Home Work 6

1)

$$\frac{dy(t)}{dt}+\int_0^t 2y(t-\tau)e^{-\tau}d\tau=\frac{dx(t)}{dt}-x(t),\ t>0$$

$y(0)=0, x(t)=y(t)=0$ for $t<0$

a) We have:

$$L_s\left\{\frac{dy(t)}{dt}\right\}=sY(s)-y(0^-)=sY(s) \quad (1)$$

$$(*)\ \int_0^t 2y(t-\tau)e^{-\tau}d\tau=\int_{-\infty}^{+\infty}\underbrace{2y(t-\tau)u(t-\tau)}\,\underbrace{e^{-\tau}u(\tau)}d\tau$$

Let $\begin{cases} h(t-\tau)=2y(t-\tau)u(t-\tau)\\ x(\tau)=e^{-\tau}u(\tau)\end{cases}$

$$\Rightarrow f(t)=\int_0^t 2y(t-\tau)e^{-\tau}d\tau=\int_{-\infty}^{+\infty}h(t-\tau)\,x(\tau)\,d\tau$$

Since LTI causal system, then

$$f(t)=x(t)*h(t)=[\underbrace{e^{-t}u(t)}_{x(t)}]*[\underbrace{2y(t)u(t)}_{h(t)}]$$

$$F(s)=L_s\left\{\int_0^t 2y(t-\tau)e^{-\tau}d\tau\right\}=X(s).H(s)$$

(+) Check $x(t) = e^{-t} u(t)$

$$\Rightarrow x(t) = e^{-t} u(t) \xrightarrow{L_s} X(S) = \frac{1}{S+1} \qquad Re\{S\} > -1$$

(*) Check $h(t) = 2y(t) \cdot u(t)$

$$\Rightarrow H(S) = 2Y(S)$$

$$\Rightarrow F(S) = X(S) \cdot H(S) = \frac{1}{S+1} \cdot 2 \cdot Y(S) = \frac{2}{S+1} \cdot Y(S) \quad (2)$$

$Re\{S\} > -1$

From (1) & (2), We have:

$$L_S\{LHS\} = SY(S) + \frac{2}{S+1} \cdot Y(S)$$

$$= \left[S + \frac{2}{S+1}\right] Y(S) = \frac{S^2+S+2}{S+1} \cdot Y_S$$

(*) Also, $L_S\left\{\frac{dx(t)}{dt} - x(t)\right\} = SX(S) - X(0^-) - X(S)$
$(t>0)$

$$= SX(S) - X(S) = (S-1)X(S) \quad (4)$$

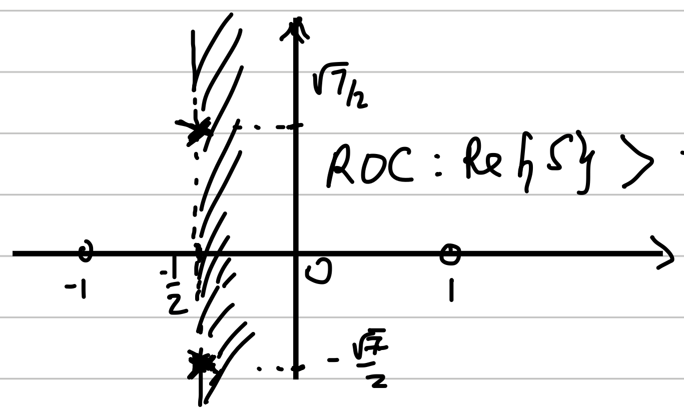
From ③ & ④, we have:

$$L_s\{LHS\} = L_s\{RHS\}$$

$$\Rightarrow \frac{s^2+s+2}{s+1} \cdot Y_s = (s-1)\,X(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{(s-1)}{\dfrac{s^2+s+2}{s+1}} = \frac{(s-1)(s+1)}{s^2+s+2}$$

* Check $s^2+s+2=0 \Leftrightarrow \left[\begin{array}{l} s = -\frac{1}{2} + \frac{\sqrt{7}}{2}j \\ s = -\frac{1}{2} - \frac{\sqrt{7}}{2}j \end{array}\right.$



ROC: $Re\{s\} > -\frac{1}{2}$ included $j\omega$ axis

$\Rightarrow$ this system is BIBO system

$$\boxed{\Rightarrow H(s) = \frac{(s-1)(s+1)}{s^2+s+2} \quad Re\{s\} > -\frac{1}{2}}$$

b) Applied $x(t) = |\cos^2(t)| + 1 = \cos^2(t) + 1$

(since $\cos^2(t) \geq 0 \ \forall t$)

$$= \frac{1+\cos(2t)}{2} + 1 = \frac{1}{2}\cos(2t) + \frac{3}{2}$$

We can see $x(t)$ is periodic with $\omega_0 = 2$

Also, $x(t) = \frac{1}{2}\left[\frac{1}{2}\left(e^{j2t} + e^{-j2t}\right)\right] + \frac{3}{2}$

$$= \frac{3}{2} + \frac{1}{4}e^{j2t} + \frac{1}{4}e^{-2jt}$$

Generally, $x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{j\omega_0 kt} = \sum_{k=-\infty}^{+\infty} X_k e^{j2kt}$

$$\Rightarrow X_0 = \frac{3}{2}, \quad X_1 = \frac{1}{4}, \quad X_{-1} = \frac{1}{4}$$

Applying for LTI causal - BIBO system, we have:

$$\tilde{y}(t) = \sum_{k=-\infty}^{+\infty} X_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} X_k H(2jk) e^{2jkt}$$

$$\Rightarrow Y_0 = X_0 H(0) = \frac{3}{2}\left[\left.\frac{(S-1)(S+1)}{S^2+S+2}\right|_{S=0}\right]$$

$$= \frac{3}{2} \cdot \frac{-1}{2} = \boxed{\frac{-3}{4}}$$

$$Y_1 = X_1 H(j\omega_0) = X_1 H(2j)$$

We have $H(2j) = \left.\frac{(s-1)(s+1)}{s^2+s+2}\right|_{s=2j} = \left.\frac{s^2-1}{s^2+s+2}\right|_{s=2j}$

$$= \frac{(2j)^2-1}{(2j)^2+2j+2} = \frac{-4-1}{-4+2j+2} = \frac{-5}{-2+2j} = \frac{5}{2-2j}$$

$$\Rightarrow Y_1 = \frac{1}{4}\cdot\frac{5}{2-2j} = \frac{5}{8(1-j)} = \boxed{\frac{5+5j}{16}}$$

$$Y_{-1} = X_{-1} H(-2j) = \frac{1}{4}\cdot H(-2j).$$

Also: $H(-2j) = \left.\frac{s^2-1}{s^2+s+2}\right|_{s=-2j} = \frac{(-2j)^2-1}{(-2j)^2-2j+2}$

$$= \frac{-4-1}{-4-2j+2} = \frac{-5}{-2-2j} = \frac{5}{2+2j}$$

$$\Rightarrow Y_{-1} = \frac{1}{4}\cdot\frac{5}{2+2j} = \frac{5}{8(1+j)} = \boxed{\frac{5-5j}{16}}$$

