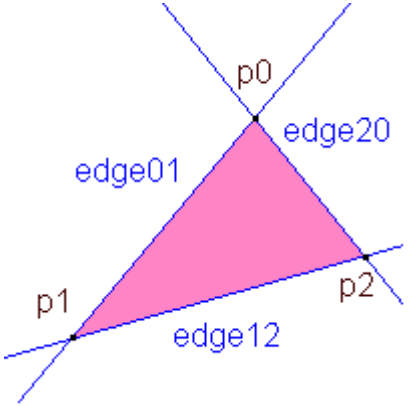
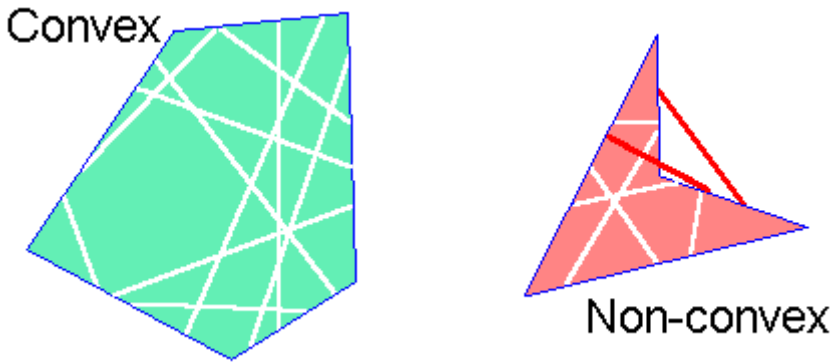
Rasterizing Triangles

Triangles are perhaps the most important filled primitive. Some reasons for this are:

- **Triangles are minimal:** They are determined by just 3 points or 3 edges.

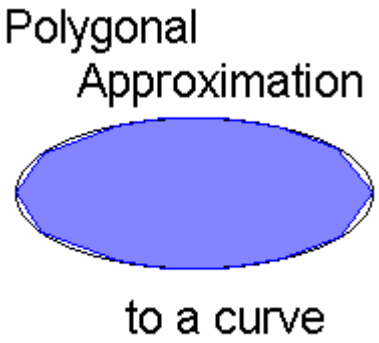


- **Triangles are always *convex* polygons** What does it mean to be a convex polygon?

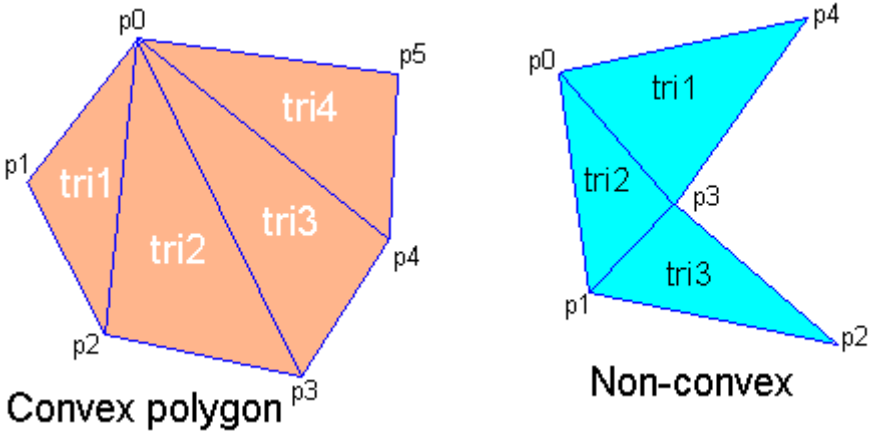


  A polygon is convex if and only if any line segment connecting two points on its boundary is contained entirely within the polygon or one of its edges.

- **Triangles are mathematically very simple** The math involved in scan converting a triangle involves only simple linear equations.
- **Arbitrary shapes can be approximated with triangles** Any 2-dimensional shape can be approximated by a polygon using a locally linear approximation to the surface. To improve the quality of fit we need only increase the number edges.



