Home Work 5

1) Given $x(t)$ is periodic with:

+ Period T_0

+ Fundamental frequency $\omega_0 = \frac{2\pi}{T_0}$

+ a_k coefficient for k^{th} harmonic

$$\Rightarrow x(t) = \sum_{-\infty}^{+\infty} X_n e^{jn\omega_0 t}, \quad a_k = X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

a) $$y(t) = x(2t-3) + 4\frac{d^2x(t)}{dt^2}$$

Let $y_1(t) = x(2t-3)$, then $y_1(t)$ has period $= T_{01} = \frac{T_0}{2}$

$$\omega_{01} = \frac{2\pi}{T_{01}} = \frac{2\pi}{T_0}\cdot 2 = \frac{4\pi}{T_0} = 2\omega_0$$

$$\Rightarrow a_{k1} = \frac{1}{T_{01}}\int_{T_{01}} y_1(t) e^{-jk\omega_{01}t} dt = \frac{1}{T_{01}}\int_{\tau}^{\tau+T_{01}} x(2t-3) e^{-jk\omega_{01}t} dt$$

$$= \frac{2}{T_0}\int_{\tau}^{\tau+T_0/2} x(2t-3) e^{-jk.2\omega_0 t} . dt$$

Let $\lambda = 2t-3 \Rightarrow t = \frac{\lambda+3}{2}$ & $dt = \frac{d\lambda}{2}$

$t = \tau \longrightarrow \tau + T_0/2$

$\Rightarrow r = 2\tau - 3 \longrightarrow 2\tau + T_0 - 3$

$$\Rightarrow a_{k1} = \frac{2}{T_0} \int_{2\tau-3}^{2\tau-3+T_0} x(r)\, e^{-jk2w_0\left(\frac{r+3}{2}\right)} \frac{dr}{2}$$

$$\Rightarrow a_{k1} = \frac{1}{T_0} \int_{2\tau-3}^{2\tau-3+T_0} x(r)\, e^{-jkw_0 r}\, e^{-jkw_0 3}\, dr$$

$$= e^{-jk3w_0} \cdot \frac{1}{T_0} \int_{2\tau-3}^{2\tau-3+T_0} x(r)\, e^{-jkw_0 r}\, dr$$

$$\Rightarrow \boxed{a_{k1} = e^{-jk3w_0} \cdot a_k} \quad \& \quad T_{01} = \frac{T_0}{2}$$

$$\Rightarrow y_1(t) = x(2t-3) = \sum_{-\infty}^{+\infty} e^{-jk3w_0} a_k e^{jk2w_0 t}$$

$*$ let $y_2(t) = 4\dfrac{d^2x(t)}{dt^2}$

$$x(t) = \sum_{-\infty}^{+\infty} X_n e^{jnw_0 t} \Rightarrow \frac{dx(t)}{dt} = \sum_{-\infty}^{+\infty} X_n (jnw_0) e^{jnw_0 t}$$

$$\Rightarrow \frac{d^2x(t)}{dt^2} = \sum_{-\infty}^{+\infty} X_n (jnw_0)^2 e^{jnw_0 t}$$ $\Rightarrow$ $y_2(t)$ also has period T_0 and W_0.

$$y_2(t) = 4\sum_{-\infty}^{+\infty} x_n (jn\omega_0)^2 e^{jn\omega_0 t} = \sum_{n=-\infty}^{+\infty} \left(-4n^2\omega_0^2\right) \underbrace{x_n}_{a_k} e^{jn\omega_0 t}$$

$$\Rightarrow \boxed{a_{2k} = -4k^2\omega_0^2 \; a_k}$$

Then, applying Linearity $y(t) = y_1(t) + y_2(t)$

$$\Rightarrow y(t) \text{ has coefficient } a_{yk} = a_{1k} + a_{2k} = e^{-3jk\omega_0} a_k - 4k^2\omega_0^2 a_k$$

$$\Rightarrow \boxed{b_k = \left(e^{-3jk\omega_0} - 4k^2\omega_0^2\right) a_k}$$

b) $$y(t) = \int_{-\infty}^{t+2\alpha} e^{j\omega_0 \tau} x(\tau+1)\, d\tau$$

$$x(t) = \sum_{-\infty}^{+\infty} X_k e^{jk\omega_0 t} \quad \& \quad a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

Let $y_1(t) = e^{j\omega_0 t} x(t)$ with $T_{01} = T_0$

$$\Rightarrow a_{1k} = \frac{1}{T_0}\int_{\tau}^{\tau+T_0} y_1(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{\tau}^{\tau+T_0} x(t) e^{j\omega_0 t} e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{\tau}^{\tau+T_0} x(t) e^{j\omega_0 t(1-k)}\, dt = \frac{1}{T_0}\int_{\tau}^{\tau+T_0} x(t) e^{-j\omega_0 t(k-1)}\, dt$$

$$= a_{k-1}$$

$$\Rightarrow a_{1k} = a_{k-1} \quad (1)$$

or $\quad e^{j\omega_0 t} x(t) \longrightarrow a_{k-1}$

Let $y_2(t) = x(t+1)$ period T_0 & ω_0

$$\Rightarrow a_{2\_k} = \frac{1}{T_0} \int_{\tau}^{\tau+T_0} x(t+1)\, e^{-j\omega_0 k t}\, dt$$

let $\lambda = t+1 \Rightarrow d\lambda = dt$ & $t = \lambda - 1$

$t: \tau \longrightarrow \tau + T_0$

$\Rightarrow \lambda = \tau+1 \longrightarrow \tau + T_0 + 1$

$$\Rightarrow a_{2\_k} = \frac{1}{T_0} \int_{\tau+1}^{\tau+1+T_0} x(\lambda)\, e^{-j\omega_0 k(\lambda-1)}\, d\lambda$$

$$= e^{j\omega_0 k} \frac{1}{T_0} \int_{T_0} x(\lambda)\, e^{-j\omega_0 k \lambda}\, d\lambda = e^{j\omega_0 k} a_k$$

$$\Rightarrow a_{2\_k} = e^{j\omega_0 k} a_k$$

$$\Rightarrow y_2(t) = x(t+1) \longrightarrow a_k e^{j\omega_0 k}$$

$\Rightarrow y_3(t) = e^{j\omega_0 t} y_2(t) \longrightarrow a_{2\_k-1}$ (Based on 1)

$$\Rightarrow \quad y_3(t) = e^{j\omega_0 t} y_2(t) \text{ has } a_{3\_k} = a_{2\_k-1} = e^{j\omega_0 (k-1)} a_{k-1}$$

$$\Rightarrow y(t) = \int_{-\infty}^{t+2\alpha} y_3(\tau)\, d\tau = \int_{-\infty}^{t+2\alpha} \sum_{-\infty}^{+\infty} a_{3\_k} e^{jk\omega_0 \tau} d\tau$$

$$= \sum_{-\infty}^{+\infty} a_{3\_k} \int_{-\infty}^{t+2\alpha} e^{jk\omega_0 \tau} d\tau = \sum_{-\infty}^{+\infty} a_{3\_k} \left. \frac{e^{jk\omega_0 \tau}}{jk\omega_0} \right|_{-\infty}^{t+2\alpha}$$

$$= \sum_{-\infty}^{+\infty} \frac{a_{3\_k}}{jk\omega_0} \left[e^{jk\omega_0 (t+2\alpha)} - 0 \right]$$

$$= \sum_{-\infty}^{+\infty} \left(\frac{a_{3\_k}}{jk\omega_0} e^{jk\omega_0 2\alpha} e^{jk\omega_0 t} \right)$$

$$\Rightarrow b_k = \frac{a_{3\_k} e^{j2\alpha k\omega_0}}{jk\omega_0} = \frac{e^{j\omega_0 (k-1)} e^{j\omega_0 2\alpha k} a_{k-1}}{jk\omega_0}$$

$$\Rightarrow \boxed{b_k = \frac{e^{j\omega_0 (2\alpha k + k - 1)}}{jk\omega_0} a_{k-1}}$$

c) $y(t) = \dfrac{d}{dt}(x^3(t))$

$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j\omega_0 k t}$ has coefficient $a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$

Using multiplication of F.S property, we have already:

$$y(t) = \underbrace{a(t)}_{a_k} b(t) \quad \text{with } b(t) = \sum_{l=-\infty}^{+\infty} B_l e^{jl\omega_0 t}$$

then $Y_k = \sum_{l=-\infty}^{+\infty} A_{k-l} B_l$.

So, apply this property for $y_1(t) = x(t).x(t)$, then we have the $y_1(t)$ will have the coefficient

$$Y_k = \sum_{l=-\infty}^{+\infty} a_{k-l}\, a_k$$

Continuing applying this for $z(t) = \underbrace{x(t)}_{a_k} \underbrace{y_1(t)}_{Y_k}$, then

the $z(t)$ will have the coefficient z_k

$$z_k = \sum_{j=-\infty}^{+\infty} a_{k-j} (Y_k)_j = \sum_{j=-\infty}^{+\infty} a_{k-j} \left[\sum_{l=-\infty}^{+\infty} a_{k-l}\, a_k\right]_j$$

So, $z(t) = x(t)\, x(t)\, x(t) = [x(t)]^3$ has the coefficient

$$z_k = \sum_{j=-\infty}^{+\infty} a_{k-j} \left[\sum_{l=-\infty}^{+\infty} a_{k-l}\, a_k\right]_j$$

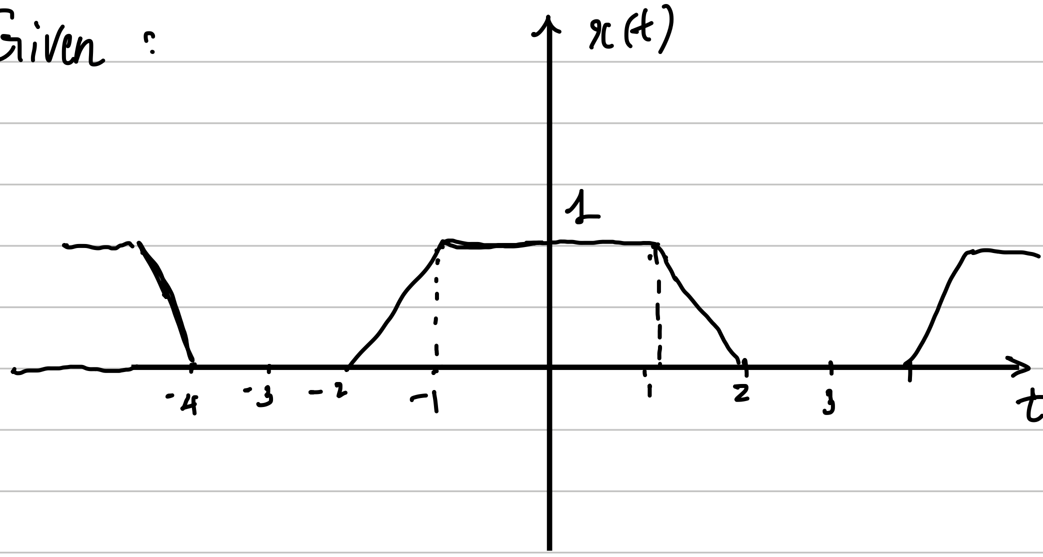
Also, $y(t) = \frac{d}{dt}[x(t)]^3 = \frac{d}{dt}[z(t)]$.