$\boxed{\text{(and other } Y_k = 0,\ k\in\mathbb{Z},\ k\neq 0,\pm 1)}$

$$\tilde{y}(t) = -\frac{3}{4} + \frac{5+5j}{16}\cdot e^{2jt} + \frac{5-5j}{16}\, e^{-2jt}$$

c) We have:

$$x(t) = \sum_{-\infty}^{+\infty} X_k e^{jw_0kt} \longrightarrow \boxed{\text{LTI causal}} \longrightarrow y(t) = \sum_{-\infty}^{+\infty} X_k H(jkw_0) e^{jkw_0t}$$

$$= \sum_{-\infty}^{+\infty} Y_k e^{jkw_0t}, \quad W_0 = 2$$

$$\Rightarrow \tilde{y}(t) = \sum_{-\infty}^{+\infty} Y_k e^{j2kt} = -\frac{3}{4} + \frac{5+5j}{16} e^{2jt} + \frac{5-5j}{16} e^{-2jt}$$

$\tilde{y}(t)$ has coefficient $Y_k = X_k H(jkw_0)$

Applying the Fourier propeties for coefficient, we have:

$\tilde{y}(t-3)$ has coefficient $e^{-jw_0k3} Y_k = e^{-j6k} \cdot Y_k = Y_{k1}$

& $\tilde{y}(t-3) = \sum_{k=-\infty}^{+\infty} Y_k e^{jkw_0(t-3)} \Rightarrow \tilde{y}(t-3)$ has $w_{01} = 2$

$\Rightarrow$ period $T_{01} = \frac{2\pi}{w_{01}} = \pi$

$\tilde{y}(2t)$ has the same coefficient with $\tilde{y}(t)$, $Y_k = Y_{k2}$

and $\tilde{y}(2t) = \sum_{k=-\infty}^{+\infty} Y_k e^{j2kw_0t} = \sum_{k=-\infty}^{+\infty} Y_k^{\,jk4t}$ has $W_{02} = 4$

$\Rightarrow T_{02} = \frac{2\pi}{4} = \frac{\pi}{2}$

We have $\frac{T_{01}}{T_{02}} = \frac{\pi}{\pi/2} = \frac{2}{1}$

$\Rightarrow$ both $\tilde{y}(t-3)$ & $\tilde{y}(2t)$ are two periodic signal with

$T = 2T_{0_2} = 1 . T_{0_1} = \pi .$

$\Rightarrow$ the Fourier coefficient of $z(t) = \tilde{y}(t-3) * \tilde{y}(2t)$

is $z_K = T \, Y_{K1} . Y_{K2} = \pi . e^{-6kj} Y_K . Y_K$

$$\Rightarrow Z_K = \pi e^{-6kj} Y_k . Y_k$$

$$\Rightarrow z_0 = \pi Y_0 . Y_0 = \pi . \left(\frac{-3}{4} \right)^2 = \boxed{\frac{9\pi}{16}}$$

$$\Rightarrow z_1 = \pi e^{-6j} Y_1 . Y_1 = \pi e^{-6j} \left(\frac{5+5j}{16} \right)^2 = \pi . e^{-6j} \frac{25j}{128}$$

$$\Rightarrow \boxed{z_1 = \frac{25 \pi e^{-6j} j}{128}}$$

$$\Rightarrow z_{-1} = \pi e^{6j} Y_{-1} . Y_{-1} = \pi e^{6j} \left[\frac{5-5j}{16} \right]^2 = \pi e^{6j} \left(\frac{-25j}{128} \right)$$

$$= \boxed{\frac{-25 \pi e^{6j} j}{128}}$$

, and other coefficient

$$\boxed{z_k = 0 , \; k \in \mathbb{Z} \; \& \; k \neq 0, \pm 1}$$

2) a) $x_1(t) = \begin{cases} t^2, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$

We have:

$$X_1(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_0^1 t^2 e^{-j\omega t}\, dt$$

let $u = t^2 \Rightarrow du = 2t\,dt$

$$e^{-j\omega t}\, dt = dv \Rightarrow v = -\frac{1}{j\omega} e^{-j\omega t}$$

$t = 0 \longrightarrow 1$

$u = 0 \longrightarrow 1$

$$\Rightarrow X_1(\omega) = \underbrace{t^2\left(-\frac{1}{j\omega}\right) e^{-j\omega t}\Big|_0^1}_{A} + \underbrace{\int_0^1 \frac{1}{j\omega} e^{-j\omega t}\, 2t\, dt}_{B}$$

$$A = \frac{-1}{j\omega} e^{-j\omega} - 0 = \frac{-1}{j\omega} e^{-j\omega}$$

$$B = \frac{2}{j\omega} \int_0^1 t e^{-j\omega t}\, dt$$

Let $C = \int_0^1 t e^{-j\omega t}\, dt = \int_0^1 t e^{at}\, dt \;, a = -j\omega$

We have: $\int t e^{at}\, dt = \left(\frac{t}{a} - \frac{1}{a^2}\right) e^{at} \;, a = -j\omega$

$\Rightarrow \int_0^1 t e^{at}\, dt = \left(\frac{t}{a} - \frac{1}{a^2}\right) e^{at} \Big|_0^1$

$= \left(\frac{1}{a} - \frac{1}{a^2}\right) e^{a} - \left(\frac{-1}{a^2}\right)$

$= \left(\frac{1}{a} - \frac{1}{a^2}\right) e^{a} + \frac{1}{a^2} \;, \quad a = -j\omega \Rightarrow a^2 = (-j\omega)^2$

