Lecture 5

Now, focus on <u>Linear System</u>

$$x(t) \rightarrow \overset{L}{\boxed{S}} \rightarrow y(t) = S\{x(t)\}$$

TI, TV
C, NC

$$\delta(t-\tau) \rightarrow \boxed{h(t,\tau)} \rightarrow S\{\delta(t-\tau)\} = \underline{\underline{h(t,\tau)}}$$

$$x(t) \rightarrow \boxed{h(t,\tau)} \rightarrow \underline{\underline{y(t)}} = ?$$

Generic representation of $x(t)$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = S\{x(t)\}$$

$$= S\left\{ \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \right\}$$

$$\overset{L}{=} \int_{-\infty}^{+\infty} S\{x(\tau)\,\delta(t-\tau)\}\,d\tau$$

$$\overset{L}{=} \int_{-\infty}^{+\infty} x(\tau)\,\underbrace{S\{\delta(t-\tau)\}}_{h(t,\tau)}\,d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, \underline{h(t,\tau)}\, d\tau}$$

Convolution Integral for linear systems.

If S is L + $\underline{\text{TI}}$ $h(t)$

because of TI $h(t,\tau) = \underline{h(t-\tau)}$

$$\boxed{y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau}$$

conv. int. for LTI systems.

If S is L + $\underline{\text{TI}}$ + $\underline{\text{C}}$

$$h(t,\tau) = h(t-\tau)\cdot u(t-\tau)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, u(t-\tau)\, \cancel{dt}$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

Conv. Int. for LTIC sys.

For LTI

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t) \triangleq \underline{x}(t) * \underline{h}(t)$$

convolution

Property:

$$x(t) * h(t) = h(t) * x(t)$$

$$h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$\parallel$ $\qquad\qquad\qquad\qquad$ $\parallel$? proof.

$$x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Proof.

$$\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau =$$

$t - \tau = \sigma \qquad d\tau = -d\sigma$

$\tau = t - \sigma \qquad \tau \to -\infty \quad \sigma \to +\infty$

$\qquad\qquad\qquad \tau \to +\infty \quad \sigma \to -\infty$

$$= \int_{+\infty}^{-\infty} x(t-\sigma)\, h(\sigma)\, (-d\sigma)$$

$$= \int_{-\infty}^{+\infty} h(\sigma)\, x(t-\sigma)\, d\sigma$$

$$= h(t) * x(t)$$

Back to System Properties

- Linear
- TI
- C

\+ 1 more

Bounded Input Bounded Output Stability

BIBO Stability

Bounded Input

$$|x(t)| < M < +\infty \ \forall t$$

LTI

$$x(t) \longrightarrow \overset{S}{\boxed{h(t)}} \longrightarrow y(t)$$

if $|x(t)| < M < +\infty$

results in $|y(t)| < K < +\infty$

$\Rightarrow$ S is BIBO stable.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

