Fourier Transform Properties

8) Differentiation in time

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$\frac{d^n x(t)}{dt^n} \xrightarrow{\mathcal{F}} (j\omega)^n X(\omega)$$

Proof:

1st order derivative.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} \, d\omega$$

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}\,d\omega\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\frac{d}{dt}\left\{e^{j\omega t}\right\}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{X(\omega)\,j\omega}\,e^{j\omega t}\,d\omega$$

Looks like $\mathcal{F}^{-1}\left\{j\omega X(\omega)\right\} = \frac{dx(t)}{dt}$

$$\boxed{\mathcal{F}\left\{\frac{dx(t)}{dt}\right\} = j\omega X(\omega)}$$

$$\mathcal{F}\left\{\frac{d^2x(t)}{dt^2}\right\} = j\omega\,\mathcal{F}\left\{\frac{dx(t)}{dt}\right\}$$

$$= (j\omega)(j\omega)\cdot X(\omega)$$

$$= (j\omega)^2 X(\omega)$$

Integration property:

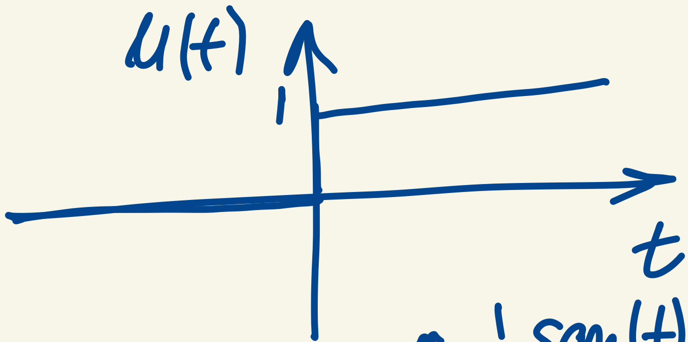
$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{+\infty} x(\tau)\cdot u(t-\tau)\,d\tau$$

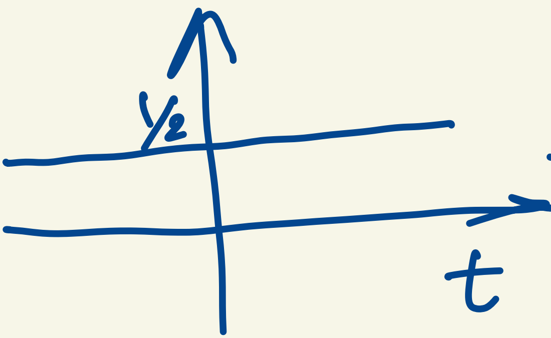
$$= x(t) * u(t)$$

To find $\mathcal{F}\left\{\int_{-\infty}^{\tau} x(\tau)\,d\tau\right\}$ we will need

$\mathcal{F}\{u(t)\}$ and $\mathcal{F}\{x(t)*y(t)\}$

$\mathcal{F}\{u(t)\}$.





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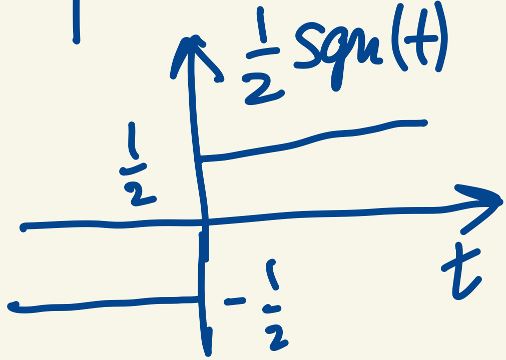
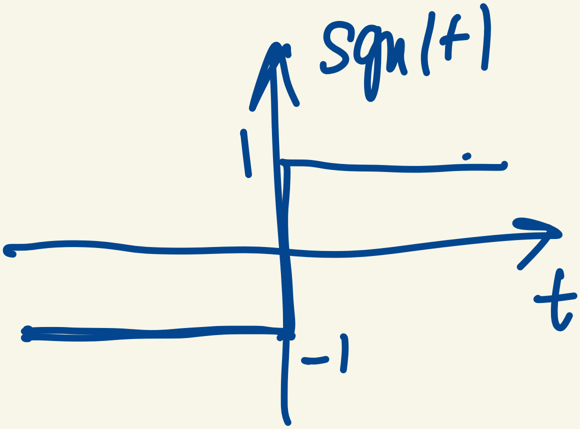
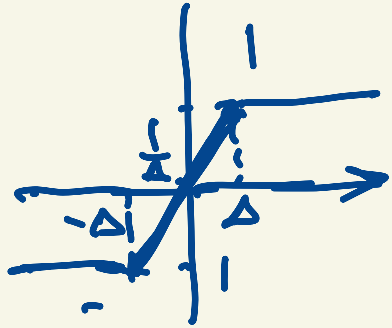


