Lecture 8

- Inverse Laplace Transform
- Solving Linear Differential Equation with constant coefficients using Laplace T
- Analysis of LTI, C using Laplace Transform

$\delta(t) \longrightarrow 1$ All s

$u(t) \longrightarrow \frac{1}{s}$ $Re\{s\}>0$

$r(t) \longrightarrow \frac{1}{s^2}$ $Re\{s\}>0$

$$\cos(\Omega_0 t)u(t) \longrightarrow \frac{s}{s^2+\Omega_0^2} \quad Re\{s\}>0$$

$$\sin(\Omega_0 t)u(t) \longrightarrow \frac{\Omega_0}{s^2+\Omega_0^2} \quad Re\{s\}>0$$

$$\cdot\, t^n u(t) \longrightarrow \frac{n!}{s^{n+1}} \quad Re\{s\}>0$$

Properties of Laplace T.

① $\alpha f(t)+\beta g(t) \longrightarrow \alpha F(s)+\beta G(s)$

② $e^{-at} f(t) \longrightarrow F(s+a)$

③ $\dfrac{df(t)}{dt} \longrightarrow sF(s)-f(0^-)$

$$\frac{d^2 f(t)}{dt^2} \rightarrow s^2 F(s) - s f(0^-) - f'(0^-)$$

④ $$\int_0^t f(\tau)\, d\tau \rightarrow \frac{F(s)}{s}$$

⑤ $$f(t-d)\, u(t-d) \rightarrow e^{-sd} F(s)$$

⑥ $$t^n f(t) \rightarrow (-1)^n \frac{d^n F(s)}{ds^n}$$

⑦ $$x(t) * h(t) \rightarrow X(s) \cdot H(s)$$

Inverse Laplace Transform

* $X(s) = \frac{N(s)}{D(s)}$.

$N(s)$, $D(s)$ are polynomials in s.

** $X(s) = e^{-as} \frac{N(s)}{D(s)}$

*** $X(s) = \frac{N(s)}{D(s)} \cdot \frac{1}{1-e^{-as}}$

① $\deg(N(s)) \geq \deg(D(s))$

X(S) is not a proper fractional polynomial in s, $\Rightarrow$ we shoud divide N(S) with D(S).

e.g. $X(s) = \dfrac{s^2+1}{s+2}$

$$(s^2+1):(s+2) = s-2+\frac{5}{s+2}$$

$$\begin{array}{r} -(s^2+2s) \\ \hline -2s+1 \\ +2s+4 \\ \hline 5 \end{array}$$

$$X(s) = \frac{s^2+1}{s+2} = \frac{s^2+2s-2s+1}{s+2}$$

$$= s + \frac{-2s+1}{s+2}$$

$$= s + \frac{-2s-4+4+1}{s+2}$$

$$= s - 2 + \frac{5}{s+2}$$

$$X(s) = \frac{s^2+1}{s+2} = s - 2 + \frac{5}{s+2}$$

$$x(t) = \frac{d}{dt}\delta(t) - 2\delta(t) + 5e^{-2t}u(t)$$

② $X(s)$: $\deg(N(s)) < \deg(D(s))$

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-b_0)(s-b_1)\cdots(s-b_n)}$$

Case 1: All poles $\{b_0, \dots, b_1\}$ are real and distinct

Case 2: There are complex poles that are distinct

Case 3: There are repeated poles.

Case 1: Real & distinct poles

$$X(s) = \frac{N(s)}{(s-b_0)(s-b_1)+\cdots(s-b_n)}$$

$$= \frac{A_0}{s-b_0} + \frac{A_1}{s-b_1} + \cdots + \frac{A_n}{s-b_n}$$

$$A_0 = X(s)\cdot(s-b_0)\Big|_{s=b_0}$$

$$A_1 = X(s)\cdot(s-b_1)\Big|_{s=b_1}$$

$$x(t) = A_0 e^{b_0 t} u(t) + A_1 e^{b_1 t} u(t) + \cdots$$

$$+ \dots + A_n e^{bn} u(t)$$

Ex. $F(s) = \dfrac{s}{s^2 + 3s + 2}$

$$s^2 + 3s + 2 = 0$$

Cheet sheat $as^2 + bs + c = 0$

$$s_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$s_{1/2} = \frac{-3 \pm \sqrt{9 - 4 \cdot 2}}{2}$$

