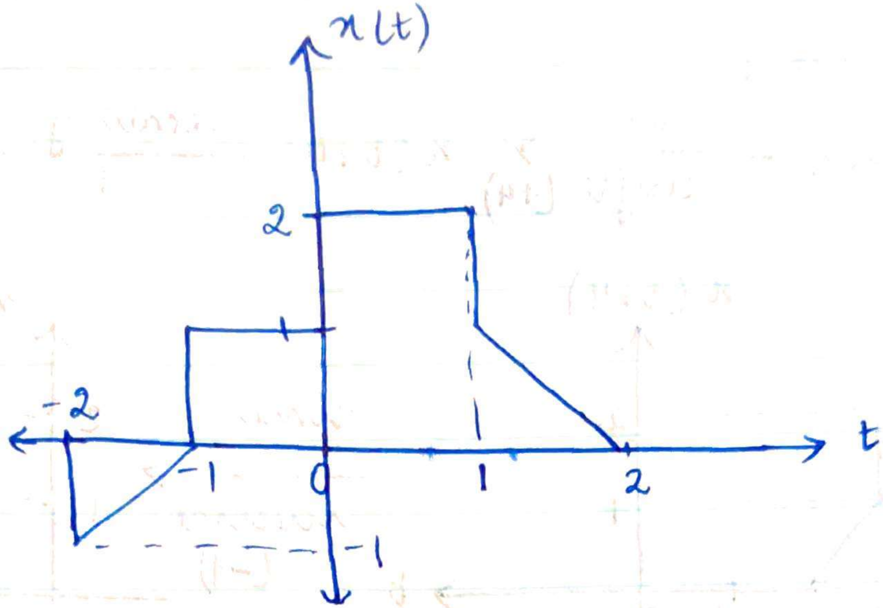
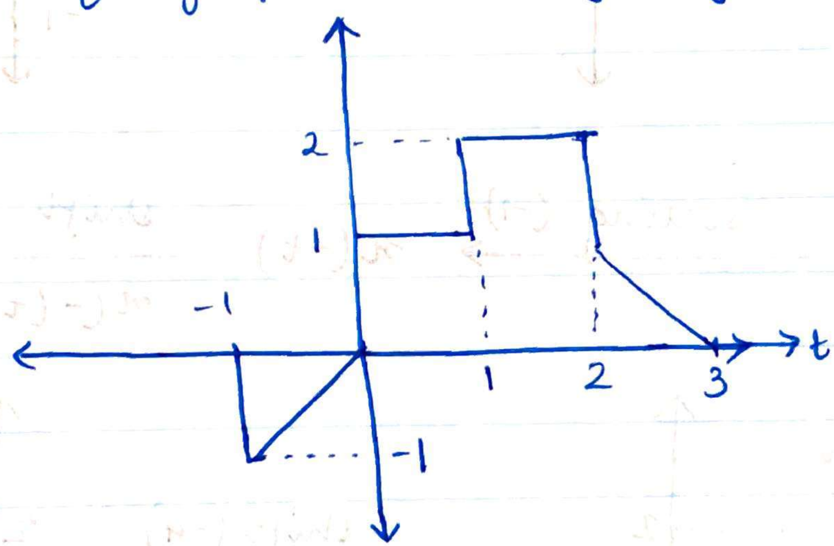
HOMEWORK (1)

Question (1)



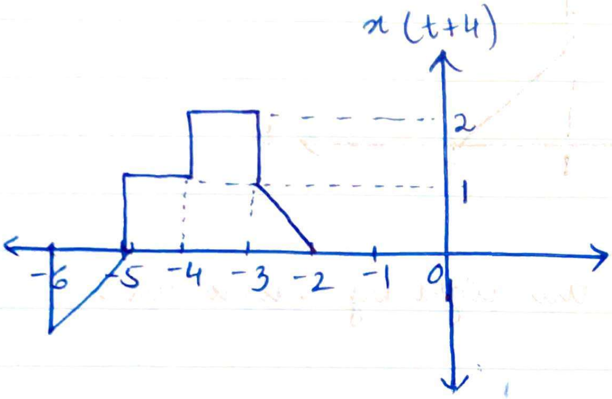
(a) $x(t-1)$: shift graph to the right by 1 time unit.