reminder: $|a+b| \leq |a| + |b|$

$$|a \cdot b| = |a| \cdot |b|$$

$$|y(t)| \leq \int_{-\infty}^{+\infty} |h(\tau)\, x(t-\tau)|\, d\tau$$

$$= \int_{-\infty}^{+\infty} |h(\tau)|\, \underbrace{|x(t-\tau)|}_{<M}\, d\tau$$

$$\leq M \int_{-\infty}^{+\infty} |h(\tau)| \, d\tau$$

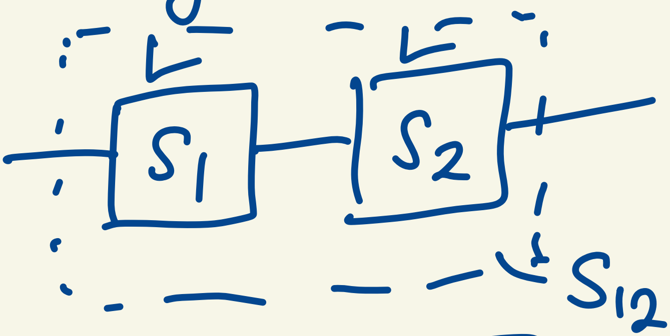
if BIBO stable
then

$$< +\infty$$

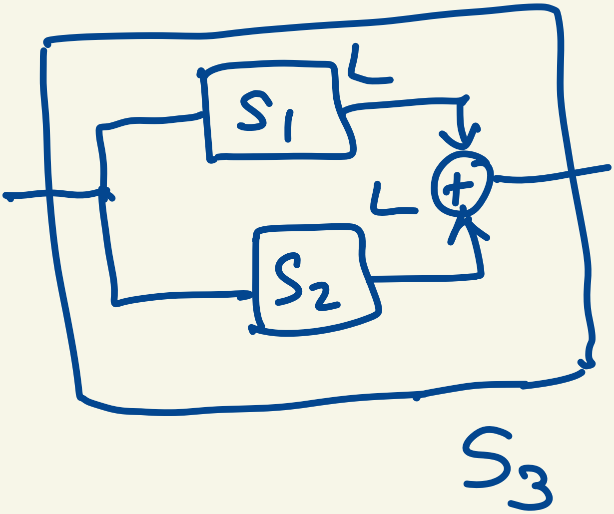
$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} |h(\tau)| \, d\tau < +\infty \quad \text{LTI \& BIBO stable}}$$

h(t) is absolutely summable.

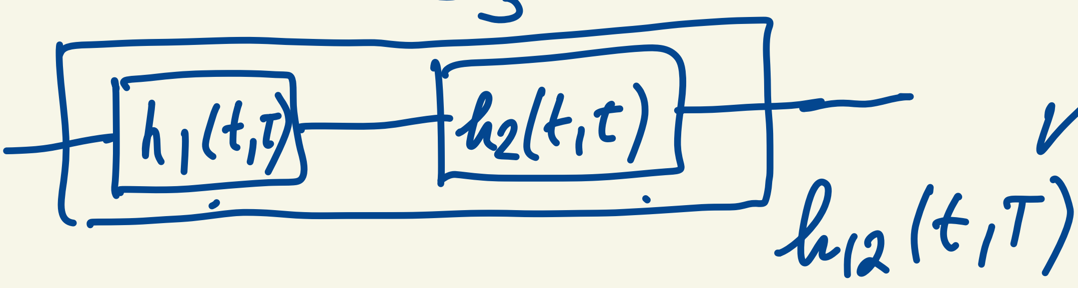
System connections.

