Lecture 6.

* In Person class next Monday Jan 31.
* HW #3 2 problems w/ MATLAB

Laplace Transform (LTI, C)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

Not multiplication

$*$ is just a notation for convolution integral.

Last time:

<u>complex</u> exponential signal

$$e^{s_0 t} \longrightarrow \boxed{h(t)} \longrightarrow e^{s_0 t} \cdot H(s_0)$$

$s_0 \in \mathbb{C}$

$s_0 = \sigma_0 + j\Omega_0$

from convol.: $H(s_0) = \int_{-\infty}^{+\infty} h(\tau) e^{-s_0 \tau} d\tau$

$$\boxed{h(t) \longrightarrow H(s)}$$

LTI

$$\sum_k A_k e^{s_k t} \longrightarrow \boxed{H(s)} \longrightarrow \sum_k A_k H(s_k) e^{s_k t}$$

Laplace - Transform

$$f(t) \xrightarrow{\mathcal{L}_s} F(s) \triangleq \int_{-\infty}^{+\infty} f(t) e^{-st} dt$$

$t \in \mathbb{R}$

Two-sided Laplace Transf.

Is this Transform defined for all $s \in \mathbb{C}$

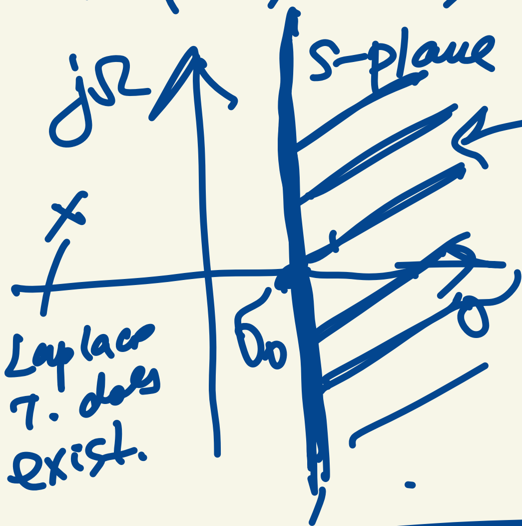
Not always: $F(s)$ needs to converge (integral needs to converge).

$s \in ROC$ Region of convergence.

Inverse Laplace Transform

"maping $F(s)$ back to $f(t)$"

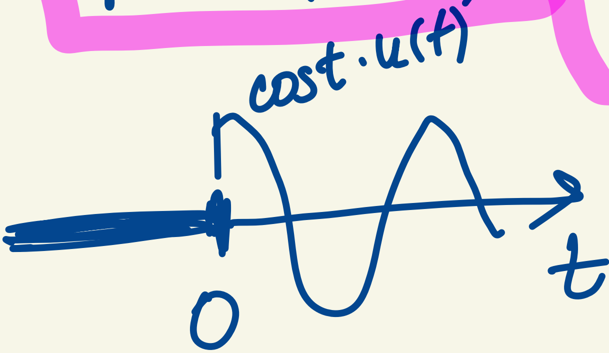
$$F(s) \xrightarrow{\mathcal{L}_s^{-1}} f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)\,e^{st}\,ds$$

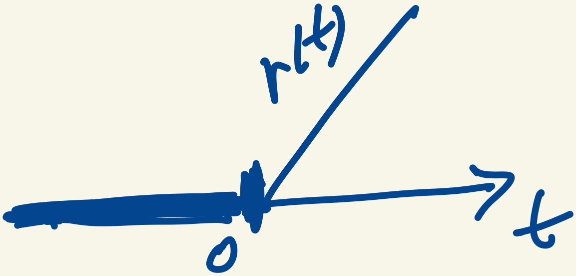


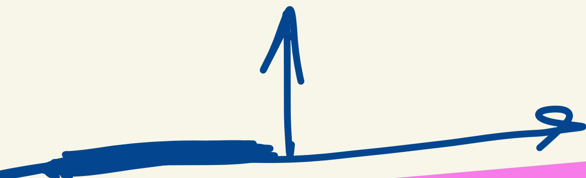
$\sigma_0 \in ROC$

In this class we are going to use one-sided Laplace Transform Applies to signals that are of the form.

