UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Final Examination

17^{th} March, 2022

Your name : ______________________________

Instructions

- This exam has 5 questions and 8 pages.
- The exam is closed book. Two double-sided A4 sized cheat sheets are allowed. The use of calculators is permitted.
- All steps and working must be shown. No marks will be awarded for answers without math steps and/or an explanation.
- Write legibly and clearly! Any illegible work will not be graded.
- All plots must be neatly drawn and completely labelled (axes, intercepts, amplitudes) for full credit.

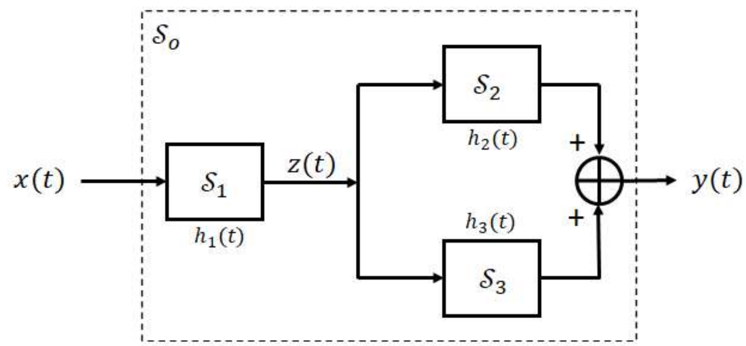
Good Luck!

Table 1: Score Table

Question	Total	Break up	Marks scored	Total score
1	20	5 + 7 + 8		
2	20	10 + 10		
3	20	8 + 5 + 7		
4	20	3 + 3 + 4 + 10		
5	20	12 + 8		
Total	100			

Question 1 (20 marks)

Given below is the block diagram of a cascaded LTI causal system $\mathcal{S}_o$, comprising of three system blocks: $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, with impulse response functions $h_1(t)$, $h_2(t)$ and $h_3(t)$ respectively.



- System $\mathcal{S}_o$ is given by the input output relation shown below, where $x(t)$ is the input and $y(t)$ is the output.

$$y(t) = x(t) - 9\int_{-\infty}^{t} x(\sigma)(t-\sigma)e^{-2(t-\sigma)}d\sigma$$

- When an input $x(t) = e^{-5(t-3)}u(t-3)$ is applied to the block $\mathcal{S}_1$, we get the output as $z(t) = \delta(t-3) + 2e^{-2(t-3)}u(t-3)$.

- Further, $\mathcal{S}_2$ and $\mathcal{S}_3$ are single pole systems with no zeros. $\mathcal{S}_2$ has the higher magnitude pole.

(a) Find the transfer function $H_o(s)$ and indicate its ROC. (5 marks)

(b) Find the transfer functions $H_1(s)$, $H_2(s)$ and $H_3(s)$. (7 marks)

(c) Find the transfer function $\tilde{H}(s)$ of a system whose impulse response function is given by

$$\tilde{h}(t) = \int_{-\infty}^{\infty} e^{-(4t+\tau)}h_2(\tau)h_3(t-\tau)d\tau$$

Indicate its ROC. (8 marks)

Solution:

(a) The overall system $\mathcal{S}_\text{o}$ is given by the input output relation

$$\begin{aligned} y(t) &= x(t) - 9\int_{-\infty}^{t} x(\sigma)(t-\sigma)e^{-2(t-\sigma)}d\sigma \\ &= x(t) - 9\int_{-\infty}^{\infty} x(\sigma)(t-\sigma)e^{-2(t-\sigma)}u(t-\sigma)d\sigma \\ &= x(t) - 9x(t) * e^{-2t}tu(t) \end{aligned}$$

Assuming that the input $x(t)$ is causal, the input-output relation can be given by the Laplace transform

$$Y(s) = X(s) - 9X(s) \, . \, \mathcal{L}\left\{e^{-2t}tu(t)\right\} = X(s)\left[1 - \frac{9}{(s+2)^2}\right]$$

$$\Longrightarrow H_\text{o}(s) = 1 - \frac{9}{(s+2)^2} = \boxed{\frac{(s+5)(s-1)}{(s+2)^2}} \qquad \text{ROC: } \mathcal{R}e\{s\} > -2$$

