· Final info

March 17th 3:00 - 6:00pm
in the classroom.

- Closed book, closed notes
- 2 cheat sheets double-sided (any format)
- Calculators are allowed

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① $x(t) = \sin(2t) \xrightarrow{\mathcal{F}} X(\omega)$ exists.

not causal

There is no $X(s)$

② $h(t) = e^{-2t} \cdot u(t)$ $H(\omega) \to$ freq. response of system with $h(t)$

$$H(s) = \frac{1}{s+2}$$

ROC $Re\{s\} > -2$ $\Rightarrow$ BIBO stable.
includes $j\omega$ axis.

$$H(s)\Big|_{s=j\omega} = \int_0^{+\infty} h(t) e^{-st}\, dt \Big|_{s=j\omega}$$

$$= \int_0^{+\infty} e^{-2t} u(t) e^{-j\omega t}\, dt = \mathcal{F}\{e^{-2t} u(t)\}$$

$$H(s)\Big|_{s=j\omega} = \underline{\underline{H(\omega)}}$$

$$H(\omega) = \frac{1}{j\omega + 2}$$

3 $h(t) = u(t)$ $\underline{H(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)}$
causal

$H(s) = \frac{1}{s}$ → not BIBO stable

$Re\{s\} > 0$ does not include $j\omega$.

$h(t)$ → LTI systems for FT $\underline{\underline{H(\omega)}}$

→ LTI,C system for LT $\underline{\underline{H(s)}}$

$h(t) \cdot u(t)$

$x(t)$ any signals → FT

$y(t)$

⋮ causal signals → LT

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one sided Laplace

$$x(t) \cdot \underline{\underline{u(t)}} \xrightarrow{LT} X(s) = \int_{0^-}^{+\infty} x(t) e^{-st} dt$$

Convolution property.

LT:

$$\begin{tikzcd}
\underset{\text{causal signal}}{x(t)\,u(t)} \arrow[r] \arrow[d] & \boxed{h(t)\cdot u(t)} \arrow[r] \arrow[d] & y(t) = x(t) * h(t) \arrow[d] \\
X(s) & H(s) & Y(s)
\end{tikzcd}$$

LTI, C

causal system

causal signal

$$\underline{Y(s) = H(s)\cdot X(s) \quad s \in \mathbb{C}}$$

FT

$$\begin{tikzcd}
\overset{\text{any}}{x(t)} \arrow[r] \arrow[d, "\mathcal{F}"'] & \overset{\text{LTI}}{\boxed{h(t)}} \arrow[r] \arrow[d, "\mathcal{F}"] & \overset{\text{any}}{y(t) = x(t) * h(t)} \arrow[d, "\mathcal{F}"] \\
X(\omega) & H(\omega) & Y(\omega)
\end{tikzcd}$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$Y(\omega) = H(\omega) \cdot X(\omega) \quad \underline{\omega \in \mathbb{R}}$$

Practice Final Problems

1. $$h(t,\tau) = e^{-2t-2\tau}\big(\sin(t)\cos(\tau) - \cos(t)\sin(\tau)\big)u(t-\tau)$$

is this System Time-invar.

$$h(t,\tau) = e^{-2\underline{(t+\tau)}}\underline{\sin(\underline{t-\tau})}\,\underline{u(t-\tau)}$$

it cannot be written as $h(t-\tau)$

$\Rightarrow$ TV.

(2) $y(t) = x(t-3) + \int_{t-3}^{3t} e^{-(t-\sigma)} \underline{\underline{u(t-\sigma)}} \cdot x(\sigma)\, d\sigma$

is the system causal?

