Home Work 1

Nhat Ho

1) Given:

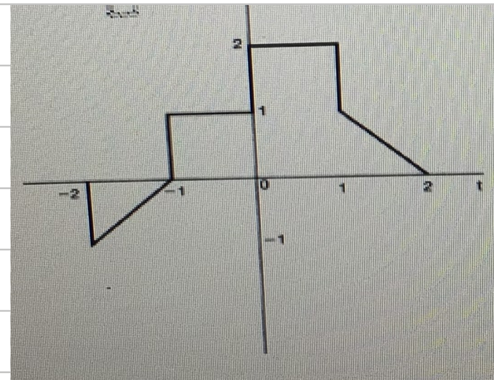
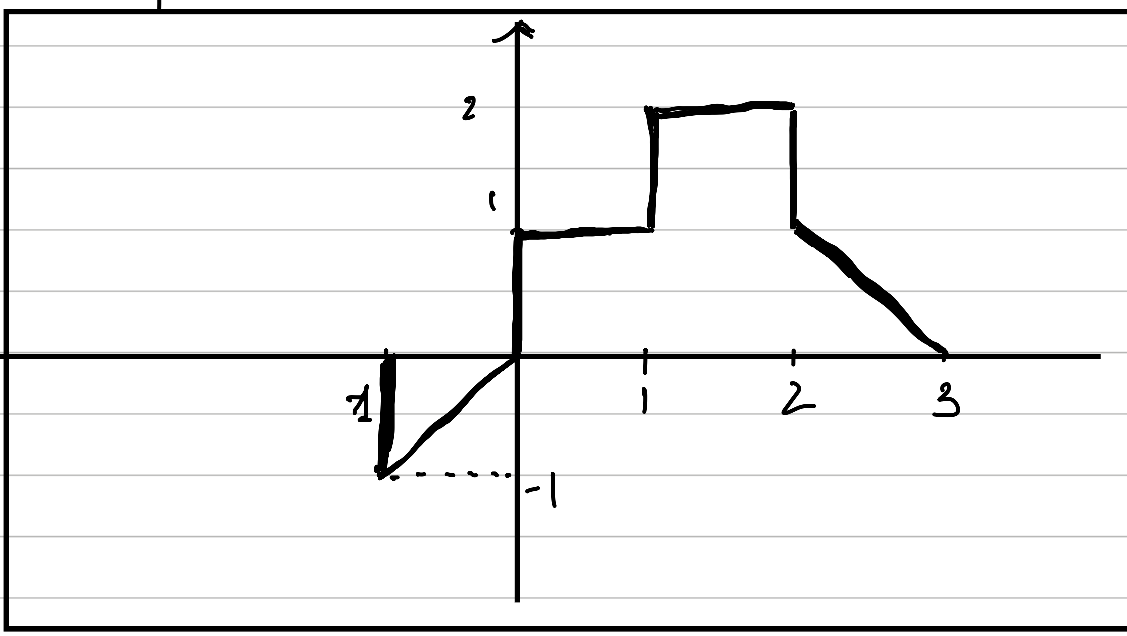


Figure 1: $x(t)$

a) $x(t-1)$

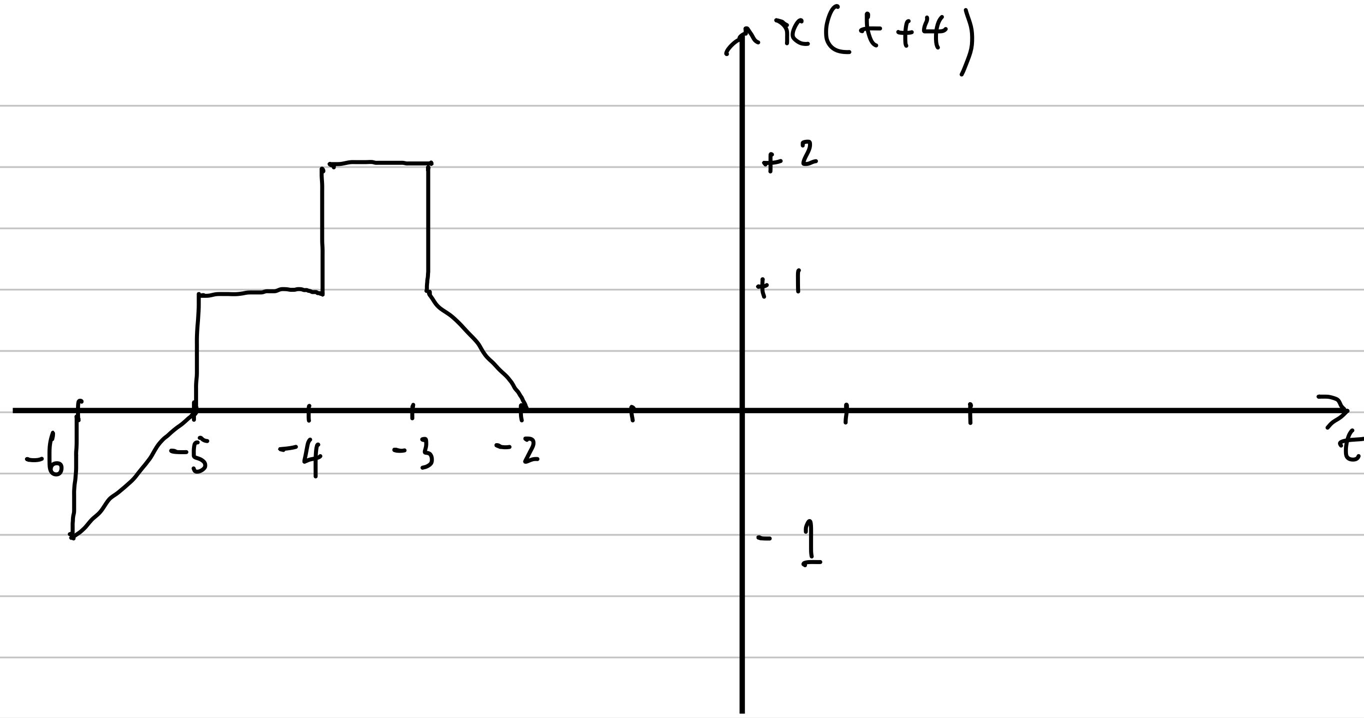
We obtain $x(t-1)$ by time shifting to the right 1 unit, then we have:

