HOMEWORK (5)

Question (1)

(a) $$y(t) = \int_{-\infty}^{t-1} e^{t} \cos(2(\tau - t) + 2)\, x(\tau)\, e^{-\tau + 2}\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{t-\tau} \cdot e^{2} \cdot \cos(2(1-(t-\tau)))}_{h(t-\tau)}\, x(\tau)\, \cancel{d\tau} \cdot u(-\tau + t - 1)\, d\tau$$

$$\Rightarrow \quad h(t) = e^{2} \cdot e^{t} \cos(2-t)\, u(t-1)$$

i) Causality:

$$h(t)\, u(t) = e^{2+t} \cos(2-t)\, u(t-1)\, u(t)$$

$$= e^{2+t} \cos(2-t)\, u(t-1)$$

$$= h(t)$$

$\therefore$ $\boxed{\text{Causal system}}$

ii) BIBO stability: The term e^{2+t} ~~is not as~~ does not converge upon integration from $t=1$ to $t=\infty$

$$\int_{+1}^{\infty} e^{2+t}\, dt \rightarrow \infty$$

$\therefore$ $\boxed{\text{Not BIBO stable}}$

(b)

$$y(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}[\cos t\cos\tau - \sin t\sin\tau]\,x(\tau)\,d\tau$$

$$= e^{-t}\int_{-\infty}^{t} e^{\tau}[\cos(t+\tau)]\,x(\tau)\,d\tau$$

$$\therefore\ h(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}\cos(t+\tau)\,\delta(\tau)\,d\tau$$

$$= e^{-t}\int_{-\infty}^{t} e^{\tau}\cos(t+\tau)\,u(-\tau+t)\,\delta(\tau)\,d\tau$$

$$\boxed{h(t) = e^{-t}\cos(t)\,u(t)}$$

i) $h(t) = h(t)\,u(t) \Rightarrow$ system is $\boxed{\text{causal}}$

ii)

$$\int_{-\infty}^{\infty}|e^{-t}\cos(t)\,u(t)|\,dt \leq \int_{-\infty}^{\infty}|e^{-t}|\,|\cos t|\,|u(t)|\,dt$$

$$\leq \int_{-\infty}^{\infty}|e^{-t}|\,u(t)\,dt$$

$$= \int_{0}^{\infty}e^{-t}\,dt < \infty$$

$$\therefore \int_{-\infty}^{\infty}|h(t)|\,dt < \infty \Rightarrow \boxed{\text{BIBO stable}}$$

$$y(t) = \int_{-\infty}^{t-1} e^{-(t-\tau)} \, x(\tau-2) \, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-(t-\tau)} \, x(\tau-2) \, u(-\tau+t-1) \, d\tau$$

$$h(t) = \int_{-\infty}^{\infty} e^{-(t-\tau)} \, \delta(\tau-2) \, u(-\tau+t-1) \, d\tau = \boxed{e^{-(t-2)} \, u(t-3)}$$

i) $h(t)\,u(t) = e^{-(t-2)} u(t-3) u(t) = e^{-(t-2)} u(t-3) = h(t)$

$\therefore$ $\boxed{\text{Causal}}$

ii) $$\int_{-\infty}^{\infty} \left| e^{-(t-2)} \, u(t-3) \right| dt = \int_{\cancel{-\infty}\ 0}^{\infty} e^{-(t-2)} \, dt < \infty$$

$\Rightarrow$ $\boxed{\text{BIBO stable}}$

Question (2)

$$y(t) = x(t)u(t) - \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)u(\tau)\, d\tau$$

$$w(t) = \int_{-\infty}^{t} y(\sigma)u(\sigma)\, d\sigma$$

(a) $\underline{S_1}$:

$$h_1(t,\tau) = \delta(t-\tau)u(t) - \int_{-\infty}^{t-2} e^{-(t-\sigma)} \underbrace{x(\sigma)}_{\delta(\sigma-\tau)} u(\sigma)\, d\sigma$$

$$= \delta(t-\tau)u(t) - \int_{-\infty}^{\infty} e^{-(t-\sigma)} \delta(\sigma-\tau)u(\sigma)u(-\sigma+t-2)\, d\sigma$$

$$= \delta(t-\tau)u(t) - e^{-(t-\tau)} u(\tau)u(t-\tau-2)$$

$$=$$

b) $\underline{S_2}$:

$$h_2(t,\tau) = \int_{-\infty}^{t} \cancel{y(\sigma)}\, \delta(\sigma-\tau)u(\sigma)\, d\sigma = \int_{-\infty} \delta(\sigma-\tau)u(\sigma)u(-\sigma+t)$$

$$\boxed{h_2(t,\tau) = u(t-\tau)u(\tau)}$$

(b)

$$h_{12}(t,\tau) = \int_{-\infty}^{\infty} h_1(\sigma,\tau)\, h_2(t,\sigma)\, d\sigma$$