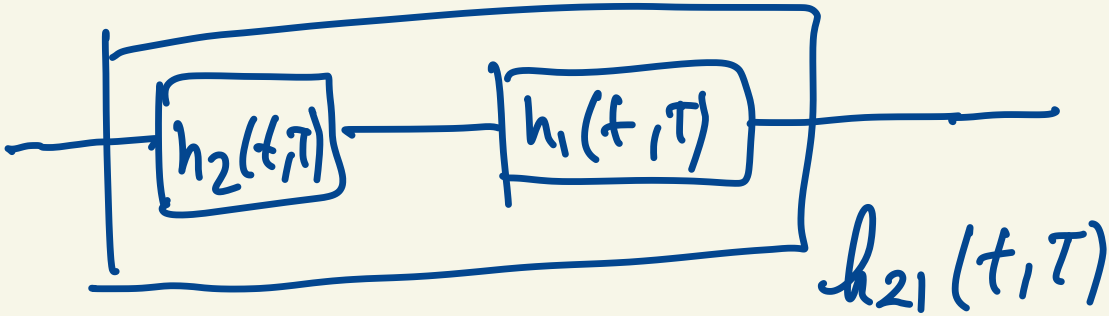


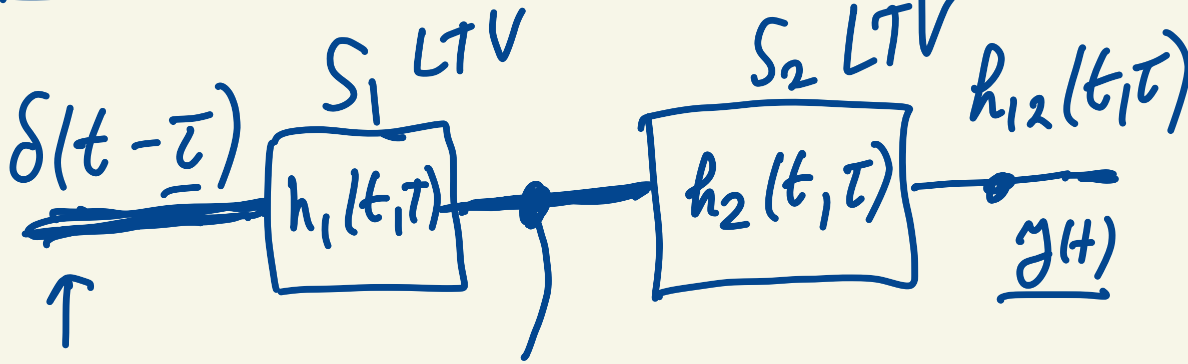
cascaded system.



parallel system connection.



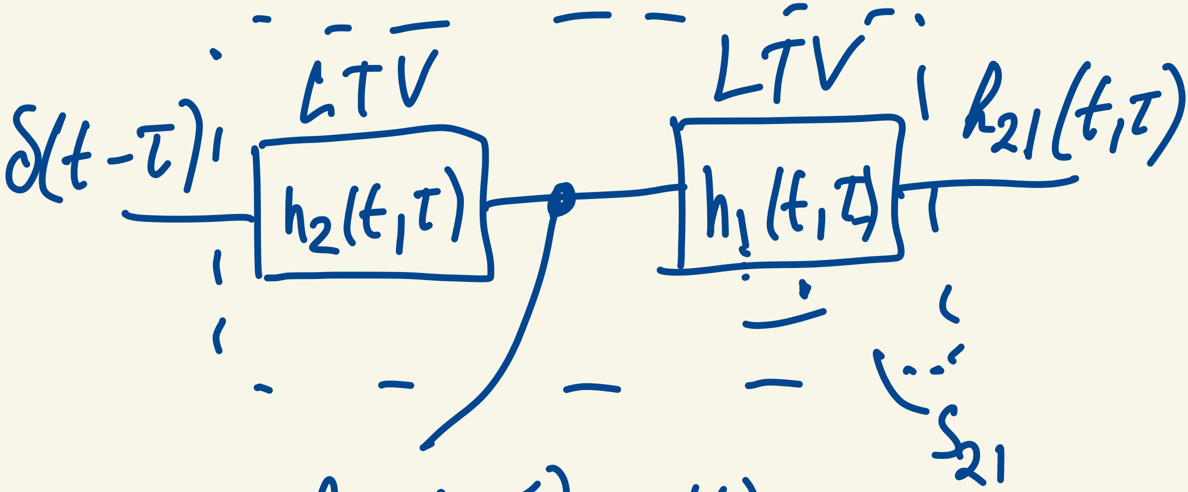




$x(t) = h_1(t,\tau)$

$$S_2: \quad y(t) = \int_{-\infty}^{+\infty} x(\sigma)\, h_2(t,\sigma)\, d\sigma$$

$$S_{12} \quad h_{12}(t,\tau) = \int_{-\infty}^{+\infty} h_1(\sigma,\tau)\, h_2(t,\sigma)\, d\sigma$$



$$h_2(t,\tau) = x(t)$$

$$h_{21}(t,\tau) = \int_{-\infty}^{+\infty} x(\sigma)\, h_1(t,\sigma)\, d\sigma$$