$f(t)\cdot u(t)$ ← causal signals





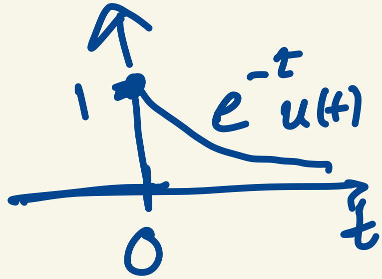


$$f(t)\cdot u(t) \xrightarrow{L_s} F(s)=\int_{0^-}^{+\infty} f(t)e^{-st}\,dt$$

$s \in ROC$

Example.

$$f(t) = e^{-t} \cdot u(t)$$



$$F(s) = \int_{0^-}^{+\infty} e^{-t} u(t) \cdot e^{-st} \, dt$$

$$\left\{ \int e^{at} \, dt = \frac{1}{a} e^{at} \right|$$

$$= \int_{0^-}^{+\infty} e^{\underbrace{-(s+1)}_{a} t} \, dt$$

detour

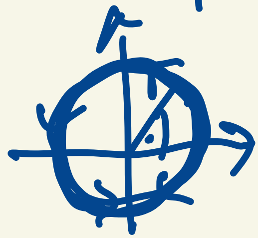
ROC check $s = \sigma + j\Omega$

$$= \int_{0^-}^{+\infty} e^{-(\sigma + j\Omega + 1)t} \, dt$$

$$= \int_{0^-}^{+\infty} \underbrace{e^{-(\sigma+1)t}} \cdot \underbrace{e^{-j\Omega t}} \, dt$$

converges
"decaying exponential"

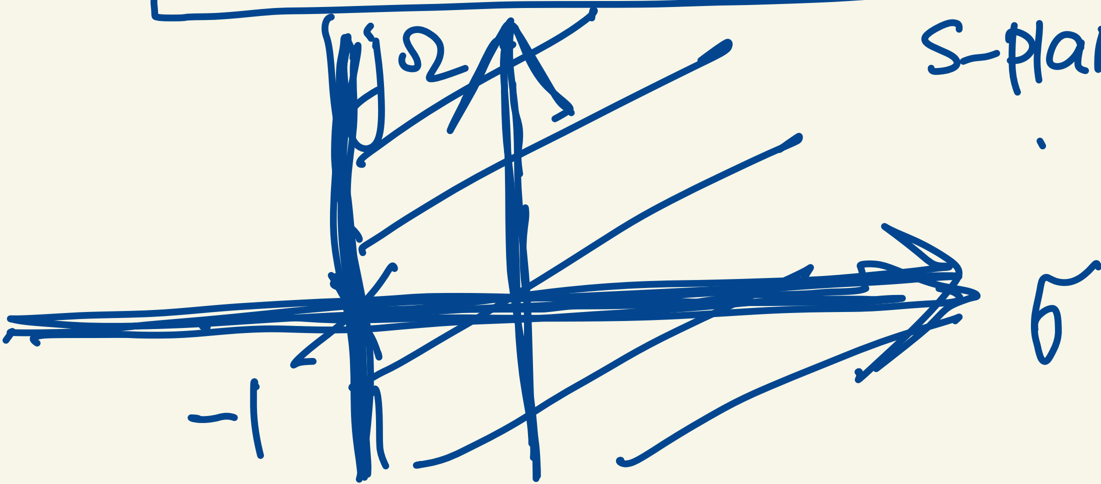
$$|e^{-j\Omega t}| = 1$$



if $\sigma + 1 > 0$

$$\mathrm{Re}\{s\} + 1 > 0$$

$$\boxed{\mathrm{Re}\{s\} > -1 \quad \text{ROC}}$$



coming back to solve $\int$

$$= -\frac{1}{s+1} e^{-(s+1)t} \Big|_{0^-}^{+\infty}$$

$$= 0 - \left(-\frac{1}{s+1} e^{-(s+1)\cdot 0^-} \right)$$

$$= \frac{1}{s+1}$$

$$e^{-t} u(t) \xrightarrow{\mathcal{L}s} \frac{1}{s+1}$$

$$Re\{s\} > -1$$

$$e^{-2t}u(t) \xrightarrow{LS} \frac{1}{s+2}$$

$$Re\{s\} > -2$$

$$e^{-at}u(t) \to \frac{1}{s+a}$$

$$Re\{s\} > -a$$

Causal signals in 102 will have Laplace T. in the form of fractional polynomial.

$$f(t)\cdot u(t) \longrightarrow F(s) = A\frac{(s-a_0)\cdots(s-a_n)}{(s-b_0)\cdots(s-b_m)}$$

(hint review polynomial factorization)

$$e^{-t}u(t) \rightarrow F(s) = \frac{1}{s+1}$$

$a_0, a_1, \ldots, a_n$ are zeros of Laplace T.

