Home Work 3

1) Compute the impulse response function $h(t)$

a) S_1: $y(t) = \int_{-\infty}^{t-1} e^{t} \cos(2\tau + 2 - 2t)\, x(\tau)\, e^{-\tau+2}\, d\tau$

Since $h(t) = S\{\delta(t)\}$, then

$$h(t) = \int_{-\infty}^{t-1} e^{t} \cos(2\tau + 2 - 2t)\, \delta(\tau)\, e^{-\tau+2}\, d\tau$$

$$= \underset{\text{const}}{e^{t}} \cdot \int_{-\infty}^{t-1} e^{-\tau+2} \cos(2\tau + 2 - 2t)\, \delta(\tau)\, d\tau$$

Let $f(\tau) = e^{-\tau+2} \cos(2\tau + 2 - 2t)$

Then, using the shifting property of impulse, we have

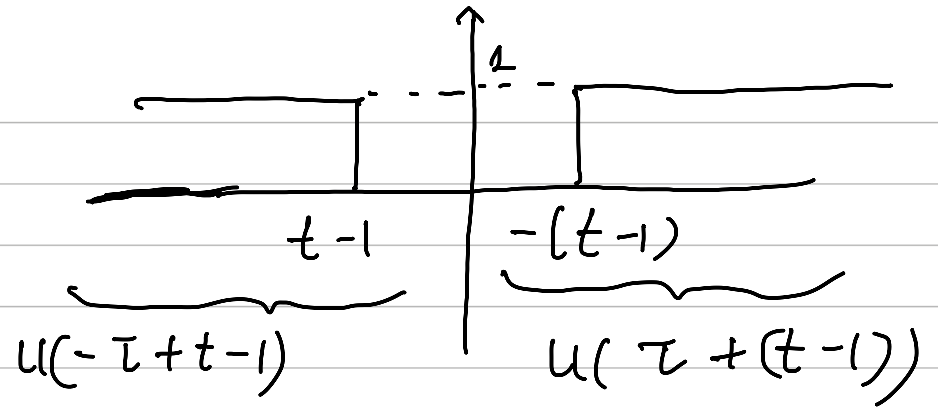
$$f(\tau)\,\delta(\tau) = f(\tau)\,\delta(\tau - 0) = f(0)\,\delta(\tau)$$

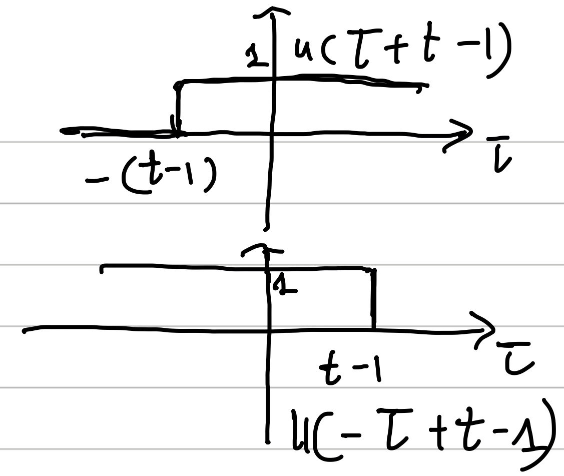
$$\Rightarrow f(0) = e^{-2} \cos(2 - 2t)$$

$$\Rightarrow h(t) = e^{t} \int_{-\infty}^{t-1} e^{-2} \cos(2 - 2t)\, \delta(\tau)\, d\tau$$

$$= e^{t-2} \cos(2 - 2t) \int_{-\infty}^{t-1} \delta(\tau)\, d\tau$$

<u>We have:</u>





$$\Rightarrow h(t) = e^{t-2}\cos(2t-2)\int_{-\infty}^{+\infty} \delta(\tau)\, u[-\tau+t-1]\, d\tau$$

Continue applying the shifting property for impulse.

$$u(-\tau+t-1)\,\delta(\tau) = u(-0+t-1)\,\delta(\tau)$$

$$= u(t-1)\,\delta(\tau)$$

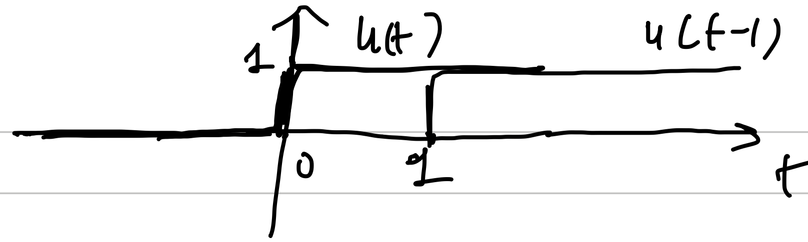
$$\Rightarrow h(t) = e^{t-2}\cos(2t-2)\int_{-\infty}^{+\infty} u(t-1)\,\delta(\tau)\, d\tau$$

$$\Rightarrow h(t) = e^{t-2}\cos(2t-2)\, u(t-1) \underbrace{\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau}_{=1}$$

$$\Rightarrow \boxed{h(t) = e^{t-2}\cos(2t-2)\, u(t-1)}$$

* Check $h(t).\, u(t)$.

We have $h(t)\, u(t) = e^{t-2}\cos(2t-2)\, u(t-1)\, u(t)$



$$\Rightarrow u(t)\,u(t-1) = u(t-1)$$

Since $t < 1 \Rightarrow u(t)\,u(t-1) = 0$

$$\Rightarrow h(t).u(t) = h(t)$$

$\Rightarrow$ $\boxed{\text{This LTI system is Causal}}$

* We also have:

$$h(t) = e^{t-2}\cos(2t-2)\,u(t-1) = \begin{cases} e^{t-2}\cos(2t-2) & t \geq 1 \\ 0, & t < 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{1}^{+\infty} |e^{t-2}\cos(2t-2)|\,dt$$

$$= \int_{1}^{+\infty} |e^{t-2}|\,|\cos(2t-2)|\,dt \quad \text{Since } |\cos(2t-2)| \leq 1$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(t)|\,dt \leq \int_{1}^{+\infty} |e^{t-2}|\,dt = \int_{1}^{+\infty} e^{t-2}\,dt$$

$$(\text{Sin } e^{t-2} > 0 \ \forall t \in \mathbb{R})$$

$$= e^{t-2}\Big|_1^{+\infty} = +\infty - e^{-1} = +\infty$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(t)|\,dt < +\infty$$

$$\Rightarrow \boxed{\text{this LTI system is BIBO stable}}$$

b) $$y(t) = e^{-t}\int_{-\infty}^{t} e^{\tau}\left[\cos(t)\cos(\tau) - \sin(t)\sin(\tau)\right]x(\tau)\,d\tau$$

$$h(t) = S\{\delta(t)\} = e^{-t}\int_{-\infty}^{t} e^{\tau}\left[\cos(t)\cos(\tau) - \sin t \sin\tau\right]\delta(\tau)\,d\tau$$

$$= e^{-t}\int_{-\infty}^{+\infty} e^{\tau}\left[\cos(t)\cos(\tau) - \sin t \sin\tau\right]\delta(\tau)\,u(-\tau+t)\,d\tau$$

Let $f(\tau) = e^{\tau}\left[\cos(t)\cos(\tau) - \sin t \sin\tau\right]u(-\tau+t)$

Apply the shifting property of impulse function.