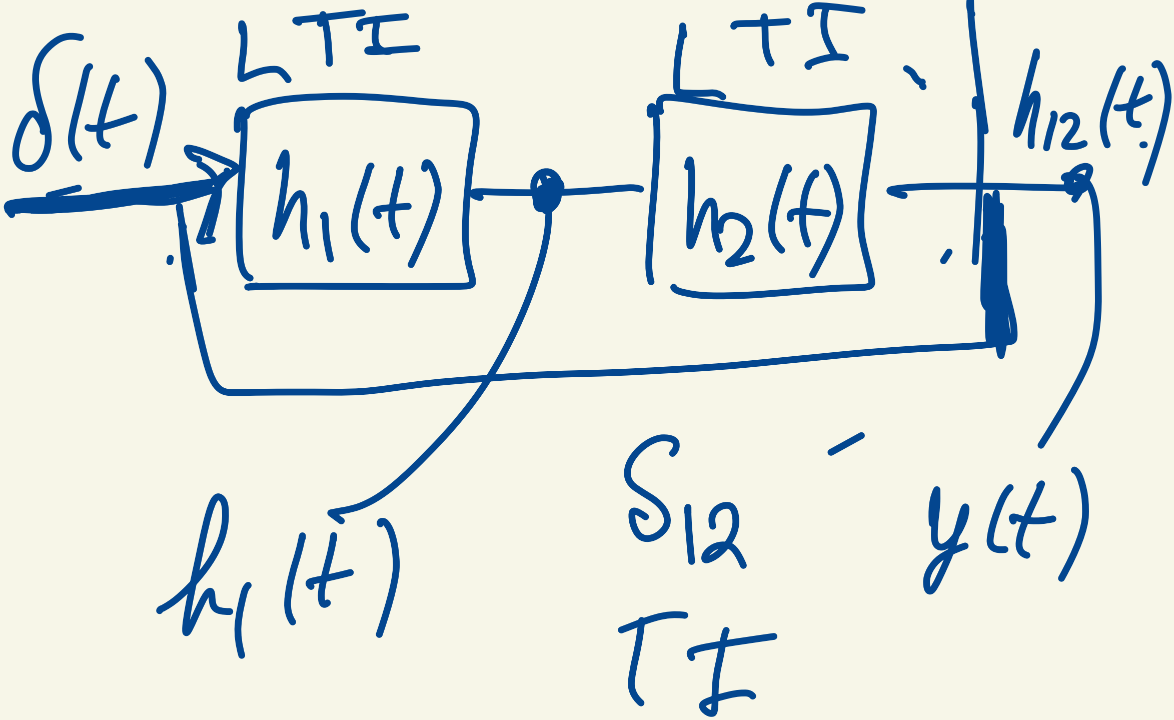
$$S_{21} \quad h_{21}(t,\tau) = \int_{-\infty}^{+\infty} h_2(\sigma,\tau)\, h_1(t,\sigma)\, d\sigma$$

$$\underline{h_{12}(t,\tau)} \neq \underline{h_{21}(t,\tau)}$$

S_{12} is different S_{21}
from
order of cascading
LTV systems matters

What if S_1 and S_2
are LTI?

