Discussion 9

ECE 102: Systems and Signals

Winter 2022

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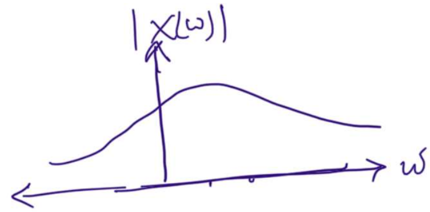
Review of Fourier Transforms



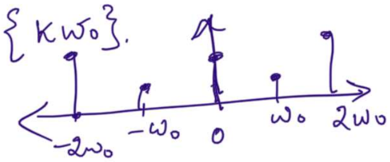
(1) For a continuous time $x(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt.$$

continuous in frequency

Compare this with Fourier series : discrete frequency space.

$$X_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt \qquad \{k\omega_0\}.$$



(2). Existence of Fourier Transform (Dirichlet).

1. Iff. $x(t)$ is absolutely integrable.

$$\text{ie} \quad \int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

2. If $x(t)$ has a finite no. of maxima/minima/discontinuities.

(3) Fourier Transform of Periodic signals

$\sin(2\pi t)$, $\cos(2\pi t)$.

— Infinitely long.

— $\int_{-\infty}^{\infty} |x(t)|dt$ does not converge

} $x(t)$: periodic T_0, $\omega_0 = \frac{2\pi}{T_0}$

F.T. can be computed from the Fourier series rep. of $x(t)$.

Step. i) : $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$

Step ii) $\mathcal{F}\{x(t)\} = \mathcal{F}\{\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}\}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}\right).e^{-j\omega t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} X_k \int_{-\infty}^{\infty} e^{-j(\omega - k\omega_0)t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} X_k \underbrace{\int_{-\infty}^{\infty} e^{jk\omega_0 t}.e^{-j\omega t}\,dt}_{\mathcal{F}\{e^{jk\omega_0 t}\} = 2\pi.\delta(\omega - k\omega_0)}$$

$$\boxed{X(\omega) = \sum_{k=-\infty}^{\infty} X_k\, 2\pi\, \delta(\omega - k\omega_0)}$$

$$X(\omega). = \sum_{k=-\infty}^{\infty} 2\pi X_k\, \delta(\omega - k\omega_0)$$

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(4) Finding F.T. from Laplace Transform

$x(t)$.

L.T $\quad X(s). = \int_{-\infty}^{\infty} x(t).e^{-st}\,dt = \int_{-\infty}^{\infty} x(t).e^{-(\sigma + j\omega)t}\,dt.$

$s = \sigma + j\omega$

F.T. $\quad X(\omega) = \int_{-\infty}^{\infty} x(t).e^{-j\omega t}\,dt$

$$\boxed{X(\omega) = X(s)\Big|_{s = 0 + j\omega}}$$

$X(\omega)$ can be obtained as the Laplace Transform evaluated on the ~~jω~~ $j\omega$ axis.
$\{s = j\omega\}$

Provided: The ROC contains the $j\omega$ axis entirely.

You can use the substitution $s = j\omega$ to find F.T.

as $X(\omega) = X(s)\Big|_{s=j\omega}$ only if $j\omega$ axis is contained in the ROC.

Properties of Fourier Transform :-

1. Linearity: $a x(t) + b y(t) \xleftrightarrow{F.T} a X(\omega) + b Y(\omega)$.

2. Time shifting: $x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$

3. Freq. shift: $e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$.

4. Conjugation: $x^*(t) \longleftrightarrow X^*(-\omega)$

5. $x(-t) \longleftrightarrow X(-\omega)$.

6. Time scaling $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

7. Convolution:

$$x(t) * y(t) \longleftrightarrow X(\omega) \cdot Y(\omega)$$

8. Multiplication

$$x(t) \cdot y(t) \longleftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) Y(\omega - \theta)\, d\theta$$

$$= \frac{1}{2\pi} X(\omega) * Y(\omega)$$

9. Differentiation $\frac{d x(t)}{dt} \longleftrightarrow j\omega X(\omega)$

10. Integration. $\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$.

11. $t \cdot x(t) \longleftrightarrow j \frac{d}{d\omega} X(\omega)$.

* 12. Parseval's identity: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

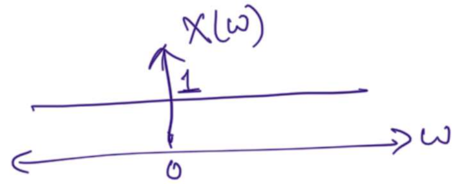
13 : Freq- time duality :-

If $x(t) \xleftrightarrow{F.T.} X(\omega)$.

