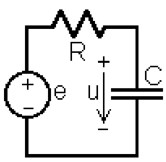


Fig.2A

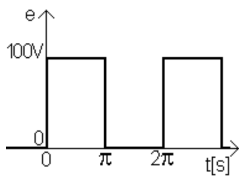


Fig.2B

Step 1: Expand e(t) into Fourier series

$$e(t) = E_o + \sum_{k=1}^{\infty}\left(C_k\cos(k\omega_o t) + S_k\sin(k\omega_o t)\right)$$

$$\omega_o = \frac{2\pi}{T_o} = 1 \text{ rad/s}$$

$$E_o = \frac{1}{T_o}\int_0^{T_o} e(t)dt = \frac{1}{2\pi}\int_0^{\pi} 100dt = 50 \text{ V}$$

$$C_k = \frac{2}{T_o}\int_0^{T_o} e(t)\cos(k\omega_o t)dt = \frac{1}{\pi}\int_0^{\pi} 100\cos(kt)dt = 0$$

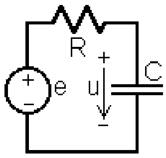
$$S_k = \frac{2}{T_o}\int_0^{T_o} e(t)\sin(k\omega_o t)dt = \frac{1}{\pi}\int_0^{\pi} 100\sin(kt)dt = -\frac{100}{\pi k}\cos(kt)\Big|_0^{\pi} = -\frac{100}{\pi k}\left(\cos(k\pi)-1\right)$$

$$= -\frac{100}{\pi k}\left((-1)^k - 1\right) = \begin{cases} 0 & \text{for k even} \\ \frac{200}{\pi k} & \text{for k odd} \end{cases}$$

$$e(t) = 50 + \sum_{k=1}^{\infty}\frac{100}{k\pi}\left(1-(-1)^k\right)\sin(kt) = 50 + \sum_{k=1,3,5,...}^{\infty}\frac{200}{k\pi}\sin(kt) \ [V]$$

Step 2: Find the DC component

$$\omega = 0 \text{ [rad/s]} \Rightarrow \frac{1}{j\omega C} = \infty \ [\Omega] \Rightarrow \text{C is open circuit} \Rightarrow U_o = 50 \text{ [V]}$$



Step 3: Find the kth harmonic

$$e_k(t) = \frac{200}{k\pi}\sin(kt) \qquad \text{k odd}$$

$$\omega = k \text{ [rad/s]} \Rightarrow Z = R + \frac{1}{j\omega C} = 1 + \frac{1}{jk} = 1 - j\frac{1}{k} = \sqrt{1+\frac{1}{k^2}}\angle\text{-arctg}(\frac{1}{k}) \ [\Omega]$$

$$\Rightarrow i_k(t) = \frac{200}{k\pi\sqrt{1+\frac{1}{k^2}}}\sin(kt + \text{arctg}(\frac{1}{k}))\,[A]$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{jk} = \frac{1}{k}\angle\text{-}90°$$

$$\Rightarrow u_k(t) = \frac{200}{k^2\pi\sqrt{1+\frac{1}{k^2}}}\sin(kt + \text{arctg}(\frac{1}{k}) - 90°)\,[V]$$

Step 4: Apply the superposition principle

$$u(t) = 50 + \sum_{k=1,3,5}^{+\infty}\frac{200}{k^2\pi\sqrt{1+\frac{1}{k^2}}}\sin(kt + \text{arctg}(\frac{1}{k}) - 90°) \ [V]$$

$$i(t) = \sum_{k=1,3,5}^{+\infty}\frac{200}{k\pi\sqrt{1+\frac{1}{k^2}}}\sin(kt + \text{arctg}(\frac{1}{k})) \ [V]$$

---

5. Consider the circuit in Fig.3 with E = 100 V, r = 10 Ω, R = 100 Ω, L = 0.1 H. Initially the switch K is closed for very long time and suddenly open at t = 0. Find u(t) and i(t).