$b_0, b_1, \ldots, b_m$ are poles of Laplace T

Properties of ROC for Laplace T in fractional polynomial form.

① ROC is always a plane bounded by a line parallel to $j\Omega$ axis!

② no poles are inside ROC

③ If there are multiple poles $(\sigma_1, \Omega_1) \ldots (\sigma_m, \Omega_m)$

$$ROC = \{(\sigma, \Omega) : \sigma > \max \{\sigma_1, \ldots, \sigma_m\}\}$$

ROC is to the right of the pole with largest σ

$$F(S) = \frac{(s-2)(S+1)}{(\underline{s^2+1})(\underline{s-1})(\underline{s+2})}$$

$a_1 = 2$

$a_2 = -1$

poles:

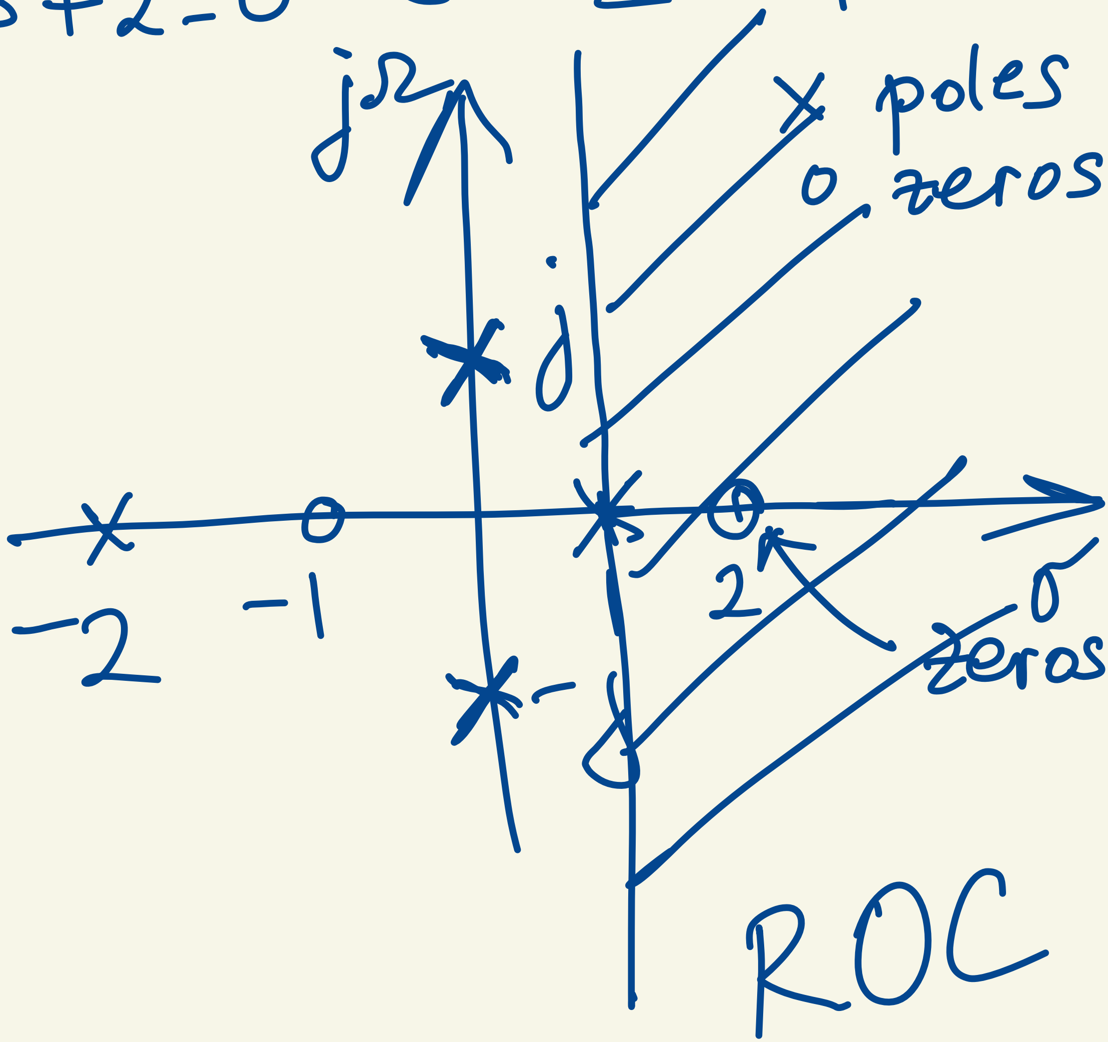
$s^2 + 1 = 0 \Rightarrow s^2 = -1$

$s_{1/2} = \pm\sqrt{-1}$

$b_1 = j \quad b_2 = -j \quad s_{1/2} = \pm j$

$s-1=0 \quad s=1 \quad b_3=1$

$s+2=0 \quad s=-2 \quad b_4=-2$



Some basic Laplace T.

(we will use Table of Laplace T.)

$\delta(t)\cdot u(t) = \delta(t)$

$\delta(t)$ is causal signal.

$\delta(t) \xrightarrow{L_S}$

$$L_S\{\delta(t)\} = \int_{0^-}^{+\infty} \delta(t) e^{-st} \, dt$$