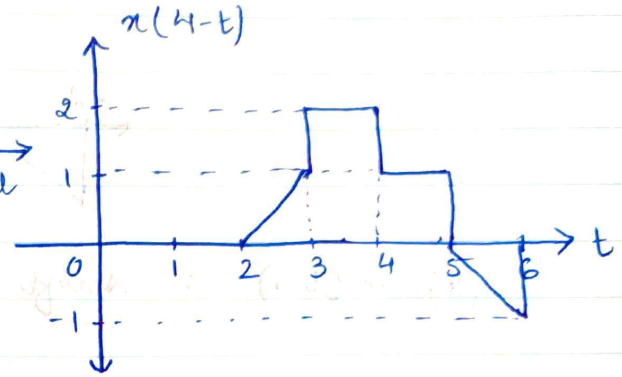
(b) $x(4-t)$:

Two approaches :

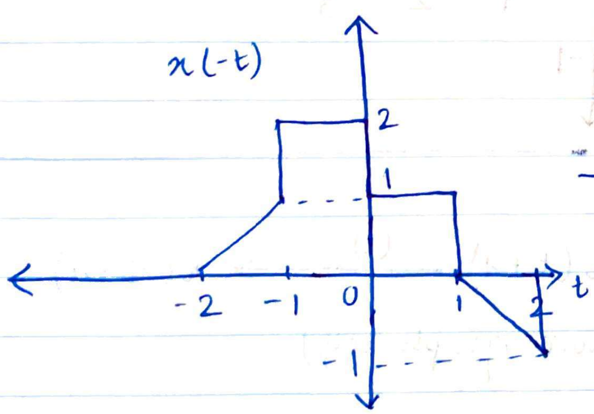
1) $x(t) \xrightarrow[\text{shift } (+4)]{\text{time}} x(t+4) \xrightarrow[-1]{\text{scaling}} x(-t+4)$



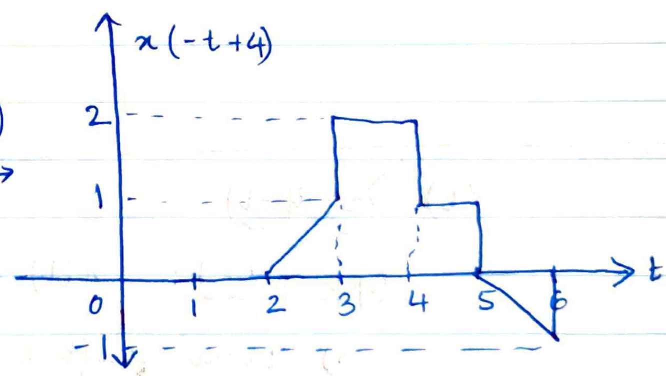
$\xrightarrow[\text{reversal } (-1)]{\text{time}}$



2) $x(t) \xrightarrow{\text{scaling } (-1)} x(-t) \xrightarrow[x(-(t-4))]{\text{shift } (-4)} x(4-t)$



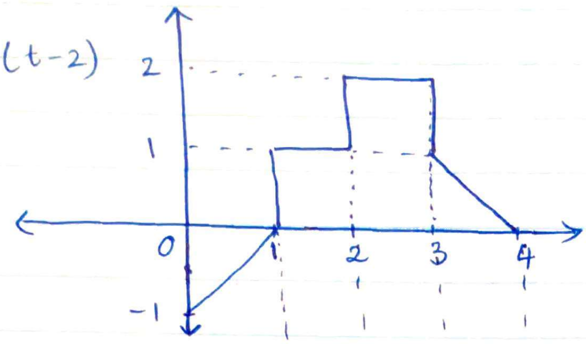
$\xrightarrow{\text{shift } (-4)}$



Both approaches give the same result. Doing time shifting or time scaling is a matter of preference. However, if scaling first, the shift needs to be modified to account for the scaling factor.