- **Any polygon can be decomposed into triangles**

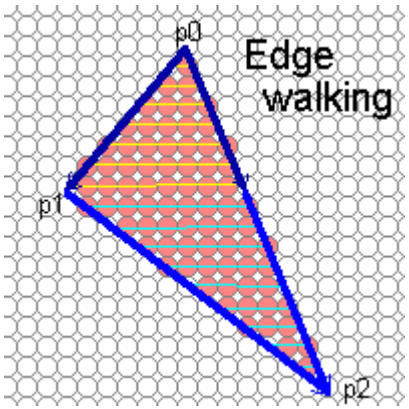


A convex n-sided polygon, with ordered vertices $\{v_0, v_1, \ldots v_n\}$ along the perimeter, can be trivially decomposed into triangles $\{(v_0,v_1,v_2), \{v_0,v_2,v_3), (v_0,v_i,v_{i+1}), \ldots (v_0,v_{n-1},v_n)\}$.

You can usually decompose a non-convex polygon into triangles, but it is non-trivial, and in some overlapping cases you have to introduce a new vertex.

There are two common strategies for scan-converting a triangle. The first uses *edge walking* and the second uses *edge equations*.

Edge-walking



Notes on edge walking:

- Sort the vertices in both x and y
- Determine if the middle vertex, or *breakpoint* lies on the left or right side of the polygon. If the trianlge has an edge parallel to the scanline direction then there is no breakpoint.
- Determines the left and right edge for each scanline (called *spans*).
- Walk down the left and right edges filling the pixels in-between until either a breakpoint or the bottom vertex is reached.
- Exit of change active walking edges.

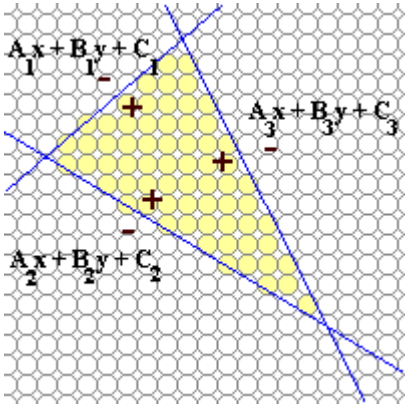
Advantages and Disadvantages:

- Loaded with special cases (left and right breakpoints, no breakpoints)
- Difficult to get right
- Requires computing fractional offsets when interpolating parameters across the triangle
- Generally very fast

The algorithm described in the book is an edge walking algorithm.

Edge equations

Another approach to rasterizing triangles uses edge equations to determine which pixels to fill. An edge equation is another name for the discriminating fuction that we used in our curve and line-drawing algorithms. An edge equation segments a planar region into three parts, a boundary, and two half-spaces. The boundary is identified by points where the edge equation is equal to zero. The half-spaces are distiguished by differences in the edge equation's sign. We can choose which half-space gives a positive sign by multiplication by -1.



Notes on using edge equations to scan-convert triangles:

- Compute edge equations from vertices
- Orient edge equations so that their positive-half spaces are in the triangle's interior
- Compute a bounding box
- Scan through pixels in bounding box evaluating the edge equations. When all three are positive then draw the pixel.

Here's is my example implementation of a triangle rasterizer that uses edge equations.

Before starting we will define a few useful objects.

First here is the representation of a vertex

```
public class Vertex2D {
    public float x, y;              // coordinate of vertex
    public int argb;                // color of vertex

    public Vertex2D(float xval, float yval, int cval)
    {
        x = xval;
        y = yval;
        argb = cval;
    }
}
```

Next we define an *EdgeEquation* object

```
class EdgeEqn {
    public final static int FRACBITS = 12;
    public int A, B, C;
```

```
    public int flag;

    public EdgeEqn(Vertex2D v0, Vertex2D v1)
    {
        double a = v0.y - v1.y;
        double b = v1.x - v0.x;
        double c = -0.5f*(a*(v0.x + v1.x) + b*(v0.y + v1.y));

        A = (int) (a * (1<<FRACBITS));
        B = (int) (b * (1<<FRACBITS));
        C = (int) (c * (1<<FRACBITS));
        flag = 0;
        if (A >= 0) flag += 8;
        if (B >= 0) flag += 1;
    }

    public void flip()
    {
        A = -A;
        B = -B;
        C = -C;
    }

    public int evaluate(int x, int y)
    {
        return (A*x + B*y + C);
    }
}
```

Notice that I'm using integers for my coefficients. This implementation uses 12 fractional bits, thus its practical use will be limited to screens with resolutions of 4096 by 4096 or less.