$$= \int_{0^-}^{+\infty} \delta(t) \cdot e^{-s \cdot 0} \, dt$$

1

for all S

$$= \int_{0^-}^{+\infty} \delta(t)\,dt = 1$$

$$\delta(t) \xrightarrow{L_s} 1 \quad \text{all } s$$

$u(t) \rightarrow$ causal signal

$$L_s\{u(t)\} = \int_{0^-}^{+\infty} u(t)\, e^{-st}\,dt$$

$$= \int_{0^-}^{+\infty} e^{-st}\,dt \Rightarrow \text{detour}$$

$s = \sigma + j\Omega$

$$= -\frac{1}{s} e^{-st} \Big|_{0^-}^{+\infty}$$ do as HW. pract.

$$= 0 - \left(-\frac{1}{s} e^{-s \cdot 0} \right) = \frac{1}{s}$$

pole $s = 0$

ROC : $Re\{s\} > 0$

$$u(t) \xrightarrow{Ls} \frac{1}{s} \quad Re\{s\} > 0$$

$\cos(\Omega_0 t)u(t)$ causal signal.

Ω_0 is a real constant.

$$\mathcal{L}_S\{\cos(\Omega_0 t)u(t)\} =$$

$$= \int_{0^-}^{\infty} \cos(\Omega_0 t)e^{-st}\,dt$$

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

$$= \int_{0^-}^{+\infty} \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} e^{-st} \, dt$$

$$= \frac{1}{2}\int_{0^-}^{+\infty} e^{-(s-j\Omega_0)t} \, dt + \frac{1}{2}\int_{0^-}^{+\infty} e^{-(s+j\Omega_0)t} \, dt$$

