Problem 1: Refer to Figure 1 for this problem. The switch changes from position 1 to position 2 at time t = 0, after the circuit having already achieved steady state before t = 0.

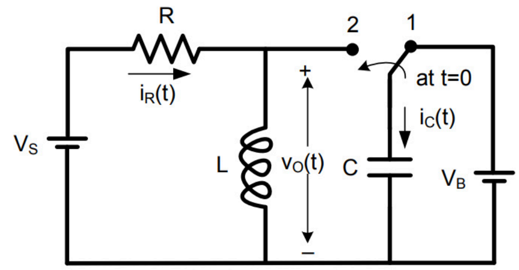


Figure 1.

For parts (a) through (d), express your answers symbolically, in terms of V_S, V_B, R, L and C.

a. Derive a differential equation in terms of $v_O(t)$ that characterizes circuit behavior for $t \geq 0$. **Hint**: Apply KCL at the node that is the junction of R, L, and C.

b. Determine the values of $v_O(t)$ and $i_R(t)$ just after t = 0.

c. Determine the value of $\left.\frac{dv_O(t)}{dt}\right|_{t=0+}$. **Hint**: Note that $C\left.\frac{dv_O(t)}{dt}\right|_{t=0+}$ equals $i_C(0+)$.

For the rest, use numerical values, V_S = 10V, V_B = 5V, C = 4nF, L = 400nH, and R = 5 Ohms.

d. Determine the resonant frequency and the damping factor of the circuit. What kind of damping does this circuit exhibit?

e. Derive an expression for the complete solution of $v_O(t)$ in this circuit.

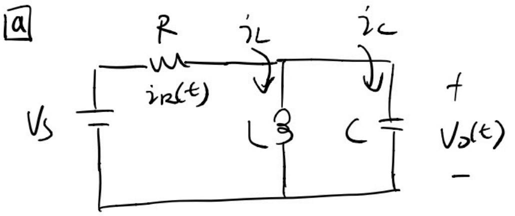
f. Draw a rough sketch of the solution, $v_O(t)$. Do not forget to show the initial and final values of $v_O(t)$. If $v_O(t)$ shows any ringing, mark the period and the time constant of the envelope as well.

Note: For parts (e) and (f), you may need various boundary conditions i.e. answers from parts (b), and/or (c). If you are unsure of your answers, proceed symbolically i.e. derive your answers in terms of symbols such as $v_O(0+)$ etc., for partial credit. You will be penalized a maximum of 5 points for this.

(8 + 4 + 4 + 4 + 8 + 4 = 32 points)

Solution:

a



KCL: $i_R = i_L + i_C$

$$i_R = \frac{V_s - V_o}{R}$$

$$V_L = L\frac{di_L}{dt} = V_o \Rightarrow i_L = \frac{1}{L}\int_{-\infty}^{t} V_o\,dt$$

$$i_C = C\frac{dV_o}{dt}$$

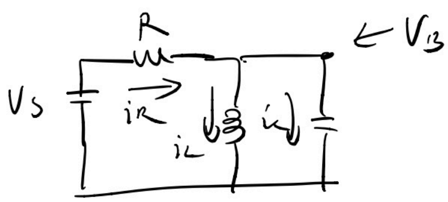
$\therefore$ KCL: $$\frac{V_s - V_o(t)}{R} = \frac{1}{L}\int_{-\infty}^{t} V_o(t)\,dt + C\frac{dV_o(t)}{dt}$$

(diff) $\Rightarrow$ $$C\frac{d^2V_o(t)}{dt^2} + \frac{1}{R}\frac{dV_o(t)}{dt} + \frac{1}{L}V_o(t) = 0$$

$$\Rightarrow \boxed{\frac{d^2V_o(t)}{dt^2} + \frac{1}{RC}\frac{dV_o(t)}{dt} + \frac{1}{LC}V_o(t) = 0}$$

b $\boxed{V_o(t=0^+) = V_B}$ $\boxed{i_R(t=0^+) = \frac{V_s - V_B}{R}}$

c



KCL @ $t=0^+$: $i_R = i_L + i_C$

$$\Rightarrow \frac{V_s - V_B}{R} = \frac{V_s}{R} + i_C$$

$$\Rightarrow i_C = -\frac{V_B}{R} = C\frac{dV_o}{dt}\Big|_{t=0^+} \Rightarrow \boxed{\frac{dV_o}{dt}\Big|_{t=0^+} = -\frac{V_B}{RC}}$$

d

$$\begin{cases} s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \\ \frac{1}{RC} = \frac{10^9}{5\times 4} = 5\times10^7 \\ \frac{1}{LC} = \frac{(10^9)^2}{400\times 4} = (2.5\times10^7)^2 \end{cases}$$