We determine the coefficients of an edge equation using two points on the edge. Each point determines an equation in terms of our three unknowns, *A*, *B*, and *C*.

$$Ax_0 + By_0 + C = 0$$
$$Ax_1 + By_1 + C = 0$$

We can solve for A and B in terms of C by setting up the following homogeneous linear system.

$$\begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = -C \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Multiplying both sides by the matrix inverse.

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{-C}{x_0 y_1 - x_1 y_0} \begin{bmatrix} y_1 - y_0 \\ x_1 - x_0 \end{bmatrix}$$

If we choose $C = x_0 y_1 - x_1 y_0$, then we get $A = y_0 - y_1$ and $B = x_1 - x_0$. The equations for *A* and *B* match those that appear in the method definition.

In order to understand the expression used for *C* we'll need to discuss the numerical precision of the floating point calculations used by computers. Computers represent floating-point number internally in a format similar to scientific notation. Each number is stored with fractional part having a fixed number of significant digits along with an exponent. If you remember back to you chemistry or physics classes, the very worse thing that you can do with numbers represented in scientific notation is subtract number of similar magnitude. Here is what happens. Suppose we have four significant digits in our notation. If we subtract numbers of similar magnitudes as shown below:

$$\underline{1.234} \times 10^2 - \underline{1.233} \times 10^2 = \underline{1.000} \times 10^0$$

We loose most of the significant digits in our result.

In the case of triangles, we can expect these sort of precision problems to occur frequently, because in general the vertices of a triangle are usually relatively close to each other.

$$x_0 \approx x_1 \text{ and } y_0 \approx y_1 \text{ thus } x_0 y_1 - x_1 y_0 \approx 0$$

Thankfully, we can avoid this subtraction of large numbers when computing an expression for C. Given that we know *A* and *B* we can solve for *C* as follows:

$$C_0 = -Ax_0 - By_0 \text{ or } C_1 = -Ax_1 - By_1$$

In order to eliminate any unnecessary bias toward either vertex in our calculation we can compute the average of these C values as follows.

$$C_{ave} = \frac{-(A(x_0 + x_1) + B(y_0 + y_1))}{2}$$

This is the expression for *C* that appears in our method, and it avoids many of the numerical problems that plague other approaches.

