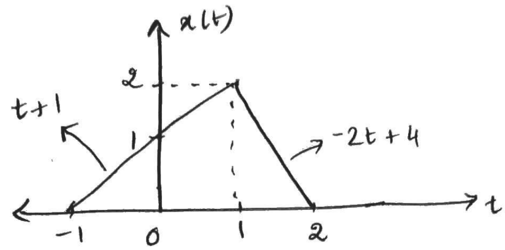
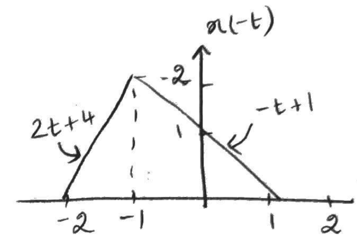
MIDTERM REVIEW PROBLEMS

SOLUTIONS

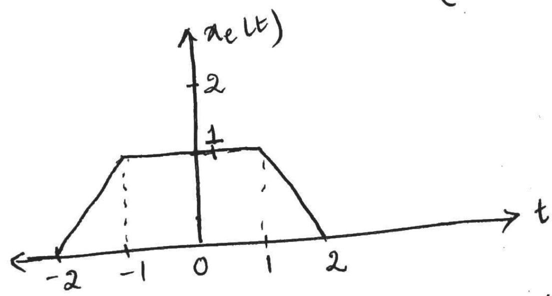
Question 1

(a)

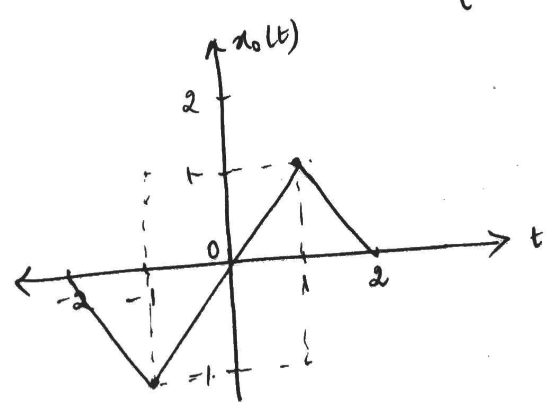




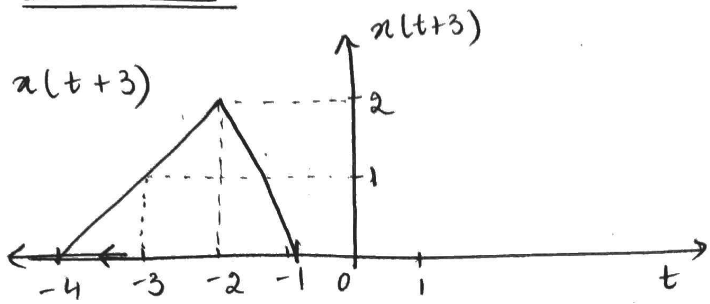
$$x_e(t) = \frac{x(t)+x(-t)}{2} = \begin{cases} t+2 & -2 \leq t < -1 \\ \frac{(t+1)+(-t+1)}{2} = 1 & -1 \leq t < 1 \\ -t+2 & 1 \leq t < 2 \end{cases}$$

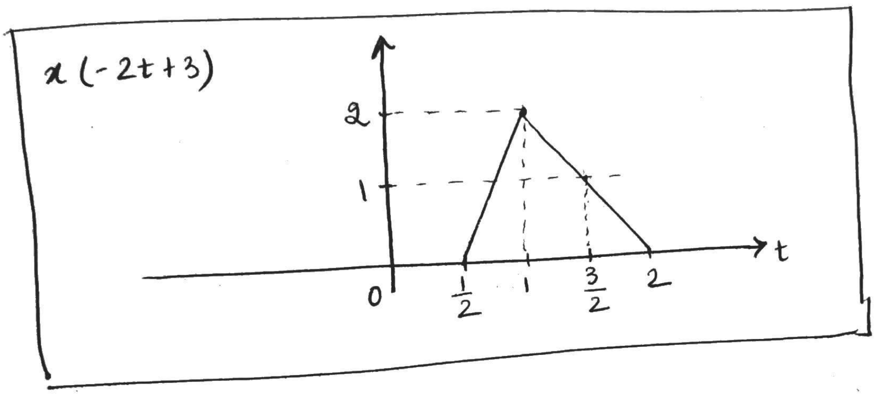


$$x_o(t) = \frac{x(t)-x(-t)}{2} = \begin{cases} -(t+2) & -2 \leq t < -1 \\ t & -1 \leq t < 1 \\ -t+2 & 1 \leq t < 2 \end{cases}$$

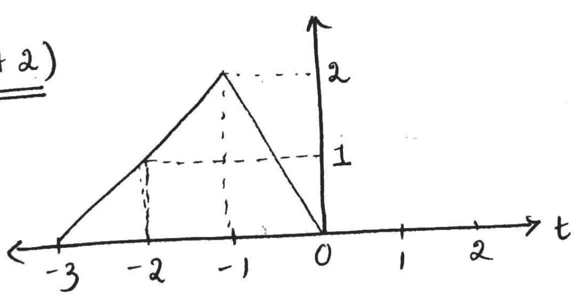


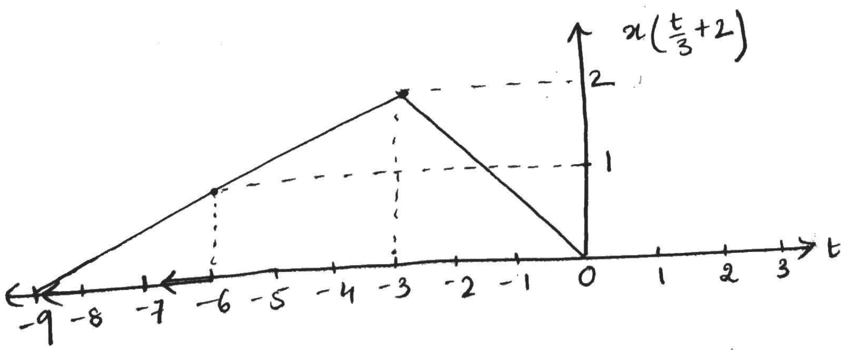
(b) $x(-2t+3)$;


