HOMEWORK ②

Solutions.

1 – (1-b)

Question ①

a) $y(t) = \int_{-\infty}^{2t} x(\tau+3)\,d\tau$

Linearity $x_1(t) \rightarrow y_1(t) = \int_{-\infty}^{2t} x_1(\tau+3)\,d\tau$

$x_2(t) \rightarrow y_2(t) = \int_{-\infty}^{2t} x_2(\tau+3)\,d\tau$

$\therefore \int_{-\infty}^{2t} [\alpha x_1(\tau+3) + \beta x_2(\tau+3)]\,d\tau = \alpha \int_{-\infty}^{2t} x_1(\tau+3)\,d\tau + \beta \int_{-\infty}^{2t} x_2(\tau+3)\,d\tau$

$= \alpha y_1(t) + \beta y_2(t).$

$\therefore$ $\boxed{\text{Linear}}$

Time Invariance $x(t-t_0) \rightarrow \boxed{S} \rightarrow y_0(t)$

$y_0(t) = S\{x(t-t_0)\} = \int_{-\infty}^{2t} x(\tau - t_0 + 3)\,d\tau$

Let $\tau - t_0 = \tilde{\tau}$ $\Rightarrow$ $d\tau = d\tilde{\tau}$ $\Rightarrow$ $\tau = 2t$ $\Rightarrow$ $\tilde{\tau} = 2t - t_0$

$\therefore y_0(t) = \int_{-\infty}^{2t-t_0} x(\tilde{\tau}+3)\,d\tilde{\tau}$

$y(t-t_0) = \int_{-\infty}^{2(t-t_0)} x(\tau+3)\,d\tau$

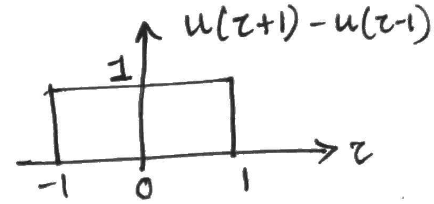
$\boxed{\text{T.V}}$

Causality $y(t) = \int_{-\infty}^{t-3} x(\tau+3)\,d\tau + \underbrace{\int_{t-3}^{2t} x(\tau+3)\,d\tau}_{\text{depends on future inputs}}$

$\boxed{\text{Non Causal}}$

o/p to $x(t) = u(t-2) - u(t-4)$

$$y(t) = \int_{-\infty}^{2t} \{u(\tau+1) - u(\tau-1)] \, d\tau$$



$$y(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ \int_{-1}^{2t} 1 . dt & -\frac{1}{2} \leq t < \frac{1}{2} \\ \int_{-1}^{1} . 1 \, dt & t \geq \frac{1}{2} \end{cases} \Rightarrow \boxed{y(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 2t+1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 2 & t \geq \frac{1}{2} \end{cases}}$$

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b) $y(t) = x(t) \sin(\pi t)$

i) Linearity

$$S\{\alpha x_1(t) + \beta x_2(t)\} = (\alpha x_1(t) + \beta x_2(t)) \sin(\pi t)$$

$$= \alpha y_1(t) + \beta y_2(t)$$

$\therefore$ $\boxed{\text{Linear}}$

ii) Time Invariance

$$\left.\begin{aligned} S\{x(t-t_0)\} &= x(t-t_0) \sin(\pi t) \\ y(t-t_0) &= x(t-t_0) \sin(\pi t - \pi t_0) \end{aligned}\right\} \boxed{\text{Time varying}}$$

iii) Causality:

For All instants of time t, $y(t)$ depends on present values of $x(t)$.

