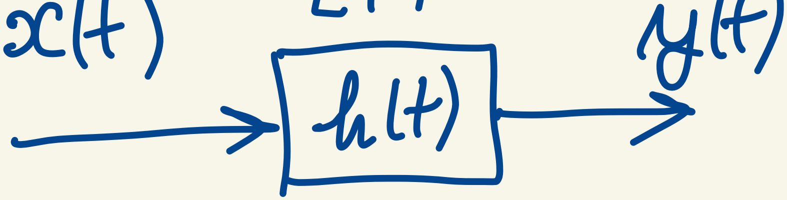
- Final Review Monday March 14
  class time 2:00-4:00pm.

  over zoom only
  recorded.

- We will post Practice
  Final this week.

Analysis of <u>LTI</u> systems
using Fourier Transform



$$y(t) = x(t) * h(t)$$

- Causal signals $x(t)\cdot u(t)$
- Causal systems $\underline{LTI, C}$

  $h(t)\cdot u(t)$

Laplace Transform was used to convert convolution integral

$$y(t) = x(t) * h(t)$$

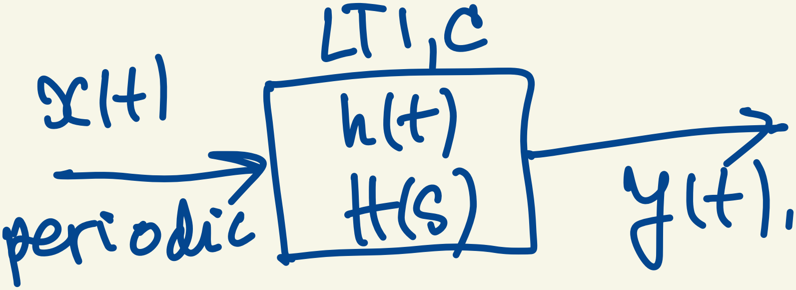
$$\Big\downarrow \mathcal{L}_s$$

$$Y(s) = X(s)\cdot H(s)$$

$$H(s) = \mathcal{L}_s\{h(t)\}$$

"transfer" system function

• if $x(t)$ is periodic and S is LTI, C



Fourier series of $x(t)$

$$x(t)=\sum_{k=-\infty}^{+\infty} \underline{X_k}\, e^{jk\omega_0 t}$$

$\Downarrow$ Linearity + eigenfunction property of $e^{j\omega_0 t}$

$$y(t)=\sum_{k=-\infty}^{+\infty} X_k \cdot H(jk\omega_0)\, e^{jk\omega_0 t}$$

→ this output holds for BIBO Stable Systems.

$H(s)$ with ROC includes $j\omega$ axis.

What if $x(t)$ is neither causal nor periodic and S is LTI

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- $y(t) = x(t) * h(t)$

$$X(\omega) = \mathcal{F}\{x(t)\}$$

$$Y(\omega) = \mathcal{F}\{y(t)\}$$

$$H(\omega) = \mathcal{F}\{h(t)\}.$$

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\}$$

convolution property from last lecture.

$$\underline{Y(\omega) = X(\omega) \cdot H(\omega)}$$

$$\omega \in \mathbb{R}$$

$$H(\omega) = \mathcal{F}\{h(t)\}$$ IRF

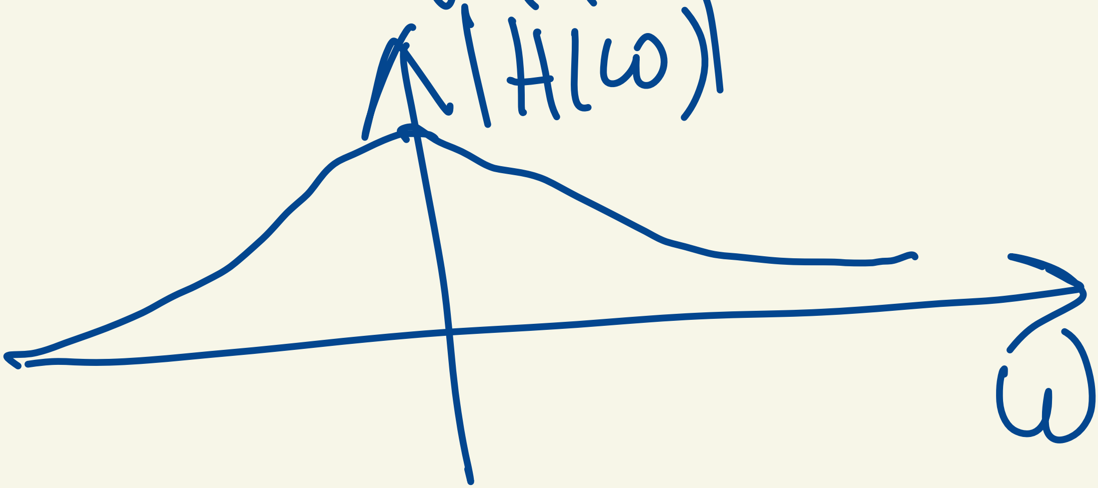
called Frequency Response Function (FRF)

$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$$

$$h(t) = e^{-2t} u(t)$$ ✓

$$H(\omega) = \frac{1}{2 + j\omega}$$ ✓

$$|H(\omega)| = \frac{1}{\sqrt{4+\omega^2}}$$



$x(t) = \cos(3t)$ ✓

$h(t) = e^{-2t}u(t)$

$y(t) = ?$

$X(\omega) = \mathcal{F}\{\cos(3t)\}$.

$$= \mathcal{F}\left\{ \frac{e^{j3t} + e^{-j3t}}{2} \right\}$$

$$= \frac{1}{2}\mathcal{F}\{e^{j3t}\} + \frac{1}{2}\mathcal{F}\{e^{-j3t}\}$$

$$= \frac{1}{2}\cdot\left[2\pi\delta(\omega-3)\right] + \frac{1}{2}\left[2\pi\delta(\omega+3)\right]$$