We apply the differential property for Fourier Series, then the $y(t)$ will have the coefficience $Y_k = b_k$:

$$b_k = jk\omega_0 Z_k.$$

$$\Rightarrow \boxed{b_k = jk\omega_0 \sum_{j=-\infty}^{+\infty}\left[a_{k-j}\left(\sum_{l=-\infty}^{+\infty} a_{k-l}\, a_k\right)_j\right]}$$

2) Given :



a) We have with $t \in [-3, 3]$

$$x(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases} \qquad T_0 = 6 \qquad \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

With $x(t) = \sum_{-\infty}^{+\infty} X_n e^{j\omega_0 n t}$ with $X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 n t}$

$$\Rightarrow X_n = \frac{1}{6}\left[\underbrace{\int_{-2}^{-1} (t+2) e^{-j\omega_0 n t} dt}_{A} + \underbrace{\int_{-1}^{1} e^{-j\omega_0 n t} dt}_{B} + \underbrace{\int_{1}^{2} (2-t) e^{-j\omega_0 n t} dt}_{C} \right]$$

We have: $\int t e^{at} dt = \left(\frac{t}{a} - \frac{1}{a^2}\right) e^{at}$, $a = -jn\omega_0$

$$\int e^{at} dt = \frac{1}{a} e^{at}$$

$$A = \int_{-2}^{-1} (t+2) e^{-j\omega_0 n t} dt = \int_{-2}^{-1} t e^{-j\omega_0 n t} dt + 2\int_{-2}^{-1} e^{-j\omega_0 n t} dt.$$

$$* \int_{-2}^{-1} t e^{-j\omega_0 n t} dt = \int_{-2}^{-1} t e^{at} dt = \left(\frac{t}{a} - \frac{1}{a^2}\right) e^{at} \Big|_{-2}^{-1}$$

$$(a = -j\omega_0 n) \quad (a^2 = -\omega_0^2 n^2)$$

$$= \left(\frac{-1}{a} - \frac{1}{a^2}\right) e^{-a} - \left(\frac{-2}{a} - \frac{1}{a^2}\right) e^{-2a}$$

$$= \left(\frac{1}{j\omega_0 n} + \frac{1}{\omega_0^2 n^2}\right) e^{-a} - \left(\frac{2}{j\omega_0 n} + \frac{1}{\omega_0^2 n^2}\right) e^{-2a}$$

$$= \left(\frac{-j}{\omega_0 n} + \frac{1}{\omega_0^2 n^2}\right) e^{-a} - \left(\frac{-2j}{\omega_0 n} + \frac{1}{\omega_0^2 n^2}\right) e^{-2a}$$

$$* \int_{-2}^{-1} e^{-j\omega_0 n t} dt = \int_{-2}^{-1} e^{at} dt = \frac{1}{a} e^{at} \Big|_{-2}^{-1}$$

$$= \frac{1}{a}\left(e^{-a} - e^{-2a}\right) = \frac{1}{-j\omega_0 n}\left(e^{-a} - e^{-2a}\right) = \frac{j}{\omega_0 n}\left(e^{-a} - e^{-2a}\right)$$

$$\Rightarrow A = \left(\frac{-j}{w_0 n} + \frac{1}{w_0^2 n^2}\right) e^{-a} - \left(\frac{-2j}{w_0 n} + \frac{1}{w_0^2 n^2}\right) e^{-2a} + \frac{2j}{w_0 n} e^{-a}$$

$$- \frac{2j}{w_0 n} e^{-2a} = \left(\frac{j}{w_0 n} + \frac{1}{w_0^2 n^2}\right) e^{-a} - \frac{1}{w_0^2 n^2} e^{-2a}$$

$$B = \int_{-1}^{1} e^{-jnw_0 t} dt = \int_{-1}^{1} e^{at} dt = \frac{1}{a} e^{at} \Big|_{-1}^{1}$$

$$= \frac{1}{a}\left[e^{a} - e^{-a}\right] = \frac{1}{-jw_0 n}\left[e^{a} - e^{-a}\right] = \frac{j}{w_0 n}\left[e^{a} - e^{-a}\right]$$

$$\Rightarrow A + B = \left(\frac{j}{w_0 n} + \frac{1}{w_0^2 n^2}\right) e^{-a} - \frac{1}{w_0^2 n^2} e^{-2a} + \frac{j}{w_0 n} e^{a} - \frac{j}{w_0 n} e^{-a}$$

$$= \frac{1}{w_0^2 n^2} e^{-a} - \frac{1}{w_0^2 n^2} e^{-2a} + \frac{j}{w_0 n} e^{a}$$

$$C = \int_{1}^{2} (2-t) e^{-jw_0 nt} dt = 2\int_{1}^{2} e^{-jw_0 nt} dt - \int_{1}^{2} t e^{-jw_0 nt} dt$$

$$* \quad 2\int_{1}^{2} e^{-jw_0 nt} dt = 2\int_{1}^{2} e^{at} dt = \frac{2}{a} e^{at} \Big|_{1}^{2} = \frac{2}{a}\left[e^{2a} - e^{a}\right]$$

$$= \frac{2}{-j\omega_0 n}\left[e^{2a} - e^{a}\right] = \frac{2j}{\omega_0 n}\left[e^{2a} - e^{a}\right]$$

$$* \int_1^2 t e^{-j\omega_0 n t}\, dt = \int_1^2 t e^{at}\, dt = \left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at}\Big|_1^2$$

$$= \left(\frac{2}{a} - \frac{1}{a^2}\right)e^{2a} - \left(\frac{1}{a} - \frac{1}{a^2}\right)e^{a}$$

$$\Rightarrow C = \frac{2}{a}\left[e^{2a} - e^{a}\right] - \left(\frac{2}{a} - \frac{1}{a^2}\right)e^{2a} + \left(\frac{1}{a} - \frac{1}{a^2}\right)e^{a}$$

$$= \left(\frac{2}{a} - \frac{2}{a} + \frac{1}{a^2}\right)e^{2a} + e^{a}\left[\frac{1}{a} - \frac{1}{a^2} - \frac{2}{a}\right]$$

$$= \frac{1}{a^2}e^{2a} - \left(\frac{1}{a} + \frac{1}{a^2}\right)e^{a} = \frac{1}{a^2}e^{2a} - \frac{e^{a}}{a} - \frac{e^{a}}{a^2}$$

$$(a = -j\omega_0 n)$$

$$(a^2 = -\omega_0^2 n^2)$$

$$= \frac{-1}{\omega_0^2 n^2}e^{2a} + \frac{e^{a}}{j\omega_0 n} + \frac{e^{a}}{\omega_0^2 n^2}$$

$$= \frac{-1}{\omega_0^2 n^2}e^{2a} - \frac{j e^{a}}{\omega_0 n} + \frac{e^{a}}{\omega_0^2 n^2}$$

$$\Rightarrow A + B + C = \frac{1}{\omega_0^2 n^2}e^{-a} - \frac{1}{\omega_0^2 n^2}e^{-2a} + \frac{j}{\omega_0 n}e^{a}$$

$$-\frac{1}{\omega_0^2 n^2} e^{2a} - \frac{j e^{a}}{\omega_0 n} + \frac{e^{a}}{\omega_0^2 n^2}$$

$$= \frac{e^{-a}+e^{a}}{\omega_0^2 n^2} - \frac{e^{-2a}+e^{2a}}{\omega_0^2 n^2} = \frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{\omega_0^2 n^2} - \frac{e^{2j\omega_0 n}+e^{-2j\omega_0 n}}{\omega_0^2 n^2}$$

$$= \frac{2\cos\omega_0 n}{\omega_0^2 n^2} - \frac{2\cos(2\omega_0 n)}{\omega_0^2 n^2}$$

$$\Rightarrow x_n = \frac{1}{6}\left[A+B+C\right] = \frac{1}{3\omega_0^2 n^2}\left[\cos\omega_0 n - \cos 2\omega_0 n\right]$$

$$= \frac{1}{3\omega_0^2 n^2}\left[\cos\frac{\pi}{3}n - \cos\frac{2\pi}{3}n\right]$$

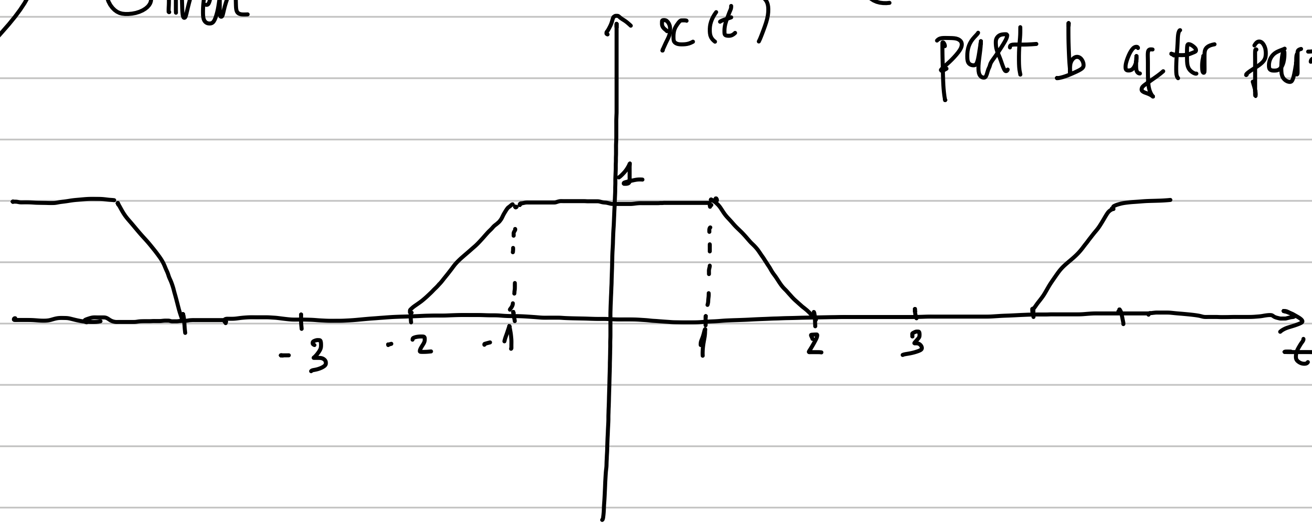
$$\Rightarrow x(t) = \sum_{-\infty}^{+\infty} \frac{1}{3\omega_0^2 n^2}\left[\cos\frac{\pi n}{3} - \cos\frac{2\pi n}{3}\right] e^{j\frac{\pi}{3}nt}$$

Sin $\omega_0 = \frac{\pi}{3} \Rightarrow 3\omega_0^2 = 3\frac{\pi^2}{9} = \frac{\pi^2}{3}$

$$\boxed{\Rightarrow x(t) = \sum_{n=-\infty}^{+\infty} \frac{3}{\pi^2 n^2}\left[\cos\frac{\pi n}{3} - \cos\frac{2\pi n}{3}\right] e^{j\frac{\pi}{3}nt}}$$

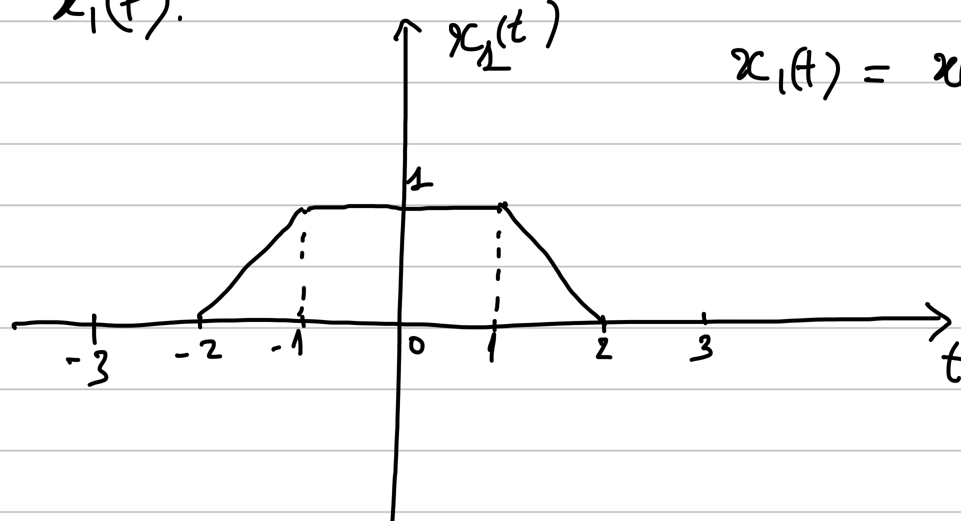
c) Given

(I forget b then I did part b after part c)



Let $x_1(t)$.