$$\mathrm{sgn}(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$

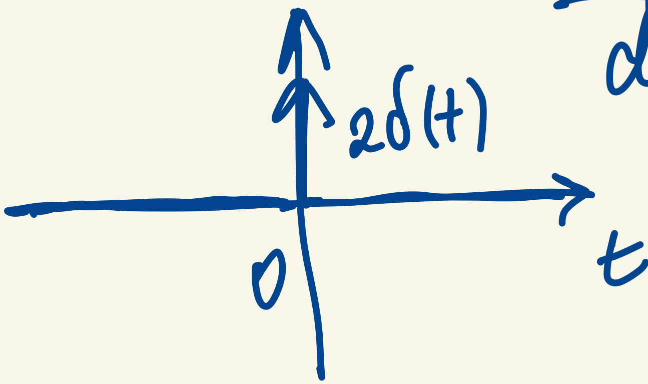
$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{1}{2}\right\} + \frac{1}{2}\mathcal{F}\{\underline{\mathrm{sgn}(t)}\}$$





$$\frac{d\,\mathrm{sgn}(t)}{dt} = 2\delta(t)$$



$$\frac{d\,\mathrm{sgn}(t)}{dt} = 2\delta(t)$$

$$\mathcal{F}\left\{\frac{d\,\mathrm{sgn}(t)}{dt}\right\} = \mathcal{F}\{2\delta(t)\}$$

$$j\omega\,\mathcal{F}\{\mathrm{sgn}(t)\} = 2$$

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \frac{2}{j\omega}$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\left\{\frac{1}{2}\right\} + \frac{1}{2}\mathcal{F}\{\operatorname{sgn}(t)\}$$

$$= \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{j\omega}$$

$$= \frac{1}{\cancel{2}}\cancel{2}\pi\delta(\omega) + \frac{1}{j\omega}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

$$u(t) \xrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega)$$

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⑨ Convolution Property.

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$h(t) \xrightarrow{\mathcal{F}} H(\omega)$$

$$y(t) = x(t) * h(t) = \underbrace{\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

$$\mathcal{F}\{x(t) * h(t)\} = \int_{-\infty_t}^{+\infty} \int_{-\infty_\tau}^{+\infty} x(\tau)\, h(t-\tau)\, \underline{\underline{d\tau}}\, e^{-j\omega t}\, \underline{\underline{dt}}$$

$$= \int_{-\infty}^{+\infty} x(\tau) \underbrace{\int_{-\infty}^{+\infty} h(t-\tau) e^{-j\omega t}\, dt}_{}\, d\tau$$

τ $\quad$ t $\quad$ change of variable.

$$t - \tau = \sigma$$

$$t = \sigma + \tau$$

$$dt = d\sigma$$

$$= \int_{-\infty}^{+\infty} x(\tau) \int_{-\infty}^{+\infty} h(\sigma) e^{-j\omega(\sigma+\tau)}\, d\sigma\, d\tau$$