delay

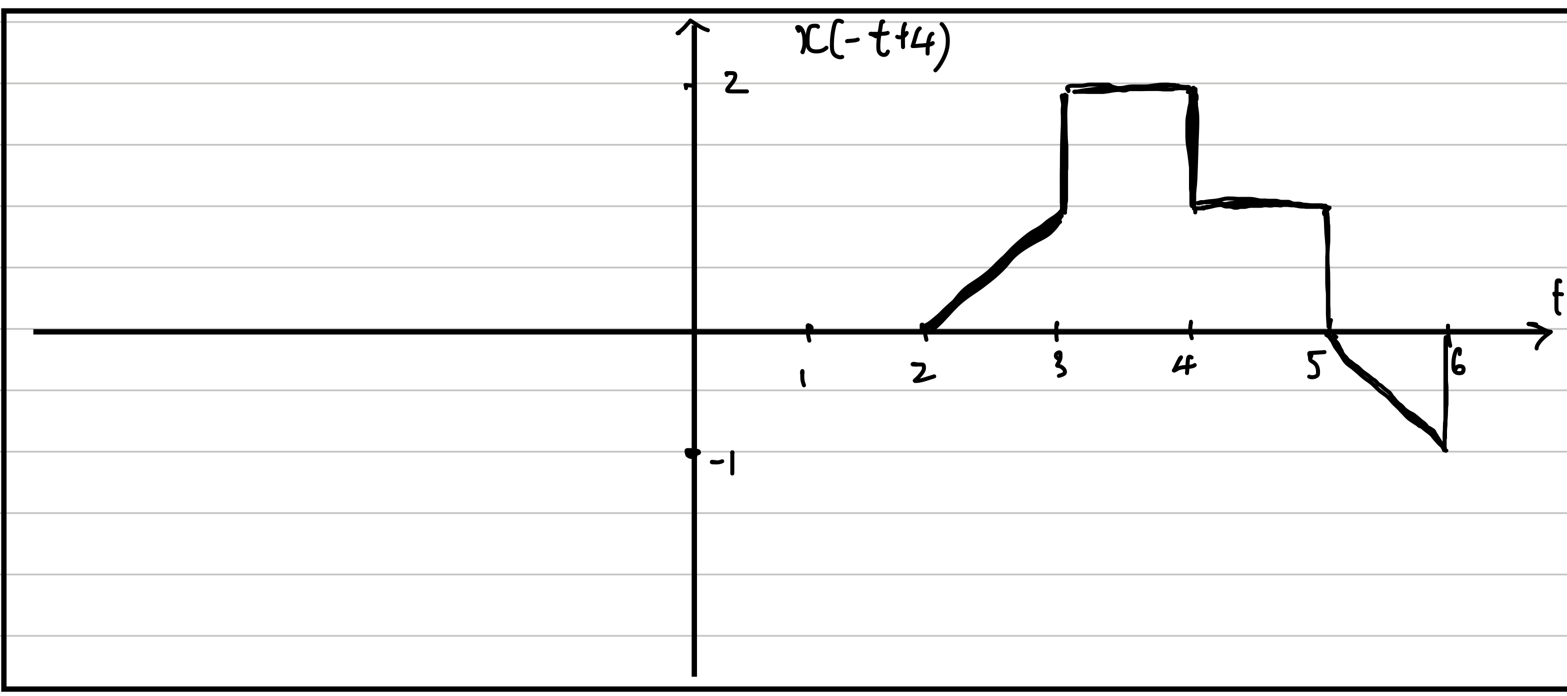


b) $x(4-t) = x(-t+4)$

Firstly, we obtain $x(t+4)$ by time shifting to the left (advanced), then we have:



⇒ get $x(-t+4)$ by scaling -1 (reflex), then we have $x(4-t)$ or $x(-t+4)$.



c) $\left[x(t-2) - 2x(-t+1)\right] u(t-\frac{1}{2})$ (1)

We have $u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$

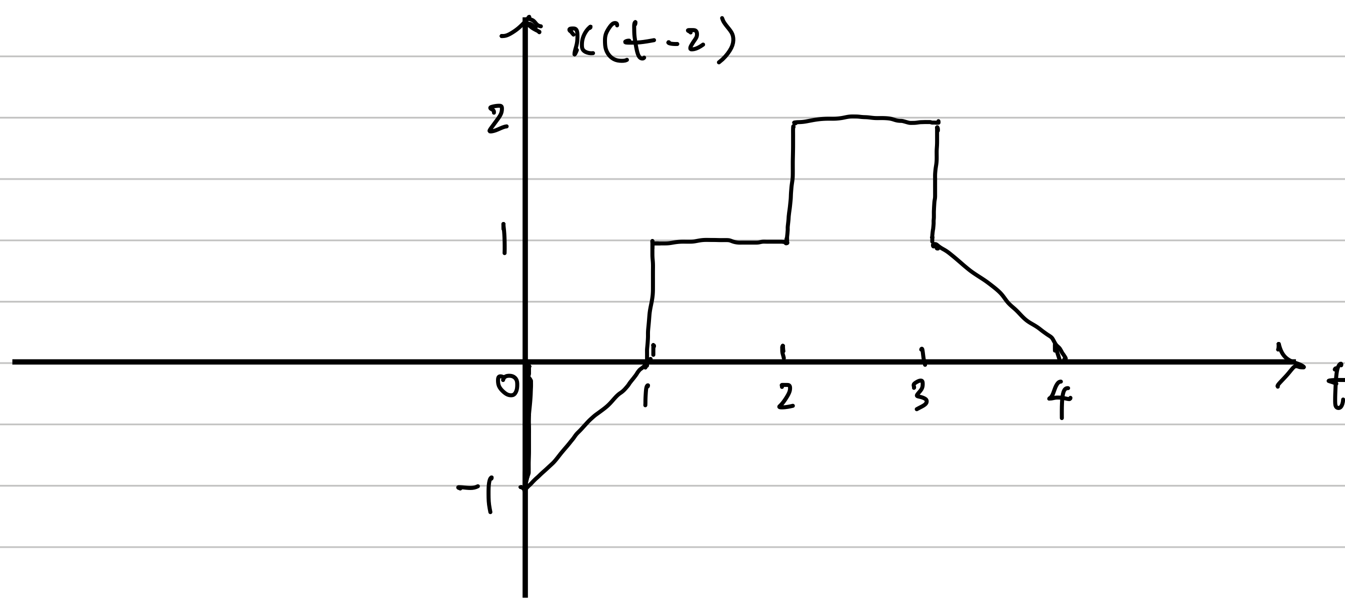
$$\Rightarrow \quad u(t-\tfrac{1}{2}) = \begin{cases} 1 & t > \frac{1}{2} \\ 0 & t < \frac{1}{2} \end{cases}$$

Also, ① $\Leftrightarrow$ $\underbrace{x(t-2)\,u(t-\tfrac{1}{2})}_{A} - \underbrace{2x(-t+1)\,u(t-\tfrac{1}{2})}_{B}$

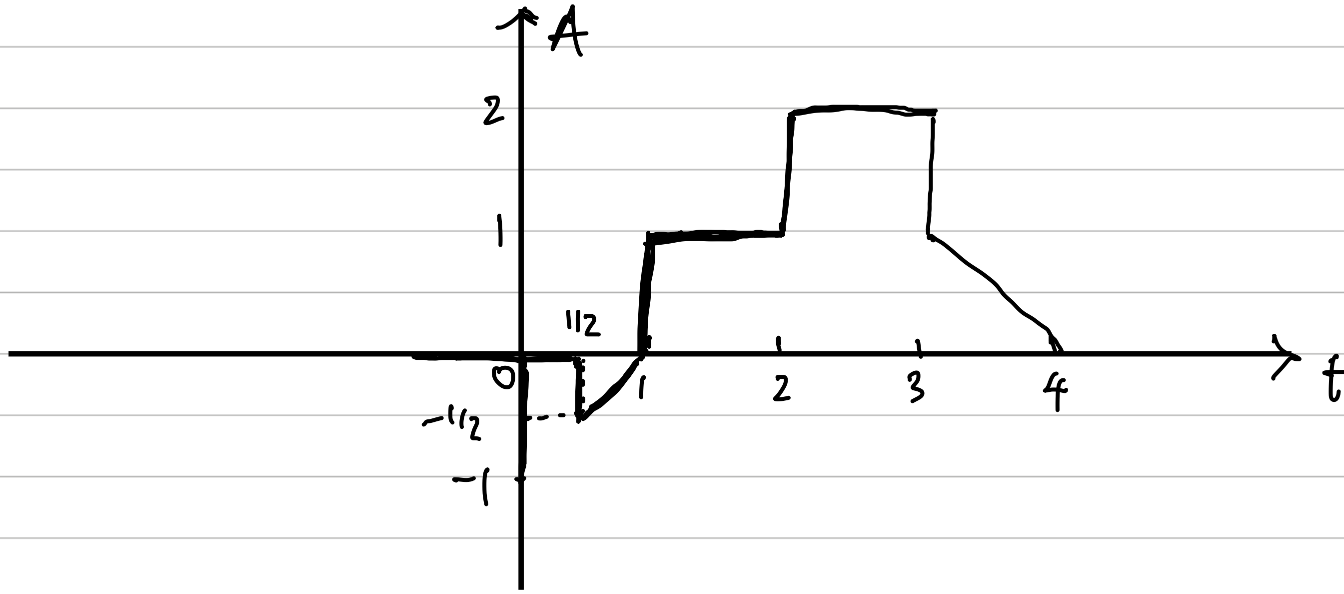
$$\Rightarrow A = \begin{cases} x(t-2) & t > \frac{1}{2} \\ 0 & t < \frac{1}{2} \end{cases}$$