$= x(t-3) + \int_{t-3}^{\min(3t,t)} e^{-(t-\sigma)} x(\sigma)\, d\sigma$

$\min(3t,t) \le t$ ∴ S is $\underline{\underline{C}}$

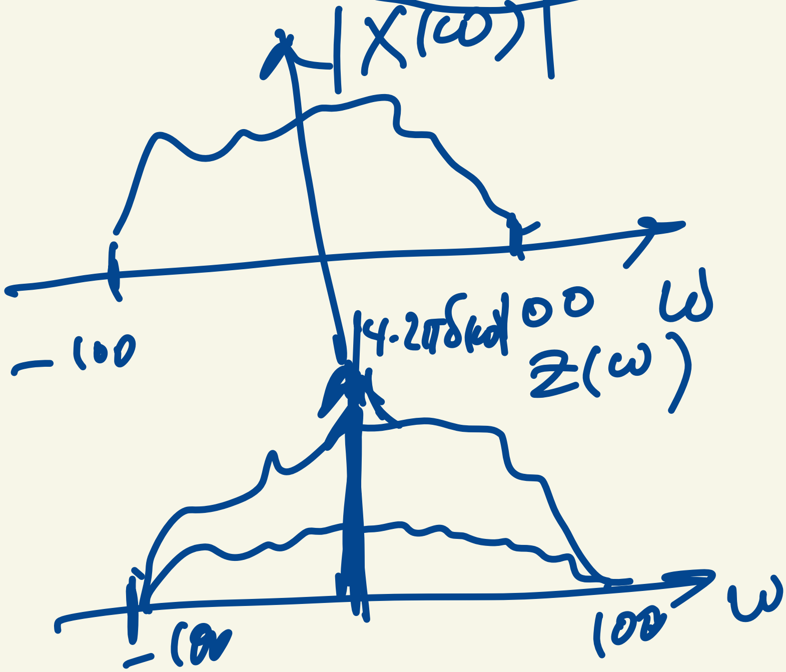
(3) Let $z(t) = \frac{1}{3} x(t-2) + 4$

where $x(t)$ is band-limited with maximum freq. of $100 \frac{rad}{s}$

The minimum sampling freq. of $z(t)$ is

$$\omega_s = 2 \cdot \frac{1}{3} \times 100 \ \frac{rad}{s}.$$

TRUE or FALSE ?

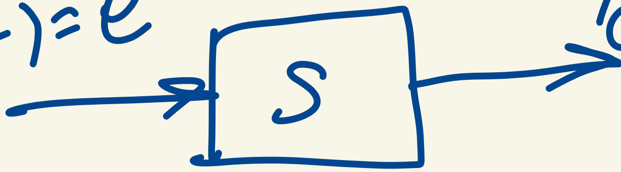


$|Z(\omega)| = 0$ for $|\omega| > 100 \frac{r}{s}$

$$\omega_s = 2\omega_c = 2 \times 100 = 200 \frac{rad}{s}$$

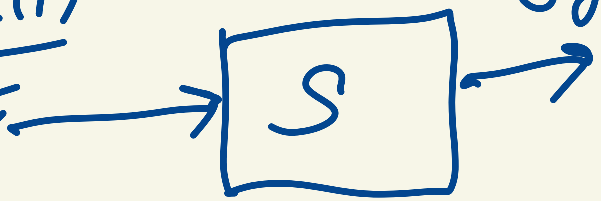
Prob. 2.

(1) $x(t)=e^{-5t}u(t-1)$



$y(t)$

(2) $\frac{dx(t)}{dt}$



$-5y(t)+e^{-2t}u(t)$

① What is $H(s)$ and $h(t)$?

② What is $y_1(t)$ for

$x_1(t)=e^{-2t}\cos(3t)u(t)$.

(1) $Y(s)=H(s)\cdot X(s)$

(2) $L_s\{-5y(t)+e^{-2t}u(t)\}=$

$$H(s) \cdot \mathcal{L}_s\left\{\frac{dx(t)}{dt}\right\}.$$

$$-5Y(s) + \frac{1}{s+2} = H(s) \cdot sX(s)$$

$$-5H(s) \cdot X(s) + \frac{1}{s+2} = sH(s) \cdot X(s)$$

$$\frac{1}{s+2} = H(s) \cdot X(s) \cdot (s+5)$$

$$H(s) = \frac{1}{(s+2)(s+5) \cdot \underline{X(s)}} \leftarrow$$

$$X(s) = \mathcal{L}_s\left\{e^{-5t} u(t-1)\right\}$$

$$= \mathcal{L}_s\left\{e^{-5(t-1)} \cdot e^{-5} u(t-1)\right\}$$

$$= e^{-5} \mathcal{L}_s\{e^{-5(t-1)}u(t-1)\}$$

$$= e^{-5}\,\frac{e^{-s}}{s+5}$$

$$X(s) = \frac{e^{-(s+5)}}{s+5}$$

$$H(s) = \frac{\cancel{(s+5)}}{(s+2)\cancel{(s+5)}\,e^{-(s+5)}}$$

$$\boxed{H(s) = \frac{e^{s+5}}{s+2}}$$

$$h(t) = e^{5}\cdot e^{-2(t+1)}\,u(t+1)$$

$$h(t) = e^{-2t+3} u(t+1)$$

b) $\underline{x_1(t)} = e^{-2t} \cos(3t) u(t)$

$$X_1(s) = \frac{s+2}{(s+2)^2+9}$$

$$Y_1(s) = H(s) \cdot X_1(s)$$

$$= \frac{e^{s+5}}{\cancel{s+2}} \cdot \frac{\cancel{s+2}}{(s+2)^2+9}$$

$$= \frac{e^{s+5}}{(s+2)^2+9} = e^{s} \cdot e^{5} \cdot \left[\frac{1}{(s+2)^2+9} \right]$$