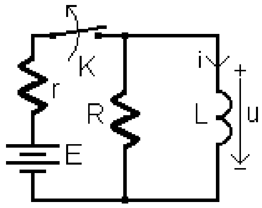
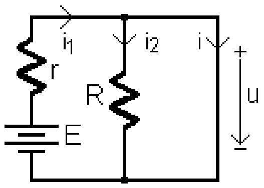


Fig.3



<u>**t < 0**</u>:

K is closed for very long time $\Rightarrow$ the circuit is operating in DC steady-state $\Rightarrow$ the inductor acts as a short circuit $\Rightarrow$ u = 0 V

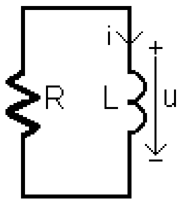
$i_2 = u/R = 0/100 = 0$ A

KCL: $i_1 - i_2 - i = 0 \Rightarrow i_1 = i$

KVL: $-E + ri_1 + u = 0 \Rightarrow i_1 = E/r$

$\Rightarrow$ i = E/r = 10 A

<u>**t > 0**</u>: K is open



KVL: $L\frac{di}{dt} + Ri = 0$

Natural response:

The characteristic equation: pL + R = 0

$\Rightarrow\ p = -\frac{R}{L} = -1000 \ \Rightarrow i_n(t) = Ce^{-1000t}$ [A]

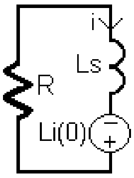
Forced response: $i_f(t) = 0$ [A]

Complete response: $i(t) = i_f(t) + i_n(t) = Ce^{-1000t}$ [A]

Initial conditions: $i(0^+) = i(0^-) = 10 = C$ (current through inductor is continuous)

$\Rightarrow \qquad i(t) = 10e^{-1000t}$ [A]

$\Rightarrow \qquad u(t) = L\frac{di}{dt} = -1000e^{-1000t}$ [V]



KVL: $RI(s) + LsI(s) – Li(0) = 0$

$I(s) = Li(0)/(Ls + R) = 1/(0.1s + 100) = 10/(s + 1000)$

$\Rightarrow i(t) = 10e^{-1000t}$ A

$U(s) = LsI(s) - Li(0) = s/(s + 1000) - 1 = 1000/(s + 1000)$

$\Rightarrow u(t) = 1000e^{-1000t}$ V

6.The circuit in Fig.4 has E = 100V (DC), R = 100 Ω, L = 100 mH, C = 10 μF.The switch K is open for very long time and is closed at t = 0. Find u(t).

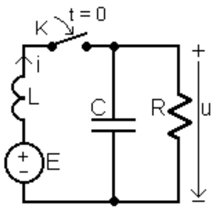
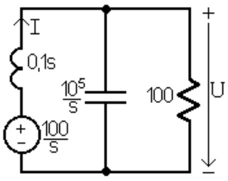
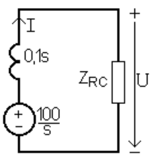


Fig.4



$$Z_{RC} = \frac{\frac{10^5}{s}100}{\frac{10^5}{s}+100}$$

$$= \frac{10^5}{s+10^3}$$



KVL:

$$-\frac{100}{s} + (0.1s+Z_{RC})I(s) = 0$$

$U(s) = Z_{RC}I(s)$

t < 0: K is open for very long time$\Rightarrow$ i = 0 [A], u = 0 [V]

t > 0: K is closed

$$U\left(\frac{10}{s}+\frac{s}{10^5}+\frac{1}{100}\right)=\frac{10^3}{s^2}$$

$$U\left(\frac{10^6+s^2+10^3s}{10^5s}\right)=\frac{10^3}{s^2}$$

$$\Rightarrow\ U = \frac{10^8}{s(s^2+10^3s+10^6)} = \frac{A}{s} + \frac{Bs+C}{s^2+10^3s+10^6}$$

$$= \frac{(A+B)s^2+(10^3A+C)s+10^6A}{s(s^2+10^3s+10^6)}$$

$\Rightarrow$ A + B = 0

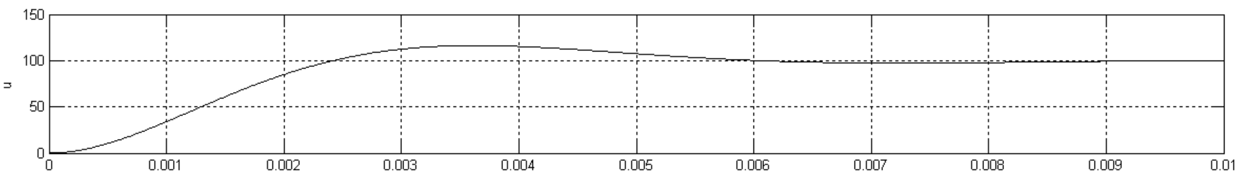
$10^3A + C = 0$

$10^6A = 10^8$

$$U(s) = \frac{100}{s} - \frac{100s+10^5}{s^2+10^3s+10^6}$$

$$= \frac{100}{s} - \frac{100(s+500)}{(s+500)^2+(500\sqrt{3})^2} - \frac{5x10^4}{(s+500)^2+(500\sqrt{3})^2}$$