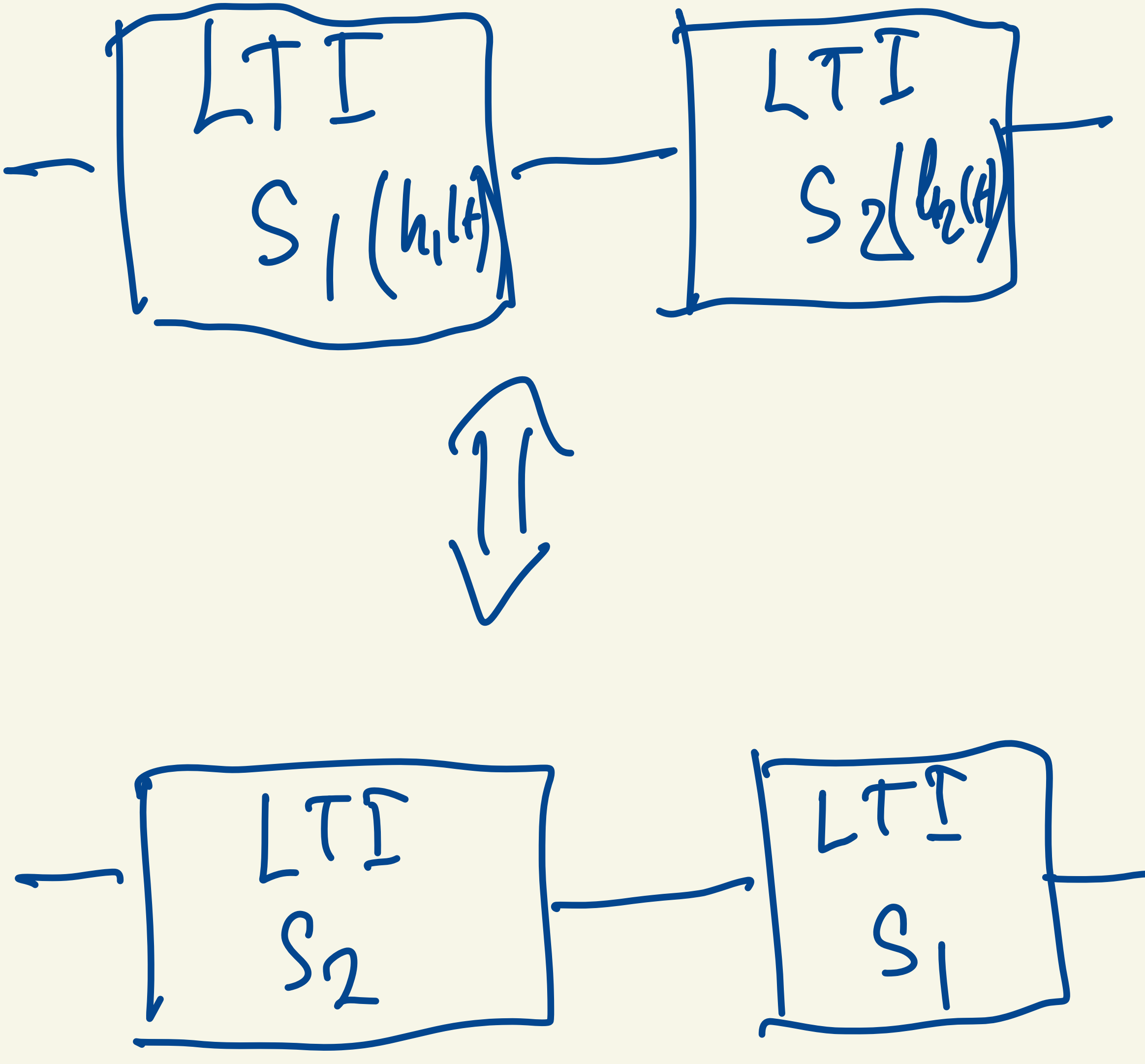
$$h(t) = S\{\delta(t)\}$$



$$y(t) = \underline{h_{12}(t)} = h_1(t) * h_2(t)$$

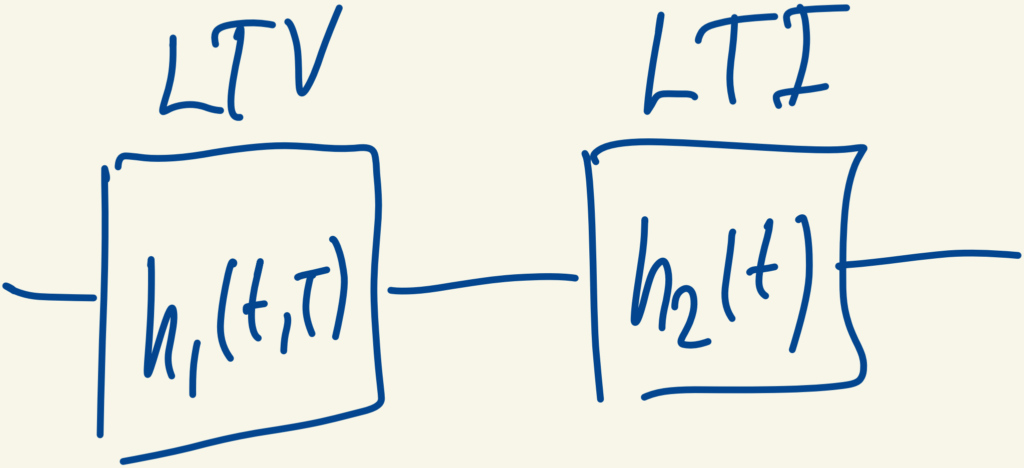
$$= h_2(t) * h_1(t)$$

$$= h_{21}(t)$$



if S_1 and S_2 are LTI
order of cascading does

not matter.



use LTV cascade.

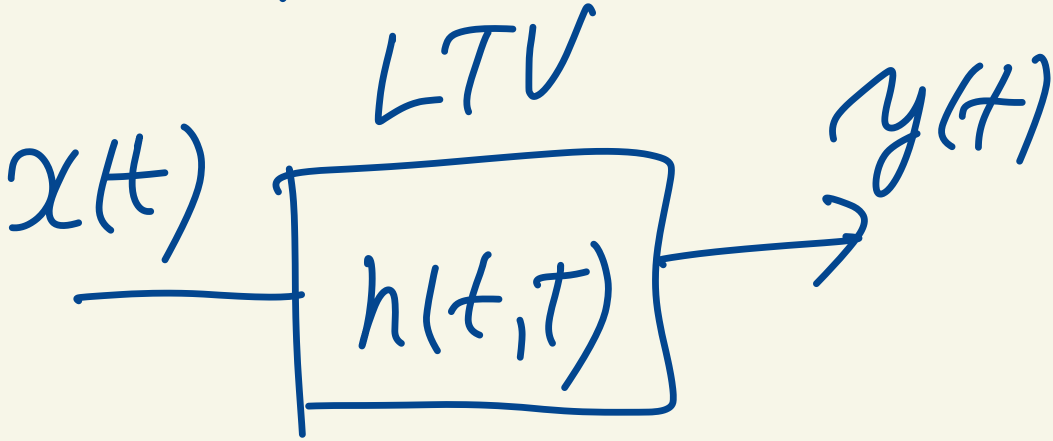
So far we were able to solve

<u>system analysis</u>

questions using convolution integral

given $x(t)$ input and $S(L)$ $h(t,\tau)$ IRF of S.

we could predict $y(t)$ output.



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t,\tau)\, d\tau$$

LTI

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

How about
given $x(t)$ and $y(t)$
find $h(t)$

$$x(t) \longrightarrow \overset{\text{LTI}}{\boxed{\begin{matrix} ? \\ h(t) \end{matrix}}} \longrightarrow y(t)$$

not easy! deconvolution

$$? \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

what is $x(t)$?

We will learn transformations to solve these types of problems.

Laplace Transform.

$$x(t) \longrightarrow X(s)$$

s-domain

Fourier Transforms

$x(t) \longrightarrow X(f)$

$X(\omega)$

f (or ω) domain

$X(S) \rightarrow$ polynomials

operations

factoring polynomials

$$y(t) = x(t) * h(t).$$

we are heading here.

