Discussion 5

ECE 102: Systems and Signals

Winter 2022

Properties of Continuous time Fourier series

$x(t)$: period T_0; fundamental freq: $\omega_0 = \frac{2\pi}{T_0}$

X_k: $$X_k = \frac{1}{T_0}\int_{T_0} x(t).e^{-j\omega_0 kt}\,dt.$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{+j\omega_0 nt}$$

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① **Linearity**: $x_1(t)$: T_1

$x_2(t)$: T_2.

$$y(t) = Ax_1(t) + Bx_2(t).$$

$$\boxed{Y_k = AX_{1_k} + BX_{2_k}}$$

Proof:

$$Y_k = \frac{1}{T_y}\int_{T_y}(Ax_1(t) + Bx_2(t))\,e^{-j\omega_y kt}\,dt$$

$$T_y = \text{LCM}\{T_1, T_2\}.$$

$$\left.\begin{aligned} T_y &= nT_1 \\ &= mT_2 \end{aligned}\right\} n, m \in \mathbb{Z}.$$

ECE 236 Project Report

ay.wadaskar

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$$Y_k = \frac{1}{T_y}\int_{T_y}\left(A x_1(t) + B x_2(t)\right)dt \cdot e^{-j\left(\frac{2\pi}{T_y}\right)kt}.$$

$$= \frac{A}{T_y}\int_{T_y} x_1(t)\cdot e^{-j\left(\frac{2\pi}{T_y}\right)kt} dt + \frac{B}{T_y}\int_{T_y} x_2(t)\, e^{-j\left(\frac{2\pi}{T_y}\right)kt} dt.$$

$$T_y = nT_1 = mT_2 \quad ; n, m \in \mathbb{Z}.$$

$$Y_k = \frac{A}{nT_1}\int_{nT_1} x_1(t)\, e^{-j\left(\frac{2\pi}{nT_1}\right)kt} dt + \frac{B}{mT_2}\int_{mT_2} x_2(t)\, e^{-j\left(\frac{2\pi}{mT_2}\right)kt} dt.$$

$$\rightarrow \frac{t}{n} = \tau_1 \;; \quad \frac{t}{m} = \tau_2.$$

$$\Rightarrow dt = n\,d\tau_1 \quad dt = m\,d\tau_2$$

$$y_k = \frac{A}{nT_1}\int_{T_1} x_1(n\tau_1)\, e^{-j\left(\frac{2\pi}{T_1}\right)k\tau_1} \cdot n\, d\tau_1$$

$$+ \frac{B}{mT_2}\int_{T_2} x_2(m\tau_2)\, e^{-j\left(\frac{2\pi}{T_2}\right)k\tau_2} \cdot m\, d\tau_2$$

$$= \frac{A}{T_1}\int_{T_1} x_1(n\tau_1)\, e^{-j\left(\frac{2\pi}{T_1}\right)k\tau_1}\, d\tau_1$$

$$+ \frac{B}{T_2}\int_{T_2} x_2(m\tau_2)\, e^{-j\left(\frac{2\pi}{T_2}\right)k\tau_2}\, d\tau_2.$$

$n, m \in \mathbb{Z}.$

$$= \frac{A}{T_1}\int_{T_1} x_1(\tau_1)\, e^{-j\left(\frac{2\pi}{T_1}\right)k\tau_1}\, d\tau_1$$

$$+ \frac{B}{T_2}\int_{T_2} x_2(\tau_2)\, e^{-j\left(\frac{2\pi}{T_2}\right)k\tau_2}\, d\tau_2$$

$$= \boxed{A X_{1k} + B X_{2k}.}$$

② Time shift : $y(t) = x(t-t_0)$. $\longrightarrow$

$$Y_k = \frac{1}{T_0}\int_0^{T_0} x(t-t_0)\cdot e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt.$$

$$\tau = t - t_0 \Rightarrow d\tau = dt.$$

$$= \frac{1}{T_0}\int_{-t_0}^{T_0-t_0} x(\tau)\cdot e^{-j\omega_0 k(\tau+t_0)}\,d\tau.$$