$= -\omega^2$

$= \left(\frac{1}{-j\omega} - \frac{1}{-\omega^2}\right) e^{a} + \frac{1}{-\omega^2}$

$= \left(\frac{1}{\omega^2} - \frac{1}{j\omega}\right) e^{-j\omega} - \frac{1}{\omega^2} = \frac{e^{-j\omega}}{\omega^2} - \frac{e^{-j\omega}}{j\omega} - \frac{1}{\omega^2}$

$\Rightarrow B = \frac{2}{j\omega} \cdot C = \frac{2}{j\omega}\left[\frac{e^{-j\omega}}{\omega^2} - \frac{e^{-j\omega}}{j\omega} - \frac{1}{\omega^2}\right]$

$\Rightarrow X_1(\omega) = A + B = \frac{-1}{j\omega} e^{-j\omega} + \frac{2}{j\omega}\left[\frac{e^{-j\omega}}{\omega^2} - \frac{e^{-j\omega}}{j\omega} - \frac{1}{\omega^2}\right]$

$$= \frac{-e^{-jw}}{jw} + \frac{2e^{-jw}}{jw^3} + \frac{2e^{-jw}}{w^2} - \frac{2}{jw^3}$$

$$= \frac{je^{-jw}}{w} - \frac{2je^{-jw}}{w^3} + \frac{2e^{-jw}}{w^2} + \frac{2j}{w^3}$$

$$\boxed{\Rightarrow X_1(w) = \frac{jw^2e^{-jw} - 2je^{-jw} + 2we^{-jw} + 2j}{w^3}}$$

b) $x_2(t) = [u(t+2) - u(t-2)]\cos(100t) + 1$

$= rec(t, 2)\cos(100t) + 1$

let $x(t) = rect(t, 2) \xrightarrow{F} X(w) = 4\,Sinc(2w)$

Also, $\cos(100t) = \dfrac{e^{j100t} + e^{-j100t}}{2}$, then applying the Frequency property of Fs, we have:

$$x(t)\cos(100t) = \frac{1}{2}[X(w-100) + X(w+100)]$$

$$= \frac{1}{2}[4\,Sinc\,2(w-100) + 4\,Sinc\,2(w+100)]$$

$$= 2[Sinc(2w-200) + Sinc(2w+200)]$$

Also, $1 \xrightarrow{F} 2\pi\,\delta(w)$

$$\Rightarrow X_2(\omega) = 2[\text{Sinc}(2\omega - 200) + \text{Sinc}(2\omega + 200)] + 2\pi\delta(\omega)$$

$$= 2\left[\frac{\sin(2\omega - 200)}{2(\omega - 100)} + \frac{\sin(2\omega + 200)}{2(\omega + 100)}\right] + 2\pi\delta(\omega)$$

$$\boxed{\Rightarrow X_2(\omega) = \frac{\sin(2\omega - 200)}{\omega - 100} + \frac{\sin(2\omega + 200)}{\omega + 100} + 2\pi\delta(\omega)}$$

c) $x_3(t) = \int_{-\infty}^{t} \cos 5(t - \sigma)\, \delta(\sigma - 2)\, d\sigma$

$= \int_{-\infty}^{t} \cos 5(t - 2)\, \delta(\sigma - 2)\, d\sigma$ (Shifting property of impulse signal)

$$= \cos(5t - 10) \int_{-\infty}^{t} \delta(\sigma - 2)\, d\sigma$$

$$= \cos(5t - 10) \int_{-\infty}^{+\infty} \delta(\sigma - 2)\, u(-\sigma + t)\, d\sigma$$

$$= \cos(5t - 10) \int_{-\infty}^{+\infty} \delta(\sigma - 2)\, u(t - 2)\, d\sigma$$

$$= \cos(5t - 10)\, u(t - 2) \underbrace{\int_{-\infty}^{+\infty} \delta(\sigma - 2)\, d\sigma}_{1}$$

$$= \cos[5(t - 2)]\, u(t - 2)$$

$\Rightarrow x_3(t) = \cos[5(t-2)]\, u(t-2)$

Let $y(t) = \cos 5t \cdot u(t) = \dfrac{e^{j5t} + e^{-j5t}}{2} \cdot u(t)$

$\Rightarrow y(t) = \cos(5t)\, u(t) \xrightarrow{F} Y(w) = \frac{1}{2}\left[U(w-5) + U(w+5)\right]$

with $u(t) \xrightarrow{F} U(w) = \dfrac{1}{jw} + \pi\, \delta(w)$

Also $x_3(t) = y(t-2) \xrightarrow{F} e^{-j2w}\, Y(w) = X_3(w)$

$\Rightarrow X_3(w) = e^{-2jw}\, \frac{1}{2}\left[U(w-5) + U(w+5)\right]$

$= \frac{1}{2} e^{-2jw}\left[\dfrac{1}{j(w-5)} + \pi\, \delta(w-5) + \dfrac{1}{j(w+5)} + \pi\, \delta(w+5)\right]$

$= \frac{1}{2} e^{-2jw}\left[\dfrac{1}{j(w-5)} + \dfrac{1}{j(w+5)}\right] + \frac{1}{2} e^{-2jw}\, \pi\, \delta(w-5)$

$+ \frac{1}{2} e^{-2jw}\, \pi\, \delta(w+5)$

* Check $\dfrac{1}{j(w-5)} + \dfrac{1}{j(w+5)} = \dfrac{j(w+5) + j(w-5)}{j^2(w-5)(w+5)}$

$$= -\frac{j(w+5+w-5)}{w^2-25} = \frac{-2jw}{w^2-25}$$

$$\circledast\ \frac{1}{2}e^{-2jw}\pi\delta(w-5) = \frac{1}{2}e^{-2j.5}\pi\delta(w-5) = \frac{1}{2}e^{-10j}\pi\delta(w-5)$$