$$= \pi\delta(\omega-3) + \pi\delta(\omega+3)$$

$$\mathcal{F}\{\cos(\omega_0 t)\} =$$

$$= \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

$$Y(\omega) = H(\omega) \cdot X(\omega)$$

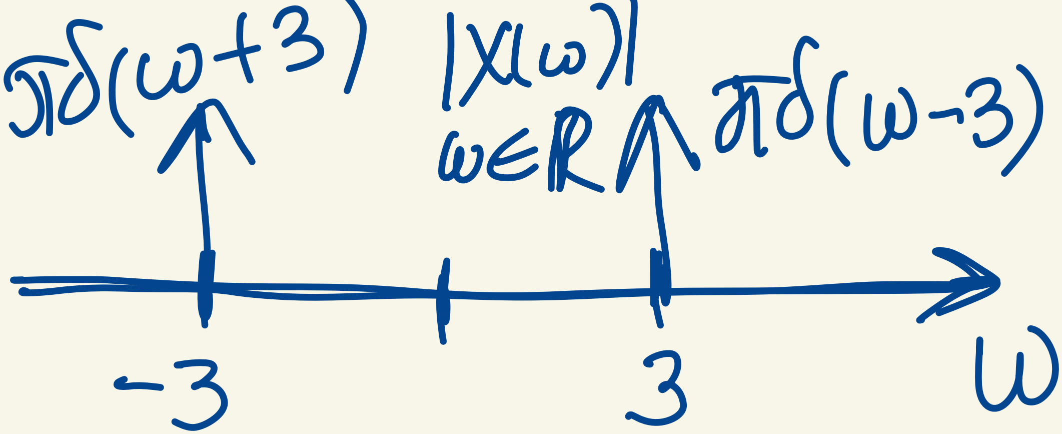
$$= H(\omega) \cdot \left[\pi \delta(\omega - 3) + \pi \delta(\omega + 3) \right]$$

$$= H(3) \cdot \pi \delta(\omega - 3)$$

$$+ H(-3) \pi \delta(\omega - 3)$$

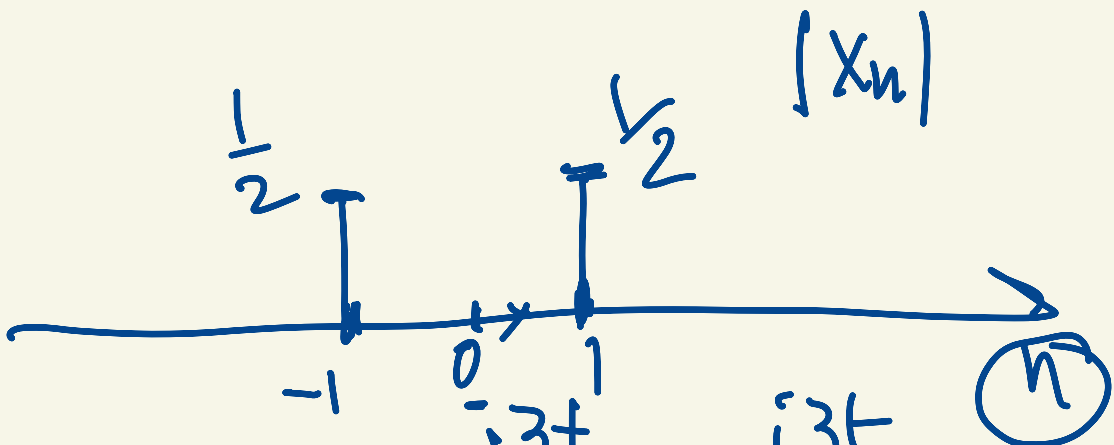
$$y(t) = \mathcal{F}^{-1} \{ Y(\omega) \}.$$

$\cos(3t) \longrightarrow \pi\delta(\omega-3)+\pi\delta(\omega+3)$



$\cos(3t) \rightarrow$ magnitude line spectra

$\omega_0 = 3$



$$\cos(3t) = \frac{e^{j3t}+e^{-j3t}}{2}$$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

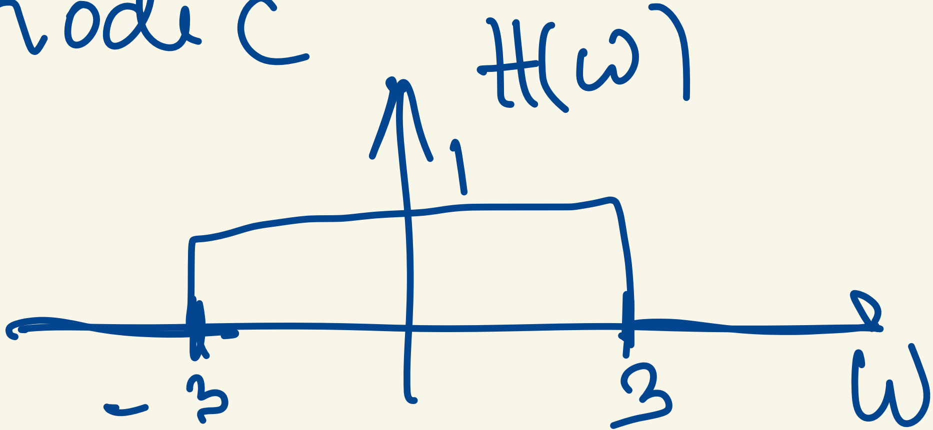
e.g

$$H(\omega) = \underline{rec(\omega, 3)} \rightarrow$$

$$x(t) = 3 + \cos(\underbrace{2t}_{\omega_0 = 2}) + \sin(\underbrace{4t}_{\omega_1 = 4})$$

