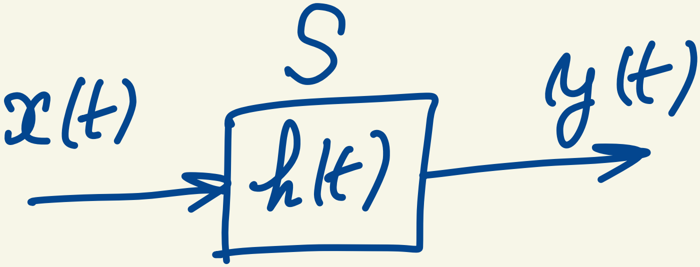
Analysis Of LTI, C Systems using Laplace Transform.



LTI, C : $h(t) \cdot u(t)$ causal system.

input, output signals are causal signals
$x(t) \cdot u(t)$, $y(t) \cdot u(t)$.

Main result from time analysis :

$y(t) = x(t) * h(t)$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \mathcal{L}_s \\ Y(s) & X(s) & H(s) \end{array}$$

$$\boxed{H(s) = \mathcal{L}_s\{h(t)\}}$$

$$X(s) = \mathcal{L}_s\{x(t) \cdot u(t)\}$$

$$Y(s) = \mathcal{L}_s\{y(t)\, u(t)\}.$$

$$\mathcal{L}_s\{x(t) * h(t)\} = X(s) \cdot H(s)$$

$$Y(s) = H(s) \cdot X(s)$$

H(s) is called system (transfer) function

Example:

$$\text{LTI, C}$$

$$x(t)=u(t-1) \longrightarrow \boxed{h(t)} \longrightarrow y(t)=e^{-(t-1)}u(t-1)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$X(s)=e^{-s}\cdot\frac{1}{s} \qquad\qquad\qquad Y(s)$$

$$Y(s)=\mathcal{L}_s\left\{e^{-(t-1)}u(t-1)\right\}=$$

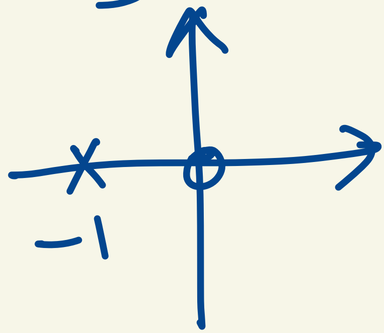
$$=e^{-s}\cdot\mathcal{L}_s\left\{e^{-t}u(t)\right\}$$

$$=e^{-s}\,\frac{1}{s+1}$$

$$Y(s)=H(s)\cdot X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\cancel{e^{-s}}\,\frac{1}{s+1}}{\cancel{e^{-s}}\,\frac{1}{s}} = \frac{s}{s+1}$$

$$H(s) = \frac{s}{s+1}$$



$$H(s) = \frac{s+1-1}{s+1} = 1 - \frac{1}{s+1}$$

$$\boxed{h(t) = \delta(t) - e^{-t}u(t)}$$

Example:

Solve this integral using LT.

$$y(t) = \int_{-\infty}^{+\infty} \underbrace{\left[\delta(t-\tau) - e^{-(t-\tau)} u(t-\tau)\right]}_{h(t-\tau)\,u(t-\tau)} \underbrace{\tau u(t)}_{x(\tau)u(\tau)} d\tau$$

Try to express it as

$$y(t) = \int_{-\infty}^{+\infty} \underbrace{h(t-\tau)u(t-\tau)} \cdot \underbrace{x(\tau)u(\tau)}$$

$$Y(s) = H(s) \cdot X(s)$$

$$X(s) = \mathcal{L}_s\{x(t)u(t)\}$$

$$= \mathcal{L}_s\{t\,u(t)\}$$

$$= \frac{1}{s^2}$$

$$H(s) = L_s\{h(t)\cdot u(t)\}$$

$$= L_s\{\delta(t) - e^{-t}u(t)\}$$

$$= 1 - \frac{1}{s+1} = \frac{s+1-1}{s+1} = \frac{s}{s+1}$$

$$Y(s) = H(s)\cdot X(s) = \frac{\cancel{s}}{s+1}\cdot\frac{1}{s^{\cancel{2}}}$$

$$Y(s) = \frac{1}{(s+1)\cdot s} = \frac{-1}{s+1} + \frac{1}{s}$$

$$\boxed{y(t) = u(t) - e^{-t}u(t)}$$

System that is described by LDE w/ constant coeff.
you can find IRF or System Function by applying Laplace T.