$$\downarrow \mathcal{L}^{-1}$$

$$\frac{1}{3} e^{-2t} \cdot \sin(3t) u(t)$$

$$y_1(t) = e^{9} \cdot \frac{1}{3} e^{-2(t+1)} \sin(3(t+1)) \cdot u(t+1)$$

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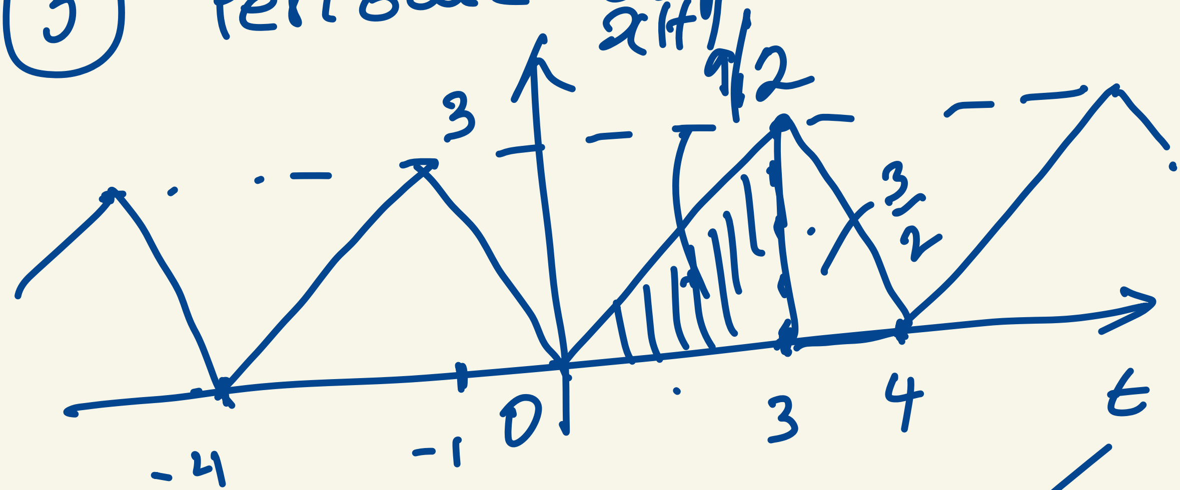
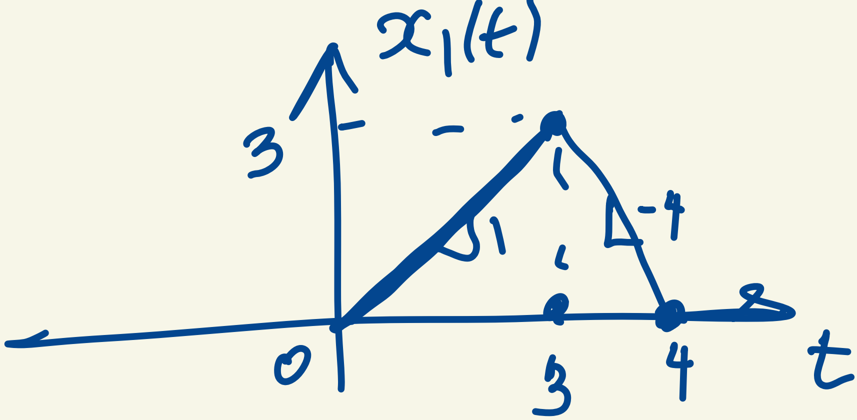
3) Periodic signal



$$T_0 = 4 \qquad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{4}\int_0^{4} x(t)\,dt$$

$$= \frac{1}{4}\left(\frac{9}{2} + \frac{3}{2}\right) = \frac{1}{4}\cdot\frac{12}{2} = \frac{3}{2}$$



