Lecture 11

$x(t)$ periodic w/ T_0 (fundamental period)

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j \underline{n w_0} t}$$

$w_0 = \frac{2\pi}{T_0}$ fundamental freq.

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) \cdot e^{-j n w_0 t} dt$$

$n = \pm 1, \pm 2, \ldots$

$$X_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

Recap: $x(t) = t \quad -1 \le t \le 1$

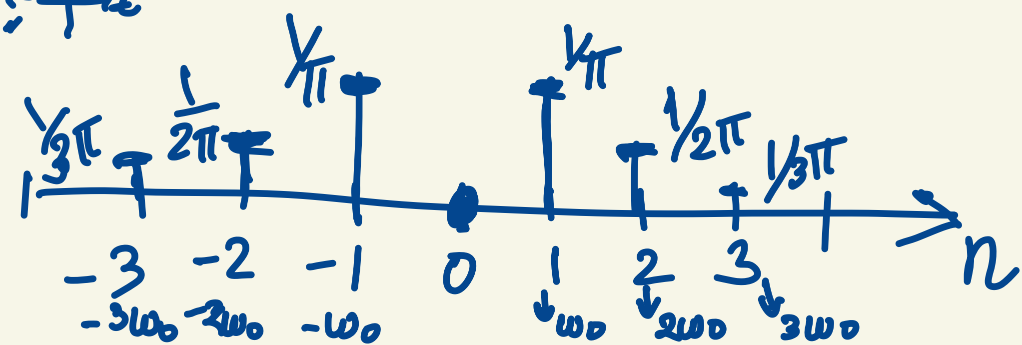
$X_0 = 0$

$X_n = (-1)^n \dfrac{j}{n\pi} \quad n = \pm 1, \pm 2, \ldots$

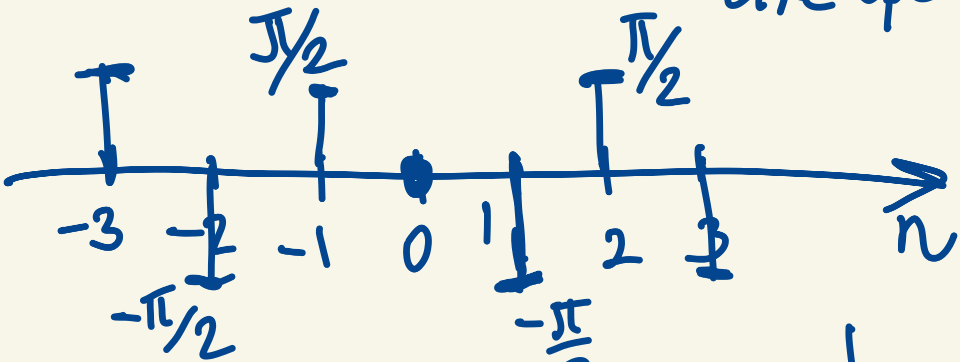
$X_n = |X_n| e^{j \measuredangle X_n}$

visualization of FS coeff - using magnitude and phase

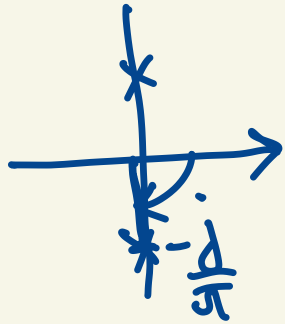
magnitude line spectrum



phase
line spectrum



$$X_1 = (-1)^1 \frac{j}{1\cdot\pi} = \frac{-j}{\pi}$$



$$|X_1| = \frac{1}{\pi}$$

$$\angle X_1 = -\frac{\pi}{2}$$

$$X_{-1} = (-1)^{-1} \frac{j}{-1\cdot\pi} = \frac{j}{\pi}$$

$$|X_{-1}| = \frac{1}{\pi} \qquad \angle X_{-1} = \frac{\pi}{2}$$

Fourier series coefficient (magnitude) have interpretation in terms of power.

$\Rightarrow$ Parseval's Theorem.

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |X_n|^2$$

power of periodic signal!

$|X_n|^2$ is power of n^{th} harmonic.

periodic signal power is contained in power of its harmonics.