LTI, C

$$S: \; x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad t>0$$

$$\underbrace{s^2Y(s)}_{\frac{d^2y(t)}{dt^2}} + \underbrace{2sY(s)}_{2\frac{dy(t)}{dt}} + \underbrace{Y(s)}_{y(t)} = \underbrace{sX(s)}_{\frac{dx(t)}{dt}}$$

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}$$

initial condition $= 0$

$$\underline{y'(0^-) = 0} \qquad \underline{y(0^-) = 0}$$

Take the LT of both sides.

$$s^2Y(s) - sy(0^-) - y'(0^-) + 2\left(sY(s) - y(0^-)\right) + Y(s) = sX(s) \cancel{+\frac{1}{s}}$$

$$Y(s)\left[s^2+2s+1\right] - \underbrace{\cancel{2s+2}} = sX(s) \cancel{+\frac{1}{s}}$$

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s)\left[s^2+2s+1\right]=sX(s) \quad /:X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{s}{s^2+2s+1}$$

$$H(s)=\frac{s}{s^2+2s+1}=\frac{s}{(s+1)^2}$$

$$=\frac{A_0}{s+1}+\frac{A_1}{(s+1)^2}$$

$$A_1=H(s)\cdot(s+1)^2\Big|_{s=-1}=-1$$

$$A_0 = \frac{d}{ds}\left\{ H(s)(s+1)^2 \right\}\Big|_{s=-1}$$

$$= \frac{d}{ds}\left\{ s \right\}\Big|_{s=-1} = 1$$

$$H(s) = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$h(t) = e^{-t}u(t) - te^{-t}u(t)$$

BIBO stability for LTI, C

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < +\infty$$

Can I infer BIBO stability from $H(s)$?

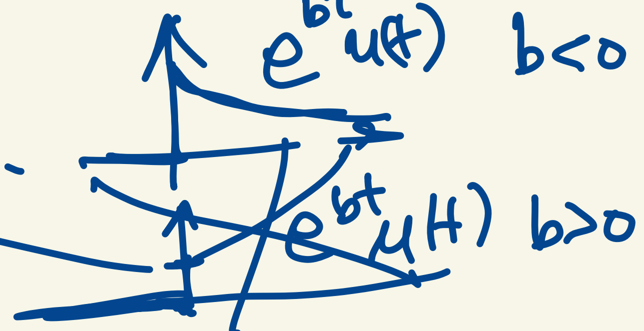
$$H(s)=\frac{(s-a_0)(s-a_1)\cdots(s-a_n)}{(s-b_0)(s-b_1)\cdots(s-b_m)}$$

$$=\frac{A_0}{s-b_0}+\frac{A_1}{s-b_1}+\cdots\frac{A_m}{s-b_m}$$

if all poles are real.

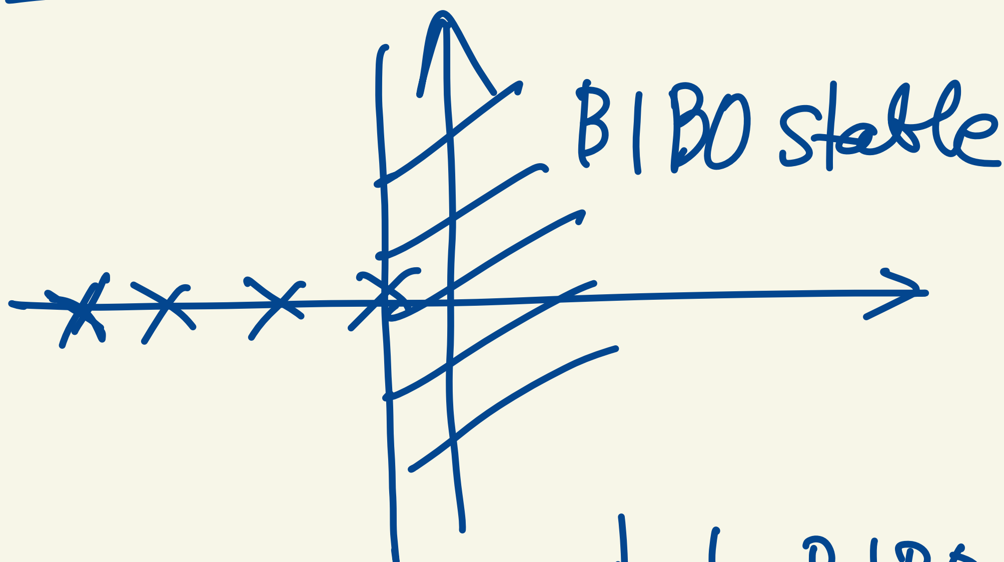
$$h(t)=A_0e^{+b_0t}u(t)+A_1e^{+b_1}u(t)+\cdots A_m e^{+b_m}u(t)$$

