Home Work 2

Nhat Ho

1) $x(t) = u(t-2) - u(t-4)$

a) $y(t) = \int_{-\infty}^{2t} x(\tau+3)\, d\tau$

Considering an arbitary signal $x_1(t)$ & $x_2(t)$ such that

$$\begin{cases} x_1(t) \to \boxed{S} \to y_1(t) \\ x_2(t) \to \boxed{S} \to y_2(t) \end{cases}$$

Where $\alpha x_1(t) + \beta x_2(t) \longrightarrow \boxed{S} \to y_3(t)$

$$\alpha, \beta \in \mathbb{C}$$

* We have $y_1(t) = S\{x_1(t)\} = \int_{-\infty}^{2t} x_1(\tau+3)\, d\tau$

$$y_2(t) = S\{x_2(t)\} = \int_{-\infty}^{2t} x_2(\tau+3)\, d\tau$$

Also, we have $y_3(t) = S\{\alpha x_1 + \beta x_2\}$

$$= \int_{-\infty}^{2t} (\alpha x_1 + \beta x_2)(\tau+3)\, d\tau = \int_{-\infty}^{2t} [\alpha x_1(\tau+3) + \beta x_2(\tau+3)]\, d\tau$$

$$= \int_{-\infty}^{2t} \alpha x_1(\tau+3)\, d\tau + \int_{-\infty}^{2t} \beta x_2(\tau+3)\, d\tau$$

$$= \alpha y_1(t) + \beta y_2(t) \Rightarrow y_3(t) = \alpha y_1(t) + \beta y_2(t)$$

$\Rightarrow$ $\boxed{\text{System is Linear}}$

* Let t_0 is constant, &.

$x(t-t_0) \to \boxed{S} \to z(t) = S\{x(t-t_0)\}$

Also have: $x(t) \to \boxed{S} \to y(t) = \int_{-\infty}^{2t} x(\tau+3)\,d\tau$

We have $z(t) = S\{x(t-t_0)\} = \int_{-\infty}^{2t} x(\tau+3-t_0)\,d\tau$

While $y(t-t_0) = \int_{-\infty}^{2t} x(\tau-t_0+3)\,d\tau$

Since $z(t) = y(t-t_0)$

$\Rightarrow$ $\boxed{\text{System is Time - Invariance}}$

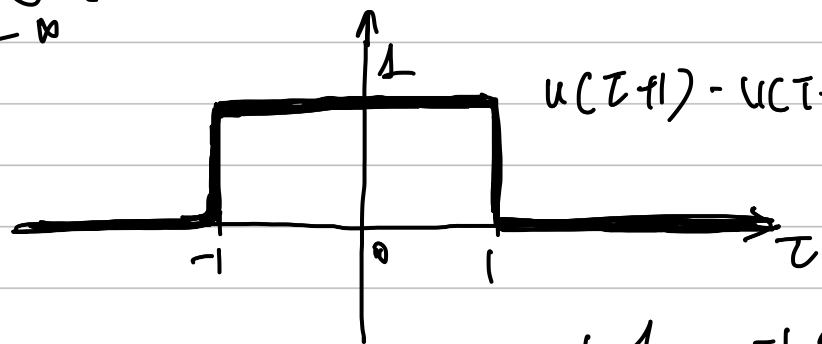
* Since $y(t) = \int_{-\infty}^{2t} x(\tau+3)\,d\tau$, the output $y(t)$ also depends on the future of the input $x(t)$ For example, with $t=2$ for $y(2)$, we need to know the input from $-\infty \to 4$

$t=4>2.$ $\Rightarrow$ $\boxed{\text{System is not Causal}}$

* Find output $y(t)$ with $x(t) = u(t-2) - u(t-4)$

$x(t+3) = u(t+3-2) - u(t+3-4) = u(t+1) - u(t-1)$

$$\Rightarrow y(t) = \int_{-\infty}^{2t} [u(\tau+1) - u(\tau-1)]\, d\tau$$



Also, $u(t+1) - u(t-1) = \begin{cases} 1 & -1 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$

When $2t < -1 \Leftrightarrow t < -\frac{1}{2} \Rightarrow u(\tau+1) - u(\tau-1) = 0 \Rightarrow y(t) = 0$

When $-1 \le 2t \le 1 \Leftrightarrow -1/2 \le t \le 1/2 \Rightarrow y(t) = \int_{-1}^{2t} 1\, d\tau = \tau \Big|_{-1}^{2t}$

$= 2t + 1$

When $2t > 1 \Leftrightarrow t > \frac{1}{2},\ y(t) = \int_{-1}^{1} 1\, d\tau = 1 + 1 = 2$

Combine three cases, we have:

$$\boxed{y(t) = \begin{cases} 2t+1, & -1/2 \le t \le 1/2 \\ 2, & t > \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}}$$

b) $y(t) = x(t)\sin(\pi t)$

Let $x_1(t)$ & $x_2(t)$ with

$x_1(t) \rightarrow \boxed{S} \rightarrow y_1(t) = S\{x_1(t)\} = x_1(t)\sin \pi t$

$x_2(t) \rightarrow \boxed{S} \rightarrow y_2(t) = S\{x_2(t)\} = x_2(t)\sin\pi t$

And: $\alpha x_1 + \beta x_2 \rightarrow \boxed{S} \rightarrow z(t) = S\{\alpha x_1 + \beta x_2\}$

$\Rightarrow z(t) = (\alpha x_1 + \beta x_2)(t) \sin\pi t = [\alpha x_1(t) + \beta x_2(t)] \sin\pi t$

$= \alpha x_1(t)\sin\pi t + \beta x_2(t) \sin\pi t = \alpha y_1(t) + \beta y_2(t)$

$\Rightarrow z(t) = S\{\alpha x_1(t) + \beta x_2(t)\} = \alpha y_1(t) + \beta y_2(t)$

$\Rightarrow$ $\boxed{\text{System is linear.}}$

* Let t_0 is constant, we have:

$x(t-t_0) \rightarrow \boxed{S} \rightarrow z(t) = S\{x(t-t_0)\}$

$\Rightarrow z(t) = S\{x(t-t_0)\} = x(t-t_0)\sin\pi t$

While $y(t-t_0) = x(t-t_0)\sin\pi(t-t_0)$

Since $z(t) \neq y(t-t_0) \Rightarrow$ $\boxed{\text{System is Time Variant}}$

* Since $y(t) = x(t)\sin\pi t$, with output $y(t)$ at any time instant depend only on present value of input $x(t) \Rightarrow$ $\boxed{\text{System is Causal \& memoryless}}$ (since only depend on present)

(*) $x(t) \rightarrow \boxed{S} \rightarrow y(t) = x(t)\sin \pi t$

$\Rightarrow$ with $x(t) = u(t-2) - u(t-4)$

$$\Rightarrow \boxed{y(t) = [u(t-2) - u(t-4)] \sin(\pi t)}$$

c) $y(t) = \dfrac{dx(t)}{dt}$

* Let $x_1(t)$ & $x_2(t)$ such that

$$x_1(t) \rightarrow \boxed{S} \rightarrow y_1(t) = \frac{dx_1(t)}{dt}$$

$$x_2(t) \rightarrow \boxed{S} \rightarrow y_2(t) = \frac{dx_2(t)}{dt}$$

Where $\alpha x_1(t) + \beta x_2(t) \rightarrow \boxed{S} \rightarrow z(t) = S\{\alpha x_1 + \beta x_2\}$

$$\Rightarrow z(t) = \frac{d}{dt}(\alpha x_1 + \beta x_2)(t) = \frac{d}{dt}[\alpha x_1(t) + \beta x_2(t)]$$

$$= \frac{d}{dt}\alpha x_1(t) + \frac{d}{dt}\beta x_2(t) = \alpha \frac{d}{dt} x_1(t) + \beta \frac{d}{dt} x_2(t)$$

$$= \alpha y_1(t) + \beta y_2(t)$$

$$\Rightarrow z(t) = \alpha y_1(t) + \beta y_2(t)$$

$\Rightarrow$ $\boxed{\text{System is linear}}$

* Let t_0 is constant, with

$$x(t-t_0) \rightarrow \boxed{S} \rightarrow z(t) = S\{x(t-t_0)\}$$

$$\Rightarrow z(t) = \frac{d\,x(t-t_0)}{dt}$$

$$\text{While } y(t-t_0) = \frac{d\,x(t-t_0)}{dt}$$

$\Leftrightarrow z(t) = y(t-t_0)$

$\Rightarrow$ $\boxed{\text{System is Time-Invariance}}$

* Since $y(t) = \frac{d\,x(t)}{dt}$ with output $y(t)$ at anytime instant depend only on present value of input $x(t)$

$\Rightarrow$ $\boxed{\text{System is Causal}}$ (memoryless)

