Second-order Differential Equations:

* generally circuits with two energy storage devices (e.g. both a capacitor and an inductor)

Refresher (Math 33B)

$$\frac{d^2x(t)}{dt^2} + a_1\frac{dx(t)}{dt} + a_2x(t) = \underbrace{f(t)}_{\text{forcing function (excitation or stimulus)}}$$

we often use:

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2x(t) = f(t)$$

note: same equation as above

$a_1 = 2\zeta\omega_0$ OR $\omega_0 = \sqrt{a_2}$

$a_2 = \omega_0^2$ $\quad\zeta = \frac{a_1}{2\sqrt{a_2}}$

ω_0 = "resonant frequency"

ζ = "damping factor"

Homogenized Equation: $(f(t)=0)$

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_0\frac{dx(t)}{dt} + \omega_0^2x(t) = 0$$

- Characteristic Equation:
  - ↳ obtained assuming $x = e^{st}$ is a solution

$$s^2e^{st} + 2\zeta\omega_0 s\,e^{st} + \omega_0^2e^{st} = 0$$

$$\Rightarrow \boxed{s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0}$$

Quadratic eq. w/ 2 roots, s_1, s_2:

$$s_1, s_2 = \frac{-2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}}{2}$$

$$\Rightarrow \boxed{s_1, s_2 = \left[-\zeta \pm \sqrt{\zeta^2-1}\right]\omega_0} \quad ☆$$

↳ solution depends on value of ζ

• **Case #1 : Over-damped ($|\xi| > 1$)**

↳ roots (s_1, s_2) are real and distinct

general solution : $\boxed{x_g(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}}$

K_1, K_2 are integration constants, to be determined using boundary coefficients

• **Case #2 : critically-damped ($|\xi| = 1$)**

↳ roots are real but equal

$$s_1, s_2 = \left[-\xi \pm \sqrt{\xi^2 - 1}\right] \cdot \omega_0 = \underline{\underline{-\xi\omega_0}} \qquad (s_1 = s_2)$$

$$x_g(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$

$$\Rightarrow \boxed{x_g(t) = K_1 e^{-\xi\omega_0 t} + K_2 t e^{-\xi\omega_0 t}}$$

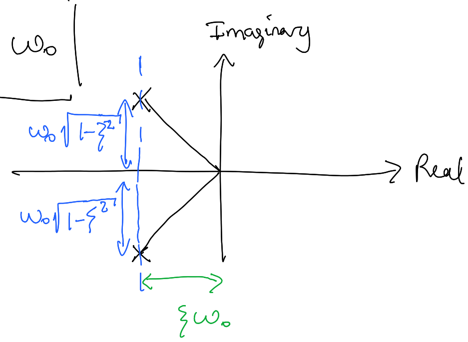
If you are curious why the 2nd solution is of the form $t e^{s_1 t}$, you can review this resource on "repeated roots":

https://www2.kenyon.edu/Depts/Math/Paquin/RepeatedRoots.pdf

• **Case #3 : under-damped ($|\xi| < 1$)**

$\Rightarrow \xi^2 - 1 < 0 \Rightarrow$ implies roots are complex conjugates

$$\Rightarrow \boxed{s_1, s_2 = \left[-\xi \pm j\sqrt{1-\xi^2}\right] \omega_0}$$



Alternatively, we can write the roots as:

$$\boxed{s_1, s_2 = -\sigma \pm j\omega_n}$$

where:

$$\sigma = \xi\omega_0$$

$$\omega_n = \sqrt{1-\xi^2}\,\omega_0 \qquad (\omega_n : \text{known as the "natural frequency"})$$

general solution:

$$x_g(t) = K_1 e^{(-\sigma + j\omega_n)t} + K_2 e^{(-\sigma - j\omega_n)t}$$

$$\Rightarrow \boxed{x_g(t) = K_1 e^{-\sigma t}\cos(\omega_n t) + K_2 e^{-\sigma t}\sin(\omega_n t)}$$

OR...

$$\boxed{x_g(t) = 2Ke^{-\sigma t}\cos(\omega_n t + \phi)}$$

K, ϕ are integration constants

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* For all 3 cases (except $\zeta \le 0$), the general solution decays and vanishes to 0 as $t \to \infty$

$$\left(\text{e.g. } x_g(t\to\infty) \sim e^{-\zeta\omega_0 t} \to 0\right)$$

Aside: If $\zeta < 0$:

unstable $\left\{ x_g(t\to\infty) \sim e^{-\zeta\omega_0 t} \to \infty \quad (\text{since } \zeta < 0)\right.$