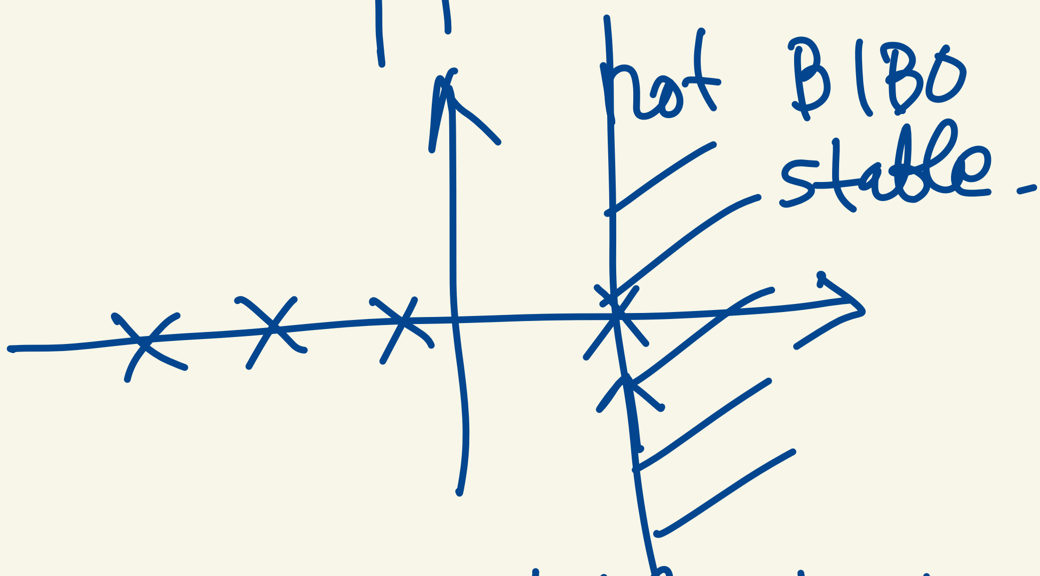
$$\int|h(t)|dt<+\infty.$$



BIBO stability requires

$b_0, b_1, b_2, \ldots, b_m < 0$





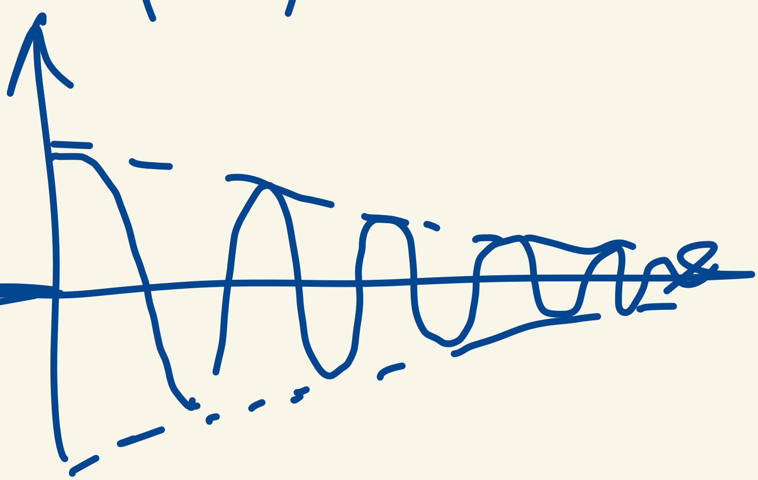
For BIBO stable LTI, C

system, ROC of H(S) has to include imaginary axis.

