Lecture 12

No class on Monday.

- Review of Fourier Series
- Compute FS coeff. using Laplace T.
- Response of LTI, C system due to periodic inputs.

periodic signals

$x(t) = x(t+T_0) = x(t+2T_0) = \cdots$

T_0 fundamental period

$\omega_0 = \frac{2\pi}{T_0}$ fundamental frequency $\left[\frac{rad}{s}\right]$

$f_0 = \frac{1}{T_0}$ [Hz] fundamental freq.

periodic w/ T_0

$\{e^{jn\omega_0 t}\} \rightarrow$ orthogonal basis. (Fourier basis)

$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle$

$$= \int_{T_0} e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t} \, dt$$

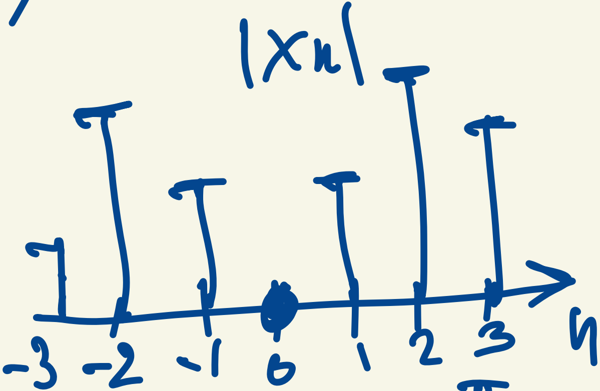
$$= \begin{cases} 0 & m \neq n \quad \text{orthog.} \\ T_0 & m = n \end{cases}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

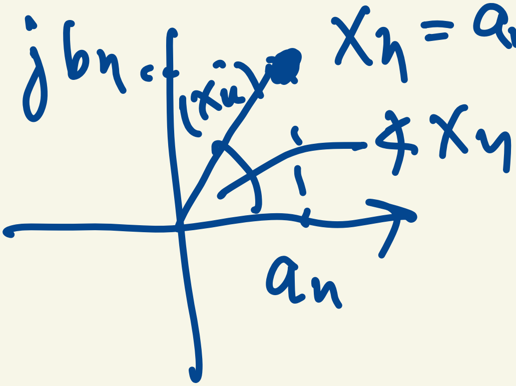
$$X_n = \langle x(t), e^{jn\omega_0 t} \rangle$$

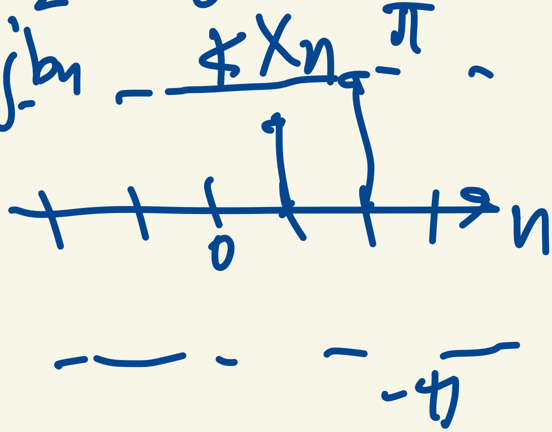
$$= \frac{1}{T_0} \int_{T_0} x(t) \cdot e^{-jn\omega_0 t} \, dt$$

$|X_n|$, $\measuredangle X_n$



$X_n = a_n + jb_n$





$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \quad \underline{\underline{DC}}$$

$|X_n|^2 \rightarrow$ power of nth harmonic

Parseval's Theorem for Fourier Series.

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt$$

$$= \sum_{n=-\infty}^{+\infty} |X_n|^2$$

Real signals $x(t) = x^*(t)$

$$\left.\begin{aligned} X_n^* &= X_{-n} \\ X_n &= X_{-n}^* \end{aligned}\right\}$$ complex conjugate symmetry.

$|X_n| = |X_{-n}|$ even symmetry in mag.

$\angle X_n = -\angle X_{-n}$ odd symmetry in phase.

Real FS.