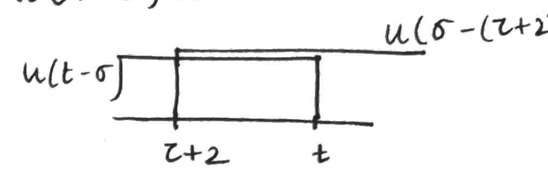
$$= \int_{-\infty}^{\infty} \left[\delta(\sigma-\tau)u(\sigma) - e^{-(\sigma-\tau)} u(\tau)u(\sigma-\tau-2)\right] \cdot \cancel{u(t-\sigma)u(\sigma)\, d\sigma}$$

$$\times\ u(t-\sigma)u(\sigma)\, d\sigma$$

$$= \int_{-\infty}^{\infty} \delta(\sigma-\tau)u(\sigma)u(t-\sigma)\, d\sigma - \int_{-\infty}^{\infty} e^{-(\sigma-\tau)} u(\tau)u(\sigma-\tau-2)\, u(t-\sigma)u(\sigma)\, d\sigma$$

$$h_{12} = u(\tau)\,u(t-\tau) - \underbrace{\int_{-\infty}^{\infty} e^{-(\sigma-\tau)}\, u(\tau)\, u(\sigma-\tau-2)\, u(t-\sigma)\, u(\sigma)\, d\sigma}_{I_0}$$

$$I_0 = -\int_{0}^{\infty} e^{-(\sigma-\tau)}\, u(\tau)\, u(\sigma-(\tau+2))\, u(t-\sigma)\, d\sigma$$



$u(\sigma-(\tau+2))\,u(t-\sigma) = 1$ only when $t \geq \tau+2$.

ie $\underline{\tau \leq t-2}$

and $\underline{\underline{\tau \geq 0}}$

$$= -\int_{\tau+2}^{t} e^{-(\sigma-\tau)}\, u(\tau)\, d\sigma = -\int_{\tau+2}^{t} e^{\tau} \cdot e^{-\sigma}\, d\sigma\, u(\tau)$$

$$= e^{\tau} u(\tau)\left[e^{-t} - e^{-\tau-2}\right] = u(\tau)\left[e^{-(t-\tau)} - e^{-2}\right]$$

$$\therefore \boxed{h_{12} = u(\tau)\left[u(t-\tau) + e^{-(t-\tau)} - e^{-2}\right]}$$

(c) <u>For BIBO stability</u>:-

$$\int_{-\infty}^{\infty} \left| u(\tau)\left[u(t-\tau) + e^{-(t-\tau)} - e^{-2}\right]\right| dt$$

$$\leq |u(\tau)| \int_{-\infty}^{\infty} \left| u(t-\tau) + e^{-(t-\tau)} - e^{-2}\right| dt$$

$$= \underbrace{u(\tau) \int_{-\infty}^{\infty} u(t-\tau)\, dt}_{\text{converges only when } \cancel{\tau \leq 0}\ \tau<0} + \underbrace{\int_{-\infty}^{\infty} u(\tau)\left(e^{-(t-\tau)} - e^{-2}\right) dt}_{\text{converges as long as } \cancel{\tau \leq t}\ \tau<t}$$