(b) For system $\mathcal{S}_1$: Input $x(t) = e^{-5(t-3)}u(t-3)$ $X(s) = \frac{e^{-3s}}{s+5}$.
Output $z(t) = \delta(t-3) + 2e^{-2(t-3)}u(t-3) \Longrightarrow Z(s) = e^{-3s}\left(1 + \frac{2}{s+2}\right) = e^{-3s}\left(\frac{s+4}{s+2}\right)$.
Thus, the transfer function $H_1(s)$ is given as follows:

$$\boxed{H_1(s) = \frac{Z(s)}{X(s)} = \frac{(s+4)(s+5)}{(s+2)}}$$

From the system block diagram, we can infer that

$$H_\text{o}(s) = H_1(s)\left[H_2(s) + H_3(s)\right]$$

$$H_2(s) + H_3(s) = \frac{H_\text{o}(s)}{H_1(s)} = \frac{s-1}{(s+2)(s+4)}$$

Since we know that $\mathcal{S}_2$ and $\mathcal{S}_3$ are single pole systems with no zeros, we can obtain $H_2(s)$ and $H_2(s)$ by resolving $\frac{s-1}{(s+2)(s+4)}$ into fractional parts.

$$\frac{s-1}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}$$

$$A = \left.\frac{s-1}{s+4}\right|_{s=-2} = -\frac{3}{2} \quad ; \quad B = \left.\frac{s-1}{s+2}\right|_{s=-4} = \frac{5}{2}$$

We are given that $\mathcal{S}_2$ has the larger magnitude pole. Therefore,

$$\boxed{H_2(s) = \frac{5}{2}\left(\frac{1}{s+4}\right)} \quad ; \quad \boxed{H_3(s) = -\frac{3}{2}\left(\frac{1}{s+2}\right)}$$

(c) We are given that

$$\begin{aligned}\tilde{h}(t) &= \int_{-\infty}^{\infty} e^{-(4t+\tau)} h_2(\tau) h_3(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} e^{-5\tau} h_2(\tau) e^{-4(t-\tau)} h_3(t-\tau) d\tau \\ &= e^{-4t} h_3(t) * e^{-5t} h_2(t)\end{aligned}$$

Thus, the transfer function $\tilde{H}(s)$ is given by

$$\tilde{H}(s) = H_3(s+4)\ .\ H_2(s+5) = -\frac{15}{4} \frac{1}{(s+6)(s+9)}$$

Question 2 (20 marks)

An LTI causal system $\mathcal{S}$ has impulse response $h(t)$ given by

$$h(t) = \int_0^t \sin(3\tau)e^{-3(t-\tau)}d\tau$$

(a) Find the Frequency response function $H(\omega)$ of the system. (10 marks)

(b) An input $x(t) = 1 + 3\cos(3t)$ applied to the system results in output $y(t)$. Sketch the magnitude response $|Y(\omega)|$. (10 marks)

Solution:

(a)

$$\begin{aligned} h(t) &= \int_0^t \sin(3\tau)e^{-3(t-\tau)}d\tau \\ &= \int_{-\infty}^{\infty} \sin(3\tau)u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau \\ &= \{\sin(3t)u(t)\} * \{e^{-3t}u(t)\} \end{aligned}$$

Thus, the frequency response function $H(\omega)$ is given by the product of the Fourier Transforms as follows:

$$H(\omega) = \mathcal{F}\{\sin(3t)u(t)\} \,.\, \mathcal{F}\{e^{-3t}u(t)\}$$

$$\begin{aligned} \mathcal{F}\{\sin(3t)u(t)\} &= \mathcal{F}\left\{\frac{1}{2j}(e^{j3t} - e^{-3jt})u(t)\right\} \\ &= \mathcal{F}\left\{\frac{1}{2j}e^{j3t}u(t)\right\} - \mathcal{F}\left\{\frac{1}{2j}e^{-j3t}u(t)\right\} \\ &= \frac{1}{2j}\left[\frac{1}{j(\omega-3)} + \pi\delta(\omega-3)\right] - \frac{1}{2j}\left[\frac{1}{j(\omega+3)} + \pi\delta(\omega+3)\right] \\ &= \frac{3}{9-\omega^2} + \frac{\pi}{2j}\left[\delta(\omega-3) - \delta(\omega+3)\right] \end{aligned}$$

$$\mathcal{F}\{e^{-3t}u(t)\} = \int_{-\infty}^{\infty} e^{-3t}u(t)e^{-j\omega t}dt \;=\; \int_0^{\infty} e^{-(3+j\omega)t}dt \;=\; \frac{1}{j\omega+3}$$