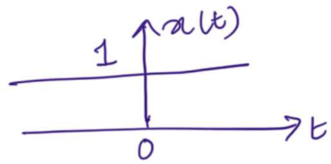
then $X(t) \xleftrightarrow{F.T.} 2\pi\, x(\omega)$

Fourier Transform pairs

1. $\delta(t) \longleftrightarrow 1 \rightarrow$ (freq).



2. $1 \longleftrightarrow 2\pi\delta(\omega)$.



3. $\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0}$

4. $e^{+j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$

$e^{j\omega_0 t}\cdot \underline{1} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$.

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* 5. $u(t) \longleftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$.

$= \int_{-\infty}^{t} \delta(\tau)\,d\tau$ (use Integrⁿ property).

6. $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$ ∀ $Re\{a\} > 0$.

7. $t.\,e^{-at}u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2}$ ∀ $Re\{a\} > 0$.

8. $\cos(\omega_0 t) \longleftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

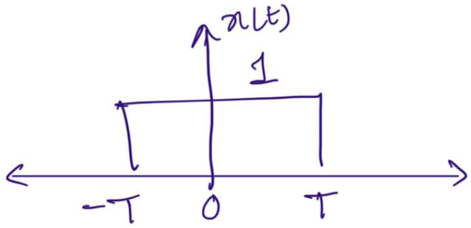
derive i) Euler's identity.
ii) time shift prop.

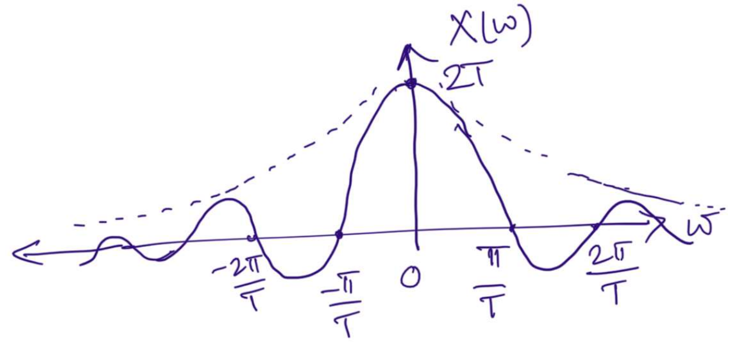
9. $\sin(\omega_0 t) \longleftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

10. $x(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases} \longleftrightarrow X(\omega) = 2T\,\mathrm{sinc}(\omega T)$

$$= 2T\left(\frac{\sin \omega T}{\omega T}\right)$$

$$= \mathrm{rect}(t, T)$$

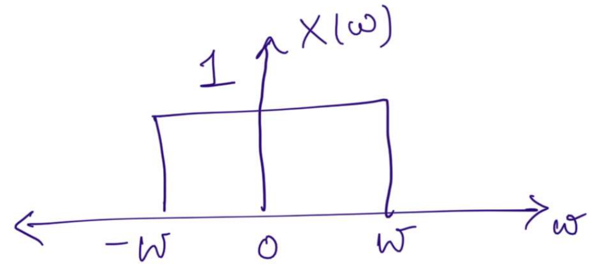


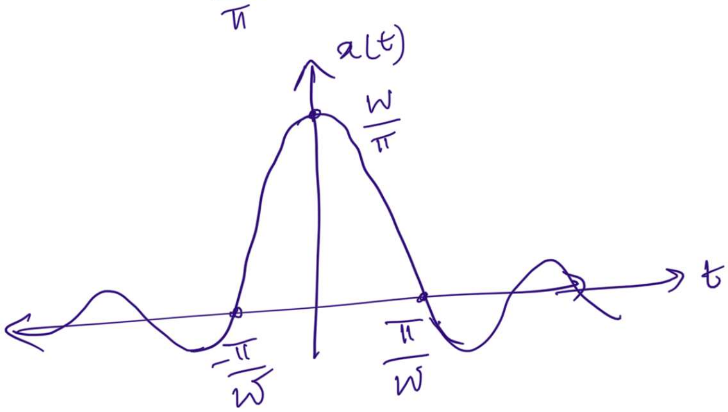


11. $x(t) = \frac{1}{2\pi} 2W\mathrm{sinc}(tW) \longleftrightarrow X(\omega) = \mathrm{rect}(\omega, W)$.

$$= \frac{2}{2\pi}\,\frac{\sin(Wt)}{t}$$

$$= \frac{W}{\pi}\,\mathrm{sinc}(Wt).$$





Question 1: Laplace Transform vs. Fourier Series vs. Fourier Transform

$x(t)$ is defined as one period of the periodic signal $y(t)$ (over time range $[0, 4]$):

$$x(t) = 2r(t) - 4r(t-1) + 4r(t-3) - 2r(t-4) + 1$$

Find the following transforms/representations. If the representation is not possible, specify why.