$$\Rightarrow u(t) = 100 - 100e^{-500t}\cos(500\sqrt{3}\,t) - \frac{100}{\sqrt{3}}e^{-500t}\sin(500\sqrt{3}\,t)\ [V]$$



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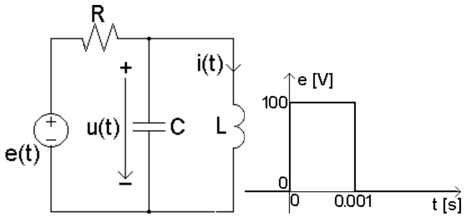
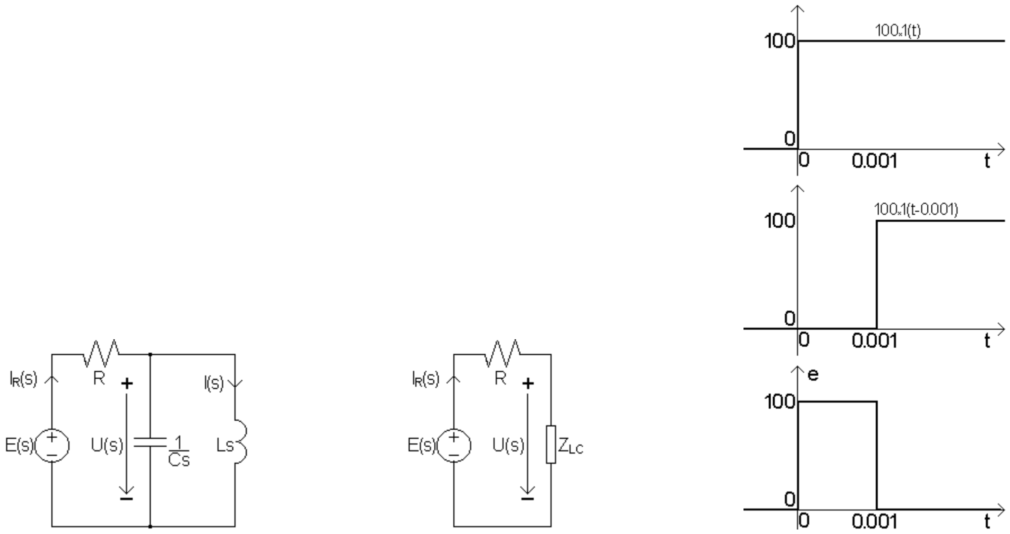
7. Given the circuit in Fig.E4.5 with R = 100 Ω, L = 100 mH, C = 10 μF. Find u(t) and i(t).



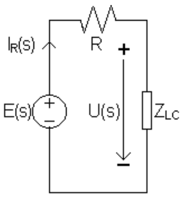
t < 0: e = 0 V ⇒ u(t) = 0 V and i(t) = 0 A

t > 0:



1st method: $e(t) = 100.1(t) - 100.1(t-0.001) \Rightarrow E(s) = \frac{100}{s} - \frac{100}{s}e^{-0.001s} = \frac{100}{s}(1 - e^{-0.001s})$

2nd method: $E(s) = \int_0^{+\infty} e(t)e^{-st}dt = \int_0^{0.001} 100e^{-st}dt = \frac{100e^{-st}}{-s}\Big|_0^{0.001} = \frac{100(1-e^{-0.001s})}{s}$