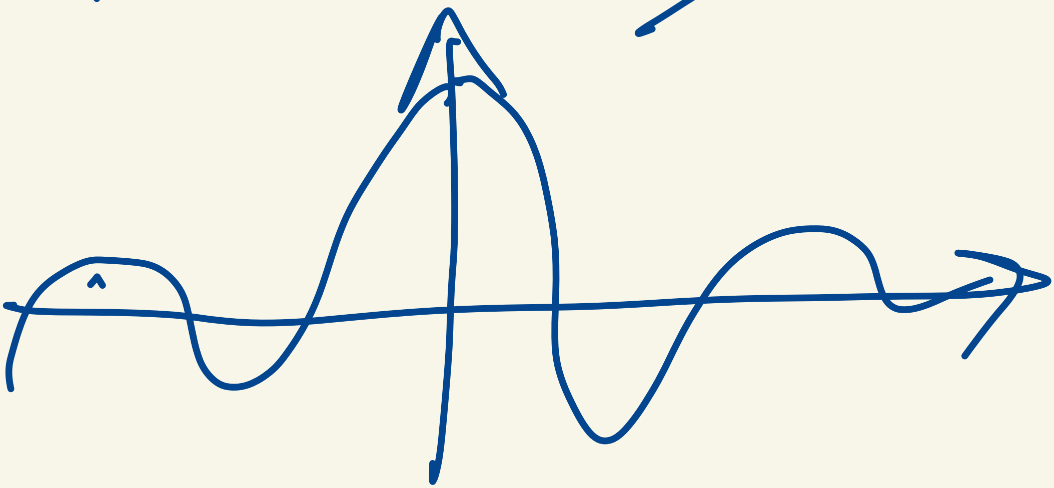
↓

periodic



$$h(t) = \frac{3}{\pi}\,\text{sinc}(3t)$$

not causal. ~~H(s)~~



$$\underline{Y(\omega)} = \underline{X(\omega)} \cdot \underline{H(\omega)}$$

$X(\omega) = \mathcal{F}\{3 + \cos(2t)$

$+ \sin(4t)\}$

$= 6\pi\delta(\omega) + \pi\delta(\omega - 2)$

$+ \pi\delta(\omega + 2) + \mathcal{F}\{\sin(4t)\}$

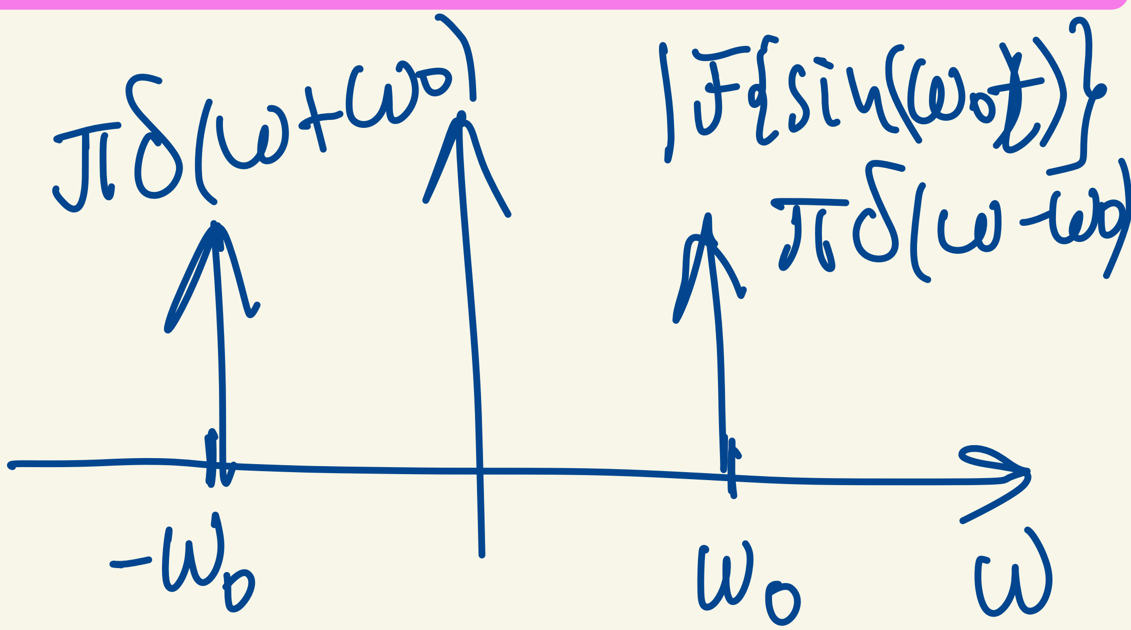
~~de~~tor.

$\underline{\mathcal{F}\{\sin(4t)\}} = \mathcal{F}\left\{\dfrac{e^{j4t} - e^{-j4t}}{2j}\right\}$

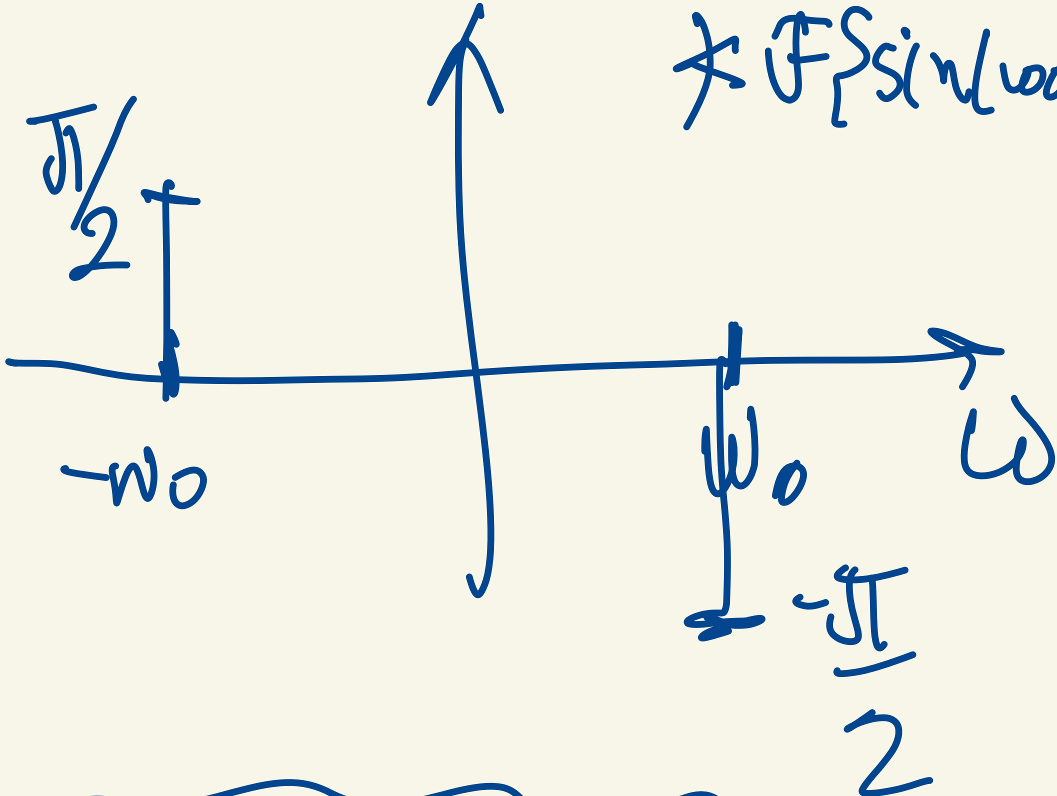
$= \frac{1}{2j} 2\pi\delta(\omega - 4) - \frac{1}{2j} 2\pi\delta(\omega + 4)$

$$= j\pi\delta(\omega+4) - j\pi\delta(\omega-4)$$

$$\mathcal{F}\{\sin(\omega_0 t)\} = j\pi\delta(\omega+\omega_0) - j\pi\delta(\omega-\omega_0)$$