* With $x(t) = u(t-2) - u(t-4)$

$$\Rightarrow y(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left[u(t-2) - u(t-4)\right]$$

$$= \frac{d}{dt}u[t-2] - \frac{d}{dt}u(t-4) \Rightarrow \boxed{y(t) = \delta(t-2) - \delta(t-4)}$$

d) $y(t) = x(2-t) + x(2+t)$

Let $x_1(t)$ & $x_2(t)$ such that

$$x_1(t) \to \boxed{S} \to y_1(t) = S\{x_1(t)\} = x_1(2-t) + x_1(2+t)$$

$$x_2(t) \to \boxed{S} \to y_2(t) = S\{x_2(t)\} = x_2(2-t) + x_2(2+t)$$

Where $\alpha x_1 + \beta x_2 \rightarrow \boxed{S} \rightarrow z(t) = S\{\alpha x_1 + \beta x_2\}$

$$\Rightarrow z(t) = (\alpha x_1 + \beta x_2)(2-t) + (\alpha x_1 + \beta x_2)(2+t)$$

$$= \alpha x_1(2-t) + \beta x_2(2-t) + \alpha x_1(2+t) + \beta x_2(2+t)$$

$$= \alpha[x_1(2-t) + x_1(2+t)] + \beta[x_2(2-t) + x_2(2+t)]$$

$$= \alpha y_1(t) + \beta y_2(t)$$

$\Rightarrow z(t) = \alpha y_1(t) + \beta y_2(t) \Rightarrow$ $\boxed{\text{System is Linear}}$

* Let t_0 is constant,

$$x(t-t_0) \longrightarrow \boxed{S} \longrightarrow z(t) = S\{x(t-t_0)\}$$

$$\Rightarrow z(t) = x(2-t-t_0) + x(2+t-t_0)$$

while $y(t-t_0) = x(2-(t-t_0)) + x(2+t-t_0)$

$$= x(2-t+t_0) + x(2+t-t_0)$$

Because $z(t) \neq y(t-t_0) \Rightarrow$ System is $\boxed{\text{Time Variant}}$

* We have: $y(t) = x(2-t) + x(2+t)$

the system is causal when any value of t, we have:

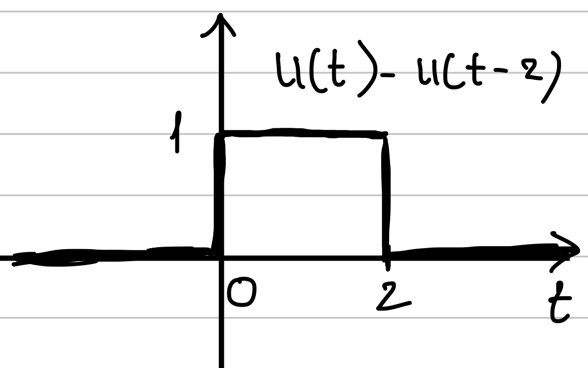
$$\begin{cases} 2-t \le t \\ 2+t \le t \end{cases} \Leftrightarrow \begin{cases} 2t \ge 2 \\ 0 \ge 2 \end{cases} \text{(Reject)}$$

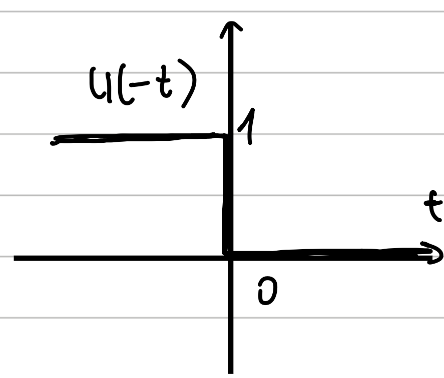
$\Rightarrow$ System is not causal. Furthermore, we can see with the

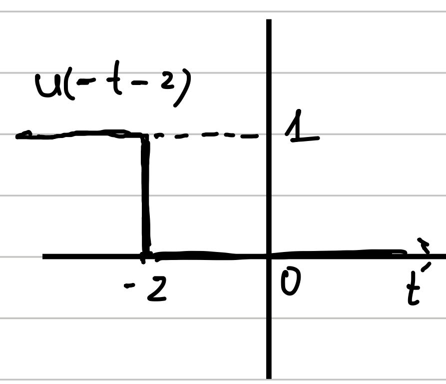
Output $y(t)$, it depends on the future time of the value of input with $x(t+2)$ $\Rightarrow$ $\boxed{\text{System is not Causal}}$

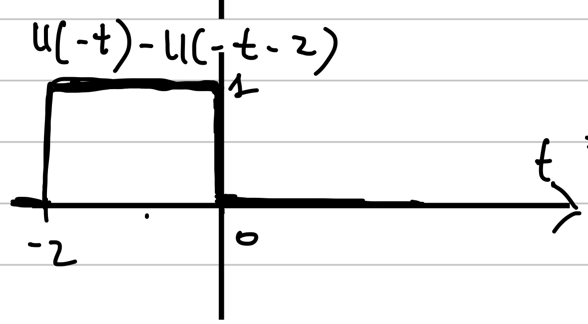
* With $x(t) = u(t-2) - u(t-4)$

$$\Rightarrow x(2-t) = u(2-t-2) - u(2-t-4)$$
$$= u(-t) - u(-t-2)$$
$$x(2+t) = u(t+2-2) - u(t+2-4)$$
$$= u(t) - u(t-2)$$
$$\Rightarrow y(t) = x(2-t) + x(t+2)$$
$$= u(-t) - u(-t-2) + u(t) - u(t-2)$$
$$= u(t) - u(t-2) + u(-t) - u(-t-2)$$

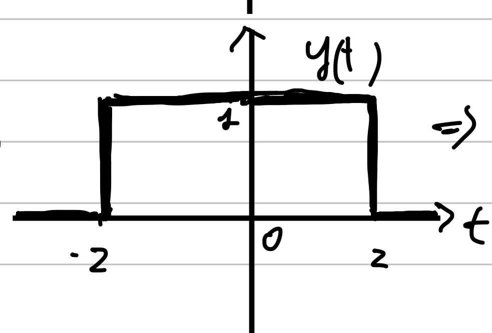








$\Rightarrow$

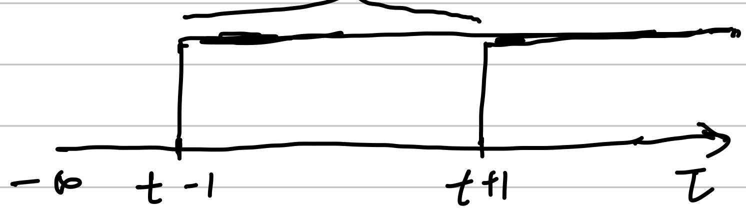


$\Rightarrow$ $\boxed{y(t) = u(t+2) - u(t-2)}$

2) Given: $$y(t) = x(t) - \int_{t-1}^{t+1} e^{|t-\tau|} x(\tau)\, d\tau$$

a) We have:

$$\underbrace{u(\tau-(t-1)) - u(\tau-(t+1))}$$



$$= u(\tau - t + 1) - u(\tau - t - 1)$$

$$\Rightarrow y(t) = x(t) - \int_{-\infty}^{+\infty}$$

Also, $$x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau - \int_{-\infty}^{+\infty} e^{|t-\tau|} \left[u(\tau-t+1) - u(\tau-t-1)\right] x(\tau)\, d\tau$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} \left(\delta(t-\tau) - e^{|t-\tau|} \left[u(\tau-t+1) - u(\tau-t-1)\right]\right) x(\tau)\, d\tau$$

So, for now, $y(t)$ can be rewrote:

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau) \left[\delta(t-\tau) - e^{|t-\tau|}\left[u(\tau-t+1) - u(\tau-t-1)\right]\right] d\tau}$$

Furthermore, if this system is Linear, we can rewrote $y(t)$ by applying the CONVOLUTION that shows:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t,\tau)\, d\tau.$$ We will discuss in part B