$$s_{1/2} = \frac{-3 \pm 1}{2}$$

$$s_1 = -1$$

$$s_2 = -2$$

$$F(s) = \frac{s}{(s+1)(s+2)}$$

$$= \frac{A_1}{s+1} + \frac{A_2}{s+2} \quad \left\{ \begin{aligned} & A_1(s+2) + A_2(s+1) \\ & \quad = s \\ & (A_1+A_2)s + 2A_1 + A_2 \\ & \quad = s \end{aligned} \right.$$

$$A_1 = F(s)\cdot(s+1)\Big|_{s=-1} = \frac{s}{s+2}\Big|_{s=-1} = -1$$

$$A_2 = F(s)(s+2)\Big|_{s=-2} = \frac{s}{s+1} = 2$$

$$A_1 + A_2 = 1$$

$$2A_1 + A_2 = 0$$

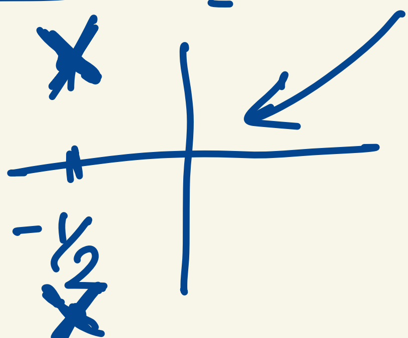
$$F(s) = -\frac{1}{s+1} + \frac{2}{s+2}$$

$$f(t) = -e^{-t}u(t) + 2e^{-2t}u(t)$$

Case 2: Complex roots.

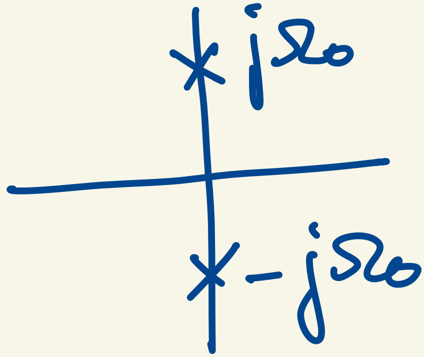
$$F(s) = \frac{s}{s^2+s+1} \cdot$$

$$s_{1/2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm j\sqrt{3}}{2}$$



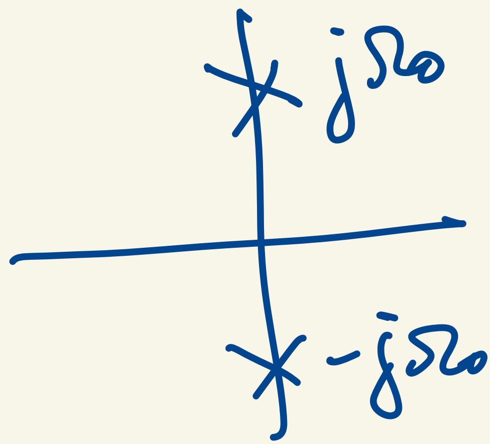
$\cos(\Omega_0 t)u(t)$

$$\frac{s}{s^2+\Omega_0^2}$$



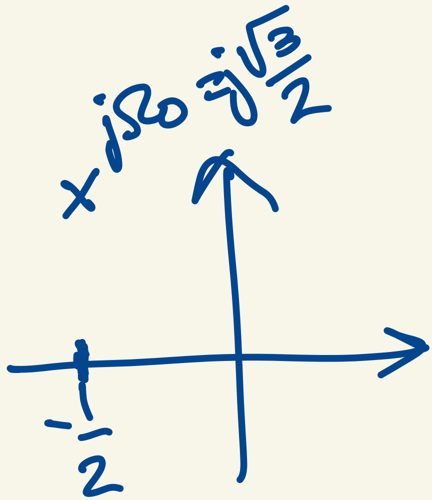
$\sin(\Omega_0 t)u(t)$

$$\frac{\Omega_0}{s^2+\Omega_0^2}$$



$e^{-\frac{1}{2}t}\cos(\Omega_0 t)u(t)$

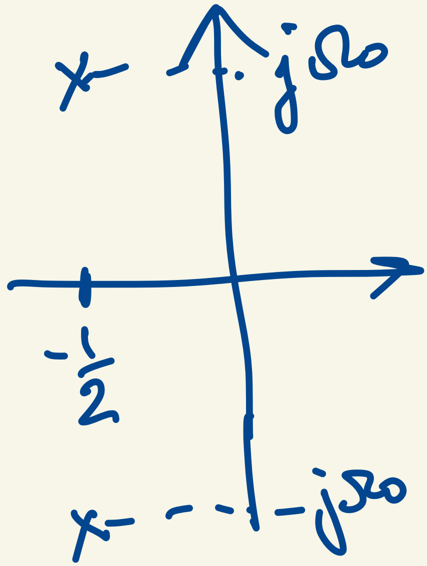
$$\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\Omega_0^2}$$