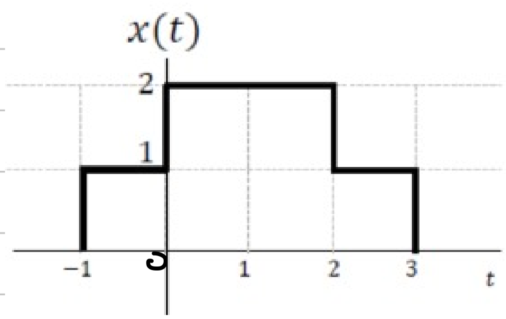
$$\circledast\ \frac{1}{2}e^{-2jw}\pi\delta(w+5) = \frac{1}{2}e^{-2j(-5)}\pi\delta(w+5) = \frac{1}{2}e^{10j}\pi\delta(w+5)$$

$$\Rightarrow X_3(w) = \frac{1}{2}e^{-2jw}\cdot\frac{-2jw}{w^2-25} + \frac{1}{2}e^{-10j}\pi\delta(w-5)$$

$$+\frac{1}{2}e^{10j}\pi\delta(w+5)$$

$$\Rightarrow \boxed{X_3(w) = \frac{jw\,e^{-2jw}}{25-w^2} + \frac{1}{2}\pi\left[e^{-10j}\delta(w-5) + e^{10j}\delta(w+5)\right]}$$

3) Given real signal, $x(t) \xrightarrow{\mathcal{F}} X(\omega)$



a) $$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-1}^{0} e^{-j\omega t}\, dt + \int_{0}^{2} 2e^{-j\omega t}\, dt$$

$$+ \int_{2}^{3} e^{-j\omega t}\, dt = \frac{-1}{j\omega} e^{-j\omega t}\Big|_{-1}^{0} + \frac{(-2)}{j\omega} e^{-j\omega t}\Big|_{0}^{2}$$

$$+ \frac{(-1)}{j\omega} e^{-j\omega t}\Big|_{2}^{3} = \left(\frac{-1}{j\omega} + \frac{1}{j\omega} e^{j\omega}\right) + \left(\frac{-2}{j\omega} e^{-2j\omega} + \frac{2}{j\omega}\right)$$

$$+ \left(-\frac{1}{j\omega} e^{-3j\omega} + \frac{1}{j\omega} e^{-2j\omega}\right)$$

$$= -\frac{1}{j\omega} + \frac{1}{j\omega} e^{j\omega} - \frac{2}{j\omega} e^{-2j\omega} + \frac{2}{j\omega} - \frac{1}{j\omega} e^{-3j\omega} + \frac{1}{j\omega} e^{-2j\omega}$$

$$= \frac{1}{j\omega} - \frac{1}{j\omega} e^{-2j\omega} + \frac{1}{j\omega} e^{j\omega} - \frac{1}{j\omega} e^{-3j\omega}$$

$$\Rightarrow \boxed{X(\omega) = \frac{1}{j\omega}\left[1 - e^{-2j\omega} + e^{j\omega} - e^{-3j\omega}\right]}$$

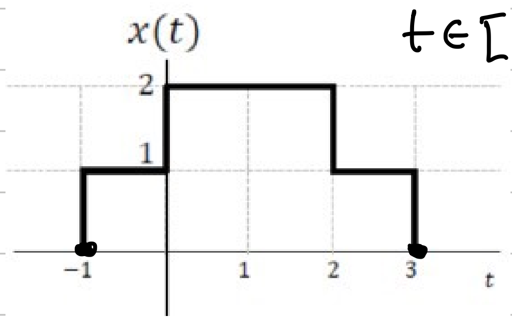
$$\circledast \quad X(0) = \int_{-\infty}^{+\infty} x(t)\,dt = \int_{-1}^{0} dt + \int_{0}^{2} 2\,dt + \int_{2}^{3} dt$$

$$= t\Big|_{-1}^{0} + 2t\Big|_{0}^{2} + t\Big|_{2}^{3} = 1 + 2(2-0) + (3-2)$$

$$= 1 + 4 + 1$$

$$\Rightarrow \boxed{X(0) = 6}$$

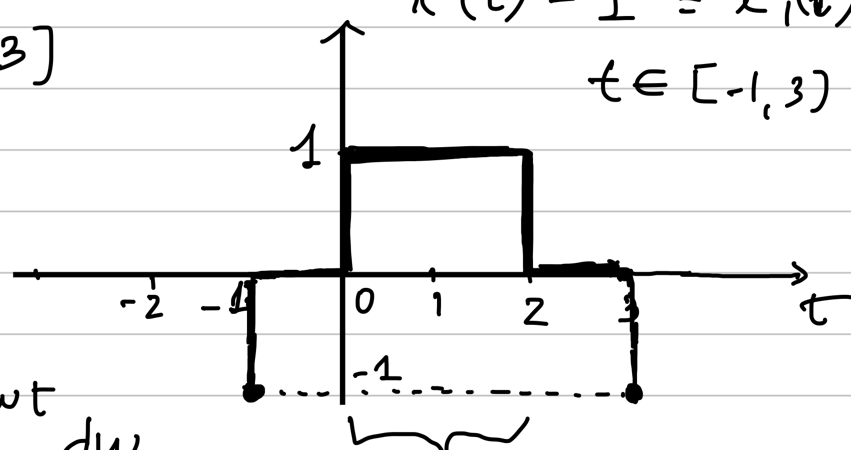
We also have:



$t \in [-1, 3]$

$x(t) - 1 = x_1(t)$

$t \in [-1, 3]$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}\,d\omega \,,$$

$$= \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow x_1(0) = 1$$

$$\left.\begin{aligned} x(t) &\xrightarrow{F} X(\omega) \\ 1 &\xrightarrow{F} 2\pi\delta(\omega) \end{aligned}\right\} \Rightarrow x(t) - 1 \xrightarrow{F} X(\omega) - 2\pi\delta(\omega)$$

$$\Rightarrow x_1(t) \longrightarrow X_1(\omega) = X(\omega) - 2\pi\delta(\omega)$$

$$\Rightarrow x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[X(\omega) - 2\pi\,\delta(\omega)\right]e^{j\omega t}\,d\omega.$$

$$\Rightarrow x_1(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[X(\omega) - 2\pi\,\delta(\omega)\right]d\omega$$

$$\Rightarrow 1 = \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} X(\omega)\,d\omega - \int_{-\infty}^{+\infty} 2\pi\,\delta(\omega)\,d\omega\right]$$

