Discussion 1
ECE 102: Systems and Signals
Winter 2022

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1 Review

1.1 Integration by parts

Find the following definite integrals – (i) $\int xe^{-x}dx$ (ii) $\int x^2e^{-x}dx$
Hint: $\int udv = uv - \int vdu$
Solution:

(i) To pick dv, we want to choose the part of the integrand that is easy to integrate, leaving the rest as u Thus, :

$$u = x \quad ; \quad dv = e^{-x}dx$$

By differentiating u and integrating dv, we get $du = dx$ and $v = -e^{-x}$. Finally, by substituting our differential results into the integration by parts formula:

$$\int xe^{-x}dx = -xe^{-x} - e^{-x}$$

(ii) We select $u = x \quad ; \quad dv = xe^{-x}dx$. By differentiating u, we get $du = dx$. The indefinite integral of xe^{-x} has been computed in part (i) as $-xe^{-x} - e^{-x}$. We substitute these in the integration by parts formula:

$$\begin{aligned}
\int x^2e^{-x}dx = \int x.xe^{-x}dx &= -xe^{-x} - e^{-x} \\
&= x\left(-xe^{-x} - e^{-x}\right) - \int \left(-xe^{-x} - e^{-x}\right)dx \\
&= -x^2e^{-x} - xe^{-x} + \int xe^{-x}dx + \int e^{-x}dx \\
&= -\left(x^2e^{-x} + 2xe^{-x} + 2e^{-x}\right)
\end{aligned}$$

2 Problems

2.1 Euler's identity and trigonometric identities

Use Euler's identity to obtain an expression for $e^{j(\alpha-\beta)} = e^{j\alpha}e^{-j\beta}$. Obtain its real and imaginary components and show that

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

Hint: Expand $e^{j(\alpha-\beta)}$ and $e^{j\alpha}e^{-j\beta}$ *, and equate their real and imaginary parts.*

Solution:

Euler's identity is given by $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, where $j = \sqrt{-1}$. The real and imaginary parts of $e^{j\theta}$ can be expressed as $Re\{e^{j\theta}\} = \cos\theta$, and $Im\{e^{j\theta}\} = \sin\theta$.

$$e^{j(\alpha-\beta)} = \cos(\alpha-\beta) + j\sin(\alpha-\beta) \tag{1}$$

$$\begin{aligned} e^{j\alpha}e^{-j\beta} &= (\cos\alpha + j\sin\alpha)(\cos\beta - j\sin\beta) \\ &= (\cos\alpha\cos\beta + \sin\alpha\sin\beta) + j(\sin\alpha\cos\beta - \cos\alpha\sin\beta) \end{aligned} \tag{2}$$

Equating the real and imaginary parts in (1) and (2), we get $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$. We thus compute $\tan(\alpha-\beta)$ as

$$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

Dividing the numerator and denominator by $\cos\alpha\cos\beta$ yields $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$ upon further simplification

2.2 Reflection, time shifting, and time scaling

1. Consider a continuous time signal $x(t)$ as described below

$$x(t) = \begin{cases} t+1, & -1 \leq t \leq 0 \\ -\frac{t}{2}+1, & 0 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

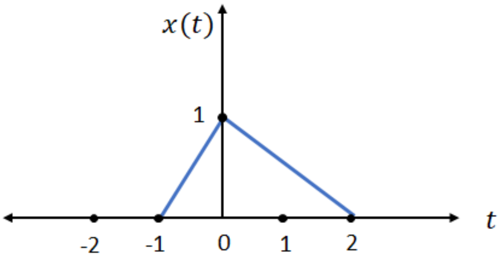
   a) Plot $y(t) = x(t+1) + x(-t+1)$

   b) Is $y(t)$ an even or odd signal? Express $y(t)$ analytically.

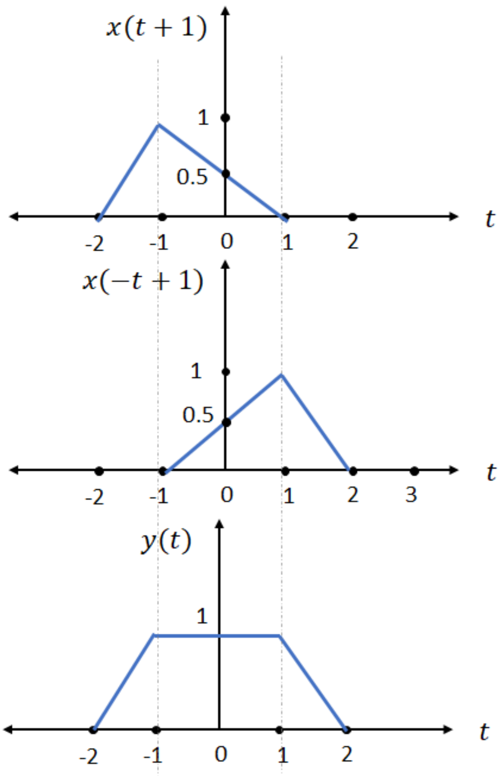
   c) Plot $x\left(-2t + \frac{3}{2}\right)$

Solution:

a) The signal $x(t)$ can be given by



We always perform time shifting first, followed by time scaling. Thus, we first obtain $x(t+1)$, and then $x(-t+1)$. Signal $y(t)$ can be obtained by addingthe two as shown below.



