Discussion 2

ECE 102: Systems and Signals

Winter 2022

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1 Problems

1.1 System Linearity, Time Invariance, and Causality

The system S has the following IPOP:

$$y(t) = x(3t-2) + x(2-t), \quad -\infty < t < \infty$$

(a) Is the system S linear? Provide a proof.

(b) Is the system S time variant (TV) or time invariant (TI)? Justify your answer.

(c) Is the system S causal (C) or non-causal (NC) ? Justify your answer.

The system S has the following IPOP: $-2,\ 3t$ $\quad x(t) \to \boxed{h(t)} \to y(t)$

$$\boxed{y(t) = x(3t-2) + x(2-t),} \quad -\infty < t < \infty$$

$+2,\ -1$

(a) Is the system S linear? Provide a proof.

Consider arbitrary signals $x_1(t), x_2(t)$

s.t. $\quad x_1(t) \to \boxed{h(t)} \to y_1(t)$

$\quad x_2(t) \to \boxed{h(t)} \to y_2(t)$

linear combination: $\alpha x_1(t) + \beta x_2(t) \to \boxed{h(t)} \to y_3(t)$

$\alpha, \beta \in \mathbb{C}$

(*) The system $h(t)$ is linear if

$$\boxed{y_3(t) = \alpha y_1(t) + \beta y_2(t)}$$

i) $y_1(t) = x_1(3t-2) + x_1(2-t)$

$y_2(t) = x_2(3t-2) + x_2(2-t)$

$(\alpha x_1 + \beta x_2) \to \boxed{h(t)} \to y_3(t)$

$$\begin{aligned} y_3(t) &= (\alpha x_1 + \beta x_2)(3t-2) + (\alpha x_1 + \beta x_2)(2-t) \\ &= \alpha x_1(3t-2) + \beta x_2(3t-2) + \alpha x_1(2-t) + \beta x_2(2-t) \\ &= \alpha\left(x_1(3t-2) + x_1(2-t)\right) + \beta\left(x_2(3t-2) + x_2(2-t)\right) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

$\therefore$ System is linear.

The system S has the following IPOP:

$$y(t) = x(3t-2) + x(2-t), \quad -\infty < t < \infty$$

(b) Is the system S time variant (TV) or time invariant (TI)? Justify your answer.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

System represented by $h(t)$ is time invariant

if $\underline{x(t-t_0)} \rightarrow \boxed{h(t)} \rightarrow \underline{y(t-t_0)} \quad \forall\, t_0 \in \mathbb{R}$

i.e. the system is invariant to time delays / shifts

Let $x_4(t) = x(t-t_0)$

i) $x(t-t_0) \rightarrow \boxed{h(t)} \rightarrow y_4(t)$

Find $y_4(t)$: o/p obtained when $x(t-t_0)$ is passed through S.

ii) Find $y(t-t_0)$.

Compare $y(t-t_0)$ & $y_4(t)$.

i) $$\left.\begin{aligned} y_4(t) &= x((3t-2)-t_0) + x((2-t)-t_0) \\ &= x(3t-2\underline{\underline{-t_0}}) + x(2-t\underline{\underline{-t_0}}) \end{aligned}\right\} \text{o/p with } x(t-t_0).$$

ii) $$y(t) = x(3t-2) + x(2-t)$$

$$\left.\begin{aligned} y(t-t_0) &= x(3(t-t_0)-2) + x(2-(t-t_0)) \\ &= x(3t-2\underline{\underline{-3t_0}}) + x(2-t\underline{\underline{+t_0}}) \end{aligned}\right\} y(t-t_0).$$

$$\boxed{y_4(t) \neq y(t-t_0)}$$ $\therefore$ Time variant

(*) In general, any system that introduces time scaling is time variant

eg. $y(t) = x(3t),\ x(1-t),\ x[t/4]$.

The system S has the following IPOP:

$$y(t) = x(3t-2) + x(2-t), \quad -\infty < t < \infty$$

(c) Is the system S causal (C) or non-causal (NC) ? Justify your answer.