OR, $\zeta = 0$:

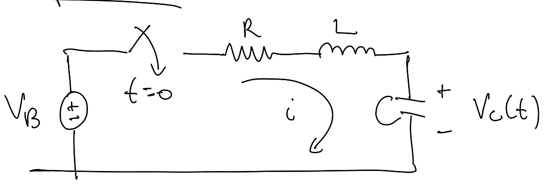
marginally stable $\left\{ x_g(t) \sim e^{-\zeta\omega_0 t}\cos(\omega_n t + \phi) \sim \cos(\omega_n t + \phi)\right.$

↳ implies: oscillating behavior

* we call these two scenarios:

1) $\zeta < 0$: unstable
2) $\zeta = 0$: marginally stable

Example #1: RLC



initial conditions provided:

(1) $V_c(0^+)$: voltage across the capacitor at $t=0$

(2) $i_L(0^+)$: current through the inductor @ $t=0$

KVL ($t \ge 0$):

$$V_B = \underbrace{i(t)R}_{V_R(t)} + \underbrace{L\frac{di(t)}{dt}}_{V_L(t)} + V_c(t)$$

note: $i(t) = C\dfrac{dV_c(t)}{dt}$

$$\Rightarrow V_B = RC\frac{dV_c(t)}{dt} + LC\frac{d^2V_c(t)}{dt^2} + V_c(t)$$

$$\Rightarrow \frac{d^2 V_C(t)}{dt^2} + \left(\frac{R}{L}\right)\frac{dV_C(t)}{dt} + \left(\frac{1}{LC}\right)V_C(t) = \frac{V_B}{LC}$$

① $2\zeta\omega_0 = \frac{R}{L}$ ② $\omega_0^2 = \frac{1}{LC}$

$$\Rightarrow \boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \qquad \boxed{\zeta = \frac{R/L}{2\omega_0} = \frac{1}{2}\frac{R}{\sqrt{L/C}}}$$

Example 1A: $(V_B = 10V,\ L = 1H,\ C = 0.25F,\ R = 5\Omega)$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1H \times 0.25F}} = 2 \text{ radians/s}$$

$$\zeta = \frac{1}{2}\frac{R}{\sqrt{L/C}} = \frac{1}{2}\frac{5\Omega}{\sqrt{\frac{1H}{0.25F}}} = \frac{5}{4} > 1$$

$\Rightarrow$ overdamped (case #1) $\zeta > 1$

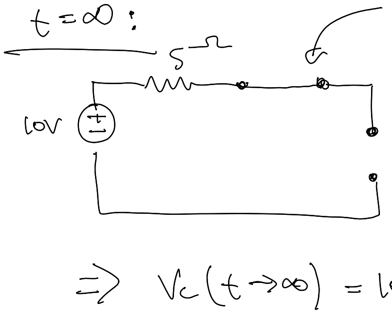
$$S_1, S_2 = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\omega_0 = \left[-\frac{5}{4} \pm \sqrt{\frac{25}{16} - 1}\right]2$$

$$= \left[-\frac{5}{4} \pm \frac{3}{4}\right] \times 2 = -1, -4$$

$$V_C(t) = \underbrace{K_1 e^{-t} + K_2 e^{-4t}}_{\text{general solution}} + \underbrace{K_3}_{\text{particular solution}}$$

* to solve for particular solution: look at DC steady-state

$t = \infty$:



inductor acts as short @ $t = \infty$

capacitor acts as open @ $t = \infty$

$$\Rightarrow V_C(t \to \infty) = 10V$$

$$V_C(t \to \infty) \approx K_1 e^{-\infty} + K_2 e^{-\infty} + K_3 = 10V$$

$\hookrightarrow K_3 = 10V$

- To find K_1, K_2 we use boundary conditions:

(1) $V_c(0^+) = K_1 + K_2 + \underbrace{K_3}_{10V}$

$\Rightarrow K_1 + K_2 + 10V = \underbrace{V_c(0^+)}$

<u>initial condition:</u>
initial voltage on the capacitor
(e.g. could be 0)

(2) $i_L(0^+) = C \left.\dfrac{dV_c(t)}{dt}\right|_{t=0^+}$

$\Rightarrow \left.\dfrac{dV_c(t)}{dt}\right|_{t=0^+} = \dfrac{i_L(0^+)}{C} \leftarrow$