$Z_{LC} = \frac{Ls\frac{1}{Cs}}{Ls+\frac{1}{Cs}} = \frac{Ls}{LCs^2+1}$

KVL: $-E(s) + (R + Z_{LC})I_R(s) = 0 \Rightarrow I_R(s) = \frac{E(s)}{R + Z_{LC}}$

$$U(s) = Z_{LC}I_R(s) = \frac{Z_{LC}E(s)}{R+Z_{LC}} = \frac{\frac{Ls}{LCs^2+1}E(s)}{R+\frac{Ls}{LCs^2+1}} = \frac{\frac{0.1s}{10^{-6}s^2+1}E(s)}{100+\frac{0.1s}{10^{-6}s^2+1}} = \frac{\frac{10^5 s}{s^2+10^6}E(s)}{100+\frac{10^5 s}{s^2+10^6}}$$

$$= \frac{10^3 sE(s)}{s^2+10^3 s+10^6} = \frac{10^5}{s^2+10^3 s+10^6}(1 - e^{-0.001s}) = \frac{10^5}{(s+500)^2+(500\sqrt{3})^2}(1 - e^{-0.001s})$$

$$\frac{10^5}{(s+500)^2+(500\sqrt{3})^2} \leftrightarrow \frac{200}{\sqrt{3}}e^{-500t}\sin(500\sqrt{3}\,t)$$

$$\frac{10^5}{(s+500)^2+(500\sqrt{3})^2}e^{-0.001s} \leftrightarrow \frac{200}{\sqrt{3}}e^{-500(t-0.001)}\sin(500\sqrt{3}\,(t-0.001))1(t-0.001)$$

$$\Rightarrow u(t) = \frac{200}{\sqrt{3}}e^{-500t}\sin(500\sqrt{3}\,t) - \frac{200}{\sqrt{3}}e^{-500(t-0.001)}\sin(500\sqrt{3}\,(t-0.001))1(t-0.001)\text{ V}$$

$$I(s) = \frac{U(s)}{Ls} = \frac{10^6}{s(s^2+10^3 s+10^6)}(1 - e^{-0.001s}) = \left(\frac{1}{s} - \frac{10^3}{s^2+10^3 s+10^6}\right)(1 - e^{-0.001s})$$

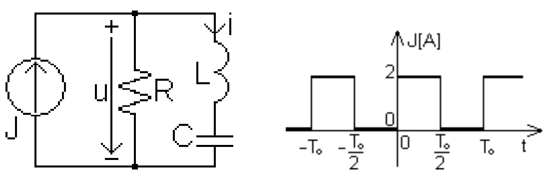
$$= \left(\frac{1}{s} - \frac{10^3}{(s+500)^2+(500\sqrt{3})^2}\right)(1 - e^{-0.001s})$$

$$\frac{1}{s} - \frac{10^3}{(s+500)^2+(500\sqrt{3})^2} \leftrightarrow 1 - \frac{2}{\sqrt{3}}e^{-500t}\sin(500\sqrt{3}\,t)$$

$$\left(\frac{1}{s} - \frac{10^3}{(s+500)^2+(500\sqrt{3})^2}\right)e^{-0.001s} \leftrightarrow \left(1 - \frac{2}{\sqrt{3}}e^{-500(t-0.001)}\sin(500\sqrt{3}\,(t-0.001))\right)1(t-0.001)$$

$$\Rightarrow i(t) = 1 - \frac{2}{\sqrt{3}}e^{-500t}\sin(500\sqrt{3}\,t) - \left(1 - \frac{2}{\sqrt{3}}e^{-500(t-0.001)}\sin(500\sqrt{3}\,(t-0.001))\right)1(t-0.001)\text{ A}$$

8.The circuit in Fig.E3.4 has R = 1Ω, L = 1H, C = 1F. J is a periodic current source with period $T_o = 2\pi$ [s]. Expand J(t) into Fourier series. Find an expression of u(t) and i(t).



