Discussion 10

ECE 102: Systems and Signals

Winter 2022

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Question 1:

Consider a cascade of LTI systems S_1 and S_2, shown below.

$$x(t) \rightarrow \boxed{S_1} \xrightarrow[z(t)]{} \boxed{S_2} \rightarrow y(t)$$

$$S_1: \quad h_1(t) = \frac{\sin(\frac{2\pi t}{3})}{\pi t} \quad ; \quad S_2: \quad \text{given by IPOP } \; y(t) + 2\frac{dy(t)}{dt} = z(t) \quad .$$

(a) Find and sketch the frequency response $H_o(\omega)$ for the cascaded system $S_1 S_2$.

(b) Consider the input $x(t) = e^{j2\pi t/3}\sin(\pi t/6) + \cos(\pi t/3)$. Find the fourier series representation of the output $y(t)$ obtained when $x(t)$ is passed through the cascaded system $S_1 S_2$.

(a) S_1 : $\underbrace{h_1(t) = \frac{\sin\left(\frac{2\pi}{3}t\right)}{\pi t}}$ $= \frac{2}{3}\cdot\frac{\pi}{\pi}\cdot\frac{\sin\left(\frac{2\pi}{3}t\right)}{\frac{2}{3}\pi t}$

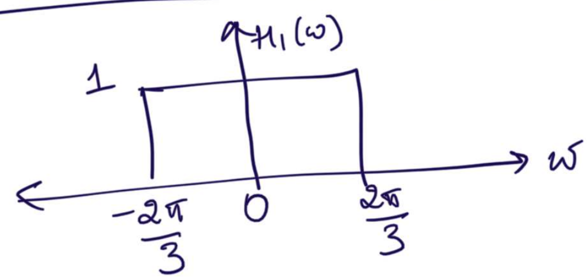
$$= \frac{2}{3}\,\text{sinc}\left(\frac{2\pi}{3}t\right).$$

$$\boxed{h(t) = \frac{W}{\pi}\,\text{sinc}(Wt)} \longleftrightarrow \text{rec}(\omega, W).$$

$$h_1(t) = \frac{2\pi}{3}\cdot\frac{1}{\pi}\,\text{sinc}\left(\frac{2\pi}{3}t\right).$$

$$W = 2\pi/3$$

$$\therefore \boxed{H_1(\omega) = \begin{cases} 1 & |\omega| < 2\pi/3 \\ 0 & |\omega| > 2\pi/3. \end{cases}}$$



S_2 : $y(t) + 2\dfrac{dy(t)}{dt} = z(t)$.

$\mathcal{F}\{y(t) + 2\dot{y} = z(t)\} \longleftrightarrow Y(\omega) + 2j\omega\, Y(\omega) = Z(\omega)$

$$\Rightarrow \frac{Y(\omega)}{Z(\omega)} = \frac{1}{1+2j\omega}$$

$$\boxed{H_2(\omega) = \frac{1}{1+2j\omega} \quad \forall\ \omega \in \mathbb{R}}$$

$H_0(\omega) = \underline{H_1(\omega)}\, H_2(\omega)$.

$$\underline{H_0(\omega)} = \begin{cases} \left\{\dfrac{1}{1+2j\omega}\right\} & |\omega| \leq \dfrac{2\pi}{3} \\ 0 & |\omega| > \dfrac{2\pi}{3} \end{cases}$$

i) Magnitude response $|H_0(\omega)|$ vs. ω.

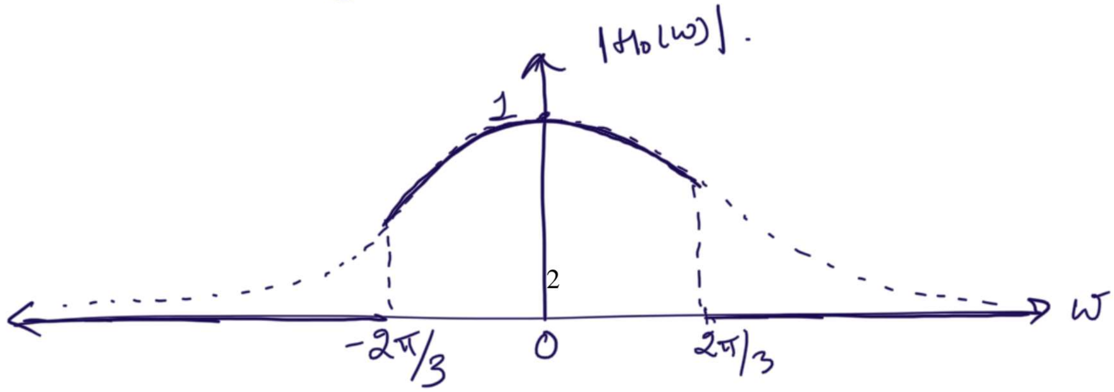
ii) Phase response. $\angle H_0(\omega)$ vs. ω.