x(t+3)
x(t+3)
2
1
-4 -3 -2 -1 0 1
t


x(-2t+3)
2
1
0
1/2 1 3/2 2
t

(c) $x(t+2)$

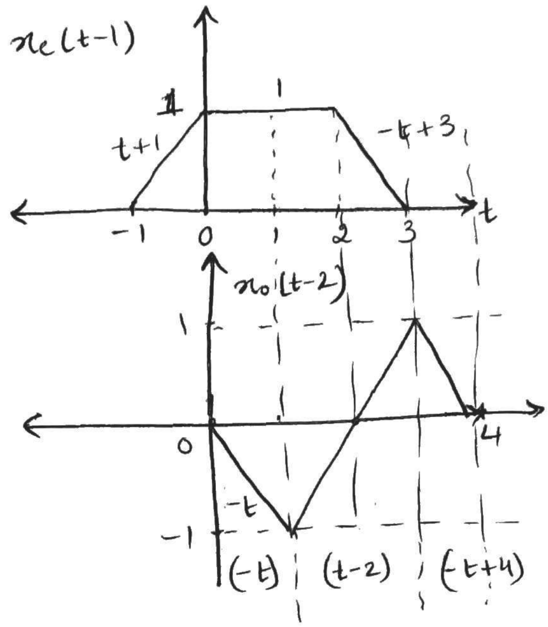

2
1
-3 -2 -1 0 1 2
t


x(t/3+2)
2
1
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3
t

(4)

(2)

(c) $y(t) = \frac{3}{5} x\left(\frac{t}{3}+2\right) - j\, x_e(t-1) \cdot x_o(t-2)$.



$(3-t)(t-2)$.

$$x_e(t-1)\, x_o(t-2) = \begin{cases} -t & 0 \le t < 1 \\ t-2 & 1 \le t < 2 \\ (-t+3)(t-2) & 2 \le t < 3 \\ 0 & \text{elsewhere} \end{cases}$$

Energy of $y(t)$

E_y = Energy of real part + energy of Img. part

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} \frac{9}{25} x^2\left(\frac{t}{3}+2\right) dt + \int_{-\infty}^{\infty} \left(x_e(t-1)\, x_o(t-2)\right)^2 dt$$

$\therefore$ Energy of Img part $= \int_{-\infty}^{\infty} \left(x_e(t-1)\, x_o(t-2)\right)^2 dt$

$$= \int_0^1 t^2 dt + \int_1^2 (t-2)^2 dt + \int_2^3 (t^2-5t+6)^2 dt$$

$$= \frac{1}{3} + \frac{1}{3}(+1) +$$

$$= \underline{0.7}.$$

Energy of real part:

$$\int_{-\infty}^{\infty} \frac{9}{25} x^2\left(\frac{t}{3}+2\right) dt$$

$$= \cancel{\frac{9}{25}\int_{-\infty}^{\infty}\left(\frac{t}{3}+3\right)} = \frac{9}{25}\int_{-9}^{-3}\left(\frac{t}{3}+3\right)^2 dt + \frac{9}{25}\int_{-3}^{0}\frac{4}{9}t^2 dt$$

$-\frac{2t}{3}$

$$= \frac{9}{25}\left[\int_{-9}^{-3}\left(\frac{t}{3}+3\right)^2 dt. + \frac{4}{25}\int_{-3}^{0} t^2 dt\right]$$

$$= \frac{9}{25}\cdot 3 \cdot \frac{1}{3}\left(\frac{t}{3}+3\right)^3\Big|_{-9}^{-3} + \frac{4}{25}\cdot\frac{1}{3}\left(\frac{t^3}{3}\right)_{-3}^{0}$$

$$= \frac{9}{25}(8) + \frac{4}{25}\cdot\frac{1}{3}\left(\frac{27}{3}\right)$$

$$= \frac{72+12}{25} = \frac{84}{25}$$

$\therefore$ Energy of $y(t)$

$$E_{y(t)} = 3.36 + 0.7$$
$$= \boxed{4.06.}$$

Question 2

$$x(t) = \sin\left(\omega_0 t + \frac{\pi}{6}\right)\cos\left(\gamma_0 t - \frac{\pi}{2}t\right) - \cos^2\left((\omega_0 - \gamma_0)t\right)$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos(2A) = 2\cos^2 A - 1$$

$$\therefore x(t) = \frac{1}{2}\left[\sin\left(\left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right)t + \frac{\pi}{6}\right) + \sin\left(\left(\omega_0 - \gamma_0 + \frac{\pi}{2}\right)t + \frac{\pi}{6}\right)\right]$$
$$- \frac{1}{2}\left(1 + \cos\left(2(\omega_0 - \gamma_0)t\right)\right)$$

$$\Rightarrow x(t) = \frac{1}{2}\left[\sin\left(\left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right)t + \frac{\pi}{6}\right) + \sin\left(\left(\omega_0 - \gamma_0 + \frac{\pi}{2}\right)t + \frac{\pi}{6}\right) - \cos\left(2(\omega_0 - \gamma_0)t\right) - 1\right]$$

$$\sin\left(\left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right)t + \frac{\pi}{6}\right) : \text{period} = \frac{2\pi}{\omega_0 + \gamma_0 - \frac{\pi}{2}}$$

$$\sin\left(\left(\omega_0 - \gamma_0 + \frac{\pi}{2}\right)t + \frac{\pi}{6}\right) : \text{period} = \frac{2\pi}{\omega_0 - \gamma_0 + \frac{\pi}{2}}$$

$$\cos\left(2(\omega_0 - \gamma_0)t\right) : \text{period} = \frac{2\pi}{2(\omega_0 - \gamma_0)}$$

Thus, for $x(t)$ to be periodic,

$$\left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right) : \left(\omega_0 - \gamma_0 + \frac{\pi}{2}\right) : 2(\omega_0 - \gamma_0) = M : N : K$$

where, M, N, K are coprime

ie $$\frac{\omega_0 + \gamma_0 - \pi/2}{\omega_0 - \gamma_0 + \pi/2} = \frac{M}{N} \; ; \; \frac{\omega_0 + \gamma_0 - \frac{\pi}{2}}{2(\omega_0 - \gamma_0)} = \frac{M}{K}$$ and so on