$\Rightarrow$ $\boxed{\text{Causal}}$

iv) $x(t) = u(t-2) - u(t-4)$

$$y(t) = [u(t-2) - u(t-4)] \sin(\pi t)$$

$$\boxed{y(t) = \begin{cases} \sin(\pi t) & 2 \leq t < 4 \\ 0 & \text{elsewhere} \end{cases}}$$

$y(t) = \frac{d}{dt} x(t) = \dot{x}(t)$

i) Linearity

$S\{\alpha x_1(t) + \beta x_2(t)\} = \frac{d}{dt}\{\alpha x_1(t) + \beta x_2(t)\}$

$= \alpha \dot{x}_1(t) + \beta \dot{x}_2(t)$; $\dot{x} = \frac{dx}{dt}$

$= \alpha y_1(t) + \beta y_2(t)$

$\therefore \boxed{\text{Linear}}$

ii) Time Invariance :

$S\{x(t-t_0)\} = \frac{d}{dt} x(t-t_0) = y(t-t_0)$ $\therefore \boxed{TI}$

iii) Causality : $\boxed{\text{Causal}}$ $\because$ depends on present values of input $\forall\, t \in (-\infty, \infty)$

iv) $x(t) = u(t-2) - u(t-4)$.

$y(t) = \frac{d}{dt}\{u(t-2) - u(t-4)\}$

$$\boxed{y(t) = \delta(t-2) - \delta(t-4)}$$

(d) $y(t) = x(2-t) + x(2+t)$

i) Linear : $S\{\alpha x_1(t) + \beta x_2(t)\} = \alpha S\{x_1(t)\} + \beta S\{x_2(t)\}$

ii) Time varying :

$S\{x(t-t_0)\} = x(2-t-t_0) + x(2+t-t_0)$ $\boxed{T.V}$

$y(t-t_0) = x(2-t+t_0) + x(2+t-t_0)$

iii) Non-causal : $y(t)$ depends on $x(t+2)$ $\forall\, t$.

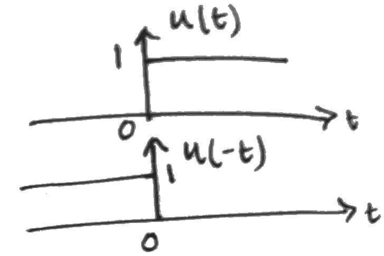
iv) $x(t) = u(t-2) - u(t-4)$

$x(2-t) = u(-t) - u(-t-2)$;

$x(2+t) = u(t) - u(t-2)$

$\Rightarrow y(t) = u(-t) - u(-t-2) + u(t) - u(t-2)$

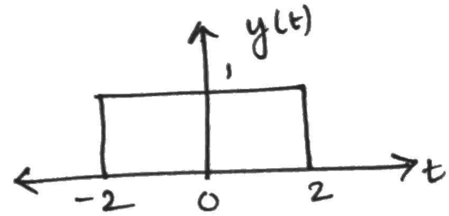
$$y(t) = \underbrace{u(t) + u(-t)} - u(-t-2) - u(t-2)$$



$$= \underbrace{1 - u(-t-2)} - u(t-2)$$

$$= u(t+2) - u(t-2)$$

$$\boxed{y(t) = u(t+2) - u(t-2)}$$



uestion (2)

$$y(t) = x(t) - \int_{t-1}^{t+1} e^{|t-\tau|} x(\tau)\, d\tau$$

Write $x(t)$ as $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau = x(t) * \delta(t)$

(a) $$y(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau - \underbrace{\int_{t-1}^{t+1} e^{|t-\tau|} x(\tau)\,d\tau}_{I_2:\ \text{Integral over } (t-1,\,t+1).}$$