Proof. $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$

$$\boxed{\begin{aligned} a \cdot a^* &= |a|^2 \\ |a|e^{j\angle a} \cdot |a| \cdot e^{-j\angle a} &= \end{aligned}}$$

$$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{T_0} \underline{x(t)} \cdot \underline{x^*(t)}\, dt$$

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t} \right) \underline{\left(\sum_{m=-\infty}^{+\infty} X_m e^{jm\omega_0 t} \right)^*} dt$$

$$(a+b)^* = a^* + b^*$$

$$(a \cdot b)^* = a^* \cdot b^*$$ (prove during break)

$$= \frac{1}{T_0} \int_{T_0} \left(\sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t} \right) \cdot \left(\sum_{m=-\infty}^{+\infty} X_m^* e^{-jm\omega_0 t} \right) dt$$

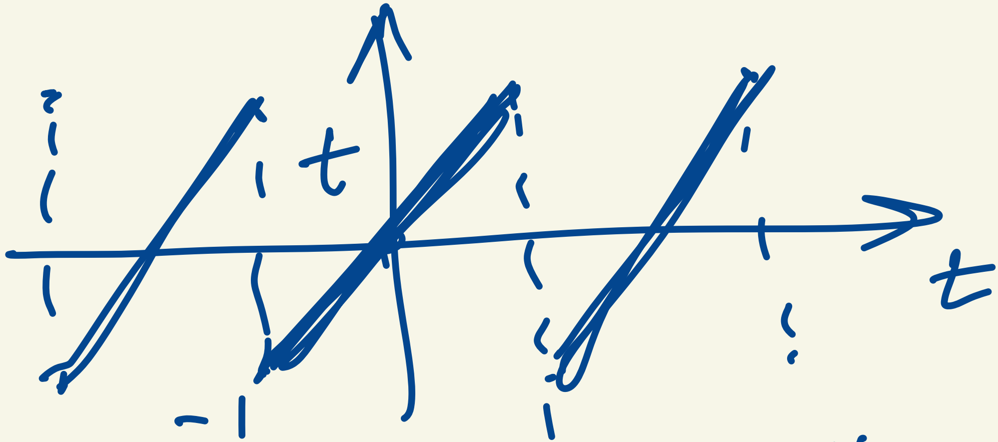
$$= \frac{1}{T_0} \int_{T_0} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} X_n \cdot X_m^* e^{j(n-m)\omega_0 t} dt$$

$$= \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} X_n X_m^* \underbrace{\int_{T_0} e^{j(n-m)\omega_0 t} dt}_{= \begin{cases} T_0 & n=m \\ 0 & n \neq m \end{cases}}$$

$$= \frac{1}{\cancel{T_0}} \sum_{n=-\infty}^{+\infty} |X_n|^2 \cdot \cancel{T_0}$$

$$= \sum_{n=-\infty}^{+\infty} |X_n|^2$$

Example. $x(t)$



$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \underbrace{e^{jn\omega_0 t}}$$

if you have a source (generator) of $e^{jn\omega_0 t}$

The goal is to create

$$\tilde{x}(t) = \sum_{n=-\underline{\underline{N}}}^{N} X_n e^{jn\omega_0 t}$$

such that

$$P_{\tilde{x}} = 0.99 \, P_{\bar{x}}$$

$$\underline{\underline{\sum_{n=-N}^{N} |X_n|^2 = 0.99 \cdot P_x}}$$

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

$$= \frac{1}{2} \int_{-1}^{1} t^2 \, dt$$

$$= \frac{1}{2} \frac{t^3}{3} \Big|_{-1}^{1}$$

$$= \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3} \right) = \frac{1}{3}$$

$$\sum_{n=-N}^{N} |X_n|^2 = 0.99 \cdot \frac{1}{3} = 0.33$$

$$\cancel{X_0}^{0} + 2\sum_{n=1}^{N} \frac{1}{n^2\pi^2} \leq 0.33$$

$N=1$ $\quad 2 \cdot \frac{1}{\pi^2} \leq 0.33$

$\underline{N=2}$ $\quad 2 \cdot \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2}\right)^2 < 0.33$

$N=3$ $\quad 2\cdot\left(\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}\right) \overset{?}{<} 0.33$

Error: $\frac{1}{T_0}\int |x(t)|^2 dt$

$$- \frac{1}{T}\int |\tilde{x}(t)|^2 dt$$

$$= \frac{1}{T_0}\int_{T_0}\left(|x(t)|^2 - |\tilde{x}(t)|^2\right) dt$$

Properties of X_n for real sig.

$x(t) = x^*(t) \Rightarrow$ real sig.