$x_1(t) = x(t)$ with $t \in [-3, 3]$



We have $T_0 = 6(s)$

With $T_0 = 6s$, then $\Rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

$$x_1(t) = \begin{cases} t+2, & -2 < t < -1 \\ 1, & -1 < t < 1 \\ 2-t, & 1 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

We have: $x_1(t) = (t+2)[u(t+2) - u(t+1)] + u(t+1) - u(t-1)$

$$+ (2-t)[u(t-1) - u(t-2)]$$

$$\Rightarrow x_1(t-2) = (t-2+2)[u(t-2+2) - u(t-2+1)]$$

$$+ u(t-2+1) - u(t-2-1) + (2-t+2)[u(t-2-1) - u(t-2-2)]$$

$$= t[u(t) - u(t-1)] + u(t-1) - u(t-3) + (4-t)[u(t-3) - u(t-4)]$$

$$= t\,u(t) - t\,u(t-1) + u(t-1) - u(t-3) + 4u(t-3) - 4u(t-4)$$

$$- t\,u(t-3) + t\,u(t-4)$$

$$= t\,u(t) - (t-1)u(t-1) - (t-3)u(t-3) + (t-4)u(t-4)$$

Using LapLace:

$$t\,u(t) \rightarrow \frac{1}{s^2} \quad \Rightarrow (t-1)u(t-1) \rightarrow e^{-s}\frac{1}{s^2}$$

$$(t-3)u(t-3) \rightarrow e^{-3s}\frac{1}{s^2}, \quad (t-4)u(t-4) \rightarrow e^{-4s}\frac{1}{s^2}$$

$$x_1(t-2) \rightarrow e^{-2s}X_1(s)$$

$$\Rightarrow e^{-2s}X_1(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}$$

$$\Rightarrow X_1(s) = \frac{e^{-4s} - e^{-3s} - e^{-s} + 1}{s^2 \cdot e^{-2s}} = \frac{e^{-2s} - e^{-s} - e^{s} + e^{2s}}{s^2}$$

$$\Rightarrow X_n = \frac{1}{T_0} X_1(s)\Big|_{s=jn\omega_0} \qquad \text{with } T_0 = 6,\ \omega_0 = \pi/3$$

$$= \frac{1}{6}\left[\frac{\left(e^{-2jn\omega_0} + e^{2jn\omega_0}\right) - \left(e^{-jn\omega_0} + e^{jn\omega_0}\right)}{(jn\omega_0)^2}\right]$$

$$= \frac{-1}{3n^2\omega_0^2}\left[\frac{e^{-2jn\omega_0} + e^{2j\omega_0 n}}{2} - \frac{e^{-jn\omega_0} + e^{jn\omega_0}}{2}\right]$$

$$= \frac{-1}{3n^2\omega_0^2}\left[\cos 2n\omega_0 - \cos(n\omega_0)\right]$$

$$= \frac{1}{3n^2\omega_0^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]$$

$$\Rightarrow x(t) = \sum_{-\infty}^{+\infty} \frac{1}{3n^2\omega_0^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right] e^{j\frac{\pi}{3}nt}$$

$$\omega_0 = \frac{\pi}{3} \Rightarrow 3\omega_0^2 = 3\frac{\pi^2}{9} = \frac{\pi^2}{3}$$

$$\Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{+\infty} \frac{3}{\pi^2 n^2}\left[\cos\frac{\pi n}{3} - \cos\frac{2\pi n}{3}\right] e^{j\frac{\pi}{3}nt}}$$

b) Since $X_n = \frac{1}{3n^2\omega_0^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]$ is real

$\Rightarrow \measuredangle X_n = \tan^{-1}\left(\frac{Im\{X_n\}}{Re\{X_n\}}\right) = 0$; $X_n = \frac{3}{n^2\pi^2}\left[\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right]$

Also, $X_0 = \frac{1}{T_0}\int_{-3}^{3} x(t)\,dt = \frac{1}{6}\left[\int_{-2}^{-1}(t+2)\,dt + \int_{-1}^{1}dt + \int_{1}^{2}(2-t)\,dt\right]$

$= \frac{1}{6}\left[\left(\frac{t^2}{2}+2t\right)\Big|_{-2}^{-1} + t\Big|_{-1}^{1} + \left(2t - \frac{t^2}{2}\right)\Big|_{1}^{2}\right]$

$= \frac{1}{6}\left[\left(\frac{1}{2}-2\right) - (2-4) + 1 + 1 + (4-2) - \left(2-\frac{1}{2}\right)\right]$

$= \frac{1}{6} \times 3 = \frac{1}{2}$

$n = 1 \Rightarrow |X_1| = \left|\frac{3}{\pi^2}\left[\cos\frac{\pi}{3} - \cos\frac{2\pi}{3}\right]\right| = \frac{3}{\pi^2}$

$n = -1 \Rightarrow |X_{-1}| = \left|\frac{3}{\pi^2}\left[\cos\frac{\pi}{3} - \cos\frac{2\pi}{3}\right]\right| = \frac{3}{\pi^2}$

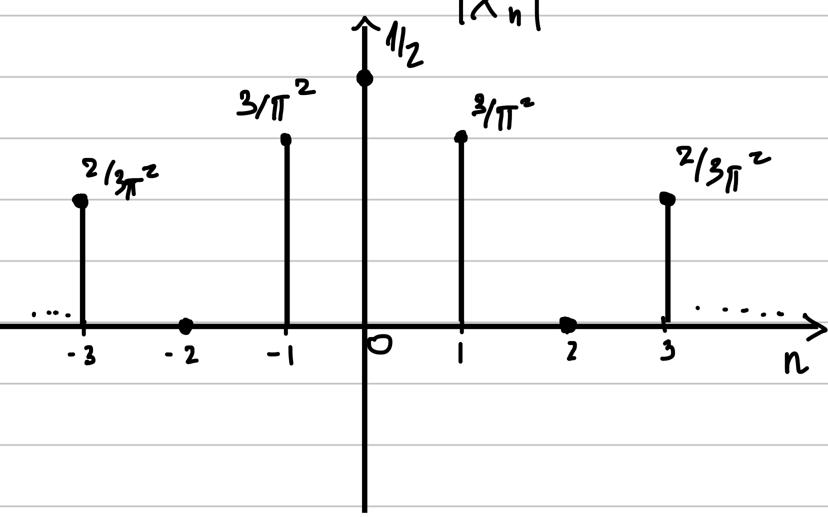
$n = 2 \Rightarrow |X_2| = \left|\frac{3}{4\pi^2}\left[\cos\frac{2\pi}{3} - \cos\frac{4\pi}{3}\right]\right| = 0$

$n = -2 \Rightarrow |X_{-2}| = 0$

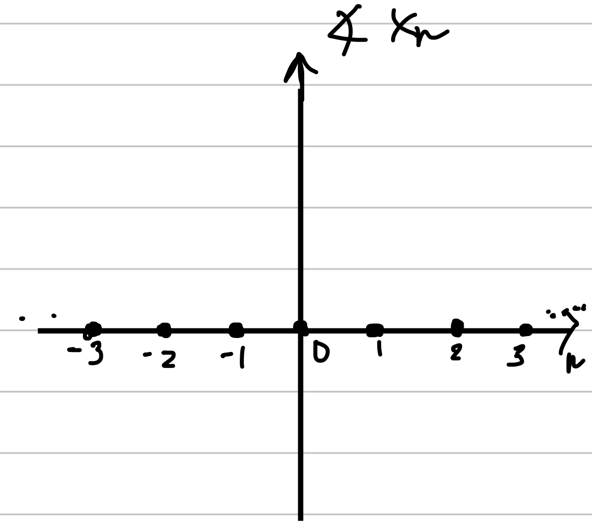
$n = 3 \Rightarrow |X_3| = \left| \frac{1}{3\pi^2}\left[\cos\pi - \cos 2\pi\right] \right| = \frac{2}{3\pi^2}$

$n = -3 \Rightarrow |X_{-3}| = \frac{2}{3\pi^2} \quad \cdots\cdots$

Magnitude Spectrum



Phase Spectrum



d) $x(t) = X_0 + 2\sum_{k=1}^{\infty} a_k \cos(k\Omega_0 t) - 2\sum_{k=1}^{\infty} b_k \sin(k\Omega_0 t)$

We have already $X_0 = \frac{1}{2}$, $w_0 = \Omega_0 = \pi/3$, $T_0 = 6$

Also, $a_k = \frac{1}{6} \int_{T_0} x(t) \cos(k w_0 t)\, dt$

$= \frac{1}{6} \int_{-3}^{3} x(t) \cos(k w_0 t)\, dt = \frac{1}{6}\Big[\int_{-2}^{-1} \underbrace{(t+2)\cos(k w_0 t)\, dt}_{A}$

$+ \underbrace{\int_{-1}^{1} \cos(k w_0 t)\, dt}_{B} + \underbrace{\int_{1}^{2} (2-t)\cos(k w_0 t)\, dt}_{C} \Big]$

We have $\int t \cos at\, dt$ let $u = t \Rightarrow du = dt$

$dv = \cos at\, dt$

$\Rightarrow v = \frac{1}{a} \sin at$

$\Rightarrow \int t \cos at\, dt = t \frac{1}{a} \sin at - \int \frac{1}{a} \sin at\, dt$

$= \frac{t}{a} \sin at + \frac{1}{a^2} \cos at$

Then $A = \int_{-2}^{-1} (t+2)\cos(k\omega_0 t)\,dt = \int_{-2}^{-1} t\cos(k\omega_0 t)\,dt$

$+ \int_{-2}^{-1} 2\cos(k\omega_0 t)\,dt$, $a = k\omega_0$

$= \int_{-2}^{-1} t\cos at\,dt + \int_{-2}^{-1} 2\cos at\,dt$

$= \left(\frac{t}{a}\sin at + \frac{1}{a^2}\cos at\right)\Big|_{-2}^{-1} + \frac{2}{a}\sin at\Big|_{-2}^{-1}$

$= \left(\frac{+1}{a}\sin a + \frac{1}{a^2}\cos a\right) - \left(\frac{+2}{a}\sin 2a + \frac{1}{a^2}\cos 2a\right)$