We can change limits of I_2 to $(-\infty,\infty)$ if we restrict the integrand to the range $(t-1, t+1)$ by multiplying with rectangular block

$$\left[u(\tau-(t-1)) - u(\tau-(t+1))\right]$$

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau - \int_{-\infty}^{\infty} e^{|t-\tau|} x(\tau)\left[u(\tau-t+1) - u(\tau-t-1)\right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[\delta(t-\tau) - e^{|t-\tau|}u(\tau-(t-1)) + e^{|t-\tau|}u(\tau-(t+1))\right]}_{h(t,\tau)} d\tau$$

$$\boxed{h(t,\tau) = \delta(t-\tau) - e^{|t-\tau|}u(1-(t-\tau)) + e^{|t-\tau|}u(1+(t-\tau))}$$

$$= \delta(t-\tau) - e^{t-\tau}u(\tau-(t-1))\,u(t-\tau) - e^{\tau-t}u(-\tau+(t+1))\,u(\tau-t)$$

(b) → $h(t,\tau)$ is a f^n of $(t-\tau)$ only

$\therefore$ System is time Invariant (TI).

$$\boxed{h(t,\tau) = h(t-\tau, 0)}$$

→ Linearity can be confirmed from the input/output relation

~~Th~~ Further, since $h(t,\tau) = h(t-\tau, 0) = h(t-\tau)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$ → This re-affirms that the system is linear

Proof of Linearity:

Define $y_3(t) = S\{\alpha x_1(t) + \beta x_2(t)\}$

$$= \int_{-\infty}^{\infty} (\alpha x_1(\tau) + \beta x_2(\tau))\, h(t-\tau)\, d\tau$$

$$= \alpha \int_{-\infty}^{\infty} x_1(\tau)\, h(t-\tau)\, d\tau + \beta \int_{-\infty}^{\infty} x_2(\tau)\, h(t-\tau)\, d\tau$$

$$= \alpha y_1(t) + \beta y_2(t).$$

Proof of Time Invariance:

Define $y_4(t) = S\{x(t-t_0)\}$

$$\Rightarrow y_4(t) = x(t-t_0) - \int_{t-1}^{t+1} e^{|t-\tau|} x(\tau - t_0).$$

or

$$y_4(t) = \int_{-\infty}^{\infty} x(\tau - t_0)\, h(t-\tau)\, d\tau \quad — (1)$$

$$\cancel{y(t)}\ y(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(t-t_0-\tau)\, d\tau$$

~~Let $t - t_0 = \tilde{t}$~~

~~$\Rightarrow dt = d\tilde{t}$~~

Let $\tau + t_0 = \tilde{\tau}$

$\Rightarrow d\tau = d\tilde{\tau}$

$$\text{Thus, } y(t-t_0) = \int_{-\infty}^{\infty} x(\tilde{\tau} - t_0) . h(t - \tilde{\tau})\, d\tilde{\tau} \quad — (2)$$

Thus $y_4(t) = \boxed{S\{x(t-t_0)\} = y(t-t_0)}$

$\Rightarrow$ **Time Invariant**

Causality :

Since the system ~~is~~ is LTI,

we can express $h(t,\tau) = h(t-\tau,0)$

$\therefore$ ~~g.~~ System can be expressed as $h(\tilde{t})$. ; $\tilde{t} = t-\tau$

Condition for causality for LTV system:-

$$h(t,\tau) = h(t,\tau)\,u(t-\tau)$$

now reduces to $\boxed{h(\tilde{t}) = h(\tilde{t})\,u(\tilde{t})}$

$\therefore$ RHS $= h(\tilde{t})\,u(\tilde{t})$

$$= \left[\delta(\tilde{t}) - e^{|\tilde{t}|}u(1-\tilde{t}) + e^{|\tilde{t}|}u(1+\tilde{t})\right]u(\tilde{t})$$

