Lecture 7

OH for Prof. Cabric on Friday
01/28 to 12:30 - 1:30pm.

Laplace Transform

causal signals

$x(t) \cdot u(t)$

one-sided Laplace Transform

$$x(t) \xrightarrow{\mathcal{L}_s} X(s) = \mathcal{L}_s\{x(t)\}$$

$$\mathcal{L}_s\{x(t)\} \triangleq \int_{0^-}^{+\infty} x(t) e^{-st} dt$$

$s = \sigma + j\Omega \in ROC$

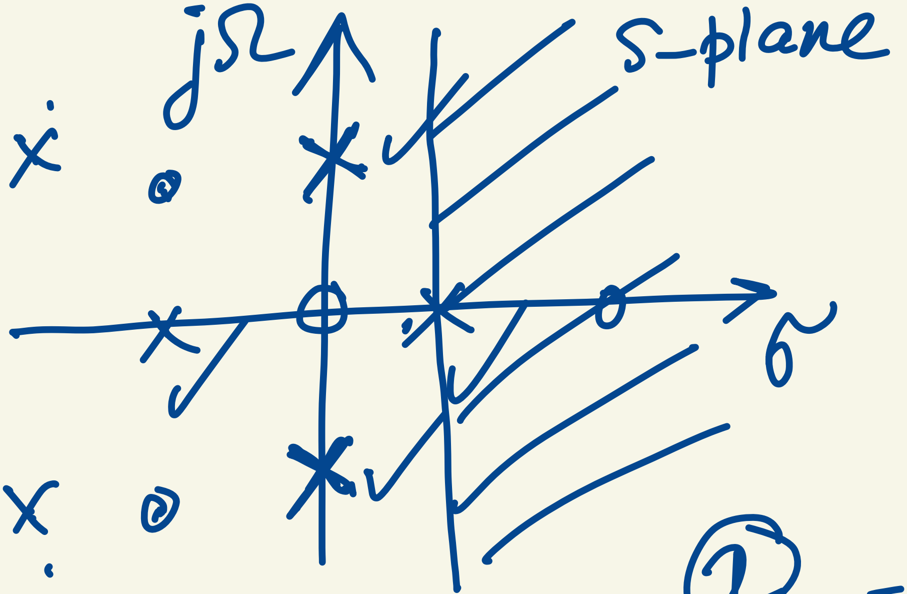
Table of Laplace Transforms

		ROC
$\delta(t) \longrightarrow$	1	All s
$u(t) \longrightarrow$	$\frac{1}{s}$	$Re\{s\}>0$
$e^{at}u(t) \longrightarrow$	$\frac{1}{s-a}$	$Re\{s\}>a$
$\cos(\Omega_0 t)u(t) \longrightarrow$	$\frac{s}{s^2+\Omega_0^2}$	$Re\{s\}>0$
$\sin(\Omega_0 t)u(t) \longrightarrow$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$Re\{s\}>0$
$t^n u(t) \longrightarrow$	$\frac{n!}{s^{n+1}}$	$Re\{s\}>0$

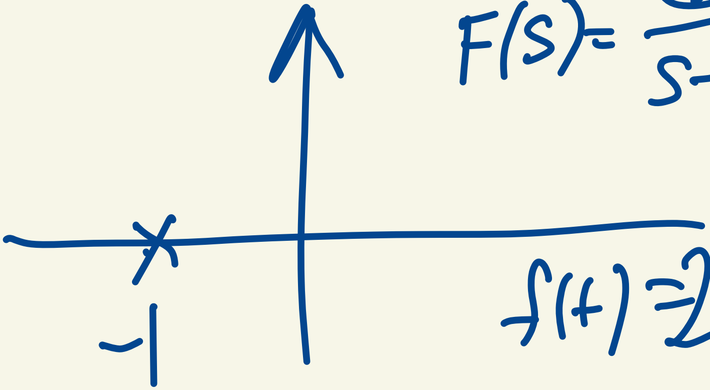
$$\underline{F(s)} = \frac{(s-a_0)(s-a_1)\cdots\cdots(s-a_m)}{(s-b_0)(s-b_1)\cdots(s-b_n)}$$