Therefore, $H(\omega)$ can be given by the product:

$$H(\omega) = \left[\frac{3}{9-\omega^2} + \frac{\pi}{2j}\left[\delta(\omega-3) - \delta(\omega+3)\right]\right]\frac{1}{j\omega+3}$$

$$\implies H(\omega) = \begin{cases} \frac{3(3-j\omega)}{81-\omega^4} & \omega \neq 3, -3 \\ -\frac{\pi}{12}(1+j) & \omega = 3 \\ -\frac{\pi}{12}(1-j) & \omega = -3 \end{cases}$$

(b) Input $x(t) = 1 + 3\cos(3t)$.Thus, the Fourier transform is

$$X(\omega) = 2\pi\delta(\omega) + 3\pi\left[\delta(\omega - 3) + \delta(\omega + 3)\right]$$

The output $Y(\omega)$ can be found as follows:

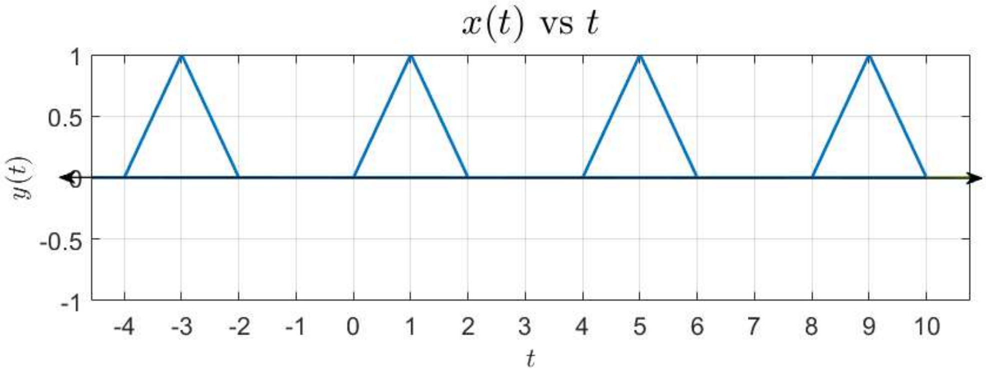
$$\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= H(\omega)\left[2\pi\delta(\omega) + 3\pi\left[\delta(\omega - 3) + \delta(\omega + 3)\right]\right] \\
&= 2\pi H(0)\delta(\omega) + 3\pi H(3)\delta(\omega - 3) + 3\pi H(-3)\delta(\omega + 3) \\
&= \frac{2\pi}{9}\delta(\omega) - \frac{\pi^2}{4}(1 + j)\delta(\omega - 3) - \frac{\pi^2}{4}(1 - j)\delta(\omega + 3)
\end{aligned}$$

Therefore, the magnitude response of the output $|Y(\omega)|$ can be obtained as follows:

$$|Y(\omega)| = \begin{cases} \frac{2\pi}{9} & \omega = 0 \\ \frac{\pi^2}{2\sqrt{2}} & \omega = \pm 3 \\ 0 & otherwise \end{cases}$$

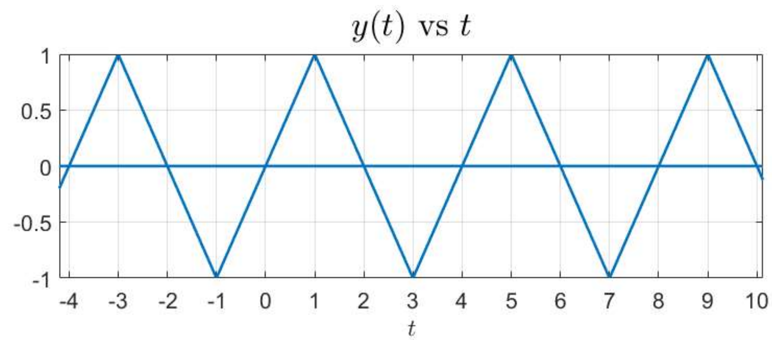
Question 3 (20 marks)

Consider a periodic signal $x(t)$ with period $T_o = 4$.



