Lecture 4

Systems Properties

- Linearity → non Linear
- Time Invariance → Time Varying
- Causal → Non Causal

Some nice properties of LTI systems

$$x(t) \longrightarrow \overset{LTI}{\boxed{S}} \longrightarrow y(t) = S[x(t)]$$

$$\sum_k A_k\, x(t - t_k) \longrightarrow \boxed{S} \longrightarrow z(t) = S\left\{\sum_k A_k\, x(t - t_k)\right\}$$

$$z(t) = \sum_k A_k\, y(t - t_k)$$

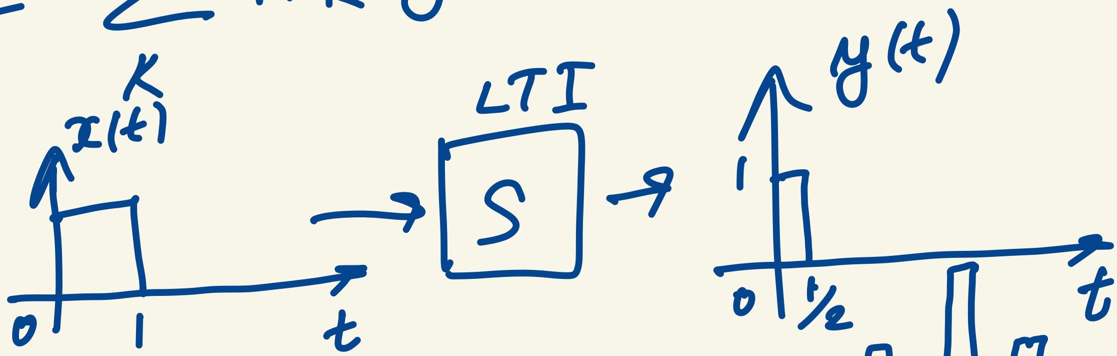
Proof:

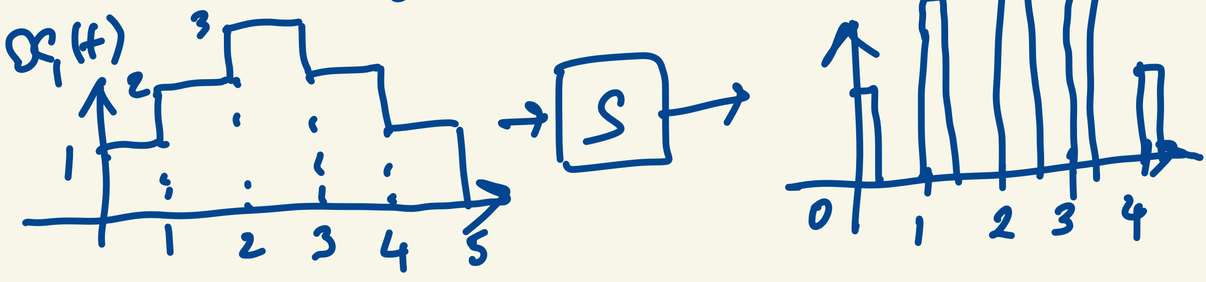
$$z(t) = S\left\{\sum_k A_k\, x(t - t_k)\right\}$$

$$\overset{\text{L superposition}}{=} \sum_k S\{\underline{A_k}\, x(t - t_k)\}$$

$$\overset{\text{L scaling}}{=} \sum_k A_k\, S\{x(t - \underline{t_k})\}$$

$$\overset{\text{TI}}{=} \sum_k A_k\, y(t - t_k)$$





$$x_1(t) = x(t) + 2x(t-1) + 3x(t-2)$$
$$+ 2x(t-3) + x(t-4)$$

$$y_1(t) = y(t) + 2y(t-1) + 3y(t-2)$$
$$+ 2y(t-3) + y(t-4)$$

LTI

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t)$$

given a function $g(t)$

e.g. $g(t) = \cos(\Omega_0 t)$

$g(t) = e^{-j\Omega_0 t}$

$\vdots$

$$x_1(t) = \int \underline{g(\tau)}\, x(t-\tau)\, d\tau$$

$$y_1(t) = S\{x_1(t)\} =$$

$$= \int g(\tau)\, y(t-\tau)\, d\tau$$

Proof:

$$y_1(t) = S\{x_1(t)\}$$

$$= S\left\{\int g(\tau)\, x(t-\tau)\, d\tau\right\}$$

$$\overset{\text{superpos.}\ L}{=} \int S\{g(\tau)\, x(t-\tau)\, d\tau\}$$

$$\overset{\text{scaling}\ L}{=} \int g(\tau)\, S\{x(t-\tau)\}\, d\tau$$

$$\overset{TI}{=} \int g(\tau)\, y(t-\tau)\, d\tau$$

Impulse Response Funct.