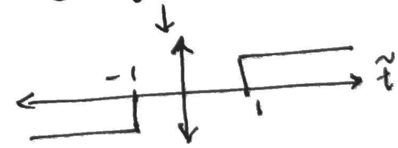
$$= \delta(\tilde{t})\,u(\tilde{t}) - e^{|\tilde{t}|}u(1-\tilde{t})\,u(\tilde{t}) + e^{|\tilde{t}|}u(\tilde{t})\,u(1+\tilde{t})$$

$$= \cancel{\delta(\tilde{t}).1 - e^{|\tilde{t}|}\left[u(\tilde{t}) - u(\tilde{t}-1)\right]} + \cancel{e^{|\tilde{t}|}u(\tilde{t})}$$

$$= \delta(\tilde{t}) \cancel{+ e^{|\tilde{t}|}u(\tilde{t}-1)}$$

$$= \delta(\tilde{t}).1 - e^{|\tilde{t}|}u(\tilde{t})\left[u(1-\tilde{t}) - u(1+\tilde{t})\right]$$

$$= \delta(\tilde{t}) + e^{|\tilde{t}|}u(\tilde{t})\left[\underbrace{u(\tilde{t}+1) - u(1-\tilde{t})}\right]$$



$\therefore \neq \delta(\tilde{t}) + e^{|\tilde{t}|}\left[u(\tilde{t}+1) - u(1-\tilde{t})\right]$

$\therefore$ $\boxed{h(\tilde{t})\,u(\tilde{t}) \neq h(\tilde{t})}$

$\therefore$ System is $\boxed{\text{Non-Causal}}$

$$
\begin{aligned}
h(t,\tau) &= \delta(t,\tau) - e^{|t-\tau|}\left[u(1-t+\tau) - u(1+t-\tau)\right] \\
&= \delta(t-\tau) - e^{t-\tau}\left[u(1-t+\tau) - u(1+t-\tau)\right] u(t-\tau) \\
&\qquad - e^{\tau-t}\left[u(1-t-\tau) - u(1+t-\tau)\right] u(\tau-t)
\end{aligned}
$$

5.

$$h(t,\tau) = \delta(t-\tau) - e^{t-\tau}u(\tau-(t-1))u(t-\tau) - e^{\tau-t}u(-\tau+(t+1))u(\tau-t)$$

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau+2)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau}u(\tau+2)\,\delta(t-\tau)\,d\tau - \int_{-\infty}^{\infty} e^{-\tau}u(\tau+2).e^{t-\tau}u(\tau-(t-1))u(t-\tau)\,d\tau$$

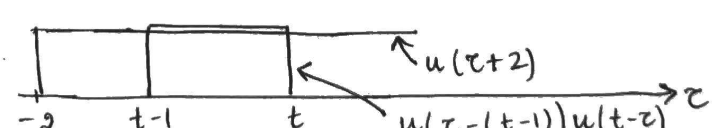
$$- \int_{-\infty}^{\infty} e^{-\tau}e^{\tau-t}u(\tau+2)u(-\tau+(t+1))u(\tau-t)\,d\tau$$

$$= e^{-t}u(t+2) - \underbrace{\int_{-\infty}^{\infty} e^{t}.e^{-2\tau}u(\tau+2)u(\tau-(t-1))u(t-\tau)\,d\tau}_{y_1(t)}$$

$$- \underbrace{\int_{-\infty}^{\infty} e^{-\tau}e^{\tau-t}u(\tau+2)u(-\tau+(t+1))u(\tau-t)\,d\tau}_{y_2(t)}$$

$$y_1(t) = \int_{-\infty}^{\infty} e^{t}.e^{-2\tau}u(\tau+2)u(\tau-(t-1))u(t-\tau)\,d\tau$$

$$u(\tau+2)\underbrace{u(\tau-(t-1))u(t-\tau)}_{\text{non-zero only in } \tau\in[t-1,t]}$$



3 cases i) $-2 < t-1 \Rightarrow \boxed{t > -1}$

$$\Rightarrow y_1(t) = e^{t}\int_{t-1}^{t} e^{-2\tau}\,d\tau = \frac{1}{2}\left(e^{-t} - e^{-t+2}\right)$$