Let $x_1(t)$ & $x_2(t)$ such that

$$x_1(t) \to \boxed{S} \to y_1(t) = x_1(t) - \int_{t-1}^{t+1} e^{|t-\tau|} x_1(\tau)\,d\tau$$

$$x_2(t) \to \boxed{S} \to y_2(t) = x_2(t) - \int_{t-1}^{t+1} e^{|t-\tau|} x_2(\tau)\,d\tau$$

where: $\alpha x_1 + \beta x_2 \to \boxed{S} \to z(t) = S\{\alpha x_1 + \beta x_2\}$

$$z(t) = (\alpha x_1 + \beta x_2)(t) - \int_{t-1}^{t+1} e^{|t-\tau|} (\alpha x_1 + \beta x_2)(\tau)\,d\tau$$

$$= \alpha x_1(t) + \beta x_2(t) - \int_{t-1}^{t+1} e^{|t-\tau|} [\alpha x_1(\tau) + \beta x_2(\tau)]\,d\tau$$

$$= \alpha y_1(t) + \beta y_2(t)$$

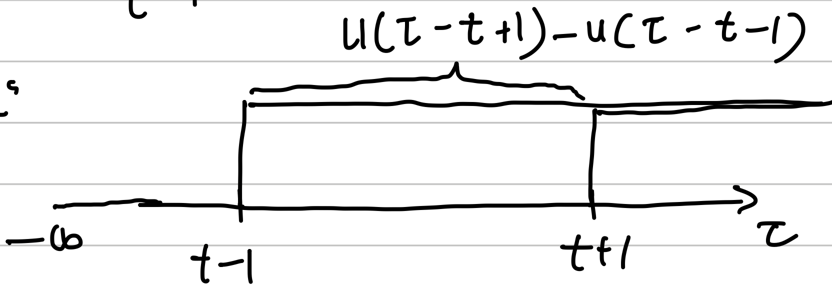
$\Rightarrow \quad z(t) = \alpha y_1(t) + \beta y_2(t)$

$\Rightarrow \quad \boxed{\text{System is Linear}}$

(*)

$$= \delta(t-\sigma) - \int_{t-1}^{t+1} e^{|t-\tau|} \delta(\tau-\sigma)\,d\tau$$

we have:



$$\Rightarrow h(t,\sigma) = \delta(t-\sigma) - \int_{-\infty}^{+\infty} e^{|t-\tau|}\delta(\tau-\sigma)\left[u(\tau-t+1) - u(\tau-t-1)\right] d\tau$$

$$= \delta(t-\sigma) - \int_{-\infty}^{+\infty} e^{|t-\tau|}\left[u(\tau-t+1)\right] - u(\tau-t-1)\right]\delta(\tau-\sigma)d\tau$$

Using shifting property of impulse:

$$f(\tau)\,\delta(\tau-\sigma) = f(\sigma)\,\delta(\tau-\sigma) \quad (\sigma \text{ is constant})$$

with $f(\tau) = e^{|t-\tau|}\left[u(\tau-t+1) - u(\tau-t-1)\right]$

$$\Rightarrow f(\sigma) = e^{|t-\sigma|}\left[u(\sigma-t+1) - u(\sigma-t-1)\right]$$

$$\Rightarrow h(t,\sigma) = \delta(t-\sigma) - \int_{-\infty}^{+\infty} e^{|t-\sigma|}\left[u(\sigma-t+1) - u(\sigma-t-1)\right]\delta(\tau-\sigma)d\tau$$

$$= \delta(t-\sigma) - e^{|t-\sigma|}\left[u(\sigma-t+1) - u(\sigma-t-1)\right]\underbrace{\int_{-\infty}^{+\infty}\delta(\tau-\sigma)d\tau}_{1}$$

$$= \delta(t-\sigma) - e^{|t-\sigma|}\left[u(\sigma-t+1) - u(\sigma-t-1)\right]$$

$$= \delta(t-\sigma) - e^{|t-\sigma|}\left[u(-(t-\sigma)+1) - u(-(t-\sigma)-1)\right]$$

Since $h(t,\sigma)$ only depend on $(t-\sigma)$

$\Rightarrow$ System is $\boxed{\text{Time - Invariant}}$

Furthermore, we can check if $h(t,\sigma) = h(t-\sigma, 0)$

We can have:

$$h(t-\sigma,0) = \delta(t-\sigma-0) - e^{|t-\sigma-0|}\left[u(-(t-\sigma-0)+1) - u(-(t-\sigma-0)-1)\right]$$

$$= \delta(t-\sigma) - e^{|t-\sigma|}\left[u[-(t-\sigma)+1] - u(-(t-\sigma)-1)\right]$$

$= h(t,\sigma) \Rightarrow$ this system must be $\boxed{\text{Time - Invariant}}$

(*) Just go back to part a. So far, we have prove the system is LTI, and also have:

$$h(t,\tau) = \delta(t-\tau) - e^{|t-\tau|}\left[u(\tau-t+1) - u(\tau-t-1)\right]$$

$\Rightarrow y(t)$ can be rewrote:

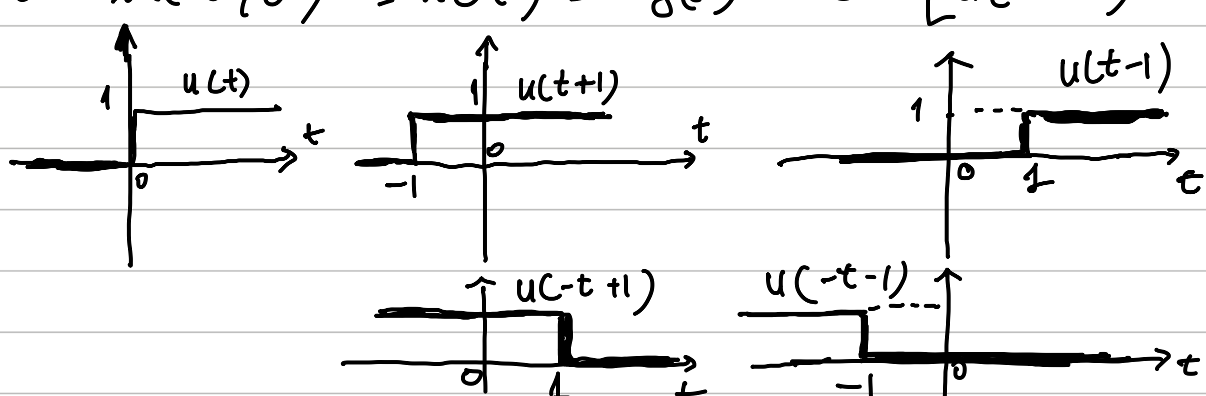
$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\left[\delta(t-\tau) - e^{|t-\tau|}\left[u(\tau-t+1) - u(\tau-t-1)\right]\right]d\tau$$

that is exactly the same with what we have in part a.

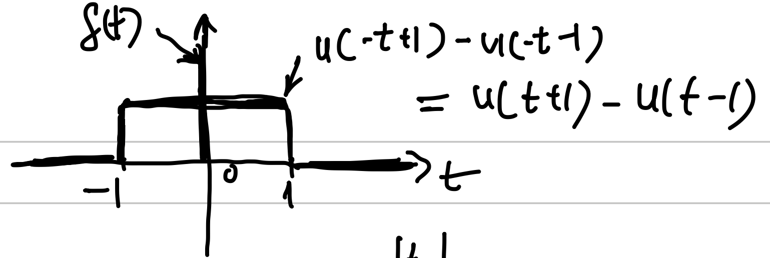
(*) We have;

$$h(t,\sigma) = \delta(t-\sigma) - e^{|t-\sigma|}\left[u(-t+\sigma+1) - u(-t+\sigma-1)\right]$$

Based on above result, we have S is LTI, we only need $h(t,0) = h(t) = \delta(t) - e^{|t|}\left[u(-t+1) - u(-t-1)\right]$



$\Rightarrow$



Since $h(t) = \delta(t) - e^{|t|}[u(t+1) - u(t-1)] \neq 0$ with $t < 0$

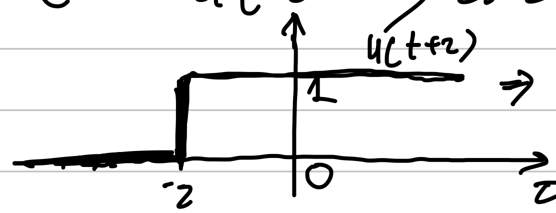
Or $h(t) \neq h(t)\,u(t)$. Also, from the relationship, at any instant time of output $y(t)$, it also depends on the future time of input $x(t)$ from $t-1 \to t+1$.