$$\Rightarrow 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\,d\omega - \underbrace{\int_{-\infty}^{+\infty}\delta(\omega)\,d\omega}_{1}$$

$$\Rightarrow 2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\,d\omega$$

$$\Rightarrow 4\pi = \int_{-\infty}^{+\infty} X(\omega)\,d\omega \quad \text{or} \quad \boxed{\int_{-\infty}^{+\infty} X(\omega)\,d\omega = 4\pi}$$

b) $\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$. Applying Parseval's theorem; we have

$$\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega = 2\pi \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

$$= 2\pi \left[\int_{-1}^{0} dt + \int_{0}^{2} 4\, dt + \int_{2}^{3} dt \right]$$

$$= 2\pi \left[t\Big|_{-1}^{0} + 4t\Big|_{0}^{2} + t\Big|_{2}^{3} \right] = 2\pi (1 + 4.(2-0) + 3-2)$$

$$= 2\pi (1 + 8 + 1) = \boxed{20\pi}$$

c) $\int_{-\infty}^{+\infty} X(\omega) Y(\omega) d\omega$, $Y(\omega) = \frac{2\sin(\omega)}{\omega} e^{j2\omega}$

We have $\int_{-\infty}^{+\infty} X(\omega) Y(\omega) d\omega = 2\pi \int_{-\infty}^{+\infty} x(t) y(-t) dt$

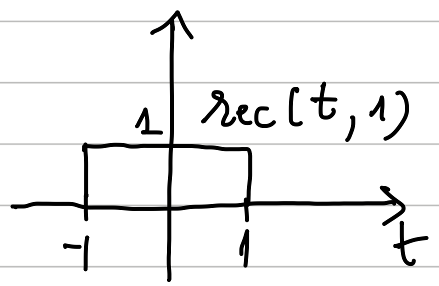
with $\begin{cases} x(t) \xrightarrow{F} X(\omega) \\ y(t) \xrightarrow{F} Y(\omega) \end{cases}$, $x(t)$ is real.

We have $Y(\omega) = \frac{2\sin\omega}{\omega} e^{2j\omega} = 2\,\mathrm{Sinc}(\omega)\, e^{2j\omega}$

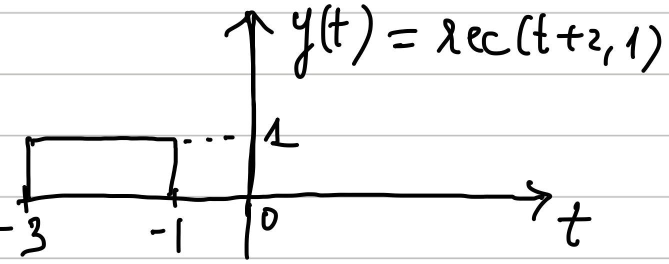
Also, $rec(t,1) \longrightarrow 2\,Sinc(\omega)$

and $rec(t+2,1) \longrightarrow e^{2j\omega} \cdot 2\,Sinc(\omega)$

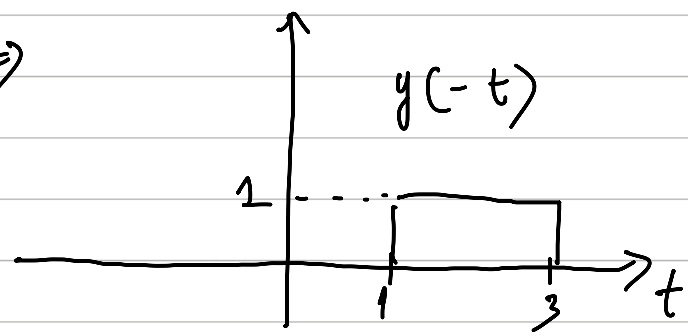
$\Rightarrow y(t) = rec(t+2,1) \Rightarrow y(t)$ is real signal



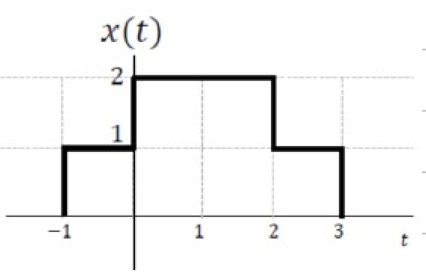
$\Rightarrow$



$\Rightarrow$



We also have $x(t)$:



Since $x(t)$ & $y(t)$ are real signal, then:

$$\Rightarrow \int_{-\infty}^{+\infty} X(\omega)\,Y(\omega)\,d\omega = 2\pi \int_{-\infty}^{+\infty} x(t)\,y(-t)\,dt$$

$$= 2\pi\left[\int_1^2 2.1\,dt + \int_2^3 1.1\,dt\right]$$

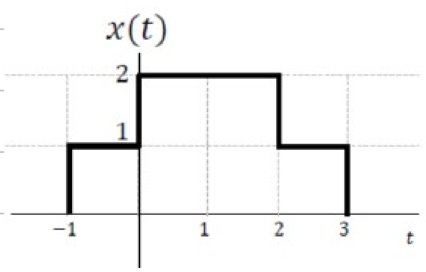
$$= 2\pi\left[2t\Big|_1^2 + t\Big|_2^3\right] = 2\pi\left[2.1 + 1\right] = \boxed{6\pi}$$

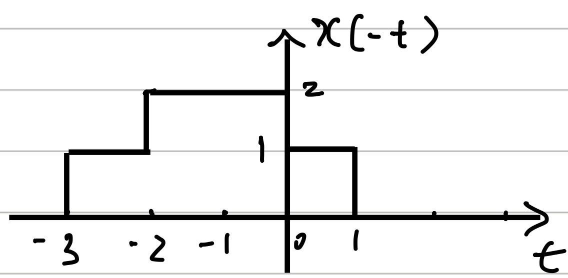
d) Sketch the inverse Fourier transform of $Re\{X(\omega)\}$

