Quiz #2 Solutions

Friday, May 21, 2021 7:07 AM

Problem 1

(a) $i_T(t) = i_1(t) + i_2(t)$

$$i_1(t) = 4\cos(\omega t + 30°)$$
$$i_2(t) = 5\sin(\omega t - 20°) = 5\cos(\omega t - 0° - 90°) = 5\cos(\omega t - 110°)$$

$$I_1 = 4\angle 30° = 4\cos(30°) + j4\sin(30°) = \frac{4\sqrt{3}}{2} + j\frac{1}{2}(4)$$
$$I_2 = 5\angle 110° = 5\cos(110°) + j5\sin(110°) \cong -4.7 - j1.7$$
$$I_T = I_1 + I_2 \cong (2\sqrt{3} + j2) + (-4.7 - j1.7) = 1.75 - j2.7$$
$$I_T = \sqrt{1.75^2 + 2.7^2} \angle \tan^{-1}\left(\frac{-2.7}{1.75}\right)$$

$$\boxed{\begin{aligned} I_T &= 3.218\angle -57° \\ i_T(t) &= 3.218\cos(\omega t - 57°) \end{aligned}}$$

(b) **(b) Using the phasor approach, determine the current i(t) in a circuit described by the following 2nd order differential equation.** *(5 points)*

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75°)$$

Let's recall:

for an inductor:

$$V = L\frac{di}{dt} \rightarrow V_L = j\omega L I_L$$

for a capacitor:

$$V = \frac{1}{C}\int i\,dt \qquad V_C = \frac{1}{j\omega C} I_C$$

transform eq. into phasor domain:

$$4i(t) + 8\int i(t)\,dt - 3\frac{di(t)}{dt} = 50\cos(2t + 75°)$$

$$\rightarrow \quad 4I + 8\frac{1}{j\omega} I - 3j\omega I = 50\angle 75^\circ$$

note: $8 = \frac{1}{C_{eq}}$ $\qquad$ $L_{eq} = 3H$

$\Rightarrow C_{eq} = \frac{1}{8} = 0.25H$

given $\omega = 2$:

$$I\left(4 - j\frac{8}{2} - j3(2)\right) = 50\angle 75^\circ$$

$$I = \frac{50\angle 75^\circ}{4 - j10} = \frac{50\angle 75^\circ}{\sqrt{4^2+10^2}\angle \tan^{-1}\left(\frac{-10}{4}\right)}$$

$$= \frac{50\angle 75^\circ}{10.77\angle -68.2^\circ} = \boxed{4.64\angle 143.2^\circ}$$

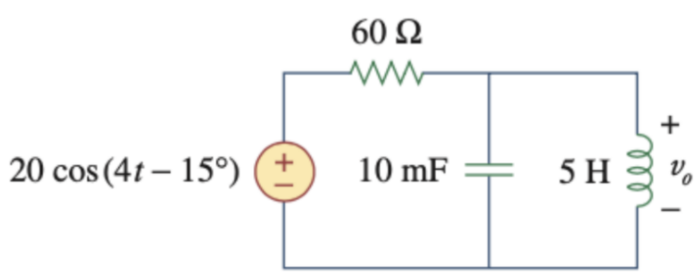
$$\rightarrow \boxed{i(t) = 4.64\cos(2t + 143.2^\circ)}$$

Note: this is known as the particular solution

OR steady-state solution

(does not require knowledge of initial conditions)

(c) **(c) Determine $v_o(t)$ in the following circuit.** *(5 points)*



* solve using voltage divider

$$v_o(t) = \frac{\left(\frac{1}{j\omega C}\right) \| j\omega L}{60\Omega + \left(\frac{1}{j\omega C} \| j\omega L\right)} \quad 20\angle -15^\circ$$

* $\omega = 4$:

$$\frac{1}{j\omega C} = \frac{1}{j(4)(0.01F)} = -j25$$

$$j\omega L = j(4)(5H) = j20$$

$$\frac{1}{j\omega C} \| j\omega L = \frac{(-j25)(j20)}{-j25 + j20} = \frac{500}{-j5} = j100$$

$$\rightarrow \quad v_o(t) = \left(\frac{j100}{60\Omega + j100}\right) 20\angle{-15^\circ}$$