The flag is used in computing bounding boxes. We'll consider that later. Next, let's look at the main loop of the rasterizer.

```
public class FlatTri implements Drawable {
    protected Vertex2D v[];
    protected int color;

    public FlatTri()
    {
    }

    public FlatTri(Vertex2D v0, Vertex2D v1, Vertex2D v2)
    {
        v = new Vertex2D[3];
        v[0] = v0;
        v[1] = v1;
        v[2] = v2;

        /*
            ... Our policy is to assign a triangle
            the average of it's vertex colors ...
        */
        int a = ((v0.argb >> 24) & 255) + ((v1.argb >> 24) & 255) + ((v2.argb >> 24) & 255);
        int r = ((v0.argb >> 16) & 255) + ((v1.argb >> 16) & 255) + ((v2.argb >> 16) & 255);
        int g = ((v0.argb >> 8) & 255) + ((v1.argb >> 8) & 255) + ((v2.argb >> 8) & 255);
        int b = (v0.argb & 255) + (v1.argb & 255) + (v2.argb & 255);

        a = (a + a + 3) / 6;
        r = (r + r + 3) / 6;
        g = (g + g + 3) / 6;
        b = (b + b + 3) / 6;

        color = (a << 24) | (r << 16) | (g << 8) | b;
    }


    protected EdgeEqn edge[];
    protected int area;
    protected int xMin, xMax, yMin, yMax;
    private static byte sort[][] = {
        {0, 1}, {1, 2}, {0, 2}, {2, 0}, {2, 1}, {1, 0}
    };


    public void Draw(Raster r)
    {
        if (!triangleSetup(r)) return;

        int x, y;
        int A0 = edge[0].A;
        int A1 = edge[1].A;
```

```
        int A2 = edge[2].A;

        int B0 = edge[0].B;
        int B1 = edge[1].B;
        int B2 = edge[2].B;

        int t0 = A0*xMin + B0*yMin + edge[0].C;
        int t1 = A1*xMin + B1*yMin + edge[1].C;
        int t2 = A2*xMin + B2*yMin + edge[2].C;

        yMin *= r.width;
        yMax *= r.width;

        /*
             .... scan convert triangle ....
        */
        for (y = yMin; y <= yMax; y += r.width) {
                int e0 = t0;
                int e1 = t1;
                int e2 = t2;
                int xflag = 0;
                for (x = xMin; x <= xMax; x++) {
                    if ((e0|e1|e2) >= 0) {      // all 3 edges must be >= 0
                            r.pixel[y+x] = color;
                            xflag++;
                    } else if (xflag != 0) break;
                    e0 += A0;
                    e1 += A1;
                    e2 += A2;
                }
                t0 += B0;
                t1 += B1;
                t2 += B2;
        }
    }
```

Most everything here is straight forward, with two exceptions.

- All three edges are tested with a single comparison by oring together the three edges and checking if the result is positive. If any one of the thre is negative then its sign-bit will be set to a 1, and the result of the or will be negative.
- Since triangles are convex, we can only be inside for a single interval on any given scanline. The *xflag* variable is used to keep track of when we exit the triangle's interior. If ever we find ourselves outside of the triangle having already set some pixels on the span then we can skip over the remainder of the scanline.

All the dirty work is done by the setup method.

```
    protected boolean triangleSetup(Raster r)
```

```
{
    if (edge == null) edge = new EdgeEqn[3];

    /*
        Compute the three edge equations
    */
    edge[0] = new EdgeEqn(v[0], v[1]);
    edge[1] = new EdgeEqn(v[1], v[2]);
    edge[2] = new EdgeEqn(v[2], v[0]);

    /*
        Trick #1: Orient edges so that the
        triangle's interior lies within all
        of their positive half-spaces.

        Assuring that the area is positive
        accomplishes this
    */
    area = edge[0].C + edge[1].C + edge[2].C;
    if (area == 0) return false;                // degenerate triangle
    if (area < 0) {
        edge[0].flip();
        edge[1].flip();
        edge[2].flip();
        area = -area;
    }

    /*
        Trick #2: compute bounding box
    */
    int xflag = edge[0].flag + 2*edge[1].flag + 4*edge[2].flag;
    int yflag = (xflag >> 3) - 1;
    xflag = (xflag & 7) - 1;

    xMin = (int) (v[sort[xflag][0]].x);
    xMax = (int) (v[sort[xflag][1]].x + 1);
    yMin = (int) (v[sort[yflag][1]].y);
    yMax = (int) (v[sort[yflag][0]].y + 1);

    /*
        clip triangle's bounding box to raster
    */
    xMin = (xMin < 0) ? 0 : xMin;
    xMax = (xMax >= r.width) ? r.width - 1 : xMax;
    yMin = (yMin < 0) ? 0 : yMin;
    yMax = (yMax >= r.height) ? r.height - 1 : yMax;
    return true;
}
```

In this method we do two critical things. We orient the edge equations, and we compute the bounding box.

ToDo: Positive area means positive half spaces.

ToDo: Sorting with the minimum number of compares.

Here is a demonstration of to edge equation based rasterizer described. Click anywhere below to see an example:

That way easy wasn't it! Go back to the index.

Last updated: Monday, September 30, 1996