① You still have to find X_n

$|X_n|$, $\measuredangle X_n$

$$x(t) = X_0 + 2\sum_{n=1}^{+\infty} |X_n| \cos(n\omega_0 t + \measuredangle X_n)$$

$Re\{X_n\}$

② $$a_n = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{1}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

$x(t)$ even $\quad b_n = 0$

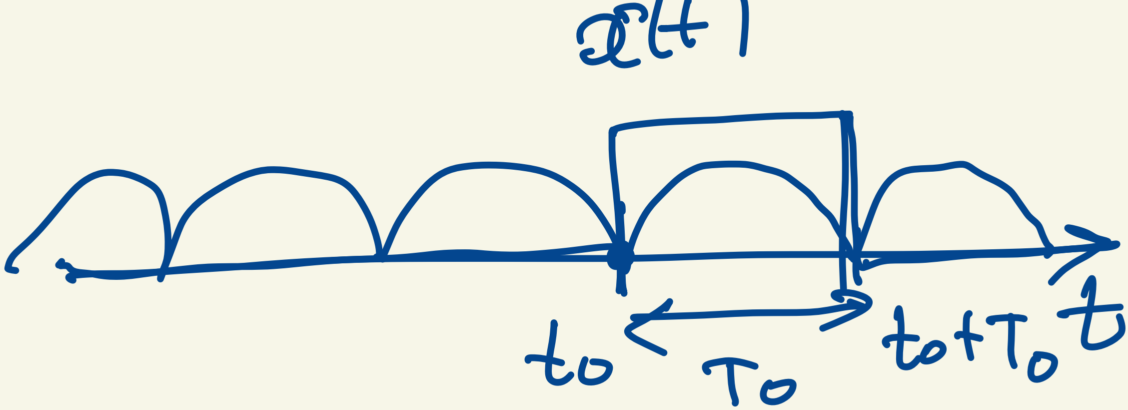
$x(t)$ odd $\quad a_n = 0$

$$x(t) = X_0 + 2\sum_{n=1}^{+\infty} a_n \cos(n\omega_0 t)$$

$$- 2\sum_{n=1}^{+\infty} b_n \sin(n\omega_0 t)$$

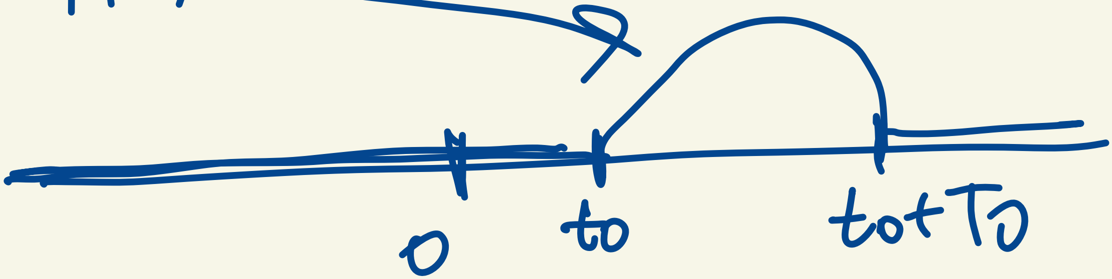
Using Laplace Transf.

to find FS coeff.