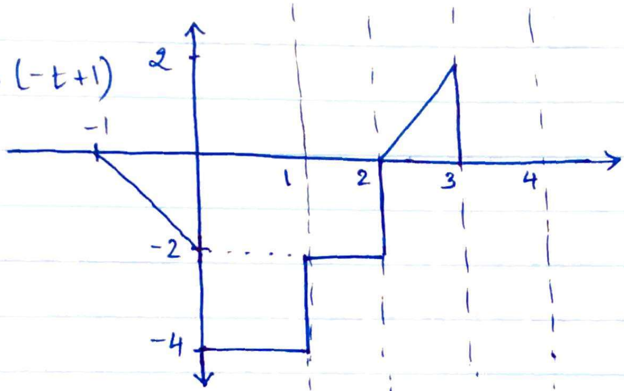
(c) $\left[x(t-2) - 2x(-t+1)\right] u\left(t-\frac{1}{2}\right)$

$x(t-2)$



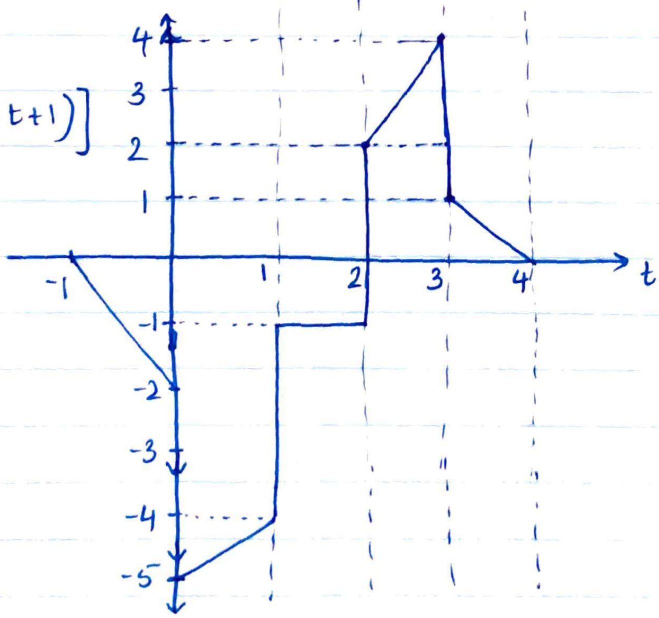
$$x(t-2) = \begin{cases} 0 & t<0,\ t\geq 4 \\ t-1 & 0\leq t<1 \\ 1 & 1\leq t<2 \\ 2 & 2\leq t<3 \\ -t+4 & 3\leq t<4 \end{cases}$$

$-2x(-t+1)$

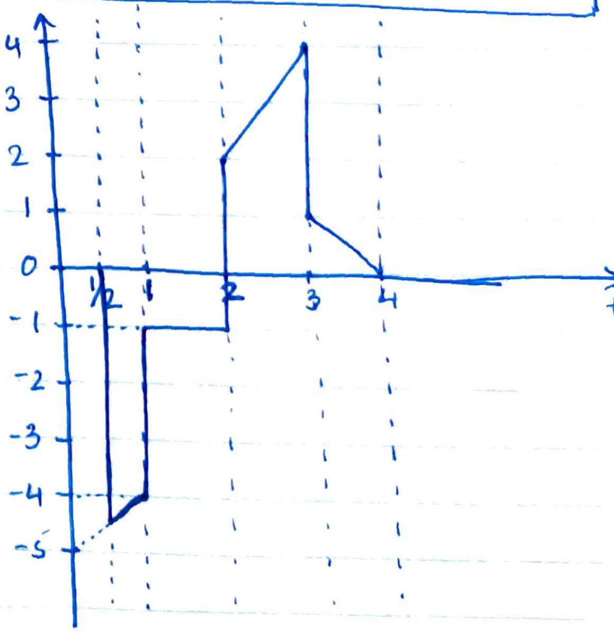


$$-2x(-t+1) = \begin{cases} 0 & t<-1,\ t\geq 3 \\ -2t-2 & -1\leq t<0 \\ -4 & 0\leq t<1 \\ -2 & 1\leq t<2 \\ 2t-4 & 2\leq t<3 \end{cases}$$