$$\Rightarrow B = \begin{cases} 2x(-t+1) & t > \frac{1}{2} \\ 0 & t < \frac{1}{2} \end{cases}$$

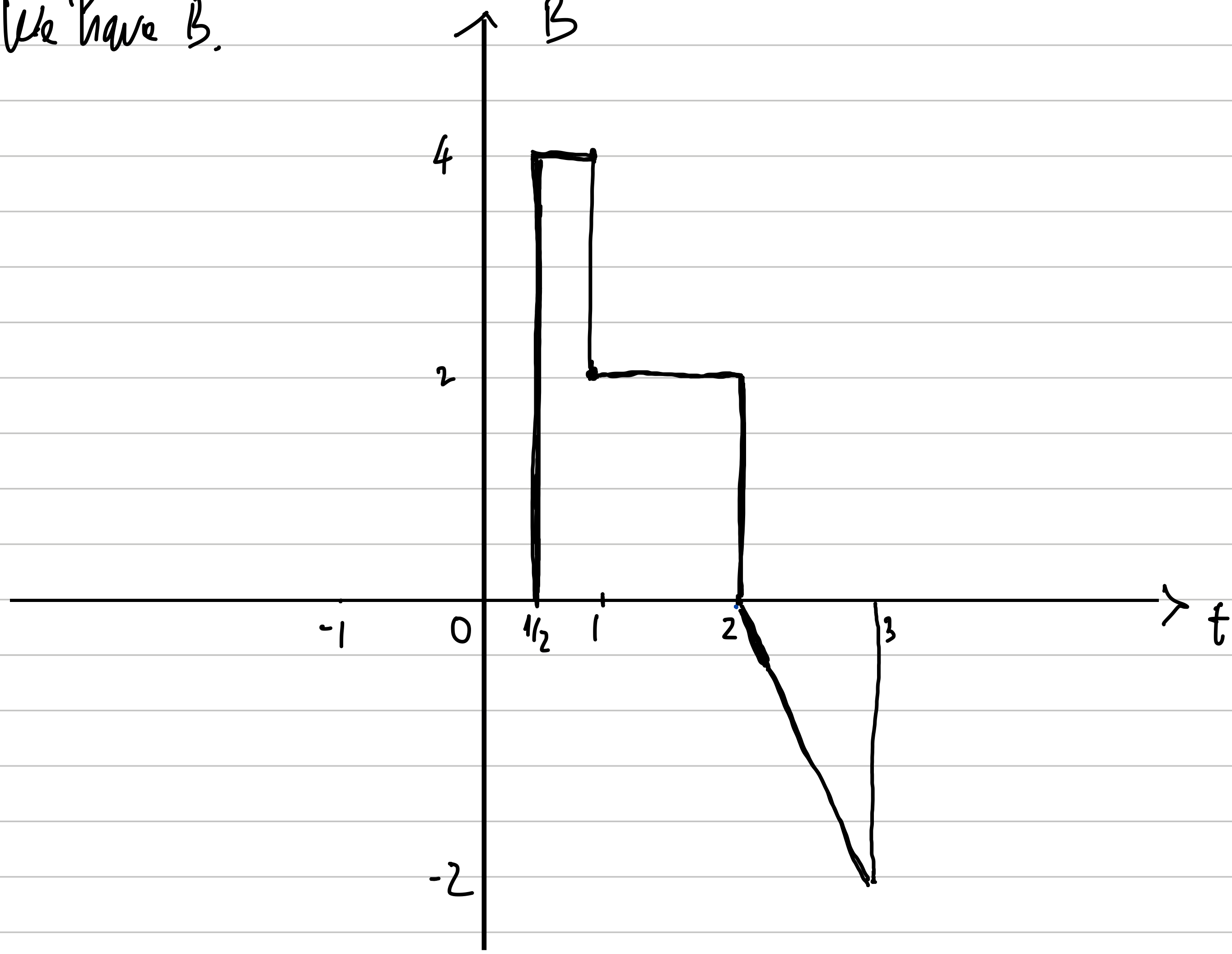
(*) We obtain $x(t-2)$ by time shifting to the right



Then we have A:

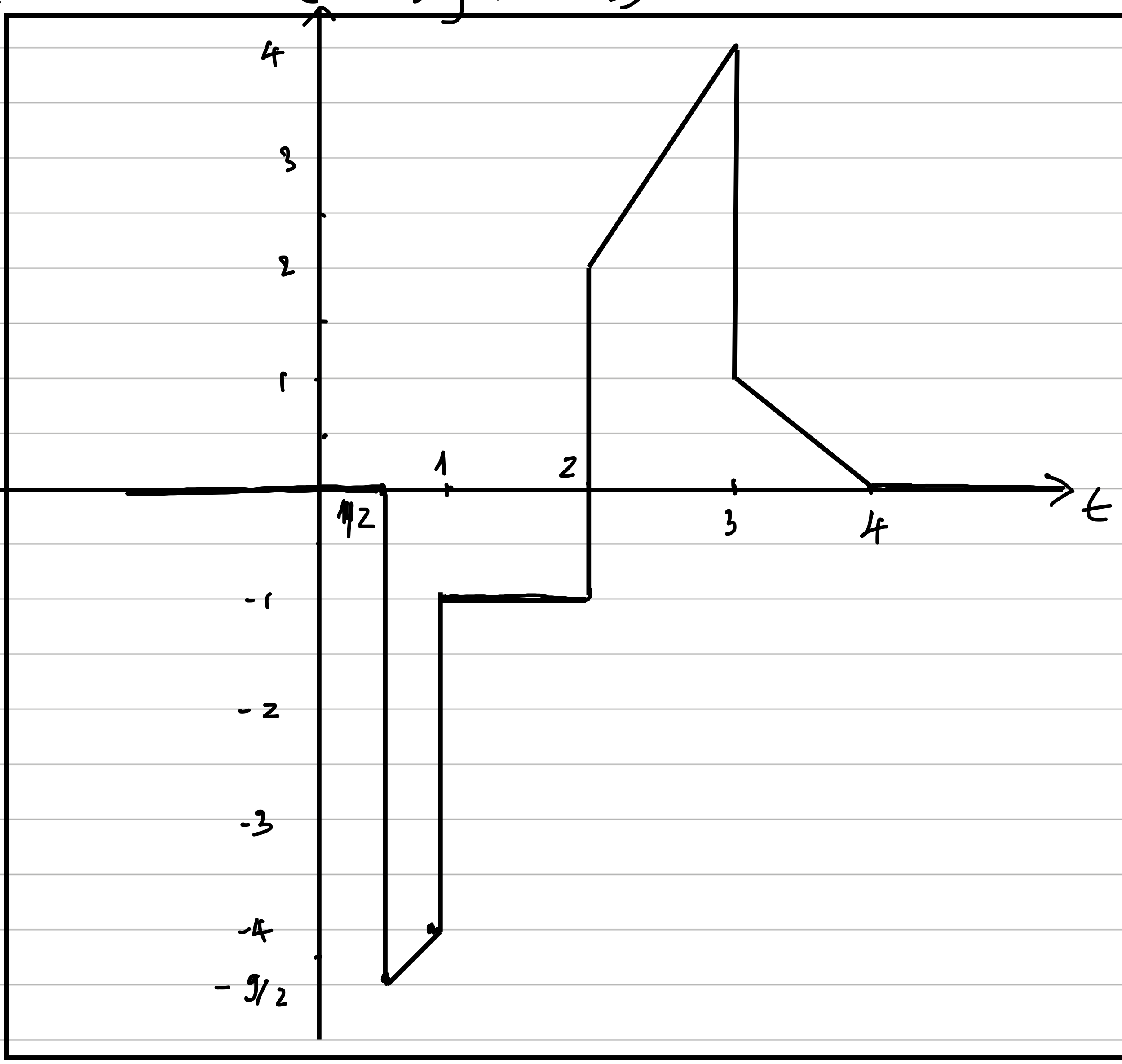


(*) We obtain $2x(-t+1)$ by time shifting to the left followed by scale -1. Then multiple for 2. And then we have B.



Based on A & B, we have the final signal

$$\left[x(t-2) - 2x(-t+1)\right]u(t-\tfrac{1}{2}) = A - B$$



Besides, we can recheck by using the definition of signal.

Based on $x(t)$, we have:

$$x(t) = \begin{cases} t+1 & -2 \leq t \leq -1 \\ 1 & -1 \leq t \leq 0 \\ 2 & 0 \leq t \leq 1 \\ -t+2 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t-2)u(t-1/2) = \begin{cases} t-1 & \frac{1}{2} \leq t \leq 1 \\ 1 & 1 \leq t \leq 2 \\ 2 & 2 \leq t \leq 3 \\ -t+4 & 3 \leq t \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$2x(-(t-1))u(t-\tfrac{1}{2}) = \begin{cases} 4 & 1/2 \leq t \leq 1 \\ 2 & 1 \leq t \leq 2 \\ -2t+4 & 2 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \text{Signal } (c) = \begin{cases} t-5 & 1/2 \leq t \leq 1 \\ -1 & 1 \leq t \leq 2 \\ 2t-2 & 2 \leq t \leq 3 \\ -t+4 & 3 \leq t \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

This is matched with the diagram we sketched above.

2) a) left side $= \cos(\theta t)\sin(\psi t)$

We have $e^{j\theta t} = \cos(\theta t) + j\sin\theta t$

$\Rightarrow e^{-j\theta t} = \cos(\theta t) - j\sin\theta t$

$\Rightarrow \begin{cases} \cos\theta t = (e^{j\theta t} + e^{-j\theta t})/2 \\ \sin\psi t = (e^{j\psi t} - e^{-j\psi t})/(2j) \end{cases}$

$\Rightarrow a(t) = \text{left side} = \dfrac{e^{j\theta t} + e^{-j\theta t}}{2} \times \dfrac{e^{j\psi t} - e^{-j\psi t}}{2j}$

$$= \frac{e^{j\theta t}e^{j\psi t} - e^{j\theta t}e^{-j\psi t} + e^{-j\theta t}e^{j\psi t} - e^{-j\theta t}e^{-j\psi t}}{4j}$$

$$= \frac{e^{jt(\theta+\psi)} - e^{jt(\theta-\psi)} + e^{-jt(\theta-\psi)} - e^{-jt(\theta+\psi)}}{4}$$

$$= \frac{\left[e^{jt(\theta+\psi)} - e^{-jt(\theta+\psi)}\right] - \left[e^{jt(\theta-\psi)} - e^{-jt(\theta-\psi)}\right]}{4j}$$

$$= \frac{2j\sin(\theta+\psi)t - 2j\sin(\theta-\psi)t}{4j}$$

$$= \frac{1}{2}\left[\sin(\theta+\psi)t - \sin(\theta-\psi)t\right] = \text{right side}$$

$\Rightarrow$ $\boxed{\text{We have proven successfully for part A.}}$

b) We have $a(t) = \frac{1}{2}\left[\sin(\theta+\psi)t - \sin(\theta-\psi)t\right]$

$$\Rightarrow a(t) = \underbrace{\frac{1}{2}\sin(\theta+\psi)t}_{x(t)} - \underbrace{\frac{1}{2}\sin(\theta-\psi)t}_{y(t)}$$

⊛ When $\theta = \psi$

$\Rightarrow a(t) = \frac{1}{2}\sin 2\theta t$ has period $T_0 = \frac{2\pi}{2\theta} = \frac{\pi}{\theta}$

$\Rightarrow a(t)$ is periodic with period $T_0 = \frac{\pi}{\theta}$

+ When $\theta = 2\pi \Leftrightarrow \psi = 2\pi$, then period $T_0 = \frac{\pi}{2\pi} = \frac{1}{2}$ is not equal 3.

$\Rightarrow$ When $\theta = \psi$, $a(t)$ can be periodic $T_0 = \frac{\pi}{\theta}$, but with $\theta = 2\pi$, could not find ψ to $a(t)$ has period of 3. (I)

⊛ When $\psi = 0 \Rightarrow a(t) = 0 \Rightarrow$ not periodic (II)

⊛ When $\theta \neq \psi$, $\psi \neq 0$

(next page)

With $x(t) = \frac{1}{2}\sin(\theta+\psi)t$ has $\omega_1 = \frac{2\pi}{T_1} = \theta+\psi$

$\Rightarrow T_1 = \frac{2\pi}{\theta+\psi} \Rightarrow x(t)$ is periodic with $T_1 = \frac{2\pi}{\theta+\psi}$ |

$y(t) = \frac{1}{2}\sin(\theta-\psi)t$ has $\omega_2 = \frac{2\pi}{T_2} = \theta-\psi$

$\Rightarrow T_2 = \frac{2\pi}{\theta-\psi} \Rightarrow y(t)$ is periodic with $T_2 = \frac{2\pi}{\theta-\psi}$

We have $\frac{T_1}{T_2} = \frac{2\pi}{\theta+\psi}\cdot\frac{\theta-\psi}{2\pi} = \frac{\theta-\psi}{\theta+\psi}$

Case 1: $\frac{T_1}{T_2} = N$ with N is an integer number

$\Rightarrow a(t) = x(t) - y(t)$ can be periodic with $T = T_1 = NT_2$.