$$x(t) \longrightarrow \boxed{S} \longrightarrow y(t) = S\{x(t)\}$$

Def. Impulse Response Function (IRF)

$$x(t) = \delta(t) \longrightarrow \boxed{S} \longrightarrow S\{\delta(t)\} = h(t,0)$$

$$\delta(t-\tau) \longrightarrow \boxed{S} \longrightarrow h(t,\tau)$$

General IRF is $h(t,\tau)$

if S is TI:

$z(t) = y(t-\tau)$ (Last lecture)

$$h(t,\tau) = h(\underline{\underline{t-\tau}},0)$$

All IRFs are related (time-shifted versions) of $\underline{\underline{h(t,0)}}$ is S is TI

For LTI we only need $h(t,0) = h(t)$

For TV: $h(t,\tau) \overset{TV}{=} S\{\delta(t-\tau)\}$

For TI: $h(t) = S\{\delta(t)\}$

$$S\{\delta(t-\tau)\} \overset{TI}{=} h(t-\tau)$$

Example

$$y(t) = \int_0^{+\infty} e^{-(t-\sigma)} u(t-\sigma) x(\sigma) d\sigma$$

a) Find IRF $h(t,\tau)$

b) Check if the system is TI

$$h(t,\tau) = S\{\delta(t-\tau)\} \qquad \underline{\tau > 0}$$

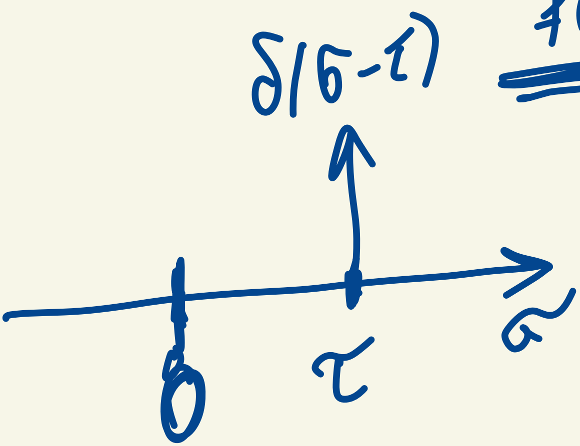
$$= \int_0^{+\infty} \underbrace{e^{-(t-\sigma)} u(t-\sigma)}_{\text{evaluate at } \sigma=\tau} \delta(\sigma-\tau)\,d\sigma$$

sits at τ

Hint

Last lecture: $\underline{f(t)}\,\underline{\delta(t)} = f(0)\delta(t)$

$$\underline{\underline{f(t)}}\,\underline{\underline{\delta(t-\sigma)}} = f(\sigma)\,\underline{\underline{\delta(t-\sigma)}}$$



1) evaluate f where δ sits!

2) δ stays!!

$$= \int_0^{+\infty} \underline{e^{-(t-\tau)} u(t-\tau)}\,\delta(\sigma-\tau)\,d\sigma$$

$$= e^{-(t-\tau)} u(t-\tau) \underbrace{\int_{0}^{+\infty} \delta(\sigma-\tau)\, d\sigma}_{=1 \quad \tau > 0}$$

$$= e^{-(t-\tau)} u(t-\tau)$$

$$\boxed{h(t,\tau) = e^{-(t-\tau)} u(t-\tau)}$$

b) Is S TI?

Use property of IRF? for TI

For TI $h(t,\tau) = h(t-\tau)$

IRF depends only on $t-\tau$, not t and τ independently

$$h(t,\tau) = e^{-\underset{\sim}{(t-\tau)}}\, u(\underset{\sim}{t-\tau})$$

$\Longrightarrow$ S is TI.