$\therefore$ $s^2 + 5\times10^7 s + (2.5\times10^7)^2 = 0$

$s^2 + 2\xi\omega_0 s + \omega_0^2 = 0$

$$\therefore \begin{cases} \boxed{\omega_0 = 2.5\times10^7} \\ \boxed{\xi = 1} \longrightarrow \boxed{\text{critical damp}} \end{cases}$$

e $V_0(t) = k_1 e^{s_1 t} + t k_2 e^{s_2 t}$

① $V_0(t=0) = V_B = \boxed{k_1 = 5}$

② $\left.\frac{dv_0}{dt}\right|_{t=0^+} = -\frac{V_B}{RC} = -\frac{5}{5\times4}\times10^9 = -2.5\times10^8 = s_1 k_1 + k_2$

$\Rightarrow -2.5\times10^8 = 5\times(-2.5\times10^7) + k_2$

$\Rightarrow \underline{k_2 = 1.25\times10^8 - 2.5\times10^8 = -1.25\times10^8}$

$$\therefore \boxed{\begin{aligned} V_0(t) &= 5e^{-2.5\times10^7 t} - t\cdot(1.25\times10^8)\, e^{-2.5\times10^7 t} \\ &= (5 - 1.25\times10^8 t)\, e^{-2.5\times10^7 t} \end{aligned}}$$

f)

$$V_o(t) = (5 - 1.25\times10^{8}t)\, e^{-2.5\times10^{7}t}$$

To find its minimum value, take derivative:

$$V_o'(t) = (-1.25\times10^{8})\, e^{-2.5\times10^{7}t} - 2.5\times10^{7}(5 - 1.25\times10^{8}t)\, e^{-2.5\times10^{7}t}$$

$V_o'(t) = 0$, get:

$$-1.25\times10^{8} = 2.5\times10^{7}(5 - 1.25\times10^{8}t)$$

$$\Rightarrow \quad 1.25\times10^{8} = 2.5\times1.25\times10^{15}t - 1.25\times10^{8}$$

$$\Rightarrow \quad 2.5\times10^{8} = 2.5\times1.25\times10^{15}t$$

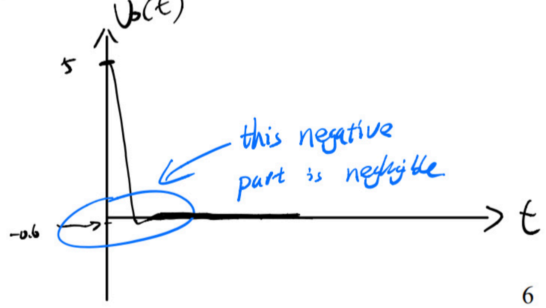
$$\Rightarrow \quad t = \frac{1}{1.25\times10^{7}} = 8\times10^{-8}$$

$\therefore$ Minimum of $V_o(t)$ is at $t = 8\times10^{-8}$.

$$\therefore \; V_{o,min} = V_o(t = 8\times10^{-8}) = \left.\frac{(5 - 1.25\times10^{8}t)}{e^{2.5\times10^{7}t}}\right|_{t=8\times10^{-8}}$$

$$= \frac{5 - 1.25\times8}{e^{2.5\times8\times10^{-1}}} = \frac{-5}{e^{2}} \approx -0.677 \ll 5.$$

$\therefore$ plot



Problem 2:
(a) Determine the Thevenin's equivalent of the circuit shown in Figure 2, looking into 1-1'.
(b) If a 10 Ohm resistance is connected between the terminals 1 and 1', how much current will flow through it, and in which direction? **Hint**: Use the Thevenin's equivalent.