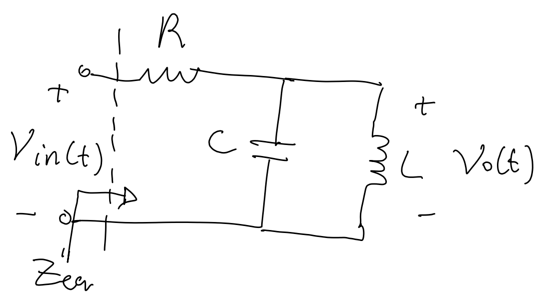
$$= \left(\frac{100\angle 90^\circ}{\sqrt{60^2+100^2}\angle\left(\tan^{-1}\left(\frac{100}{60}\right)\right)}\right) 20\angle{-15^\circ}$$

$$= \left(\frac{100\angle 90^\circ}{116.62\angle 59^\circ}\right) 20\angle{-15^\circ}$$

$$= (0.857\angle 31^\circ)(20\angle{-15^\circ}) = \boxed{17.15\angle 16^\circ}$$

$$\Rightarrow \quad \boxed{V_o(t) = 17.15\cos(4t + 16^\circ)\ [V]}$$

(d) **(d) What type of filter can be constructed using the circuit topology shown in part (c)?** *(1 point)*



① $$Z_{eq} = R + j\omega L \| \frac{1}{j\omega C} = R + \frac{\frac{L}{C}}{j\omega L - j\frac{1}{\omega C}}$$

$$Z_{eq} = R + j\frac{L}{C}\left(\frac{1}{\frac{1}{\omega C} - \omega L}\right)$$

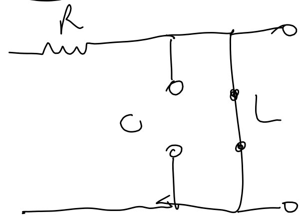
@ $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ (resonance):

$$Z_{eq} = R + jX \quad \text{where} \quad X \rightarrow \infty$$

$Z_{eq} \approx j\infty$

$$|V_{out}| = |V_{in}|$$

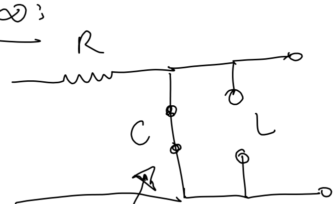
(2) $\underline{\omega = 0:}$



inductor acts as a short

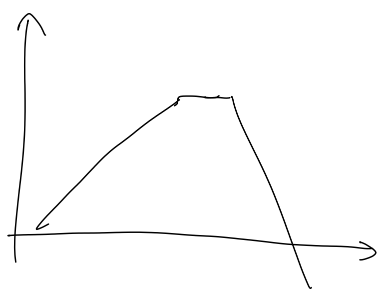
$\boxed{V_{out} \approx 0}$

(3) $\underline{\omega = \infty:}$



$\boxed{V_{out} \approx 0}$

capacitor acts as a short



$\boxed{\therefore \text{ bandpass filter}}$

(a) $V_S = V_R + V_L + V_C$

$$R i + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V_S$$

$$\Rightarrow L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dV_S}{dt}$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = V_m \cos(\omega t)\cdot \omega$$

$$\boxed{i_p = a e^{j\omega t}} \quad (1)$$

$$\Rightarrow L i'' + R i' + \frac{i}{C} = V_m \omega e^{j\omega t} \quad (4)$$

$$i_p' = a e^{j\omega t} j\omega \quad (2)$$

$$i_p'' = -a\omega^2 e^{j\omega t} \quad (3)$$

(1), (2), (3) in (4)

$$\Rightarrow -La\omega^2 e^{j\omega t} + j\omega R a e^{j\omega t} + \frac{a e^{j\omega t}}{C} = V_m \omega e^{j\omega t}$$

$$\Rightarrow a\left(-L\omega^2 + j\omega R + \frac{1}{C}\right) = V_m \omega$$

$$a\left(-L\omega + jR + \frac{1}{\omega C}\right) = V_m$$

$$a\left(R + j\left(L\omega - \frac{1}{\omega C}\right)\right) = -jV_0$$

$$\boxed{a = \frac{V_m \angle -90^\circ}{R + j\left(L\omega - 1/\omega C\right)}}$$

$$\therefore \boxed{V_R = aR = \frac{V_m R \angle -90^\circ}{R + j\left(L\omega - 1/\omega C\right)}}$$

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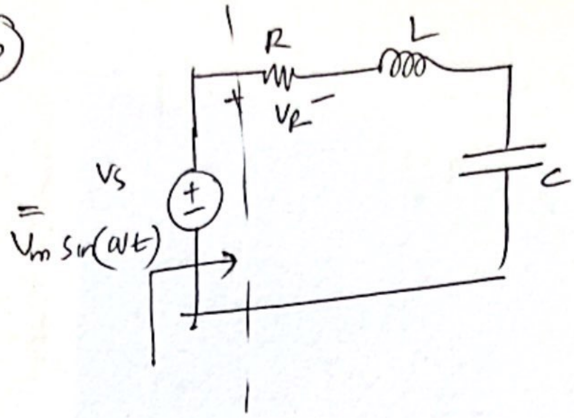
(b)