$$f(\tau)\,\delta(\tau) = f(0)\,\delta(\tau)$$

$$\Rightarrow f(\tau)\,\delta(\tau) = e^{0}\left[\cos t \cos 0 - \sin t \sin 0\right]u(t)\,\delta(\tau)$$

$$\Rightarrow h(t) = e^{-t}\int_{-\infty}^{+\infty} f(\tau)\,\delta(\tau)\,d\tau = e^{-t}\int_{-\infty}^{+\infty}\cos t\, u(t)\,\delta(\tau)\,d\tau$$

$$= e^{-t}\cos(t)\,u(t)\underbrace{\int_{-\infty}^{+\infty}\delta(\tau)\,d\tau}_{=1}$$

$$\Rightarrow \boxed{h(t) = e^{-t}\cos(t)\,u(t)}$$

* Check $h(t)\,u(t) = e^{-t}\cos(t)\,u(t)\,.\,u(t)$

Since $u(t)\,.\,u(t) = u(t)$

$\Rightarrow h(t) = h(t).u(t)$ or $h(t) = 0$ with $t < 0$

$$\Rightarrow \boxed{\text{this LTI system is Causal.}}$$

* Check $\displaystyle\int_{-\infty}^{+\infty} \left| e^{-t}\cos(t)\,u(t) \right| dt$ Since $u(t) = \begin{cases} 1 & t \geq 0 \\ 0, & t < 0 \end{cases}$

$$= \int_0^{+\infty} \left| e^{-t}\cos(t) \right| dt = \int_0^{+\infty} \left| e^{-t} \right| \left| \cos(t) \right| dt.$$

$$\leq \int_0^{+\infty} e^{-t}\,dt \quad (\text{Since } |\cos t| \leq 1$$

and $e^{-t} > 0 \ \forall t \in \mathbb{R}$)

$$= -e^{-t}\Big|_0^{+\infty} = e^{-t}\Big|_{+\infty}^{0} = 1 - 0 = 1 < \infty$$

$$\Rightarrow \int_{-\infty}^{+\infty} |h(t)|\,dt < +\infty$$

$$\Rightarrow \boxed{\text{this LTI System is BIBO stable}}$$

c) S_3: $y = \int_{-\infty}^{t-1} e^{-(t-\tau)} x(\tau-2)\,d\tau$

$$h(t) = S\{\delta(t)\} = \int_{-\infty}^{t-1} e^{-(t-\tau)} \delta(\tau-2)\,d\tau$$

$$= \int_{-\infty}^{+\infty} e^{-(t-\tau)} \delta(\tau-2)\, u(-\tau+t-1)\,d\tau$$

$$= \int_{-\infty}^{+\infty} e^{-(t-\tau)} u(-\tau+t-1)\, \delta(\tau-2)\,d\tau$$

Apply shift property of impulse:

$$f(\tau)\,\delta(\tau-2) = f(2)\,\delta(\tau-2)$$

with $f(\tau) = e^{-(t-\tau)} u(-\tau+t-1)$

$$\Rightarrow f(2)\,\delta(\tau-2) = e^{-(t-2)} u(-2+t-1)\,\delta(\tau-2)$$

$$= e^{-(t-2)} u(t-3)\,\delta(\tau-2)$$

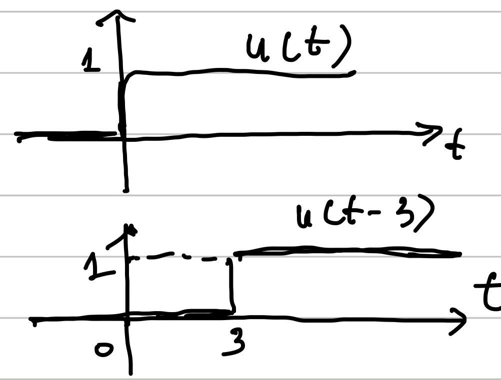
$$\Rightarrow h(t) = \int_{-\infty}^{+\infty} e^{-(t-2)} u(t-3)\,\delta(\tau-2)\,d\tau$$

$$= e^{-(t-2)} u(t-3) \underbrace{\int_{-\infty}^{+\infty} \delta(\tau-2)\,d\tau}_{=1}$$

$$\Rightarrow \boxed{h(t) = e^{-(t-2)} u(t-3)}$$

* Check $h(t)u(t) = e^{-(t-2)}u(t-3)u(t)$

Since $u(t-3)u(t) = u(t-3)$



$\Rightarrow h(t) = h(t)u(t)$

$\Rightarrow$ $\boxed{\text{this LTI system is Causal}}$

* Check $\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} |e^{-(t-2)}u(t-3)|\, dt$

$= \int_{3}^{+\infty} |e^{-(t-2)}|\, dt = \int_{3}^{+\infty} e^{-(t-2)}\, dt = -e^{-(t-2)}\Big|_{3}^{+\infty}$

$= e^{-(t-2)}\Big|_{+\infty}^{3} = e^{-1} - 0 = e^{-1} < +\infty$

$\Rightarrow$ $\boxed{\text{this LTI system is BIBO stable}}$

2) $x(t) \xrightarrow{} \boxed{S_1} \xrightarrow{y(t)} \boxed{S_2} \xrightarrow{} w(t)$

LTV LTV

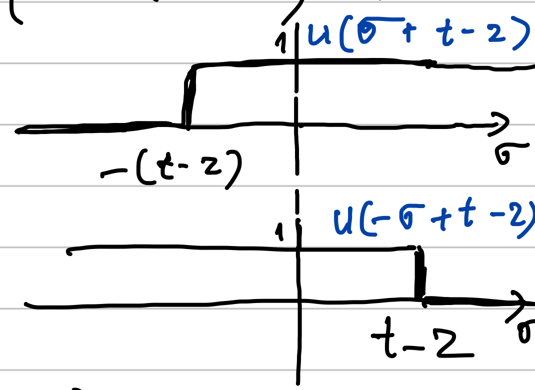
a) given: $y(t) = x(t)\,u(t) - \int_{-\infty}^{t-2} e^{-(t-\tau)} x(\tau)\,u(\tau)\,d\tau$

Rewrite the IPOP interm of σ.

$$y(t) = x(t)\,u(t) - \int_{-\infty}^{t-2} e^{-(t-\sigma)} x(\sigma)\,u(\sigma)\,d\sigma$$

$$\Rightarrow h_1(t,\tau) = S\{\delta(t-\tau)\} = \delta(t-\tau)\,u(t) - \int_{-\infty}^{t-2} e^{\sigma-t} u(\sigma)\,\delta(\sigma-\tau)\,d\sigma$$

$$= \delta(t-\tau)\,u(t) - \int_{-\infty}^{+\infty} e^{\sigma-t} u(\sigma)\,\delta(\sigma-\tau) \cdot u(-\sigma+t-2)\,d\sigma$$



Applying the shifting property of impulse function:

$$f(\sigma)\,\delta(\sigma-\tau) = f(\tau)\,\delta(\sigma-\tau)$$

With $f(\sigma) = e^{\sigma-t} u(\sigma)\,u(-\sigma+t-2)$

$$\Rightarrow f(\tau) = e^{\tau-t} u(\tau)\,u(-\tau+t-2)$$

$$\Rightarrow h_1(t,\tau) = \delta(t-\tau)\,u(t) - \int_{-\infty}^{+\infty} e^{\tau-t} u(\tau)\,u(-\tau+t-2)\,\delta(\sigma-\tau)\,d\sigma$$

$$= \delta(t-\tau)\,u(t) - e^{\tau-t} u(\tau)\,u(-\tau+t-2) \underbrace{\int_{-\infty}^{+\infty} \delta(\sigma-\tau)\,d\sigma}_{=1}$$

$$\Rightarrow \boxed{h_1(t,\tau) = u(t)\,\delta(t-\tau) - e^{-t+\tau}\,u(\tau)\,u(t-2-\tau)}$$