i) Magnitude resp: $|H_0(\omega)| = \sqrt{H_0(\omega) \cdot H_0^*(\omega)}$

$A \in \mathbb{C}$; then $|A| = \sqrt{A \cdot A^*}$

$$\therefore\ |H_0(\omega)| = \sqrt{\frac{1}{(1+2j\omega)(1-2j\omega)}} = \sqrt{\frac{1}{1+4\omega^2}}$$

$$\underline{|H_0(\omega)|} \begin{cases} = \dfrac{1}{\sqrt{1+4\omega^2}} & \forall\ |\omega| \leq \dfrac{2\pi}{3} \\ = 0 & |\omega| > \dfrac{2\pi}{3} \end{cases}$$



$$\measuredangle H_0(\omega) = \tan^{-1}\left\{\frac{\mathrm{Im}\{\}}{\mathrm{Re}\{\}}\right\}$$

$$\left\{\frac{1}{\underline{\underline{1+2j\omega}}}\right\} = \left\{\frac{1-2j\omega}{(1+2j\omega)(1-2j\omega)}\right\}$$

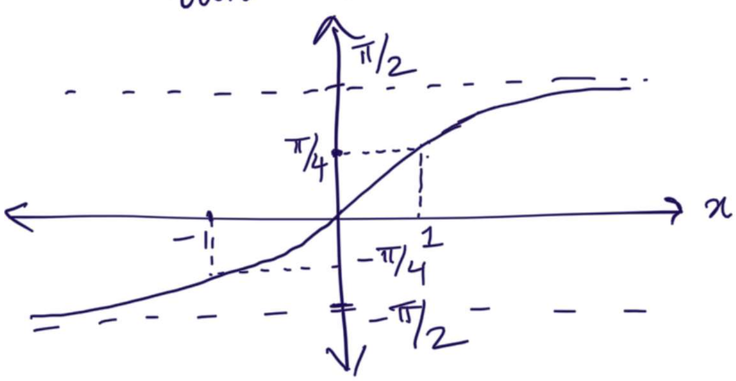
$$= \left\{\frac{1-2j\omega}{1+4\omega^2}\right\}.$$

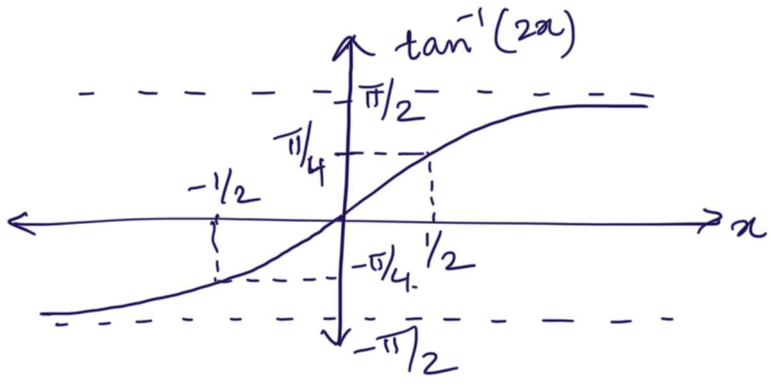
$$\therefore \measuredangle H_0(\omega) = \tan^{-1}\left\{\frac{-2\omega/(1+4\omega^2)}{1/(1+4\omega^2)}\right\}$$