The time period T_0 = LCM of $\left\{ \frac{2\pi}{\omega_0 - \gamma_0 + \frac{\pi}{2}}, \frac{2\pi}{\omega_0 + \gamma_0 - \frac{\pi}{2}}, \frac{2\pi}{2(\omega_0 - \gamma_0)} \right\}$.

$$\Rightarrow \quad T_0 \geq \max \left\{ \frac{2\pi}{\omega_0 - \gamma_0 + \frac{\pi}{2}}, \frac{2\pi}{\omega_0 + \gamma_0 - \frac{\pi}{2}}, \frac{2\pi}{2(\omega_0 - \gamma_0)} \right\}$$

$\therefore$ ~~The~~ We need to order the 3

i) we find that $\left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right) \geq \omega_0 - \gamma_0 + \frac{\pi}{2}$ if $\gamma_0 > \frac{\pi}{2}$

ii) $2(\omega_0 - \gamma_0) \geq \omega_0 - \gamma_0 + \frac{\pi}{2}$ if $\omega_0 > \pi$, $\gamma_0 > \frac{\pi}{2}$

iii) $2(\omega_0 - \gamma_0) \geq \left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right)$ if $\omega_0 > \pi$, $\gamma_0 > \frac{\pi}{2}$

The above inequalities are consistent with the assumption $\omega_0 > \pi$; $\gamma_0 > \frac{\pi}{2}$

$\therefore \forall \; \omega_0 > \pi$; $\gamma_0 > \frac{\pi}{2}$

$$2(\omega_0 - \gamma_0) \geq \left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right) \geq \left(\omega_0 - \gamma_0 + \frac{\pi}{2}\right)$$

$\therefore$ max time period $= \dfrac{2\pi}{\omega_0 - \gamma_0 + \frac{\pi}{2}}$

$$\therefore \quad T_0 \geq \frac{2\pi}{\omega_0 - \gamma_0 + \pi/2}$$

Thus, we can conclude that $\boxed{T_0 = \dfrac{2\pi k}{\omega_0 - \gamma_0 + \pi/2} \; ; \; k \in \mathbb{N}}$ $\forall \; \omega_0 > \pi$, $\gamma_0 > \frac{\pi}{2}$

$-T_0/2$

$$P_x = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sin^2\left(\left(\omega_0 + \gamma_0 - \frac{\pi}{2}\right)t + \frac{\pi}{6}\right)dt + \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sin^2\left(\left(\omega_0 - \gamma_0 + \frac{\pi}{2}\right)t + \frac{\pi}{6}\right)dt$$

$$+ \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left(\cos^2(2(\omega_0 - \gamma_0)t) + 2\cos(2(\omega_0 - \gamma_0)) + 1\right.$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1$$

$$= \boxed{\frac{5}{2}}.$$

Question 1

(a) $$y(t) = e^{-t} x(t-2) - \int_0^{t/2} x(\tau) e^{2\tau - t} d\tau$$

$$h(t,\tau) = e^{-t} \delta(t-\tau-2) - \int_0^{t/2} \underbrace{x(\sigma)}_{\to \delta(\sigma-\tau)} e^{2\sigma - t} d\sigma$$

$$= e^{-t} \delta(t-\tau-2) - \int_{-\infty}^{\infty} \delta(\sigma-\tau) e^{2\sigma-t} u(\sigma) u\left(\tfrac{t}{2}-\sigma\right) d\sigma$$

$$= e^{-t}\delta(t-\tau-2) - e^{2\tau-t} u(\tau) u\left(\tfrac{t}{2}-\tau\right)$$

$$\boxed{h(t,\tau) = e^{-t} \delta(t-\tau-2) - e^{2\tau-t} u(\tau) u\left(\tfrac{t}{2}-\tau\right)}$$

(b) Time Invariance

$$h(t,\tau) \neq h(t-\tau, 0).$$

$$\because h(t-\tau,0) = e^{-(t-\tau)} \delta(t-\tau-2) - e^{-(t-\tau)} u\left(\tfrac{t-\tau}{2}\right)$$

$$\neq h(t,\tau).$$

$\therefore$ $\boxed{\text{Time Varying}}$

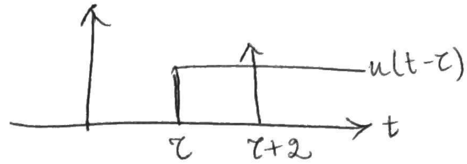
(c) Causality:

$$h(t,\tau) u(t-\tau) = \left(e^{-t} \delta(t-\tau-2) - e^{2\tau-t} u(\tau) u\left(\tfrac{t}{2}-\tau\right)\right) u(t-\tau)$$

$$= \underbrace{e^{-t} \delta(t-\tau-2) u(t-\tau)} - e^{2\tau-t} u(\tau) u\left(\tfrac{t}{2}-\tau\right) u(t-\tau)$$

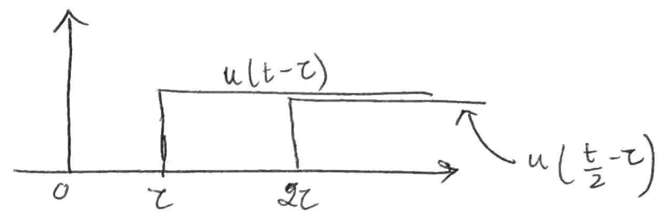
$$= e^{-t} \delta(t-\tau-2) - e^{2\tau-t} u(\tau) u\left(\tfrac{t}{2}-\tau\right) u(t-\tau)$$

i) $\delta(t-(\tau+2))\,u(t-\tau)$.



$\Rightarrow \quad \delta(t-\tau-2)\,u(t-\tau) = \delta(t-\tau-2)$.

ii) $u\left(\frac{t}{2}-\tau\right)u(t-\tau) =$



Note $\tau > 0 \quad \because u(\tau)$

$\therefore \quad u\left(\frac{t}{2}-\tau\right)u(t-\tau).u(\tau) = u\left(\frac{t}{2}-\tau\right)u(\tau)$.