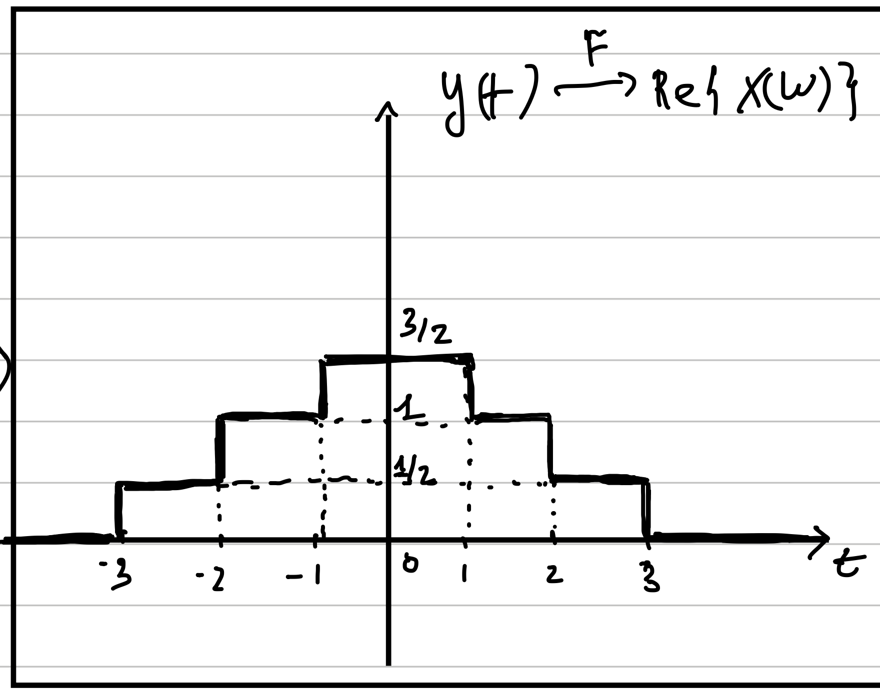
Since the $Re\{X(\omega)\}$ give even part of the signal? $x(t)$ is real

Let $y(t) \xrightarrow{F} Re\{X(\omega)\}$

$$\Rightarrow y(t) = \frac{x(t) + x(-t)}{2}$$







Recheck: $Re\{X(\omega)\} = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)\,dt$

$$= \int_{-1}^{0} \cos(\omega t)\,dt + \int_{0}^{2} 2\cos(\omega t)\,dt + \int_{2}^{3} \cos(\omega t)\,dt$$

$$= \frac{\sin\omega t}{\omega}\Big|_{-1}^{0} + \frac{2}{\omega}\sin(\omega t)\Big|_{0}^{2} + \frac{\sin\omega t}{\omega}\Big|_{2}^{3}$$

$$= \frac{\sin\omega}{\omega} + \frac{2}{\omega}\sin 2\omega + \frac{\sin 3\omega - \sin 2\omega}{\omega}$$

$$= \frac{\sin w}{w} + \frac{2\sin 2w}{2w} + \frac{3\sin 3w}{3w} = \mathrm{Sinc}(w) + 2\,\mathrm{Sinc}(2w) + 3\,\mathrm{Sinc}(3w)$$

$$\mathrm{rec}(t,1) \xrightarrow{F} 2\,\mathrm{Sinc}(w) \Rightarrow \frac{1}{2}\mathrm{rec}(t,1) \xrightarrow{F} \mathrm{Sinc}(w)$$

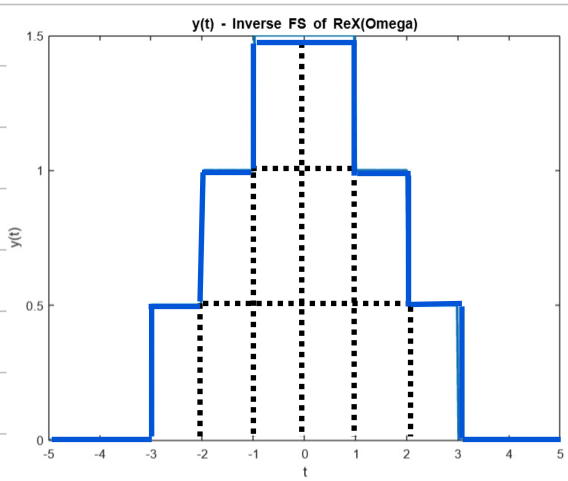
$$\mathrm{rec}(t,2) \longrightarrow 4\,\mathrm{Sinc}(2w) \Rightarrow \frac{1}{2}\mathrm{rec}(t,2) \xrightarrow{F} 2\,\mathrm{Sinc}(2w)$$

$$\mathrm{rec}(t,3) \longrightarrow 6\,\mathrm{Sinc}(3w) \Rightarrow \frac{1}{2}\mathrm{rec}(t,3) \xrightarrow{F} 3\,\mathrm{Sinc}(3w)$$

$$\Rightarrow y(t) = \frac{1}{2}\left[\mathrm{rec}(t,1) + \mathrm{rec}(t,2) + \mathrm{rec}(t,3)\right] \xrightarrow{F} \mathrm{Re}\{X(w)\}$$

$$\Rightarrow y(t) = \frac{1}{2}\left[u(t+1) - u(t-1) + u(t+2) - u(t-2) + u(t+3) - u(t-3)\right]$$

Using matlab to draw this y(t), we also have:



This is matched with one we sketched above => it is corrected

4) Given:

i. $x(t)$ is real and non-negative.

ii. $\mathcal{F}^{-1}\{(1+j\omega)X(j\omega)\} = Ae^{-2t}u(t)$, where A is a constant.

iii. $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = 2\pi$

From ii) $\mathcal{F}^{-1}\{(1+jw)X(jw)\} = Ae^{-2t}u(t)$

$\Rightarrow (1+wj)X(jw) = \mathcal{F}\{Ae^{-2t}u(t)\}$

Check $Ae^{-2t}u(t) \xrightarrow{\mathcal{L}_s} \dfrac{A}{s+2}$ $\quad Re\{s\} > -2$

Since ROC : $Re\{s\} > -2$ included jw axis

$\Rightarrow \mathcal{F}\{Ae^{-2t}u(t)\} = \dfrac{A}{s+2}\Big|_{s=jw} = \dfrac{A}{2+jw}$

$\Rightarrow (1+jw)X(jw) = \dfrac{A}{2+jw}$

$\Rightarrow X(jw) = \dfrac{A}{(1+jw)(2+jw)} = A\left(\dfrac{1}{1+jw} - \dfrac{1}{2+jw}\right)$

$\Rightarrow X(jw) = X(s)\Big|_{s=jw} = \underbrace{A\left(\dfrac{1}{s+1} - \dfrac{1}{s+2}\right)}\Big|_{s=jw}$

ROC : $Re\{s\} > -1$ includes jw axis.

$$\Rightarrow x(t) = A[e^{-t}u(t) - e^{-2t}u(t)]$$

$$= A[e^{-t} - e^{-2t}]\, u(t)$$

Since $(e^{-t} - e^{-2t})u(t) = \left(\frac{1}{e^{t}} - \frac{1}{e^{2t}}\right)u(t)$

$$= \frac{e^{t}-1}{e^{2t}} \cdot u(t) = \begin{cases} \frac{e^{t}-1}{e^{2t}}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Also with $t \geq 0$, $\frac{e^{t}-1}{e^{2t}} \geq 0$.