$$= \frac{1}{T_0}\int_{-t_0}^{T_0-t_0} x(\tau)\cdot e^{-j\omega_0 k\tau}\cdot \underbrace{e^{-j\omega_0 k t_0}}\,d\tau.$$

$$= e^{-j\omega_0 k t_0}\underbrace{\frac{1}{T_0}\int_{-t_0}^{T_0-t_0} x(\tau)\, e^{-j\omega_0 k\tau}\,d\tau}_{X_k}$$

$$\boxed{Y_k = e^{-j\omega_0 k t_0} X_k}$$

③ Frequency shift

$$y(t) = e^{j\underbrace{M\omega_0 t}}\cdot x(t). \quad ; M \in \mathbb{Z}$$

$$M\omega_0 t$$

$$Y_k = \frac{1}{T_0}\int_{T_0} \underbrace{e^{jM\omega_0 t}\cdot x(t)}\cdot e^{-j\omega_0 kt}\,dt$$

$$\underline{1} \quad Y_k = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-j\omega_0 t(k-M)}\,dt.$$

2 $$X_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-j(k\omega_0)t}\, dt$$

$$\boxed{Y_k = X_{k-M}}$$

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④ Time Reversal : $y(t) = x(-t)$.

$$Y_k = \frac{1}{T_0} \int_{0}^{T_0} x(-t)\, e^{-j\omega_0 k t}\, dt$$

$$\tau = -t$$

$$\Rightarrow d\tau = -dt.$$

$$Y_k = \frac{1}{T_0} \int_{0}^{-T_0} x(\tau)\, e^{jk\omega_0 \tau}\, (-d\tau).$$

$$= \frac{1}{T_0} \int_{-T_0}^{0} x(\tau)\, e^{-j\tau(-k\omega_0)}\, d\tau$$

$$= X_{-k}.$$

$$\boxed{x(-t) \longleftrightarrow X_{-k}.}$$

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⑤ Conjugation : $y(t) = x^*(t)$.

$$Y_k = \frac{1}{T_0} \int_{T_0} x^*(t)\, e^{-j\omega_0 k t}\, dt$$

$$= \left(\frac{1}{T_0} \int_{T_0} x(t)\, e^{+j\omega_0 k t}\, dt \right)^*$$

$$= \left(\frac{1}{T_0} \int_{T_0} x(t)\, e^{j\omega_0 k t}\, dt \right)^*$$

$$e^{jk\omega_0 t} = e^{-jt(-k\omega_0)}.$$

$$= \boxed{(X_{-k})^*}$$

$$\boxed{x^*(t) \longleftrightarrow (X_{-k})^*}$$

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(6) <u>Time scaling</u> : $y(t) = x(at)$. $(a > 0)$. $x(t) \longrightarrow T_0.$ ω_0

$$Y_k = \frac{1}{1} \int$$

Period of $y(t) = \dfrac{T_0}{a}$; $\omega_y = \dfrac{2\pi}{T_0} \cdot a$

$$= a \cdot \omega_0$$

$$Y_k = \frac{a}{T_0} \int_0^{T_0/a} x(at) \cdot e^{-jka\omega_0 t}\, dt$$

$$\tau = at \Rightarrow d\tau = a\,dt.$$

$$\therefore Y_k = \frac{\cancel{a}}{T_0} \int_0^{T_0} x(\tau)\, e^{-jk\cancel{a}\omega_0 \left(\frac{\tau}{\cancel{a}}\right)} \frac{d\tau}{\cancel{a}}$$

$$= \frac{1}{T_0} \int_0^{T_0} x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

$$= X_k.$$

Time scaling does not alter the F.S. coeff.

* F.S. coeff are invariant to time scaling.