+ With $\theta = 2\pi$ & $T = 3 \Rightarrow$ We have:

$T_1 = \frac{2\pi}{2\pi+\psi} = 3$ & $NT_2 = \frac{2\pi}{2\pi-\psi}N = 3$

$T_1 = \frac{2\pi}{2\pi+\psi} = 3 \Rightarrow 2\pi = 6\pi + 3\psi \Rightarrow \psi = \frac{-4\pi}{3}$

$\Rightarrow T_2 = \frac{2\pi}{2\pi-\psi} = \frac{2\pi}{2\pi+\frac{4\pi}{3}} = \frac{2\pi}{\frac{10\pi}{3}} = \frac{3}{5}$

$\Rightarrow \frac{T_1}{T_2} = \frac{3.5}{3} = 5 \Rightarrow N=5$ is a integer

So, in case 1, $a(t)$ can be periodic, and then with $\theta = 2\pi$, ψ will be $\boxed{-\frac{4\pi}{3}}$ to make $a(t)$ has a period of 3.

Case 2: $\frac{T_2}{T_1} = N$ with N is an integer number

Then $a(t)$ can be periodic with period $T = T_2 = NT_1$

+ With $\theta = 2\pi$ & $T = 3$

$$\Rightarrow T = T_2 = \frac{2\pi}{\theta - \psi} = \frac{2\pi}{2\pi - \psi} = 3 \Rightarrow 2\pi = 6\pi - 3\psi$$

$$\Rightarrow \psi = \frac{4\pi}{3} \quad \Rightarrow T_1 = \frac{2\pi}{\theta + \psi} = \frac{2\pi}{2\pi + \frac{4\pi}{3}} = \frac{2\pi}{10\pi/3} = \frac{3}{5}$$

$$\Rightarrow \frac{T_2}{T_1} = \frac{3}{3/5} = 5 \Rightarrow N = 5 \text{ is an integer number}$$

So, in case 2, $a(t)$ can be periodic, and then with $\theta = 2\pi$ & $T = 3$, ψ will be $\frac{4\pi}{3}$

Case 3: $\frac{T_1}{T_2} = \frac{M}{N}$, with $T_1 = \frac{2\pi}{\theta + \psi}$, $T_2 = \frac{2\pi}{\theta - \psi}$

$$\Rightarrow \frac{T_1}{T_2} = \frac{\theta - \psi}{\theta + \psi} = \frac{M}{N}, \quad \theta \neq \psi, \ \psi \neq 0$$

$\Rightarrow$ a(t) can be periodic with $T_0 = MT_2 = NT_1$ if M & N are integer number or $M \& N \in \mathbb{Z}$ $(M, N \neq 0)$

Let take an example while $\theta = 4\pi$ & $\psi = \pi \Rightarrow T_1 = \frac{2}{5}, T_2 = \frac{2}{3}$

$\Rightarrow \frac{T_1}{T_2} = \frac{3}{5} = \frac{M}{N} \Rightarrow M = 3, N = 5$ are both integer numbers.

Also, M & N are non-divisible integer $\Rightarrow$ the fraction

$\frac{T_1}{T_2} = \frac{\theta - \psi}{\theta + \psi} = \frac{M}{N}$, with M, & N are co-prime or have no common factor except 1. And $T_0 = MT_2 = NT_1 \Rightarrow T_0$ is divisible by T_1 & T_2 means $T_0 \% T_2 = 0$, $T_0 \% T_1 = 0 \Rightarrow T_0$ is a least Common Multiple of T_1 & $T_2 \Rightarrow$ the fundamental period of a(t):

$$T_0 = LCM(T_1, T_2) = \frac{LCM(\text{numerator of } T_1, \text{numerator of } T_2)}{HCF(\text{denominator of } T_1, \text{denominator of } T_2)}$$

$$= \frac{LCM(2\pi, 2\pi)}{HCF(\theta + \psi, \theta - \psi)} = \frac{2\pi}{HCF(\theta + \psi, \theta - \psi)}$$

* Use $\theta = 2\pi$, $T_0 = 3s$

$$\rightarrow T_0 = 3 = \frac{2\pi}{HCF(2\pi + \psi, 2\pi - \psi)} \Rightarrow HCF(2\pi + \psi, 2\pi - \psi) = \frac{2\pi}{3}$$

So, in case 3, a(t) can be periodic, and with $\theta = 2\pi$, $T_0 = 3$ the value of ψ satisfies $HCF(2\pi + \psi, 2\pi - \psi) = \frac{2\pi}{3}$ with $\psi \neq 0$, $\psi \neq 2\pi$

+ We realize that case 1 & 2 are special cases of case 3 when $\psi = \pm \frac{4\pi}{3}$. (III)

Then combine (I) → (III), we have a final conclusion.

$$\boxed{\begin{array}{l} a(t) = \frac{1}{2}\left[\sin(\theta+\psi)t - \sin(\theta-\psi)t\right] \text{ can be periodic. Then, } \theta = 2\pi \text{ \& } T = 3, \text{ the value} \\ \text{of } \psi \text{ need to satisfy } HCF(2\pi+\psi, 2\pi-\psi) = \frac{2\pi}{3} \\ \text{with } \psi \neq 0 \text{ \& } \psi \neq 2\pi \end{array}}$$

c) Given signal $x(t) = \underbrace{2\cos(10t+1)}_{x_1(t)} - \underbrace{\sin(4t-1)}_{x_2(t)}$

For periodicity, we don't need to worry about constant. We can ignore the phase also. So, the fundamental period T_0 of $2\cos(10t+1)$: $T_{01} = \frac{2\pi}{10} = \frac{\pi}{5}$

+ The fundamental period T_{02} of $\sin(4t-1)$:

$$T_{02} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Since $\frac{T_{01}}{T_{02}} = \frac{\pi}{5} \times \frac{2}{\pi} = \frac{2}{5} = \frac{M}{N}$, $M, N \in \mathbb{Z}$, $M, N \neq 0$

$\Rightarrow \begin{cases} M = 2 \\ N = 5 \end{cases} \Rightarrow$ $x(t)$ is periodic with fundamental

period $MT_{02} = NT_{01} = 2 \times \frac{\pi}{2} = 5 \times \frac{\pi}{5} = \pi$

$\Rightarrow$ the fundamental period of the signal

$x(t) = 2\cos(10t + 1) - \sin(4t - 1)$

is $T_0 = \pi \, (s)$

3) Given $x(t) = \cos(3\Omega_0 t) + 5\cos(\Omega_0 t)$, $-\infty < t < +\infty$, $\Omega_0 = \pi$

$$\Rightarrow x(t) = \underbrace{\cos(3\pi t)}_{x_1(t)} + \underbrace{5\cos(\pi t)}_{x_2(t)}$$

a) we have $x_1(t)$ is periodic with $T_{01} = \frac{2\pi}{3\pi} = \frac{2}{3}$

$x_2(t)$ is periodic with $T_{02} = \frac{2\pi}{\pi} = 2$

$$\Rightarrow \frac{T_{01}}{T_{02}} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} = \frac{M}{N} \quad \text{with } M = 1, N = 3 \in \mathbb{Z}$$

$$\rightarrow \boxed{x(t) \text{ is periodic with } T_0 = 3M = N = 3 \text{ (s)}}$$

b) $$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) \, dt$$

with $T = N T_0$

$$\Rightarrow P_x = \lim_{N \to \infty} \frac{1}{T_0} \int_0^{T_0} x^2(t) \, dt = \frac{1}{T_0} \int_0^{T_0} x^2(t) \, dt$$