How about complex poles

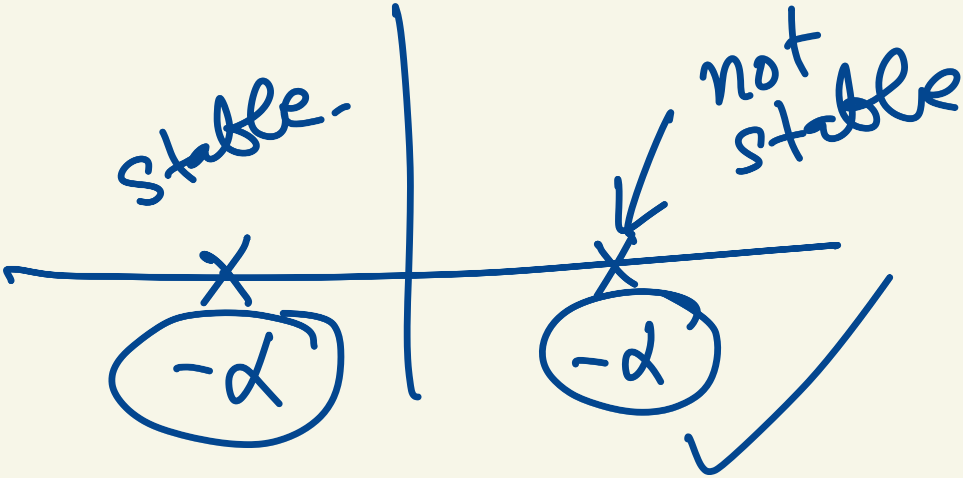
$e^{-\alpha t}\cos(\Omega_0 t) \leftarrow$ terms inside $h(t)$

$e^{-\alpha t}\sin(\Omega_0 t)$



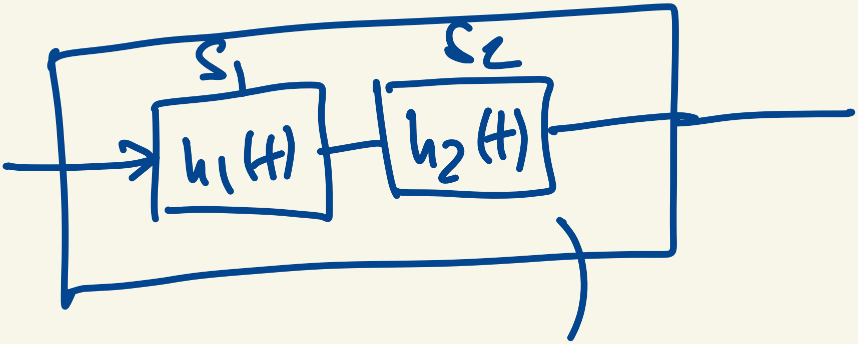
$$s \rightarrow A_0 \frac{s+\alpha}{(s+\alpha)^2+\Omega_0^2} + A_1 \frac{\Omega_0}{(s+\alpha)^2+\Omega_0^2}$$

$-\alpha \pm j\Omega_0 \qquad -\alpha < 0$



Cascaded Systems

LTI, C + LTI, C



$S_{12} = S_{21}$

$$h_{21}(t) = h_1(t) * h_2(t)$$

$$\underline{H_{12}(s)} = H_1(s) \cdot H_2(s)$$

Find Laplace Transform and ROC. Show all of your work and state properties.

a) $f(t) = \sin(\underbrace{5t-15}_{\theta})\sin(\underbrace{3t-9}_{\phi}) \cdot u(t-3) + u(t)$

Hint.

$\sin\theta\sin\phi = \frac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right]$

$f(t) = \frac{1}{2}\left[\cos(2t-6) - \cos(8t-24)\right]$

$\cdot\, u(\underline{t-3}) + u(t)$

$$f(t) = \frac{1}{2}\left[\cos(2(t-3)) - \cos(8(t-3))\right]$$

$$\cdot\, u(t-3] + u(t)$$

$$f(t) = \frac{1}{2}\cos(2(t-3))u(t-3) -$$

$$-\frac{1}{2}\cos(8(t-3))u(t-3)$$

$$+ u(t).$$

$$\mathcal{L}_s\{f(t)\} = \frac{1}{2}\mathcal{L}_s\{\underline{\cos(2(t-3))u(t-3)}\}$$

$$-\frac{1}{2}\mathcal{L}_s\{\cos(8(t-3))u(t-3)\}$$

$$+ u(t)$$

Linearity

$$= \frac{1}{2} e^{-3s} L_s\{\cos(2t)u(t)\}$$

$$- \frac{1}{2} e^{-3s} L_s\{\cos(8t)u(t)\}$$

$$+ L_s\{u(t)\}$$

~~T~~ Time shift

$$= \frac{1}{2} e^{-3s} \cdot \frac{s}{s^2+4}$$

From Table.