$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$x_1(t)$

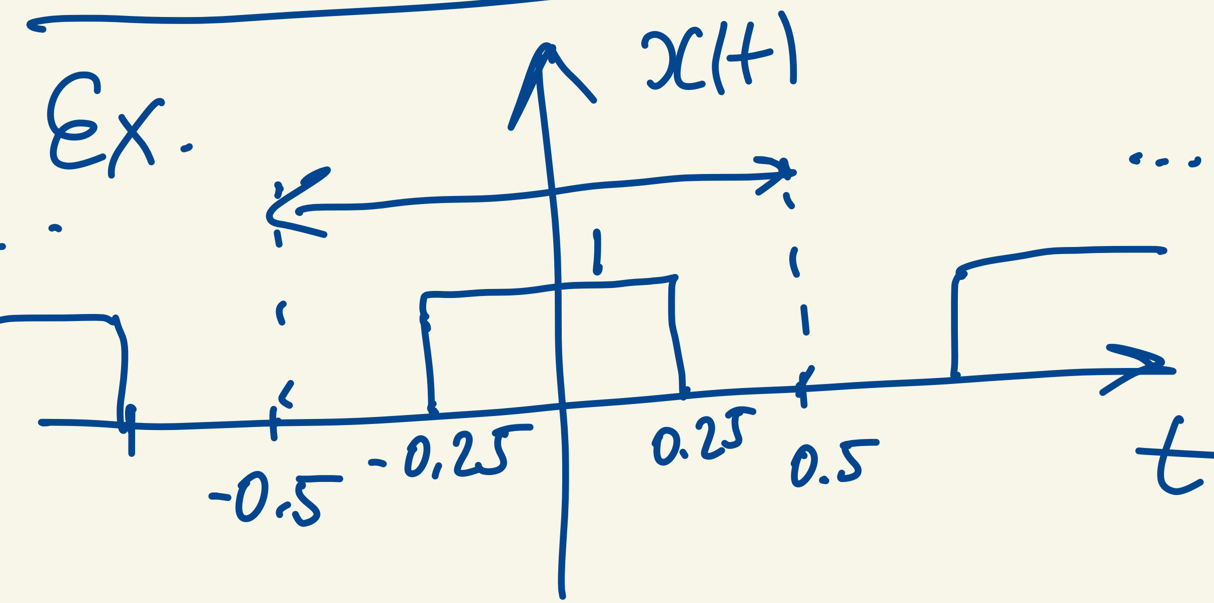


$$x_1(t) = x(t) \cdot [u(t-t_0) - u(t-t_0-T_0)]$$

$$X_1(s) = \int_{t_0}^{t_0+T_0} x(t) e^{-st} dt$$

$$X_n = \frac{1}{T_0} X_1(s)\Big|_{s=jn\omega_0}$$

Ex.