$* \mathcal{F}\{\sin(\omega_0 t)\}$



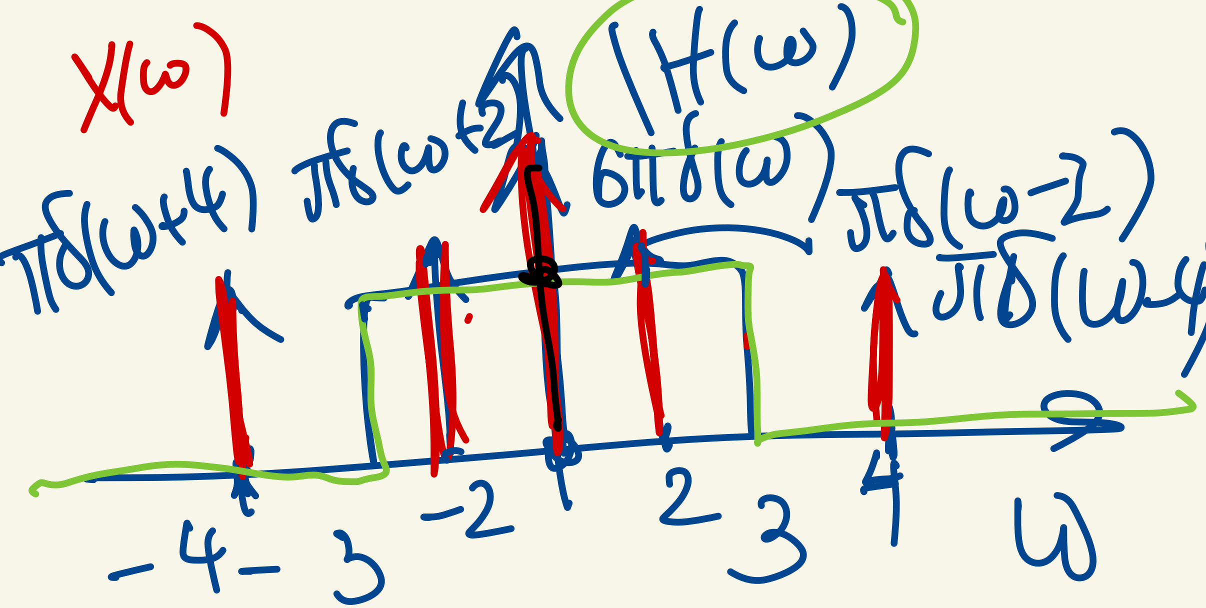
come back.

$$X(\omega) = 6\pi\delta(\omega) + \pi\delta(\omega-2)$$
$$+ \pi\delta(\omega+2) + j\pi\delta(\omega+4)$$
$$- j\pi\delta(\omega-2)$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$= \text{rec}(\omega, 3) \cdot \left[6\pi\delta(\omega) + \underline{\pi\delta(\omega+2)} + \pi\delta(\omega-2) + \underline{j\pi\delta(\omega+4)} - j\pi\delta(\omega-4) \right]$$

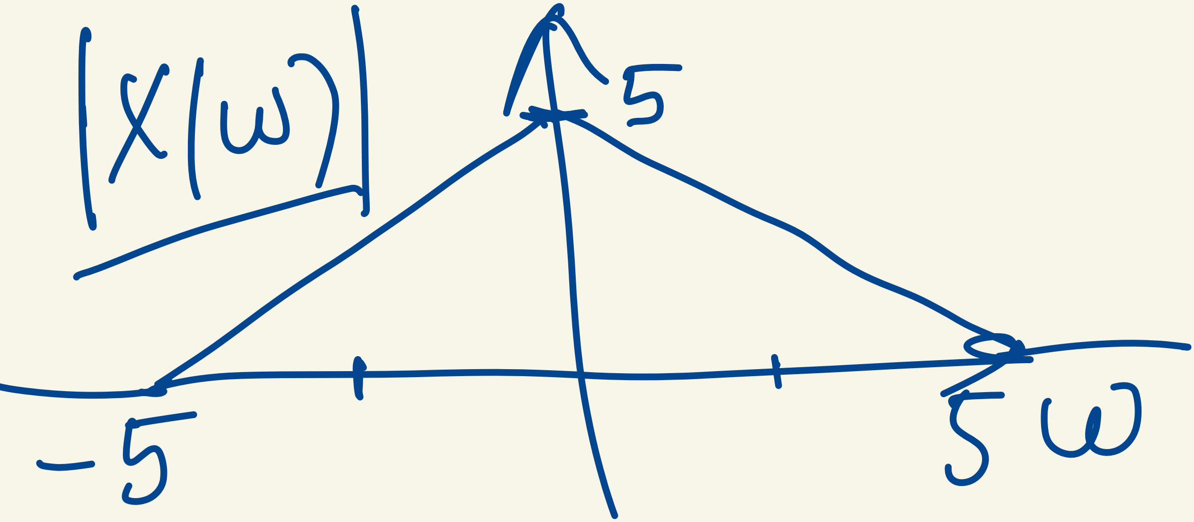
$$= 1 \cdot 6\pi\delta(\omega) + 1 \cdot \pi\delta(\omega+2) + 1 \cdot \pi\delta(\omega-2) + 0 \cdot (\quad) + 0 \cdot (\quad)$$



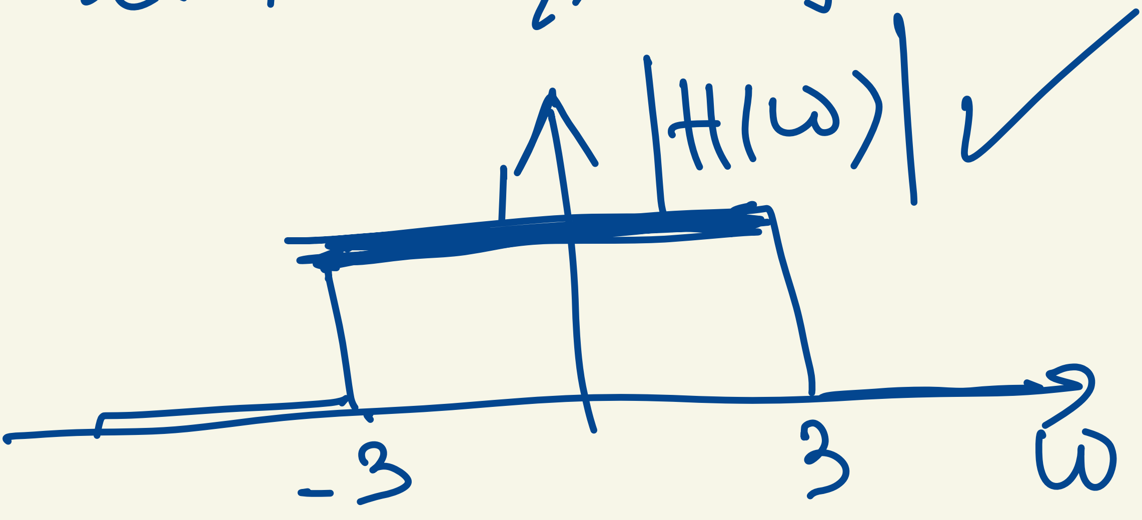
$$= 6\pi\delta(\omega) + \underbrace{\pi\delta(\omega+2)} + \underbrace{\pi\delta(\omega-2)}$$