$\therefore$ <u>Not BIBO stable</u>

)

$$h_{21}(t,\tau) = \int_{-\infty}^{\infty} h_2(\sigma,\tau)\, h_1(t,\sigma)\, d\sigma$$

$$= \int_{-\infty}^{\infty} u(\sigma-\tau)u(\tau)\left[\delta(t-\sigma)u(t) - e^{-(t-\sigma)}u(\sigma)u(t-\sigma-2)\right] d\sigma$$

$$= \int_{-\infty}^{\infty} u(\sigma-\tau)\, u(\tau)\, u(t)\, \delta(t-\sigma)\, d\sigma$$

$$- \int_{-\infty}^{\infty} u(\sigma-\tau)\, u(\tau)\, e^{-(t-\sigma)} u(\sigma)\, u(t-\sigma-2)\, d\sigma$$

$$= u(t-\tau)u(\tau)u(t) - e^{-t}\int_{-\infty}^{\infty} e^{\sigma}\, u(\sigma-\tau)\, u(\tau)\, u(\sigma)\, u(t-\sigma-2)\, d\sigma$$

$$= u(t-\tau)\, u(\tau) - I_0$$

$\because$ $u(t)u(t-\tau) = u(t-\tau)$ $\forall\ \tau \geq 0$.

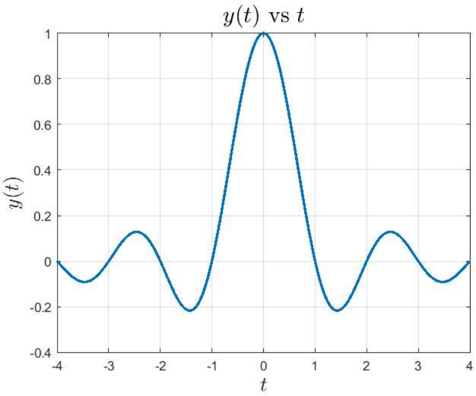
$$I_0 = e^{-t}\int_{-\infty}^{\infty} e^{\sigma}\, u(\tau)\, u(\sigma-\tau)\, u(t-\sigma-2)\, d\sigma$$

$$= e^{-t}\int_{\tau}^{t-2} e^{\sigma}\, d\sigma \cdot u(\tau) \qquad \text{if } t-2$$

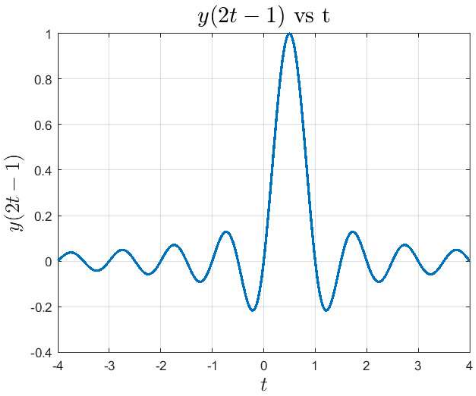
$$= e^{-t}\cdot\left(e^{t-2} - e^{\tau}\right)\cancel{u(\tau)} = \left(e^{-2} - e^{\tau-t}\right) u(\tau)$$

$$\therefore\quad h_{21}(t,\tau) \begin{cases} = u(\tau)u(t-\tau) + u(\tau)\left[e^{-2} - e^{\tau-t}\right] & t-2 \geq \tau \\ = u(\tau)u(t-\tau) & t-2 < \tau \end{cases}$$

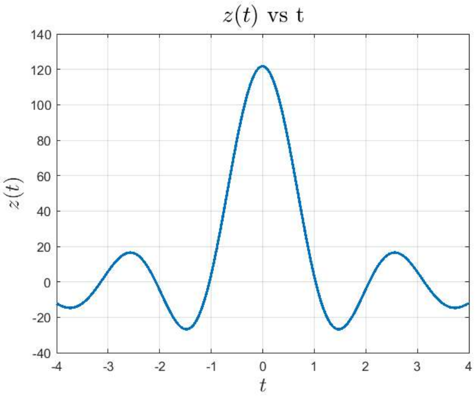
Matlab assignment 1



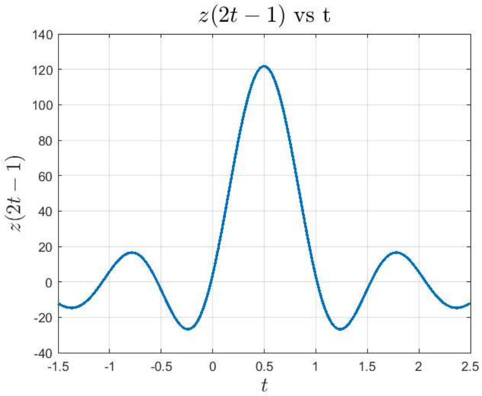
(a)