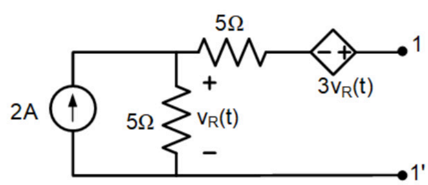
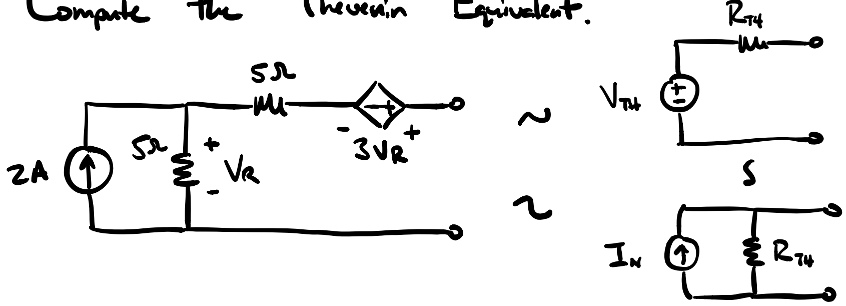


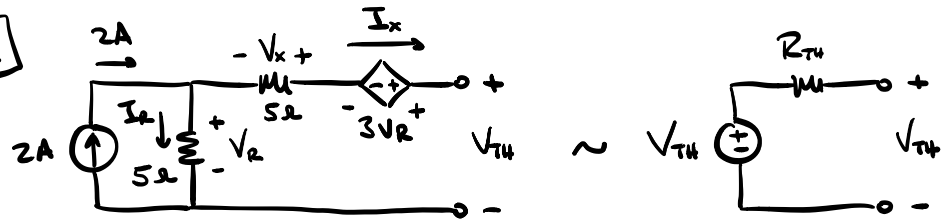
Figure 2

(12 + 6 = 18 points)
Solution:

(a) Compute the Thevenin Equivalent.



$\boxed{V_{TH}}$



$I_x = 0\ A$ because no current flows through an open circuit.

Via Ohm's Law, $V_x = -5I_x = 0$.

Applying KVL, it follows that:

$$V_{TH} = 3V_R + \cancel{V_x}^{0} + V_R = 4V_R$$

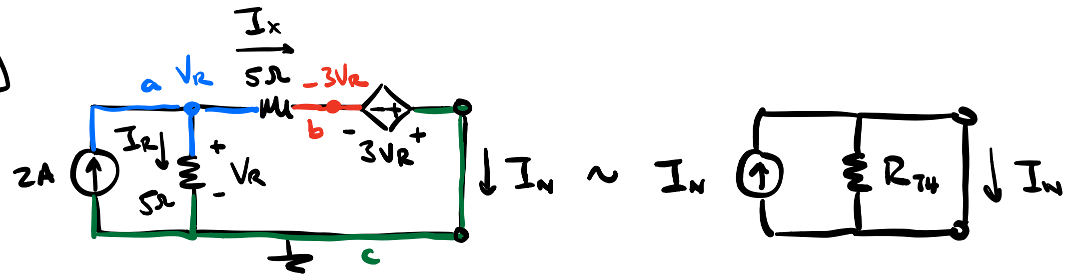
Applying KCL, it follows that:

$$2\ A = I_x + I_R = I_R$$

$$V_R = 5I_R = 10\ V$$

Hence, $V_{TH} = 40V$.

$\boxed{I_N}$



Apply KVL. (Definition of voltage difference.)

$$V_{cb} = \cancel{V_c}^{\,0} - V_b = 3V_R \longrightarrow V_b = -3V_R$$

Apply KCL. $\quad V_a - 0 = V_R \qquad V_b - 0 = -3V_R$

$$2 = I_R + I_x$$

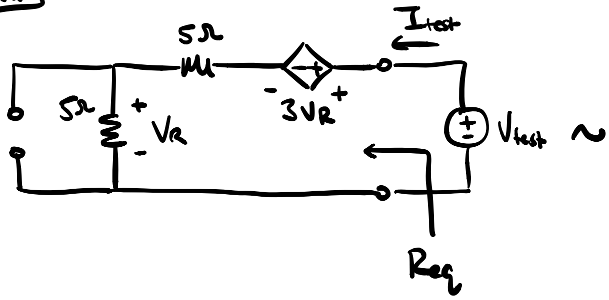
$$= \frac{V_R}{5} + \frac{V_R - (-3V_R)}{5} = V_R$$

and $I_x = I_N$ implies that:

$$I_N = \frac{V_R - (-3V_R)}{5} = \frac{8}{5}\text{ A}$$

$$R_{TH} = \frac{V_{TH}}{I_N} = \frac{40\text{V}}{8/5\text{A}} = 25\,\Omega$$

$\boxed{R_{TH}}$



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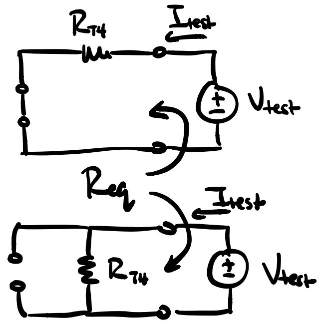