$+ \frac{2}{a}\Big[\underbrace{\sin(-a) - \sin(-2a)}_{\sin 2a - \sin a}\Big]$

$= \frac{1}{a}\sin a + \frac{1}{a^2}\cos a - \frac{2}{a}\sin 2a - \frac{1}{a^2}\cos 2a + \frac{2}{a}\sin 2a$

$- \frac{2}{a}\sin a = \frac{1}{a^2}\cos a - \frac{1}{a^2}\cos 2a - \frac{1}{a}\sin a$

* $B = \int_{-1}^{1} \cos(k\omega_0 t)\,dt = \int_{-1}^{1} \cos at\,dt$, $a = k\omega_0$

$$= \frac{1}{a}\sin at \Big|_{-1}^{1} = \frac{1}{a}(\sin a + \sin a) = \frac{2}{a}\sin a$$

$$* \; C = \int_{1}^{2} (2-t)\cos(k\omega_0 t)\,dt = \int_{1}^{2} 2\cos at\,dt - \int_{1}^{2} t\cos at\,dt$$

$$\ominus \int_{1}^{2} 2\cos at\,dt = 2\times\frac{1}{a}\sin at\Big|_{1}^{2} = \frac{2}{a}(\sin 2a - \sin a)$$

$$\oplus \int_{1}^{2} t\cos at\,dt = \left(\frac{t}{a}\sin at + \frac{1}{a^2}\cos at\right)\Big|_{1}^{2}$$

$$= \left(\frac{2}{a}\sin 2a + \frac{1}{a^2}\cos 2a\right) - \left(\frac{1}{a}\sin a + \frac{1}{a^2}\cos a\right)$$

$$\Rightarrow C = \frac{2}{a}\sin 2a - \frac{2}{a}\sin a - \frac{2}{a}\sin 2a - \frac{1}{a^2}\cos 2a$$

$$+ \frac{1}{a}\sin a + \frac{1}{a^2}\cos a = \frac{1}{a^2}\cos a - \frac{1}{a^2}\cos 2a - \frac{1}{a}\sin a$$

$$\Rightarrow A+B+C = \underset{\times}{\frac{1}{a^2}\cos a} - \underset{\circ}{\underline{\frac{1}{a^2}\cos 2a}} - \underline{\frac{1}{a}\sin a} + \underline{\frac{2}{a}\sin a}$$

$$+ \underset{\times}{\frac{1}{a^2}\cos a} - \underset{\circ}{\underline{\frac{1}{a^2}\cos 2a}} - \underline{\frac{1}{a}\sin a} = \frac{2}{a^2}\cos a - \frac{2}{a^2}\cos 2a$$

$$= \frac{2}{a^2}(\cos a - \cos 2a) = \frac{2}{(k\omega_0)^2}\left[\cos(k\omega_0) - \cos(2k\omega_0)\right]$$

$$\Rightarrow a_k = \frac{1}{6}[A+B+C] = \frac{1}{6}\cdot\frac{2}{(k\omega_0)^2}\left[\cos(k\omega_0) - \cos(2k\omega_0)\right]$$

$$a_k = \frac{1}{3(k\omega_0)^2}\left[\cos(k\omega_0) - \cos(2k\omega_0)\right], \quad \omega_0 = \pi/3$$

Also, $3(k\omega_0)^2 = 3k^2\omega_0^2 = 3k^2\frac{\pi^2}{9} = \frac{k^2\pi^2}{3}$

$$\Rightarrow \boxed{a_k = \frac{3}{k^2\pi^2}\left[\cos\frac{k\pi}{3} - \cos\frac{2k\pi}{3}\right]}$$

Then, $b_k = -\frac{1}{T_0}\int_{T_0} x(t)\sin(k\omega_0 t)\,dt = -\frac{1}{6}\int_{-3}^{3} x(t)\sin(k\omega_0 t)\,dt$

$$= -\frac{1}{6}\Big[\underbrace{\int_{-2}^{-1}(t+2)\sin(k\omega_0 t)\,dt}_{A} + \underbrace{\int_{-1}^{1}\sin(k\omega_0 t)\,dt}_{B} + \underbrace{\int_{1}^{2}(2-t)\sin k\omega_0 t\,dt}_{C}$$

We have: $\int t\sin at\,dt$ $\quad u = t \Rightarrow du = dt$

$\quad dv = \sin at\,dt \Rightarrow v = -\frac{1}{a}\cos at$

$$= -t\cdot\frac{1}{a}\cos at + \int\frac{1}{a}\cos at\,dt$$

$$= -\frac{t}{a}\cos at + \frac{1}{a^2}\cdot\sin at$$

$\circledast$ $A = \int_{-2}^{-1}(t+2)\sin k\omega_0 t\,dt = \int_{-2}^{-1}(t+2)\sin at\,dt, \quad a = k\omega_0$

$$= \int_{-2}^{-1} t \sin at \, dt + \int_{-2}^{-1} 2 \sin at \, dt$$

$$\oplus \int_{-2}^{-1} t \sin at \, dt = \left(-\frac{t}{a} \cos at + \frac{1}{a^2} \sin at \right) \Big|_{-2}^{-1}$$

$$= \left(\frac{1}{a} \cos a - \frac{1}{a^2} \sin a \right) - \left(\frac{2}{a} \cos 2a - \frac{1}{a^2} \sin 2a \right)$$

$$\oplus \int_{-2}^{-1} 2 \sin at \, dt = -\frac{2}{a} \cos at \Big|_{-2}^{-1} = \frac{2}{a} \cos at \Big|_{-1}^{-2}$$

$$= \frac{2}{a} \left(\cos 2a - \cos a \right)$$

$$\Rightarrow A = \frac{1}{a} \cos a - \frac{1}{a^2} \sin a - \frac{2}{a} \cos 2a + \frac{1}{a^2} \sin 2a + \frac{2}{a} \cos 2a$$

$$- \frac{2}{a} \cos a = \frac{1}{a^2} \sin 2a - \frac{1}{a^2} \sin a - \frac{1}{a} \cos a$$

$$\oplus B = \int_{-1}^{1} \sin(k\omega_0 t) \, dt = \int_{-1}^{1} \sin at \, dt, \quad a = k\omega_0$$

$$= -\frac{1}{a} \cos at \Big|_{-1}^{1} = \frac{1}{a} \cos at \Big|_{1}^{-1} = \frac{1}{a} \left[\cos a - \cos a \right]$$

$$= 0$$

$$\oplus C = \int_{1}^{2} (2-t) \sin k\omega_0 t \, dt = \int_{1}^{2} (2-t) \sin at \, dt, \quad a = k\omega_0$$

$$= \int_1^2 2\sin at\,dt - \int_1^2 t\sin at\,dt$$

⊕ $\int_1^2 2\sin at\,dt = -\frac{2}{a}\cos at\Big|_1^2 = \frac{2}{a}\cos at\Big|_2^1$

$$= \frac{2}{a}(\cos a - \cos 2a) = \frac{2}{a}\cos a - \frac{2}{a}\cos 2a$$

⊕ $\int_1^2 t\sin at\,dt = \left(-\frac{t}{a}\cos at + \frac{1}{a^2}\cdot\sin at\right)\Big|_1^2$

$$= \left(-\frac{2}{a}\cos(2a) + \frac{1}{a^2}\sin 2a\right) - \left(\frac{-1}{a}\cos a + \frac{1}{a^2}\sin a\right)$$

$$= -\frac{2}{a}\cos 2a + \frac{1}{a^2}\sin 2a + \frac{1}{a}\cos a - \frac{1}{a^2}\sin a$$

$$\Rightarrow C = \frac{2}{a}\cos a - \frac{2}{a}\cos 2a + \frac{2}{a}\cos 2a - \frac{1}{a^2}\sin 2a - \frac{1}{a}\cos a$$

$$+ \frac{1}{a^2}\sin a = \frac{1}{a}\cos a + \frac{1}{a^2}\sin a - \frac{1}{a^2}\sin 2a$$

$$\Rightarrow A+B+C = \frac{1}{a^2}\sin 2a - \frac{1}{a^2}\sin a - \frac{1}{a}\cos a + \frac{1}{a}\cos a + \frac{1}{a^2}\sin a$$

$$- \frac{1}{a^2}\sin 2a = 0 \Rightarrow b_k = 0$$

Therefore: $\boxed{x(t) = \frac{1}{2} + 2\sum_{k=1}^{\infty}\left[\frac{3}{k^2\pi^2}\left(\cos\frac{k\pi}{3} - \cos\frac{2k\pi}{3}\right)\right]\cos\left(k\frac{\pi}{3}t\right)}$

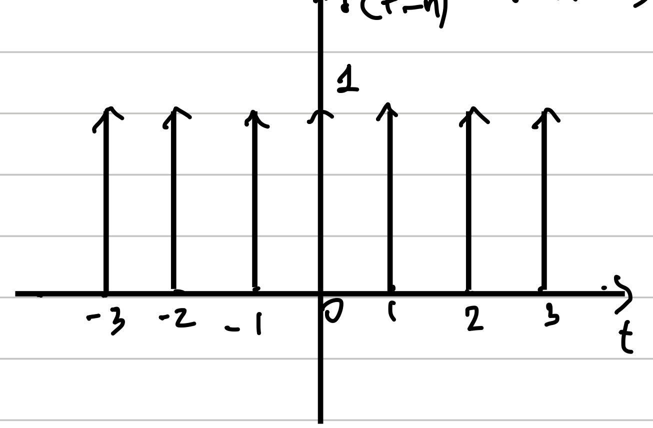
3) LTI system, $h(t) = e^{-4t} u(t)$

Find FS of $y(t)$.

We have $h(t)$ is real & $H(s) = \frac{1}{s+4}$ $Re\{s\} > -4$

$H(jk\omega_0) = H(s)\big|_{s=jk\omega_0}$, $jk\omega_0 \in$ ROC of $H(s)$

a) $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$



Firstly, we need to find

$x(t) = X_0 + 2\sum_{k=1}^{+\infty} |X_k| \cos(k\omega_0 t + \measuredangle X_k)$

$T_0 = 1s \Rightarrow \omega_0 = 2\pi$

$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{1}\int_{T_0} \delta(t)\,dt = 1$

$X_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\,dt = \frac{1}{1}\int_0^1 \delta(t)\, e^{-jk\omega_0 t}\,dt$

$= \int_0^1 \delta(t)\, e^{0}\,dt = \int_0^1 \delta(t)\,dt = 1$

$\Rightarrow \measuredangle X_k = 0°$ & $|X_k| = 1$

$\Rightarrow x(t) = 1 + 2\sum_{k=1}^{+\infty} \cos(k2\pi t)$

Since $h(t)$ & $x(t)$ is real, then $y(t)$:

$$y(t) = Y_0 + 2\sum_{k=1}^{+\infty} |Y_k| \cos(k\omega_0 t + \measuredangle Y_k)$$

$\oplus$ $Y_0 = H(0) X_0 = 1 . \left.\frac{1}{s+4}\right|_{s=0} = \frac{1}{4}$

$\oplus$ $|Y_k| = |X_k| \, |H(jk\omega_0)|$

Since $H(jk\omega_0) = H(jk.2\pi) = \left.\frac{1}{s+4}\right|_{s=jk2\pi}$

$= \frac{1}{2k\pi j + 4}$

$\Rightarrow |Y_k| = 1. \left|\frac{1}{2k\pi j + 4}\right| = \frac{1}{|2k\pi j + 4|}$