$\Rightarrow$ $\boxed{\text{System is NOT Causal.}}$

c) With input $x(t) = e^{-t}u(t+2)$

$$\Rightarrow y(t) = e^{-t}u(t+2) - \int_{t-1}^{t+1} e^{|t-\tau|}\left[e^{-\tau}u(\tau+2)\right]d\tau$$

$$\text{Let } A = \int_{t-1}^{t+1} e^{|t-\tau|}\, e^{-\tau}\, u(\tau+2)\, d\tau$$



$$\Rightarrow u(t+2) = \begin{cases} 1, & t \geq -2 \\ 0, & t < -2 \end{cases}$$

* When $t+1 < -2 \iff t < -3 \Rightarrow A = 0$

$\Rightarrow y(t) = e^{-t}u(t+2)$, and $u(t+2) = 0 \Rightarrow y(t) = 0$ (I)

* When $\begin{cases} t-1 < -2 \\ t+1 \geq -2 \end{cases} \iff -3 \leq t < -1$

$$\Rightarrow A = \int_{-2}^{t+1} e^{|t-\tau|}\, e^{-\tau}\, d\tau \;;\; e^{|t-\tau|} = \begin{cases} e^{t-\tau} & t \geq \tau \\ e^{-t+\tau} & t \leq \tau \end{cases}$$

$$= \int_{-2}^{t} e^{t-\tau}\, e^{-\tau}\, d\tau + \int_{t}^{t+1} e^{-t+\tau}\, e^{-\tau}\, d\tau$$

$$= \int_{-2}^{t} e^{t-2\tau}\, d\tau + \int_{t}^{t+1} e^{-t}\, d\tau$$

$$= e^{t} \int_{-2}^{t} e^{-2\tau}\, d\tau + e^{-t} \int_{t}^{t+1} 1 . d\tau$$

$$= e^{t} \left(\frac{1}{-2}\right) e^{-2\tau}\Big|_{-2}^{t} + e^{-t} . \tau\Big|_{t}^{t+1}$$

$$= \frac{1}{2} e^{t} e^{-2\tau}\Big|_{t}^{-2} + e^{-t}(t+1-t)$$

$$= \frac{1}{2} e^{t}\left[e^{4} - e^{-2t}\right] + e^{-t} = \frac{1}{2} e^{t+4} - \frac{1}{2} e^{-t} + e^{-t}$$

$$= \frac{1}{2} e^{t+4} + \frac{1}{2} e^{-t} \Rightarrow \text{ With } -3 \leq t < -1 ,\, A = \frac{1}{2} e^{t+4} + \frac{1}{2} e^{-t}$$

$$u(t+2) = \begin{cases} 1, & t \geq -2 \\ 0, & t < -2 \end{cases} \Rightarrow y(t) = e^{-t} u(t+2) - A$$

$$= \begin{cases} -\frac{1}{2} e^{t+4} - \frac{1}{2} e^{-t} & -3 \leq t < -2 \\ e^{-t} - \frac{1}{2} e^{t+4} - \frac{1}{2} e^{-t} & -2 \leq t < -1 \end{cases}$$

II

* When $t-1 \geq -2 \Leftrightarrow t \geq -1$

$$A = \int_{t-1}^{t+1} e^{|t-\tau|} e^{-\tau}\, d\tau = \int_{t-1}^{t} e^{t-\tau} e^{-\tau}\, d\tau + \int_{t}^{t+1} e^{-t+\tau} e^{-\tau}\, d\tau$$

$$= \int_{t-1}^{t} e^{t-2\tau} d\tau + \int_{t}^{t+1} e^{-t} d\tau$$

$$= e^{t} \int_{t-1}^{t} e^{-2\tau} d\tau + e^{-t} \int_{t}^{t+1} 1 d\tau$$

$$= \frac{e^{t}}{2} e^{-2\tau} \Big|_{t}^{t-1} + e^{-t} \tau \Big|_{t}^{t+1} = \frac{1}{2} e^{t} \left(e^{-2(t-1)} - e^{-2t} \right)$$

$$+ e^{-t} (t+1-t) = \frac{1}{2} e^{t} \left(e^{-2t+2} - e^{-2t} \right)$$

$$+ e^{-t} = \frac{1}{2} e^{-t+2} - \frac{1}{2} e^{-t} + e^{-t}$$

$$= \frac{1}{2} e^{-t+2} + \frac{1}{2} e^{-t} \Rightarrow \text{with } t \geq -1, A = \frac{1}{2} e^{-t+2} + \frac{1}{2} e^{-t}$$

Also, $u(t+2) = \begin{cases} 1, & t \geq -2 \\ 0, & t < -2 \end{cases} \Rightarrow y(t) = e^{-t} u(t+2) - A =$

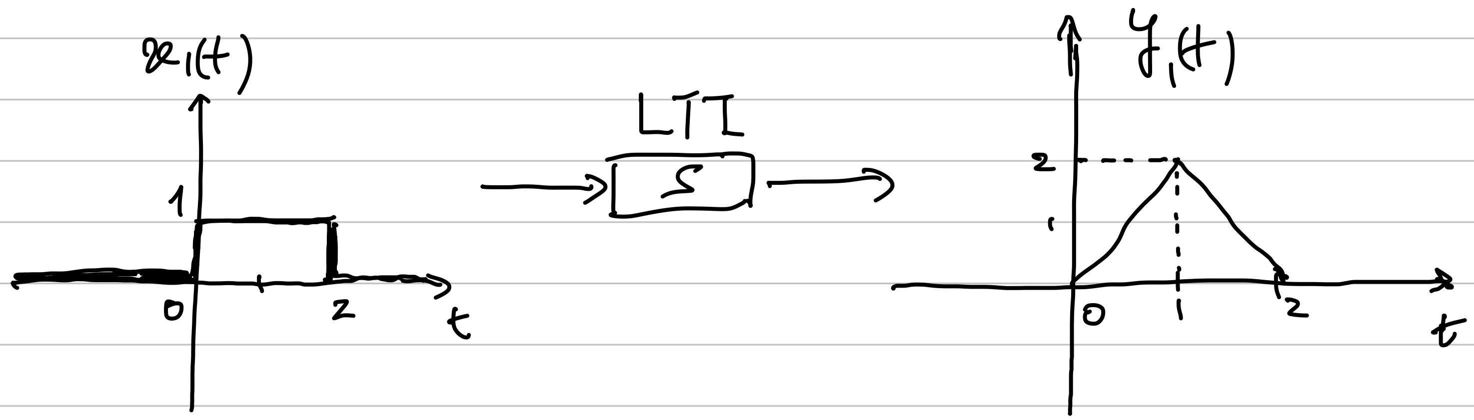
$$y(t) = e^{-t} - \frac{1}{2} e^{-t+2} - \frac{1}{2} e^{-t} \quad \text{with } t \geq -1 \quad \text{III}$$

Combining 3 cases I, II, & III

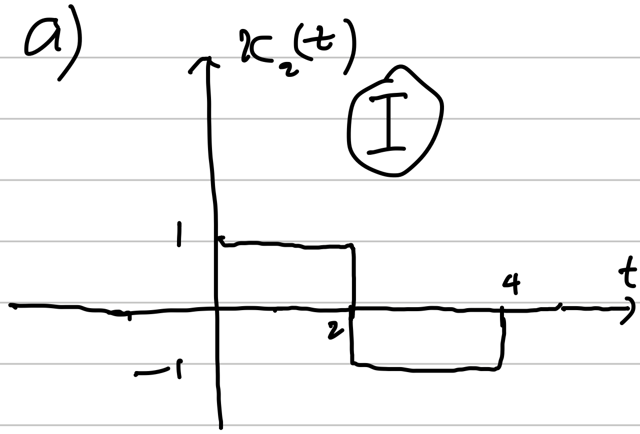
$$\boxed{y(t) = \begin{cases} 0, & t < -3 \\ -\frac{1}{2} e^{t+4} - \frac{1}{2} e^{-t}, & -3 \leq t < -2 \\ e^{-t} - \frac{1}{2} e^{t+4} - \frac{1}{2} e^{-t}, & -2 \leq t < -1 \\ e^{-t} - \frac{1}{2} e^{-t+2} - \frac{1}{2} e^{-t}, & t \geq -1 \end{cases}}$$

3) Given: consider an LTI system

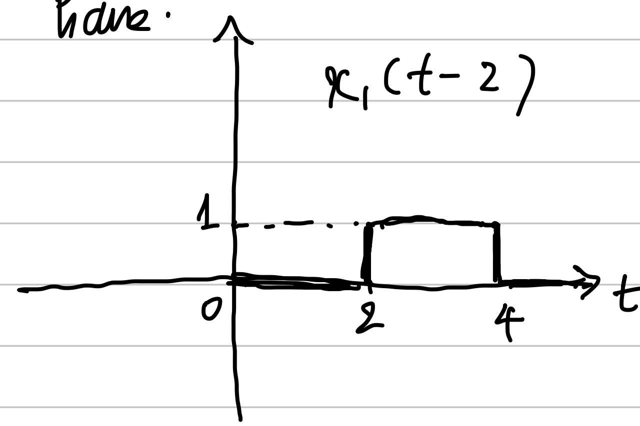
$$x_1(t) \longrightarrow \underset{}{\overset{LTI}{\boxed{S}}} \longrightarrow y_1(t)$$

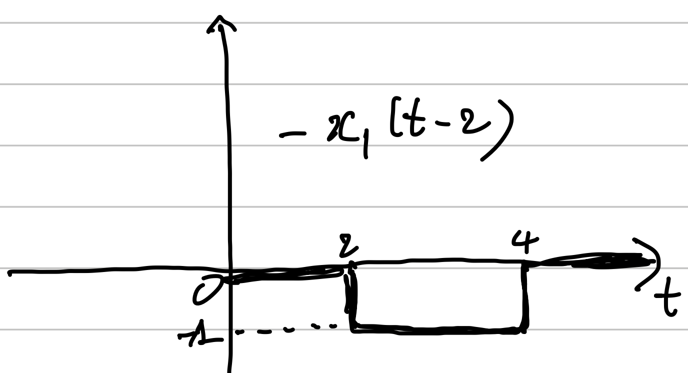


a)

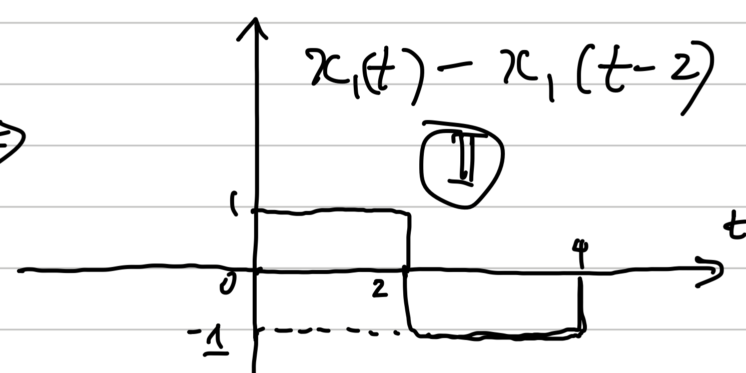


We have.