τ $\quad$ σ

$$= \int_{-\infty}^{+\infty} x(\tau) \int_{-\infty}^{+\infty} h(\sigma) e^{-j\omega\sigma} \cdot e^{-j\omega\tau}\, d\sigma\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} \cdot \underbrace{\int_{-\infty}^{+\infty} h(\sigma) e^{-j\omega\sigma} d\sigma}_{\mathcal{F}\{h(t)\} = H(\omega)} d\tau$$

$$= H(\omega) \cdot \underbrace{\int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau}_{\mathcal{F}\{x(t)\} = X(\omega)}$$

$$= H(\omega) \cdot X(\omega)$$

$$\mathcal{F}\{x(t) * h(t)\} = X(\omega) \cdot H(\omega)$$

10 Integration Property.

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} =$$

$$= \mathcal{F}\left\{x(t) * u(t)\right\}$$

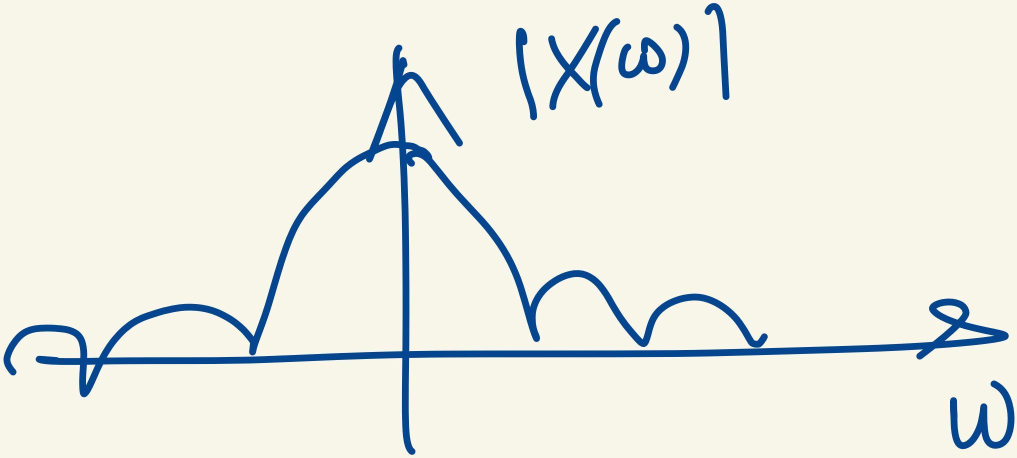
$$= \mathcal{F}\left\{x(t)\right\} \cdot \mathcal{F}\left\{u(t)\right\}.$$

$$= X(\omega) \circ \left[\frac{1}{j\omega} + \pi\delta(\omega) \right]$$

$$= \frac{1}{j\omega} X(\omega) + \pi \underline{\delta(\omega)} \cdot \underline{X(\omega)}$$

$$= \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$$

$$X(0) = \int_{-\infty}^{+\infty} x(t)\, dt$$



Parseval's Theorem for Fourier Transf.

Reminder:

We had Parseval's Theorem for Fourier Series (periodic)