$$- \frac{1}{2} e^{-3s} \frac{s}{s^2+64}$$

$$+ \frac{1}{s}$$

$\mathrm{Re}\{s\} > 0$

b) $f(t) = \int_{0^-}^{+\infty} e^{-\sigma} \sin(2\sigma - 8)\, u(\sigma - 4) \cdot$
$\cdot\, \underline{u(t-\sigma)}\, d\sigma$

$$f(t) = \int_{0^-}^{t} e^{-\sigma} \sin(2\sigma-8)\, u(\sigma-4)\, d\sigma$$

$$\mathcal{L}_s\{f(t)\} = \frac{1}{s}\left[\mathcal{L}_s\left\{e^{-\sigma} \sin(2\sigma-8)\, u(\sigma-4)\right\}\right]$$

Integration property.

$$= \frac{1}{s}\left[\mathcal{L}_s\left\{\sin(2(\sigma-4))\, u(\sigma-4)\right\}\Big|_{s'}\right]$$

Frequency shift prop. $s' = s+1$

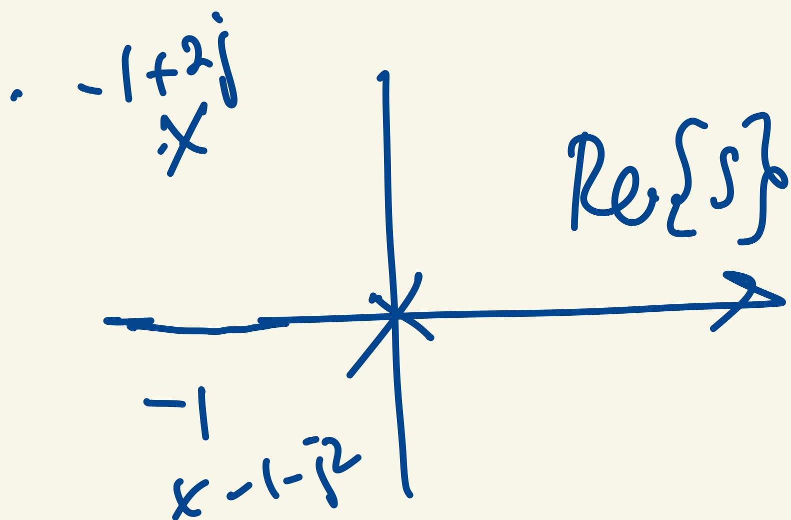
$$= \frac{1}{s}\left[e^{-4s} \mathcal{L}_s\{\sin(2t)u(t)\} \Big|_{s'=s+1} \right]$$

Time shift.

$$= \frac{1}{s}\left[e^{-4s} \cdot \frac{2}{s^2+4} \Big|_{s'=s+1} \right]$$

Table

$$= \frac{1}{s} e^{-4(s+1)} \frac{2}{(s+1)^2+4}$$



$Re\{s\} > 0.$

Example.

$$H(s) = \frac{3s^2+12s+15}{(s^2+2s+5)(s+3)}$$

a) Sketch zero-pole plot of $H(s)$ ✓

b) Is S BIBO stable

c) Find $h(t)$.

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zeros: $3s^2+12s+15=0$

$$3(s^2+4s+5)=0$$

$$s_{1/2} = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$s_{1/2} = \frac{-4 \pm j2}{2}$$