$$\left(s+\frac{1}{2}\right)^2+\Omega_0^2=0$$

$$\left(s+\frac{1}{2}\right)^2=-\Omega_0^2$$

$$s=-\frac{1}{2}\pm j\Omega_0$$



$$e^{-\frac{1}{2}t}\sin(\Omega_0 t)u(t)$$

$$F(s)=\frac{s}{s^2+s+1}=\frac{s}{\underbrace{s^2+2\cdot\frac{1}{2}\cdot s+\left(\frac{1}{2}\right)^2}-\left(\frac{1}{2}\right)^2+1}$$

$$=\frac{s}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\frac{1}{\cancel{2}}\frac{\sqrt{3}}{2}\cdot\frac{\cancel{2}}{\sqrt{3}}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)\cdot u(t) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$$

Case 3: $D(s)$ has repeated poles.

$$F(s) = \frac{N(s)}{(s-b_0)(s-b_1)^2}$$

$$= \frac{A_0}{s-b_0} + \frac{B_0 s + C_0}{(s-b_1)^2} \leftarrow \text{Method } 2$$

$$= \frac{A_0}{s-b_0} + \frac{B_1}{s-b_1} + \frac{B_2}{(s-b_1)^2}$$

$$A_0 = F(s)\cdot(s-b_0)\Big|_{s=b_0}$$

$$B_2 = F(s)\,(s-b_1)^2\Big|_{s=b_1}$$

$$B_1 = \frac{d}{ds}\left\{F(s)\cdot(s-b_1)^2\right\}\Big|_{s=b_1}$$

Ex. $F(s) = \dfrac{1}{(s+2)(s+1)^2}$

$$= \frac{A_0}{s+2} + \frac{B_1}{s+1} + \frac{B_2}{(s+1)^2}$$

$$A_0 = F(s)\cdot(s+2)\Big|_{s=-2} = \frac{1}{(s+1)^2}\Big|_{s=-2} = 1$$

$$B_2 = F(s)(s+1)^2\Big|_{s=-1} = \frac{1}{s+2}\Big|_{s=-1} = 1$$

$$B_1 = \frac{d}{ds}\left\{F(s)\cdot(s+1)^2\right\}\Big|_{s=-1} = \frac{d}{ds}\left\{\frac{1}{s+2}\right\}\Big|_{s=-1}$$

$$= -\frac{1}{(s+2)^2}\Big|_{s=-1} = -1$$

$$F(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$f(t) = e^{-2t}u(t) - e^{-t}u(t) + \underline{\underline{te^{-t}u(t)}}$$

$$\frac{1}{s^2} \rightarrow t\,u(t)$$

$$\frac{1}{(s+1)^2} \rightarrow t e^{-t} u(t)$$

$$X(s) = \boxed{\frac{N(s)}{D(s)}} \cdot \frac{1}{1-e^{-\alpha s}}$$

$$F(s) \xrightarrow{\mathcal{L}_s^{-1}} \underline{\underline{f(t)}}$$

$$\frac{1}{1-q} = \sum_{n=0}^{+\infty} q^n$$

$$\frac{1}{1-e^{-\alpha s}} = \sum_{n=0}^{+\infty} e^{-\alpha n s}$$

$$X(s) = F(s) \cdot \sum_{n=0}^{+\infty} e^{-\alpha n s}$$

$$= \sum_{n=0}^{+\infty} F(s) e^{-\alpha n s}$$

$$x(t) = \sum_{n=0}^{+\infty} f(t - n\alpha)$$

Example:

$$X(s) = \underbrace{\frac{1-e^{-s}}{(s+1)}}_{F(s)} \frac{1}{(1-e^{-2s})}$$

$$X(s) = F(s) \cdot \frac{1}{1-e^{-2s}}$$

$$= F(s) \cdot \sum_{n=0}^{+\infty} e^{-2ns}$$

$\mathcal{L}^{-1}$

$$F(s) = \frac{1-e^{-s}}{s+1} = \frac{1}{s+1} - \frac{1}{s+1}e^{-s}$$

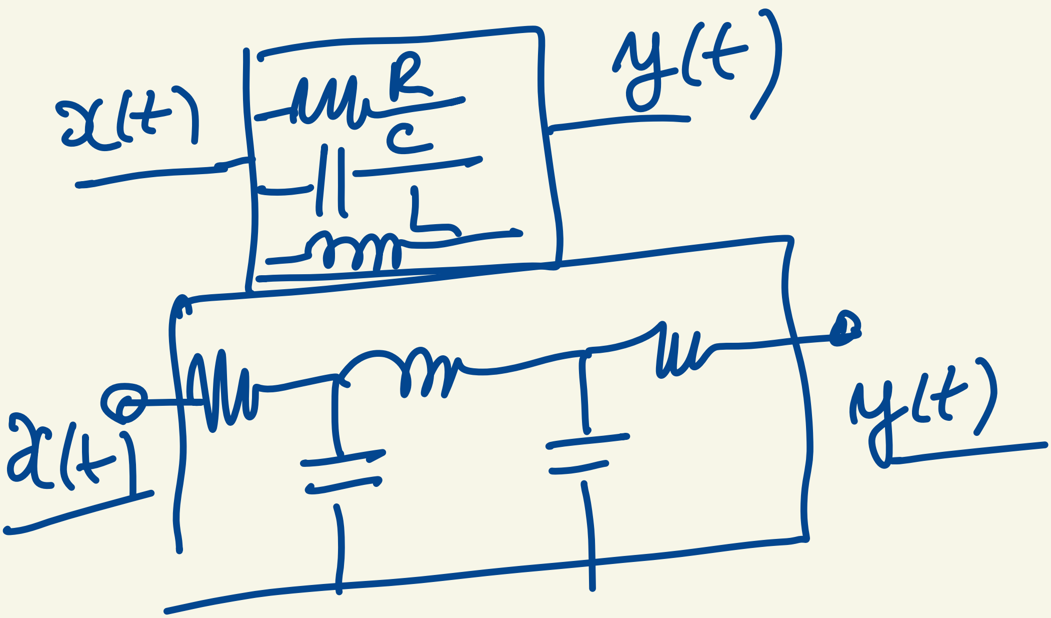
$$f(t) = e^{-t}u(t) - e^{-(t-1)}u(t-1)$$

$$x(t) = \sum_{n=0}^{+\infty} f(t-2n)$$

$$x(t) = \sum_{n=0}^{+\infty} \left\{ e^{-(t-2n)} u(t-2n) - \right.$$

$$- e^{-(t-2n-1)} u(t-2n-1) \}$$

Application of Laplace Trans. to Solving Linear Constant Coefficients Differential Equations



order(n)

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} y(t) + \dots$$

$$+ \dots + a_1 y(t) = x(t) \quad \underline{t>0}$$

$$= \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1}}{dt^{m-1}} \dots + b_1 x(t)$$

n initial conditions

$$y(0^-), y'(0^-), \dots, y^{(n-1)}(0^-)$$

1. Take Laplace Transf. of both sides

Y(S) in terms of X(S)

(2) Given $x(t) \longrightarrow X(S)$

+ get inverse Laplace T. of Y(S).

Ex. $\frac{d^2 y(t)}{dt^2} + y(t) = 1 \quad \underline{\underline{t > 0}}$

" u(t)

$y(\underline{\underline{0^-}}) = 1 \quad y'(\underline{\underline{0^-}}) = 2$

$\mathcal{L}_s\left\{\frac{d^2 y(t)}{dt^2} + y(t)\right\} = \mathcal{L}_s\left\{u(t)\right\}$

$$s^2 Y(s) - s\,y(0^-) - y'(0^-)$$
$$+ Y(s) = \frac{1}{s}$$

$$Y(s)(1+s^2) - s - 2 = \frac{1}{s}$$

$$Y(s)(1+s^2) = \frac{1}{s} + s + 2$$

$$Y(s)(1+s^2) = \frac{s^2+2s+1}{s}$$

$$Y(s) = \frac{s^2 + 2s + 1}{s(s^2+1)}$$

$$= \frac{s^2+1 + 2s}{(s^2+1)s}$$

$$= \frac{1}{s} + \frac{2}{s^2+1}$$

$$\boxed{y(t) = u(t) + 2\sin(t)u(t)}$$

Analysis of LTI, C System using Laplace Transform

$$x(t) \longrightarrow \boxed{h(t)\cdot u(t)} \longrightarrow y(t)$$

S: LTI, C

$x(t)$ → input has to be causal

① given $x(t)$ and $h(t)$

$$\Rightarrow \quad y(t) = x(t) * h(t)$$

$$\begin{array}{ccccc} y(t) & = & x(t) & * & h(t) \\ \downarrow L_s & & \downarrow L_s & & \downarrow L_s \\ Y(s) & & X(s) & & H(s) \end{array}$$

$$Y(s) = \underline{H(s)} \cdot \underline{X(s)}$$

$$\Rightarrow \quad y(t) = \mathcal{L}_s^{-1}\{Y(s)\}$$