$$y(t) = \mathcal{F}^{-1}\{Y(\omega)\}$$

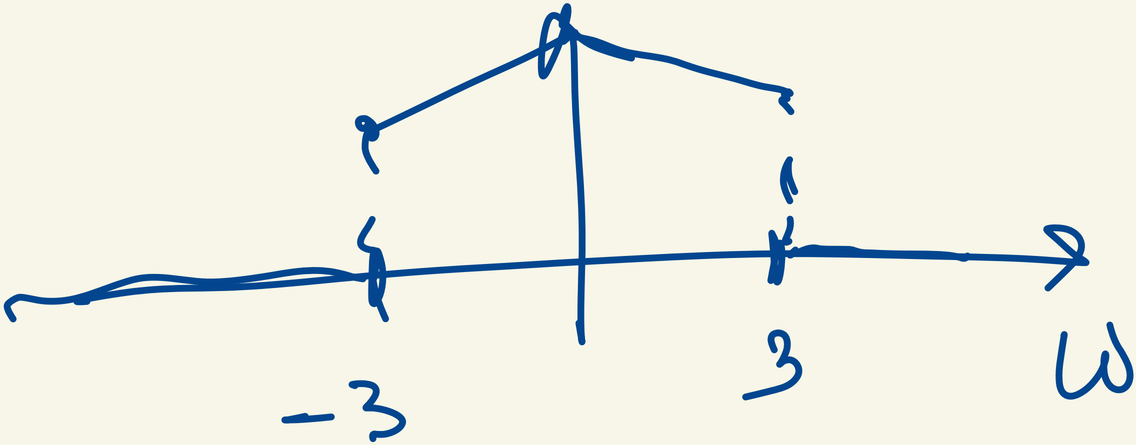
$$= 3 + \cos(2t)$$



$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$



$$Y(\omega) = X(\omega)\cdot H(\omega)$$



$$x(t) = 2 + \cos(2.5t) + 5\sin(3.9t) + 2\cos(9t) + 3\cdot\sin(15t)$$

$$\cos(\omega_0 t) \longrightarrow \underset{\text{LTI}}{\boxed{H(\omega)}} \longrightarrow \; ?$$

$h(t) = h^*(t)$

system is real

$$\sin(\omega_0 t) \longrightarrow \boxed{H(\omega)} \longrightarrow \; ?$$

$$\underline{H(\omega) = H^*(-\omega)}$$

$$H(\omega) = \mathrm{Re}\{H(\omega)\} + j\,\mathrm{Im}\{H(\omega)\}$$

$$= R(\omega) + jI(\omega).$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$= \underbrace{\left[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right]}_{F\{\cos(\omega_0 t)\}} \cdot H(\omega)$$

$$= \pi\delta(\omega-\omega_0)\cdot H(\omega_0)$$

$$+ \pi\delta(\omega+\omega_0)\cdot H(-\omega_0)$$

$$= \pi\delta(\omega-\omega_0)\cdot H(\omega_0)$$

$$+ \pi\delta(\omega+\omega_0)\cdot H^*(\omega_0)$$

$$= \pi\delta(\omega-\omega_0)\left[R(\omega_0)+jI(\omega_0)\right]$$

$$+ \pi\delta(\omega+\omega_0)\left[R(\omega_0)+jI(\omega_0)\right]^*$$

$$= \pi\delta(\omega-\omega_0)\left[R(\omega_0)+jI(\omega_0)\right]$$

$$+ \pi\delta(\omega+\omega_0)\left[R(\omega_0)-jI(\omega_0)\right]$$

$$= R(\omega_0)\left[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\right]$$

$$+ jI(\omega_0)\left[\pi\delta(\omega-\omega_0)-\pi\delta(\omega+\omega_0)\right]$$

$$\mathcal{F}^{-1}\left\{R(\omega_0)\cdot\left[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\right]\right.$$

$$\left. + j\underbrace{I(\omega_0)}\left[\pi\delta(\omega-\omega_0)-\pi\delta(\omega+\omega_0)\right]\right\}$$

$$= R(\omega_0)\cos(\omega_0 t)$$

$$+ I(\omega_0)\mathcal{F}^{-1}\{ -j\pi\delta(\omega-\omega_0) - j\pi\delta(\omega+\omega_0) \}$$

$$\underbrace{-j\pi\delta(\omega-\omega_0) - j\pi\delta(\omega+\omega_0)}_{-\sin(\omega_0 t)}$$

$$= R(\omega_0)\cos(\omega_0 t) - I(\omega_0)\sin(\omega_0 t)$$

$$\cos(\omega_0 t) \longrightarrow \underbrace{\boxed{H(\omega)}}_{\text{real}} \longrightarrow R(\omega_0)\cos(\omega_0 t) - I(\omega_0)\sin(\omega_0 t)$$

$$R(\omega_0) = \mathrm{Re}\{H(\omega)\}\Big|_{\omega=\omega_0}$$

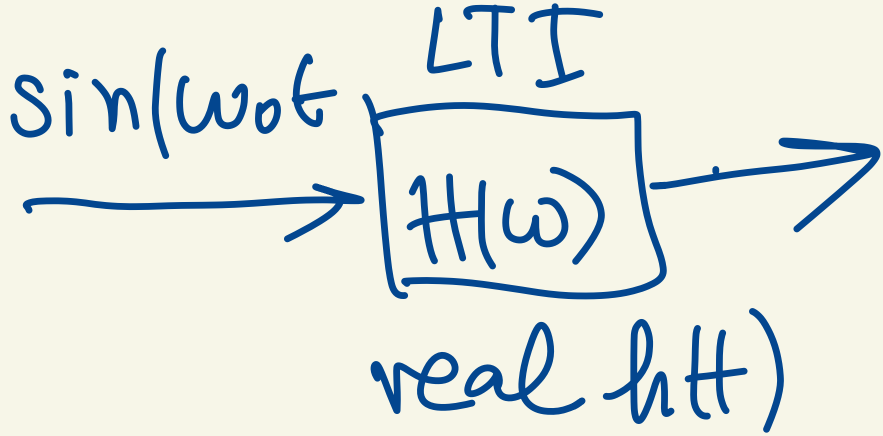
$$I(\omega) = \mathrm{Im}\{H(\omega)\}\Big|_{\omega=\omega_0}$$