$\Rightarrow$

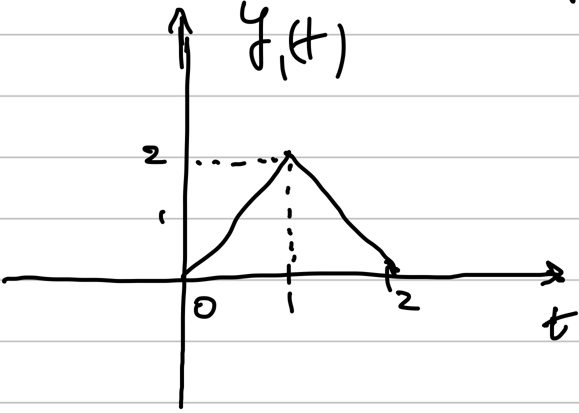


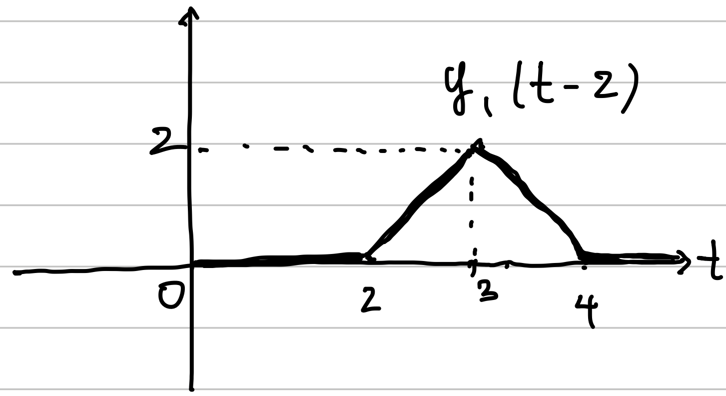
Based on I & II, we conclude $x_2(t) = x_1(t) - x_1(t-2)$

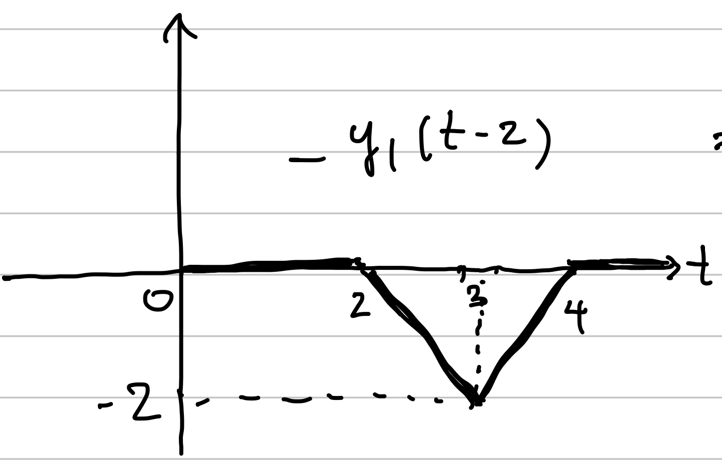
Since the system is LTI, so,

$$x_2(t) = x_1(t) - x_1(t-2) \longrightarrow \overset{LTI}{\boxed{S}} \longrightarrow y_2(t) = y_1(t) - y_1(t-2)$$

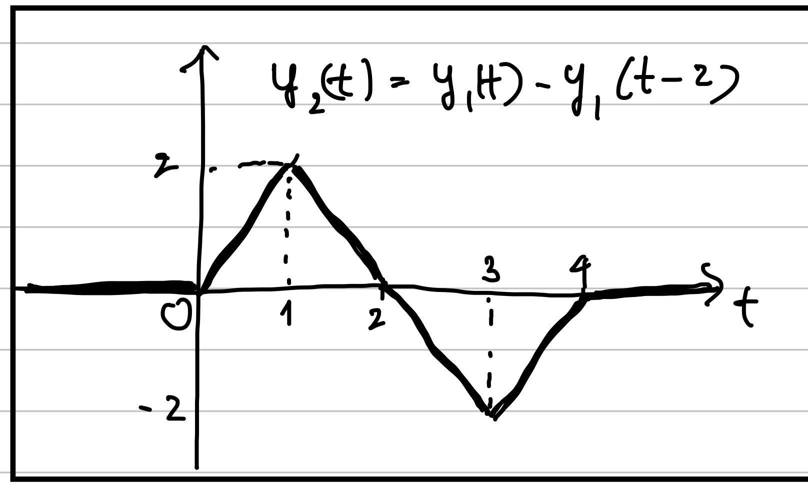
Now, we sketch $y_2(t)$ based on $y_1(t)$







$\Rightarrow$



b) As $x_1(t)$, we can represent it:

$$x_1(t) = u(t) - u(t-2)$$

Also, for $x_3(t)$, we can also take an advantage of unit step function, then we have:

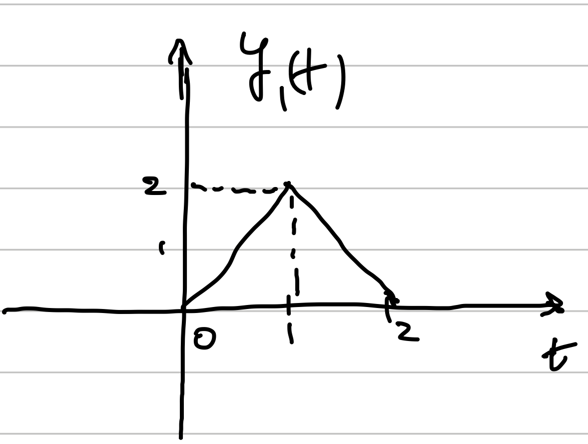
$$x_3(t) = u(t+1) + u(t) - u(t-1) - u(t-2)$$

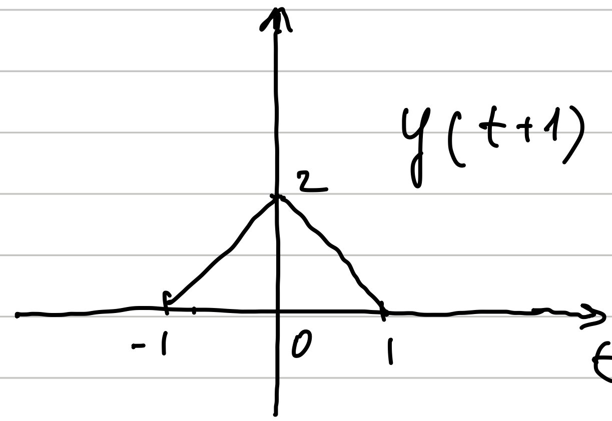
$$= u(t+1) - u(t-1) + u(t) - u(t-2)$$

$$= x_1(t+1) + x_1(t)$$

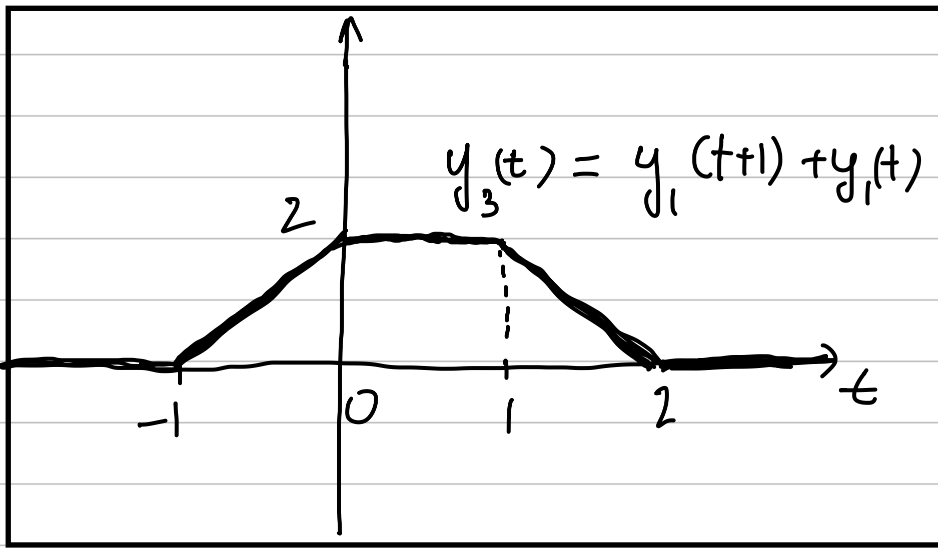
Since the system is LTI, we also have

$$x_3(t) = x_1(t+1) + x_1(t) \longrightarrow \overset{LTI}{\boxed{S}} \longrightarrow y_3(t) = y_1(t+1) + y_1(t)$$