$= \frac{1}{\sqrt{4k^2\pi^2 + 16}} = \frac{1}{2\sqrt{k^2\pi^2 + 4}}$

$\oplus$ $\measuredangle Y_k = \measuredangle X_k + \measuredangle H(jk\omega_0)$

Since $H(jk\omega_0) = \frac{1}{2k\pi j + 4} = \frac{2k\pi j - 4}{-(4k^2\pi^2 + 16)}$

$$= \frac{4 - 2k\pi j}{4k^2\pi^2 + 16} = \frac{2 - k\pi j}{2k^2\pi^2 + 8}$$

$$= \frac{1}{k^2\pi^2 + 4} - \frac{k\pi}{2k^2\pi^2 + 8} j$$

$$\Rightarrow \measuredangle H(jk\omega_0) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(jk\omega_0)\}}{\mathrm{Re}\{H(jk\omega_0)\}} \right)$$

$$= \tan^{-1}\left(\frac{-k\pi / (2k^2\pi^2 + 8)}{\frac{1}{k^2\pi^2 + 4}} \right)$$

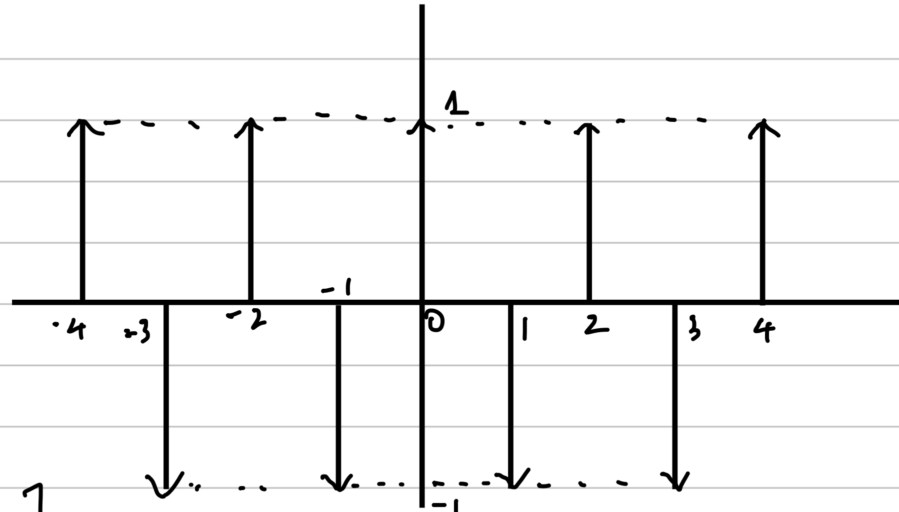
$$= -\tan^{-1}\left(\frac{k\pi}{2(k^2\pi^2 + 4)} \cdot (k^2\pi^2 + 4) \right)$$

$$= -\tan^{-1}\left(\frac{k\pi}{2} \right)$$

$$\Rightarrow \measuredangle Y_k = -\tan^{-1}\left(\frac{k\pi}{2} \right)$$

$$\Rightarrow \boxed{y(t) = \frac{1}{4} + 2\sum_{k=1}^{+\infty} \frac{1}{2\sqrt{k^2\pi^2 + 4}} \cos\left(2\pi k t - \tan^{-1}\left(\frac{k\pi}{2} \right) \right)}$$

b) $x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \delta(t-n)$



We have $T_0 = 2$, $[0,2)$

$$X_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

$$= \frac{1}{2}\left[\int_0^1 x(t)\, dt + \int_1^2 x(t)\, dt \right]$$

$$= \frac{1}{2}\left[\int_0^1 \delta(t)\, dt - \int_1^2 \delta(t-1)\, dt \right] = \frac{1}{2}(1-1) = 0$$

$$X_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt, \quad \omega_0 = \frac{2\pi}{T_0} = \pi$$

$$= \frac{1}{2}\left[\int_0^1 \delta(t)\, e^{-jk\omega_0 t}\, dt - \int_1^2 \delta(t-1) e^{-jk\omega_0 t}\, dt \right]$$

$$\circledast \int_0^1 \delta(t) e^{-jk\omega_0 t}\, dt = \int_0^1 \delta(t)\, e^0\, dt = 1$$

$$\circledast \int_1^2 \delta(t-1) e^{-jk\omega_0 t}\, dt = \int_1^2 \delta(t-1) e^{-jk\omega_0 \cdot 1}\, dt$$

$$= e^{-jk\pi} \int_1^2 \delta(t-1)\, dt = e^{-jk\pi}$$

$$\Rightarrow X_k = \frac{1}{2}\left[1 - e^{-jk\pi}\right] = \frac{1}{2} - \frac{1}{2}e^{-jk\pi} \quad (k \neq 0, k \in \mathbb{Z})$$

$$\circledast \; e^{-jk\pi} = \cos(k\pi) - j\sin(k\pi) = \cos(k\pi)$$

$$\Rightarrow X_k = \frac{1}{2} - \frac{1}{2}\cos(k\pi) \quad (\text{real})$$

$$\Rightarrow |X_k| = \frac{1}{2}[\underbrace{1 - \cos(k\pi)}_{\geq 0}] \quad \& \quad \measuredangle X_k = 0 \text{ since } X_k \text{ is real}$$

Since $h(t)$ & $x(t)$ are real, then $y(t)$ is real and

$$y(t) = Y_0 + 2\sum_{k=1}^{\infty} |Y_k| \cos(k\omega_0 t + \measuredangle Y_k)$$

with:

$$Y_0 = X_0 \cdot H(0) = 0 \cdot H(0) = 0$$

$$|Y_k| = |X_k|\,|H(j\omega_0 k)| = \frac{1}{2}[\underbrace{1 - \cos k\pi}_{\geq 0}]\,|H(j\pi k)|$$

$$H(jk\pi) = \left.\frac{1}{s+4}\right|_{s=jk\pi} = \frac{1}{k\pi j + 4}$$

$$\Rightarrow |H(jk\pi)| = \frac{1}{|k\pi j + 4|} = \frac{1}{\sqrt{k^2\pi^2 + 16}}$$

$$\Rightarrow |Y_k| = \frac{1}{2}(1 - \cos k\pi)\frac{1}{\sqrt{k^2\pi^2 + 16}}$$

(f) $\measuredangle Y_k = \measuredangle X_k + \measuredangle H(jk\pi) = \measuredangle H(jk\pi)$

$$H(jk\pi) = \frac{1}{k\pi j + 4} = \frac{k\pi j - 4}{-(k^2\pi^2 + 16)} = \frac{4 - k\pi j}{k^2\pi^2 + 16}$$

$$= \frac{4}{k^2\pi^2 + 16} - \frac{k\pi}{k^2\pi^2 + 16} j$$

$$\Rightarrow \measuredangle H(jk\pi) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(jk\pi)\}}{\mathrm{Re}\{H(jk\pi)\}}\right)$$

$$= \tan^{-1}\left(\frac{-k\pi}{k^2\pi^2 + 16} \cdot \frac{k^2\pi^2 + 16}{4}\right) = -\tan^{-1}\left(\frac{k\pi}{4}\right)$$

$$\Rightarrow \measuredangle Y_k = -\tan^{-1}\left(\frac{k\pi}{4}\right)$$

$$\Rightarrow y(t) = 2\sum_{k=1}^{+\infty} \frac{1}{2}(1 - \cos k\pi)\frac{1}{\sqrt{k^2\pi^2 + 16}} \cdot \cos\left(k\pi t - \tan^{-1}\left(\frac{k\pi}{4}\right)\right)$$

$$\Rightarrow \boxed{y(t) = \sum_{k=1}^{+\infty} \frac{(1 - \cos k\pi)}{\sqrt{k^2\pi^2 + 16}} \cos\left(k\pi t - \tan^{-1}\left(\frac{k\pi}{4}\right)\right)}$$

<u>Note:</u> $\left|\frac{1 - \cos k\pi}{\sqrt{k^2\pi^2 + 16}}\right| = \frac{1 - \cos k\pi}{\sqrt{k^2\pi^2 + 16}}$ Since $\cos k\pi \in [-1, 1]$

$\Rightarrow 1 - \cos k\pi \geq 0$

& $\sqrt{k^2\pi^2 + 16} > 0$.

4) a) Given : Real Periodic signal.

- $T_0 = 6$
- $x(t) = -x(t-3)$
- $X_0 = 0$
- $X_k = 0$ for $k > 2$
- $X_1 > 0$ & real number
- $\int_{-3}^{3} |x(t)|^2 dt = 3$

We have $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{6} = \frac{\pi}{3}$

Generally, $x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$

$= X_0 + \sum_{k=-\infty}^{+\infty} X_k e^{jn\omega_0 t}$ ($k = \pm 1, \pm 2, \ldots$ & $k \neq 0$)

$= 0 + \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$

$= \sum_{k=-\infty}^{+\infty} X_k e^{jk\omega_0 t}$ ($k \neq 0$, & $k = \pm 1, \pm 2, \ldots$)

$= \sum_{k=-\infty}^{2} X_k e^{jk\omega_0 t}$ Since $X_k = 0$ with $k > 2$ (1)

- Also, we have : $\frac{1}{T_0} \int_{T_0} |x(t)|^2 dt = \sum_{-\infty}^{+\infty} |X_k|^2$

(Parseval's Theorem)

$\Rightarrow \frac{1}{6} \int_{-3}^{3} |x(t)|^2 dt = \sum_{-\infty}^{+\infty} |X_k|^2$

$$\Rightarrow \int_{-3}^{3} |x(t)|^2 dt = 6 \sum_{-\infty}^{+\infty} |X_k|^2 = 3$$

$$\Rightarrow \sum_{-\infty}^{+\infty} |X_k|^2 = \frac{1}{2} \quad (2)$$

$$(*) \quad X_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{0}^{T_0} -x(t-3) e^{-jn\omega_0 t} dt$$

let $\tau = t - 3 = t - \frac{T_0}{2} \Rightarrow t = \tau + \frac{T_0}{2}$

& $d\tau = dt$

$\Rightarrow t = 0 \longrightarrow T_0$

$\Rightarrow \tau = -\frac{T_0}{2} \longrightarrow T_0/2$

$$\Rightarrow X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} -x(\tau) e^{-jk\omega_0 (\tau + \frac{T_0}{2})} d\tau$$

We have: $e^{-jk\omega_0 (\tau + \frac{T_0}{2})} = e^{-jk\omega_0 \tau} e^{-jk\omega_0 \frac{T_0}{2}}$

$= e^{-jk\omega_0 \tau} e^{-jk\frac{\pi}{3} \cdot 3} = e^{-jk\omega_0 \tau} e^{-jk\pi}$

also, $e^{-jk\pi} = \cos(k\pi) - j\sin(k\pi) = \cos(k\pi)$

$= (-1)^k$

$\Rightarrow \quad e^{-jkw_0(\tau + T_0/2)} = (-1)^k e^{-jkw_0\tau}$

$$\Rightarrow X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} -x(\tau)(-1)^k e^{-jkw_0\tau} \, d\tau$$

$$\Rightarrow X_k = \frac{1}{T_0} \int_{T_0} x(\tau)\left(-(-1)^k\right) e^{-jkw_0\tau} \, d\tau$$

$$= \left[-(-1)^k\right] \underbrace{\frac{1}{T_0} \int x(\tau) e^{-jkw_0\tau} \, d\tau}_{X_k}$$