$\left[x(t-2) - 2x(-t+1)\right]$



$\left[x(t-2) - 2x(-t+1)\right] u\left(t-\frac{1}{2}\right)$



$$[x(t-2) - 2x(-t+1)]\,u(t-\tfrac{1}{2}) = \begin{cases} t-5 & \frac{1}{2} \leq t < 1 \\ -1 & 1 \leq t < 2 \\ 2t-2 & 2 \leq t < 3 \\ -t+4 & 3 \leq t < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Question ②

(a) Using Euler's theorem : $\left(\cos(\theta t) = \dfrac{e^{j\theta t} + e^{-j\theta t}}{2}\right)$ and $\left(\sin(\psi t) = \dfrac{e^{j\psi t} - e^{-j\psi t}}{2j}\right)$

$$a(t) = \cos(\theta t)\sin(\psi t)$$

$$= \left(\frac{e^{j\theta t} + e^{-j\theta t}}{2}\right)\left(\frac{e^{j\psi t} - e^{-j\psi t}}{2j}\right)$$

$$= \frac{1}{4j}\left(e^{j(\theta+\psi)t} - e^{j(\theta-\psi)t} - e^{-j(\theta+\psi)t} + e^{-j(\theta-\psi)t}\right)$$

$$= \frac{1}{2}\left[\left(\frac{e^{j(\theta+\psi)t} - e^{-j(\theta+\psi)t}}{2j}\right) - \left(\frac{e^{j(\theta-\psi)t} - e^{-j(\theta-\psi)t}}{2j}\right)\right]$$

$$= \frac{1}{2}\left[\sin(\theta+\psi)t - \sin(\theta-\psi)t\right]$$

$$\therefore \boxed{a(t) = \frac{1}{2}\left[\sin(\theta+\psi)t - \sin(\theta-\psi)t\right]}$$ Hence proved

(b) For $a(t)$ to be periodic, the ratio of periods of $\sin(\theta+\psi)t$ and $\sin(\theta-\psi)t$ must be rational.

$$\left.\begin{array}{l} T_1 : \sin(\theta+\psi)t \longrightarrow T_1 = \dfrac{2\pi}{\theta+\psi} \\ T_2 : \sin(\theta-\psi)t \qquad\ \ T_2 = \dfrac{2\pi}{\theta-\psi} \end{array}\right\} \quad \frac{T_2}{T_1} = \frac{\theta+\psi}{\theta-\psi}$$

If $\dfrac{\theta+\psi}{\theta-\psi} = \dfrac{M}{N}$; $M, N \in \mathbb{N}$; M and N are coprime, then $a(t)$ is periodic.

period of $a(t) = T_0 = \text{LCM}\left\{ \frac{2\pi}{\theta+\psi}, \frac{2\pi}{\theta-\psi} \right\}$

For period of $a(t) = 3$:

Note : $\boxed{\text{LCM}\{a,b\} \geq \max\{a,b\}}$

The least common multiple (LCM) of 2 no's is greater than or equal to the larger no.

* Equality holds when $\max\{a,b\}$ is an ~~inte~~ integer multiple of $\min\{a,b\}$

$\therefore$ The period T_0 must be $\geq \frac{2\pi}{\theta-\psi}$ (the greater period) $T_2 > T_1$

$\therefore$ let $T_0 = \frac{2\pi}{\theta-\psi} k$; where $k \in \mathbb{N}$

i) We want to find ψ, given $\theta = 2\pi$, s.t. $T_0 = 3$

$\therefore \quad \frac{2\pi k}{2\pi - \psi} = 3 \Rightarrow \boxed{\psi = (6-2k)\frac{\pi}{3}}$ ensures period of $a(t)$ is 3

ii) We still need to ensure that $\frac{M}{N}$ is rational. ~~Thereafter,~~

Thereby, we find the smallest value of $k \in \mathbb{N}$ ~~such~~ for which $\frac{M}{N}$ is rational and positive, for $\theta = 2\pi$ and $\psi = (6-2k)\frac{\pi}{3}$

$$\frac{M}{N} = \frac{\theta+\psi}{\theta-\psi} = \frac{2\pi + (6-2k)\frac{\pi}{3}}{2\pi - (6-2k)\frac{\pi}{3}} = \boxed{\frac{6-k}{k}}$$