$T_0 = 1$ $\quad\omega_0 = 2\pi$

$$X_n = \frac{1}{1}\int_{-0.5}^{0.5} x(t)\, e^{-j2\pi nt}$$

$$= \int_{-0.25}^{0.25} e^{-j2\pi nt}\, dt$$

$$= -\frac{1}{j2\pi n} e^{-j2\pi nt}\Big|_{-0.25}^{0.25}$$

$$= -\frac{1}{j2\pi n}\left[e^{-j\frac{\pi}{2}n} - e^{j\frac{\pi}{2}n}\right]$$

$$= \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}$$

$$\boxed{X_n = \frac{\sin\left(\frac{\pi}{2}n\right)}{\pi n}}$$

$$x_1(t) = x(t)$$

$$-0.5 \leq t < 0.5$$

$$x_1(t - 0.25) =$$

$$= u(t) - u(t - 0.5)$$

$$X_n = \frac{1}{T_0} \underbrace{\int x_1(t) e^{-jn\omega_0 t} \, dt}$$

$$= \frac{1}{T_0} \underline{\underline{X_1(s)}} \Big|_{s = jn\omega_0}$$

$$e^{-0.25s} \quad X_1(s) = \frac{1}{s} - \frac{1}{s} e^{-0.5s}$$

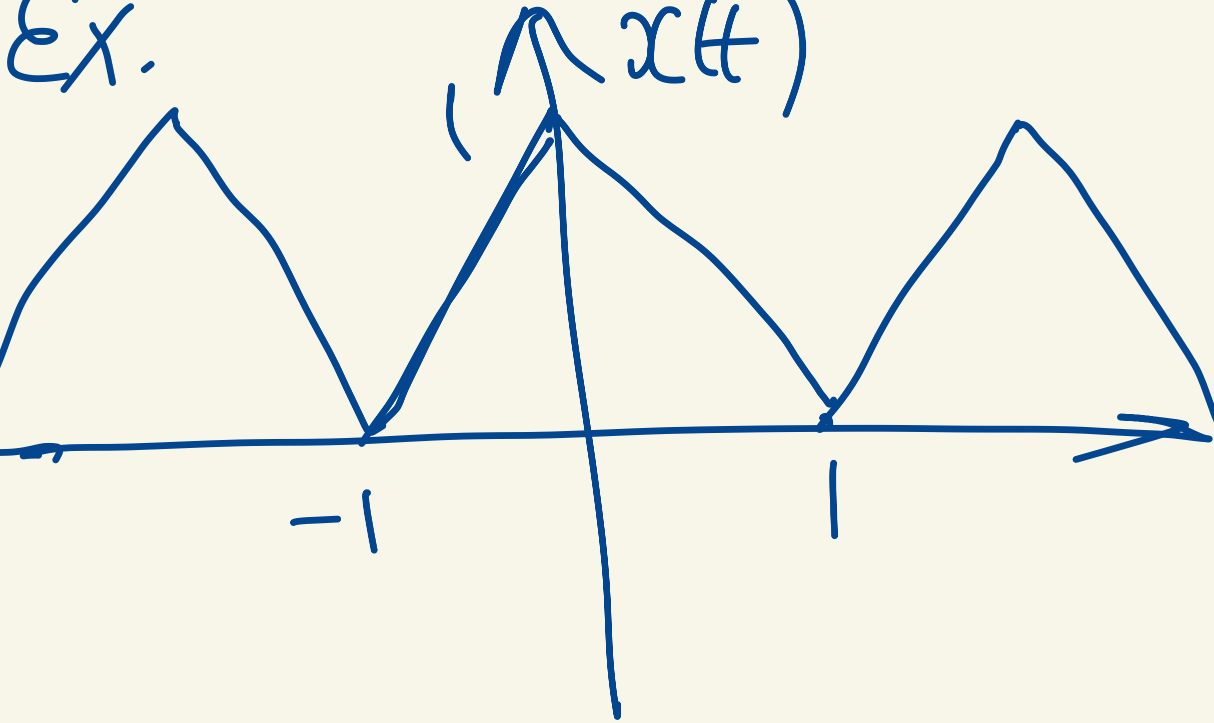
$$X_1(s) = \frac{e^{0.25s} - e^{-0.25s}}{s}$$

$$X_n = \frac{1}{1} \frac{e^{0.25 \cdot s} - e^{-0.25s}}{s} \Bigg|_{s = jn2\pi}$$

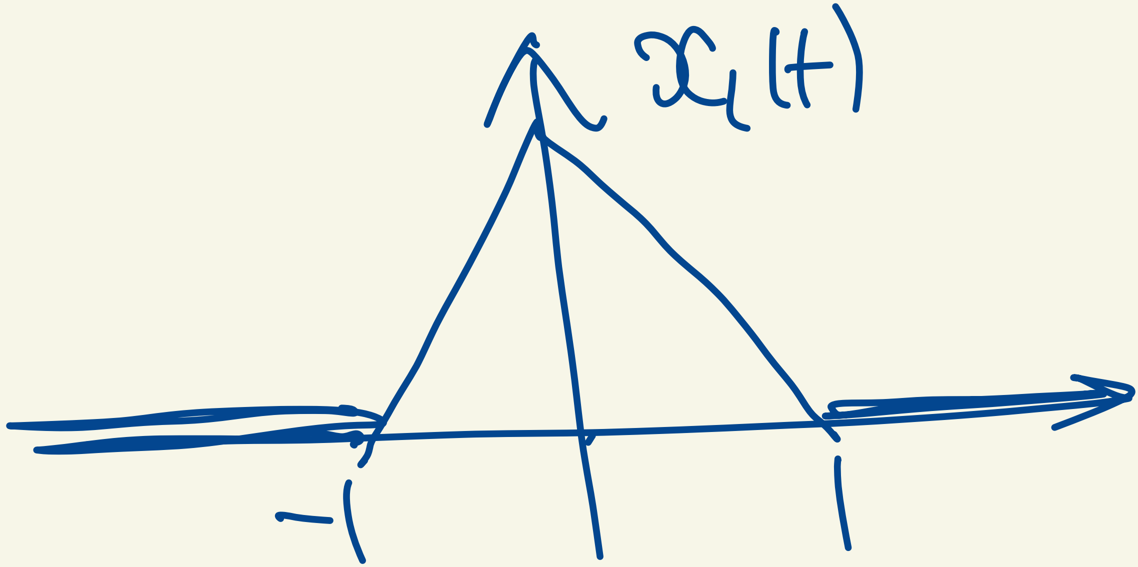
$$= \frac{e^{jn\frac{\pi}{2}} - e^{-jn\frac{\pi}{2}}}{jn2\pi}$$