remember example

$$H(\omega) = \mathrm{rec}(\omega, 3)$$

$$R(\omega) = \mathrm{rec}(\omega, 3)$$

$$I(\omega) = 0$$

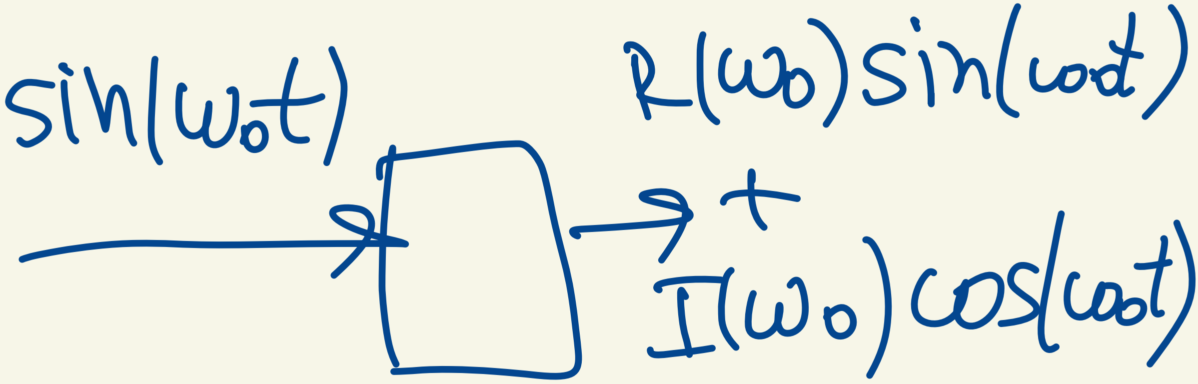


$$Y(\omega) = H(\omega) \cdot X(\omega)$$

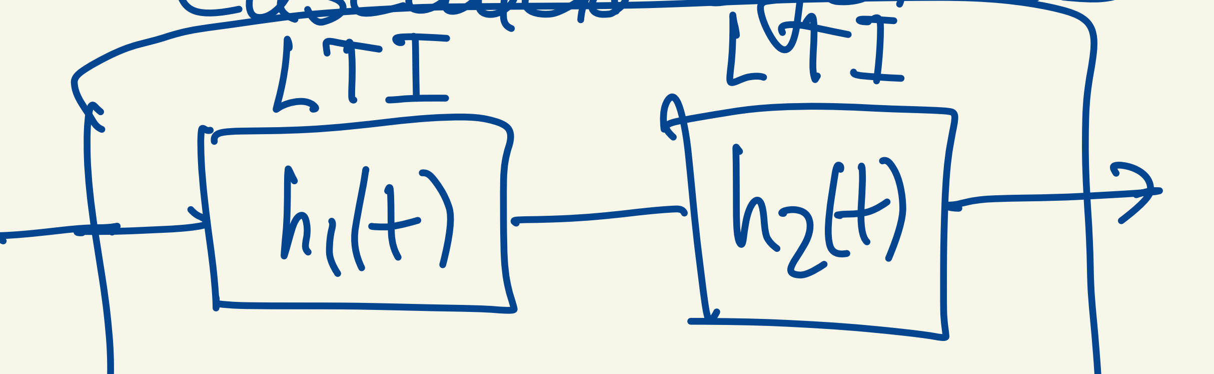
$$X(\omega) = j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0)$$

do the same steps as for $\cos(\omega_0 t)$

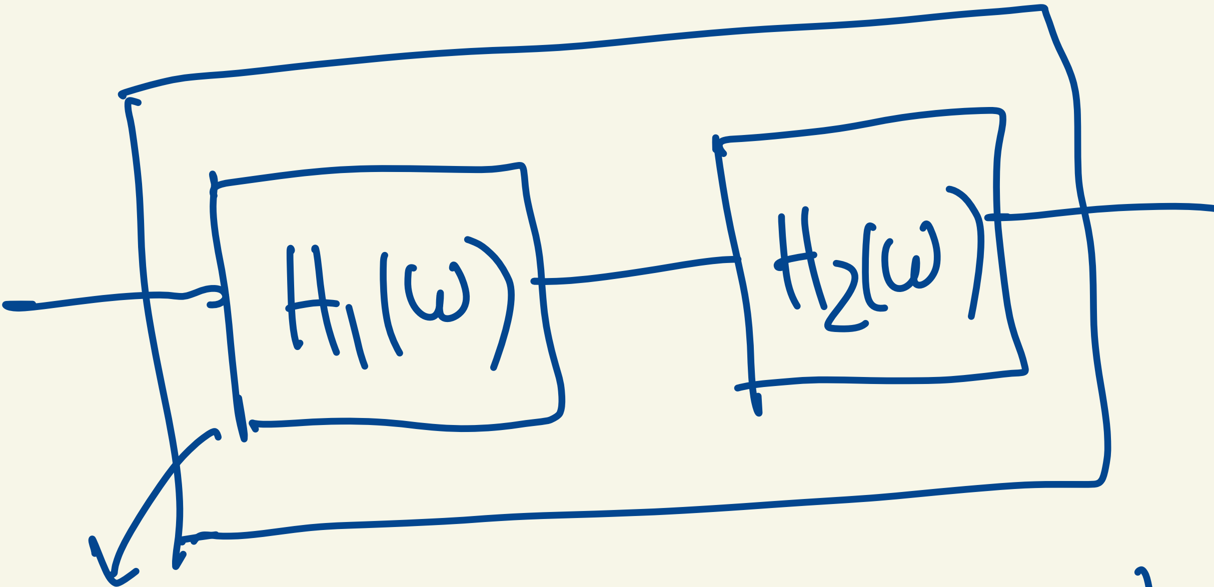
$$y(t) = R(\omega_0)\sin(\omega_0 t) + I(\omega_0)\cos(\omega_0 t)$$