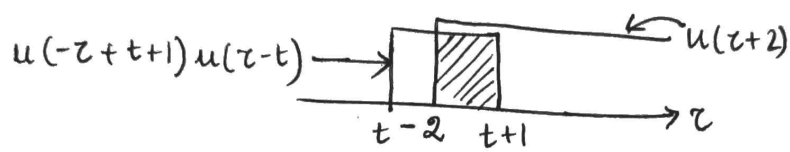
ii) $t-1 \leq -2 \leq t$ ie $\boxed{-2 \leq t \leq -1}$

$$\Rightarrow y_1(t) = e^{t}\int_{-2}^{t} e^{-2\tau}\,d\tau = \frac{1}{2}\left(e^{-t} - e^{t+4}\right)$$

iii) $\boxed{t<-2}$ $\quad y_1(t) = 0.$ $\quad$ Thus, $\quad y_1(t) = \begin{cases} 0 & t<-2 \\ \frac{1}{2}\left(e^{-t}-e^{t+4}\right) & -2\leq \\ \frac{1}{2}\left(e^{-t}-e^{-t+2}\right) & t\geq - \end{cases}$

Consider $y_2(t)$:

$$y_2(t) = \int_{-\infty}^{\infty} e^{-\tau}\, u(\tau+2)\, \underbrace{u(-\tau+(t+1))\, u(\tau-t)}_{\text{non-zero in } \tau\in[t,t+1]}\, d\tau.$$



If $\boxed{t\geq-2}$ $\Rightarrow$ $y_2(t) = \int_{t}^{t+1} e^{-\tau}\, d\tau = e^{-t}(1)$

If $t\in[-3,-2]$ $\Rightarrow$ $y_2(t) = \int_{-2}^{t+1} e^{-\tau}\, d\tau = e^{-t}(t+3)$

If $t\leq-3$ $\Rightarrow$ $y_2(t) = 0.$

$$y_2(t) = \begin{cases} 0 & t\leq-3 \\ e^{-t}(t+3) & -3<t\leq-2 \\ e^{-t} & t\geq-2 \end{cases}$$

We have:

$$y(t) = e^{-t}u(t+2) - y_1(t) - y_2(t)$$

$$y(t) = \begin{cases} 0 & t<-3 \\ -e^{-t}(t+3) & -3\leq t<-2 \\ e^{-t}-\frac{1}{2}\left(e^{-t}-e^{t+4}\right)-e^{-t} & -2\leq t<-1 \\ e^{-t}-\frac{1}{2}\left(e^{-t}-e^{-t+2}\right)-e^{-t} & t\geq-1 \end{cases}$$

Thus, $\quad y(t) = \begin{cases} 0 & t\in(-\infty,-3) \\ -e^{-t}(t+3) & t\in[-3,-2) \\ -\frac{1}{2}\left(e^{-t}-e^{t+4}\right) & t\in[-2,-1) \\ -\frac{1}{2}e^{-t}\left(1-e^{2}\right) & t\in[-1,\infty) \end{cases}$

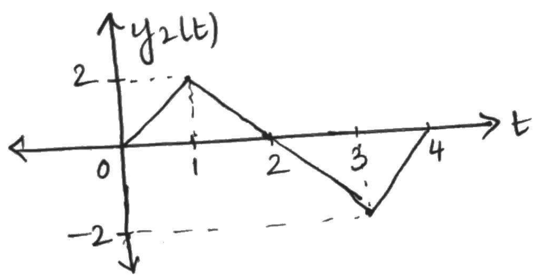
uestion (3)

a) $x_2(t) = x_1(t) - x_1(t-2)$

$\therefore$ By linearity: $S\{x_1(t) - x_1(t-2)\} = S\{x_1(t)\} - S\{x_1(t-2)\}$

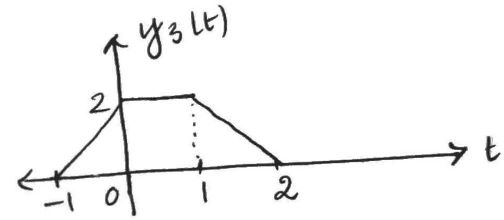
By time invariance: $S\{x_1(t-2)\} = y_1(t-2)$