$$= \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi}$$

Ex.



$T_0 = 2 \qquad \omega_0 = \pi$



$$x_1(t) = r(t+1) - 2r(t) + r(t-1)$$

$$x_1(t-1) = r(t) - 2r(t-1) + r(t-2)$$

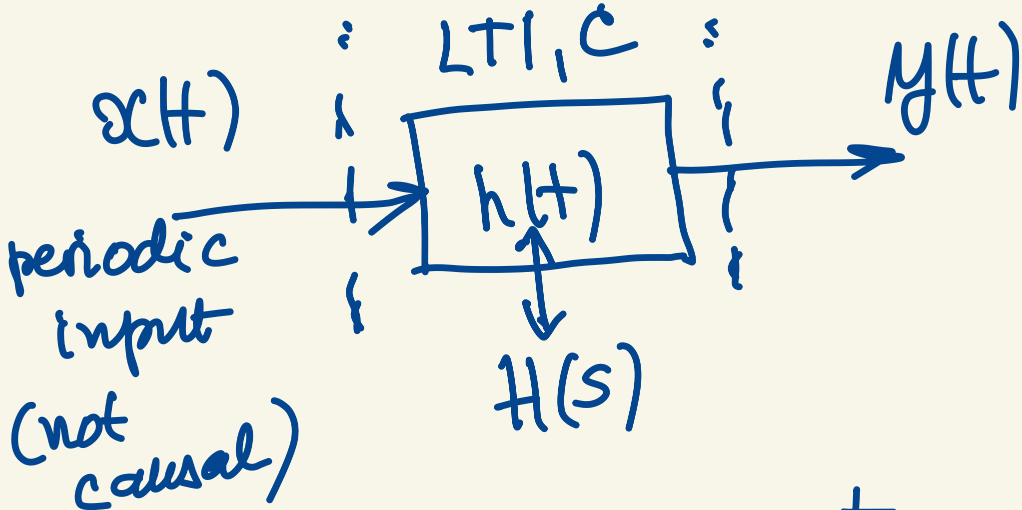
$$e^{-s} X_1(s) = \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}$$

$$X_1(s) = \frac{e^{s} - 2 + e^{-s}}{s^2}$$

$$X_n = \frac{1}{T_0} X_1(s) \Big|_{s = jn\pi}$$

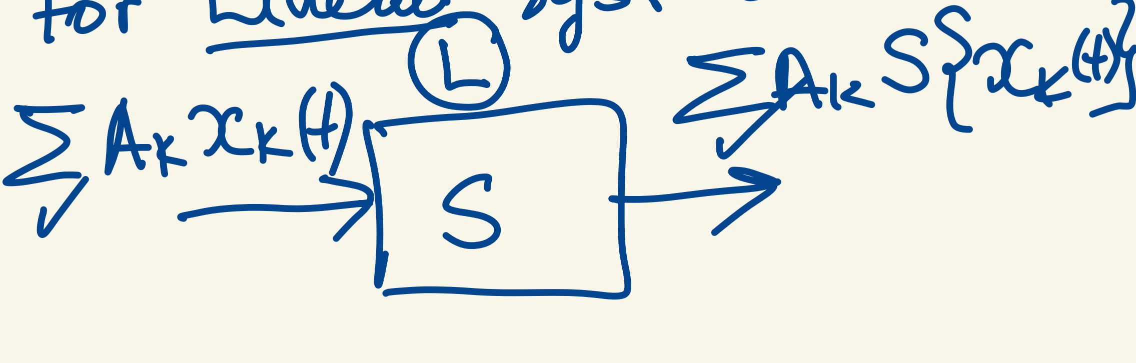
$$X_n = \frac{1}{2} \frac{e^{jn\pi} + e^{-jn\pi} - 2}{(-1)n^2\pi^2}$$

Response of LTI, C system to periodic inputs.