Cascaded Systems



$$h_{12}(t) = h_1(t) * h_2(t)$$



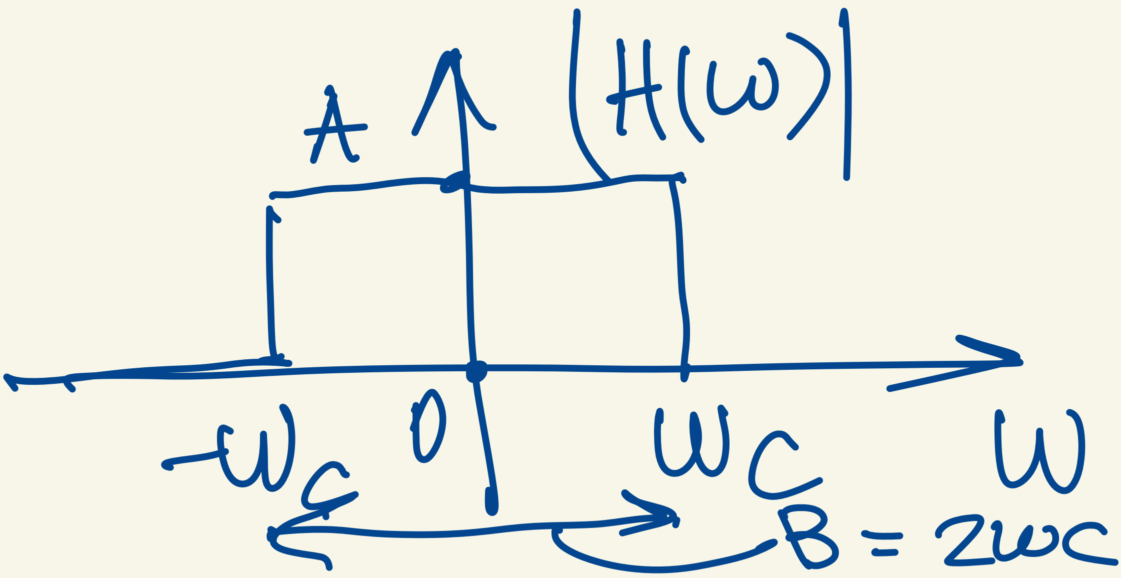
$$H_{12}(\omega) = H_1(\omega) \cdot H_2(\omega)$$

LTI is often called

Filter

Because we are using its FRF to analyze its output.

Classification of Filters. (Ideal)

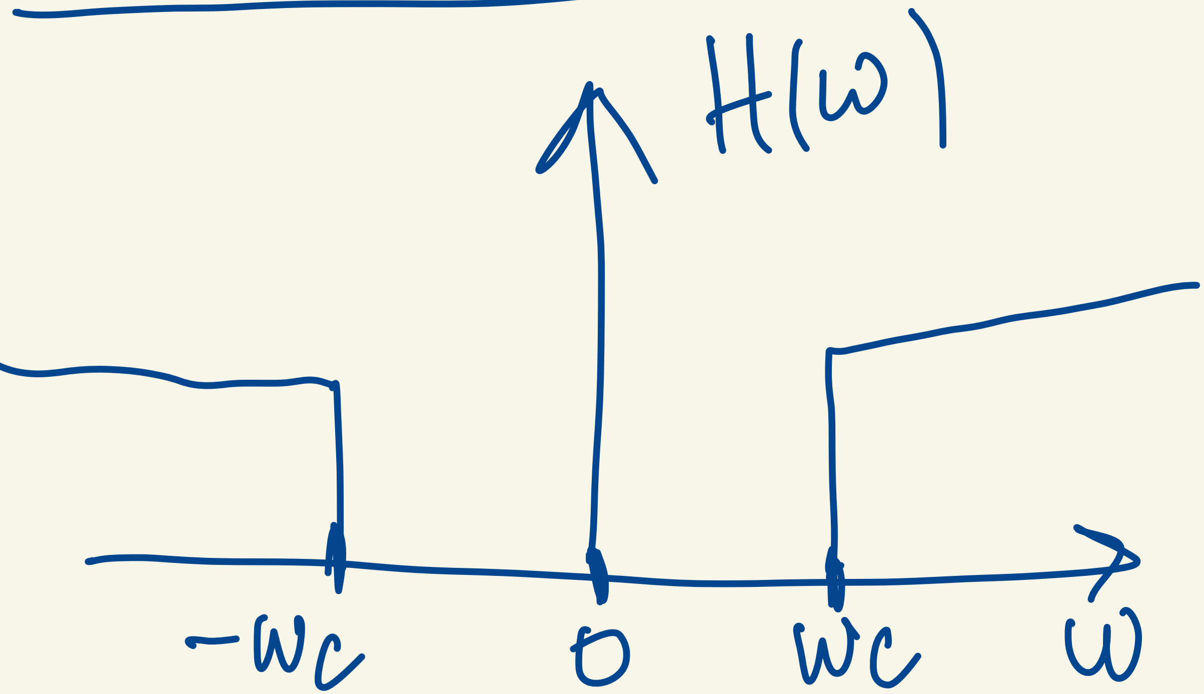


Low Pass Filter

center freq = 0

W_c — cut off freq.

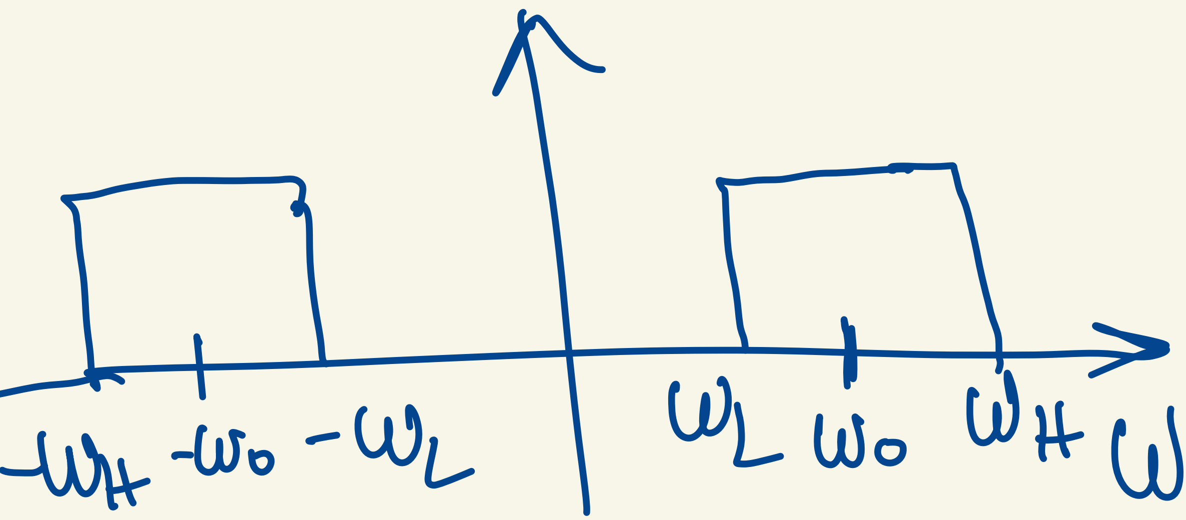
Bandwith = $2W_c$



High Pass Filter.