(a) Find the complex Fourier series coefficients (X_k) of $x(t)$. (8 marks)

(b) Consider the periodic signal $y(t)$ with period $T_o = 4$. (5 marks)



Use properties of Fourier Series to find the complex Fourier Series coefficients (Y_k) of $y(t)$ from X_k computed in part (a). Verify that the even harmonics of signal $y(t)$ are zero.

(c) Sketch the magnitude response $|Z(\omega)|$ of $z(t)$, where (7 marks)

$$z(t) = \left\{ y(t) \; * \; 7sinc\left(\frac{7\pi t}{4}\right)\right\}$$

$\tilde{x}(t) =$

(a) $x(t)[u(t) - u(t-4)] = t u(t) - 2(t-1)u(t-1) + (t-2)u(t-2)$.

$$\mathcal{L}_s\{\tilde{x}(t)\} = \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right]$$

$$\therefore X_k = \frac{1}{T_0} \tilde{X}(s)\Big|_{s = jk\omega_0}$$

$$T_0 = 4 \Rightarrow \omega_0 = \frac{\pi}{2}$$

$$\therefore X_k = \frac{1}{4} \tilde{X}(s)\Big|_{s = jk\frac{\pi}{2}}$$

$$= \frac{1}{4} \frac{4}{j^2 k^2 \pi^2}\left[1 - 2e^{-jk\frac{\pi}{2}} + e^{-jk\pi}\right]$$

$$\boxed{X_k = \frac{-1}{k^2\pi^2}\left[1 + (-1)^k - 2(-j)^k\right] \quad \forall\ k \neq 0}$$

$$X_0 = \frac{1}{4}\int_0^4 x(t)\,dt = \frac{1}{4}\int_0^1 t\,dt + \frac{1}{4}\int_1^2 (2-t)\,dt = \boxed{\frac{1}{4}}$$

(b) $y(t) = x(t) - x(t-2)$; $y(t)$ periodic with $T_0 = 4$; $\omega_0 = \frac{\pi}{2}$

$$\therefore Y_k = X_k - e^{-jk\left(\frac{\pi}{2}\right)\cdot 2} X_k$$

$$= X_k - e^{-jk\pi} X_k$$

$$= X_k - (-1)^k X_k$$

$$= \frac{2(j)^k}{k^2\pi^2}\left[(-1)^k - 1\right]$$

$$\therefore \boxed{Y_k = \begin{cases} -\dfrac{4(j)^k}{k^2\pi^2} & ;\ k \text{ odd} \\ 0 & k \text{ even} \end{cases}} \qquad \boxed{Y_0 = 0}$$

(c) $z(t) = y(t) * 7\,\mathrm{sinc}\left(\frac{7\pi t}{4}\right)$

$\Rightarrow Z(\omega) = Y(\omega) \cdot 4\,\mathrm{rec}\left(\omega, \frac{7\pi}{4}\right)$

$\because\ 7\,\mathrm{sinc}\left(\frac{7\pi t}{4}\right) \xleftrightarrow{F.T.} 4\,\mathrm{rec}\left(\omega, \frac{7\pi}{4}\right)$

- $y(t)$ is periodic with $\omega_0 = \pi/2$

$\Rightarrow Y(\omega) = \sum_{K=-\infty}^{\infty} 2\pi Y_K \delta\left(\omega - K\frac{\pi}{2}\right)$

- $4\,\mathrm{rec}\left(\omega, \frac{7\pi}{4}\right) = 0 \quad \forall\ |\omega| > \frac{7\pi}{4}$

$\therefore Z(\omega)$ will be non-zero only for $|\omega| < \frac{7\pi}{4}$

$\Rightarrow Z(\omega)$ will only contain frequencies corresponding to $\tilde{k}^{th}$ harmonics of $y(t)$,

s.t. $\left|\tilde{k}\frac{\pi}{2}\right| < \frac{7\pi}{4}$

$\Rightarrow \tilde{k} = \{-3, -2, -1, 0, 1, 2, 3\}$

- Let $\tilde{y}(t) = \sum_{K=-3}^{3} Y_K e^{jK\frac{\pi}{2}}$

$\therefore \tilde{Y}(\omega) = 2\pi\Big[Y_1\delta\left(\omega - \frac{\pi}{2}\right) + Y_{-1}\delta\left(\omega + \frac{\pi}{2}\right) + Y_3\delta\left(\omega - \frac{3\pi}{2}\right) + Y_{-3}\delta\left(\omega + \frac{3\pi}{2}\right)\Big]$