$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

For Linear systems.



$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{jn\omega_0 t}$$

$$\downarrow$$

$$\boxed{h(t),\ H(s)} \qquad \text{LTI, C}$$

$$\downarrow$$

$$y(t) = \sum_{k=-\infty}^{+\infty} X_k \cdot S\{e^{jk\omega_0 t}\}$$

$$e^{jk\omega_0 t} \xrightarrow{S} \boxed{\begin{matrix}\text{LTI, C} \\ h(t),\ H(s)\end{matrix}} \longrightarrow e^{jk\omega_0 t} \cdot H(jk\omega_0)$$

$$e^{s_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{s_0 t}\, \underline{\underline{H(s_0)}}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} X_k H(jk\omega_0) e^{jk\omega_0 t}$$

① output is periodic signal

② output period is the same as input.

ω_0 also fundamental freq.

③ Y_K are FS of $y(t)$.

$$Y_K = H(jk\omega_0) \cdot X_K$$

if $x(t)$ is real.

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\omega_0 t + \angle X_k)$$

✓

if $h(t)$ is real.

e.g. ✓ $h(t) = e^{-2t} u(t)$

then $y(t)$ is real.

$$y(t) = Y_0 + 2 \sum_{k=1}^{+\infty} |Y_k| \cos(k\omega_0 t + \angle Y_k)$$

$$Y_0 = H(0) \cdot X_0$$

$$|Y_k| = | H(jk\omega_0) \cdot X_k |$$

$$= |H(jk\omega_0)| \, |X_k|$$

$$Y_k = H(jk\omega_0) \cdot X_k$$

$$|Y_k| e^{j\angle Y_k} = |H(jk\omega_0)| \cdot e^{j\angle H(jk\omega_0)} \cdot |X_k| e^{j\angle X_k}$$

$$\angle Y_k = \angle X_k + \angle H(jk\omega_0)$$

$$y(t) = \underline{X_0 \cdot H(0)} + 2\sum_{k=1}^{+\infty} \underline{|X_k|} \cdot \underline{|H(jk\omega_0)|} \cdot \cos\left(k\omega_0 t + \measuredangle X_k + \measuredangle H(jk\omega_0)\right)$$

Example:

$$x(t) = 3 + 3\cos\left(2\pi t + \frac{\pi}{3}\right) + 4\cos\left(6\pi t + \frac{\pi}{4}\right)$$

$$h(t) = e^{-t} u(t)$$

$$y(t) = ?$$

$$H(s) = \frac{1}{s+1} \quad \text{ROC} \quad \text{Re}\{s\} > -1$$

Break lights.

$H(s)$ it has ROC.

$$H(jk\omega_0) = H(s)\Big|_{s=jk\omega_0}$$

$jk\omega_0$ has to be inside ROC of $H(s)$

$\Rightarrow$ BIBO stability.

$$x(t) = \overset{2|X_1|}{2} + 3\cos(\overset{1\cdot\omega_0}{2\pi t} + \frac{\pi}{3}) + 4\cos(\overset{3\cdot\omega_0}{6\pi t} + \frac{\pi}{4})$$