(a) Laplace transform of $x(t)$: can be found after subtracting DC component.

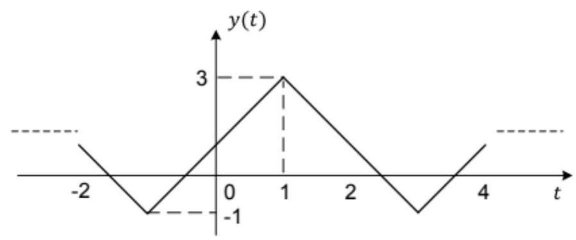
(b) Laplace transform of $y(t)$: cannot be found, since $y(t)$ not causal.

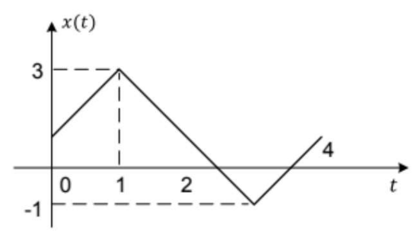
(c) Fourier series of $x(t)$: cannot be found $\because x(t)$ not periodic.

(d) Fourier series of $y(t)$: yes $\because$ periodic.

(e) Fourier transform of $x(t)$:

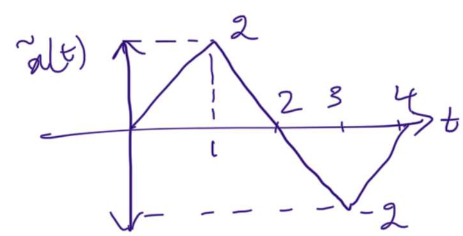
(f) Fourier transform of $y(t)$: $\Big\}$ yes, can be found.

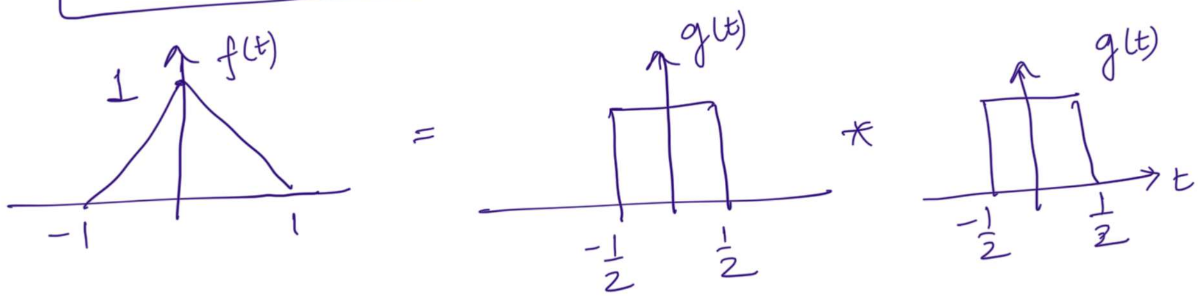




(e). $x(t) = \tilde{x}(t) + 1$

$$\boxed{X(\omega) = \tilde{X}(\omega) + 2\pi\delta(\omega).}$$





$f(t) = g(t) * g(t).$

$$\boxed{F(\omega) = (G(\omega))^2.}$$

$\tilde{x}(t) = 2f(t-1)$
$\quad - 2f(t-3)$

① How does $F(\omega)$ relate to $\tilde{X}(\omega)$:

$$\tilde{x}(t) = 2f(t-1) - 2f(t-3).$$

$$\tilde{X}(\omega) \quad \cancel{\tilde{X}(f)} = 2F(\omega)\left[e^{-j\omega} - e^{-j3\omega}\right]$$

② Find $G(\omega)$:

$$g(t) = \cancel{u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2})} . \text{ rect}(t, \tfrac{1}{2}).$$

$\cancel{\mathcal{F}\{u(t)\}}$

$$G(\omega) = 2T \operatorname{sinc}(\omega T)$$

$$= 2 \cdot \frac{1}{2} \cdot \frac{\sin(\omega/2)}{(\omega/2)}$$

$$= \operatorname{sinc}(\omega/2).$$

$$F(\omega) = (G(\omega))^2 = \operatorname{sinc}^2\left(\frac{\omega}{2}\right).$$

$$\tilde{X}(\omega) = 2\operatorname{sinc}^2\left(\frac{\omega}{2}\right)\left[e^{-j\omega} - e^{-j3\omega}\right].$$

$$\boxed{X(\omega) = 2\operatorname{sinc}^2\left(\frac{\omega}{2}\right)\left[e^{-j\omega} - e^{-j3\omega}\right] + 2\pi\delta(\omega)}$$