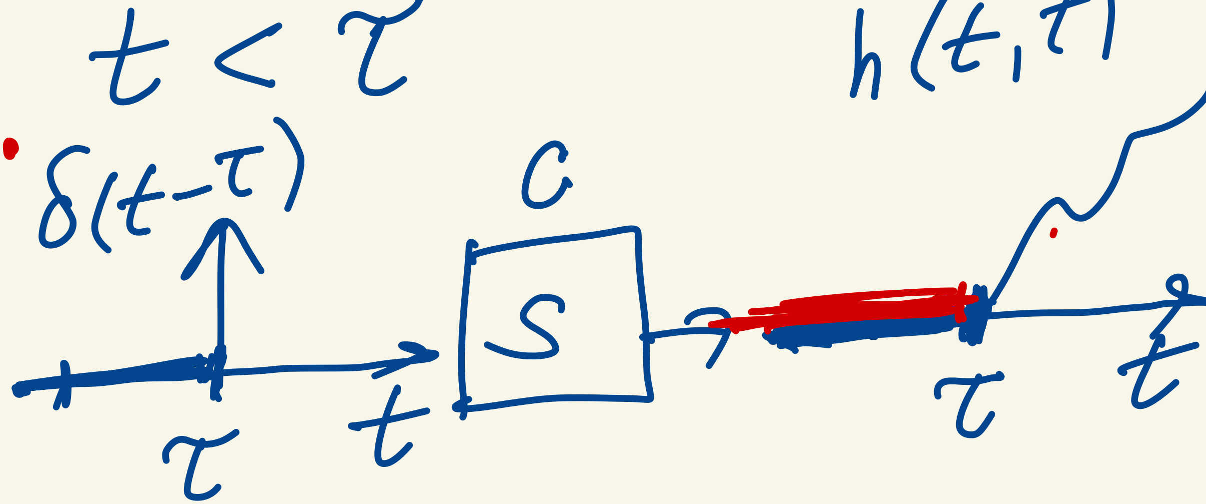
Example of IRF for $\underline{\underline{TV}}$

$$h(t,\tau) = t\, e^{-\underline{(t-\tau)}}\, u(\underline{t-\tau})$$

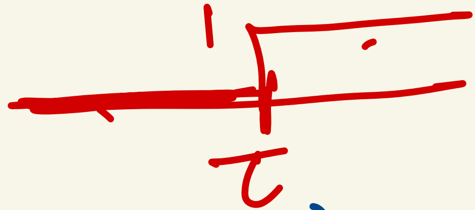
Property of IRF for causal systems.

Proposition:

If S is causal (C) then $h(t,\tau) = 0$ for $t < \tau$



$$h(t,\tau) \overset{C}{=} h(t,\tau)\cdot \underbrace{u(t-\tau)}$$



$$TI + C \rightarrow h(t) = h(t)\cdot u(t)$$

Example.

a) Find IRF ✓

b) Check if S is C.

$$y(t) = x(t) - \int_{t}^{+\infty} 2e^{-(\tau - t)}\, x(\tau)\, d\tau$$

$$\delta(t-\sigma) \longrightarrow \boxed{S} \longrightarrow h(t,\sigma)$$

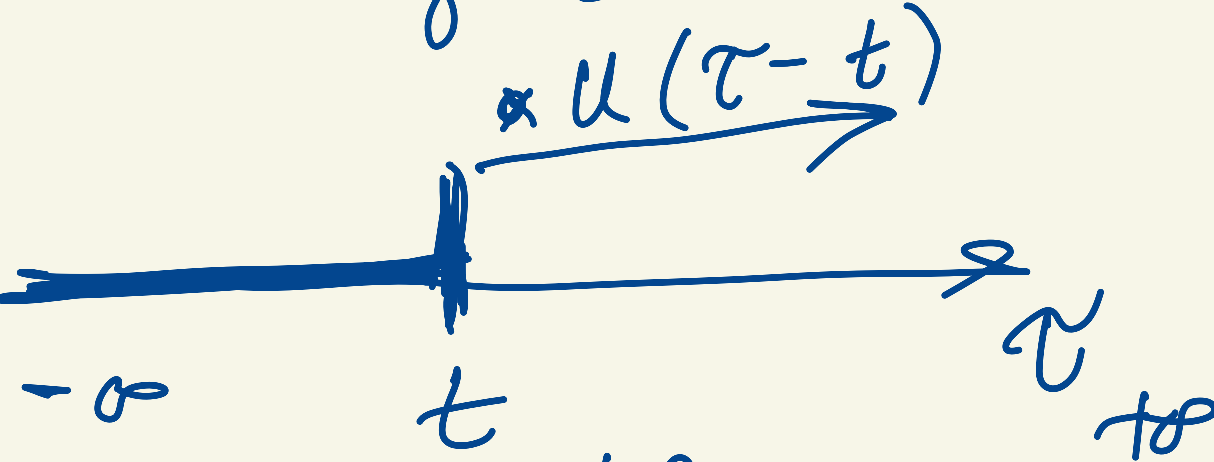
$$h(t,\sigma) = \delta(t-\sigma) - \int_{t}^{+\infty} \underbrace{2e^{-(\tau-t)}}\,\underbrace{\delta(\tau-\sigma)}\,d\tau$$