$\therefore \quad h(t,\tau)\,u(t-\tau) = e^{-t}\delta(t-\tau-2) - e^{2\tau-t}u(\tau)\,u\left(\frac{t}{2}-\tau\right)$

$= h(t,\tau)$

$\therefore$ $\boxed{\text{Causal}}$

(c). Input $x(t) = e^{-t}u(t-2)$

$$y(t) = \int_{-\infty}^{\infty} x(\sigma)\,h(t,\sigma)\,d\sigma$$

$$= \int_{-\infty}^{\infty} e^{-\sigma}u(\sigma-2)\left[e^{-t}\delta(t-\sigma-2) - e^{2\sigma-t}u(\sigma)\,u\left(\frac{t}{2}-\sigma\right)\right]d\sigma$$

$\sigma = (t-2)$

$$= \cancel{\int_{-\infty}^{\infty}} e^{-(t-2)}u(t-4).e^{-t} - \int_{-\infty}^{\infty} e^{\sigma-t}u(\sigma)\,u\left(\frac{t}{2}-\sigma\right)u(\sigma-2)\,d\sigma$$

$u\left(\frac{t}{2}-\sigma\right)u(\sigma-2) \;\forall\, \sigma \geq 0$:

non-zero product over $\sigma \in [2, 2t]$

If $\frac{2t}{2} \geq 2 \Rightarrow$

If $\frac{2t}{2} < 2 \Rightarrow 0$

$\therefore \quad u(\sigma)\, u(\sigma - 2)\, u\left(\frac{t}{2} - \sigma\right) = \begin{cases} 1 & t \geq 4 \\ 0 & t < 4. \end{cases}$

If $t \geq 4$: $\quad y(t) = e^{-(t-2)} u(t-4) e^{-t} - \int_{2}^{t/2} e^{\sigma - t} \cdot d\sigma$

$$= e^{-(t-2)} u(t-4) e^{-t} - e^{-t}\left(e^{t/2} - e^{2}\right)$$

$$= e^{-2t+2} u(t-4) - e^{-t}\left(e^{t/2} - e^{2}\right).$$

$$\therefore \boxed{y(t) = \left[e^{-2t+2} - e^{-t}\left(e^{t/2} - e^{2}\right)\right] u(t-4)}$$

Question (2)

$$x(t) \longrightarrow \boxed{S_1} \xrightarrow{y(t)} \boxed{S_2} \longrightarrow z(t)$$

(a)

$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) * \{ h_1(t) + h_2(t) \}$$ $\because$ convolution is a linear operation.

$$\therefore y(t) = x(t) * \{ \delta(t-1) + \delta(t-2) \}.$$

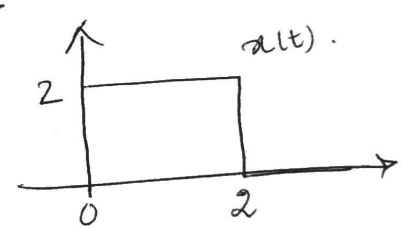
$$= \int_{-\infty}^{\infty} x(\tau) \left[\delta(t-\tau-1) + \delta(t-\tau-2) \right] d\tau .$$

$$= \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau-1) d\tau + \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau-2) d\tau$$

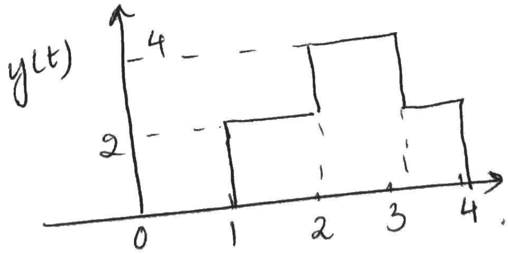
$$\boxed{y(t) = x(t-1) + x(t-2).}$$

(b)

$$x(t) = 2.(u(t) - u(t-2)).$$



$$\therefore y(t) = 2u(t-1) - 2u(t-3) + 2u(t-2) - 2u(t-4).$$



(b)

$$h_{eff}(t) = (h_1 + h_2)(t) * h_3(t).$$

$$h_3(t) = \delta(t-1) + \delta(t-2) + \delta(t-3).$$

$$h_{eff}(t) = \underbrace{\{\delta(t-1) + \delta(t-2)\}}_{\tilde{h}(t)} * \{\delta(t-1) + \delta(t-2) + \delta(t-3)\}.$$

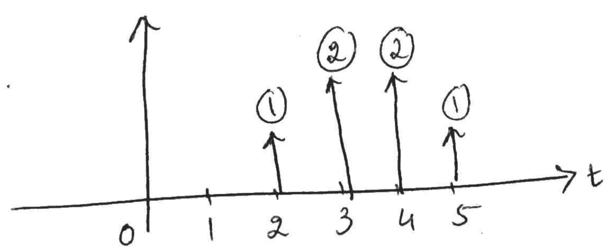
$$\tilde{h}(t) * \{\delta(t-1) + \delta(t-2) + \delta(t-3)\}$$

$$= \tilde{h}(t-1) + \tilde{h}(t-2) + \tilde{h}(t-3).$$

$$= \big(\delta(t-2) + \delta(t-3)\big) + \big(\delta(t-3) + \delta(t-4)\big) + \big(\delta(t-4) + \delta(t-5)\big).$$

$$\boxed{h_{eff}(t). = \delta(t-2) + 2\delta(t-3) + 2\delta(t-4) + \delta(t-5).}$$

(c) $h_{eff}(t)$.



$$h_{eff}(t) = h_{eff}(t).u(t)$$

$\therefore h_{eff}(t)$: represents a <u>causal system</u>.

(d)

$$z(t) = x(t) * h_{eff}(t)$$

$$= x(t) * \{2\delta(t-3) + \delta(t-2) + 2\delta(t-4) + \delta(t-5)\}.$$

$$= x(t-2) + 2x(t-3) + 2x(t-4) + x(t-5)$$

$$= 2\big[u(t-2) - u(t-4) + 2u(t-3) - 2u(t-5) + 2u(t-4) - 2u(t-6) + 2u(t-5) - u(t-7)\big]$$

Question (3)

(a) $$[x(t) * h(t)] * g(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau * g(t)$$

$$= \int_{-\infty}^{\infty} g(\sigma) \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau-\sigma)\, d\tau\, d\sigma$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} g(\sigma)\, h(t-\tau-\sigma)\, d\sigma\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) . \left\{ \int_{-\infty}^{\infty} g(\sigma)\, h(t-\tau-\sigma)\, d\sigma \right\} d\tau$$

$$= x(t) * \int_{-\infty}^{\infty} g(\sigma)\, h(t-\sigma)\, d\sigma$$

$$= x(t) * [g(t) * h(t)]$$

$$= x(t) * [h(t) * g(t)].$$

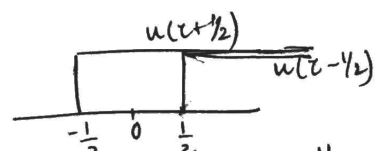
— Hence proved.