$V_m \sin(\omega t)$
$= V_m \cos(\omega t - 90^\circ)$

$$\mathbb{I} = \frac{V_s}{R + j\omega L + \frac{-j}{\omega C}} = \frac{V_m \angle -90^\circ}{\left(R + j\left(L\omega - \frac{1}{\omega C}\right)\right)}$$

$$\mathbb{V}_R = \mathbb{I} \cdot R = \frac{(V_m \angle -90^\circ) \cdot R}{\left(R + j\left(L\omega - \frac{1}{\omega C}\right)\right)} = \frac{V_m \cdot R \quad \angle -90}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2} \angle \tan^{-1}\left(\frac{L\omega - 1/\omega C}{R}\right)}$$

$$\mathbb{V}_R = \frac{V_m R}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}} \angle -90^\circ - \tan^{-1}\left(\frac{L\omega - 1/\omega C}{R}\right)$$

$$V_R(t) = \frac{V_m R}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}} \cos\left(\omega t - 90^\circ - \tan^{-1}\left(\frac{L\omega - 1/\omega C}{R}\right)\right)$$

$$V_R(t) = \frac{V_m R}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}} \sin(\omega t + \phi)$$

where $\phi = -\tan^{-1}\left(\frac{L\omega - 1/\omega C}{R}\right)$

(c) $$Z_s = R + j\omega + \frac{-j}{\omega C}.$$

(d) $$H(j\omega) = \frac{V_R(j\omega)}{V_s(j\omega)} = \frac{I \cdot R}{V_s} = \frac{V_s}{\left(R + j\omega L - \frac{j}{\omega C}\right)} \frac{R}{V_s}$$

$$H(j\omega) = \frac{R}{\left(R + j\omega L - j/\omega C\right)}$$

(e) $$|H(j\omega)| = \frac{R}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}}$$

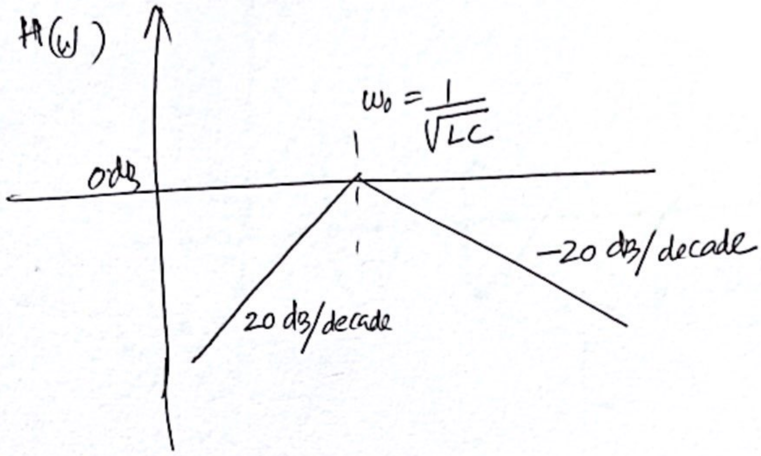
$$\angle H(j\omega) = \angle 0^\circ - \tan^{-1}\left(\frac{L\omega - \frac{1}{\omega C}}{R}\right)$$

$$= -\tan^{-1}\left(\frac{L\omega - \frac{1}{\omega C}}{R}\right).$$

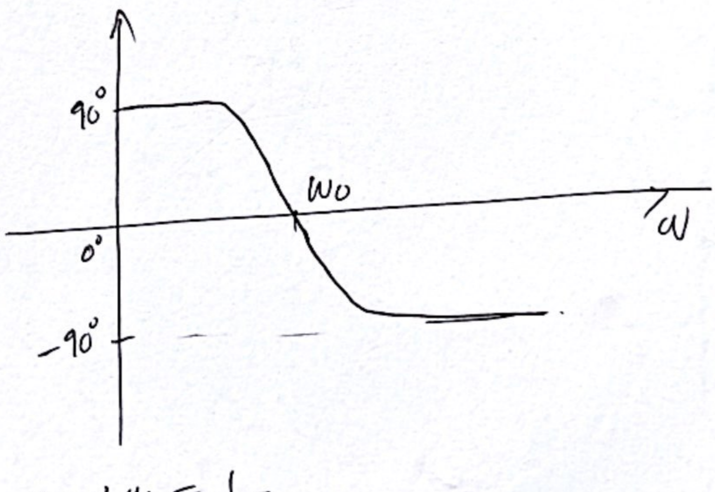
in dB

$$= 20\log\left(\frac{R}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}}\right)$$