<u>initial condition:</u>
initial current in the inductor
(e.g. could be 0)

Solution: $V_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + 10V$

<u>Differentiate:</u> $\dfrac{dV_c(t)}{dt} = K_1 s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t}$

given $\left.\dfrac{dV_c(t)}{dt}\right|_{t=0} = \dfrac{i_L(0^+)}{C}$

$\Rightarrow \boxed{K_1 s_1 + K_2 s_2 = \dfrac{i_L(0^+)}{C}}$

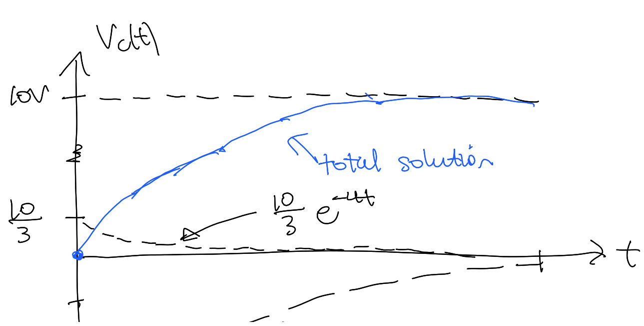
Suppose: $i_L(0^+) = 0A$ & $V_c(0^+) = 0V$

(1) $K_1 + K_2 + 10V = 0$
(2) $-K_1 - 4K_2 = 0$

$\boxed{K_2 = \frac{10}{3}, \quad K_1 = \frac{-40}{3}}$

<u>Solution:</u>

$$\boxed{V_c(t) = \frac{-40}{3} e^{-t} + \frac{10}{3} e^{-4t} + 10V}$$



<u>Example #1B</u> : what if $R = 4\Omega$, $L = 1H$, $C = 0.25F$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1H \times 0.25H}} = 2 \text{ rad/s}$$

$$\zeta = \frac{1}{2}\frac{R}{\sqrt{L/C}} = \frac{1}{2}\frac{4\Omega}{\sqrt{\frac{1H}{0.25F}}} = \underline{\underline{1}} \leftarrow \text{critically damped (case \#2)} \quad \star$$

$$S_2 = S_1 = -\zeta\omega_0 = -(1)(2 \text{ rad/s}) = -2$$

$$V_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} + K_3$$

$$V_c(t) = K_1 e^{-2t} + K_2 t e^{-2t} + K_3$$

- <u>final condition</u>:

  $V_c(\infty) = 10V$ (as shown in Example 1A)

  ↳ implies $\underline{\underline{K_3 = 10V}}$

- <u>initial conditions</u>:

  Let's assume : $V_c(0^+) = 0V$, $i_L(0^+) = 0A$

(1) $V_c(0^+) = K_1 + 0 + 10V = 0$

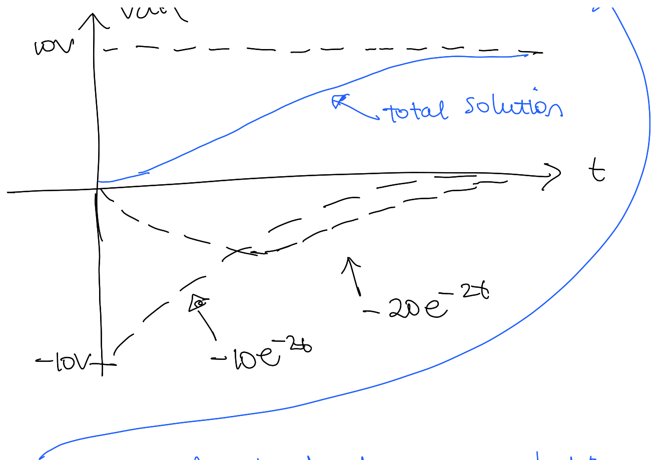
$\frac{d}{dt}(10V) = 0$

(2) $$\frac{dV_c(t)}{dt} = s_1 K_1 e^{s_1 t} + K_2\left(e^{s_1 t} + s_1 t e^{s_1 t}\right) + 0$$

$$\left.\frac{dV_c(t)}{dt}\right|_{t=0^+} = s_1 K_1 + K_2(1) + 10V = 0$$

$$\left.\begin{aligned} &(1)\ K_1 + 0 + 10V = 0 \\ &(2)\ -2K_1 + K_2 = 0 \end{aligned}\right\} \quad \begin{aligned} K_1 &= -10 \\ K_2 &= -20 \end{aligned}$$

$$\Rightarrow \boxed{V_c(t) = -10e^{-2t} - 20te^{-2t} + 10V} \quad \star$$



you can double-check your solution using wolfram alpha:

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c(t) = \frac{V_B}{LC}$$

$$\rightarrow \frac{d^2V_c}{dt^2} + 4\frac{dV_c}{dt} + 4V_c(t) = \frac{10}{0.25} = 40$$

wolframalpha.com:

search: $\boxed{V'' + 4V' + 4V = 40 \text{ and } V(0) = 0 \text{ and } V'(0) = 0}$

$\hookrightarrow$ solution: $V(x) = 10e^{-2x}(-2x + e^{2x} - 1)$

(note $t = x$)

same as derived solution ✓

Example 1C: $(R = 3\,\Omega,\ L = 1H,\ C = 0.25\,H)$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 2 \text{ rad/s}$$

$$\zeta = \frac{3}{2\sqrt{1/0.25}} = \frac{3}{4} \Rightarrow \text{underdamped } (\zeta < 1)$$

$$S_1, S_2 = \left[-\zeta \pm j\sqrt{1-\zeta^2}\right]\omega_0 = -\frac{3}{2} \pm j\underbrace{\frac{\sqrt{7}}{2}}_{\omega_n} \qquad \sigma = \frac{3}{2}$$

$$V_c(t) = K_1 e^{-\sigma t}\cos(\omega_n t) + K_2 e^{-\sigma t}\sin(\omega_n t) + K_3$$

final condition:

$V_c(\infty) = 10V$ (same as in Examples 1A & 1B)

initial conditions:

$V_c(0^+) = 0$

$i_L(0^+) = 0 \Rightarrow \left.\frac{dV_c}{dt}\right|_{t=0} = 0$

(1) $K_1 \cos(0) + K_2 \sin(0) + K_3 = 0$

$\hookrightarrow \quad K_1 + K_3 = 0 \qquad (K_3 = 10)$

$\rightarrow \boxed{K_1 = -10V}$

(2) $\frac{dV_c}{dt} = \left[K_1(-\sigma)e^{-\sigma t}\cos(\omega_n t) + K_1 e^{-\sigma t}(-\omega_n \cdot \sin(\omega_n t))\right]$

$+ \left[K_2(-\sigma)e^{-\sigma t}\sin(\omega t) + K_2 e^{-\sigma t}(\omega_n \cos(\omega_n t))\right] + \cancel{\frac{d}{dt}K_3}^{\,0}$

$\left.\frac{dV_c}{dt}\right|_{t=0} = [-\sigma K_1] + [\omega_n K_2] = 0$

$\sigma = \frac{3}{2} \qquad \omega_n = \frac{\sqrt{7}}{2} \qquad K_1 = -10$

$\Rightarrow K_2 = \frac{\sigma K_1}{\omega_n} = \frac{-15}{\left(\frac{\sqrt{7}}{2}\right)} = \frac{-30}{\sqrt{7}}$

Solution:

$$\boxed{V_c(t) = -10e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{30}{\sqrt{7}}e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right) + 10V}$$

double-check using wolframalpha.com:

$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{1}{LC}V_c(t) = \frac{V_B}{LC}$

$\rightarrow \frac{d^2V_c}{dt^2} + 3\frac{dV_c}{dt} + 4V_c(t) = \frac{10}{0.25} = 40$

wolframalpha.com:

search: $\boxed{V'' + 3V' + 4V = 40 \text{ and } V(0) = 0 \text{ and } V'(0) = 0}$

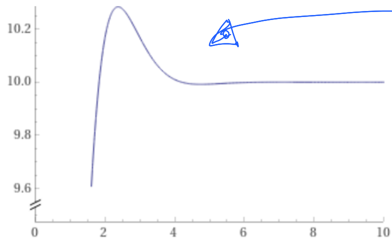
Solution:

$30\, e^{-(3x)/2} \sin\left(\frac{\sqrt{7}\,x}{2}\right) \ldots \left(\frac{\sqrt{7}\,x}{\ldots}\right)$ ✓ same as derived

$$v(x) = -\frac{[illegible]}{\sqrt{7}} - 10\,e^{-(3x)/2}\cos\left(\frac{[illegible]}{2}\right) + 10$$

V answer

$(t = x)$



underdamped case leads to what we call:

"overshoot"