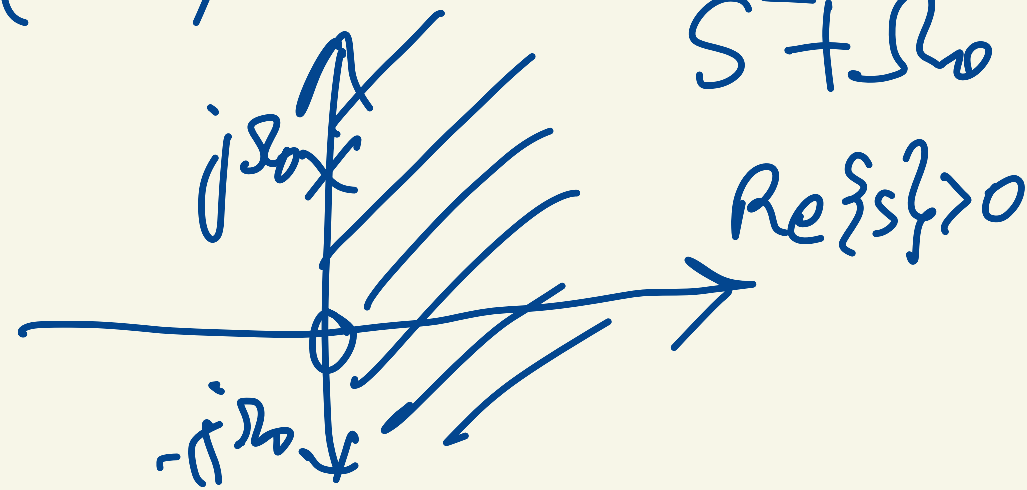
$$= \frac{1}{2} \frac{1}{s-j\Omega_0} + \frac{1}{2} \frac{1}{s+j\Omega_0}$$

$$= \frac{1}{2} \frac{s+\cancel{j\Omega_0}+s-\cancel{j\Omega_0}}{(s-j\Omega_0)(s+j\Omega_0)}$$

$$= \frac{\cancel{2}s}{\cancel{2}(s^2+\Omega_0^2)}$$

$$= \frac{s}{s^2+\Omega_0^2}$$

$$\cos(\Omega_0 t)u(t) \to \frac{s}{s^2+\Omega_0^2}$$



Practice

$\underline{\sin(\Omega_0 t)\,u(t)}$ causal signal

$$\mathcal{L}_S\{\sin(\Omega_0 t)\,u(t)\} =$$

$$= \int_{0^-}^{+\infty} \sin(\Omega_0 t)\, e^{-st}\, dt$$

$$\sin(\Omega_0 t) = \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}$$

Euler's

$$e^{j\Omega_0 t} = \cos(\Omega_0 t) + j\sin(\Omega_0 t)$$

$$e^{-j\Omega_0 t} = \cos(\Omega_0 t) - j\sin(\Omega_0 t)$$

$$= \frac{1}{2j}\int_{0^-}^{+\infty} e^{j\Omega_0 t} e^{-st}\,dt - \frac{1}{2j}\int_{0^-}^{+\infty} e^{-j\Omega_0 t} e^{-st}\,dt$$

reuse from cos

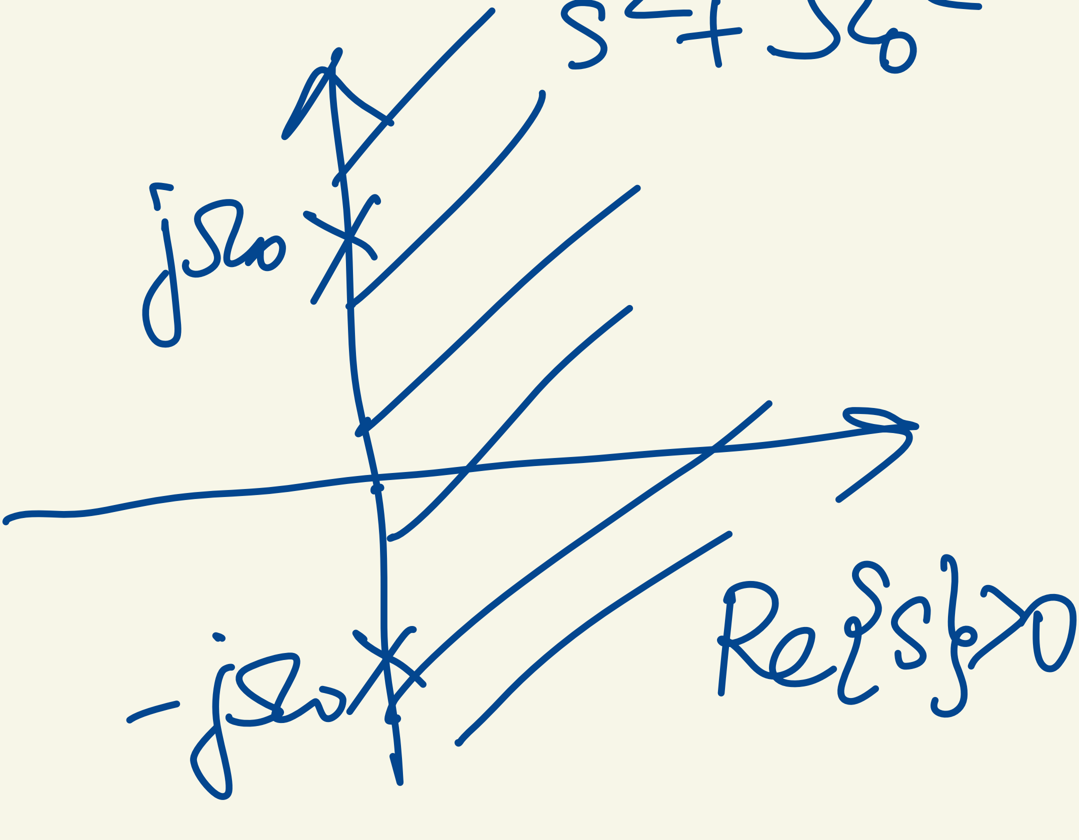
$$= \frac{1}{2j}\frac{1}{s - j\Omega_0} - \frac{1}{2j}\frac{1}{s + j\Omega_0}$$