$x(t)$ periodic /w T_0

$$P_x = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

$$= \sum_{k=-\infty}^{+\infty} |X_k|^2$$

Coming back to

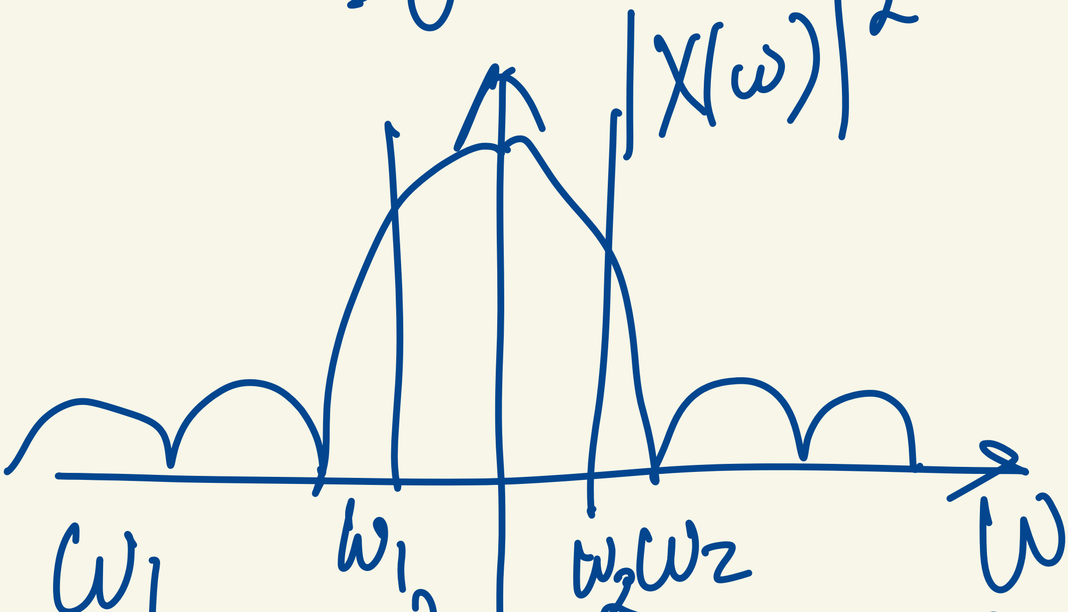
Parseval's T. for

Fourier Transform

$x(t)$

$$\mathcal{E}_x \triangleq \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega$$



$$= \frac{1}{2\pi} \int_{-\infty}^{\omega_1} |X(\omega)|^2 \, d\omega + \frac{1}{2\pi} \int_{\omega_1} |X(\omega)|^2 \, d\omega$$

$$+\frac{1}{2\pi}\int^{+\infty} |X(\omega)|^2 d\omega$$

$$\omega_2 \qquad J/Hz \quad Hz$$

$$\underset{\downarrow}{J}\ \ \mathcal{E}_x(\omega_1, \omega_2) = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} |X(\omega)|^2 d\omega$$

↓

energy of x in freq. band $[\omega_1, \omega_2]$

$$|X(\omega)|^2 \rightarrow \text{energy density}, \ \frac{J}{Hz}$$

$\Rightarrow$ // Power Spectral density !!

More general Parseval Theorem.

$$f(t) \xrightarrow{\mathcal{F}} F(\omega)$$

$$g(t) \xrightarrow{\mathcal{F}} G(\omega)$$

$$\int_{-\infty}^{+\infty} f(t) \cdot g^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot G^*(\omega)\, d\omega$$

inner product in time

inner product

Special case: in freq.

$f(t) = g(t)$

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega$$

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{+\infty} f(t) g^*(t) dt$$

$$\langle F(\omega), G(\omega) \rangle =$$

$$= \int_{-\infty}^{+\infty} F(\omega) \cdot G^*(\omega) d\omega .$$

Proof:

$$\int_{-\infty}^{+\infty} f(t) g^*(t) dt =$$

$$= \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega \right) \cdot$$

$$\cdot \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} G(\Omega) e^{j\Omega t} d\Omega \right)^* dt$$

$$= \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega \right) \cdot$$

$$\cdot \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} G^*(\Omega) e^{-j\Omega t} d\Omega \right) dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega) G^*(\Omega) e^{j(\omega-\Omega)t} \, d\Omega \, d\omega \, dt$$

$$t \qquad \omega \qquad \Omega$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\omega) \cdot G^*(\Omega) \int_{-\infty}^{+\infty} e^{j(\omega-\Omega)t} \, dt \, d\Omega \, d\omega$$

$$\omega \qquad \Omega$$

$$\int_{-\infty}^{+\infty} \underbrace{e^{-j\Omega t}}_{1} e^{j\omega t} \, dt$$

$$\int_{-\infty}^{+\infty} 1 \cdot e^{-j\omega t} \, dt = \mathcal{F}\{1\} = \underline{\underline{2\pi\delta(\omega)}}$$

$$\int_{-\infty}^{+\infty} \underbrace{e^{j\omega t}}_{\text{Freq shift}} \cdot \underbrace{e^{-j\Omega t}\, dt}_{} = 2\pi\delta(\Omega - \omega_0)$$

$$e^{j\omega_0 t} \Longleftrightarrow$$

$$x(t) \longrightarrow X(\omega - \omega_0)$$

$$= \frac{1}{2\pi}\cancel{\frac{1}{2\pi}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\omega)\cdot G^*(\omega)\,\cdot$$