$a_0, \ldots, a_m$ → zeros of $F(s)$

$b_0, \ldots, b_n$ → poles of $F(s)$



$$F(s) = \frac{\textcircled{2}}{s-(-1)} = \frac{2}{s+1}$$



$$f(t) = 2e^{-t}u(t)$$

Laplace Transform Properties

① Linearity

$$f_1(t) \longrightarrow F_1(s)$$

$$f_2(t) \longrightarrow F_2(s)$$

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) \longrightarrow$$

$$\alpha_1 F_1(S) + \alpha_2 F_2(S)$$

Proof:

$$\mathcal{L}_s\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} =$$

$$= \int_{0^-}^{+\infty} \left(\alpha_1 f_1(t) + \alpha_2 f_2(t)\right) e^{-st}\, dt$$

$$= \int_{0^-}^{+\infty} \alpha_1 f_1(t) e^{-st}\, dt + \int_{0^-}^{+\infty} \alpha_2 f_2(t) e^{-st}\, dt$$

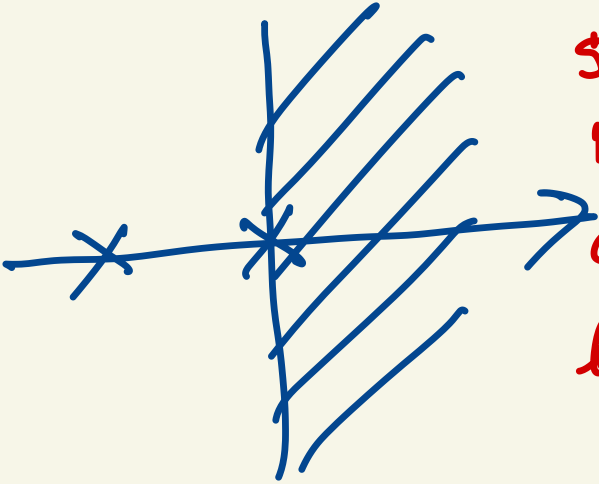
$$= \alpha_1 \int_{0^-}^{+\infty} f_1(t) e^{-st}\, dt + \alpha_2 \int_{0^-}^{+\infty} f_2(t) e^{-st}\, dt$$

$$= \alpha_1 \underbrace{F_1(s)}_{ROC_1} + \alpha_2 \underbrace{F_2(s)}_{ROC_2}$$

$$f(t) = u(t) - e^{-2t} u(t)$$

$$F(s) = \frac{1}{s} - \frac{1}{s+2} = \frac{\cancel{s}+2-\cancel{s}}{s(s+2)} = \frac{2}{s(s+2)}$$

$\uparrow$ evaluate.



some poles might be cancelled by zeros.

② $f(t) \longrightarrow F(s)$

$$e^{-\alpha t} f(t) \longrightarrow F(s+\alpha)$$

α can be real or complex constant.

$$\mathcal{L}_s\{e^{-\alpha t} f(t)\} = \int_{0^-}^{+\infty} e^{-\alpha t} f(t) e^{-st} dt$$

$$= \int_{0^-}^{+\infty} e^{-\underbrace{(s+\alpha)}t} f(t)$$

$$= \underline{F\left(\overset{s}{s}+\alpha\right)}$$

(Frequency Shifting property):

Example :

$$e^{-\alpha t} \underbrace{\cos(\Omega_0 t)\, u(t)}_{f(t)} \longrightarrow$$

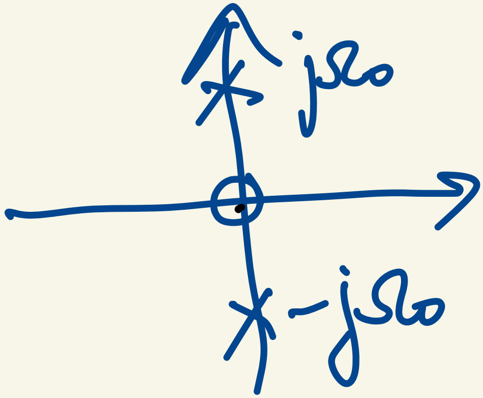
$$f(t) \longrightarrow \boxed{\frac{s}{s^2+\Omega_0^2}}$$

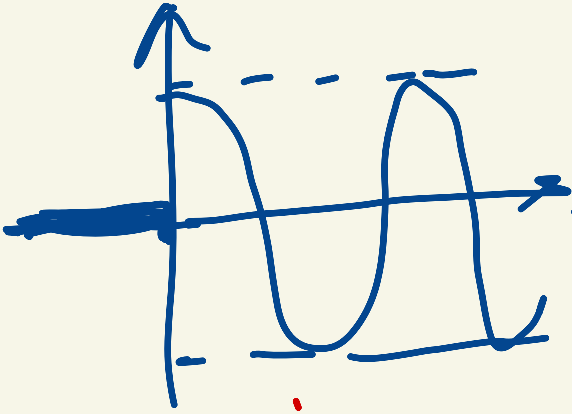
$$\longrightarrow \frac{s+\alpha}{(s+\alpha)^2+\Omega_0^2}$$

$$\underline{e^{-\alpha t} \cos(\Omega_0 t) u(t)} \longrightarrow \frac{s+\alpha}{(s+\alpha)^2+\Omega_0^2}$$