System h(t) is causal if the output y(t) at any time instant t depends only on past (and present) values of input x(t)

$$y(t) = x(3t-2) + x(2-t)$$

For S to be causal:

must hold simultaneously $\forall t \in (-\infty, \infty)$

$$\left.\begin{array}{lll} t \geq 3t-2 & \Rightarrow & t \leq 1 \\ t \geq 2-t & \Rightarrow & t \geq 1 \end{array}\right\} \underline{t=1}$$

$\therefore$ Causal relationship seen only at $t=1$
But not for all $t \in (-\infty, \infty)$

$\therefore$ Not Causal.

1.2 Unit Impulse Response Function in Integration

Given the following input-output relation (IPOP) of a system:

$$y(t) = \int_{-\infty}^{\infty} e^{-t}(t-\tau)u(\tau - t)x(\tau)d\tau, t \in (-\infty, \infty).$$

a) Find impulse response of the system $h(t, \tau)$. Is the system time variant (TV) or time invariant (TI)? Is it causal (C) or non-causal (NC)?

b) Find the corresponding output, $y(t)$, given an input of:

$$x(t) = \delta(t-2) - e^{-t}u(t), \quad t \in (-\infty, \infty)$$

$h(t,\tau)$: response to $\delta(t-\tau)$

rewrite the ip/op in terms of σ instead of τ

$$y(t) = \int_{-\infty}^{\infty} e^{-t}(t-\sigma)u(\sigma-t)x(\sigma)d\sigma \quad \forall t \in (-\infty,\infty)$$

$$x(t) = \delta(t-\tau) \rightarrow \boxed{S} \rightarrow h(t,\tau).$$

$$\therefore \quad h(t,\tau) = \int_{-\infty}^{\infty} e^{-t}(t-\sigma)u(\sigma-t)\delta(\sigma-\tau)d\sigma$$

Sifting property of impulse response

$$\int_{-\infty}^{\infty} f(\sigma)\delta(\sigma-\tau)d\sigma = f(\tau)$$

$$\therefore \quad h(t,\tau) = \int_{-\infty}^{\infty} \underbrace{e^{-t}(t-\sigma)u(\sigma-t)}\,\underbrace{\delta(\sigma-\tau)}_{\boxed{\sigma=\tau}}d\sigma$$

$$\boxed{h(t,\tau) = e^{-t}(t-\tau)u(\tau-t)}$$

Given the following input-output relation (IPOP) of a system:

$$y(t) = \int_{-\infty}^{\infty} e^{-t}(t-\tau)u(\tau - t)x(\tau)d\tau, t \in (-\infty, \infty).$$

a) Find impulse response of the system $h(t, \tau)$. Is the system time variant (TV) or time invariant (TI)? Is it causal (C) or non-causal (NC)?

$h(t,\tau)$: impulse response when a time delayed input is applied

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i) Time Invariance : $h(t,\tau)$??

If $h(t,\tau)$ is a f^n of $(t-\tau)$ alone $\Rightarrow$ Time invariant

or if $\boxed{h(t,\tau) = h(t-\tau, 0)}$

$\Rightarrow$ Time invariant.

a) $h(t,\tau) = e^{-t}(t-\tau)u(\tau - t)$.

$\hookrightarrow$ f^n of t alone $\therefore$ TV.

preferably $\longrightarrow$ b) $h(t-\tau, 0) = e^{-(t-\tau)}(t-\tau)u(0-(t-\tau))$

$\neq h(t,\tau)$.

$\therefore$ Time variant

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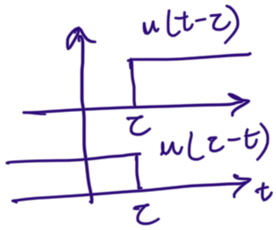
ii) Causality :

$h(t,\tau) = 0 \quad \forall\, t < \tau \Rightarrow$ Causal system

A system is causal if $\boxed{h(t,\tau) = h(t,\tau)u(t-\tau)}$

$$h(t,\tau)u(t-\tau) = e^{-t}(t-\tau)\underbrace{u(\tau-t)u(t-\tau)}_{\delta(t-\tau)}$$



$$= e^{-t}(t-\tau)\delta(t-\tau)$$

$$=$$

product : $\delta(t-\tau)$

$\boxed{h(t,\tau) \neq h(t,\tau)u(t-\tau)}$

Non causal !