$\omega \quad \Omega$

$$\cdot\, \cancel{2\pi}\,\delta(\Omega - \omega)\, d\Omega\, d\omega$$

$$= \frac{1}{2\pi} \int\limits_{\substack{-\infty \\ \omega}}^{+\infty} F(\omega) \int\limits_{\substack{-\infty \\ \Omega}}^{+\infty} G^*(\Omega) \cdot \delta(\Omega - \omega)\, d\Omega\, d\omega$$

$$= \frac{1}{2\pi} \int\limits_{\substack{-\infty \\ \omega}}^{+\infty} F(\omega) \int\limits_{\substack{-\infty \\ \Omega}}^{+\infty} \underline{\underline{G^*(\omega)}} \cdot \delta(\Omega - \omega)\, d\Omega\, d\omega$$

$$= \frac{1}{2\pi} \int\limits_{\substack{-\infty \\ \omega}}^{+\infty} F(\omega) \cdot G^*(\omega) \overbrace{\int\limits_{\substack{-\infty \\ \Omega}}^{+\infty} \delta(\Omega - \omega)\, d\Omega}^{1}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot G^*(\omega)$$

$G(\omega)$

$G^*(\omega) \neq G(-\omega)$

Ex.

$$f(t) = \frac{\omega_0}{\pi} \operatorname{sinc}(\omega_0 t)$$

$$g(t) = f(t) \cdot e^{j\omega_1 t}$$

Find ω_1 such that

$$\langle f(t), g(t) \rangle = 0$$

signals are orthogonal.

$$\int_{-\infty}^{+\infty} f(t)\cdot g^*(t) = 0$$

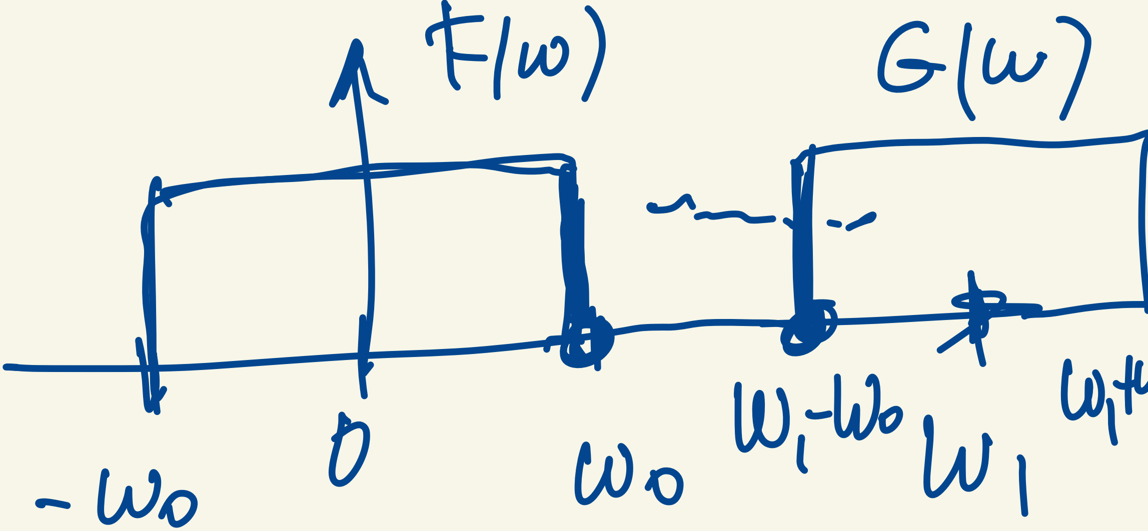
$\Rightarrow$ Parseval's

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\cdot G^*(\omega) = 0$$

$$f(t) = \frac{\omega_0}{\pi}\operatorname{sinc}(\underline{\underline{\omega_0}}\, t)$$