$\Rightarrow x(t)$ is real & non-negative $\Rightarrow x(t) \geq 0$

$\Leftrightarrow A \geq 0$. (1)

We also have $\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = 2\pi$

By Parseval theorem, $\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega = 2\pi \int_{-\infty}^{+\infty} |x(t)|^2 dt$

$$\Rightarrow 2\pi \int_{-\infty}^{+\infty} |x(t)|^2 dt = 2\pi \Leftrightarrow \int_{-\infty}^{+\infty} |x(t)|^2 dt = 1$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} \left(A[e^{-t} - e^{-2t}]\, u(t)\right)^2 dt = 1$$

$$\Leftrightarrow \quad A^2 \int_0^{+\infty} (e^{-t} - e^{-2t})^2 \, dt = 1$$

We have $\int_0^{+\infty} (e^{-t} - e^{-2t})^2 dt = \int_0^{+\infty} (e^{-2t} - 2e^{-3t} + e^{-4t}) dt$

$$= \left(-\frac{1}{2} e^{-2t} + \frac{2}{3} e^{-3t} - \frac{1}{4} e^{-4t} \right) \Big|_0^{+\infty}$$

$$= -\frac{1}{2}(0-1) + \frac{2}{3}(0-1) - \frac{1}{4}(0-1)$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

$$\Rightarrow A^2 \cdot \frac{1}{12} = 1 \Rightarrow A^2 = 12 \Rightarrow A = \pm\sqrt{12}$$

Because from (1), $A \geq 0 \Rightarrow \boxed{A = \sqrt{12} = 2\sqrt{3}}$

Combining all parts, we have:

$$x(t) = A[e^{-t} - e^{-2t}]\, u(t)$$

$$\Rightarrow \boxed{x(t) = 2\sqrt{3}\,(e^{-t} - e^{-2t})\, u(t)}$$

5) $x(t) = e^{-t}\sin(2\pi t)u(t) + \delta(t-2)$

a) * $x_1(t) = \sin(2\pi t)\,u(t) \xrightarrow{L_s} \dfrac{2\pi}{s^2+4\pi^2}$ $\mathrm{Re}\{s\} > 0$

$\Rightarrow e^{-t}\sin 2\pi t\, u(t) \xrightarrow{L_s} \dfrac{2\pi}{(s+1)^2+4\pi^2}$ $\mathrm{Re}\{s\} > -1$

Because $(s+1)^2+4\pi^2 = 0 \Rightarrow (s+1)^2 - (2\pi j)^2 = 0$

$\Leftrightarrow s = -1 \pm 2\pi j \Rightarrow$ ROC : $\mathrm{Re}\{s\} > -1$

* $x_2(t) = \delta(t-2)$

$\delta(t) \xrightarrow{L_s} 1 \Rightarrow \delta(t-2) \xrightarrow{L_s} e^{-2s}$ all s

$\Rightarrow X(s) = \dfrac{2\pi}{(s+1)^2+4\pi^2} + e^{-2s}$ with ROC: $\mathrm{Re}\{s\} > -1$

includes $j\omega$ axis.

$$\boxed{\Rightarrow X(\Omega) = X(s)\Big|_{s=j\Omega} = \frac{2\pi}{(j\Omega+1)^2+4\pi^2} + e^{-2j\Omega}}$$

b) We have: $(j\Omega+1)^2 + 4\pi^2 = -\Omega^2 + 2j\Omega + 1 + 4\pi^2$

$= 4\pi^2 + 1 - \Omega^2 + 2j\Omega$

$$\Rightarrow \frac{2\pi}{(j\Omega)^2+4\pi^2} = \frac{2\pi\left(4\pi^2+1-\Omega^2 - 2j\Omega\right)}{(4\pi^2+1-\Omega^2)^2 - (2j\Omega)^2}$$

$$= \frac{2\pi(4\pi^2+1-\Omega^2) - 4\pi\Omega j}{(4\pi^2+1-\Omega^2)^2 + 4\Omega^2}$$

Also, $e^{-2j\Omega} = \cos(2\Omega) - j\sin(2\Omega)$

$$\Rightarrow X(\Omega) = \frac{2\pi(4\pi^2+1-\Omega^2)}{(4\pi^2+1-\Omega^2)^2 + 4\Omega^2} + \cos(2\Omega) -$$

$$\frac{4\pi\Omega}{(4\pi^2+1-\Omega^2)^2 + 4\Omega^2} j - j\sin(2\Omega)$$

$$\Rightarrow X(\Omega) = A - Bj$$

$$\underbrace{\left[\frac{2\pi(4\pi^2+1-\Omega^2)}{(4\pi^2+1-\Omega^2)^2+4\Omega^2} + \cos 2\Omega\right]}_{A} - j\underbrace{\left[\frac{4\pi\Omega}{(4\pi^2+1-\Omega^2)^2 + 4\Omega^2} + \sin(2\Omega)\right]}_{B}$$

$\Rightarrow |X(\Omega)| = \sqrt{A^2+B^2}$ is amplitude of $X(\Omega)$

& $\measuredangle X(\Omega) = \tan^{-1}\left(\frac{-B}{A}\right)$ is phase of $X(\Omega)$

$$\Rightarrow |X(\Omega)| = \sqrt{\left[\frac{2\pi(4\pi^2+1-\Omega^2)}{(4\pi^2+1-\Omega^2)^2+4\Omega^2} + \cos 2\Omega\right]^2 + \left[-\frac{4\pi\Omega}{(4\pi^2+1-\Omega^2)^2+4\Omega^2} + \sin(2\Omega)\right]^2}$$