$\frac{M}{N} = \frac{6-k}{k}$ is rational $\forall\ k = 1, 2, 3, 4, 5$

~~We~~ Smallest k which ~~satisfies~~ achieves

i) periodicity of $a(t)$ (ie. rational M/N)

ii) period T_0 of $a(t) = 3$

} is $\boxed{k=1}$

$\therefore$ ~~T_0~~ $=$ $\therefore \psi = (6-2)\frac{\pi}{3} \Rightarrow \boxed{\psi = \frac{4\pi}{3}}$; $\boxed{\frac{M}{N} = \frac{5}{1}}$

(*) We select the smallest k that achieves (i) and (ii) because period T_0 is the LCM or the least common multiple

$$T_0 = \frac{2\pi k}{\theta - \psi}$$

(c)

$$x(t) = \underbrace{2\cos(10t+1)}_{T_1} - \underbrace{\sin(4t-1)}_{T_2}$$

period $T_1 = \frac{2\pi}{10} = \frac{\pi}{5}$

period $T_2 = \frac{2\pi}{4} = \frac{\pi}{2}$

$\text{LCM}\left\{\frac{\pi}{5}, \frac{\pi}{2}\right\} = \pi$

$\therefore$ $\boxed{\text{fundam}^{\text{tn}}\ \text{period} = \pi}$

Question (3)

$$x(t) = \cos(3\Omega_0 t) + 5\cos(\Omega_0 t) \quad ; \ \Omega_0 = \pi$$

(a) $x(t) = \cos(3\pi t) + 5\cos(\pi t)$

$T_1 = \frac{2\pi}{3\pi} = \frac{2}{3}$ $\quad T_2 = \frac{2\pi}{\pi} = 2$ $\quad \frac{T_1}{T_2} = \frac{1}{3}$ = rational $\therefore$ periodic

$\text{LCM}\{\frac{2}{3}, 2\} = 2$ $\quad \therefore \boxed{\text{Period } T_0 = 2}$

(b) Average power of $x(t)$:

$$P_x = \frac{1}{T_0}\int_0^{T_0} x^2(t)\,dt$$

$\because$ $x(t)$ is a real valued signal. Thus, $|x(t)|^2 = x^2(t)$

$$= \frac{1}{2}\int_0^2 \left(\cos(3\pi t) + 5\cos(\pi t)\right)^2 dt$$

$$= \frac{1}{2}\int_0^2 \left[\cos^2(3\pi t) + 25\cos^2(\pi t) + 10\cos(3\pi t)\cos(\pi t)\right] dt$$

$$= \frac{1}{2}\int_0^2 \cos^2(3\pi t)\,dt + \int_0^2 \frac{25}{2}\cos^2(\pi t)\,dt + 5\int_0^2 \cos(3\pi t)\cos(\pi t)\,dt$$

$$= \frac{1}{2}\int_0^2 \frac{\cos(6\pi t) + 1}{2}\,dt + \frac{25}{2}\int_0^2 \frac{\cos(2\pi t) + 1}{2}\,dt$$

$$+ \frac{5}{2}\int_0^2 \left(\cos(2\pi t) + \cos(4\pi t)\right) dt$$

$$\boxed{\begin{array}{l} \circledast \ \cos(2x) = 2\cos^2(x) - 1 \\ \circledast \ \cos A \cdot \cos B = \frac{1}{2}\left(\cos(A-B) + \cos(A+B)\right) \end{array}}$$

$$\therefore\ P_x = \frac{1}{4}\int_0^2 \left(\cos(6\pi t)+1\right) dt + \frac{25}{4}\int_0^2 \left(\cos(2\pi t)+1\right) dt$$

$$+ \frac{5}{2}\underbrace{\int_0^2 \left(\cos(2\pi t) + \cos(4\pi t)\right) dt}_{\text{zero}}$$