$$s_{1/2} = -2 \pm j.$$

poles: $s^2 + 2s + 5 = 0$

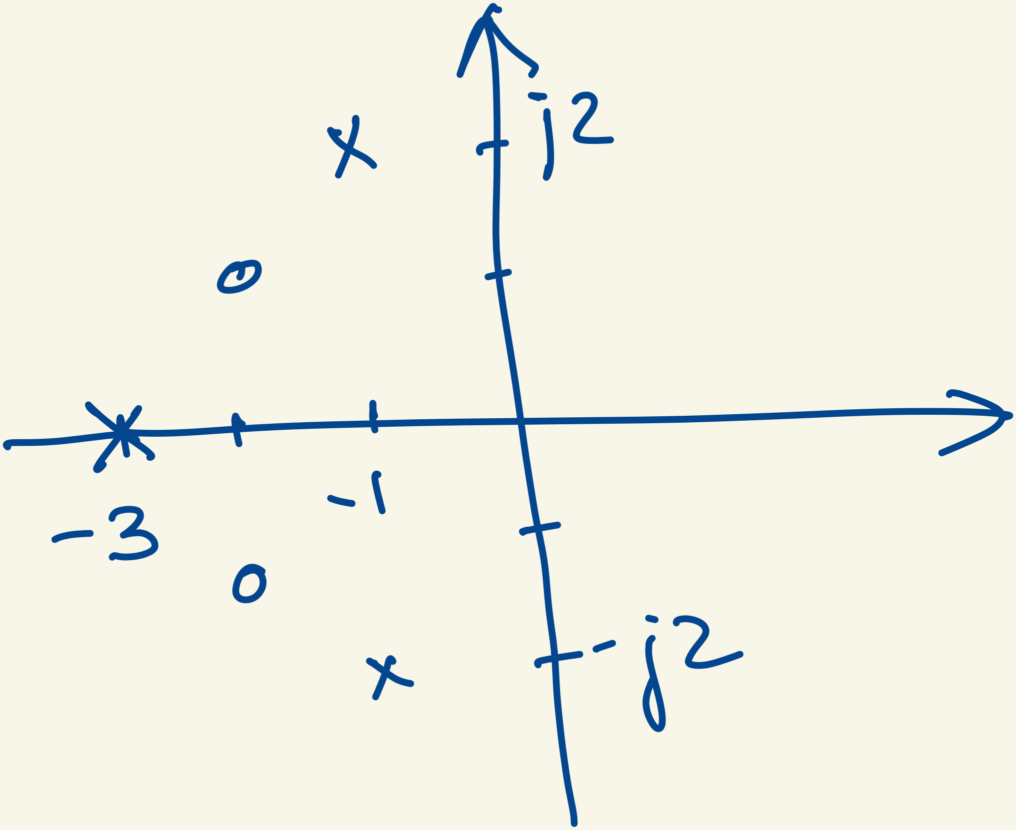
$$s_{1/2} = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$s_{1/2} = \frac{-2 \pm j4}{2}$$

$$s_{1/2} = -1 \pm j2$$

$$s + 3 = 0$$

$$s_3 = -3$$



b) Yes, ROC include $j\Omega$ axis.

c) $H(s) = \dfrac{3s^2+12s+15}{(s^2+2s+5)(s+3)}$

$$= \frac{A}{s+3} + \frac{Bs+C}{s^2+2s+5}$$

$$A = H(s)(s+3)\Big|_{s=-3} = \frac{3(-3)^2+12(-3)+15}{(-3)^2+2(-3)+5}$$

$$A = \frac{3}{4}$$

$$\frac{3s^2+12s+15}{(s^2+2s+5)(s+3)} = \frac{\frac{3}{4}(s^2+2s+5)+(Bs+C)(s+3)}{(s+3)(s^2+2s+5)}$$

$$3s^2+12s+15 = \frac{9}{4}s^2 +$$

$$B = \frac{15}{4} \qquad C = \frac{3}{4}$$

$$\frac{Bs+C}{s^2+2s+5} = \frac{Bs+C}{s^2+2s+1-1+5}$$

$$= \frac{Bs+C}{(s+1)^2+4} = \frac{B(s+1)-B+C}{(s+1)^2+4}$$

$$= \frac{B(s+1)}{(s+1)^2+4} + \left(\frac{C-B}{2}\right)\cdot\frac{2}{(s+1)^2+4}$$

$$\Rightarrow Be^{-t}\cos(2t)u(t) + \left(\frac{C-B}{2}\right)e^{-t}\sin(2t)\,u(t)$$

$$\underline{\sin(\Omega_0 t)u(t)} \rightarrow \frac{\Omega_0}{s^2+\Omega_0^2}$$

$$\sin(-\Omega_0 t)u(t) \rightarrow \frac{-\Omega_0}{s^2+\Omega_0^2}$$

$$\sin(-\Omega_0 t)u(t)$$

$$= -\sin(\Omega_0 t)u(t)$$