$\therefore$ o/p $= y_1(t) - y_1(t-2)$



(b) $x_3(t) = x_1(t) + x_1(t+1)$

$\Rightarrow \quad y_3(t) = y_1(t) + y_1(t+1)$



(c) $$y(t) = \int_{t-2}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\, u(\tau-(t-2)).\,u(t-\tau)\,d\tau$$

$$\therefore h(t) = \int_{-\infty}^{\infty} \delta(\tau)\, u(\tau-(t-2))\, u(t-\tau)\,d\tau$$

$$= \boxed{u(-t+2)\,u(t)} = u(-t+2)\,[1-u(-t)]$$

$$= u(-t+2) - \underbrace{u(-t+2)\,u(-t)}_{u(-t)}$$

$$= \boxed{u(-t+2) - u(-t)}$$

$$\boxed{\therefore h(t) = u(-t+2)\,u(t) = u(-t+2) - u(-t)}$$

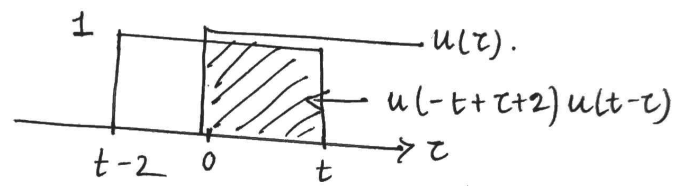
Both are correct.

Input $x(t) = u(t+2) + u(t) - 2u(t-1) + \delta(t-1)$

<u>Consider input $\tilde{x}(t) = u(t)$</u>

$$\tilde{y}(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)\, \underbrace{u(-(t-\tau)+2)\, u(t-\tau)}_{\text{non-zero in the range } \tau \in [t-2,\, t]}\, d\tau$$



If $\underline{0 \leq t-2} \Rightarrow \boxed{t \geq 2}$

$$\tilde{y}(t) = \int_{t-2}^{t} 1 \cdot d\tau = \underline{\underline{2}}$$

If $\underline{0 \leq t < 2}$ $\quad \tilde{y}(t) = \int_{0}^{t} 1 \cdot d\tau = t$

If $\underline{t < 0}$ $\quad \tilde{y}(t) = 0$

$$\tilde{y}(t) = \begin{cases} t & t \in [0,2) \\ 2 & t \in [2,\infty) \\ 0 & t \in (-\infty,0) \end{cases}$$

$\Rightarrow \tilde{y}(t) = tu(t) - (t-2)u(t-2$

$= \boxed{tu(t) - (t-2)u(t-2)}$

<u>Thus</u>, using Time-invariance property :-

$$S\{u(t+2)\} = \tilde{y}(t+2) = (t+2)u(t+2) - tu(t)$$

$$S\{u(t-1)\} = \tilde{y}(t-1) = (t-1)u(t-1) - (t-3)u(t-3)$$

$$S\{\delta(t-1)\} = h(t-1) = u(-t+3)\,u(t-1)$$

$$\therefore \; y(t) = (t+2)u(t+2) - (t-2)u(t-2) - 2(t-1)u(t-1) + 2(t-3)u(t-3) + u(t-1)\,u(-t+3)$$