$$\boxed{\measuredangle H_0(\omega) = \tan^{-1}\{-2\omega\}}$$

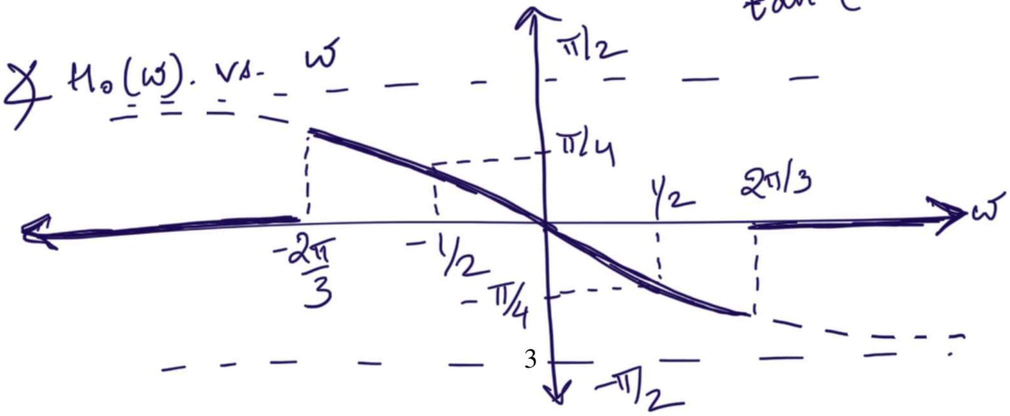
$$\tan^{-1}(-2x) = -\tan^{-1}(2x).$$

$\tan^{-1}(x).$





$\tan^{-1}(-2\omega).$



(b) $x(t) = \underbrace{e^{j\frac{2\pi}{3}t}} \sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{3}t\right).$

$$x(t) \longrightarrow \boxed{H_0(\omega)} \longrightarrow y(t).$$

$y(t) \downarrow$ F.S.

i) Find. F.S. coeff of $x(t)$.

$$x(t). = e^{j\frac{2\pi}{3}t} \cdot \left(\frac{e^{j\frac{\pi}{6}t} - e^{-j\frac{\pi}{6}t}}{2j} \right)$$

$$+ \cos\left(\frac{\pi}{3}t\right).$$

$$\underline{\underline{x(t)}}. = \frac{1}{2j}\left[e^{j\frac{5\pi}{6}t} - e^{j\frac{\pi}{2}t} \right] + \cos\left(\frac{\pi}{3}t\right).$$

$T_1 = \frac{2\pi}{5\pi/6} = \underline{\underline{\frac{12}{5}}}$; $T_2 = \frac{2\pi}{\pi/2} = \underline{\underline{4}}.$; $T_1 = \frac{2\pi}{\pi/3} = \underline{\underline{6}}$

$$T_0 = LCM\left\{ \frac{12}{5}, 4, 6 \right\} = \underline{\underline{12}}.$$

$$\omega_0 = \frac{2\pi}{T_0} = \underline{\frac{\pi}{6}}.$$

$$\rightarrow x(t) = \underline{\frac{1}{2j}} e^{j5\omega_0 t} - \frac{1}{2j} e^{j3\omega_0 t} + \frac{1}{2} e^{j2\omega_0 t} + \frac{1}{2} e^{-j2\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}.$$

$$\boxed{X_5 = \frac{1}{2j} \;;\; X_3 = -\frac{1}{2j} \qquad X_2 = \frac{1}{2} = X_{-2}.}$$

$X_0 = 0.$

$\rightarrow \quad y(t) = x(t) * h_0(t).$

$$\underline{Y_k} = X_k \cdot H_0(\omega)\big|_{\omega = k\omega_0}$$

i). $X_k = 0 \quad \forall \; k \neq 2, -2, 5, 3.$

$\therefore \quad Y_k = 0 \quad \forall \; k \neq 2, -2, 5, 3.$

$\rightarrow$ ii). since $\underline{\underline{H_0(\omega)}} = 0 \quad \forall \; |\omega| > \frac{2\pi}{3}.$

$H_0(k\omega_0) = 0 \quad \forall \; k$ s.t. $|k\omega_0| > \frac{2\pi}{3}$

$$Y_5 = \underline{\underline{X_5}} \cdot H_0\underbrace{\left(5 \cdot \frac{\pi}{6}\right)}_{0}. \qquad \frac{5\pi}{6} > \frac{2\pi}{3}.$$

$$\therefore \boxed{Y_5 = 0}.$$

$$Y_3 = X_3 \cdot H_0\left(3 \cdot \frac{\pi}{6}\right) = -\frac{1}{2j} H_0\left(\frac{\pi}{2}\right)$$

$$= -\frac{1}{2j}\left(\frac{1}{1 + 2j\,\pi/2}\right).$$

$$Y_2 = X_2 \, H_0\left(2 \cdot \frac{\pi}{6}\right) = \frac{1}{2}\left(\frac{1}{1 + 2j\,\pi/3}\right)$$