⊛ Given: $w(t) = \int_{-\infty}^{t} y(\sigma)\,u(\sigma)\,d\sigma$

$$h_2(t,\tau) = S\{\delta(t-\tau)\} = \int_{-\infty}^{t} \delta(\sigma-\tau)\,u(\sigma)\,d\sigma$$

$$= \int_{-\infty}^{+\infty} u(\sigma)\,\delta(\sigma-\tau)\,u(-\sigma+t)\,d\sigma$$

$$= \int_{-\infty}^{+\infty} u(\sigma)\,u(-\sigma+t)\,\delta(\sigma-\tau)\,d\sigma$$

Apply the shifting property of impulse:

$$f(\sigma)\,\delta(\sigma-\tau) = f(\tau)\,\delta(\sigma-\tau) \text{ with}$$

$$f(\sigma) = u(\sigma)\,u(-\sigma+t) \rightarrow f(\tau) = u(\tau)\,u(t-\tau)$$

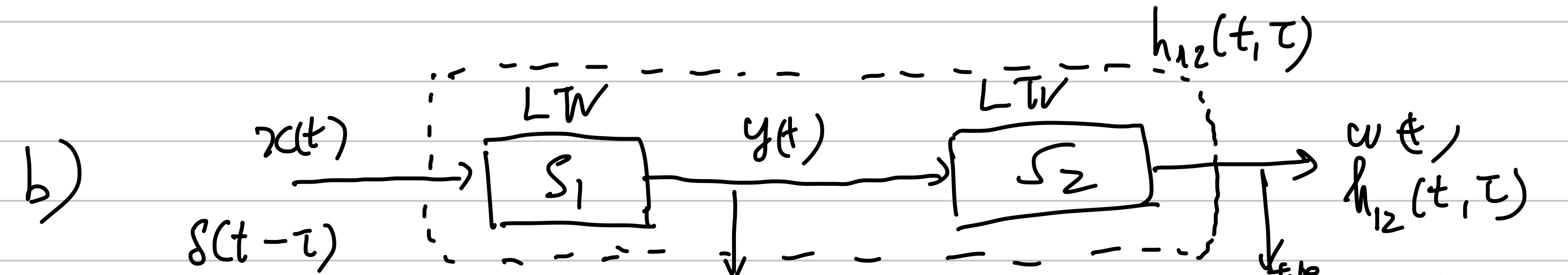
$$\rightarrow h_2(t,\tau) = \int_{-\infty}^{+\infty} u(\tau)\,u(t-\tau)\,\delta(\sigma-\tau)\,d\sigma$$

$$= u(\tau)\,u(t-\tau)\underbrace{\int_{-\infty}^{+\infty} \delta(\sigma-\tau)\,d\sigma}_{=1} = u(\tau)\,u(t-\tau)$$

$$= u(\tau)\,u(t-\tau)$$

$$\Rightarrow \boxed{h_2(t,\tau) = u(\tau)\,u(t-\tau)}$$

b)



$$y(t) = S\{\delta(t-\tau)\} = h_1(t,\tau)$$

$$w(t) = \underbrace{\int_{-\infty}^{+\infty} y(\sigma)\, h_2(t,\sigma)\, d\sigma}_{\text{CONVOLUTION}}$$

$$\Rightarrow h_{12}(t,\tau) = \int_{-\infty}^{+\infty} h_1(\sigma,\tau)\, h_2(t,\sigma)\, d\sigma$$

With $h_1(t,\tau) = u(t)\,\delta(t-\tau) - e^{-t+\tau}\, u(\tau)\, u(t-\tau-2)$

$h_2(t,\tau) = u(\tau)\, u(t-\tau)$

$$\Rightarrow h_1(\sigma,\tau) = u(\sigma)\,\delta(\sigma-\tau) - e^{-\sigma+\tau}\, \underline{u(\tau)}\, \underline{u(\sigma-\tau-2)}$$

$$\Rightarrow h_2(t,\sigma) = \underline{u(\sigma)}\, \underline{u(t-\sigma)}$$

$$\Rightarrow h_1(\sigma,\tau) . h_2(t,\sigma) = u(\sigma)\, u(t-\sigma)\, \delta(\sigma-\tau)$$
$$- \underline{u(\tau)}\, \underline{u(\sigma)}\, \underline{u(t-\sigma)}\, \underline{u(\sigma-\tau-2)}\, \underline{e^{-\sigma+\tau}}$$

Let $A = \int_{-\infty}^{+\infty} u(\sigma)\, u(t-\sigma)\, \delta(\sigma-\tau)\, d\sigma$

$$= \int_{-\infty}^{+\infty} u(\tau)\, u(t-\tau)\, \delta(\sigma-\tau)\, d\sigma$$

(Shifting property of impulse function)

$$= u(\tau)\, u(t-\tau) \underbrace{\int_{-\infty}^{+\infty} \delta(\sigma-\tau)\, d\sigma}_{=1}$$

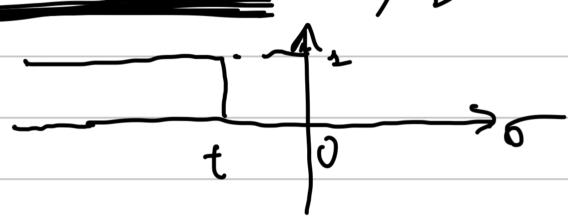
$$= u(\tau)\, u(t-\tau)$$

Let $B = \int_{-\infty}^{+\infty} u(\tau)\, u(\sigma)\, u(t-\sigma)\, u(\sigma-\tau-2)\, e^{-\sigma+\tau}\, d\sigma$

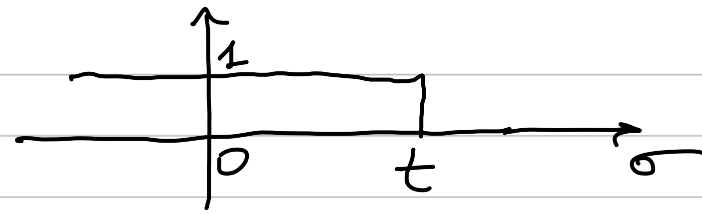
$$= \underbrace{\int_{-\infty}^{0} \cdots}_{0,\ \text{Sin } u(\sigma)=0 \text{ when } \sigma<0} + \int_{0}^{+\infty} u(\tau)\, u(t-\sigma)\, u(\sigma-\tau-2)\, e^{-\sigma+\tau}\, d\sigma$$

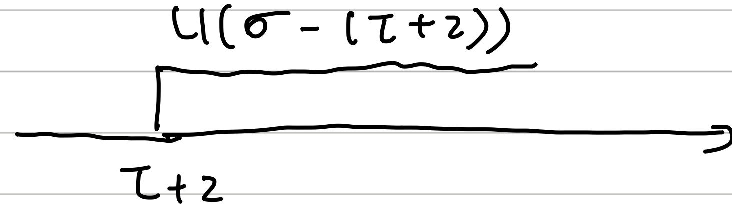
$$= u(\tau)\, e^{\tau} \int_{0}^{+\infty} u(-\sigma+t)\, u(\sigma-(\tau+2))\, e^{-\sigma}\, d\sigma$$

* **When** $t < 0 \Rightarrow B = 0$.