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{T_0} x^*(t) \left(e^{jn\omega_0 t}\right)^* dt$$

$$= \frac{1}{T_0} \int_{T_0} \left(x(t) e^{jn\omega_0 t}\right)^* dt$$

$$= \left(\frac{1}{T_0} \int_{T_0} x(t) e^{jn\omega_0 t} dt\right)^*$$

$$= \Bigg(\underbrace{\frac{1}{T_0} \int_{T_0} x(t) e^{-j(-n)\omega_0 t} dt}_{X_{-n}} \Bigg)^*$$

$$= X_{-n}^*$$

$$x(t) = x^*(t) \Rightarrow X_n = X_{-n}^*$$

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complex conjugate symmetry of FS coefficient

All real signals have it!

$$|X_n| e^{j\angle X_n} = \left(|X_{-n}| e^{j\angle X_{-n}} \right)^*$$

$$|X_n| e^{j\measuredangle X_n} = |X_{-n}| e^{-j\measuredangle X_{-n}}$$

$$\Rightarrow |X_n| = |X_{-n}|$$ even symmetry of magnitude spectrum.

$$\measuredangle X_n = -\measuredangle X_{-n}$$ odd symmetry of phase spectrum.

Real Fourier Series representation (2 versions)

→ signal is real and FS uses real coefficients and real basis.

$x(t) = x^*(t) \qquad X_n = X_{-n}^*$

$$\underline{\underline{x(t)}} = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t}$$

$$+ \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{+\infty} X_{-n} e^{-jn\omega_0 t} +$$

$$+ \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{+\infty} \left(X_{-n} e^{-jn\omega_0 t} + X_n e^{jn\omega_0 t} \right)$$

$$= X_0 + \sum_{n=1}^{+\infty} \Big(|X_{-n}| e^{j\measuredangle X_{-n}} e^{-jn\omega_0 t}$$

$$+ |X_n| e^{j\measuredangle X_n} e^{jn\omega_0 t} \Big)$$

$$= X_0 + \sum_{n=1}^{+\infty} \Big(|X_n| e^{-j\measuredangle X_n} e^{-jn\omega_0 t}$$

$$+ |X_n| e^{j\measuredangle X_n} e^{jn\omega_0 t} \Big)$$

$$2\cos\theta = e^{-j\theta} + e^{j\theta}$$

$$= X_0 + \sum_{n=1}^{+\infty} |X_n| \left(e^{-j(n\omega_0 t + \measuredangle X_n)} + e^{+j(n\omega_0 t + \measuredangle X_n)}\right)$$

$$= X_0 + 2\sum_{n=1}^{+\infty} |X_n| \cos(n\omega_0 t + \measuredangle X_n)$$

Real F.S. version!

you still need X_n

$$X_n = \underline{a_n} + j\,\underline{b_n}$$

$$\underline{X_n} = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\left[\cos(n\omega_0 t) - j\sin(n\omega_0 t)\right] dt$$

$$= \frac{1}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)\, dt$$

$$- j \frac{1}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\, dt$$

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = -\frac{1}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\, dt$$

$$x(t) = X_0 + \sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t}$$

$$+ \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{+\infty} \underline{\underline{X_{-n}}}\, e^{-jn\omega_0 t}$$

$$+ \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{+\infty} \underline{\underline{X_n^*}}\, e^{-jn\omega_0 t}$$

$$+ \sum_{n=1}^{+\infty} X_n e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{+\infty} (a_n + jb_n)^* e^{-jn\omega_0 t}$$

$$+ \sum_{n=1}^{+\infty} (a_n + jb_n) e^{jn\omega_0 t}$$

$$= X_0 + \sum_{n=1}^{+\infty} \Big[(a_n - j b_n) e^{-jn\omega_0 t} + (a_n + j b_n) e^{jn\omega_0 t} \Big]$$

$$= X_0 + \sum_{n=1}^{+\infty} a_n \Big[e^{-jn\omega_0 t} + e^{jn\omega_0 t} \Big]$$

$$+ j \cdot 2 \sum_{n=1}^{+\infty} b_n \left[\frac{-e^{-jn\omega_0 t} + e^{jn\omega_0 t}}{2j} \right]$$

$$e^{ju} - e^{-ju} / 2j$$

$$= X_0 + \sum_{n=1}^{+\infty} 2a_n \cos(n\omega_0 t)$$

$$- \sum_{n=1}^{+\infty} 2b_n \sin(n\omega_0 t)$$

Trigonometric (Real) FS (ver. 2)

Special cases.

even · even = even
signal signal

even · odd = odd
sig. sig

odd · odd = even
sig sig.

$$\int_{-a}^{a} \text{odd signal} = 0$$

$$a_n = \frac{1}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)\, dt$$

$$b_n = -\frac{1}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\, dt$$

1) $x(t)$ real and even

$b_n = 0$

2) $x(t)$ real and odd

$a_n = 0$