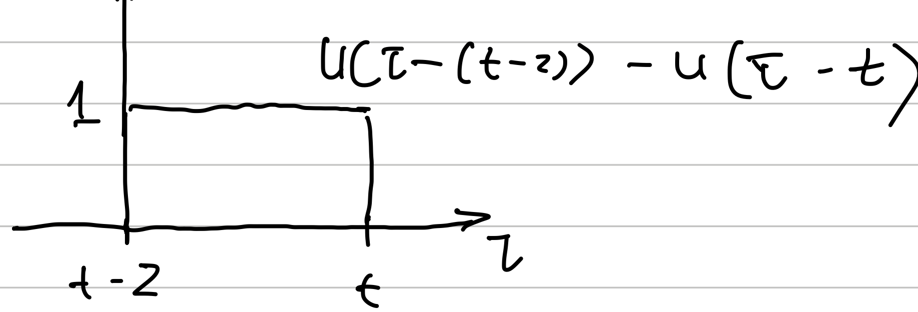
$\Rightarrow$



c) Given: $y(t) = \int_{t-2}^{t} x(\tau)\, d\tau$

Find $h(t)$.

We have:



We can rewrite $y(t) = \int_{-\infty}^{+\infty} x(\tau)\left[u(\tau-t+2) - u(\tau-t)\right] d\tau$

$$\Rightarrow \quad \delta(t) \longrightarrow \boxed{\overset{LTI}{S}} \longrightarrow y(t) = h(t) = S\{\delta(t)\}$$

$$\Rightarrow y(t) = h(t) = \int_{-\infty}^{+\infty} \delta(\tau)\left[u(\tau - t + 2) - u(\tau - t)\right] d\tau$$

Let $f(\tau) = u(\tau - t + 2) - u(\tau - t)$

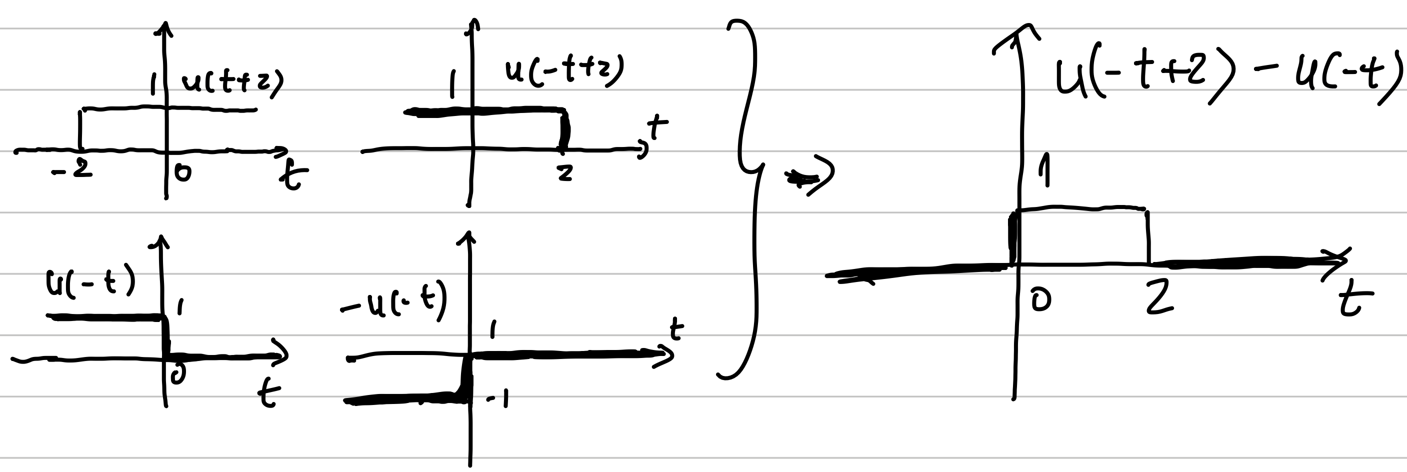
We have $h(t) = \int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau)\,d\tau$

Using the shifting property of impulse function.

$$f(\tau)\,\delta(\tau - 0) = f(0)\,\delta(\tau - 0)$$

$$\Rightarrow f(\tau)\,\delta(\tau) = \left[u(-t+2) - u(-t)\right]\delta(\tau)$$

Beside, we have



As we can see $u(-t+2) - u(-t) = u(t) - u(t-2)$

$$\Rightarrow f(\tau)\,\delta(\tau) = \left[u(t) - u(t-2)\right]\delta(\tau)$$

$$\Rightarrow h(t) = \int_{-\infty}^{+\infty} \left[u(t) - u(t-2)\right]\delta(\tau)\,d\tau$$

$$\Rightarrow h(t) = [u(t) - u(t-2)] \underbrace{\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau}_{=1}$$

$$\Rightarrow \boxed{h(t) = u(t) - u(t-2)}$$

* When input $x(t) = u(t+2) + u(t) - 2u(t-1) + \delta(t-1)$

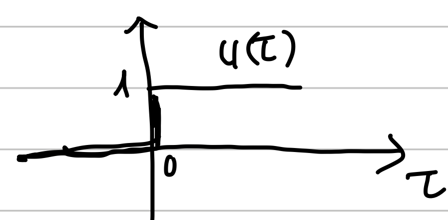
We already have:

$$\delta(t) \rightarrow \underset{LTI}{\boxed{S}} \rightarrow h(t) = u(t) - u(t-2)$$

$$\Rightarrow \delta(t-1) \rightarrow \boxed{S} \rightarrow h(t-1) = u(t-1) - u(t-3)$$

Besides, $u(t) \rightarrow \underset{LTI}{\boxed{S}} \rightarrow z(t) = S\{u(t)\}$

$$\Rightarrow z(t) = \int_{t-2}^{t} u(\tau)\, d\tau \quad \text{Since } u(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$



We have:

* When $t < 0 \Rightarrow z(t) = \int_{t-2}^{t} u(\tau)\, d\tau = 0$

* When $\begin{cases} t-2 < 0 \\ t \geq 0 \end{cases} \Leftrightarrow 0 \leq t < 2$, $z(t) = \int_{0}^{t} u(\tau)\, d\tau$

$$= 1\Big|_{0}^{t} = t$$

* when $t-2 \geq 0 \Leftrightarrow t \geq 2$

$$z(t) = \int_{t-2}^{t} u(\tau)\,d\tau = \tau \Big|_{t-2}^{t} = t - t + 2 = 2$$

So, $u(t) \longrightarrow \underset{LTI}{\boxed{S}} \longrightarrow z(t) = \begin{cases} 0, & t<0 \\ t, & 0 \leq t < 2 \\ 2 & t \geq 2 \end{cases}$

$\Rightarrow u(t+2) \rightarrow \boxed{S} \longrightarrow z(t+2) = \begin{cases} 0 & t < -2 \\ t+2 & -2 \leq t < 0 \\ 2 & t \geq 0 \end{cases}$ *

$\Rightarrow 2u(t-1) \rightarrow \boxed{S} \longrightarrow 2z(t-1) = \begin{cases} 0 & t < 1 \\ 2t-2 & 1 \leq t < 3 \\ 4 & t \geq 3 \end{cases}$

Also, $\delta(t-1) \rightarrow \boxed{S} \rightarrow u(t-1) - u(t-3) = \begin{cases} 1, & 1 \leq t < 3 \\ 0, & \text{otherwise} \end{cases}$

$\Rightarrow$ we have: $x(t) = u(t+2) + u(t) - 2u(t-1) + \delta(t-1)$

+ with $t < -2$,

$\left.\begin{cases} u(t+2) \xrightarrow{S} 0 \\ u(t) \xrightarrow{S} 0 \\ -2u(t-1) \xrightarrow{S} 0 \\ \delta(t-1) \xrightarrow{S} 0 \end{cases}\right\} \Rightarrow x(t) \xrightarrow{S} y(t) = 0$