$\Rightarrow X_k = \left[-(-1)^k\right] X_k$

If k is even $\Rightarrow X_k = -X_k \Rightarrow X_k = 0$

$\Rightarrow X_k = 0$ when k is even

From (2), $\sum_{-\infty}^{+\infty} |X_k|^2 = \frac{1}{2} = |X_0|^2 + |X_{-1}|^2 + |X_1|^2 +$

$|X_2|^2 + |X_{-2}|^2 + |X_3|^2 + |X_{-3}|^2 + \cdots$

Since $X_0 = 0$, $X_k = 0$ for $k > 2$, & $X_k = 0$ when k is even. Also $x(t)$ is real, then $|X_k| = |X_{-k}|$

$$\Rightarrow \begin{cases} |X_2| = |X_{-2}| = 0 \Rightarrow X_2 = X_{-2} = 0 \\ |X_3| = |X_{-3}| = 0 \Rightarrow X_3 = X_{-3} = 0 \\ \vdots \\ |X_k| = |X_{-k}| = 0 \quad (k > 2) \Rightarrow X_k = X_{k-1} = 0 \end{cases}$$

$$\Rightarrow \sum_{-\infty}^{+\infty} |X_k|^2 = |X_1|^2 + |X_{-1}|^2 = 2|X_1|^2 = \frac{1}{2}$$

$$\Rightarrow |X_1|^2 = \frac{1}{4} \Rightarrow |X_1| = \frac{1}{2}$$

Since X_1 is positive real number $\Rightarrow X_1 = \frac{1}{2} \Rightarrow X_{-1} = \frac{1}{2}$

From ①, $x(t) = \sum_{k=-\infty}^{2} X_k e^{jk\omega_0 t}$

$$= X_{-1} e^{-j\omega_0 t} + X_1 e^{j\omega_0 t} = \frac{1}{2}\left[e^{-j\omega_0 t} + e^{j\omega_0 t}\right]$$

(Since we have already $X_0 = 0$, $X_k = 0$ for $k > 2$ & $X_k = 0$ when k is even, $|X_k| = |X_{-k}| = 0$ with $k > 2$)

$$\Rightarrow x(t) = \cos(\omega_0 t) \Rightarrow \boxed{x(t) = \cos\left(\frac{\pi}{3} t\right)}$$

b) Given $x(t)$ is periodic signal

coefficient $a_k = \begin{cases} 2 & k=0 \\ j\left(\frac{1}{2}\right)^{|k|} & \text{else} \end{cases}$

Generally, $x(t) = \sum_{-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

i) When $k=0$, $x(t) = 2$

When $k<0$, $x(t) = a_{-1} e^{-j\omega_0 t} + a_{-2} e^{-j2\omega_0 t}$
$+ a_{-3} e^{-j3\omega_0 t} + \dots$

$= \frac{j}{2} e^{-j\omega_0 t} + \frac{j}{2^2} e^{-j2\omega_0 t} + \frac{j}{2^3} e^{-j3\omega_0 t} + \dots$

When $k>0$, $x(t) = a_1 e^{j\omega_0 t} + a_2 e^{2j\omega_0 t} + a_3 e^{3j\omega_0 t} + \dots$

$= \frac{j}{2} e^{j\omega_0 t} + \frac{j}{2^2} e^{2j\omega_0 t} + \frac{1}{2^3} e^{3j\omega_0 t} + \dots$

$\Rightarrow x(t) = 2 + \frac{j}{2} e^{-j\omega_0 t} + \frac{j}{2^2} e^{-j2\omega_0 t} + \frac{j}{2^3} e^{-j3\omega_0 t} +$

$+ \frac{j}{2} e^{j\omega_0 t} + \frac{j}{2^2} e^{2j\omega_0 t} + \frac{1}{2^3} e^{3j\omega_0 t} + \dots$

$= 2 + \frac{j}{2}\left(e^{-j\omega_0 t} + e^{j\omega_0 t}\right) + \frac{j}{2^2}\left(e^{-j2\omega_0 t} + e^{2j\omega_0 t}\right)$

$$+ \frac{j}{2^3}\left(e^{-j3\omega_0 t} + e^{3j\omega_0 t}\right) + \dots$$

$$= 2 + j\cos\omega_0 t + \frac{j}{2}\cos(2\omega_0 t) + \frac{j}{2^2}\cos(3\omega_0 t)$$

$$+ \dots + \frac{j}{2^{k-1}}\cos(k\omega_0 t)$$

$$= 2 + j\underbrace{\left(\cos\omega_0 t + \frac{1}{2}\cos(2\omega_0 t) + \frac{1}{2^2}\cos(3\omega_0 t) + \dots + \frac{1}{2^{k-1}}\cos k\omega_0 t\right)}$$

$$= 2 + Aj \qquad A, \text{ A is real}$$

$$\Rightarrow \boxed{x(t) = 2 + Aj \text{ then } x(t) \text{ is not real}}$$

* We can re-check by using the properties of real signal.

We have already $a_k = \begin{cases} 2, & k=0 \\ j\left(\frac{1}{2}\right)^{|k|}, & \text{otherwise} \end{cases}$

$$\Rightarrow a^*_{-k} = \begin{cases} 2, & k=0 \\ -j\left(\frac{1}{2}\right)^{|k|}, & \text{otherwise} \end{cases}$$

$\boxed{\text{As we can see, } a_k \neq a^*_{-k} \Rightarrow x(t) \text{ is correctly not a real signal}}$

ii) From part i), we have already:

$$x(t) = 2 + j\left[\cos(\omega_0 t) + \frac{1}{2}\cos(2\omega_0 t) + \cdots + \frac{1}{2^{k-1}}\cos(k\omega_0 t)\right]$$

$$\Rightarrow x(-t) = 2 + j\left[\cos(-\omega_0 t) + \frac{1}{2}\cos(-2\omega_0 t) + \cdots + \frac{1}{2^{k-1}}\cos(-k\omega_0 t)\right]$$

$$= 2 + j\left[\cos(\omega_0 t) + \frac{1}{2}\cos(2\omega_0 t) + \cdots + \frac{1}{2^{k-1}}\cos(k\omega_0 t)\right]$$

$$= x(t)$$

$$\Rightarrow \boxed{x(t) \text{ is even signal}}$$

(*) We can recheck by checking a_k & a_{-k}

We have $a_{-k} = \begin{cases} 2 & , k=0 \\ j\left(\frac{1}{2}\right)^{|-k|} = j\left(\frac{1}{2}\right)^{|k|} & , \text{otherwise} \end{cases}$

$$\Rightarrow a_k = a_{-k} \Rightarrow \boxed{x(t) \text{ is correctly an even signal.}}$$

iii) Let $y(t) = \frac{d\,x(t)}{dt} = j[A]'_{(t)}$

$$= j\left[\cos(\omega_0 t) + \frac{1}{2}\cos(2\omega_0 t) + \cdots + \frac{1}{2^{k-1}}\cos(k\omega_0 t)\right]'_t$$

$$= j\left[-\omega_0 \sin(\omega_0 t) - \frac{1}{2}\cdot 2\omega_0 \sin(2\omega_0 t) - \dots - \frac{1}{2^{k-1}} k\omega_0 \sin k\omega_0 t\right]$$

$$= -j\left[\omega_0 \sin(\omega_0 t) + \frac{1}{2}\cdot 2\omega_0 \sin(2\omega_0 t) + \dots + \frac{1}{2^{k-1}}\cdot k\omega_0 \sin k\omega_0 t\right]$$

$$\Rightarrow y(-t) = -j\left[\omega_0 \sin(-\omega_0 t) + \frac{1}{2} 2\omega_0 \sin(-2\omega_0 t) + \dots + \right.$$

$$\left. \frac{1}{2^{k-1}} k\omega_0 \sin(-k\omega_0 t)\right]$$

$$= j\left[\omega_0 \sin(\omega_0 t) + \frac{1}{2}\cdot 2\omega_0 \sin(2\omega_0 t) + \dots + \frac{1}{2^{k-1}} k\omega_0 \sin(k\omega_0 t)\right]$$

$$= -y(t) \Rightarrow y(-t) = -y(t)$$

$$\Rightarrow \boxed{\frac{dx(t)}{dt} \text{ is not even}}$$

* We can recheck by using:

We have $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

$$\Rightarrow y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{+\infty} \underbrace{a_k jk\omega_0}\, e^{jk\omega_0 t}$$

$$\Rightarrow y(t) = \frac{dx(t)}{dt} \text{ has } b_k = a_k jk\omega_0 .$$

$$\Rightarrow b_k = \begin{cases} 0, & k=0 \\ jk\omega_0 a_k = jk\omega_0 j\left(\frac{1}{2}\right)^{|k|}, & k \neq 0 \end{cases}$$

$$\Rightarrow b_k = \begin{cases} 0, & k=0 \\ -k\omega_0 \left(\frac{1}{2}\right)^{|k|}, & k \neq 0 \end{cases}$$

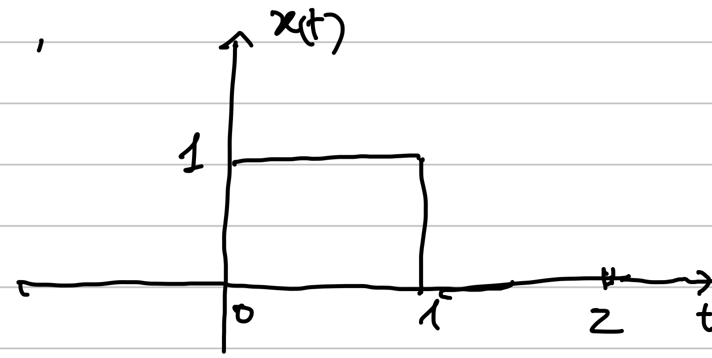
we have $b_{-k} = \begin{cases} 0, & k=0 \\ k\omega_0 \left(\frac{1}{2}\right)^{|k|}, & k \neq 0 \end{cases}$

$\boxed{\text{Since } b_k \neq b_{-k} \Rightarrow y(t) = \frac{dx(t)}{dt} \text{ is not an even signal}}$

5) Two signal have period $T_0 = 2s$ & $0 \le t \le T_0$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j w_0 n t}, \quad w_0 = \frac{2\pi}{T_0} = \pi$$

(*) $x_1(t) = u(t) - u(t-1)$,



$$X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-j w_0 n t} dt$$

$$= \frac{1}{2}\int_0^2 x(t) e^{-j\pi n t} dt = \frac{1}{2}\int_0^1 1 . e^{-j\pi n t} dt$$

$$= \frac{1}{2}\left(\frac{-1}{j\pi n} e^{-j\pi n t}\right)\Bigg|_0^1 = \frac{1}{2}\left[\frac{1}{j\pi n} e^{-j\pi n t}\right]\Bigg|_1^0$$

$$= \frac{1}{2j\pi n}\left[1 - e^{-j\pi n}\right] = \frac{1 - e^{-j\pi n}}{2j\pi n} = \frac{1 - (\cos\pi n - j\sin\pi n)}{2j\pi n}$$

$$= \frac{1 - \cos\pi n}{2j\pi n} = \frac{-j(1-\cos\pi n)}{2\pi n} = \boxed{j . \frac{\cos\pi n - 1}{2\pi n}}$$