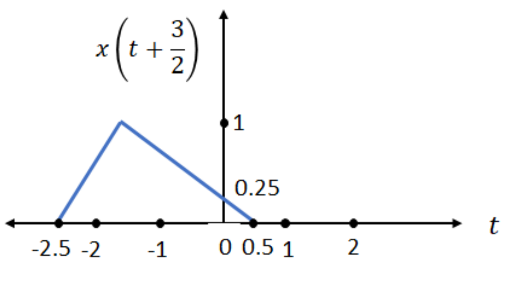
(a)

Figure 1: Caption

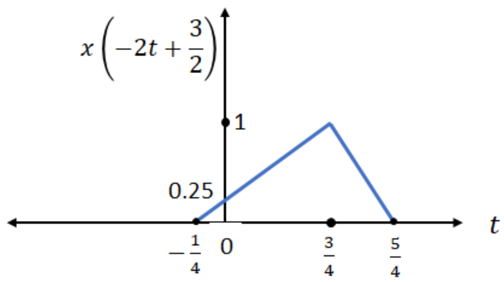
b) It can be graphically seen as well as mathematically verified that $y(t) = y(-t)$. Thus, $y(t)$ is an even signal. The odd signal component $y_o(t) = \frac{y(t)-y(-t)}{2}$ is zero.
Further, $y(t) = t+2$ for all $-2 \leq t \leq -1$ and $y(t) = -t+2$ for $1 \leq t \leq 2$. This can be captured by the condition $y(t) = -|t| + 2 \; \forall \; 1 \leq |t| \leq 2$. Thus, $y(t)$ can be analytically expressed as follows:

$$y(t) = \begin{cases} -|t|+2, & 1 \leq |t| \leq 2 \\ 1, & 0 \leq |t| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

c) We first obtain $x\left(t+\frac{3}{2}\right)$ by time shifting, and then perform time scaling to obtain $x\left(-2t+\frac{3}{2}\right)$.



(a) $x\left(t+\frac{3}{2}\right)$



(b) $x\left(-2t+\frac{3}{2}\right)$

2.3 Even & Odd signal decomposition

Consider the continuous time signal $x(t)$, defined as follows

$$x(t) = \begin{cases} 2, & 0 \leq t \leq 1 \\ t, & 1 \leq t \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

a) Plot and analytically express the even and odd components, $x_e(t)$ and $x_o(t)$ respectively, of the signal $x(t)$.

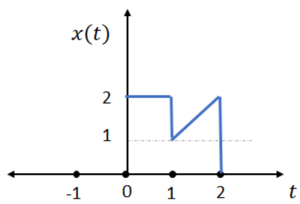
b) Verify that the energy of $x(t)$ is equal to the sum of the energies of $x_e(t)$ and $x_o(t)$.

c) Find the power of signal $x(t)$.

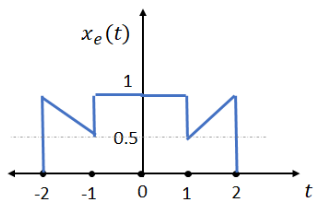
Solution:

a)

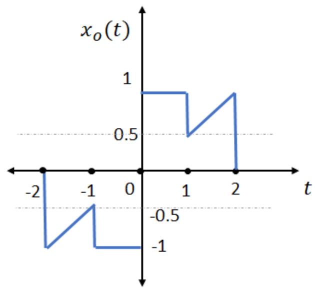
$$x_e(t) = \begin{cases} \frac{x(t)}{2}, & t \geq 0 \\ -\frac{x(t)}{2}, & t < 0 \end{cases} \quad ; \quad x_o(t) = \begin{cases} \frac{x(t)}{2}, & t \geq 0 \\ -\frac{x(-t)}{2}, & t < 0 \end{cases}$$



(c) Signal $x(t)$



(d) Even component



(e) Odd component

b) Energy of signal $x(t)$ is denoted as $E_{x(t)}$, and is computed as follows:

$$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_1^2 t^2 dt + \int_0^1 4dt = \frac{(8-1)}{3} + 4 = \frac{19}{3}$$

The energies of the even and odd components, $E_{x_e(t)}$ and $E_{x_o(t)}$ respectively, are computed as follows:

$$E_{x_e(t)} = \int_{-\infty}^{\infty} |x_e(t)|^2 dt = \int_1^2 \left(\frac{t}{2}\right)^2 dt + \int_{-2}^{-1} \left(\frac{t}{2}\right)^2 dt + \int_{-1}^{1} 1dt$$
$$= 2\int_1^2 \left(\frac{t}{2}\right)^2 dt + 2\int_0^1 1dt = 2\left(\frac{7}{12} + 1\right) = \frac{19}{6}$$
$$E_{x_o(t)} = 2\int_1^2 \left(\frac{t}{2}\right)^2 dt + 2\int_0^1 1dt = 2\left(\frac{7}{12} + 1\right) = \frac{19}{6}$$

This confirms that the total signal energy is the sum of the powers of its odd and even components. That is, $E_{x(t)} = E_{x_e(t)} + E_{x_o(t)}$.

c) Since we found that $x(t)$ has finite energy for all time, the power of this signal must be 0.