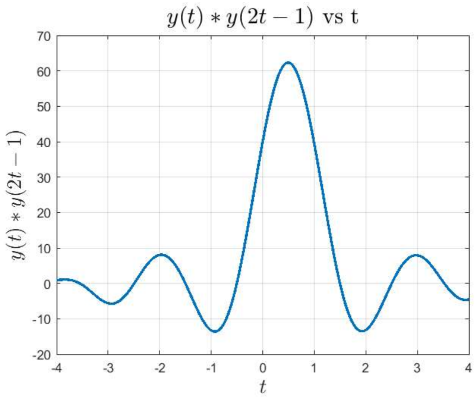
(b)



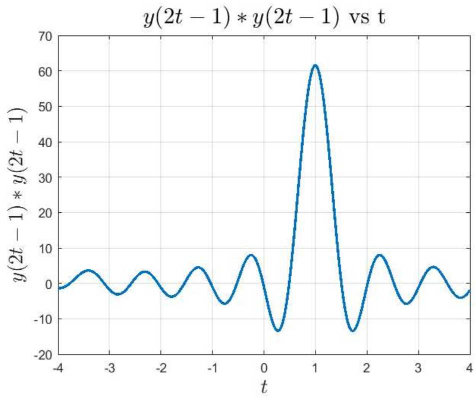
(c)



(d)



(e)



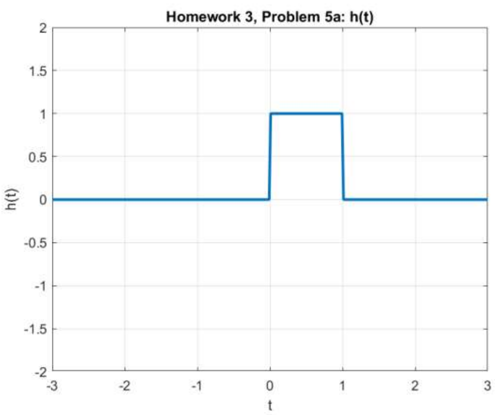
(f)

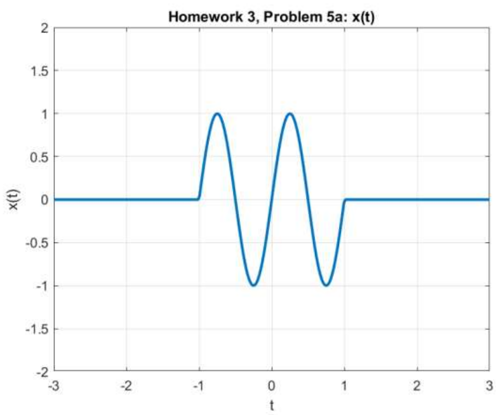
Problem 5:
For the following problems, include screenshots/images of your MATLAB figures. You should have 5 total figures for this problem. Also, copy and paste your MATLAB code into your homework.

a) Using MATLAB, plot $x(t)$ and $h(t)$ individually over the range $t \in [-3, 3]$.
$$x(t) = \sin(2\pi t)\, u(t+1)u(-t+1)$$
$$h(t) = u(t) - u(t-1)$$
b) Solve for $y_1(t) = x(t) * h(t)$ (where $*$ is the convolution operator) analytically as a piecewise function.
c) Solve for $y_2(t) = x(t) \cdot h(t)$ (where $\cdot$ is the multiplication operator) analytically.
d) Using MATLAB, plot $y_1(t)$ and $y_2(t)$ individually over the range $t \in [-3, 3]$. Are $y_1(t)$ and $y_2(t)$ the same?
e) Using MATLAB, numerically convolve $x(t)$ and $h(t)$ and plot the result over the same range. Confirm that the result is the same as $y_1(t)$.

Solution:

a) The plots are shown below:





b) We use the definition of the convolution integral to solve:
$$\begin{aligned} y_1(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} [\sin(2\pi\tau)\, u(\tau+1)u(-\tau+1)][u(t-\tau) - u(t-\tau-1)]d\tau \\ &= \int_{-1}^{1} \sin(2\pi\tau)\, [u(t-\tau) - u(t-\tau-1)]d\tau \\ &= \int_{-1}^{1} \sin(2\pi\tau)\, u(t-\tau)d\tau - \int_{-1}^{1} \sin(2\pi\tau)\, u(t-\tau-1)d\tau \end{aligned}$$
We can then split the computation into two parts: $y_1(t) = a(t) - b(t)$