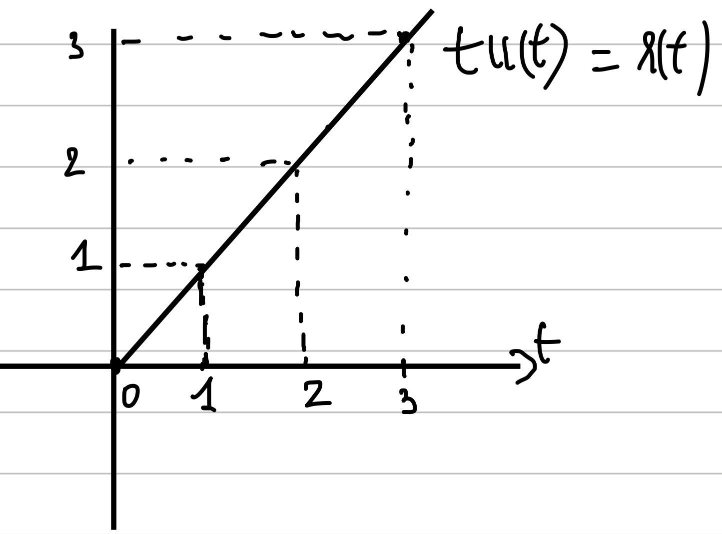
$$\Rightarrow |X_n| = \left|\frac{\cos(\pi n) - 1}{2\pi n}\right|$$

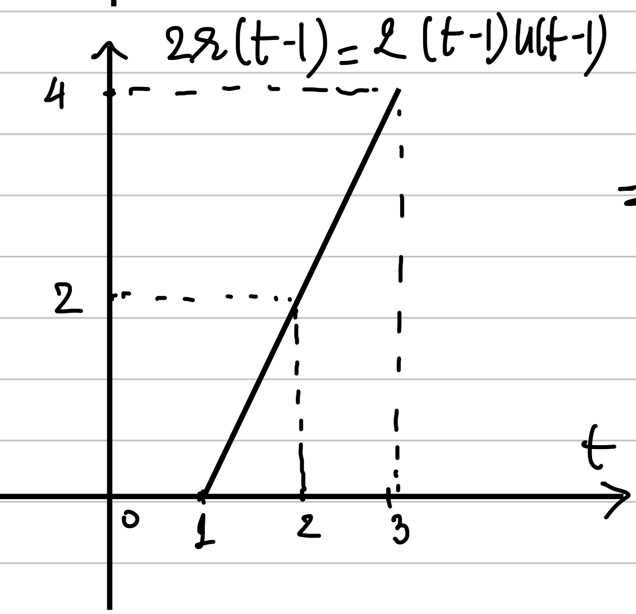
$$X_0 = \frac{1}{T_0}\int_{T_0} x(t) dt = \frac{1}{2}\int_0^2 x(t) dt = \frac{1}{2}\int_0^1 dt = \boxed{\frac{1}{2}}$$

(*) $x_2(t) = r(t) - 2r(t-1) + r(t-2)$

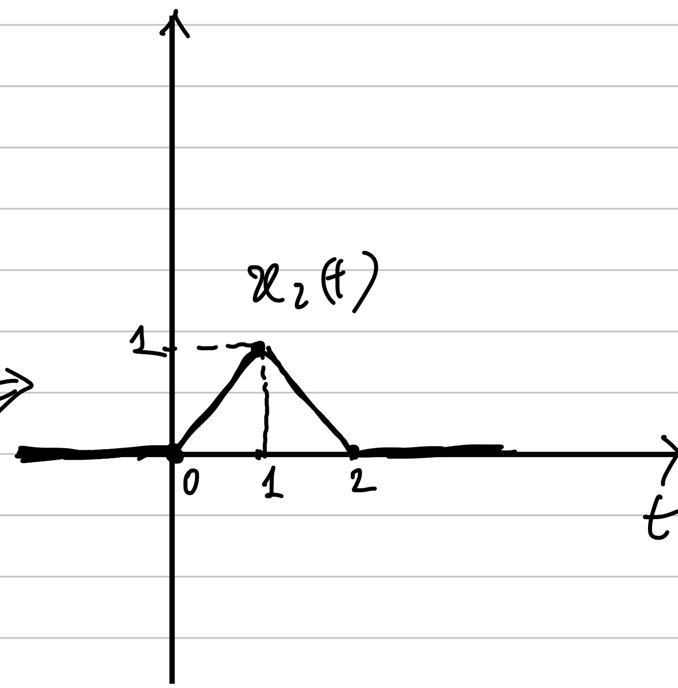
$r(t) = tu(t)$, $r(t-1) = (t-1)u(t-1)$, $r(t-2) = (t-2)u(t-2)$

$\Rightarrow x_2(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$

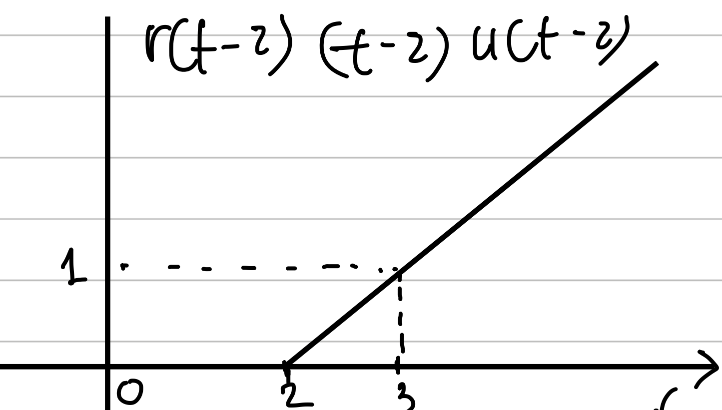




$\Rightarrow$



$$\Rightarrow x_2(t) = \begin{cases} t, & 0 \le t < 1 \\ -t+2, & 1 \le t < 2 \end{cases}$$



$$\Rightarrow X_n = \frac{1}{T_0} \int_{T_0} x_2(t) e^{-j\omega_0 n t} dt$$

$$= \frac{1}{2} \Big[\underbrace{\int_0^1 t e^{-j\pi n t} dt}_{A} + \underbrace{\int_1^2 (2-t) e^{-j\pi n t} dt}_{B} \Big]$$

$*$ We have: $\int t e^{at} dt = \left(\frac{t}{a} - \frac{1}{a^2} \right) e^{at}$

$$A = \int_0^1 t e^{-j\pi n t} dt = \int_0^1 t e^{at} dt \quad , \quad a = -j\pi n$$

$$= \left(\frac{t}{a} - \frac{1}{a^2} \right) e^{at} \Big|_0^1 = \left(\frac{1}{a} - \frac{1}{a^2} \right) e^{a} - \left(-\frac{1}{a^2} \right)$$

$$= \left(\frac{1}{a} - \frac{1}{a^2} \right) e^{a} + \frac{1}{a^2} = \boxed{\frac{e^{a}}{a} - \frac{e^{a}}{a^2} + \frac{1}{a^2}}$$

$$B = \int_1^2 (2-t) e^{-j\pi n t} dt$$

$$= 2\int_1^2 e^{-j\pi n t} dt - \int_1^2 t e^{-j\pi n t} dt = C - D$$

$$C = 2\int_1^2 e^{-j\pi n t}\,dt = 2\int_1^2 e^{at}\,dt \quad , a = -j\pi n$$

$$= 2\,\frac{1}{a} e^{at}\Big|_1^2 = \boxed{\frac{2}{a}\left(e^{2a} - e^{a}\right)}$$

$$D = \int_1^2 t e^{-j\pi n t}\,dt = \int_1^2 t e^{at}\,dt \quad , a = -j\pi n$$

$$= \left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at}\Big|_1^2 = \boxed{\left(\frac{2}{a} - \frac{1}{a^2}\right)e^{2a} - \left(\frac{1}{a} - \frac{1}{a^2}\right)e^{a}}$$

$$B = \int_1^2 (2-t)e^{at}\,dt = \frac{2}{a}\left(e^{2a} - e^{a}\right) - \left(\frac{2}{a} - \frac{1}{a^2}\right)e^{2a}$$

$$+ \left(\frac{1}{a} - \frac{1}{a^2}\right)e^{a} = \frac{2}{a}e^{2a} - \left(\frac{2}{a} - \frac{1}{a^2}\right)e^{2a}$$

$$+ \left(\frac{1}{a} - \frac{1}{a^2}\right)e^{a} - \frac{2}{a}e^{a} = \frac{1}{a^2}e^{2a} + \left(\frac{-1}{a} - \frac{1}{a^2}\right)e^{a}$$

$$= \boxed{\frac{1}{a^2}e^{2a} - \left(\frac{1}{a} + \frac{1}{a^2}\right)e^{a}}$$

$$\Rightarrow x_n = \frac{1}{2}(A+B) = \frac{1}{2}\left[\frac{e^a}{a} - \frac{e^a}{a^2} + \frac{1}{a^2} + \frac{1}{a^2}e^{2a} - \left(\frac{1}{a} + \frac{1}{a^2}\right)e^a\right]$$

Also, $A+B = \underbrace{\frac{e^a}{a}} - \underbrace{\frac{e^a}{a^2}} + \frac{1}{a^2} + \frac{1}{a^2}e^{2a} - \underbrace{\frac{e^a}{a}} - \underbrace{\frac{e^a}{a^2}}$

$$= \frac{1}{a^2}(e^{2a}+1) - \frac{2e^a}{a^2}$$

$$\Rightarrow x_n = \frac{1}{2}\left[\frac{e^{2a}+1}{a^2} - \frac{2e^a}{a^2}\right], \quad a = -j\pi n$$

and $a^2 = -\pi^2 n^2$

$$\Rightarrow x_n = \frac{1}{2a^2}\left[e^{2a} - 2e^a + 1\right]$$

$$= \frac{-1}{2\pi^2 n^2}\left[e^{-2j\pi n} - 2e^{-j\pi n} + 1\right]$$

$$\Rightarrow x_n = \frac{-1}{2\pi^2 n^2}\left[e^{-2j\pi n} - 2e^{-jn\pi} + 1\right]$$

(*) Also $e^{-2j\pi n} = \cos(2\pi n) - j\sin 2\pi n = \cos 2\pi n$

$e^{-j\pi n} = \cos(\pi n) - j\sin \pi n = \cos \pi n$

$\Rightarrow e^{-2j\pi n} - 2e^{-j\pi n} = \cos 2\pi n - 2\cos \pi n$

$\Rightarrow x_n = \frac{-1}{2\pi^2 n^2}\left[\cos 2\pi n - 2\cos \pi n + 1\right]$

$$\Rightarrow \boxed{x_n = \frac{1}{2\pi^2 n^2}\left[2\cos n\pi - \cos 2\pi n - 1\right]} \quad \text{(real)}$$