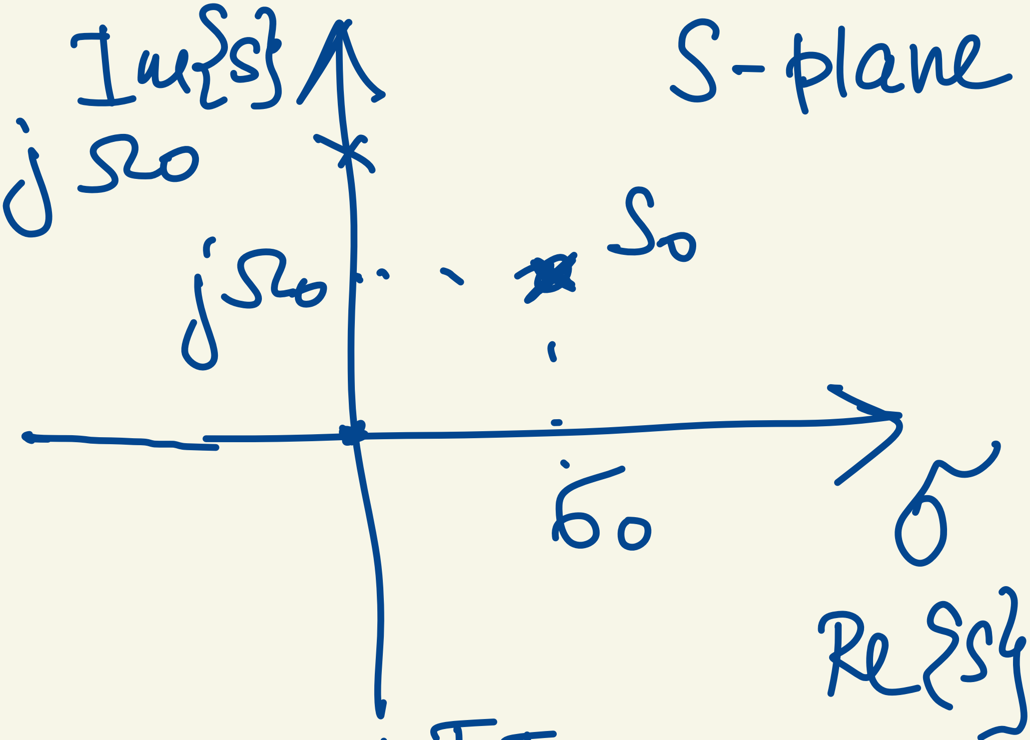
$$Y(\underline{s}) = X(\underline{s}) \cdot H(\underline{s})$$

Complex exponential as a probing signal to $\underline{\underline{LTI}}$ systems

$$x(t) = e^{s_0 t} \qquad s_0 = \sigma_0 + j\Omega_0$$

$s_0 \in \mathbb{C}$

S-plane



$$e^{s_0 t} = x(t) \longrightarrow \underset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, e^{s_0(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(t)\, \underbrace{e^{s_0 t}} \cdot e^{-s_0 \tau}\, d\tau$$

$$= e^{s_0 t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-s_0 \tau}\, d\tau$$

$$H(s_0)$$

$$= H(s_0) \cdot e^{s_0 t}$$

$$e^{s_0 t} \rightarrow \overset{\text{LTI}}{\boxed{h(t)}} \rightarrow \underline{H(s_0)}\, e^{s_0 t}$$

$$\underline{\underline{H(s_0)}} = \int_{-\infty}^{\infty} h(t)\, e^{-s_0 t}\, dt$$

$\underline{\underline{H(s_0)}} \rightarrow$ is scaling constant.

$e^{s_0 t}$ are eigenfunction of LTI systems.

Why eigen?

Remember Linear Algebra:

eigenvector

A is matrix

$$A \cdot \underline{x} = \lambda \underline{x}$$

eigen vector

eigen value.

$$\underline{x(t)} = \sum A_k e^{s_k t} \rightarrow \underset{\text{LTI}}{\boxed{h(t)}} \rightarrow$$

$$\rightarrow \sum A_k H(s_k) e^{s_k t}$$