$$\underline{X_k} = \frac{1}{T_0} X_1(s) \Big|_{s=jk\omega_0}$$

$$x_1(t) = r(t) - 4r(t-3) + 3r(t-4)$$

$$X_1(s) = \frac{1}{s^2} - \frac{4e^{-3s}}{s^2} + \frac{3e^{-4s}}{s^2}$$

$$X_1(s) = \frac{1-4e^{-3s}+3e^{-4s}}{s^2}$$

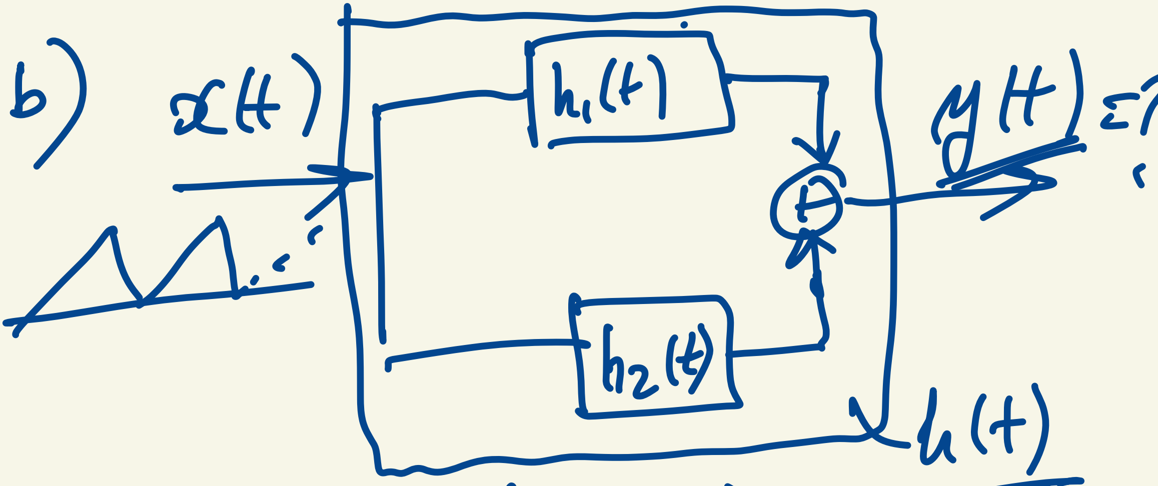
$$X_k = \frac{1}{4} X_1(s)\Big|_{s=jk\frac{\pi}{2}} \qquad e^{-jk2\pi} = 1$$

$$= \frac{1}{\cancel{4}} \; \frac{1 - 4e^{-j3k\frac{\pi}{2}} + 3\cancel{e^{-j4k\frac{\pi}{2}}}}{-\frac{k^2\pi^2}{\cancel{4}}}$$

$$= \frac{4 - 4e^{-j3k\frac{\pi}{2}}}{-k^2\pi^2}$$

$$\boxed{X_k = \frac{4e^{-3jk\frac{\pi}{2}} - 4}{k^2\pi^2}}$$

b)



$$h_1(t) = \frac{\sin(3\pi t/4)}{\pi t} \quad \Omega_1 = \frac{3\pi}{4} \quad \text{LTI}$$

table

$$h_3(t)$$

$$h_2(t) = \boxed{\frac{2\sin(\pi t/4)}{t}} \cdot \boxed{\cos(4\pi t)} \quad \Omega_2 = \frac{\pi}{4} \quad \text{LTI}$$