Step 1: Expand J(t) into Fourier series

$$J(t) = J_o + \sum_{k=1}^{\infty}\left(C_k\cos(k\omega_o t) + S_k\sin(k\omega_o t)\right)$$

$$\omega_o = \frac{2\pi}{T_o} = 1 \text{ [rad/s]}$$

$$J_o = \frac{1}{T_o}\int_0^{T_o} J(t)dt = \frac{1}{2\pi}\int_0^{\pi} 2dt = 1 \text{ [A]}$$

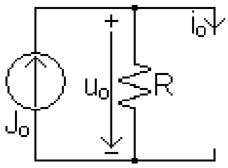
$$C_k = \frac{2}{T_o}\int_0^{T_o} J(t)\cos(k\omega_o t)dt = \frac{1}{\pi}\int_0^{\pi} 2\cos(kt)dt = 0$$

$$S_k = \frac{2}{T_o}\int_0^{T_o} J(t)\sin(k\omega_o t)dt = \frac{1}{\pi}\int_0^{\pi} 2\sin(kt)dt = -\frac{2}{\pi k}\cos(kt)\Big|_0^{\pi} = -\frac{2}{\pi k}\left(\cos(k\pi)-1\right)$$

$$= \frac{2}{\pi k}\left(1-(-1)^k\right) = \begin{cases} 0 & \text{for k even} \\ \frac{4}{\pi k} & \text{for k odd} \end{cases}$$

$$J(t) = 1 + \sum_{k=1}^{\infty}\frac{2}{k\pi}\left(1-(-1)^k\right)\sin(kt) = 1 + \sum_{k=1,3,5,...}^{\infty}\frac{4}{k\pi}\sin(kt) \text{ [A]}$$

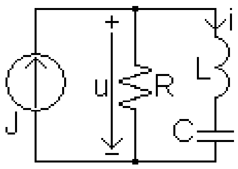
Step 2: Find the DC component



$\omega = 0$ [rad/s] $\Rightarrow \frac{1}{j\omega C} = \infty$ [Ω] $\Rightarrow$ C is open circuit $\Rightarrow I_o = 0$ [A], $U_o = RJ_o = 1$ [V]

$j\omega L = 0 \Rightarrow$ L is short circuit

Step 3: Find the k^{th} harmonic



$J_k(t) = \frac{4}{k\pi}\sin(kt) \leftrightarrow \dot{J}_k = \frac{4}{k\pi}\angle 0$ k odd

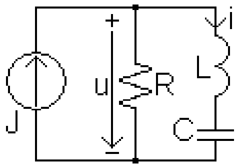
$\omega = k$ [rad/s]

$$Z = \frac{R(j\omega L + \frac{1}{j\omega C})}{R + j\omega L + \frac{1}{j\omega C}} = \frac{jk + \frac{1}{jk}}{1 + jk + \frac{1}{jk}} = \frac{1-k^2}{1-k^2+jk} = \frac{k^2-1}{k^2-1-jk}$$

$$= \frac{k^2-1}{\sqrt{(k^2-1)^2+k^2}}\angle \text{arctg}\left(\frac{k}{k^2-1}\right)$$

$$\Rightarrow \dot{U}_k = Z\dot{J}_k = \frac{k^2-1}{\sqrt{(k^2-1)^2+k^2}}\angle \text{arctg}(\frac{k}{k^2-1}).\frac{4}{k\pi}\angle 0 = \frac{4(k^2-1)}{k\pi\sqrt{(k^2-1)^2+k^2}}\angle \text{arctg}(\frac{k}{k^2-1})$$

$$\Rightarrow u_k(t) = \frac{4(k^2-1)}{k\pi\sqrt{(k^2-1)^2+k^2}}\sin(kt + \text{arctg}(\frac{k}{k^2-1}))\ \text{[V]}$$



$$\dot{I}_k = \frac{\dot{U}_k}{j\omega L+\frac{1}{j\omega C}} = \frac{\dot{U}_k}{jk+\frac{1}{jk}} = \frac{jk\dot{U}_k}{1-k^2} = \frac{k}{1-k^2}\angle 90°.\frac{4(k^2-1)}{k\pi\sqrt{(k^2-1)^2+k^2}}\angle \text{arctg}(\frac{k}{k^2-1})$$

$$= \frac{-4}{\pi\sqrt{(k^2-1)^2+k^2}}\angle 90°+\text{arctg}(\frac{k}{k^2-1})\ \text{[A]}$$

$$\Rightarrow i_k(t) = \frac{-4}{\pi\sqrt{(k^2-1)^2+k^2}}\sin(kt + 90°+ \text{arctg}(\frac{k}{k^2-1}))\ \text{[A]}$$

Step 4: Apply the superposition principle

$$u(t) = 1 + \sum_{k=1,3,5...}^{+\infty}\frac{4(k^2-1)}{k\pi\sqrt{(k^2-1)^2+k^2}}\sin(kt+\text{arctg}(\frac{k}{k^2-1}))\ \text{[V]}$$

$$i(t) = \sum_{k=1,3,5...}^{+\infty}\frac{-4}{\pi\sqrt{(k^2-1)^2+k^2}}\sin(kt+\text{arctg}(\frac{1}{k^2-1})+90°)\ \text{[A]}$$

3.10 Find the Thevenin equivalent circuit of the two-terminal networks in Fig.E3.10. Deduce the maximum power delivered by the two-terminal network.