* When $t \geq 0$,





* If $\tau + 2 < 0 \Leftrightarrow \underline{\tau < -2}$

$$B = u(\tau) e^{\tau} \int_0^t e^{-\sigma} d\sigma = u(\tau) e^{\tau} \cdot e^{-\sigma} \Big|_t^0$$

$$= u(\tau) e^{\tau} \left[1 - e^{-t} \right]$$

* If $\begin{cases} \tau + 2 \geq 0 \\ \tau + 2 < t \end{cases} \Leftrightarrow \underline{-2 \leq \tau < t - 2}$

$$B = u(\tau) e^{\tau} \int_{\tau+2}^{t} e^{-\sigma} d\sigma = u(\tau) e^{\tau} e^{-\sigma} \Big|_t^{\tau+2}$$

$$= u(\tau) e^{\tau} \left[e^{-\tau-2} - e^{-t} \right]$$

* If $\tau + 2 \geq t \Leftrightarrow \underline{\tau \geq t - 2}$

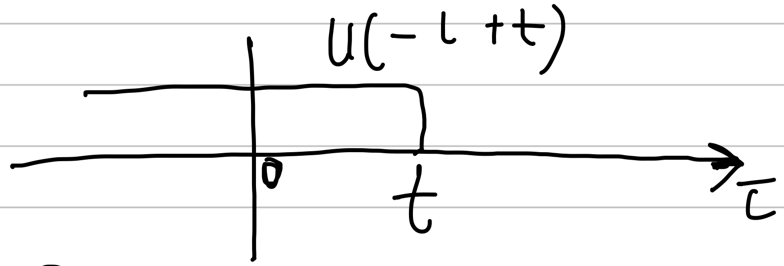
$\Rightarrow B = 0$

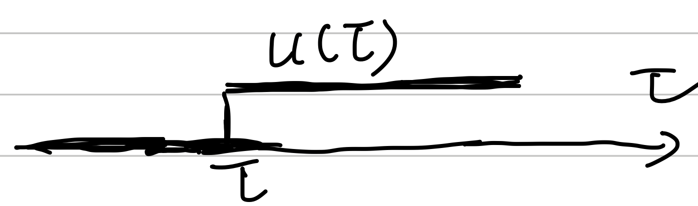
$$\Rightarrow B = \begin{cases} u(\tau)e^{\tau}\left[1-e^{-t}\right]u(t), & \tau < -2 \\ u(\tau)e^{\tau}\left[e^{-\tau-2} - e^{-t}\right]u(t), & -2 \leq \tau < t-2 \\ 0, & \tau \geq t-2 \end{cases}$$

Also, $A = u(\tau)\,u(t-\tau)$

$$\boxed{\begin{aligned} &\Rightarrow h_{12}(t,\tau) = A - B \\ &\begin{cases} u(\tau)u(t-\tau) - u(\tau)e^{\tau}\left[1-e^{-t}\right]u(t), & \tau < -2 \\ u(\tau)u(t-\tau) - u(\tau)e^{\tau}\left[e^{-\tau-2} - e^{-t}\right]u(t), & -2 \leq \tau < t-2 \\ u(\tau)u(t-\tau), & \tau \geq t-2 \ \& \ t < 0 \end{cases} \end{aligned}}$$

c) We have:





* $\tau \geq t-2 \iff t \leq \tau+2 \ \& \ t < 0$

we have:

$$\int_{-\infty}^{+\infty} |h_{12}(t,\tau)|\,dt = \int_{-\infty}^{+\infty} |u(\tau)\,u(t-\tau)|\,dt$$

if we chose $t < \tau$ then $u(\tau)u(t-\tau) = 0$

$\Rightarrow h_{12}(t,\tau) = 0 \Rightarrow \int_{\infty}^{+\infty} |h_{12}(t,\tau)| dt = 0 < +\infty.$

But if we choose $t > \tau$ then $u(\tau)u(t-\tau) = 1$

$\Rightarrow h_{12}(t,\tau) = 1 \Rightarrow \int_{-\infty}^{+\infty} dt = +\infty + \infty = +\infty$

- If $t > 0$ & $\tau < -2$

$h_{12}(t,\tau) = u(\tau)u(t-\tau) - u(\tau)e^{\tau}[1-e^{-t}]u(t)$

$= u(\tau)u(t-\tau) - u(\tau)e^{\tau}[1-e^{-t}]$

$= 0 - u(\tau)e^{\tau}[1-e^{-t}]$

$= u(\tau)e^{\tau}[e^{-t}-1] = e^{\tau}[e^{-t}-1]$

$\Rightarrow \int_{-\infty}^{+\infty} |h_{12}(t,\tau)| dt = \int_{0}^{+\infty} |u(\tau)e^{\tau}| |e^{-t}-1| dt$

$= e^{\tau} \int_{0}^{+\infty} |e^{-t}-1| dt$. We have $e^{-t} < 1$

$\Leftrightarrow \ln e^{-t} < \ln 1$

$\Leftrightarrow \underbrace{-t < 0}$

allway true since
$t \in (0, +\infty)$

$\Rightarrow e^{-t} < 1$ with $t \in (0, +\infty)$

$$\Rightarrow \int_{-\infty}^{+\infty} |h_{12}(t, \tau)| \, dt = e^{\tau} \int_{0}^{+\infty} (1 - e^{-t}) dt$$

$$= e^{\tau} \left(t + e^{-t} \right) \Big|_{0}^{+\infty}$$

$$= e^{\tau} (+\infty - 1) = +\infty$$

$\Rightarrow$ this does not converge $\Rightarrow$ system is not stable

Based on these cases, there exits a case that makes the system be stable, but also exist a case that makes the system is not stable. Therefore, generally, this system is not a stable system.

d) We have $h_{21}(t,\tau) = \int_{-\infty}^{+\infty} h_2(\sigma,\tau)\, h_1(t,\sigma)\, d\sigma$

with $h_1(t,\tau) = u(t)\,\delta(t-\tau) - e^{-t+\tau}\, u(\tau)\, u(t-\tau-2)$

$h_2(t,\tau) = u(\tau)\, u(t-\tau)$

$$h_2(\sigma,\tau) = u(\tau)\, u(\sigma-\tau)$$

$$h_1(t,\sigma) = u(t)\,\delta(t-\sigma) - e^{-t+\sigma}\, u(\sigma)\, u(t-\sigma-2)$$

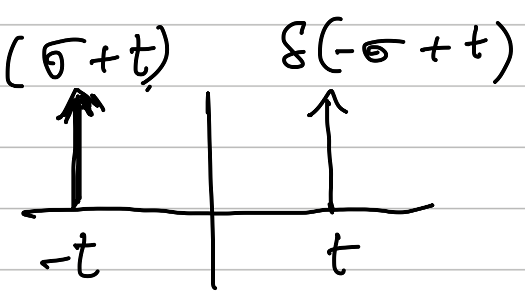
$$\Rightarrow h_2(\sigma,\tau)\, h_1(t,\sigma) = u(\tau)\, u(\sigma-\tau)\, u(t)\, \delta(t-\sigma)$$

$$- u(\tau)\, u(\sigma-\tau)\, u(\sigma)\, u(t-\sigma-2)\, e^{-t+\sigma}$$

+ We have $A = \int_{-\infty}^{+\infty} u(\tau)\, u(\sigma-\tau)\, u(t)\, \delta(t-\sigma)\, d\sigma$

$f(\sigma)\,\delta(-\sigma+t) = f(t)\,\delta(-\sigma+t)$

(Shifting property of impulse function)



$$\Rightarrow A = \int_{-\infty}^{+\infty} u(\tau)\, u(t)\, u(t-\tau)\, \delta(-\sigma+t)\, d\sigma$$

$$= u(\tau)\,u(t)\,u(t-\tau)\underbrace{\int_{-\infty}^{+\infty} \delta(-\sigma+t)\,d\sigma}_{=1}$$

$$= u(\tau)\,u(t)\,u(t-\tau)$$

$$* \text{ Let } B = \int_{-\infty}^{+\infty} u(\tau)\,u(\sigma-\tau)\,u(\sigma)\,u(t-\sigma-2)\,e^{-t+\sigma}\,d\sigma$$