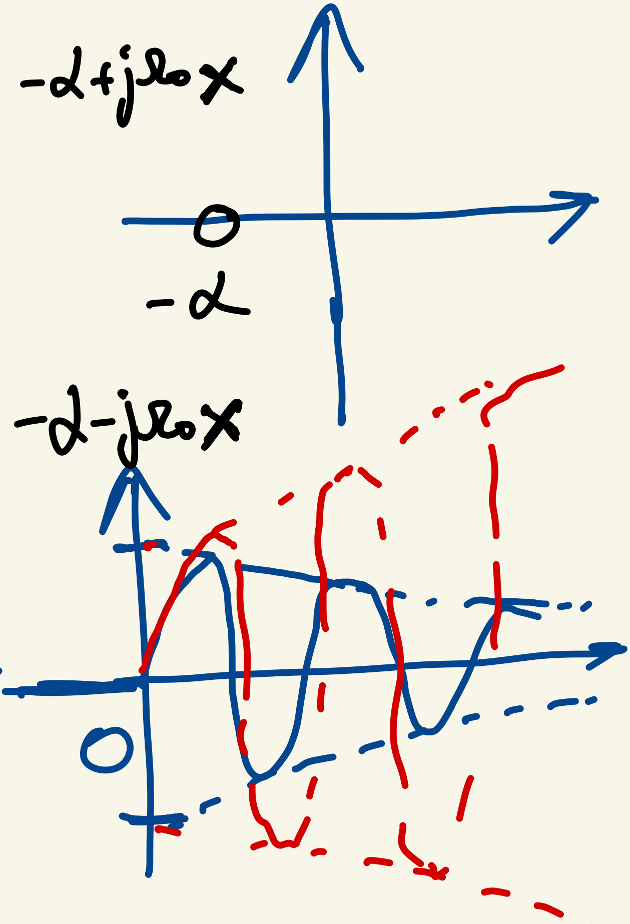
$$s_{1/2} = -\alpha \pm j\Omega_0$$

$\cos(\Omega_0 t)u(t)$





$e^{-\alpha t}\cos(\Omega_0 t)u(t)$



$\alpha > 0$

$\alpha < 0$

Example

$$e^{-dt}\underbrace{t^{n}u(t)}_{f(t)\longrightarrow\frac{n!}{s^{n+1}}}\longrightarrow$$

$$\longrightarrow\frac{n!}{(s+2)^{n+1}}$$

③ Laplace Transform of Derivatives.

$$f(t)\longrightarrow F(s)$$

$$\frac{df(t)}{dt} \longrightarrow sF(s) - f(0^-)$$

$$\int \underline{u}\,\underline{dv} = u\cdot v - \int v\,du$$

Proof:

$$\mathcal{L}_s\left\{\frac{df(t)}{dt}\right\} = \int_{0^-}^{+\infty} \left\{\frac{df(t)}{dt}\right\} e^{-st}\,dt$$

$dv = df(t) \rightarrow v = f(t)$

$u = e^{-st}$

$$= f(t)e^{-st}\Big|_{0^-}^{+\infty} - \int_{0^-}^{+\infty} f(t)(-se^{-st})\,dt$$

$$= \underbrace{\lim_{t\to\infty} f(t)e^{-st}}_{0} - f(0^-)\underbrace{e^{-0^-s}}_{1}$$