$\omega_0 = 2\pi$

$X_0 = 2$

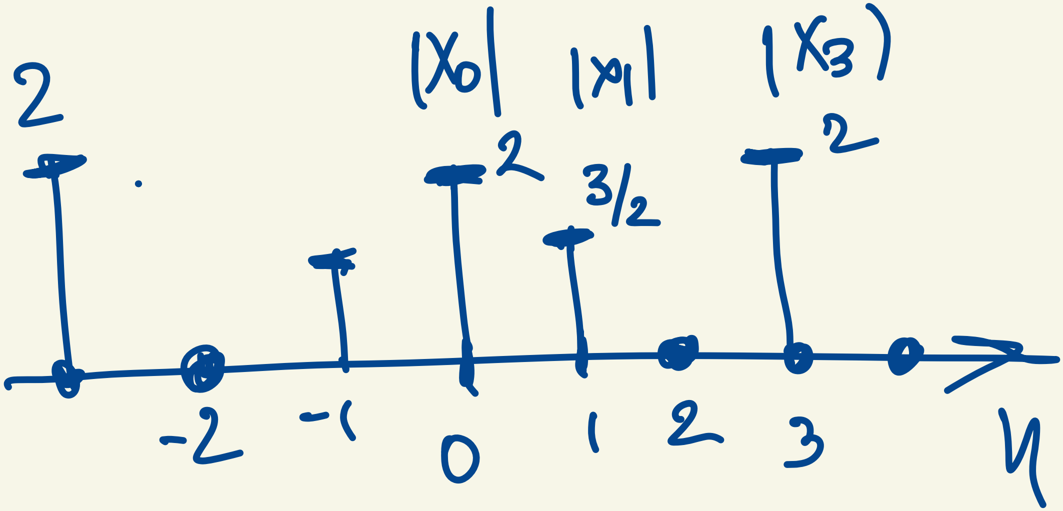
$2|X_1| = 3 \Rightarrow |X_1| = \frac{3}{2}$

$\measuredangle X_1 = \frac{\pi}{3}$

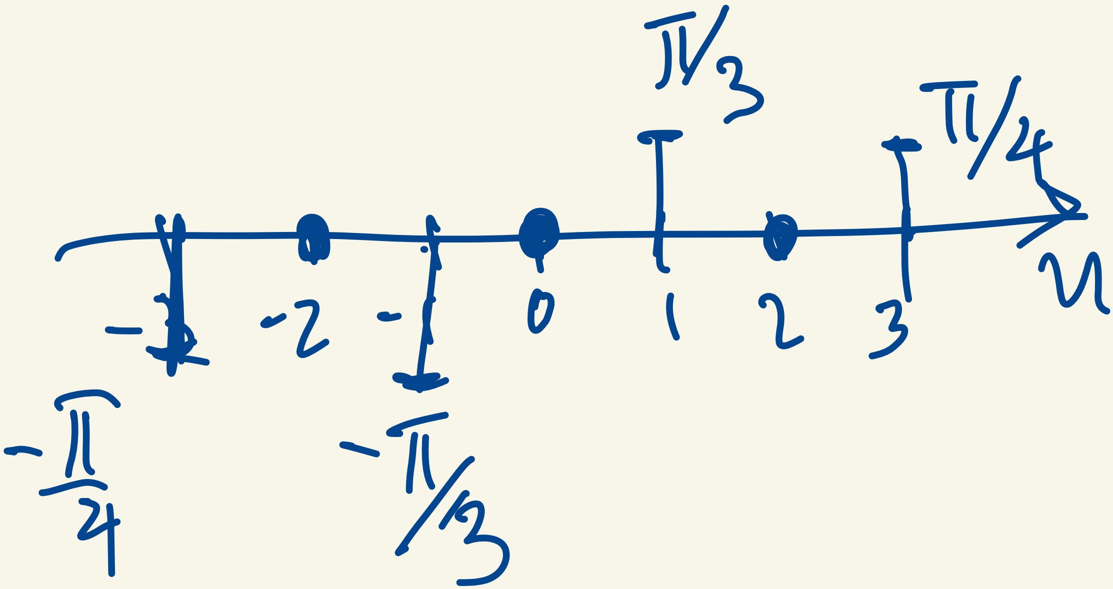
$X_2 = 0$

$2|X_3| = 4 \quad |X_3| = 2$

$\measuredangle X_3 = \frac{\pi}{4}$



|X0|
|X1|
|X3)
2
2
3/2
2
-2
-1
0
1
2
3
n



π/3
π/4
-3
-2
-1
0
1
2
3
n
-π/4
-π/3

$$Y_0 = X_0 \cdot H(0)$$

$$= 2 \cdot \frac{1}{s+1}\Big|_{s=0} = 2$$

$$\underline{|Y_1|} = \underline{|X_1|} \cdot \left|\underline{H(j\omega_0)}\right|$$