$$= \frac{1}{3} \int_0^3 \left[\cos(3\pi t) + 5\cos(\pi t) \right]^2 dt$$

$$= \frac{1}{3} \int_0^3 \left(\cos^2(3\pi t) + 10\cos 3\pi t \cos \pi t + 25\cos^2 \pi t \right) dt$$

$$= \frac{1}{3} \int_0^3 \left[\frac{1 + \cos 6\pi t}{2} + 10 \times \frac{\cos 4\pi t + \cos 2\pi t}{2} + 25 \times \frac{1 + \cos 2\pi t}{2} \right] dt$$

$$= \frac{1}{6}\int_0^3 (1+\cos 6\pi t)\,dt + \frac{5}{3}\int_0^3 \cos 4\pi t\,dt + \frac{5}{3}\int_0^3 \cos 2\pi t\,dt$$

$$+ \frac{25}{6}\int_0^3 (1+\cos 2\pi t)\,dt$$

$$= \frac{1}{6}\left(t + \frac{1}{6\pi}\sin 6\pi t\right)\Big|_0^3 + \frac{5}{3}\cdot\frac{1}{4\pi}\sin 4\pi t\Big|_0^3 + \frac{5}{3}\cdot\frac{1}{2\pi}\sin 2\pi t\Big|_0^3$$

$$+ \frac{25}{6}\left(t + \frac{1}{2\pi}\sin 2\pi t\right)\Big|_0^3$$

$$= \frac{1}{6}(3+0) + \frac{5}{12\pi}\times 0 + \frac{5}{6\pi}\times 0 + \frac{25}{6}(3+0)$$

$$= \frac{3}{6} + \frac{25}{2} = 13 \text{ (Watt)}$$

c) We have:

$$P_{x_1} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |x_1(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T x_1^2(t)\,dt$$

$$T = NT_{01} \Rightarrow P_{x_1} = \lim_{N\to\infty}\frac{1}{T_{01}}\int_0^{T_{01}} x_1^2(t)\,dt = \frac{1}{T_{01}}\int_0^{T_{01}} x_1^2(t)\,dt$$

Also, $T_{01} = \frac{2}{3}$ as part a, $P_{x1} = \frac{3}{2}\int_0^{2/3} (\cos 3\pi t)^2\,dt$

$$= \frac{3}{2}\int_0^{2/3} \frac{1+\cos 6\pi t}{2}\, dt = \frac{3}{4}\int_0^{2/3} (1+\cos 6\pi t)\, dt$$

$$= \frac{3}{4}\left(t + \frac{1}{6\pi}\sin 6\pi t\right)\Big|_0^{2/3} = \frac{3}{4} t\Big|_0^{2/3} + \frac{1}{8\pi}\sin 6\pi t\Big|_0^{2/3}$$

$$= \frac{3}{4}\left(\frac{2}{3} - 0\right) + \frac{1}{8\pi}\left(\sin 6\pi \times \frac{2}{3} - 0\right)$$

$$= \frac{1}{2} + \frac{1}{8\pi}\times(0-0) = \frac{1}{2} \text{ (Watt)}$$

$$P_{x_2} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x_2(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{T} x_2^2(t)\, dt$$

$$T = NT_{02} \Rightarrow P_{x_2} = \lim_{N\to\infty} \frac{1}{T_{02}}\int_0^{T_{02}} x_2^2(t)\, dt$$

$$= \frac{1}{T_{02}}\int_0^{T_{02}} x_2^2(t)\, dt$$, as $T_{02} = 2$ as part a, we have

$$P_{x_2} = \frac{1}{2}\int_0^{2} (5\cos \pi t)^2 dt = \frac{1}{2}\int_0^{2} 25\cos^2 \pi t\, dt$$

$$= \frac{1}{2}\int_0^{2} 25 \times \frac{1+\cos 2\pi t}{2}\, dt = \frac{25}{4}\int_0^{2} (1+\cos 2\pi t)\, dt$$

$$= \frac{25}{4}\left(t + \frac{1}{2\pi}\sin 2\pi t\right)\Big|_0^{2} = \frac{25}{4} t\Big|_0^{2} + \frac{25}{4}\cdot\frac{1}{2\pi}\sin 2\pi t\Big|_0^{2}$$

$$= \frac{25}{4}(2-0) + \frac{25}{8\pi}(\sin 4\pi - 0) = \frac{25}{2} + 0 = \frac{25}{2} \text{ Watt}$$

$$\Rightarrow \boxed{\begin{array}{l} P_{x_1} + P_{x_2} = \frac{1}{2} + \frac{25}{2} = \frac{26}{2} = 13 \;(\sim x(a+b)) = P_x \\ \Rightarrow \text{the power } P_x = \text{the sum of } P_{x_1} \;\&\; P_{x_2}. \end{array}}$$

d) $y(t) = \underbrace{\cos(t)}_{y_1(t)} + \underbrace{\cos\left(\frac{\pi}{2}t\right)}_{y_2(t)}$

$y_1(t)$ is periodic with $T_{01} = \frac{2\pi}{1} = 2\pi$

$y_2(t)$ is periodic with $T_{02} = \frac{2\pi}{\pi/2} = 4$

Since $\frac{T_{01}}{T_{02}} = \frac{\pi}{2} = \frac{M}{N}$ with M & N are not both integer number

$\Rightarrow \boxed{y(t) \text{ is not periodic}}$

* Also $P_y = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |y(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T y^2(t)\, dt$

$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\cos^2 t + 2\cos t \cos\frac{\pi}{2}t + \cos^2\frac{\pi}{2}t\right) dt$

$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[\frac{1+\cos 2t}{2} + \cos\frac{2+\pi}{2}t + \cos\frac{2-\pi}{2}t + \frac{1+\cos\pi t}{2}\right] dt$

$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{1}{2} + \frac{1}{2} + \frac{\cos 2t}{2} + \cos\frac{\pi+2}{2}t + \cos\frac{2-\pi}{2}t + \frac{\cos\pi t}{2}\right) dt$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left(1 + \frac{\cos 2t}{2} + \frac{\cos \pi t}{2} + \cos\frac{\pi+2}{2}t + \cos\frac{2-\pi}{2}t\right) dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[t + \frac{1}{4}\sin 2t + \frac{1}{2\pi}\sin\pi t + \frac{2}{\pi+2}\sin\frac{\pi+2}{2}t + \frac{2}{2-\pi}\sin\frac{2-\pi}{2}t\right]\Bigg|_0^T$$

$$= \lim_{T\to\infty}\frac{1}{T}\left[T + \frac{1}{4}\sin 2T + \frac{1}{2\pi}\sin\pi T + \frac{2}{\pi+2}\sin\frac{\pi+2}{2}T + \frac{2}{2-\pi}\sin\frac{2-\pi}{2}T\right]$$

$$= 1 + \lim_{T\to\infty}\frac{1}{T}\left(\frac{1}{4}\sin 2T + \frac{1}{2\pi}\sin\pi T + \frac{2}{\pi+2}\sin\frac{\pi+2}{2}T + \frac{2}{2-\pi}\sin\frac{2-\pi}{T}\right)$$

Since $\sin(x) \in [-1, 1]$ is finite number, with $T\to\infty$ infinite

$$\Rightarrow \lim_{T\to\infty}\frac{1}{T}\left(\frac{1}{4}\sin 2T + \frac{1}{2\pi}\sin\pi T + \frac{2}{\pi+2}\sin\frac{\pi+2}{2}T + \frac{2}{2-\pi}\sin\frac{2-\pi}{T}\right) = 0$$

$\Rightarrow P_y = 1$ is the power of $y(t)$.