$s \in$ ROC for $f(t)$

$$+s\int_{0^-}^{+\infty} f(t)e^{-st}\,dt$$

$$= sF(s) - f(0^-)$$

$$\boxed{\frac{df(t)}{dt} \longrightarrow sF(s) - f(0^-)}$$

$$\mathcal{L}_s\left\{\frac{d^2 f(t)}{dt}\right\} = \mathcal{L}_s\left\{\frac{d}{dt}\left\{\underbrace{\frac{df(t)}{dt}}\right\}\right\}$$

$$= s\cdot\mathcal{L}_s\left\{\frac{df(t)}{dt}\right\} - f'(0^-)$$

$$= s\cdot\left(sF(s) - f(0^-)\right) - f'(0^-)$$

$$= s^2F(s) - sf(0^-) - f'(0^-)$$

$$\vdots$$

$$L_s\left\{\frac{d^n f(t)}{dt}\right\} = \underline{s^n F(s)} -$$

$$- s^{n-1} f(0^-) - s^{n-2} f'(0^-) -$$

$$- \ldots - f^{n-1}(0^-)$$

④ Laplace Transform of an integral

$$f(t) \longrightarrow F(s)$$

$$g(t) = \int_0^t f(\tau)\,d\tau \longrightarrow \frac{F(s)}{s}$$

Proof:

$$f(t) = \frac{dg(t)}{dt}$$

$$\mathcal{L}_s\left\{\frac{dg(t)}{dt}\right\} = F(s)$$

by derivative property of left hand side

$$s \cdot \mathcal{L}_s\{g(t)\} - \underline{\cancelto{0}{g(0^-)}} = F(s)$$

$$\mathcal{L}_s\{g(t)\} = \frac{F(s)}{s}$$

$$\mathcal{L}_s\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

⑤ Time shifting

$$f(t)\cdot u(t) \longrightarrow F(s)$$

$$f(t-d)u(t-d) \rightarrow e^{-sd}F(s)$$

Proof.

$$\mathcal{L}_s\{f(t-d)u(t-d)\} =$$

$$= \int_{0^-}^{+\infty} f(t-d)u(t-d)\,e^{-st}\,dt$$

$$= \int_{d}^{+\infty} f\underbrace{(t-d)}\,e^{-st}\,dt$$

$$\tau = t - d \Rightarrow t = \tau + d$$

$$dt = d\tau$$

$$t = d \quad \tau = 0$$

$$t = \infty \quad \tau = \infty$$

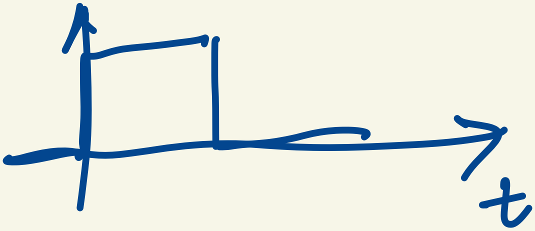
$$= \int_0^{+\infty} f(\tau) e^{-s(\tau+d)} d\tau$$

$$= \int_0^{+\infty} f(\tau) e^{-s\tau} \underbrace{e^{-sd}} d\tau$$

$$= e^{-sd} \underbrace{\int_0^{+\infty} f(\tau) e^{-s\tau} d\tau}$$

$= e^{-sd} F(s)$

$x(t) = u(t) - u(t-1)$



$X(s) = \frac{1}{s} - \frac{1}{s} e^{-s}$

⑥ $f(t) \longrightarrow F(s)$

$t^n f(t) \longrightarrow (-1)^n \frac{d^n F(s)}{ds^n}$

Proof.

$$t f(t) \longrightarrow -\frac{dF(s)}{ds}$$

$$F(s) = \int_{0^-}^{+\infty} f(t) e^{-st} dt \Big/ \frac{d}{ds}$$

$$\frac{dF(s)}{ds} = \frac{d}{ds}\left\{\int_{0^-}^{+\infty} f(t) e^{-st} dt\right\}$$

$$= \int_{0^-}^{+\infty} \frac{d}{ds}\left\{f(t) e^{-st} dt\right\}$$

$$= \int_{0^-}^{+\infty} f(t) \frac{d}{ds}\left\{ e^{-st} \right\} dt$$

$$= \int_{0^-}^{+\infty} f(t) \left(-t e^{-st} \right) dt$$

$$= - \int_{0^-}^{+\infty} \underbrace{t f(t)} e^{-st} \, dt$$

$$= - \mathcal{L}_s \{ t f(t) \}$$

$$\Rightarrow \mathcal{L}_s \{ t f(t) \} = - \frac{dF(s)}{ds}$$

$$L_s\{t^2 f(t)\} = L_s\{\underline{t}\cdot\underline{(t f(t))}\}$$

$$= -\frac{d}{ds}\{\underline{L_s\{t f(t)\}}\}$$

$$= -\frac{d}{ds}\{-\frac{d}{ds}F(s)\}$$

$$=(-1)^2\,\frac{d^2F(S)}{ds^2}$$

$$\boxed{\mathcal{L}_s\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}}$$

$$x(t) = t \underbrace{\sin(t) \cdot u(t)}_{f(t)} = t f(t) \quad \textcircled{T}$$

$$F(s) = \frac{1}{s^2+1} = \mathcal{L}_s\{\sin(t) u(t)\}$$

$$X(s) = -\frac{d F(s)}{ds}$$

↗ review of derivative of frac. poly.