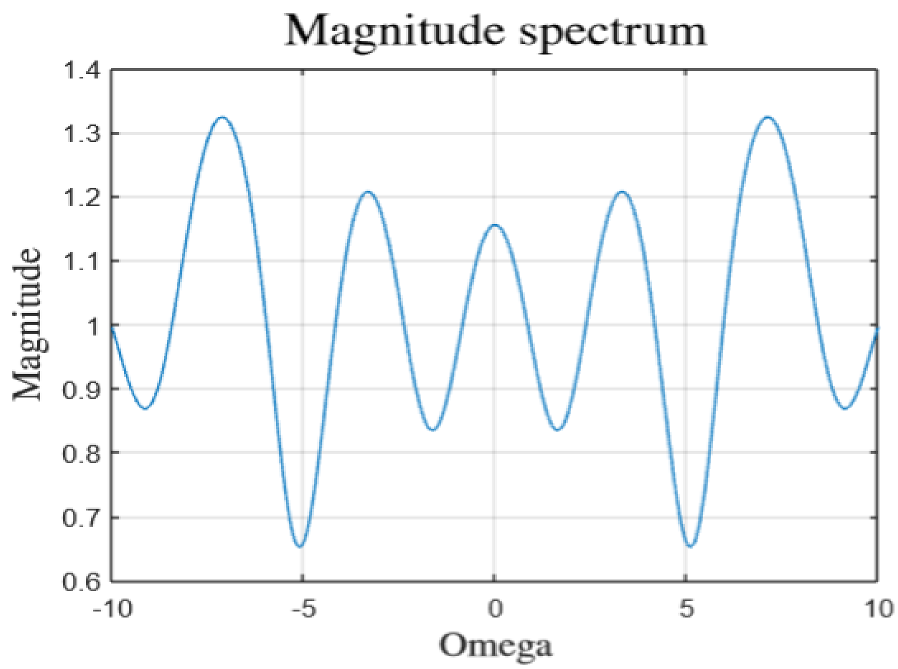
$$\measuredangle X(\Omega) = -\tan^{-1}\frac{B}{A} = -\tan^{-1}\left(\frac{-\dfrac{4\pi\Omega}{(4\pi^2+1-\Omega^2)^2+4\Omega^2} + \sin(2\Omega)}{\dfrac{2\pi(4\pi^2+1-\Omega^2)}{(4\pi^2+1-\Omega^2)^2+4\Omega^2} + \cos 2\Omega}\right)$$

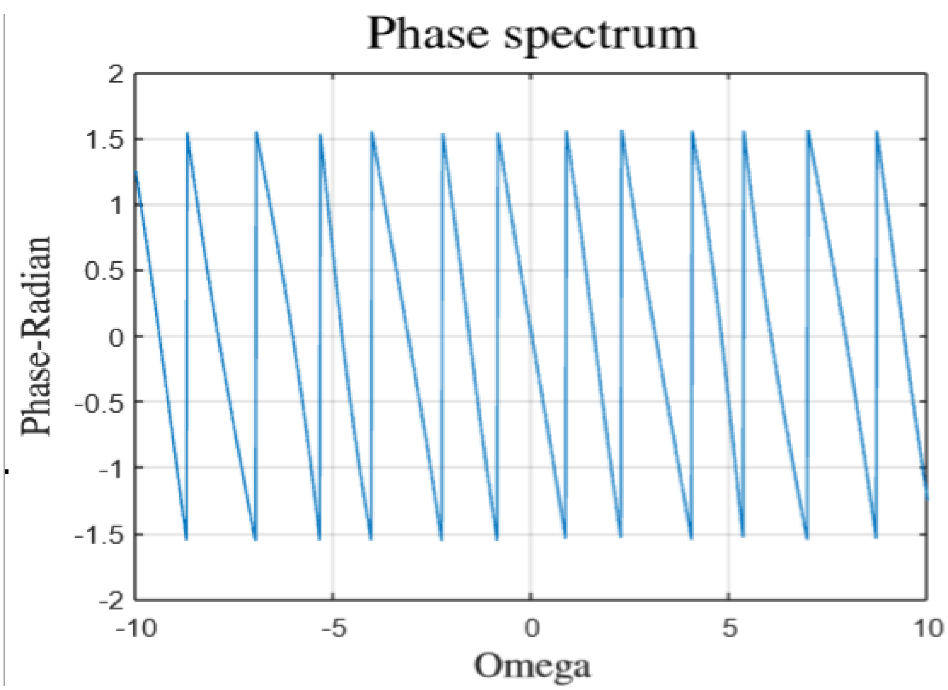
c) Used MATLAB

We have: $X(0) = \left.\dfrac{2\pi}{(j\Omega+1)^2+4\pi^2} + e^{-2j\Omega}\right|_{\Omega=0}$

$$= \frac{2\pi}{4\pi^2+1} + 1 = \frac{4\pi^2+2\pi+1}{4\pi^2+1}$$

$$\Rightarrow \measuredangle X(0) = 0.$$


Magnitude spectrum
Magnitude
Omega
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
-10
-5
0
5
10


Phase spectrum
Phase-Radian
Omega
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-10
-5
0
5
10

```
clear;
clc;
close all;
w=linspace(-10,10,1000);

Real = (2.*pi.*(4.*pi.^2-w.^2+1))./((4.*pi.^2-w.^2+1).^2 + (2.*w).^2) +
cos(2.*w);
Imaginary = (4.*pi.*w)./((4.*pi.^2-w.^2+1).^2 + (2.*w).^2) + sin(2.*w);

Magnitude = sqrt(Real.^2 + Imaginary.^2);
Phase = atan(-Imaginary./Real);

figure(1);
plot(w, Magnitude);
xlabel('Omega','Interpreter','latex','fontsize',14);
ylabel('Magnitude','Interpreter','latex','fontsize',14);
title('Magnitude spectrum','Interpreter','latex','fontsize',18);
grid on;

figure(2)
plot(w, Phase);
xlabel('Omega','Interpreter','latex','fontsize',14);
ylabel('Phase-Radian','Interpreter','latex','fontsize',14);
title('Phase spectrum','Interpreter','latex','fontsize',18);
grid on;
```