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(7) Multiplication :- ~~y(t)~~ =

$$y(t) = a(t) \cdot b(t). \quad : \text{both are periodic with period } T_0.$$

$$a(t) \downarrow A_k \qquad b(t) \downarrow B_k.$$

$$Y_k = \frac{1}{T_0} \int a(t) \cdot b(t) \cdot e^{-jk\omega_0 t} \, dt$$

$$b(t) = \sum_{l=-\infty}^{\infty} B_l \cdot e^{jl\omega_0 t}$$

$$Y_k = \frac{1}{T_0} \int a(t) \cdot \left(\sum_{l=-\infty}^{\infty} B_l \cdot e^{jl\omega_0 t} \right) \cdot e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T_0} \int_{T_0} \sum_{l=-\infty}^{\infty} B_l \cdot e^{jl\omega_0 t} \cdot a(t) \cdot e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T_0} \sum_{l=-\infty}^{\infty} \left\{ \int_{T_0} a(t) \cdot e^{-jk\omega_0 t} \, dt \; B_l \cdot e^{jl\omega_0 t} \right\}$$

$$= \sum_{l=-\infty}^{\infty} \left\{ \underbrace{\left(\frac{1}{T_0} \int_{T_0} a(t) \, e^{-j\omega_0 t(k-l)} \, dt \right)}_{} \cdot B_l \right\}$$

$$= \sum_{l=-\infty}^{\infty} \left\{ \underbrace{\left(\frac{1}{T_0} \int_{T_0} a(t) \cdot e^{-j\omega_0 t(k-l)} \, dt \right)}_{A_{k-l}} B_l \right\}$$

$$= \sum_{l=-\infty}^{\infty} A_{k-l} \cdot B_l .$$

$$a(t)\,b(t) \longleftrightarrow \boxed{\sum_{l=-\infty}^{\infty} B_l \cdot A_{k-l} = \sum_{l=-\infty}^{\infty} A_l \, B_{k-l}}$$

$$x(t) * h(t) . = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau .$$

$$y(t) = x(t) * h(t) .$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

$$c(t) . = a(t)\, b(t)$$

$$\boxed{C_k = \sum_{l=-\infty}^{\infty} A_l \, B_{k-l} .}$$

(8). Differentiation :

$$y(t) = \frac{dx(t)}{dt} \qquad Y_k.$$

$$\underbrace{\quad}_{T_0}$$

$$\underline{\underline{Y_k}} = \frac{1}{T_0}\int_{T_0} \underbrace{\frac{dx(t)}{dt}}_{u} \cdot \underbrace{e^{-jk\omega_0 t}\, dt}_{dv}$$

Laplace Transform.

$$y(t) = \frac{dx(t)}{dt}. \qquad x_1(t).$$

$$\underline{\underline{Y(s)}} = \underline{\underline{s\,X(s)}}.$$

$$Y_k = \frac{1}{T_0}\left(Y(s)\Big|_{s=jk\omega_0}\right).$$

$$= \frac{1}{T_0}\, jk\omega_0 \cdot X(s)\Big|_{s=jk\omega_0}$$

$$= jk\omega_0 \cdot \frac{1}{T_0} X(s)\Big|_{s=jk\omega_0}$$

$$= jk\omega_0 X_k.$$

$$\boxed{\frac{dx(t)}{dt} \longleftrightarrow jk\omega_0 X_k}$$

9) Integration

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau.$$

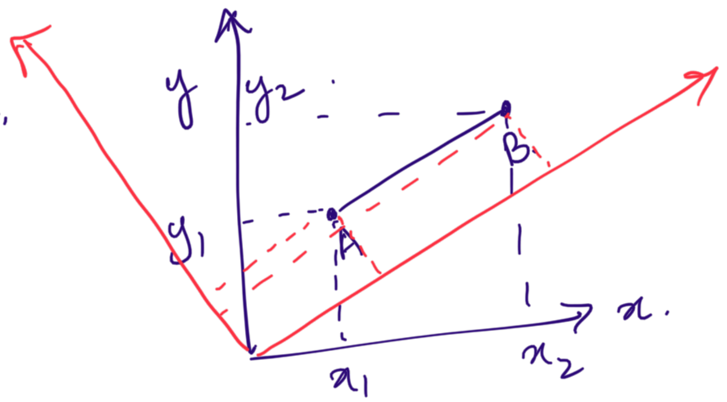
$$Y_k = \frac{1}{jk\omega_0} X_k.$$

H.W.

10). Parseval's theorem :-

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2$$

2D.



$$\overline{AB} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$