freq. shift + table

$$H(\omega) = \underline{H_1(\omega)} + \underline{H_2(\omega)}$$

$$H_1(\omega) = \mathrm{rec}\left(\omega, \frac{3\pi}{4}\right)$$

$$h_2(t) = h_3(t) \cdot \cos(4\pi t)$$

$$= h_3(t) \cdot \frac{e^{j4\pi t} + e^{-j4\pi t}}{2}$$

$$= h_3(t) \cdot \frac{1}{2} e^{j4\pi t} + \frac{1}{2} h_3(t) e^{-j4\pi t}$$

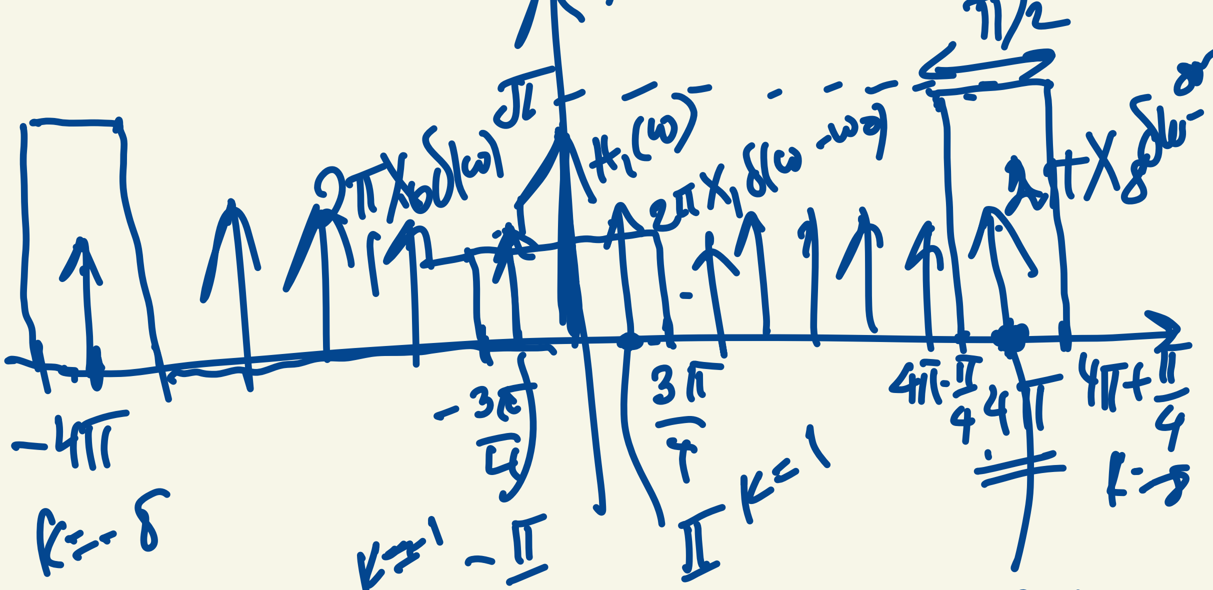
$$H_2(\omega) = \frac{1}{2} H_3(\omega - 4\pi) +$$

$$+ \frac{1}{2} H_3(\omega + 4\pi)$$

$$= \pi \operatorname{rec}\left(\omega - 4\pi, \frac{\pi}{4}\right)$$

$$+\pi\,\mathrm{rec}\left(\omega+4\pi,\ \frac{\pi}{4}\right)$$

$$H(\omega) = H_1(\omega) + H_2(\omega)$$



$$8\omega_0 = 8\cdot\frac{\pi}{2} = 4\pi$$

$$Y(\omega) = H(\omega)\cdot X(\omega) = \Sigma \cdots$$

review FT of periodic signal

$$= 2\pi X_0\,\delta(\omega)\cdot 1 +$$

$$+\,2\pi X_1\,\delta(\omega-\omega_0)\cdot 1 +$$

$$+2\pi X_{-1}\,\delta(\omega+\omega_0)\cdot 1$$

$$H(8\omega_0)$$

$$+2\pi X_{8}\,\delta(\omega+8\omega_0)\cdot \pi$$

$$+2\pi X_{-8}\,\delta(\omega-8\omega_0)\cdot \pi$$

$$X_0 = \frac{3}{2}$$

$$X_1 = \frac{4j-4}{\pi^2}$$

$$X_{-1} = \frac{-4j-4}{\pi^2}$$

$$X_8 = \frac{4\cancelto{1}{e^{-j\frac{8\pi}{2}}}-4}{64\pi^2}$$

$$X_8 = 0.$$

$$X_{-8} = 0$$

$$Y(\omega) = 2\pi \cdot \frac{3}{2}\delta(\omega) +$$

$$+ 2\pi \frac{4j-4}{\pi^2}\delta(\omega - \omega_0)$$

$$+ 2\pi \frac{(-4j-4)}{\pi^2}\delta(\omega + \omega_0)$$

$$y(t) = \overset{Y_0}{\frac{3}{2}} + \overset{Y_1}{\left(\frac{4j-4}{\pi^2}\right)} \cdot e^{j\frac{\pi}{2}t}$$

$$+ \left(\frac{-4j-4}{\pi^2}\right) e^{-j\frac{\pi}{2}t}$$

Y_{-1} periodic

$$\omega_0 = \frac{\pi}{2}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{+\infty} X_k \, \delta(\omega - k\omega_0)$$

Problem:

multiplication property of Fourier Transform.

$$x(t) \cdot y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

① What is $\mathcal{F}\{\mathrm{sinc}^2(t)\}$

where $\mathrm{sinc}(t) = \frac{\sin t}{t}$

$\mathrm{sinc}(t)\,\mathrm{sinc}(t) \xrightarrow{\mathcal{F}}$

$\frac{1}{2\pi}\left[\underline{\mathcal{F}\{\mathrm{sinc}(t)\}} * \mathcal{F}\{\mathrm{sinc}(t)\}\right]$