(f). $$Y(\omega) = \sum_{k=-\infty}^{\infty} 2\pi Y_k \,\delta(\omega - k\omega_0) \qquad \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}.$$

$$= \sum_{k=-\infty}^{\infty} 2\pi Y_k \,\delta\left(\omega - k\frac{\pi}{2}\right).$$

Question 2

Find the Fourier transform of the following signals:

(a) $f_1(t) = 4e^{-|3t|}$

(b) $f_2(t) = \cos(10t)u(t)$

(a) $f_1(t) = 4e^{-3|t|} = \begin{cases} 4e^{-3t} & t > 0 \rightarrow u(t) \\ 4e^{3t} & t < 0. \rightarrow u(-t). \end{cases}$

$= 4e^{-3t}u(t) + 4e^{3t}u(-t).$

$= 4\left[e^{-3t}u(t) + e^{3t}u(-t)\right].$

$e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$; $Re\{a\} > 0.$

$e^{-3t}u(t) \longleftrightarrow \frac{1}{j\omega+3}$

$e^{3t}u(-t) = e^{-3(-t)} \cdot u(-t) \longleftrightarrow \frac{1}{3+j(-\omega)}$

$\therefore F_1(\omega) = 4\left[\frac{1}{3+j\omega} + \frac{1}{3-j\omega}\right].$

$= \frac{4.(2\times3)}{9-(j\omega)^2} = \boxed{\frac{24}{9+\omega^2}}$

(b) $f_2(t) = \cos(10t)u(t). = \frac{1}{2}\left(e^{j10t} + e^{-j10t}\right)u(t).$

$F_2(\omega) = \int_{-\infty}^{} =$

$$F_2(\omega) = \frac{1}{2}\mathcal{F}\{e^{j10t}u(t)\} + \frac{1}{2}\mathcal{F}\{e^{-j10t}u(t)\}$$

- $\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega).$
- freq. shift $e^{j\omega_0 t}x(t) \longleftrightarrow X(\omega - \omega_0).$

$$F_2(\omega) = \frac{1}{2}\left\{\frac{1}{j(\omega-10)} + \pi\delta(\omega-10)\right\} + \frac{1}{2}\left\{\frac{1}{j(\omega+10)} + \pi\delta(\omega+10)\right\}$$

$$= \frac{1}{2}\left(\frac{2\omega}{\omega^2-100}\right) + \frac{\pi}{2}\left[\delta(\omega-10) + \delta(\omega+10)\right]$$

$$\boxed{F_2(\omega) = \frac{j\omega}{100-\omega^2} + \underbrace{\frac{\pi}{2}\left[\delta(\omega-10) + \delta(\omega+10)\right]}_{\frac{1}{2}\mathcal{F}\{\cos(10t)\}}}$$

Question 3

Consider the following function:

$$f(t) = \int_{-2}^{3} e^{-|t-\tau|} d\tau$$

(a) Find its Fourier transform $F(\omega)$

—x —— .

(b) Find a general expression for $\mathcal{F}\{f(t)\sin(\omega_k t - \theta)\}$ in terms of $F(\omega)$ (H.W).

$$f(t) = \int_{-2}^{3} e^{-|t-\tau|} d\tau.$$

$$= \int_{-\infty}^{\infty} e^{-|t-\tau|} \left[u(\tau+2) - u(\tau-3)\right] d\tau.$$

$$= e^{-|t|} * \left(u(t+2) - u(t-3)\right)$$

i) $e^{-|t|}$: $e^{-t}u(t) + e^{t}u(-t)$.

for sanity check: $\boxed{e^{-a|t|} \longleftrightarrow \dfrac{2a}{a^2+\omega^2}}$

$$\boxed{\mathcal{F}\{e^{-|t|}\} = \frac{2}{1+\omega^2}}$$

ii) $u(t+2) - u(t-3)$

$$\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega).$$

$$\Rightarrow \mathcal{F}\{u(t+2)\} = \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)e^{2j\omega}.$$

$$\boxed{F(\omega) = \left(\frac{2}{1+\omega^2}\right)\left(\frac{1}{j\omega} + \pi\delta(\omega)\right)\left(e^{2j\omega} - e^{-3j\omega}\right)}$$