$\Rightarrow$ ~~P_x~~ $=$

$\because$ period of $[\cos(2\pi t)+\cos(4\pi t)] = 1$

$\Rightarrow \int dt$ over 2 periods $= 0$.

$$\therefore\ P_x = \frac{1}{4}\int_0^2 \left(\cos(6\pi t)+1\right) dt + \frac{25}{4}\int_0^2 \left(\cos(2\pi t)+1\right) dt$$

$$= P_1 + P_2 = \frac{1}{4}(2) + \frac{25}{4}(2) = \underline{\underline{13}}$$

$$\boxed{P_x = 13}$$

(c) since $\frac{1}{T_0}\int_0^{T_0} 2\cos(3\pi t) \cdot 5\cos(\pi t)\, dt = 0$,

$$P_x = \frac{1}{T_0}\int_0^{T_0} \cos^2(3\pi t)\, dt + \frac{1}{T_0}\int_0^{T_0} \cos^2(\pi t) \cdot 25 \cdot dt$$

$$= P_1 + P_2 \qquad \therefore \boxed{P_x = P_1 + P_2}$$

$$= \frac{1}{2} + \frac{25}{2}$$

$$= 13$$

(d) Consider a general case, where

$$\gamma(t) = \cos(t) + \cos(at)$$; where a is not necessarily rational.

ie $a \in \mathbb{R}$; ~~$a \notin \mathbb{Q}$~~ ~~$a \notin \mathbb{Q}$~~

P_γ :

$\cos(t)$: period $= 2\pi$

$\cos(at)$: period $= \frac{2\pi}{a}$

$\Bigg\}$ ratio of periods $=$ ~~$a \notin \mathbb{Q}$~~

$\therefore \gamma(t)$ may not necessarily be periodic

$\therefore$ ~~$\gamma(t)$ is not periodic~~ may not be

Thus, power P_γ of $\gamma(t)$ needs to be computed on infinite support.

$$P_\gamma = \lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma^2(t)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T (\cos(t) + \cos(at))^2\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \cos^2(t)\,dt + \lim_{T\to\infty} \frac{1}{T}\int_0^T \cos^2(at)\,dt$$

$$+ \boxed{\lim_{T\to\infty} \frac{1}{T}\int_0^T 2\cos(t)\cos(at)\,dt}$$

We consider this term.

$$\int_0^T 2\cos(t)\cos(at)\,dt = \int_0^T \cos[(1+a)t]\,dt + \int_0^T \cos[(1-a)t]\,dt$$

$$\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$\therefore \int_0^T 2\cos(t)\cos(at)\,dt = \int_0^T \underbrace{\cos[(1+a)t]}_{\text{period} = \frac{2\pi}{1+a}}dt + \int_0^T \underbrace{\cos[(1-a)t]}_{\text{period} = \frac{2\pi}{1-a}}dt$$

Thus. $\lim_{T\to\infty} \frac{1}{T}\int_0^T \cos[(1+a)t]\,dt = 0$ $\because$ we are integrating a periodic f^n over a very large time interval.

$$\boxed{\therefore \lim_{T\to\infty} \frac{1}{T}\int_0^T 2\cos(t)\cos(at)\,dt = 0}$$

Thus, P_γ reduces to the following:

$$P_\gamma = \lim_{T\to\infty} \frac{1}{T}\int_0^T \cos^2(t)\,dt + \lim_{T\to\infty} \frac{1}{T}\int_0^T \cos^2(at)\,dt$$