* With $y_1(t) = \cos t$ is periodic with $T_{01} = 2\pi$

$$\Rightarrow P_1 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\cos^2 t\, dt = \frac{1}{T_{01}}\int_0^{T_{01}}\cos^2 t\, dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\cos 2t}{2}dt$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1+\cos 2t)\,dt = \frac{1}{4\pi}\left(t + \frac{1}{2}\sin 2t\right)\Bigg|_0^{2\pi} = \frac{2\pi}{4\pi}$$

$= \frac{1}{2} \Rightarrow P_{y_1} = \frac{1}{2}$ is the power of $y_1(t) = \cos(t)$

* With $r_2(t) = \cos\frac{\pi}{2}t$ is periodic with $T_{02} = \frac{2\pi}{\pi/2} = \frac{2\pi \cdot 2}{\pi} = 4$

$$\Rightarrow P_2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2\frac{\pi}{2}t\,dt = \frac{1}{T_{02}}\int_0^{T_{02}} \cos^2\frac{\pi}{2}t\,dt$$

$$= \frac{1}{4}\int_0^4 \frac{1+\cos\pi t}{2}\,dt = \frac{1}{8}\int_0^4 (1+\cos\pi t)\,dt$$

$$= \frac{1}{8}\left(T + \frac{1}{\pi}\sin\pi t\right)\Big|_0^4 = \frac{1}{8}(4-0) = \frac{4}{8} = \frac{1}{2}$$

$\Rightarrow P_{r_2} = \frac{1}{2}$ is the power of $r_2(t) = \cos\frac{\pi}{2}t$

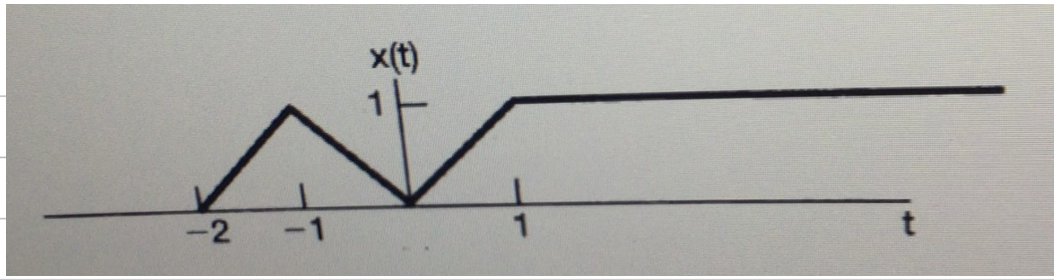
$$\Rightarrow \boxed{P_r = P_{r_1} + P_{r_2} = \frac{1}{2} + \frac{1}{2} = 1}$$

* So, for harmonically related, we use the definition for the power of periodic signal that can derived to $P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)\,dt$, that can use its T_0 directly for the integral while using the general formular for non-periodic that is $P_x = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)\,dt$

Also, based on part c & d, we can conclude that the power of a signal composed of sinusoids like $\cos(\omega t)$ is the sum of each components whether or not the frequency of the sinusoids are harmonically related.

or $P_x = \sum_i P_{x_i}$. The different thing considered is if the components sinusoids signal are harmonically related then the final signal is periodic. otherwise, it is not periodic

4) a)



We $x(t) = \begin{cases} t+2 & -2 \le t \le -1 \\ -t & -1 \le t \le 0 \\ t & 0 \le t \le 1 \\ 1 & t \ge 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow x(-t) = \begin{cases} -t+2 & 1 \le t \le 2 \\ t & 0 \le t \le 1 \\ -t & -1 \le t \le 0 \\ 1 & t \le -1 \\ 0 & \text{otherwise} \end{cases}$

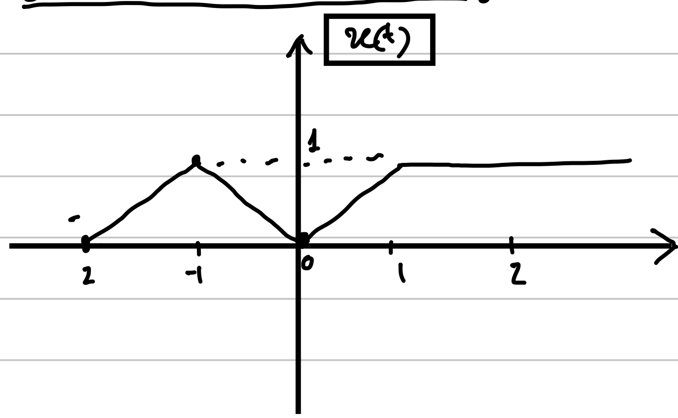
$\Rightarrow x(t) + x(-t) = \begin{cases} 0+1 & t \le -2 \\ t+2+1 & -2 \le t \le -1 \\ -t-t & -1 \le t \le 0 \\ t+t & 0 \le t \le 1 \\ 1+2-t & 1 \le t \le 2 \\ 1+0 & t \ge 2 \end{cases}$

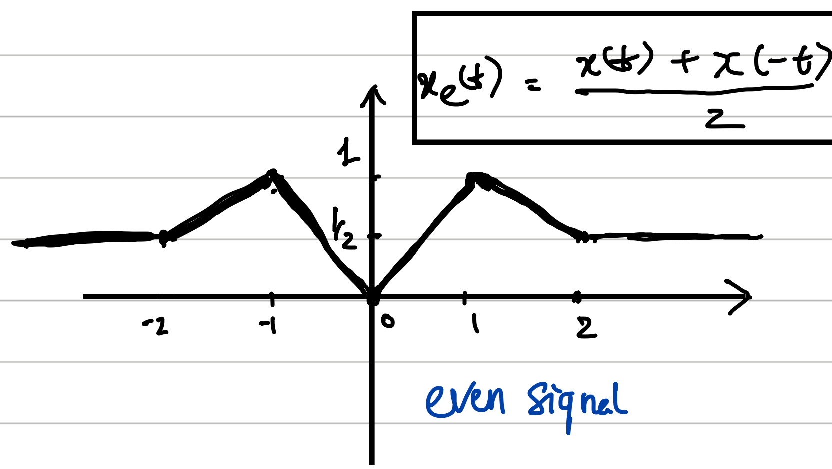
$$\Rightarrow \boxed{x_e(t) = \frac{x(t)+x(-t)}{2} = \begin{cases} 1/2 & t \le -2 \\ \frac{1}{2}t + \frac{3}{2} & -2 \le t \le -1 \\ -t & -1 \le t \le 0 \\ t & 0 \le t \le 1 \\ -\frac{1}{2}t + \frac{3}{2} & 1 \le t \le 2 \\ 1/2 & t \ge 2 \end{cases}}$$