Question (4)

a)
$$y(t) = \int_{-\infty}^{\infty} e^{-t}(t-\sigma)^2 u(\sigma+t)\, x(\sigma-2)\, d\sigma \quad ; t \in (-\infty, \infty)$$

$$h(t,\tau) = \int_{-\infty}^{\infty} e^{-t}(t-\sigma)^2 u(\sigma+t)\, \delta(\sigma-\tau-2)\, d\sigma$$

sifting property : $\sigma = \tau + 2$

$$\therefore \boxed{h(t,\tau) = e^{-t}(t-\tau-2)^2 u(\tau+2+t)}$$

i) $h(t,\tau) \neq h(t-\tau, 0) \Rightarrow$ ~~Non~~ Time Variant

ii) $h(t,\tau)\, u(t-\tau) = e^{-t}(t-\tau-2)^2 u(\tau+2+t)\, u(t-\tau)$

$\neq e^{-t}(t-\tau-2)^2 u(\tau+2+t) \Rightarrow$ Non-causal

(b)
$$y(t) = \overbrace{\int_{-\infty}^{\infty} e^{-t}(t-\sigma)^2 u(\sigma+t)\, \delta(\sigma-4)\, d\sigma}^{y_1} - \underbrace{\int_{-\infty}^{\infty} e^{-t}(t-\sigma)^2 u(\sigma+t)\, e^{-(\sigma-2)} u(\sigma-1)\, d\sigma}_{y_2}$$

$$y_1(t) = e^{-t}(t-4)^2 u(t+4)$$

$$y_2(t) = \int_{-\infty}^{\infty} e^{-t}(t-\sigma)^2 \underline{u(\sigma+t)}\, e^{-\sigma} \cdot e^{2} \cdot \underline{u(\sigma-1)}\, d\sigma$$

$$u(\sigma+t)\, u(\sigma-1) = \begin{cases} u(\sigma-1) & t > -1 \\ u(\sigma+t) & t < -1 \end{cases}$$

If $\underline{\underline{t > -1}}$ $\Rightarrow$ $y_2(t) = \int_1^{\infty} e^{-t+2} (t-\sigma)^2 \cdot e^{-\sigma} d\sigma$

If $\underline{\underline{t < -1}}$ $\Rightarrow$ $y_2(t) = \int_{-t}^{\infty} e^{-t+2} (t-\sigma)^2 e^{-\sigma} d\sigma$

$$\int e^2 \cdot e^{-t} \cdot (t-\sigma)^2 \cdot e^{-(\sigma-t)} \cdot e^{-t} \cdot d\sigma$$

$$= \int e^{2-2t} (\sigma-t)^2 e^{-(\sigma-t)} d\sigma$$

Let $\sigma - t = \gamma \Rightarrow d\sigma = d\gamma$

$$\Rightarrow \int e^{2-2t} \cdot \gamma^2 e^{-\gamma} d\gamma$$

$$= -e^{2-2t} \left[\gamma^2 e^{-\gamma} + 2\gamma e^{-\gamma} + 2e^{-\gamma} \right]$$

$$= -e^{2-2t} \left[(\sigma-t)^2 e^{-(\sigma-t)} + 2(\sigma-t) e^{-(\sigma-t)} + 2e^{-(\sigma-t)} \right]$$

$$= -e^{2-t} \cdot e^{-\sigma} \left[(\sigma-t)^2 + 2(\sigma-t) + 2 \right]$$

$\therefore$ If $\underline{\underline{t > -1}}$: $y_2(t) = +e^{2-t} \left[e^{-1} \left[(1-t)^2 + 2(1-t) + 2 \right] \right]$

$$= e^{1-t} \left[t^2 - 4t + 5 \right] = e^{1-t} \left[(t-2)^2 + 1 \right] u(t+1)$$

If $\underline{\underline{t < -1}}$: $y_2(t) = e^{2-t} \cdot e^{+t} \left[4t^2 - 4t + 2 \right]$

$$= e^2 \left[(2t-1)^2 + 1 \right] u(-t-1)$$

$$\boxed{\begin{aligned} \therefore\ y(t) = y_1(t) - y_2(t) &= e^{-t}(t-4)^2 u(t-4) - e^{1-t}\left((t-2)^2+1\right) u(t+1) \\ &\quad - e^2\left[(2t-1)^2+1\right] u(-t-1) \end{aligned}}$$