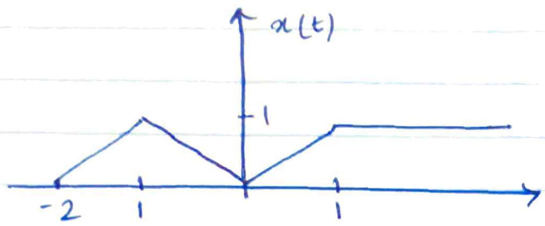
$$= P_1 + P_2$$

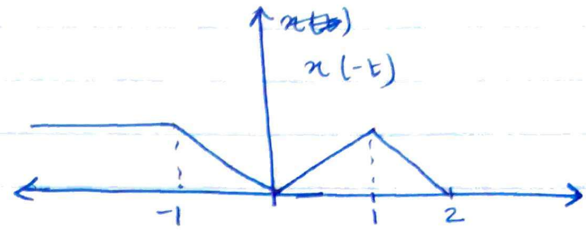
Thus, $\boxed{P_\gamma = P_1 + P_2}$ This holds for general a,

$\therefore$ The power of P_γ is the sum of its constituent sinusoids, regardless of whether or not they are harmonically related, or whether the ratio of their periods is rational.

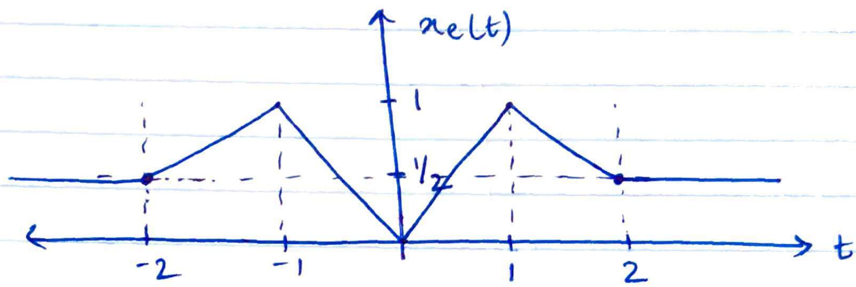
Question (4)

(a)

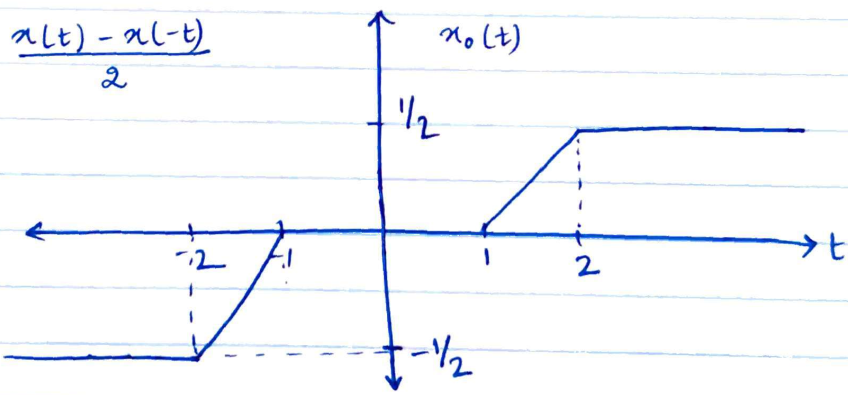




$$x_e(t) = \frac{x(t) + x(-t)}{2}$$



$$x_o(t) = \frac{x(t) - x(-t)}{2}$$



(b)

$$x(t) = x_e(t) + x_o(t)$$

$$\text{Energy of signal} = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

$$= \int_{-\infty}^{\infty} |x_e(t) + x_o(t)|^2\,dt$$

$$= \int_{-\infty}^{\infty} \left(|x_e(t)|^2 + |x_o(t)|^2 + 2\langle x_e, x_o\rangle\right) dt$$

$$= \int_{-\infty}^{\infty} |x_e(t)|^2 \, dt + \int_{-\infty}^{\infty} |x_o(t)|^2 \, dt + \underbrace{\int_{-\infty}^{\infty} 2\langle x_e(t), x_o(t)\rangle \, dt}_{\text{this is an odd function}}$$

$$\therefore \int_{-\infty}^{\infty} (\text{odd } f^n) = 0$$

$$\therefore \quad E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x_e(t)|^2 \, dt + \int_{-\infty}^{\infty} |x_o(t)|^2 \, dt$$

Hence proved.