$$F(\omega) = 1 \cdot \operatorname{rec}(\omega, \underline{\omega_0})$$

duality Table



$$g(t) = f(t)e^{jw_1 t}$$

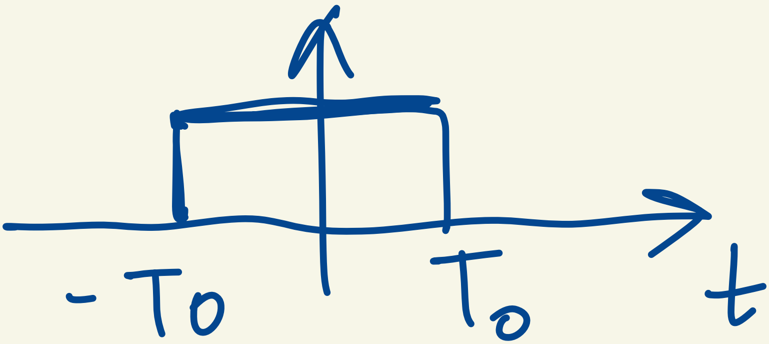
$$G(w) = F(w - w_1)$$

$$w_1 - w_0 > w_0$$

$$w_1 > 2w_0$$

Given

$f(t) = rec(t, T_0)$

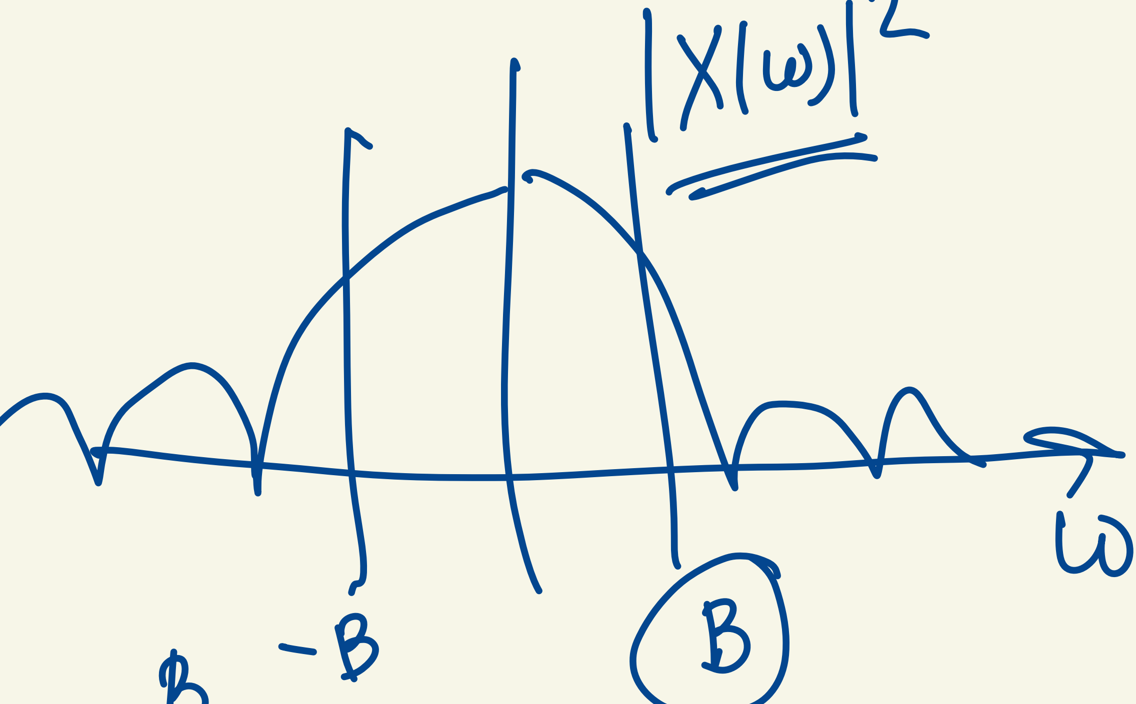


$T_0 = 100\mu s = 10^{-4} s.$

$P = 1W$

$$E = \int_{-T_0}^{T_0} P_T \, dt = 1W \cdot 10^{-4} s = 2 \cdot 10^{-4} J$$

Which frequency range contains 99% of signal Energy?



$$\int_{-B}^{B} |X(\omega)|^2 \, d\omega = 0.99\, E_x.$$