+ with $-2 \leq t < 0$

$\left.\begin{cases} u(t+2) \xrightarrow{S} t+2 \\ u(t) \xrightarrow{S} 0 \\ -2u(t-1) \xrightarrow{S} 0 \\ \delta(t-1) \xrightarrow{S} 0 \end{cases}\right\} \Rightarrow x(t) \xrightarrow{S} y(t) = t+2$

* With $0 \le t < 1$

$$\left.\begin{array}{l} u(t+2) \xrightarrow{S} 2 \\ u(t) \xrightarrow{S} t \\ -2u(t-1) \xrightarrow{S} 0 \\ \delta(t-1) \xrightarrow{S} 0 \end{array}\right\} \Rightarrow x(t) \xrightarrow{S} y(t) = t+2$$

* With $1 \le t < 2$

$$\left\{\begin{array}{l} u(t+2) \xrightarrow{S} 2 \\ u(t) \xrightarrow{S} t \\ -2u(t-1) \xrightarrow{S} -2t+2 \\ \delta(t-1) \xrightarrow{S} 1 \end{array}\right\} \Rightarrow x(t) \xrightarrow{S} y(t) = 5-t$$

* With $2 \le t < 3$

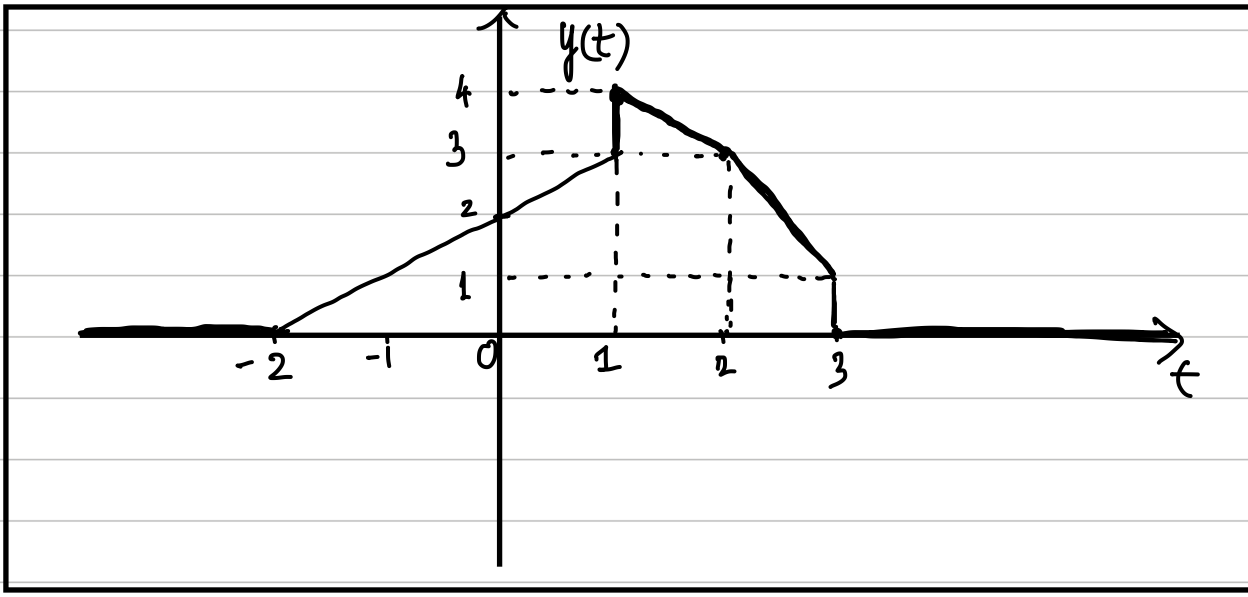
$$\left\{\begin{array}{l} u(t+2) \xrightarrow{S} 2 \\ u(t) \xrightarrow{S} 2 \\ -2u(t-1) \xrightarrow{S} -2t+2 \\ \delta(t-1) \xrightarrow{S} 1 \end{array}\right\} \Rightarrow x(t) \xrightarrow{S} y(t) = 7-2t$$

* With $t \ge 3$

$$\left.\begin{array}{l} u(t+2) \longrightarrow 2 \\ u(t) \longrightarrow 2 \\ -2u(t-1) \longrightarrow -4 \\ \delta(t-1) \longrightarrow 0 \end{array}\right\} \Rightarrow x(t) \xrightarrow{S} y(t) = 0$$

$$\rightarrow \quad x(t) \xrightarrow{S} y(t) = \begin{cases} 0, & t < -2 \\ t+2, & -2 \le t < 0 \\ t+2 & 0 \le t < 1 \\ 5-t & 1 \le t < 2 \\ 7-2t & 2 \le t < 3 \\ 0 & t \ge 3 \end{cases}$$

$$\Rightarrow \quad y(t) = \begin{cases} 0 & t < -2 \\ t+2 & -2 \leq t < 1 \\ 5-t & 1 \leq t < 2 \\ 7-2t & 2 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$$



4) $y(t) = \int_{-\infty}^{\infty} e^{-t} (t-\tau)^2 u(\tau+t)\, \underline{x(\tau-2)}\, d\tau \quad , \quad t \in (-\infty, +\infty)$

a) $h(t,\tau) = S\{\delta(t-\tau)\}$

Rewrite the IPOP interm of σ instead of τ.

$$y(t) = \int_{-\infty}^{+\infty} e^{-t} (t-\sigma)^2 u(\sigma+t)\, \underline{x(\sigma-2)}\, d\sigma$$

$$\Rightarrow h(t,\tau) = \int_{-\infty}^{+\infty} e^{-t} (t-\sigma)^2 u(\sigma+t)\, \underline{\delta(\sigma-2-\tau)}\, d\sigma$$

Using shifting property of impulse function,

$$f(\sigma)\, \delta[\sigma - (\tau+2)] = f(\tau+2)\, \delta(\sigma-(\tau+2))$$

(τ is constant)

$$f(\sigma) = e^{-t} (t-\sigma)^2 u(\sigma+t)$$

$$\Rightarrow f(\tau+2) = e^{-t} (t-\tau-2)^2 u(\tau+2+t)$$

$$\Rightarrow h(t,\tau) = \int_{-\infty}^{+\infty} e^{-t} (t-\tau-2)^2 u(\tau+2+t)\, \delta(\sigma-\tau-2)\, d\sigma$$

$$= e^{-t} (t-\tau-2)^2 u(\tau+2+t) \underbrace{\int_{-\infty}^{+\infty} \delta(\sigma-2-\tau)\, d\sigma}_{1}$$

$$= e^{-t} (t-\tau-2)\, u(\tau+2+t)$$

$$\boxed{h(t,\tau) = e^{-t}(t-\tau-2)^2\, u(t+\tau+2)}$$

Check $h(t-\tau, 0) = e^{-t+\tau}(t-\tau-0-2)\,u(t-\tau+0+2)$

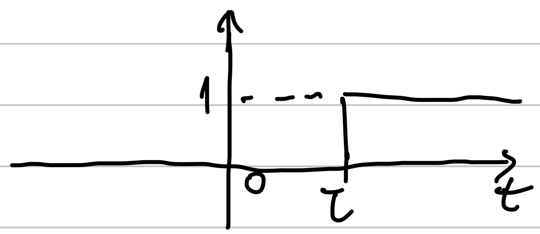
$= e^{-t+\tau}(t-\tau-2)\,u(t-\tau+2) \neq h(t,\tau)$

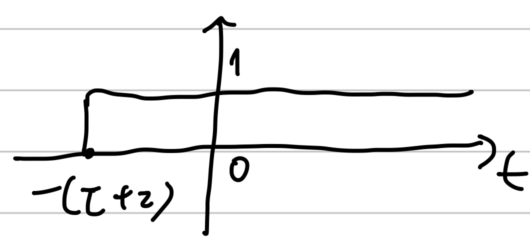
$\Rightarrow$ $\boxed{\text{This system is Time Variant - TV}}$

* Check $h(t,\tau)$ & $h(t,\tau).u(t-\tau)$

$\Leftrightarrow$ check $e^{-t}(t-\tau-2)^2\, u(t+\tau+2)$

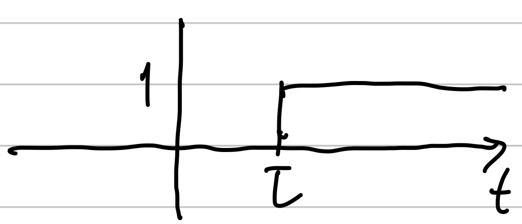
and $e^{-t}(t-\tau-2)^2\, u(t+\tau+2)\, u(t-\tau)$