$$-\infty < t < +\infty$$

$$-\infty < \sigma < +\infty$$

trick change limits

integral to $-\infty$, $+\infty$



$$= \delta(t-\sigma) - \int_{-\infty}^{+\infty} 2e^{-(\tau-t)} \cdot \underbrace{\delta(\tau-\sigma)}_{\tau=\sigma} \cdot u(\tau-t)\,d\tau$$

$$= \delta(t-\sigma) - \int_{-\infty}^{+\infty} 2e^{-(\sigma-t)} \underline{u(\sigma-t)}$$

$$\cdot\, \delta(\tau-\sigma)\,d\tau$$

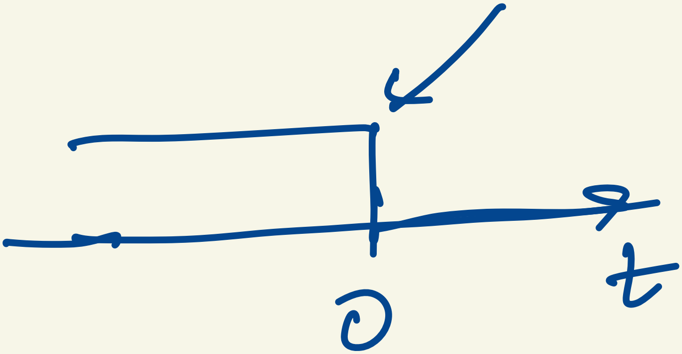
$$= \delta(t-\sigma) - 2e^{-(\sigma-t)} u(\sigma-t) \underbrace{\int_{-\infty}^{+\infty} \delta(\tau-\sigma)\,d\tau}_{=1}$$

$$h(t,\sigma) = \delta(t-\sigma) - 2e^{-(\sigma-t)} u(\sigma-t)$$

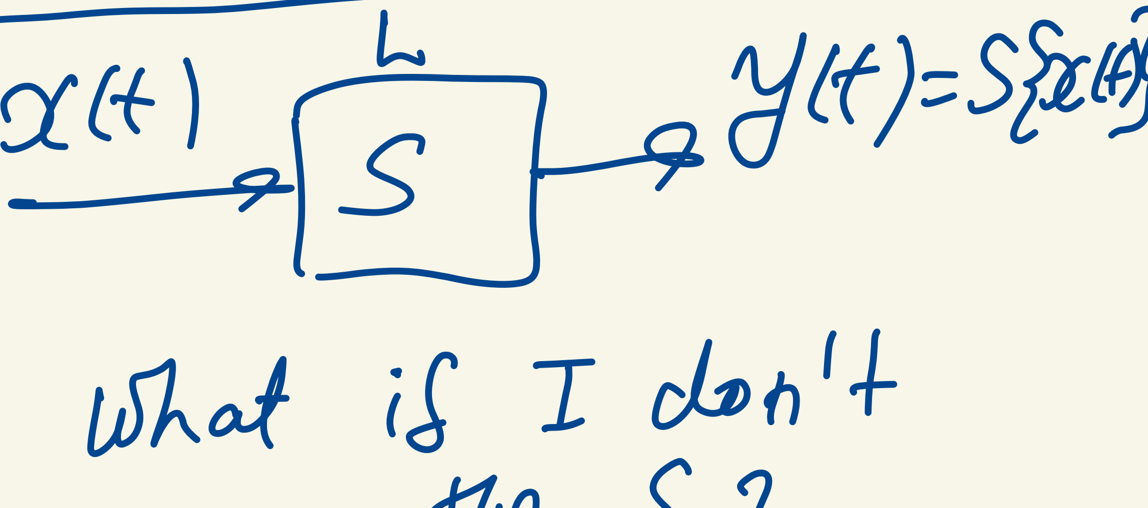
S is TI because IRF only depends on $t-\sigma$

$\sigma = 0 \qquad h(t) = h(t,0)$

$$h(t) = \delta(t) - 2e^{t}\underline{\underline{u(-t)}}$$



No! it is not C.



What if I don't know the S?

or S is too complicated.

But I know that system is L.

Big idea: Measure or compute $\underline{h(t,\tau)}$!

LTV

$$x(t) \longrightarrow \boxed{h(t,\tau)} \longrightarrow y(t)$$

LTI

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Generic Representation of Signals!

$$\underline{x(t)} = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = S\{x(t)\}$$

$$= S\left\{\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\}$$

$$\overset{L}{=} \int_{-\infty}^{+\infty} S\{x(\tau)\,\delta(t-\tau)\,d\tau\}$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, \underbrace{S\{\delta(t-\tau)\}}_{h(t,\tau)}\, d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t,\tau)\, d\tau$$

CONVOLUTION INTEGRAL!

Applies to ALL Linear Systems