$$a(t) = \int_{-1}^{1} \sin(2\pi\tau)\, u(t-\tau) d\tau = \begin{cases} \int_{-1}^{1} \sin(2\pi\tau)\, d\tau, & t > 1 \\ \int_{-1}^{t} \sin(2\pi\tau)\, d\tau, & 1 \geq t > -1 \\ 0, & t \leq -1 \end{cases}$$

$$= \begin{cases} 0, & t > 1 \\ -\frac{1}{2\pi}(\cos(2\pi t) - 1), & 1 \geq t > -1 \\ 0, & t \leq -1 \end{cases}$$

$$b(t) = \int_{-1}^{1} \sin(2\pi\tau)\, u(t-\tau-1) d\tau = \begin{cases} \int_{-1}^{1} \sin(2\pi\tau)\, d\tau, & t-1 > 1 \\ \int_{-1}^{t-1} \sin(2\pi\tau)\, d\tau, & 1 \geq t-1 > -1 \\ 0, & t-1 \leq -1 \end{cases}$$

$$= \begin{cases} 0, & t > 2 \\ -\frac{1}{2\pi}(\cos(2\pi t) - 1), & 2 \geq t > 0 \\ 0, & t \leq 0 \end{cases}$$

Combined, we find a piecewise result for the total $y_1(t)$:

$$y_1(t) = \begin{cases} 0, & t > 2 \\ \frac{1}{2\pi}(\cos(2\pi t) - 1), & 2 \geq t > 1 \\ 0, & 1 \geq t > 0 \\ -\frac{1}{2\pi}(\cos(2\pi t) - 1), & 0 \geq t > -1 \\ 0, & t \leq -1 \end{cases}$$

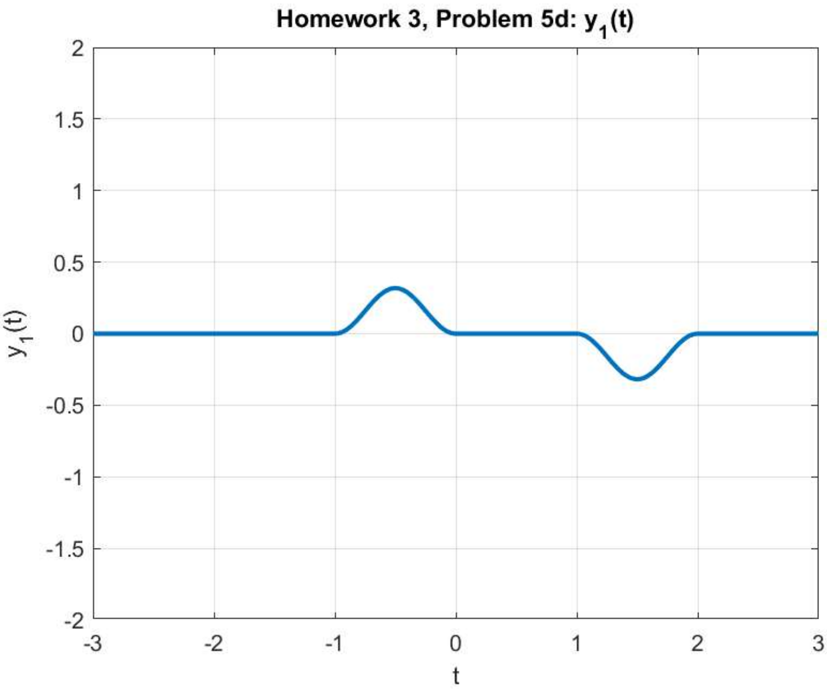
As a single equation, we can also show the result as:

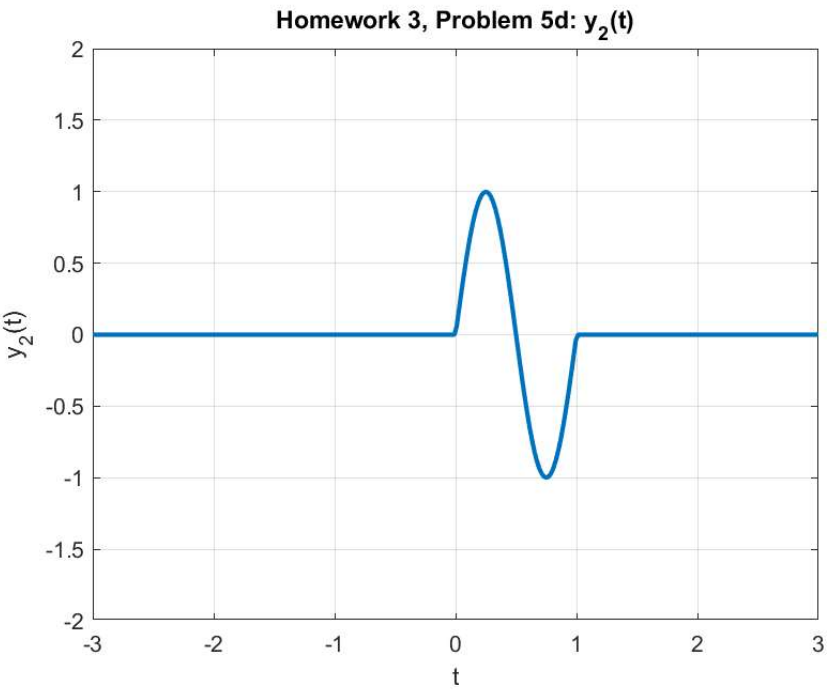
$$y_1(t) = \frac{1}{2\pi}(\cos(2\pi t) - 1)\left[\big(u(t-1) - u(t-2)\big) - \big(u(t+1) - u(t)\big)\right]$$

c) This problem is doing a simple multiplication, so the analytical result is just the two functions multiplied:

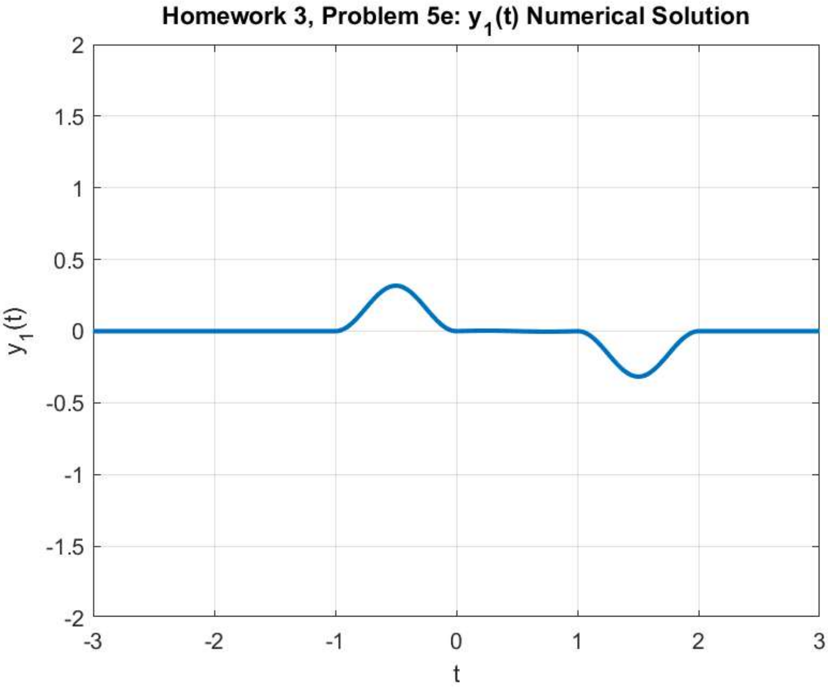
$$\begin{aligned} y_2(t) &= [\sin(2\pi t)\, u(t+1)u(-t+1)][u(t) - u(t-1)] \\ &= \sin(2\pi t)\left[[u(t+1)u(-t+1)u(t)] - [u(t+1)u(-t+1)u(t-1)]\right] \\ &= \sin(2\pi t)\left[[u(-t+1)u(t)] - 0\right] \\ &= \sin(2\pi t)\, u(-t+1)u(t) \end{aligned}$$

d) The plots are shown below. These results are not the same and usually should not be the same for general $x(t)$ and $h(t)$.





e) The plot is shown below. As expected, it is nearly the same as the result for the convolution in part d. Note that the center region (between 0 and 1) is not entirely flat, but that is likely due to small numerical inconsistencies for the evaluation of the sine function. We can reduce the impact of these numerical artifacts by using more points to evaluate each function and their convolution.


Homework 3, Problem 5e: y1(t) Numerical Solution
y1(t)
t
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2
-1
0
1
2
3