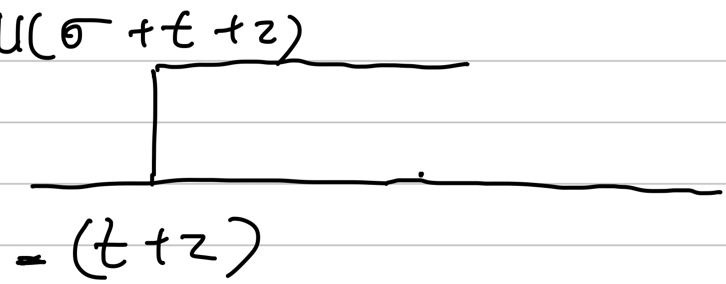
$$= u(\tau)\int_{-\infty}^{+\infty} u(\sigma)\,u(\sigma-\tau)\,u(t-\sigma-2)\,e^{-t+\sigma}\,d\sigma$$

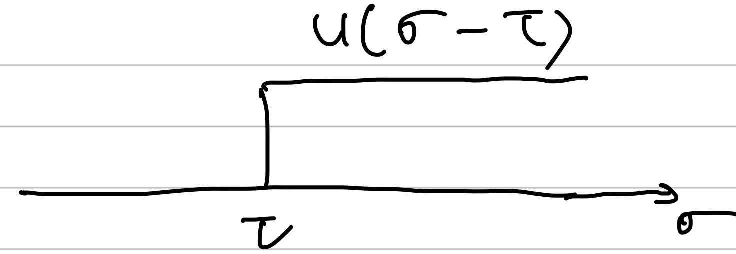
$$= u(\tau)\int_{-\infty}^{0} u(\sigma)\,u(\sigma-\tau)\,u(t-\sigma-2)\,e^{-t+\sigma}\,d\sigma$$

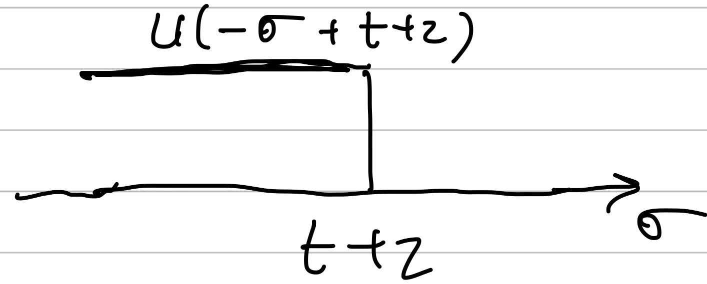
$$+ u(\tau)\int_{0}^{+\infty} u(\sigma)\,u(\sigma-\tau)\,u(t-\sigma-2)\,e^{-t+\sigma}\,d\sigma$$

$$= u(\tau)\int_{0}^{+\infty} u(\sigma-\tau)\,u(t-\sigma-2)\,e^{-t+\sigma}\,d\sigma$$

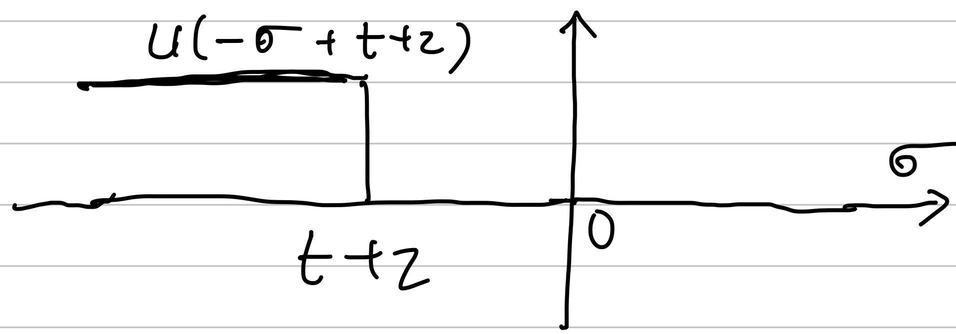
$$= u(\tau)\int_{0}^{+\infty} u(\sigma-\tau)\,u(-\sigma+t-2)\,e^{-t+\sigma}\,d\sigma$$





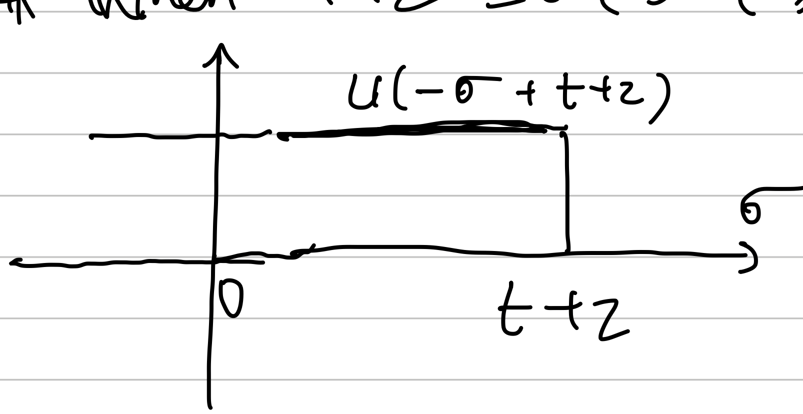


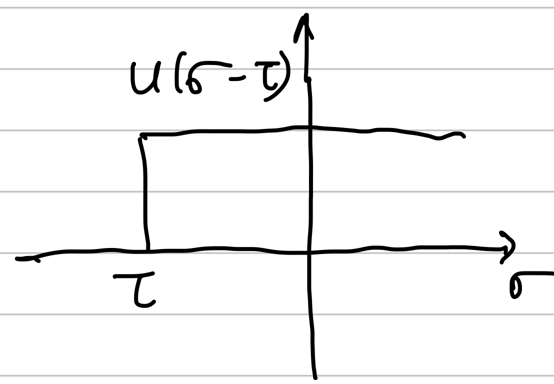
* When $t+2 < 0 \iff t < -2$



$\Rightarrow B = 0$

* When $t+2 \geq 0 \iff t \geq -2$





* If $\tau < 0$

$$\Rightarrow B = u(\tau)\, e^{-t} \int_0^{t+2} e^{\sigma}\, d\sigma$$

$$= u(\tau)e^{-t}\, e^{\sigma}\Big|_{0}^{t+2} = u(\tau)e^{-t}\left[e^{t+2}-1\right]$$

$*$ If $\tau \geq 0$ & $\tau < t+2$

$$B = u(\tau)e^{-t}\int_{\tau}^{t+2} e^{\sigma}\,d\sigma$$

$$= u(\tau)e^{-t}\left[e^{t+2} - e^{\tau}\right]$$

$*$ If $\tau \geq t+2$

$$\Rightarrow B = 0$$

$$\Rightarrow B = \begin{cases} u(\tau)e^{-t}\left[e^{t+2}-1\right]u(t+2), & \tau < 0 \\ u(\tau)e^{-t}\left[e^{t+2}-e^{\tau}\right]u(t+2), & 0 \leq \tau < t+2 \\ 0, & \tau \geq t+2 \end{cases}$$