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \qquad \frac{d}{ds}\frac{1}{s} = \frac{-1}{s^2}$$

$$= -\left(-\frac{2s}{(s^2+1)^2}\right)$$

$$= \frac{2s}{(s^2+1)^2}$$

Exercise:

$$x(t) = t\,e^{-t}\,u(t)$$

[illegible] by def. prop ② prop ⑦

② $x(t) = \underline{\underline{\overset{\alpha=1}{e^{-t}}}}\,\underbrace{t\,u(t)}_{f(t)} \xrightarrow{②} X(s) = F(s+1)$

$$\downarrow$$

$$\alpha$$

$$F(s) = \frac{1}{s^2}$$

$$X(s) = \frac{1}{(s+1)^2}$$

⑦ $x(t) = \underline{\underline{t}}\,\underbrace{e^{-t}\,u(t)}_{f(t)}$

$$F(s) = \frac{1}{s+1}$$

$$X(s) = -\frac{d}{ds} F(s)$$

$$= -\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

$$= -\left(-\frac{1}{(s+1)^2}\right)$$

$$= \frac{1}{(s+1)^2}$$

HW practice

$$\frac{d}{ds}\left(\frac{s+1}{s+2}\right)$$

① Laplace Transform of Convolution Integral.

$$y(t) = \underline{\underline{x(t)}} * \underline{\underline{h(t)}} = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Constraint:

$$x(t)\cdot u(t) \xrightarrow{L_s} X(S)$$

$$h(t)\cdot u(t) \xrightarrow{L_s} H(S)$$

$$\underline{y(t)=x(t)*h(t)}=\int_{-\infty}^{+\infty}x(\tau)\,\underline{u(\tau)}\cdot$$

$$\cdot\,h(t-\tau)\cdot\underline{u(t-\tau)}\,d\tau$$

$$=\int_{0\text{ over }\tau}^{+\infty}x(\tau)h(t-\tau)u(t-\tau)\underline{d\tau}$$

$$L_s\{y(t)\}=L_s\{x(t)*h(t)\}$$

$$=\int_{0^-\text{ over }t}^{+\infty}\int_{0\text{ over }\tau}^{+\infty}\underline{x(\tau)h(t-\tau)u(t-\tau)}e^{-st}\,dt\,d\tau$$

$\nwarrow$ ← switch order.

$$= \int_{0}^{+\infty} x(\tau) \int_{0^-}^{+\infty} h(\underbrace{t-\tau}) u(t-\tau) e^{-st}\, dt \cdot d\tau$$

over τ over t

$$t - \tau = z$$

$$t = z + \tau$$

$$dt = dz$$

$$t = 0 \quad z = -\tau$$

$$t = \infty \quad z = +\infty$$

$$= \int_{0}^{+\infty} x(\tau) \int_{-\tau}^{+\infty} h(z) \cdot \underline{u(z)} \cdot e^{-s(z+\tau)}\, dz \cdot d\tau$$

0: [illegible] $\quad -\tau \leftarrow$

$$= \int_{0 \text{ over } \tau}^{+\infty} x(\tau) \int_{0^- \text{ over } z}^{+\infty} h(z) e^{-sz} \boxed{e^{-s\tau}} \, dz \, d\tau$$

$$= \int_{0 \text{ over } \tau}^{+\infty} x(\tau) e^{-s\tau} \underbrace{\boxed{\int_{0^- \text{ over } z}^{+\infty} h(z) e^{-sz} \, dz}}_{= H(s)} \, d\tau$$

$$= H(s) \underbrace{\int_{0}^{+\infty} x(\tau) e^{-s\tau} \, d\tau}_{X(s)}$$

$$= X(s)\cdot H(s)$$

$$y(t) = x(t) * h(t) \xrightarrow{L_s} Y(s) = X(s)\cdot H(s)$$

$$\begin{tikzcd} x(t)*h(t) \arrow[d, "L_s"] \arrow[r, phantom, "="] & h(t)*x(t) \arrow[d, "L_s"] \\ X(s)\cdot H(s) \arrow[r, phantom, "="] & H(s)\cdot X(s) \end{tikzcd}$$