(f)



$$\angle H(\omega) = 0^\circ - \tan^{-1}\left(\frac{L\omega - 1/\omega C}{R}\right)$$



(g) At $\omega_0 = L\omega_0 = \frac{1}{\omega_0 C}$

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

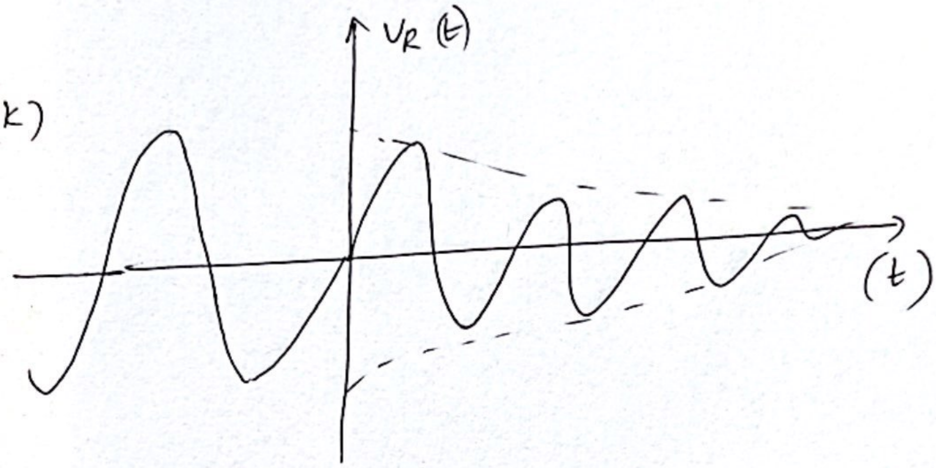
(h) ~~phase~~ o/p phase wrt to input at resonance = 0°

(i) $Q = \frac{L\omega}{R} = \frac{2\pi f_0 L}{R}$

(j) $B = \frac{f_0}{Q} = \frac{1}{2\pi\sqrt{LC}} \cdot \frac{1}{Q} = \frac{1}{2\pi\sqrt{LC}} \cdot \frac{\frac{R}{2\pi}}{\frac{1}{2\pi\sqrt{LC}}}$

$$\Rightarrow B = \frac{R}{2\pi L}$$

(k)

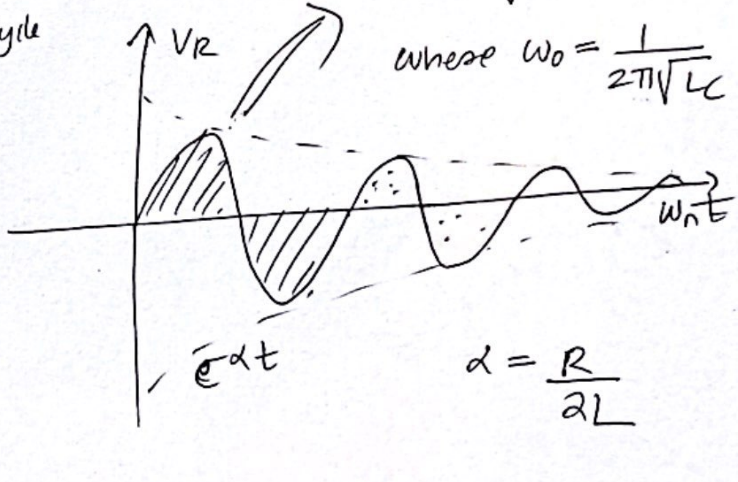


(l) Energy $E_1 = \int_{\text{first cycle}} L i\,dt = \int_{\text{first cycle}} L \frac{V_R}{R}\,di$

here $\omega_n = \sqrt{\omega_0^2 - \alpha^2}$

where $\omega_0 = \frac{1}{2\pi\sqrt{LC}}$

$E_1 = \frac{L}{R} \int_{\text{first cycle}} V_R$



$\alpha = \frac{R}{2L}$

$E_2 = \frac{L}{R} \int_{\text{second cycle}} V_R$

(m) $Q = \frac{2\pi \cdot \text{Energy stored per cycle}}{\text{Energy lost}} = \frac{2\pi E_1}{(E_1 - E_2)}$ (Note $Q = \frac{L\omega}{R} = \frac{2\pi f L}{R}$)

[$E_2 = E_1 - \Delta E$
where ΔE is the energy lost in first cycle]

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