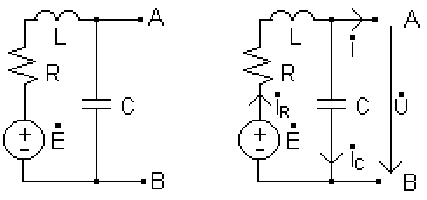


Fig.E3.10

$$\dot{I}_C = \frac{\dot{U}}{\frac{1}{j\omega C}} = j\omega C\,\dot{U}$$

KCL: $\dot{I}_R = \dot{I}_C + \dot{I} = j\omega C\,\dot{U} + \dot{I}$

KVL: $-\dot{E} + (R + j\omega L)\dot{I}_R + \dot{U} = 0$

$\Rightarrow -\dot{E} + (R + j\omega L)(j\omega C\,\dot{U} + \dot{I}) + \dot{U} = 0$

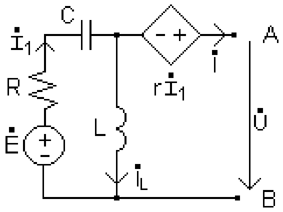
$(R + j\omega L)\dot{I} + (R + j\omega L)j\omega C\,\dot{U} + \dot{U} = \dot{E}$

$$\dot{E}_T = \dot{U}\Big|_{\dot{I}=0} = \frac{\dot{E}}{1 + j\omega C(R + j\omega L)}$$

$$\dot{J}_N = \dot{I}\Big|_{\dot{U}=0} = \frac{\dot{E}}{R + j\omega L}$$

$$Z_T = \dot{E}_T / \dot{J}_N = \frac{R + j\omega L}{1 + j\omega C(R + j\omega L)}$$

Find the Thevenin equivalent circuit of the two-terminal networks in Fig.E3.10. Deduce the maximum power delivered by the two-terminal network.



KCL: $\dot{I}_1 = \dot{I}_L + \dot{I}$

KVL: $-j\omega L\,\dot{I}_L - r\dot{I}_1 + \dot{U} = 0 \Rightarrow -j\omega L(\dot{I}_1 - \dot{I}) - r\dot{I}_1 + \dot{U} = 0 \Rightarrow \dot{I}_1 = \dfrac{\dot{U} + j\omega L\dot{I}}{r + j\omega L}$

KVL: $-\dot{E} + (R + 1/j\omega C)\dot{I}_1 - r\dot{I}_1 + \dot{U} = 0 \Rightarrow (R - r + 1/j\omega C)\dot{I}_1 + \dot{U} = \dot{E}$

$$\Rightarrow (R - r + 1/j\omega C)\frac{\dot{U} + j\omega L\dot{I}}{r + j\omega L} + \dot{U} = \dot{E}$$

$$\Rightarrow \left[\frac{j\omega L\left(R - r + \dfrac{1}{j\omega C}\right)}{r + j\omega L}\right]\dot{I} + \left[1 + \frac{\left(R - r + \dfrac{1}{j\omega C}\right)}{r + j\omega L}\right]\dot{U} = \dot{E}$$

$$\dot{E}_T = \dot{U}\Big|_{\dot{I}=0} = \frac{\dot{E}}{1 + \dfrac{\left(R - r + \dfrac{1}{j\omega C}\right)}{r + j\omega L}}$$

$$\dot{J}_N = \dot{I}\Big|_{\dot{U}=0} = \frac{(r + j\omega L)\dot{E}}{j\omega L\left(R - r + \dfrac{1}{j\omega C}\right)}$$

$$Z_T = \dot{E}_T / \dot{J}_N = \frac{j\omega L\left(R - r + \dfrac{1}{j\omega C}\right)}{R + j\omega L + \dfrac{1}{j\omega C}}$$