$$= \frac{3}{2} \cdot \left|\left|\frac{1}{s+1}\right|\right|_{s=j2\pi}$$

$$= \frac{3}{2} \left|\frac{1}{1+j2\pi}\right|$$

$$= \frac{3}{2} \frac{1}{\sqrt{1+4\pi^2}}$$

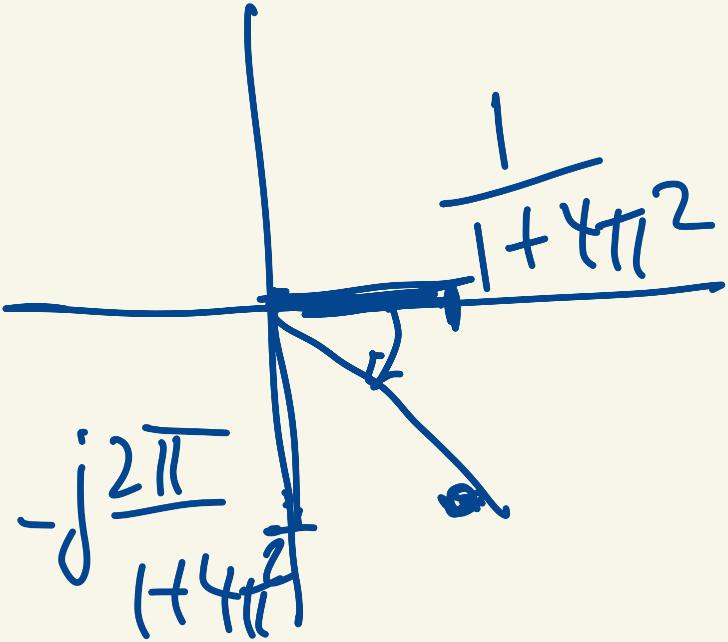
$$\cancel{x}\, Y_1 = \cancel{x}\, X_1 + \cancel{x}\, \overbrace{H(j2\pi)}^{Re+jIm.}$$

$$= \frac{\pi}{3} + \cancel{x}\, \frac{1}{1+j2\pi}$$

$H(j2\pi)$

$$\frac{1}{1+j2\pi} = \frac{1-j2\pi}{(1+j2\pi)(1-j2\pi)}$$

$$= \frac{1}{1+4\pi^2} - j\frac{2\pi}{1+4\pi^2}$$



$$tg(-\theta) = -tg\theta.$$

$$\measuredangle H_k = tg^{-1}\frac{Im\{H_k\}}{Re\{H_k\}}$$

$$= -\operatorname{tg}^{-1}(2\pi)$$

$$|Y_3| = |X_3| \cdot |H(j6\pi)|$$

$$= 2 \cdot \left| \frac{1}{1+j6\pi} \right|$$

$$= \frac{2}{\sqrt{1+36\pi^2}}$$

$$H(j6\pi) =$$

$$\frac{1}{1+j6\pi} = \frac{1-j6\pi}{1+36\pi^2}$$

$$\measuredangle H(j6\pi) = -tg^{-1}\, 6\pi$$

$$y(t) = 2 + 2|Y_1|\cos(2\pi t +$$

$$+ \measuredangle Y_1) +$$

$$+ 2|Y_3|\cos(6\pi t + \measuredangle Y_3)$$

$$\cos\left(2\pi t + \frac{\pi}{3}\right) = \frac{e^{j\left(2\pi t + \frac{\pi}{3}\right)} + e^{-j\left(2\pi t + \frac{\pi}{3}\right)}}{2}$$