(b) $y(t) = x(t) * h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left[u(\tau+\tfrac{1}{2}) - u(\tau-\tfrac{1}{2})\right] e^{j\omega_0(t-\tau)}\, d\tau$$

$u(\tau+\frac{1}{2}) - u(\tau-\frac{1}{2})$.



$$\therefore y(t) = \int_{-1/2}^{1/2} e^{j\omega_0 t} . e^{-j\omega_0 \tau}\, d\tau = \frac{e^{j\omega_0 t}}{-j\omega_0} \left[e^{-j\omega_0 \tau} \right]_{-1/2}^{1/2}$$

$$y(t) = \frac{e^{j\omega_0 t}}{-j\omega_0}\left[e^{-j\frac{\omega_0}{2}} - e^{+j\frac{\omega_0}{2}}\right]$$

$$= \frac{e^{j\omega_0 t}}{\omega_0}\left(\frac{e^{j\omega_0/2} - e^{-j\omega_0/2}}{j}\right)$$

$$\boxed{y(t) = \frac{e^{j\omega_0 t}}{\omega_0} \cdot 2\sin\left(\frac{\omega_0}{2}\right)}$$

$$y(0) = 0 \Rightarrow \frac{e^{0}}{\omega_0}\, 2 \cdot \sin\left(\frac{\omega_0}{2}\right) = 0$$

$$\sin\left(\frac{\omega_0}{2}\right) = 0 \Rightarrow \boxed{\omega_0 = 2\pi k \;;\; k \in \mathbb{Z}}$$

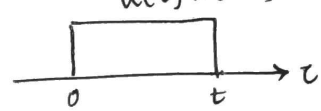
LAPLACE TRANSFORM

Question (1)

(a) $$x(t) = \int_0^t \sin(t-\tau)\, e^{-(2t+3\tau)} \cos(\tau)\, d\tau.$$

Change limits of Integration to $(-\infty$ to $\infty)$ by multiplying with $u(\tau)\,u(t-\tau)$.

$u(\tau)\,u(t-\tau)$.



$$x(t) = \int_{-\infty}^{\infty} \sin(t-\tau)\, e^{-2t+3\tau} \cos(\tau)\, d\tau$$

express $x(t)$ as a convolution integral.

$$\text{ie} \quad x(t) = \int_{-\infty}^{\infty} \tilde{x}(\tau)\, \tilde{h}(t-\tau)\, d\tau \quad = \tilde{x}(t) * \tilde{h}(t)$$

$$\therefore \quad X(s) = \tilde{X}(s) \cdot \tilde{H}(s).$$

$$x(t) = \int_{-\infty}^{\infty} \sin(t-\tau) \cdot e^{-2t+2\tau} \cdot e^{-5\tau} \cos(\tau)\, u(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{\sin(t-\tau)\, e^{-2(t-\tau)}\, u(t-\tau)}_{\tilde{h}(t-\tau)} \cdot \underbrace{e^{-5\tau} \cos(\tau)\, u(\tau)}_{\tilde{x}(\tau)}\, d\tau$$

$$\therefore x(t) = \underbrace{\{ e^{-5t} \cos(t)\, u(t) \}}_{\tilde{x}(t)} * \underbrace{\{ \sin t\, e^{-2t}\, u(t) \}}_{\tilde{h}(t)}$$

$$\tilde{x}(t) = e^{-5t} \cos(t)\, u(t)$$

$$\Rightarrow \tilde{X}(s) = \frac{(s+5)}{(s+5)^2 + 1} \qquad Re\{s\} > -5$$

$$\tilde{h}(t) = e^{-2t} \sin t\, u(t)$$

$$\Rightarrow \tilde{H}(s) = \frac{1}{(s+2)^2 + 1} \qquad Re\{s\} > -2.$$

Thus, $X(s) = \tilde{X}(s)\,\tilde{H}(s)$

$$\boxed{X(s) = \frac{s+5}{(s+5)^2+1} \cdot \frac{1}{(s+2)^2+1} \quad ; \; Re\{s\} > -2.}$$

(b) $$x(t) = \underbrace{(t^2 + t e^{-4t}) \sin(2\omega_0 t - \pi)}_{x_1(t)} + \underbrace{\int_{-\infty}^{t} \tau^2 u(\tau-2)\, d\tau}_{x_2(t)}$$

$$x(t) = x_1(t) + x_2(t).$$

$$x_2(t) = \int_{-\infty}^{t} \tau^2 u(\tau-2)\, d\tau = \int_{-\infty}^{\infty} \tau^2 u(\tau-2)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \tau^2 u(\tau-2)\; u(t-\tau)\, d\tau$$

$$= t^2 u(t-2) * u(t).$$

$\underline{t^2 u(t-2)}$: Express in terms of ~~$t^n u(t)$~~ or

$t^n u(t) \longleftrightarrow \frac{n!}{s^{n+1}}$ or $(t-a)^n u(t) \longleftrightarrow e^{-as} \cdot \frac{n!}{s^{n+1}}.$

$$t^2 = (t^2 - 4 + 4t) - 4t + 4$$

$$= (t-2)^2 - 4t + 8 - 4$$

$$t^2 = (t-2)^2 - 4(t-2) - 4$$

$$\therefore t^2 u(t-2) = (t-2)^2 u(t-2) - 4(t-2)\,u(t-2) - 4\,u(t-2)$$

$$\frac{e^{-2s} \cdot 2}{s^3} \qquad \frac{4e^{-2s}}{s^2} \qquad \frac{4e^{-2s}}{s}$$

$$\therefore \boxed{t^2 u(t-2) \longleftrightarrow e^{-2s}\left[\frac{2}{s^3} - \frac{4}{s^2} - \frac{4}{s}\right]}$$