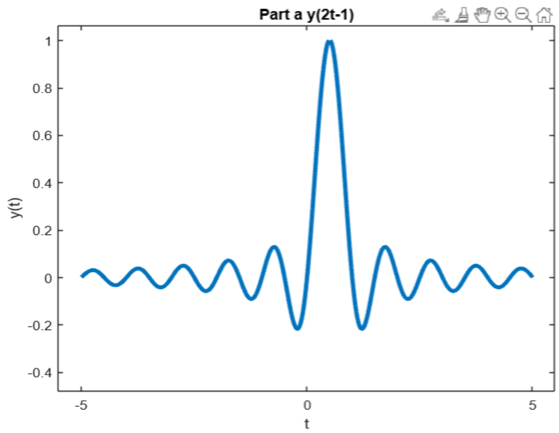
Also $A = u(\tau)u(t)u(t-\tau)$

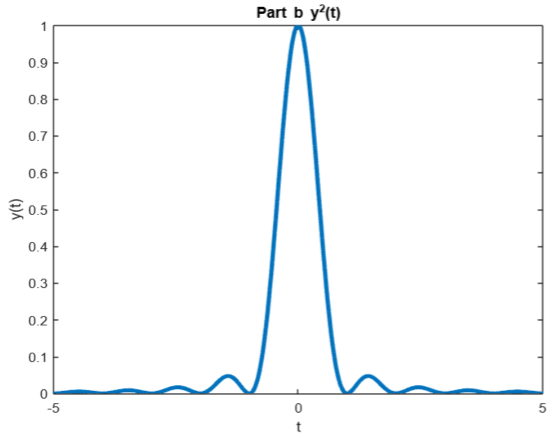
So, $h_{21}(t,\tau) = \int_{-\infty}^{+\infty} h_2(\sigma,\tau)\, h_1(t,\sigma)\, d\sigma$

$$= A - B$$

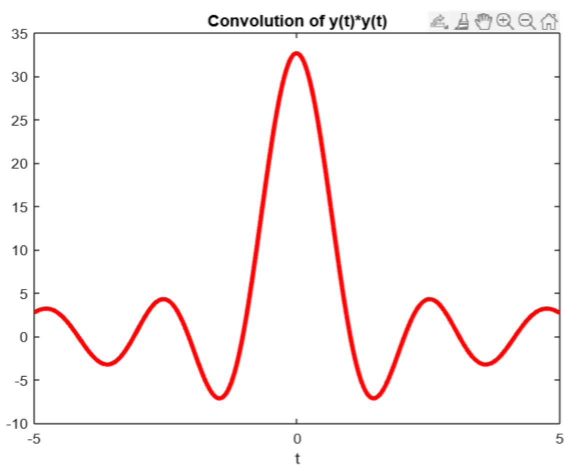
$$
\boxed{\begin{cases} u(\tau)u(t)u(t-\tau) - u(\tau)u(t+2)e^{-t}\left[e^{t+2}-1\right], & \tau<0 \\ u(\tau)u(t)u(t-\tau) - u(\tau)u(t+2)e^{-t}\left[e^{t+2}-e^{\tau}\right], & 0\le t<t+2 \\ u(\tau)u(t)u(t-\tau), & \tau \ge t+2 \ \& \ t<-2 \end{cases}}
$$

3. a) y(2t-1) b) $y^2(t)$



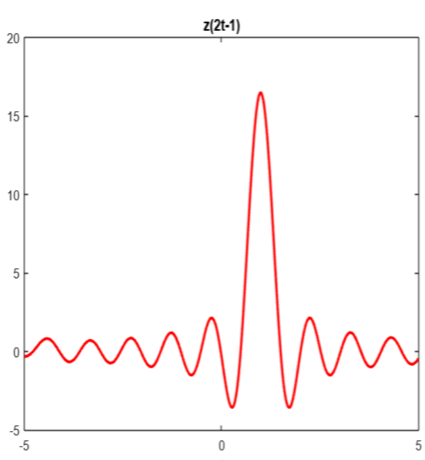


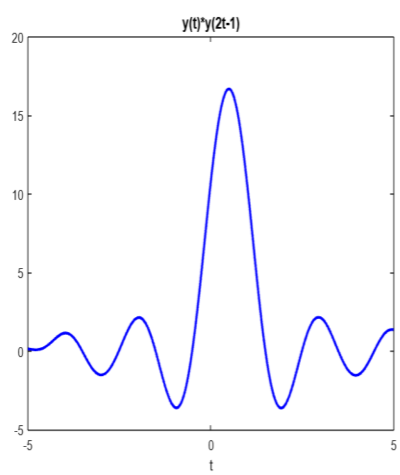
c) z(t) = y(t)*y(t)

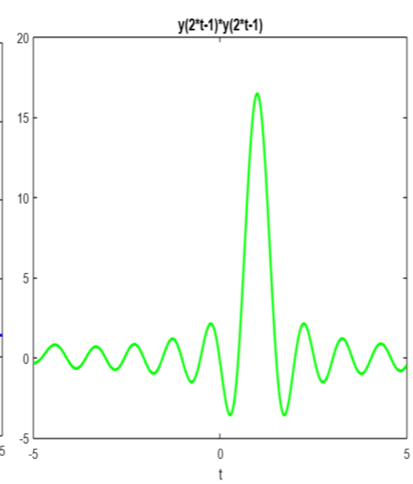


d) Based on these graphs below, we can see that z(2t-1) is equal to y(2t-1)*y(2t-1), but is different with y(t)*y(2t-1)

z(2t-1) y(t)*y(2t-1) y(2t-1)*y(2t-1)







Coding part:

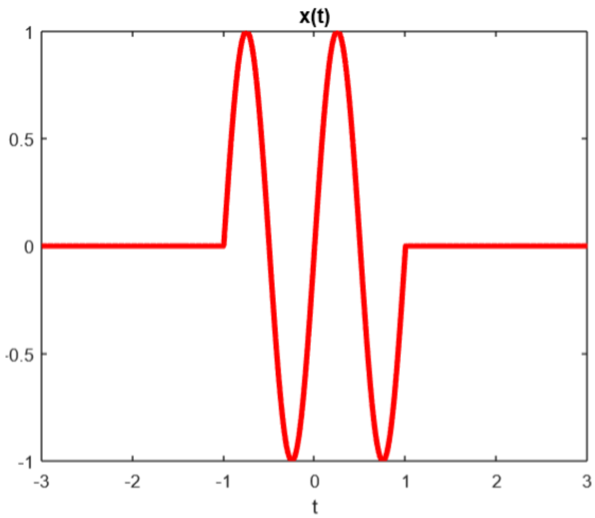
```
%% Clear Cache
clear all;
close all;
clc;

t = -5:0.03:5;
%Part a
y_a = y(2*t-1);
figure(1)
plot(t,y_a,'linewidth',2);
xlabel('t')
ylabel('y(t)')
title('Part a y(2t-1)')

%Part b
y_b = y(t).*y(t);
figure(2)
plot(t,y_b,'linewidth',2);
xlabel('t')
ylabel('y(t)')
title('Part b y^2(t)')

% Part c
conv_z = conv(y(t),y(t),'same');
figure(4)
plot(t,conv_z,'color','red','linewidth',2)
title('Convolution of y(t)*y(t)');
xlabel('t');


%Part d
z_d = z(2*t-1);
figure(5)
plot(t, z_d,'color','red', 'linewidth',2)
xlabel('t')
title('z(2t-1)')

conv_d1 = conv(y(t), y(2*t-1),'same');
figure(6)
plot(t, conv_d1,'color','blue','linewidth',2)
xlabel('t')
title('{y(t)*y(2t-1)}')

conv_d2 = conv(y(2*t-1),y(2*t-1),'same');
figure(7)
plot(t,conv_d2,'color','green','linewidth',2)
xlabel('t')
title('y(2*t-1)*y(2*t-1)')

%% Functions
```