$\Rightarrow$ $u(t+\tau+2).u(t-\tau)$ in this example

$= u(t-\tau)$



$\Rightarrow h(t,\tau)\, u(t-\tau) = e^{-t}(t-\tau-2)^2\, u(t-\tau)$

$\neq h(\tau,\tau) = e^{-t}(t-\tau-2)^2\, u(t+\tau+2)$

$\Rightarrow$ $\boxed{\text{this system is Non-Causal.}}$

b) $x(t) = \delta(t-2) - e^{-t}u(t+1) \qquad t \in (-\infty, +\infty)$

$$\Rightarrow x(t-2) = \delta(t-4) - e^{-t+2} u(t-1)$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} e^{-t} (t-\tau)^2 u(\tau+t) \left[\delta(\tau-4) - e^{-\tau+2} u(\tau-1)\right] d\tau$$

$$= \int_{-\infty}^{\infty} e^{-t} (t-\tau)^2 u(\tau+t) \delta(\tau-4) \, d\tau \qquad (A)$$

$$- \int_{-\infty}^{\infty} e^{-t} (t-\tau)^2 u(\tau+t) e^{-\tau+2} u(\tau-1) \, d\tau \qquad (B)$$

$$A = \int_{-\infty}^{\infty} e^{-t} (t-\tau)^2 u(\tau+t) \delta(\tau-4) \, d\tau$$

let $f(\tau) = e^{-t} (t-\tau)^2 u(\tau+t)$

$$\Rightarrow f(\tau) \delta(\tau-4) = f(4) \delta(\tau-4)$$

(shifting property of impulse function)

$$\Rightarrow A = \int_{-\infty}^{\infty} e^{-t} (t-4)^2 u(t+4) \delta(\tau-4) \, d\tau$$

$$= e^{-t} (t-4)^2 u(t+4) \underbrace{\int_{-\infty}^{+\infty} \delta(\tau-4) \, d\tau}_{=1}$$

$$= e^{-t} (t-4)^2 u(t+4)$$

$$= \begin{cases} e^{-t} (t-4)^2, & t \geq -4 \\ 0, & t < -4 \end{cases}$$

$$B = \int_{-\infty}^{\infty} e^{-t}(t-\tau)^2 u(\tau+t)\, e^{-\tau+2} u(\tau-1)\, d\tau$$

$$= \int_{-\infty}^{+\infty} e^{-t-\tau+2} (t-\tau)^2 u(t+\tau) u(\tau-1)\, d\tau$$

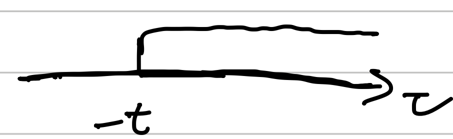
$$= \underbrace{\int_{-\infty}^{1} e^{-t-\tau+2} (t-\tau)^2 u(t+\tau) u(\tau-1)\, d\tau}_{=0}$$

$$+ \int_{1}^{+\infty} e^{-t-\tau+2} (t-\tau)^2 u(t+\tau) u(\tau-1)\, d\tau$$

Since $\begin{cases} \tau < 1 \Rightarrow u(\tau-1) = 0 \\ \tau \geq 1 \Rightarrow u(\tau-1) = 1 \end{cases}$

$$\Rightarrow B = \int_{1}^{+\infty} e^{-t-\tau+2} (t-\tau)^2 u(t+\tau)\, d\tau$$

$$= e^{-t+2} \int_{1}^{+\infty} e^{-\tau} (t-\tau)^2 u(t+\tau)\, d\tau$$



(*) If $1 < -t \Leftrightarrow t < -1$

$$B = e^{-t+2} \int_{-t}^{+\infty} e^{-\tau} (\tau - t)^2\, d\tau$$

We have $\int e^{-\tau}(\tau-t)^2 d\tau$

$$u = (\tau - t)^2, \quad dv = e^{-\tau} d\tau \Rightarrow v = -e^{-\tau}$$

$$\Rightarrow du = 2(\tau - t)\, d\tau$$

$$\Rightarrow \int \underbrace{(\tau-t)^2}_{u} \underbrace{e^{-\tau} d\tau}_{dv} = -\underbrace{(\tau-t)^2 \cdot e^{-\tau}}_{uv} \underbrace{+ 2\int e^{-\tau}(\tau-t)\, d\tau}_{-\int v\, du}$$

$$\int e^{-\tau}(\tau-t)\, d\tau = -(\tau-t)e^{-\tau} + \int e^{-\tau} d\tau$$

$$\left(\begin{array}{l} u = \tau - t \quad e^{-\tau} d\tau = dv \Rightarrow v = -e^{-\tau} \\ \Rightarrow du = d\tau \end{array} \right)$$

$$= -(\tau-t)e^{-\tau} - e^{-\tau}$$

$$\Rightarrow \int e^{-\tau}(\tau-t)^2 d\tau = -(\tau-t)^2 e^{-\tau}$$

$$+ 2\left[-(\tau-t)e^{-\tau} - e^{-\tau}\right]$$

$$= -(\tau-t)^2 e^{-\tau} - 2(\tau-t)e^{-\tau} - 2e^{-\tau}$$

$$= \boxed{-e^{-\tau}\left[(\tau-t)^2 + 2(\tau-t) + 2\right]}$$

$$\Rightarrow B = -e^{-t+2}\left[e^{-\tau}\left[(\tau-t)^2 + 2(\tau-t) + 2\right]\right]\Bigg|_{-t}^{+\infty}$$

$$\Rightarrow B = -e^{-t+2}\left[0-\left[e^{t}\left[(-t-t)^2+2(-t-t)+2\right]\right]\right]$$

$$= -e^{-t+2}\cdot\left[-e^{t}\left(4t^2-4t+2\right)\right]$$

$$= e^{2}\left(4t^2-4t+2\right) = 2e^{2}\left(2t^2-2t+1\right)$$

$$\Rightarrow \text{if } t<-1 \Rightarrow B = 2e^{2}\left(2t^2-2t+1\right)$$

$*$ If $-t \leq 1 \Leftrightarrow t \geq -1$

$$B = e^{-t+2}\int_{1}^{+\infty} e^{-\tau}(\tau-t)^2\, d\tau$$

$$= -e^{-t+2}\left[e^{-\tau}\left[(\tau-t)^2+2(\tau-t)+2\right]\right]\Big|_{1}^{+\infty}$$

$$= -e^{-t+2}\left[0-e^{-1}\left[(t-1)^2-2(t-1)+2\right]\right]$$

$$= e^{-t+1}\left[t^2-2t+1-2t+2+2\right] = e^{-t+1}\left[t^2-4t+5\right]$$

$$\Rightarrow B = \begin{cases} 2e^{2}(2t^2-2t+1), & t<-1 \\ e^{-t+1}(t^2-4t+5), & t \geq -1 \end{cases}$$

Also, $A = \begin{cases} e^{-t}(t-4)^2, & t \geq -4 \\ 0, & t<-4 \end{cases}$

Since $y(t) = A - B$

* With $t < -4$, $\begin{cases} A = 0 \\ B = 2e^2(2t^2 - 2t + 1) \end{cases}$

$\Rightarrow \underline{y(t) = -2e^2(2t^2 - 2t + 1)}$

* With $-4 \leq t < -1$ $\begin{cases} A = e^{-t}(t-4)^2 \\ B = 2e^2(2t^2 - 2t + 1) \end{cases}$

$\Rightarrow y(t) = \underline{e^{-t}(t-4)^2 - 2e^2(2t^2 - 2t + 1)}$

* With $t \geq -1$. $\begin{cases} A = e^{-t}(t-4)^2 \\ B = e^{-t+1}(t^2 - 4t + 5) \end{cases}$

$\Rightarrow y(t) = \underline{e^{-t}(t-4)^2 - e^{-t+1}(t^2 - 4t + 5)}$

Finally:

$$\boxed{y(t) = \begin{cases} -2e^2(2t^2 - 2t + 1), & t < -4 \\ e^{-t}(t-4)^2 - 2e^2(2t^2 - 2t + 1), & -4 \leq t < -1 \\ e^{-t}(t-4)^2 - e^{-t+1}(t^2 - 4t + 5), & t \geq -1 \end{cases}}$$

Note: Use L'Hospital rule to prove

$$\lim_{\tau \to +\infty} e^{-\tau}\left[(\tau-t)^2 + 2(\tau-t)+2\right] = 0$$

We have:

$$\lim_{\tau \to +\infty}\left[\frac{(\tau-t)^2 + 2(\tau-t)+2}{e^{\tau}}\right] \quad (t \text{ is const})$$

Rewrite $\tau \to x$.

$$\lim_{x \to +\infty} \frac{(x-t)^2 + 2(x-t)+2 \nearrow \infty}{e^{x} \to \infty}$$

$$= \lim_{x \to +\infty} \frac{2(x-t)+2 \nearrow +\infty}{e^{x} \to +\infty} = \lim_{x \to +\infty} \frac{2}{e^{x}} = 0$$