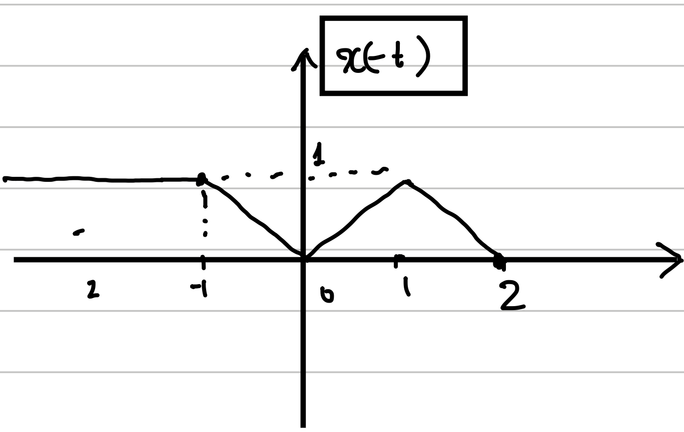
Also, $$x(t) - x(-t) = \begin{cases} 0-1 & t \leq -2 \\ t+2-1 & -2 \leq t \leq -1 \\ -t+t & -1 \leq t \leq 0 \\ t-t & 0 \leq t \leq 1 \\ 1+t-2 & 1 \leq t \leq 2 \\ 1-0 & t \geq 2 \end{cases}$$

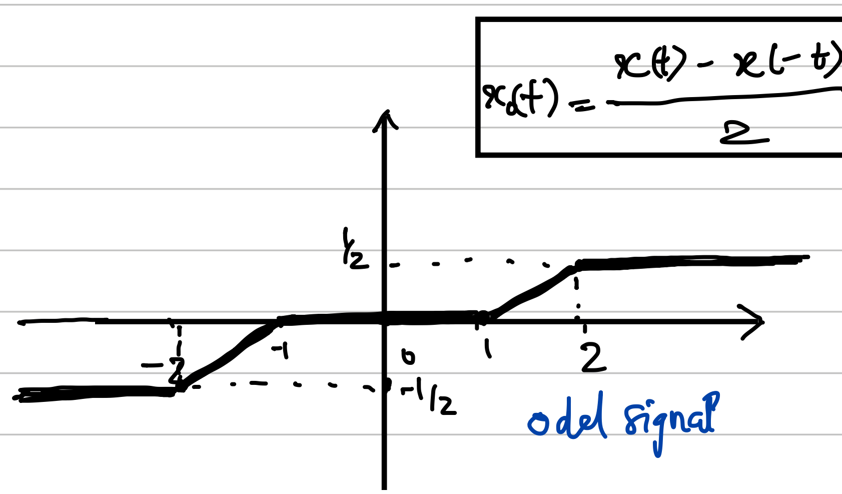
$$\Rightarrow \boxed{x_o(t) = \frac{x(t) - x(-t)}{2} = \begin{cases} -1/2 & t \leq -2 \\ \frac{1}{2}t + \frac{1}{2} & -2 \leq t \leq -1 \\ 0 & -1 \leq t \leq 1 \\ \frac{1}{2}t - \frac{1}{2} & 1 \leq t \leq 2 \\ 1/2 & t \geq 2 \end{cases}}$$

Sketch and check:









b) With the general continuous time signal $x(t)$, we need to prove $\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x_e(t)|^2 dt + \int_{-\infty}^{+\infty} |x_o(t)|^2 dt$

We have: $x(t) = x_e(t) + x_o(t)$

$$\Rightarrow x^2(t) = x_e^2(t) + 2x_e(t)\, x_o(t) + x_o^2(t)$$

$\Rightarrow$ the left side: $\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \left(x_e^2(t) + x_o^2(t) + 2x_e(t)\, x_o(t)\right) dt$

$$= \int_{-\infty}^{+\infty} x_e^2(t)\, dt + \int_{-\infty}^{+\infty} x_o^2(t)\, dt + 2\int_{-\infty}^{+\infty} x_e(t)\, x_o(t)\, dt.$$

* Let $z(t) = x_e(t)\, x_o(t)$

Since $x_e(t)$ is an even signal

$\Rightarrow x_e(t) = x_e(-t)$

$x_o(t)$ is an odd signal

$\Rightarrow x_o(t) = -x_o(-t)$

$$\Rightarrow z(t) = x_e(t)\, x_o(t) = -x_e(-t)\, x_o(-t) = -z(-t)$$

$\Rightarrow$ $\boxed{z(t) = x_e(t)\, x_o(t) \text{ is an odd signal.}}$

Now, we calculate the integral of $z(t) = x_e(t) x_o(t)$.

$$\int_{-\infty}^{+\infty} z(t)\,dt = \int_{-\infty}^{0} z(t)\,dt + \int_{0}^{+\infty} z(t)\,dt$$

For the first integral $\int_{-\infty}^{0} z(t)\,dt$, we use $\tau = -t$

$$\Rightarrow dt = -d\tau \Rightarrow \int_{-\infty}^{0} z(t)\,dt = -\int_{+\infty}^{0} z(-\tau)\,d\tau$$

$$= \int_{0}^{+\infty} z(-\tau)\,d\tau$$. Since as we proved so far, $z(\tau)$ is an odd signal $\Rightarrow z(\tau) = -z(-\tau)$

$$\Rightarrow \int_{0}^{+\infty} z(-\tau)\,d\tau = -\int_{0}^{+\infty} z(\tau)\,d\tau = -\int_{0}^{+\infty} z(t)\,dt.$$

$$\Rightarrow \int_{-\infty}^{+\infty} z(t)\,dt = \int_{-\infty}^{0} z(t)\,dt + \int_{0}^{+\infty} z(t)\,dt = -\int_{0}^{+\infty} z(t)\,dt + \int_{0}^{+\infty} z(t)\,dt$$
$$= 0$$

OR $\int_{-\infty}^{+\infty} x_e(t) x_o(t) = 0 \Rightarrow 2\int_{-\infty}^{+\infty} x_e(t) x_o(t) = 0$

Furthermore, we can use other ways to prove.

Since $x_e(t) = \dfrac{x(t) + x(-t)}{2}$

$x_o(t) = \dfrac{x(t) - x(-t)}{2}$

$$\Rightarrow x_e(t)\,x_o(t) = \frac{[x(t)]^2 - [x(-t)]^2}{4}$$

$$\Rightarrow 2\int_{-\infty}^{+\infty} x_e(t)\,x_o(t)\,dt = \frac{1}{2}\int_{-\infty}^{+\infty} x^2(t)\,dt - \frac{1}{2}\int_{-\infty}^{+\infty} x^2(-t)\,dt$$

For $\int_{-\infty}^{+\infty} x(-t)\,dt$ let $\tau = -t \Rightarrow dt = -d\tau$

$$\Rightarrow \int_{-\infty}^{+\infty} x(-t)\,dt = -\int_{+\infty}^{-\infty} x(\tau)\,d\tau = \int_{-\infty}^{+\infty} x(\tau)\,d\tau$$

$$= \int_{-\infty}^{+\infty} x(t)\,dt \;.\Rightarrow \int_{-\infty}^{+\infty} x(-t)\,dt = \int_{-\infty}^{+\infty} x(t)\,dt$$

$$\Rightarrow 2\int_{-\infty}^{+\infty} x_e(t)\,x_o(t)\,dt = \frac{1}{2}\int_{-\infty}^{+\infty} x^2(t)\,dt - \frac{1}{2}\int_{-\infty}^{+\infty} x(-t)\,dt$$

$$= 0$$

Finally, we have proven $2\int_{-\infty}^{+\infty} x_e(t)\,x_o(t)\,dt = 0$

$\Rightarrow$ the left side $= \int_{-\infty}^{+\infty} x_e^2(t)\,dt + \int_{-\infty}^{+\infty} x_o^2(t)\,dt =$ Right side

$\Rightarrow$ We can conclude: $\boxed{\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |x_e(t)|^2\,dt + \int_{-\infty}^{+\infty} |x_o(t)|^2\,dt}$