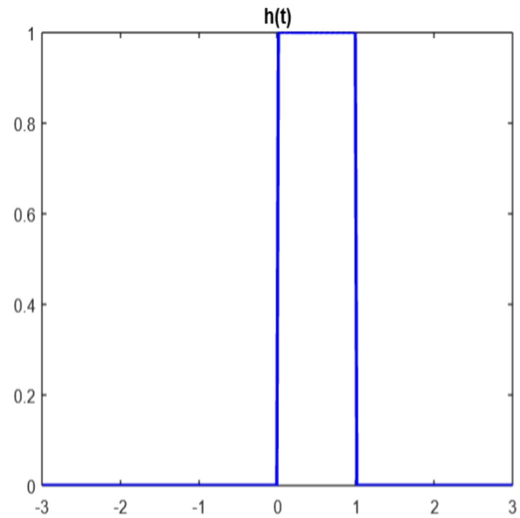
```
function y = y(t)
    y = (sin(pi.*t))./(pi.*t);
end

function z = z(t)
    z = conv(y(t),y(t), 'same');
end
```

4.

4.

a) Plot x(t) and h(t) individually over the range t ∈ [−3, 3].





Code part:

```
%% Problem 4
% Part a
t = -3:0.01:3;
figure(1)
plot(t,x(t),'color','red','DisplayName','x(t)','linewidth',3)
xlabel('t');
title('x(t)');
figure(2)
plot(t,h(t),'color','blue','DisplayName','h(t)','linewidth',2)
xlabel('t');
title('h(t)');
```

```
%% Functions
% Problem 4
function x = x(t)
    x = sin(2*pi.*t).*heaviside(t+1).*heaviside(-t+1);
end
function h = h(t)
    h = heaviside(t) - heaviside(t-1);
end
```

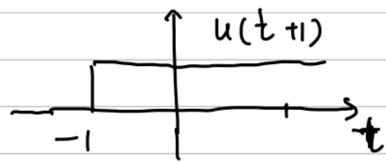
4 b) $y_1(t) = x(t) * h(t)$, We assume this is Linear system

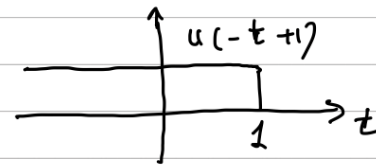
$x(t) = \sin(2\pi t)\, u(t+1)\, u(-t+1)$

$h(t) = u(t) - u(t-1)$

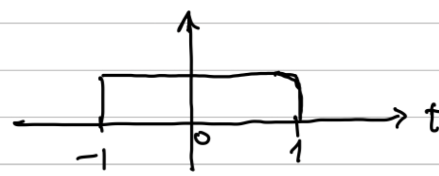
We have $y_1(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$

* With $x(t) = \sin(2\pi t)\, u(t+1)\, u(-t+1)$,



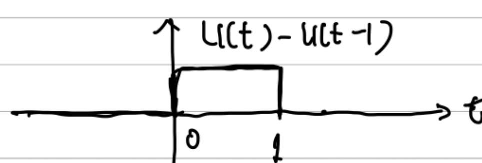


$\Rightarrow u(t+1)\, u(-t+1) = u(t+1) - u(t-1)$



$$\Rightarrow x(t) = \begin{cases} \sin 2\pi t , & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

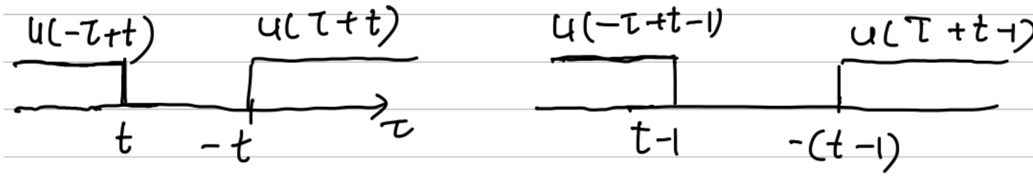
* With $h(t) = u(t) - u(t-1)$



$$\Rightarrow h(t) = \begin{cases} 1 , & 0 \leq t < 1 \\ 0 , & \text{otherwise} \end{cases} \Rightarrow h(\tau) = \begin{cases} 1 , & 0 \leq \tau < 1 \\ 0 , & \text{otherwise} \end{cases}$$

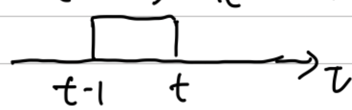
$$\Rightarrow h(t-\tau) = u(t-\tau) - u(t-1-\tau)$$

$$= u(-\tau+t) - u(-\tau+t-1)$$



$$h(t-\tau) = u(-\tau+t) - u(-\tau+t-1)$$

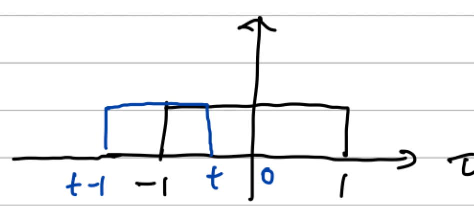
$\Rightarrow$



* When $t < -1$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = 0.$$

* When $\begin{cases} t-1 < -1 \\ t \geq -1 \end{cases} \Leftrightarrow -1 \leq t < 0$

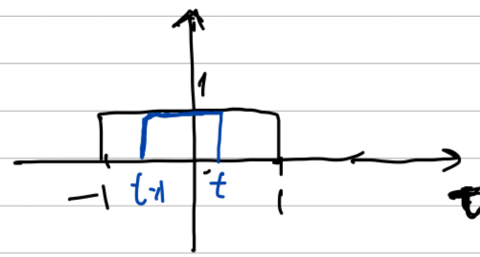


$$\Rightarrow y(t) = \int_{-1}^{t} \sin(2\pi\tau)\, d\tau = -\frac{1}{2\pi} \cos 2\pi\tau \Big|_{-1}^{t}$$

$$= \frac{1}{2\pi}\left[\cos 2\pi\tau \Big|_{t}^{-1}\right] = \frac{1}{2\pi}\left[1 - \cos 2\pi t\right]$$

* When $\begin{cases} t-1 \geq -1 \\ t < 1 \end{cases} \Leftrightarrow 0 \leq t < 1$

$$\Rightarrow y(t) = \int_{t-1}^{t} \sin(2\pi\tau)\, d\tau$$

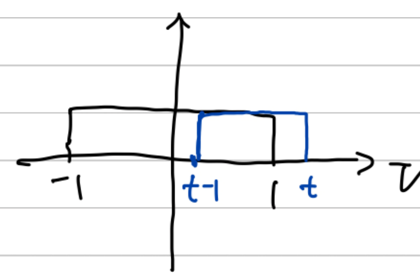


$$= -\frac{1}{2\pi}\cos 2\pi\tau \Big|_{t-1}^{t} = \frac{1}{2\pi}\cos 2\pi\tau \Big|_{t}^{t-1}$$

$$= \frac{1}{2\pi}\left[\cos 2\pi(t-1) - \cos 2\pi t\right]$$

* When $\begin{cases} t-1 \geq 0 \\ t-1 < 1 \end{cases} \Leftrightarrow 1 \leq t < 2$

$$\Rightarrow y(t) = \int_{t-1}^{1} \sin(2\pi\tau)\, d\tau$$

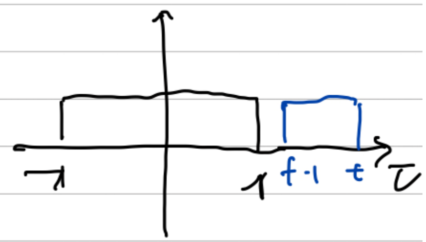


$$= -\frac{1}{2\pi}\cos 2\pi\tau \Big|_{t-1}^{1} = \frac{1}{2\pi}\cos 2\pi\tau \Big|_{1}^{t-1}$$

$$= \frac{1}{2\pi}\left[\cos 2\pi(t-1) - \cos 2\pi\right] = \frac{1}{2\pi}\left[\cos 2\pi(t-1) - 1\right]$$