Via Thevenin's Theorem and Ohm's Law,

$$R_{TH} = R_{eq} = \frac{V_{test}}{I_{test}}$$

Apply KVL (and Ohm's Law).

$$V_{test} = 3V_R + 5I_{test} + 5I_{test}$$

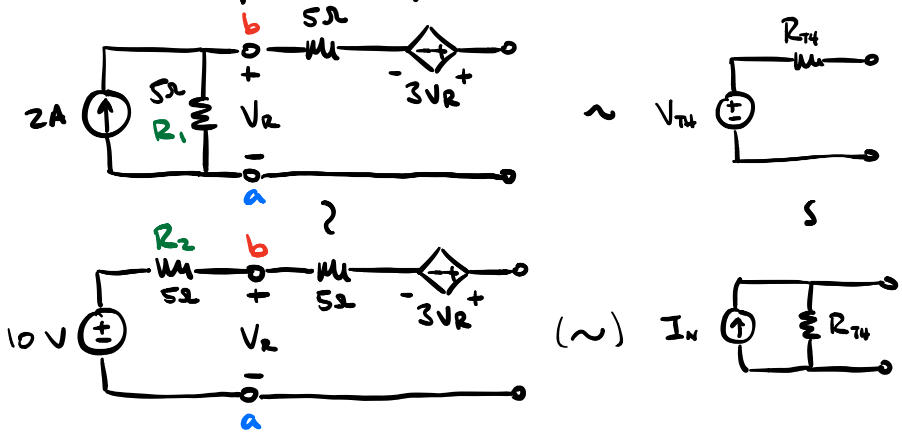
$$= 3(5I_{test}) + 5I_{test} + 5I_{test}$$

$$= 25\,I_{test}$$

$$\downarrow$$

$$R_{TH} = 25\,\Omega = \frac{V_{TH}}{I_N}$$

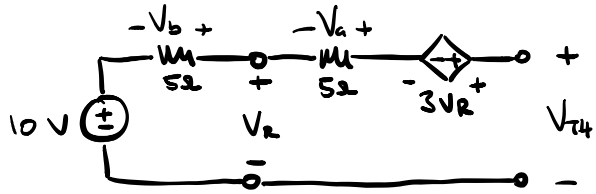
Alternatively, apply a source transform.



<u>Note.</u> $V_R = V_{ba}$ does <u>not</u> change. The original resistance ceases to exist just as the current source ceases to exist under Thevenin's transform.

Via the transitive property of the equivalence relation (∼), the source-transformed circuit can be represented by the same Thevenin / Norton equivalent.

$\boxed{V_{TH}}$



No current flows in the open circuit.

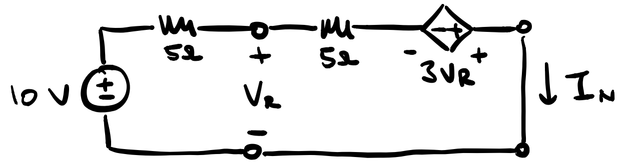
Apply KVL, we have:

$$V_R = 10 + \cancel{V_b} = 10\ V$$

$$V_{TH} = 3V_R + \cancel{V_a} + \cancel{V_b} + 10$$

$$= 40\ V$$

$\boxed{I_N}$



Applying KVL, it follows that:

$$V_R = 10 - 5I_N$$

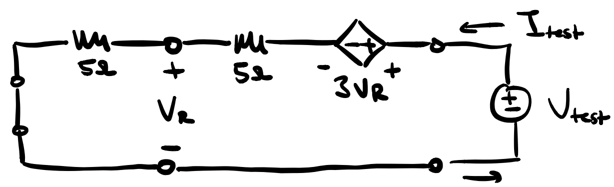
$$10 = 5I_N + 5I_N - 3V_R$$

$$= 10I_N - 3(10 - 5I_N)$$

$$= 25I_N - 30$$

$$I_N = \frac{40}{25} = \frac{8}{5}\ A$$

$\boxed{R_{TH}}$



Apply KVL (and Ohm's Law).

$$V_R = 5 I_{test}$$

$$V_{test} = 3V_R + 5I_{test} + 5I_{test}$$

$$= 25\, I_{test}$$

$$R_{TH} = 25\ \Omega = \frac{V_{TH}}{I_N}$$