$$\therefore \boxed{X_2(s) = \frac{2e^{-2s}}{s^2}\left[\frac{1}{s^2} - \frac{2}{s} - 2\right]} \quad \text{ROC}: \text{Re}\{s\} > 0.$$

$$x_1(t) = \left(t^2 + te^{-4t}\right)\sin(2\omega_0 t - \pi)$$

$$= t^2 \sin(2\omega_0 t - \pi) + te^{-4t}\sin(2\omega_0 t - \pi).$$

$$\sin\left(2\omega_0\left(t - \frac{\pi}{2\omega_0}\right)\right) \longleftrightarrow \frac{e^{-\pi/2\omega_0 s} \cdot 2\omega_0}{s^2 + 4\omega_0^2}$$

$$t\sin\left(2\omega_0\left(t - \frac{\pi}{2\omega_0}\right)\right) \longleftrightarrow -\frac{d}{ds}\left[\frac{e^{-\frac{\pi}{2\omega_0}s} \cdot 2\omega_0}{s^2 + 4\omega_0^2}\right]$$

$$-\frac{d}{ds}\left[\frac{e^{-\frac{\pi}{2\omega_0}s} \cdot 2\omega_0}{s^2 + 4\omega_0^2}\right] = \frac{(s^2+4\omega_0^2)\left(\frac{-\pi}{2\omega_0}\right)e^{-\pi/2\omega_0 s} - e^{-\pi/2\omega_0 s}(2s)}{(s^2+4\omega_0^2)^2}$$

$$= -e^{-\frac{\pi}{2\omega_0}s}\left[\frac{(s^2+4\omega_0^2)\frac{\pi}{2\omega_0} + 2s}{(s^2+4\omega_0^2)^2}\right]$$

$$t\sin\left(2\omega_0\left(t - \frac{\pi}{2\omega_0}\right)\right) \longleftrightarrow -e^{-\frac{\pi}{2\omega_0}s}\left[\frac{\pi}{s^2+4\omega_0^2} + \frac{4s\omega_0}{(s^2+4\omega_0^2)^2}\right]$$

$$\Rightarrow \quad e^{-4t}\, t\sin(2\omega_0 t - \pi) \longleftrightarrow$$

$$X_1^b(s) = -e^{-\frac{\pi}{2\omega_0}(s+4)}\left[\frac{\pi}{(s+4)^2+4\omega_0^2} + \frac{4(s+4)\omega_0}{(s+4)^2+4\omega_0^2}\right]$$

$$t \sin(2\omega_0 t - \pi) \longleftrightarrow -e^{-\pi/2\omega_0 s}\left[\frac{\pi}{s^2+4\omega_0^2} + \frac{4\omega_0 s}{(s^2+4\omega_0^2)^2}\right] \quad (11)$$

$$\Rightarrow t^2 \sin(2\omega_0 t - \pi) \longleftrightarrow -e^{-\frac{\pi}{2\omega_0}s}\left[-\frac{2\pi s}{(s^2+4\omega_0^2)^2} + \frac{\left(4\omega_0 (s^2+4\omega_0^2)^2 - 4\omega_0 s(2)(s^2+4\omega_0^2)(2s)\right)}{(s^2+4\omega_0^2)^4}\right]$$

$$t^2 \sin(2\omega_0 t - \pi) \longleftrightarrow$$

$$X_1^a(s) = -e^{-\frac{\pi}{2\omega_0}s}\left[\frac{-2\pi s}{(s^2+4\omega_0^2)^2} + \frac{4\omega_0}{(s^2+4\omega_0^2)^2} - \frac{16\omega_0 s^2}{(s^2+4\omega_0^2)^3}\right]$$

$$+ e^{-\pi/2\omega_0 s}\cdot\left(\frac{\pi}{2\omega_0}\right)\left[\frac{\pi}{s^2+4\omega_0^2} + \frac{4\omega_0 s}{(s^2+4\omega_0^2)^2}\right].$$

$$\boxed{X(s) = X_1^a(s) + X_1^b(s) + X_2(s).}$$

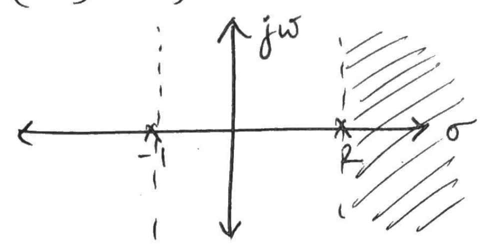
Question (3)

$$H_1(s) = \frac{e^{-2j}}{(s+1)(s-2)} \quad ; \quad H_2(s) = \frac{e^{-4s}\, s(s-2)}{(s+1)(s+2)}$$

$$H_3(s) = \frac{1}{s(s+1)}$$

(a), (b) $H_1(s) = \dfrac{e^{-2j}}{(s+1)(s-2)}$ poles : $s = -1, 2$

zeros : no zeros.



- ROC cannot contain poles
- For causal system, ROC is to the right of the rightmost pole.

ROC for causality : $Re\{s\} > 2$

BIBO stability requires that the entire $j\omega$ axis lies in the ROC.

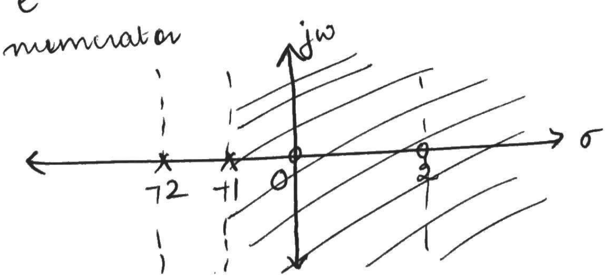
The ROC $Re\{s\} > 2$ does not contain $j\omega$ axis

$\therefore$ BIBO unstable.

(b) $H_2(s) = \dfrac{e^{-4s}\, s(s-2)}{(s+1)(s+2)}$ poles : $s = -1, -2$

zeros : $s = 0, 2$, zero at ∞.

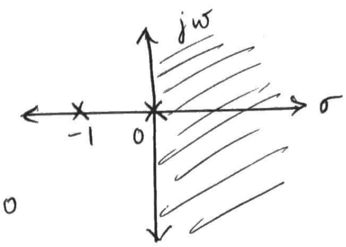
the e^{-4s} term in numerator $\Rightarrow$ zero at ∞.