$$= \frac{1}{2j}\frac{\cancel{s} + j\Omega_0 - \cancel{s} + j\Omega_0}{(s - j\Omega_0)(s + j\Omega_0)}$$

$$= \frac{\cancel{2j}\,\Omega_0}{\cancel{2j}(s^2 + \Omega_0^2)}$$

$$= \frac{\Omega_0}{s^2 + \Omega_0^2}$$

$$\sin(\Omega_0 t)u(t) \xrightarrow{\mathcal{L}_s} \frac{\Omega_0}{s^2+\Omega_0^2}$$



$t^n u(t)$ causal sig.

$n=1 \quad t \cdot u(t) = r(t)$

$n=2 \quad t^2 u(t) =$

$n=3 \quad t^3 u(t) \ldots$

$$L_s\{t^n u(t)\} = \begin{cases} u = t^n \\ \frac{du}{dt} = nt^{n-1} \end{cases}$$

$$= \int_{0^-}^{+\infty} \underbrace{t^n}_{u} \underbrace{e^{-st}\,dt}_{dv} = \underline{F_n(s)}$$

integration by parts.

$$\int_{0^-}^{+\infty} u\,dv = \underline{uv}\Big|_{0^-}^{+\infty} - \int_{0^-}^{+\infty} v\,\underline{du}$$

$$dv \Rightarrow e^{-st}\,dt$$

$$v \quad \int e^{-st}\,dt = -\frac{e^{-st}}{s}$$

$$F_n(s) = t^n \left(-\frac{e^{-st}}{s}\right)\Big|_{0^-}^{+\infty} \cancel{}$$

$$-\int_{0^-}^{+\infty} \left(-\frac{e^{-st}}{s}\right) \cdot n t^{n-1}\,\underline{dt}$$

$$= + \frac{n}{s} \underbrace{\int_{0^-}^{+\infty} t^{n-1} e^{-st}\, dt}_{F_{n-1}(s)}$$

$$F_n(s) = \frac{n}{s} F_{n-1}(s)$$

$$F_1(s) = \mathcal{L}s\{t\,u(t)\}$$

$$= \frac{1}{s} \cdot \underline{F_0(s)}$$

$$F_0(S) = \mathcal{L}_S\{t^0 \cdot u(t)\}$$

$$= \mathcal{L}_S\{u(t)\}$$

$$= \frac{1}{S}$$

$$F_1(S) = \frac{1}{S} \cdot \frac{1}{S} = \frac{1}{S^2}$$

$$F_2(S) = \mathcal{L}_S\{t^2 u(t)\}$$

$$F_2(s) = \frac{2}{s} \cdot \underline{F_1(s)}$$

$$= \frac{2}{s} \frac{1}{s^2}$$

$$= \frac{2}{s^3}$$

$$F_n(s) = \frac{n!}{s^{n+1}}$$

$$t^n u(t) \rightarrow \frac{n!}{s^{n+1}}$$

$$Re\{s\} > 0$$