$\Rightarrow \cancel{\tilde{Y}(\omega) =}$

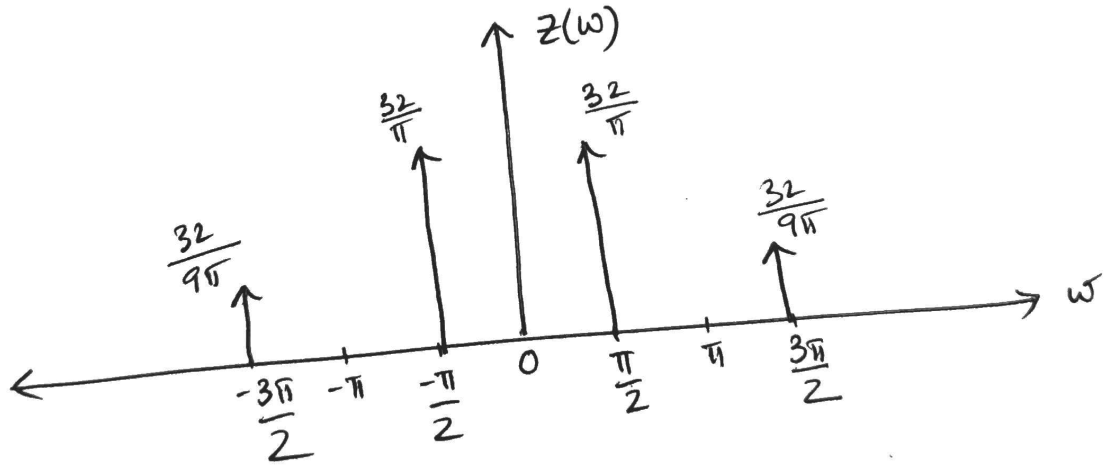
$$\Rightarrow \quad Z(\omega) = 4 \cdot 2\pi \left[Y_1 \delta\left(\omega - \frac{\pi}{2}\right) + Y_{-1}\, \delta\left(\omega + \frac{\pi}{2}\right) + Y_3\, \delta\left(\omega - \frac{3\pi}{2}\right) + Y_{-3}\, \delta\left(\omega + \frac{3\pi}{2}\right) \right]$$

$$= 4 \cdot \frac{4j}{\pi^2} \cdot 2\pi \left[\delta\left(\omega + \frac{\pi}{2}\right) - \delta\left(\omega - \frac{\pi}{2}\right) + \frac{\delta\left(\omega - \frac{3\pi}{2}\right)}{9} - \frac{\delta}{9}\left(\omega + \frac{3\pi}{2}\right) \right]$$

$$\because \; Y_1 = -\frac{4j}{\pi^2} \; ; \; Y_{-1} = \frac{4j}{\pi^2} \; ; \; Y_3 = \frac{4j}{9\pi^2} \; ; \; Y_{-3} = \frac{-4j}{9\pi^2}$$

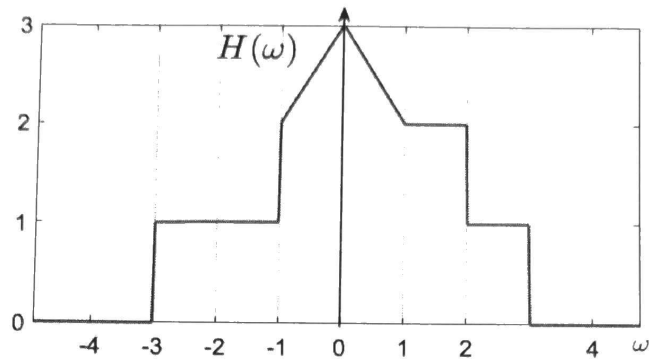
Thus,

$$|Z(\omega)| = \begin{cases} \frac{32}{\pi} & |\omega| = \frac{\pi}{2} \\ \frac{32}{9\pi} & |\omega| = \frac{3\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$



Question 4 (20 marks)

Consider a system $\mathcal{S}$ with impulse response $h(t)$ and frequency response $H(\omega)$ as shown below.