For causality : ROC is $Re\{s\} > -1$

For this ROC, the system is stable, since the $j\omega$ axis $\in$ ROC

$$H_3(s) = \frac{1}{s(s+1)}$$



ROC for causality : $Re\{s\} > 0$

S_3 is unstable

$\because$ pole on $j\omega$ axis

(c) $$H_1(s) = \frac{e^{-2j}}{(s+1)(s-2)} = e^{-2j}\left[\frac{1}{(s+1)(s-2)}\right]$$

~~Let this be $\tilde{H}_1(s)$~~.

$$\therefore \quad \frac{1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

Express a rational transfer function $H(s)$ as a sum of fractional parts of the form

$$\frac{1}{s} \; ; \; \frac{1}{s^n} \; ; \; \frac{1}{(s+a)} \; ; \; \frac{1}{(s+a)^n}$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$u(t) \qquad \frac{t^{n-1}}{(n-1)!}u(t) \qquad e^{-at}u(t) \qquad e^{-at}\frac{t^{n-1}}{(n-1)!}u(t).$$

$$\therefore \quad \frac{1}{(s+1)(s-2)} = \frac{1}{3}\,\frac{(s+1)-(s-2)}{(s+1)(s-2)}$$

$$= \frac{1}{3}\left[\frac{1}{s-2} - \frac{1}{s+1}\right]$$

$$H_1(s) = \frac{e^{-2j}}{3}\left[\frac{1}{s-2} - \frac{1}{s+1}\right]$$

$$\boxed{h_1(t) = \frac{e^{-2j}}{3}\left[e^{+2t}u(t) - e^{-t}u(t)\right]}$$

$$H_2(s) = e^{-4s} \frac{s(s-2)}{(s+1)(s+2)} = e^{-4s}\tilde{H}_2(s).$$

$$\therefore \boxed{h_2(t) = \tilde{h}_2(t-4).}$$

$\therefore$ We find $\tilde{h}_2(t) = \text{ILT}\{\tilde{H}_2(s)\}$

$$\tilde{H}_2(s) = \frac{s(s-2)}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \tilde{H}_2(s)\cdot(s+1)\Big|_{s=-1} = \frac{s(s-2)}{s+2}\Big|_{s=-1} = \cancel{8}\ -3$$

$$B = \tilde{H}_2(s)\cdot(s+2)\Big|_{s=-2} = \frac{s(s-2)}{s+1}\Big|_{s=-2} = \frac{(-2)(-4)}{-1} = -8$$

3 $$\tilde{H}_2(s) = \frac{s(s-2)}{(s+1)(s+2)} = \frac{s(s+2-4)}{(s+1)(s+2)} = \frac{s(s+2)}{(s+1)(s+2)} - \frac{4s}{(s+1)(s+2)}$$

$$= \frac{s}{s+1} - \frac{4s}{(s+1)(s+2)}$$

$$= 1 - \frac{1}{s+1} - \frac{4s}{(s+1)(s+2)}$$

$$\frac{4s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \frac{4s}{s+2}\Big|_{s=-1} = -4$$

$$B = \frac{4s}{s+1}\Big|_{s=-2} = +8$$

$$\therefore \frac{4s}{(s+1)(s+2)} = \frac{-4}{s+1} + \frac{8}{s+2}$$

$$\Rightarrow \tilde{H}_2(s) = 1 - \frac{1}{s+1} + \frac{4}{s+1} - \frac{8}{s+2}$$

$$\updownarrow$$

$$\tilde{h}_2(t) = \delta(t) + 3e^{-t}u(t) - 8e^{-2t}u(t)$$

$$\Rightarrow \boxed{h_2(t) = \tilde{h}_2(t-4) = \delta(t-4) + 3e^{-(t-4)}u(t-4) - 8e^{-2(t-4)}u(t-4)}$$

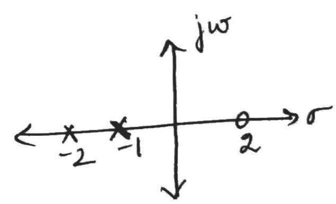
$$H_3(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$\boxed{h_3(t) = u(t)\left[1 - e^{-t}\right].}$$

(d) Cascade $S_2 S_3$

$$H_2(s)\,H_3(s) = \frac{e^{-4s}(s-2)\,s}{(s+1)(s+2)} \cdot \frac{1}{s(s+1)}$$

$$= \frac{e^{-4s}(s-2)}{(s+1)^2(s+2)}$$



poles : $s = -1, -1, -2$

zeros : $s = 2, \; s \to \infty$

ROC for causality : $Re\{s\} > -1$.

system is stable.

(e) $h_{23}(t)$:

$$H_{23}(s) = \frac{e^{-4s}(s-2)}{(s+1)^2(s+2)} = e^{-4s}\underbrace{\left[\frac{(s-2)}{(s+1)^2(s+2)}\right]}_{\tilde{H}_{23}(s)}$$

$$\tilde{H}_{23}(s) = \frac{s-2}{(s+1)^2(s+2)} = \frac{A}{s+2} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$A = \tilde{H}_{23}(s).(s+2)\Big|_{s=-2} = \frac{s-2}{(s+1)^2}\Big|_{s=-2} = -4$$

$$C = \tilde{H}_{23}(s).(s+1)^2\Big|_{s=-1} = \frac{s-2}{s+2}\Big|_{s=-1} = -3$$

$$B = \frac{d}{ds}\left[\tilde{H}_{23}(s)(s+1)^2\right]\Big|_{s=-1}$$

$$= \frac{d}{ds}\left(\frac{s-2}{s+2}\right) = \frac{d}{ds}\left\{\frac{s+2}{s+2} - \frac{4}{s+2}\right\}$$

$$= \frac{d}{ds}\left\{1 - \frac{4}{s+2}\right\} = \frac{4}{(s+2)^2}$$

$$\therefore B = \frac{4}{(s+2)^2}\Big|_{s=-1} = 4.$$

$$\therefore \tilde{h}_{23}(t) = -4e^{-2t}u(t) + 4e^{-t}u(t) - 3e^{-t}.t\,u(t).$$

$$\boxed{\begin{aligned} h_{23}(t) &= \tilde{h}_{23}(t-4) \\ &= -4e^{-2(t-4)}u(t-4) + 4e^{-(t-4)}u(t-4) - 3e^{-(t-4)}(t-4)\,u(t-4) \end{aligned}}$$