Given the following input-output relation (IPOP) of a system:

$$y(t) = \int_{-\infty}^{\infty} e^{-t}(t-\tau)u(\tau-t)x(\tau)d\tau, t \in (-\infty, \infty).$$

b) Find the corresponding output, $y(t)$, given an input of:

$$x(t) = \delta(t-2) - e^{-t}u(t), \quad t \in (-\infty, \infty)$$

$$y(t) = \int_{-\infty}^{\infty} e^{-t}(t-\tau)u(\tau-t)\left[\delta(\cancel{t}\tau-2) - e^{-\tau}u(\cancel{t}\tau)\right]d\tau$$

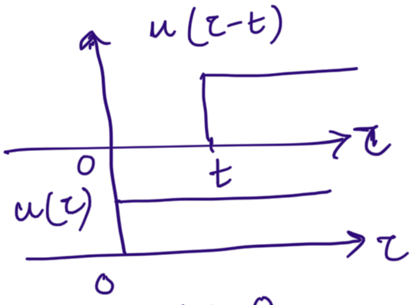
$$= \overbrace{\int_{-\infty}^{\infty} e^{-t}(t-\tau)u(\tau-t)\delta(\tau-2)d\tau}^{y_1}$$

$$- \int_{-\infty}^{\infty} \underbrace{e^{-t}\cdot e^{-\tau}(t-\tau)u(\tau-t)u(\tau)d\tau}_{y_2}$$

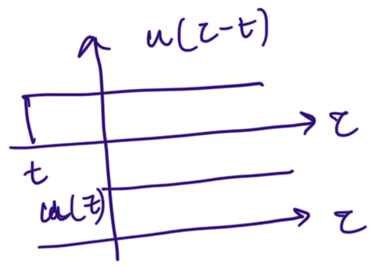
$$y_1(t) = e^{-t}(t-2)u(2-t)$$

$$y_2(t) = -\int_{-\infty}^{\infty} e^{-t}e^{-\tau}(t-\tau)\underbrace{u(\tau-t)u(\tau)}d\tau$$

$u(\tau-t)u(\tau)$.



$t \geq 0$



$t < 0$

$$\boxed{u(\tau-t)u(\tau) = \begin{cases} u(\tau-t) & t \geq 0 \\ u(\tau) & t < 0 \end{cases}}$$

solve for $y_2(t)$ as H.W.

1.3 Step Response

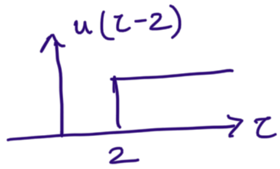
Compute the step response for following system IPOPs $y_1(t)$ and $y_2(t)$:

$$y_1(t) = \int_{-\infty}^{t} (t-\tau)e^{-2(t-\tau)}x(\tau-2)d\tau, t > -\infty$$

$$y_2(t) = \int_{-1}^{4} e^{-2t}x(t-2)dt$$

unit step response : when input $x(t) = u(t)$ is applied.

$$y_1(t) = \int_{-\infty}^{t} (t-\tau)e^{-2(t-\tau)} u(\tau-2)\, d\tau$$



$$= \int_{2}^{t} (t-\tau)e^{-2(t-\tau)} d\tau \qquad \forall\ \underline{\underline{t>2}}$$

change of variable : $2(t-\tau) = \gamma$

$$\Rightarrow \quad -2\,d\tau = d\gamma$$

$$y_1(t) = -\int_{2(t-2)}^{0} \left(\frac{\gamma}{2}\right) e^{-\gamma} \left(\frac{d\gamma}{2}\right) = \underline{\underline{\frac{1}{4}\int_{0}^{2(t-2)} \gamma e^{-\gamma} d\gamma}} \quad \underline{\underline{t>2}}$$

disc $\| \to \Gamma(a)$

$$y_1(t) = 0 \qquad t<2$$

$$y_2(t) = \int_{-1}^{4} e^{-2t} \cdot u(t-2)\, dt = \int_{-1}^{4} e^{-2t} u(t-2)\, dt$$

$$= \int_{2}^{4} e^{-2t} dt$$

$$= -\frac{1}{2}\left[e^{-8} - e^{-4}\right]$$

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