$\mathcal{F}[\mathrm{sinc}(t)] = \pi\, \underline{\mathrm{rec}(\omega, 1)}$

$\frac{\Omega}{\pi}\mathrm{sinc}(\Omega t) \rightarrow \mathrm{rec}(\omega, \Omega)$

rewrite rec as :

$$= \pi \cdot \left[u(\omega+1) - u(\omega-1) \right]$$

$$= \frac{\pi}{\cancel{2}\pi} \left[u(\omega+1) - u(\omega-1) \right] *$$

$$\left[u(\omega+1) - u(\omega-1) \right]$$

$u(t) * u(t) = ?$ proofs. practice

causal causal

$L_s \{ u(t) * u(t) \} =$

$$\underbrace{L_s\{u(t)\}}\cdot\underbrace{L_s\{u(t)\}}$$

$$\frac{1}{s}\cdot\frac{1}{s}=\frac{1}{s^2}$$

$$L_s\{u(t)*u(t)\}=\frac{1}{s^2}$$

$$u(t)*u(t)=r(t)$$

$$u(t+1)*u(t+1)\stackrel{?}{=}r(t+2)$$

$$u(t+1)*u(t-1)\stackrel{?}{=}r(t).$$

$$= \frac{\pi}{2}\Big[r(\omega+2) - r(\omega) - r(\omega) + r(\omega-2) \Big]$$

$$= \frac{\pi}{2}\Big[r(\omega+2) - 2r(\omega) + r(\omega-2) \Big] \checkmark$$

b) Compute

$$\mathcal{F}\{t\,\mathrm{sinc}^2(t)\}$$

$$= j\,\frac{d\,\mathcal{F}\{\mathrm{sinc}^2(t)\}}{d\omega}$$

$$= j \frac{d}{d\omega}\left\{ \frac{\pi}{2}\left[r(\omega+2) - 2r(\omega) + r(\omega-2) \right] \right\}$$

$$= j\frac{\pi}{2}\left[u(\omega+2) - 2u(\omega) + u(\omega-2) \right]$$

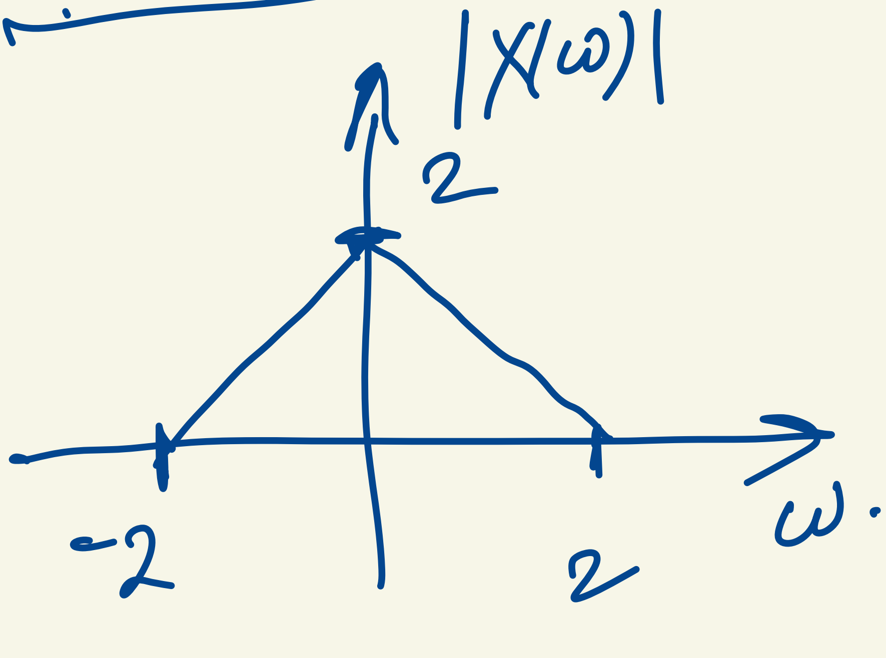
c) Evaluate

$$\int_{-\infty}^{+\infty} \text{sinc}^4(t)\,dt$$

Parseval's

$$\int_{-\infty}^{+\infty} \underline{|x(t)|^2} dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{|X(\omega)|^2} d\omega$$

$$\underline{x(t) = \operatorname{sinc}^2(t)}$$



$$\int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega$$