** $H_{12}(s)$ and $h_{12}(t)$ are left for HW

Problem 2:

Find the Inverse Laplace transform $f(t)$ for the given functions:

a) $F(s) = \frac{s^2+2s+5}{(s^2+a^2)^2(s+3)}$ where a is a real constant

b) $F(s) = \frac{2+5se^{-2s}-8e^{-4s}}{s^2+4s+3}$

Solution:

a) Using the partial expansion, we have

$$\frac{s^2+2s+5}{(s^2+a^2)^2(s+3)} = \frac{A}{s+3} + \frac{B}{s-ja} + \frac{C}{s+ja} + \frac{D}{(s-ja)^2} + \frac{E}{(s+ja)^2}$$

Taking the inverse Laplace transform of each term, we get

$$f(t) = \left(Ae^{-3t} + Be^{jat} + Ce^{-jat} + Dte^{jat} + Ete^{-jat}\right)u(t)$$

where we have used the fact that $\mathcal{L}^{-1}\left\{\frac{1}{(s+b)^{n+1}}\right\} = \frac{1}{n!}t^n e^{-bt}$. Next, we compute the coefficients

$$A = (s+3)F(s)|_{s=-3} \Rightarrow A = \frac{8}{(9+a^2)^2}$$

$$D = (s-ja)^2F(s)|_{s=ja} \Rightarrow D = \frac{(a^2-15) - j(a+a^3)}{4a^2(9+a^2)}$$

$$E = (s+ja)^2F(s)|_{s=-ja} \Rightarrow E = \frac{(a^2-15) + j(a+a^3)}{4a^2(9+a^2)}$$

Note that $= \frac{a^2-2ja-5}{4a^2(3+ja)}$, which was simplified by multiplying the numerator and denominator by $3-ja$. The same thing is done to simplify E, where we multiply the numerator and denominary by $3+ja$.

To compute B and B, we can build a system of linear equations, or alternatively, use the following

$$B = \frac{d}{ds}\left((s-ja)^2F(s)\right)|_{s=ja} \Rightarrow B = \frac{-10a + j(a^2+15)}{4a^3(a-3j)^2}$$

which can be simplified by multiplying the numerator and denominator by $(a+3j)^2$ to get

$$B = \frac{-16a^3 + j(a^4 - 54a^2 - 135)}{4a^3(9+a^2)^2}$$

Similarly,

$$C = \frac{d}{ds}\left((s+ja)^2F(s)\right)|_{s=-ja} \Rightarrow C = \frac{-16a^3 - j(a^4 - 54a^2 - 135)}{4a^3(9+a^2)^2}$$

By plugging in these coefficients in $f(t)$, we can further simplify the time-domain function. For example,

$$Dte^{jat} + Ete^{-jat} = \frac{(a^2-15)}{2a^2(9+a^2)} t \frac{(e^{jat}+e^{-jat})}{2} + \frac{(a+a^3)}{2a^2(9+a^2)} \frac{(e^{jat}-e^{-jat})}{2j}$$
$$= \frac{(a^2-15)}{2a^2(9+a^2)} t\cos(at) + \frac{(a+a^3)}{2a^2(9+a^2)} t\sin(at)$$

Similarly

$$Be^{jat} + Ce^{-jat} = -\frac{16a^3}{2a^3(9+a^2)^2}\frac{(e^{jat}+e^{-jat})}{2} - \frac{(a^4-54a^2-135)}{2a^3(9+a^2)^2}\frac{(e^{jat}-e^{-jat})}{2j}$$
$$= -\frac{8}{(9+a^2)^2}\cos(at) - \frac{(a^4-54a^2-135)}{2a^3(9+a^2)^2}\sin(t)$$

Therefore, we have for $t \geq 0$

$$f(t) = \frac{8}{(9+a^2)^2}e^{-3t} - \frac{8}{(9+a^2)^2}\cos(at) - \frac{(a^4-54a^2-135)}{2a^3(9+a^2)^2}\sin(t)$$
$$+ \frac{(a^2-15)}{2a^2(9+a^2)} t\cos(at) + \frac{(a+a^3)}{2a^2(9+a^2)} t\sin(at)$$

b) We have

$$\frac{2+5se^{-2s}-8e^{-4s}}{s^2+4s+3} = 2\left(\frac{1}{s^2+4s+3}\right) + 5se^{-2s}\left(\frac{1}{s^2+4s+3}\right) - 8e^{-4s}\left(\frac{1}{s^2+4s+3}\right)$$

Let $h(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+4s+3}\right)\right\}$. Then,

$$f(t) = 2h(t) + 5\left(h'(t-2) + h(0^+)\right) - 8h(t-4)$$

To find $h(t)$, we have

$$\frac{1}{s^2+4s+3} = \frac{A}{s+3} + \frac{B}{s+1}$$

where we can show that $A = -1/2$ and $B = 1/2$. That is,

$$h(t) = \frac{1}{2}(e^{-t} - e^{-3t})u(t)$$

Note that $h'(t) = \frac{1}{2}(3e^{-3t} - e^{-t})u(t)$. Thus

$$f(t) = (e^{-t} - e^{-3t})u(t) + \frac{5}{2}\left(3e^{-3(t-2)} - e^{-(t-2)}\right)u(t-2) - 4(e^{-3(t-4)} - e^{-(t-4)})u(t-4)$$

<u>Problem 4:</u>

Solve the following differential equation using the Laplace transform:

$$\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} - x(t), \mathrm{t} \geq 0;$$
$$x(0) = 0, y(0) = 0$$

Solution:

We start with applying Laplace operator on both sides of equations

$$sY(s) - y(0) + 4Y(s) = sX(s) - y(0) + X(s)$$

$$Y(s)(s+4) = X(s)(s-1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s+4} = 1 - \frac{5}{s+4}$$

$$h(t) = \delta(t) - 5e^{-4t}u(y)$$

$$y(t) = x(t) * h(t) = x(t) - 5\int_0^t x(\tau)e^{-4(t-\tau)}d\tau$$