$\Rightarrow |x_n| = \left|\frac{1}{2\pi^2 n^2}\left[2\cos n\pi - \cos 2\pi n - 1\right]\right|$

$$x_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 (2-t)\,dt\right]$$

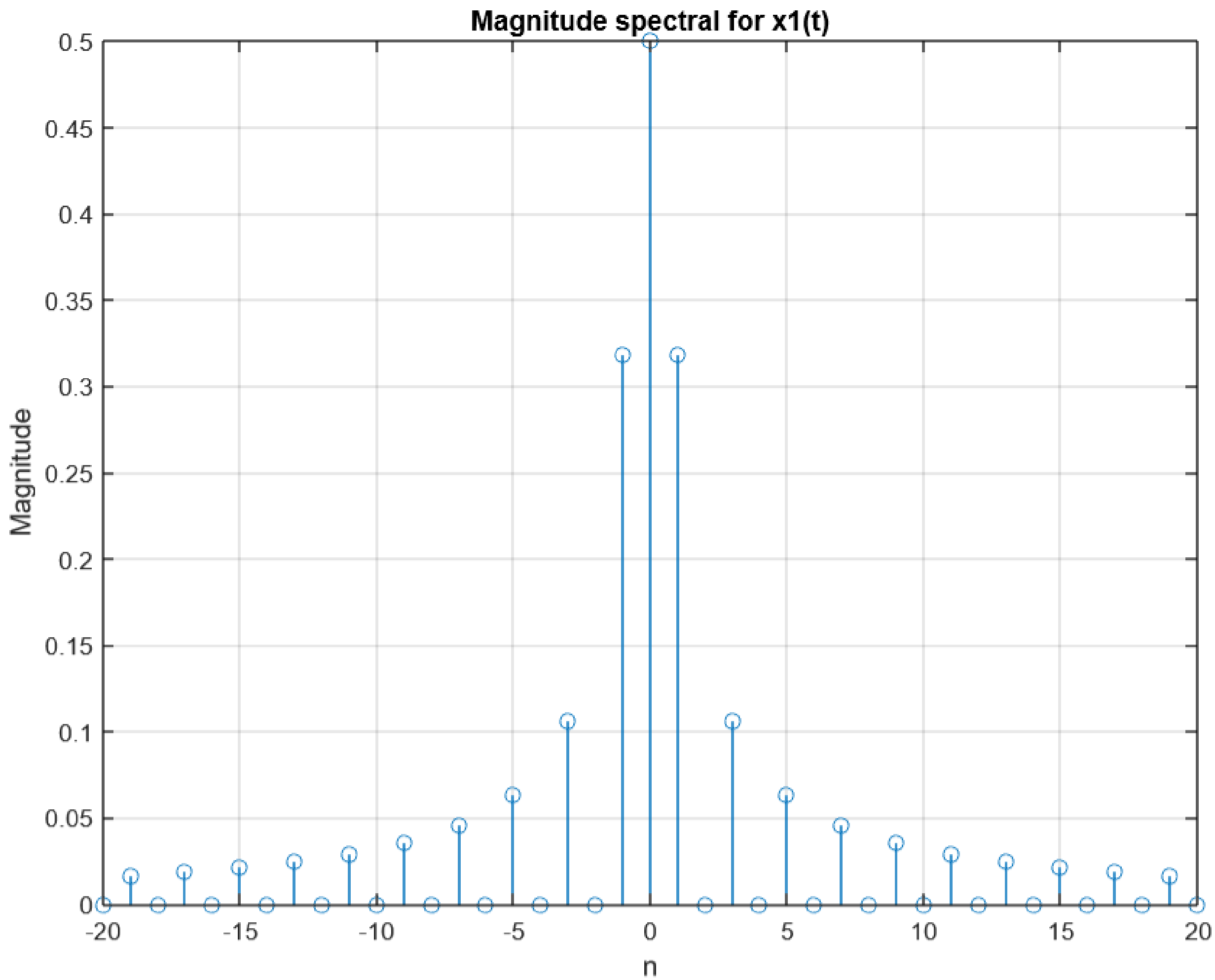
$$= \frac{1}{2}\left[\frac{t^2}{2}\Big|_0^1 + \left(2t - \frac{t^2}{2}\right)\Big|_1^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + (4-2) - \left(2 - \frac{1}{2}\right)\right] = \frac{1}{2}$$

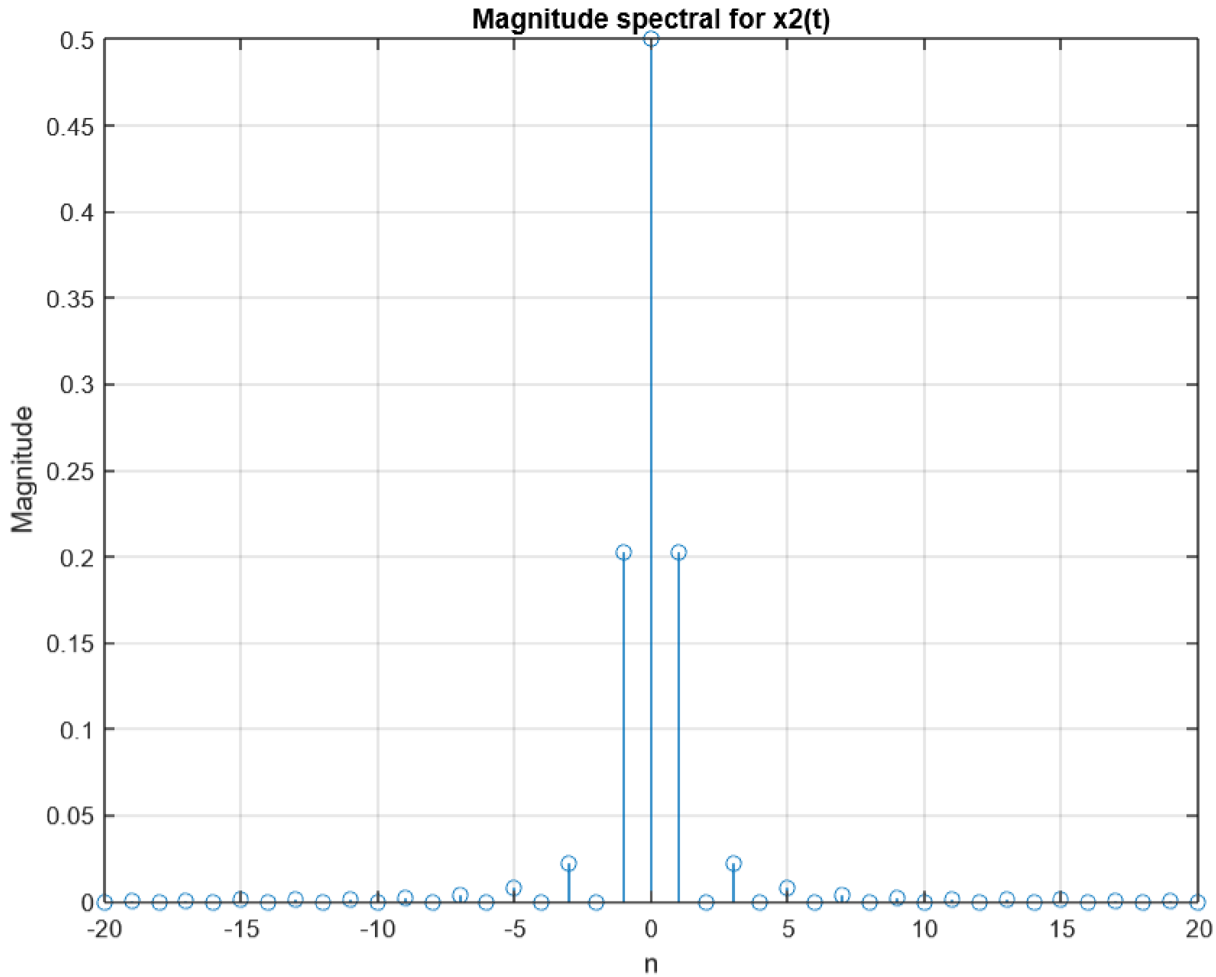
$$\Rightarrow \boxed{x_0 = \frac{1}{2}}$$

Using Matlab:

For $x_1(t)$:

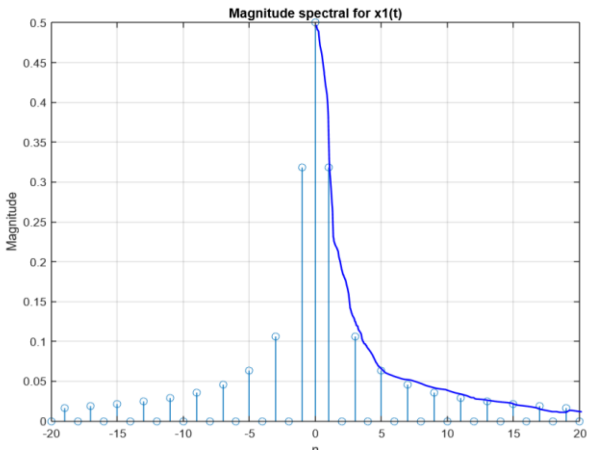


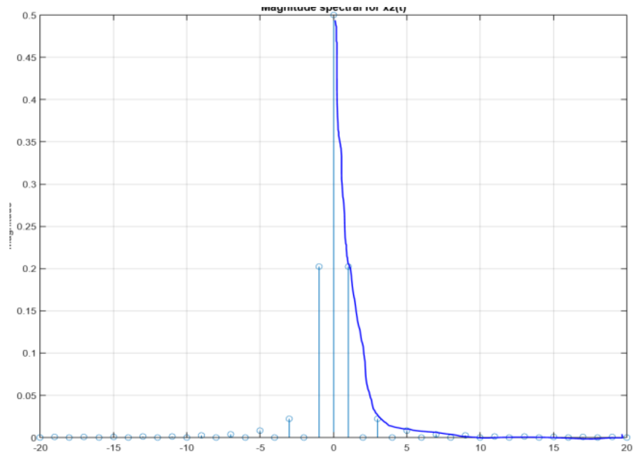
For $x_2(t)$:



Based on the paragraph, for the x2(t), the magnitude spectra decrease or decay faster than the magnitude spectra of x1(t). Specifically, $x_2(t)$'s magnitude started decreasing from ½ to 0.2 respectively n = 0, and n = 1 than immediately reach 0.025 at n = 3, and after that, the value is approximately 0. While $x_1(t)$'s magnitude spectra gradually decrease to 0 when k is increased. It started decreasing from ½ -> 0.32 -> 0.12 -> 0.07 -> 0.04 -> 0.03 -> 0.02 -> …….. -> 0. We can see the picture below to see easily the $x_2(t)$'s magnitude is decayed faster than $x_1(t)$ when k is increased.

$|X_1|$ $|X_2|$





Finally, the $|X_2|$ decay faster to 0 than the $|X_1|$, thus $x_2(t)$ is smoother than $x_1(t)$.

Coding part:

```
clc;
close all;
clear all;

%%Spectral for x1(t)
figure(1);
%Average of x1(t)
n0 = 1/2;
n1 = linspace(-20,-1,20);
n2 = linspace(1,20,20);
y1 = abs( (cos(pi*n1) - 1) ./ (2*pi*n1) );
y2 = abs( (cos(pi*n2) - 1) ./ (2*pi*n2) );
%Add array
X = [n1, 0, n2];
Y = [y1,n0,y2];

stem(X,Y);
grid;
xlabel('n');
ylabel('Magnitude');
title('Magnitude spectral for x1(t)');
```

```
%%Spectral for x2(t)
figure(2);
%Average of x2(t)
n0 = 1/2;
n1 = linspace(-20,-1,20);
n2 = linspace(1,20,20);
y1 = abs( (2*cos(pi*n1) - cos(2*pi*n1) - 1) ./ (2*(pi*n1).^2 ) );
y2 = abs( (2*cos(pi*n2) - cos(2*pi*n2) - 1) ./ (2*(pi*n2).^2 ) );
%Add array
X = [n1, 0, n2];
Y = [y1,n0,y2];

stem(X,Y);
grid;
xlabel('n');
ylabel('Magnitude');
title('Magnitude spectral for x2(t)');
```