(a) Without finding $h(t)$ explicitly, find $\int_{-\infty}^{\infty} h(2t)dt$ (3 marks)

(b) Without finding $h(t)$ explicitly, find $h(0)$ (3 marks)

(c) Without finding $h(t)$ explicitly, compute $\int_{-\infty}^{\infty} |h(t)|^2 dt$ (4 marks)

(d) Find the Inverse Fourier transform $h(t)$ without performing any integration. (10 marks)
Hint: Use linearity property of Fourier transform to decompose $H(\omega)$. Thereafter, use sinc-rec Fourier transform pairs and properties of Fourier transforms.

-- breakup. — (1)
1. sym. rect (2)
2. asym. rect (3)
3. tri = rect * rect (4)

(a) $\int_{-\infty}^{\infty} h(2t)\,dt$

$$\mathcal{F}\{h(t)\} = H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$\Rightarrow \mathcal{F}\{h(2t)\} = \frac{1}{2} H\left(\frac{\omega}{2}\right) = \int_{-\infty}^{\infty} h(2t) e^{-j\omega t} dt$$

$$\therefore \int_{-\infty}^{\infty} h(2t)\,dt = \frac{1}{2} H\left(\frac{0}{2}\right) = \frac{1}{2} H(0) = \boxed{\frac{3}{2}}$$

(b) $$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega$$

$$\Rightarrow h(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, d\omega = \frac{1}{2\pi}\left[\text{Area under } H(\omega)\right]$$

$$= \frac{1}{2\pi}\left[6+3+1\right]$$

$$= \boxed{\frac{5}{\pi}}$$

(c) $$\int_{-\infty}^{\infty} |h(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = \int_{-3}^{-1} (1)\, d\omega + \int_{2}^{3} (1)\, d\omega + \int_{1}^{2} (4)\, d\omega$$

$$+ \int_{-1}^{0} (3+\omega)^2 d\omega + \int_{0}^{1} (3-\omega)^2 d\omega$$

$$= 7 + \int_{-1}^{0} (9+\omega^2+6\omega)\, d\omega + \int_{0}^{1} (9+\omega^2-6\omega)\, d\omega$$

$$= 7 + 9 + \frac{1}{3} - 3 + 9 + \frac{1}{3} - 3$$

$$= \frac{59}{3}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{59}{6\pi}}$$

(d) $H(\omega) = \text{rec}(\omega, 3) + \text{rec}\left(\omega - \frac{1}{2}, \frac{3}{2}\right) + \text{tri}(\omega, 1)$

$= \text{rec}(\omega, 3) + \text{rec}\left(\omega - \frac{1}{2}, \frac{3}{2}\right) + \text{rec}\left(\omega, \frac{1}{2}\right) * \text{rec}\left(\omega, \frac{1}{2}\right)$

$$\boxed{\therefore\ h(t) = \frac{3}{\pi} \,\text{sinc}(3t) + \frac{3}{2\pi} \,\text{sinc}\left(\frac{3}{2}t\right) \cdot e^{\frac{jt}{2}} + \frac{1}{2\pi} \,\text{sinc}^2\left(\frac{t}{2}\right)}$$

Question 5 (20 marks)

Consider a real system with impulse response function $h(t) = 2sinc^2(\pi t)\,[1 + 2\cos(2\pi t)]$.

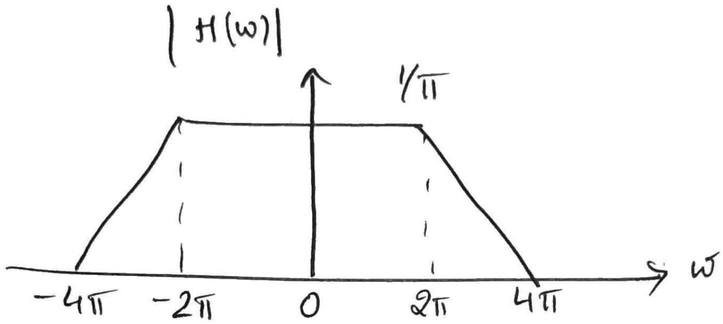
(a) Find the frequency response $H(\omega)$ and sketch its magnitude response. (12 marks)

Hint: Use multiplication property of Fourier transform.

$$x(t)y(t) \overset{\mathcal{F}}{\rightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

(b) Find the energy contained in output $y(t)$ when an input $\delta(t)$ is applied to the above system, using Parseval's theorem. (8 marks)

(a)



· correct freq.
· correct amplitude.