* When $t-1 \geq 1 \Leftrightarrow t \geq 2$

$$\Rightarrow y(t) = \int_{t-1}^{t} 0 \, d\tau = 0 .$$



Combine all cases above, we have:

$$y_1(t) = \begin{cases} 0 , & t < -1 \\ \frac{1}{2\pi}\left[1 - \cos 2\pi t\right] , & -1 \leq t < 0 \\ \frac{1}{2\pi}\left[\cos 2\pi(t-1) - \cos(2\pi t)\right] , & 0 \leq t < 1 \\ \frac{1}{2\pi}\left[\cos 2\pi(t-1) - 1\right] , & 1 \leq t < 2 \\ 0 , & t \geq 2 \end{cases}$$

Because: $\cos 2\pi(t-1) - \cos(2\pi t)$

$$= \cos(2\pi t) - \cos 2\pi t = 0$$

$$\oplus \quad \frac{1}{2\pi}\left[1 - \cos 2\pi t\right] = \frac{\sin^2 \pi t}{\pi}$$

$$\frac{1}{2\pi}\left[\cos 2\pi(t-1) - 1\right] = -\frac{\sin^2\pi(t-1)}{\pi}$$

Then, we have:

$$\boxed{y_1(t) = \begin{cases} 0, & t < -1 \\ \frac{\sin^2\pi t}{\pi}, & -1 \le t < 0 \\ 0, & 0 \le t < 1 \\ -\frac{\sin^2\pi(t-1)}{\pi}, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}}$$

To plot it easier, we can represent:

$$y_1(t) = \frac{\sin^2\pi t}{\pi}\left[u(t+1) - u(t)\right]$$

$$- \frac{\sin^2\pi(t-1)}{\pi}\left[u(t-1) - u(t-2)\right]$$

4c) $y_2(t) = x(t)\, h(t)$

We have already:

$$x(t) = \begin{cases} \sin 2\pi t, & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$$

* When $t < -1$.

$x(t) = 0,\ h(t) = 0 \Rightarrow y_2(t) = 0$

* When $-1 \le t < 0$:

$x(t) = \sin 2\pi t,\ h(t) = 0$

$\Rightarrow y_2(t) = 0$

* When $0 \le t < 1$

$x(t) = \sin 2\pi t,\ h(t) = 1$

$\Rightarrow y_2(t) = \sin 2\pi t$

* When $t \ge 1$

$x(t) = 0,\ h(t) = 0$

$\Rightarrow y_2(t) = 0$

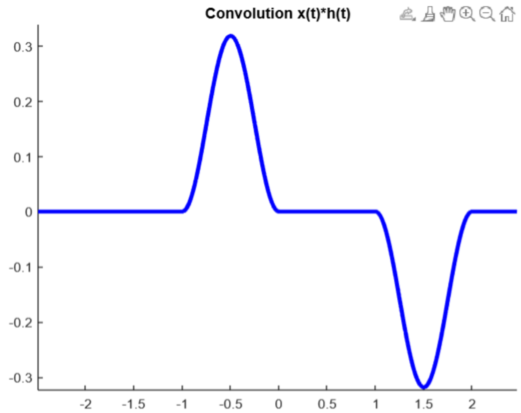
$$\Rightarrow \boxed{y_2(t) = \begin{cases} \sin 2\pi t & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}}$$

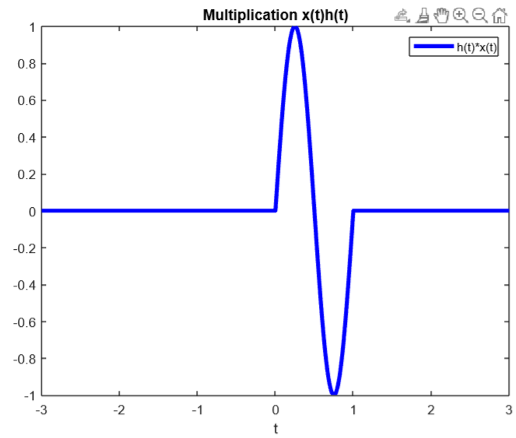
To plot easier, we can represent:

$$\boxed{y_2(t) = \sin(2\pi t)[u(t) - u(t-1)]}$$

4d)

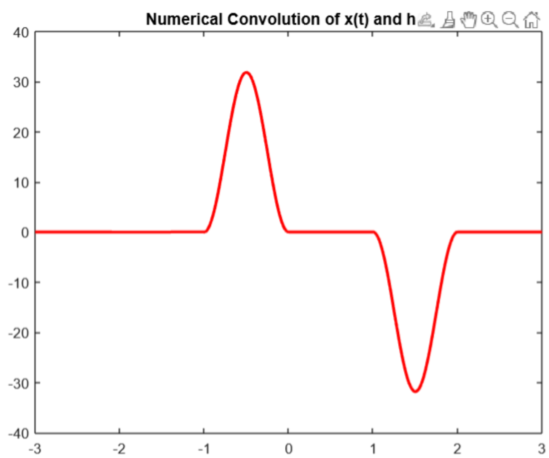
Plot y1(t) Plot y2(t)





Based on 2 graphs above, we have $y_1(t)$ is different with $y_2(t)$.

4e. Numerically convolve x(t) and h(t) and plot the result over the same range



This result is the same with $y_1(t)$ above.

Total code for question 4.

```
%% Clear Cache
clear all;
close all;
clc;

%% Problem 4
% Part a
t = -3:0.01:3;
figure(1)
plot(t,x(t),'color','red','DisplayName','x(t)','linewidth',2)
xlabel('t');
```

```matlab
title('x(t)');
figure(2)
plot(t,h(t),'color','blue','DisplayName','h(t)','linewidth',2)
xlabel('t');
title('h(t)');

% Part d
y_1 = y1(t);
figure(3)
hold on
plot(t,y_1,'linewidth',3,'color','blue')
title('Convolution x(t)*h(t)');
xlabel('t');

y_2 = y2(t);
figure(4)
plot(t,y_2,'color','blue','DisplayName','h(t)*x(t)','linewidth',2)
title('Multiplication x(t)h(t)');
xlabel('t');

% Part e
conv = conv(x(t),h(t),'same');
figure(5)
plot(t,conv,'color','red','linewidth',2)
title('Numerical Convolution of x(t) and h(t)');
xlabel('t');

%% Functions
% Problem 4
function x = x(t)
    x = sin(2*pi.*t).*heaviside(t+1).*heaviside(-t+1);
end
function h = h(t)
    h = heaviside(t) - heaviside(t-1);
end

function y1 = y1(t)
    ya = (sin(pi.*t).^2).*(1/pi).*(heaviside(t+1)-heaviside(t));
    yb = (sin(pi.*t-pi).^2).*(1/(pi)).*(heaviside(t-1) - heaviside(t-2));
    y1= ya-yb;
end

function y2 = y2(t)
    y2 = sin(2*pi*t).*(heaviside(t)-heaviside(t-1));
end
```