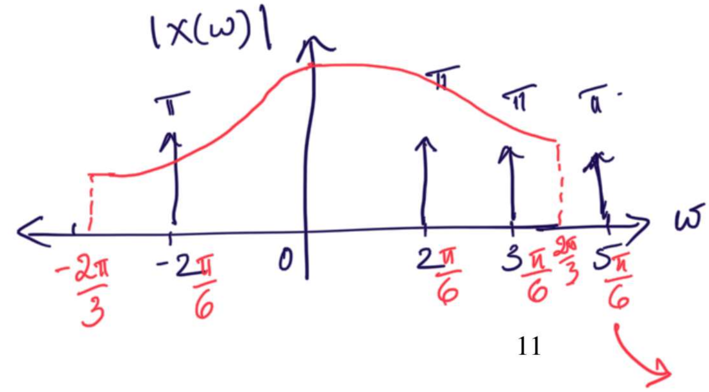
$$Y_{-2} = X_{-2} \, H_0\left(-\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{1}{1 - 2j\,\pi/3}\right).$$

X_k : $\quad \underline{\underline{X(\omega)}} = \sum_{k=-\infty}^{\infty} 2\pi X_k \, \delta(\omega - k\omega_0).$

$X_2 = X_{-2} = \frac{1}{2}$

$X_3 = -\frac{1}{2j}$

$X_5 = +\frac{1}{2j}.$



Question 2

An LTI system is given by the impulse response

$$h(t) = sinc(\pi t)$$

(a) Compute and sketch the frequency response $H(\omega)$.

(b) Consider input $x(t) = e^{-4|t|}$ applied to the system. Sketch the magnitude response of the Fourier Transform $X(\omega)$.

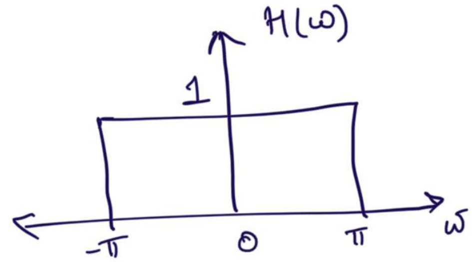
(c) Compute and sketch the magnitude response of output $Y(\omega)$ when $x(t)$ is applied to the system.

(a). $h(t) = sinc(\pi t) = \frac{\pi}{\pi} sinc(\pi t) \quad \left\{ \frac{W}{\pi} sinc(Wt) \right\}$

$$\Updownarrow$$

$$rec(\omega, W) = rec(\omega, \pi).$$

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \pi \\ 0 & |\omega| > \pi. \end{cases}$$



(b) $x(t) = e^{-4|t|}$

$= e^{-4t}u(t) + e^{4t}u(-t).$

$x(t) \longleftrightarrow X(\omega)$

$x(-t) \longleftrightarrow X(-\omega)$

$$\mathcal{F}\{\underline{e^{-4t}u(t)}\} = \frac{1}{4+j\omega}.$$

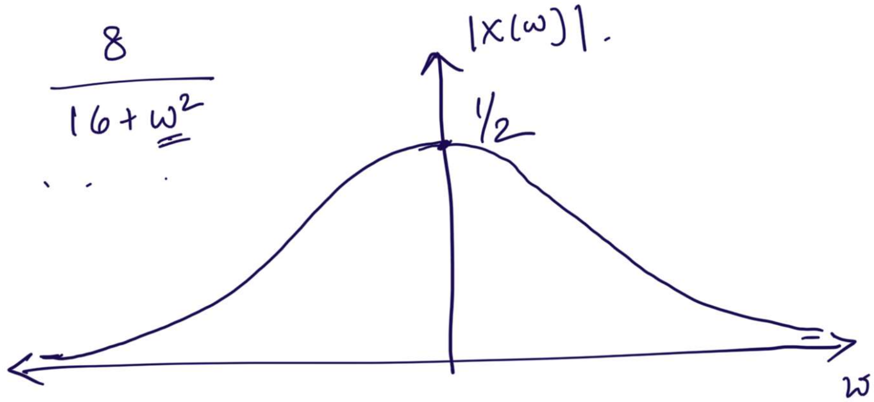
$$\mathcal{F}\{\underline{e^{4t}u(-t)}\} = \frac{1}{4-j\omega}$$

$$\therefore X(\omega) = \frac{1}{4+j\omega} + \frac{1}{4-j\omega} = \frac{2(4)}{16+\omega^2}$$

$$\boxed{X(\omega) = \frac{8}{16+\omega^2} \; ; \forall \omega \in \mathbb{R}.}$$

$\angle X(\omega) = 0.$

$$|X(\omega)| = \frac{8}{16+\omega^2}$$



(C).

$$Y(\omega) = X(\omega) \cdot H(\omega).$$