(b) $$\int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-4\pi}^{-2\pi} \left(\frac{\omega}{2\pi^2} + \frac{2}{\pi}\right)^2 d\omega + \int_{-2\pi}^{2\pi} \frac{1}{2\pi}\cdot\frac{1}{\pi^2}\cdot d\omega$$

$$+ \frac{1}{2\pi}\int_{2\pi}^{4\pi} \left(-\frac{\omega}{2\pi^2} + \frac{2}{\pi}\right)^2 d\omega$$

$$= \frac{1}{2\pi}\left[\frac{2\pi^2}{3}\left(\frac{\omega}{2\pi^2} + \frac{2}{\pi}\right)^3\Bigg|_{-4\pi}^{-2\pi} - \frac{2\pi^2}{3}\left(-\frac{\omega}{2\pi^2} + \frac{2}{\pi^2}\right)^3\Bigg|_{2\pi}^{4\pi}\right.$$

$$\left. + \frac{1}{2\pi}\cdot 4\pi \cdot \frac{1}{\pi^2}\right].$$

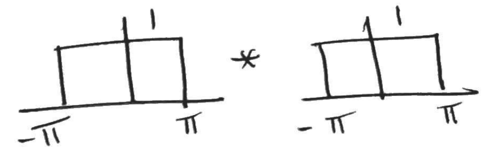
$$= \frac{\pi}{3}\left(\frac{1}{\pi^3} - 0\right) - \frac{\pi}{3}\left(0 - \frac{1}{\pi^3}\right) + \frac{2}{\pi^2}$$

$$= \frac{2\pi}{3}\cdot\frac{1}{\pi^3} + \frac{2}{\pi^2} = \boxed{\frac{8}{3\pi^2}}$$

15

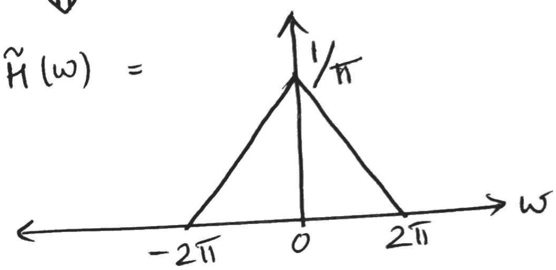
(a) . $h(t) = 2\,\text{sinc}^2(\pi t)\left[1 + 2\cos(2\pi t)\right]$

$$2\,\text{sinc}^2(\pi t) \xleftrightarrow{\text{F.T.}} \tilde{H}(\omega) = 2 \cdot \frac{1}{2\pi}\left\{\text{rec}(\omega,\pi) * \text{rec}(\omega,\pi)\right\}$$

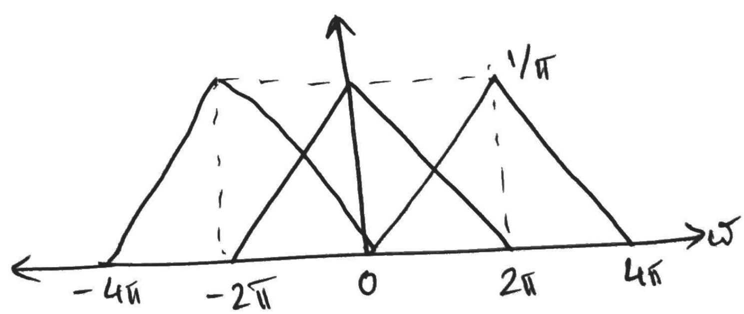


$\Downarrow$

$\tilde{H}(\omega) =$



$$\tilde{h}(t) + \underbrace{\tilde{h}(t) \cdot 2\cos(2\pi t)}_{\tilde{h}(t)\left[e^{j2\pi t} + e^{-j2\pi t}\right]} \xleftrightarrow{\text{F.T.}} \tilde{H}(\omega) + \tilde{H}(\omega - 2\pi) + \tilde{H}(\omega + 2\pi).$$



Thus,