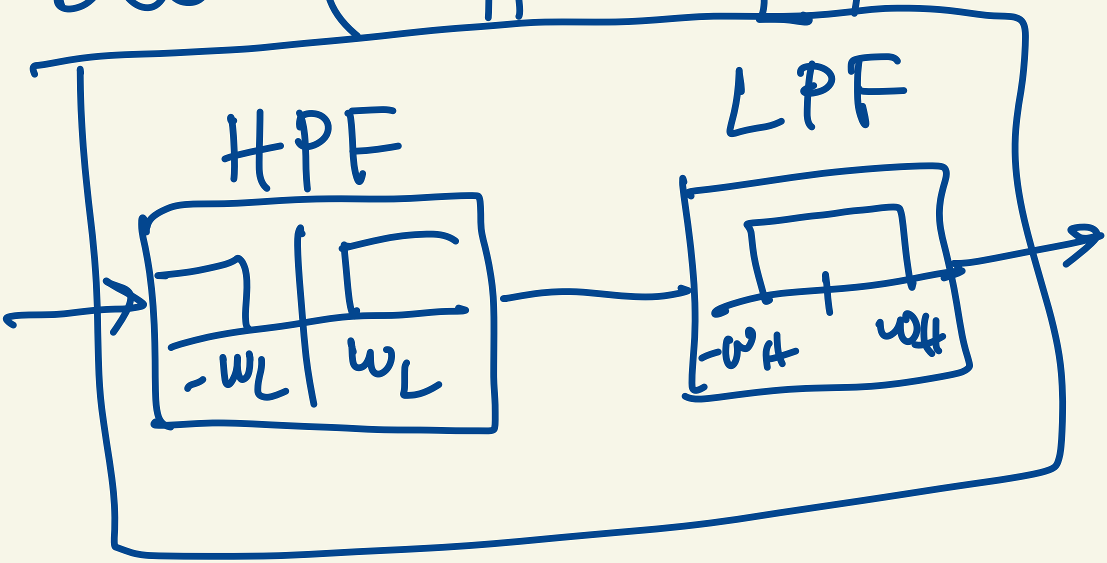
cut off $\to \omega_c$.

Band Pass Filter.

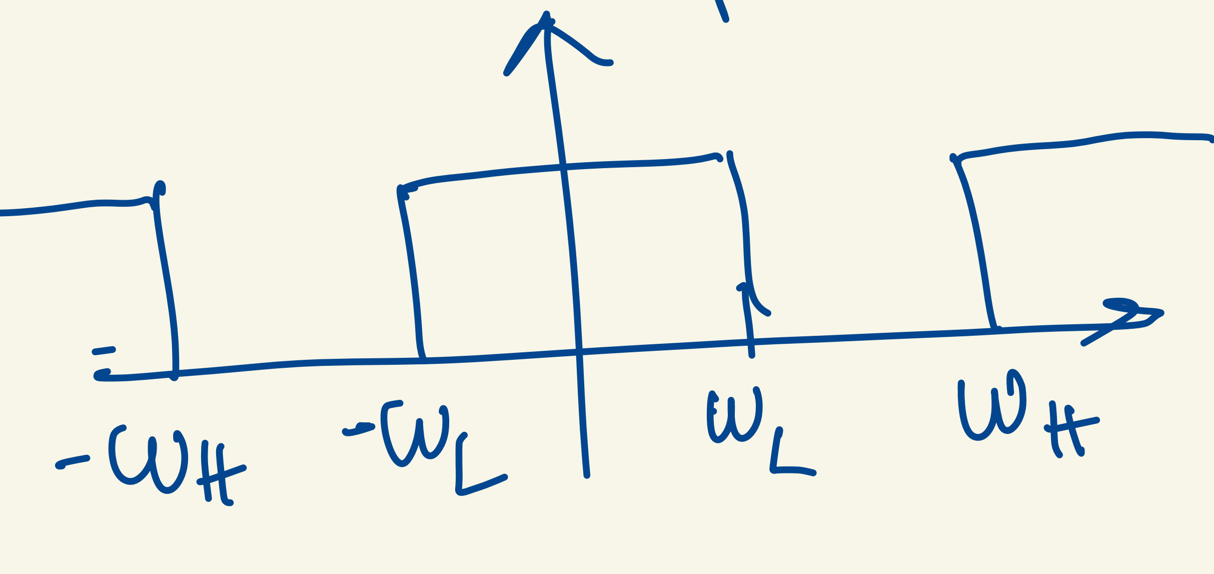


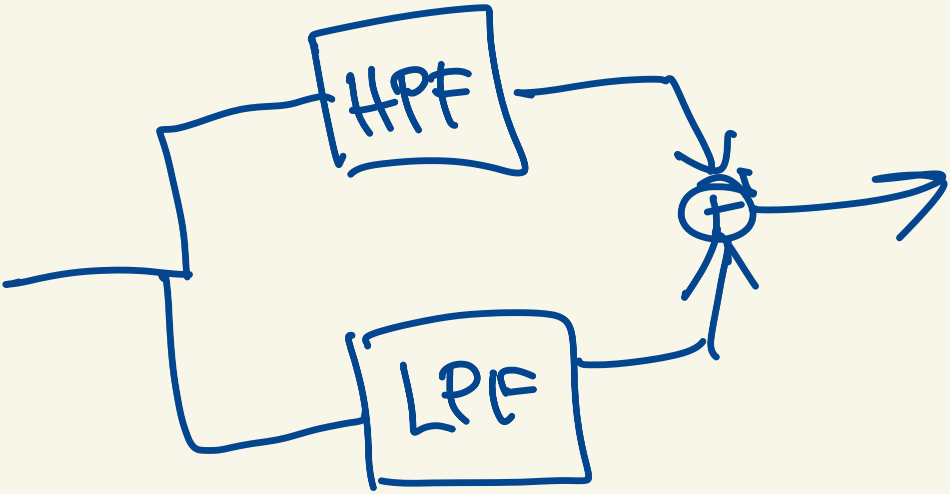
center freq ω_0

$BW \; (\omega_H - \omega_L)$

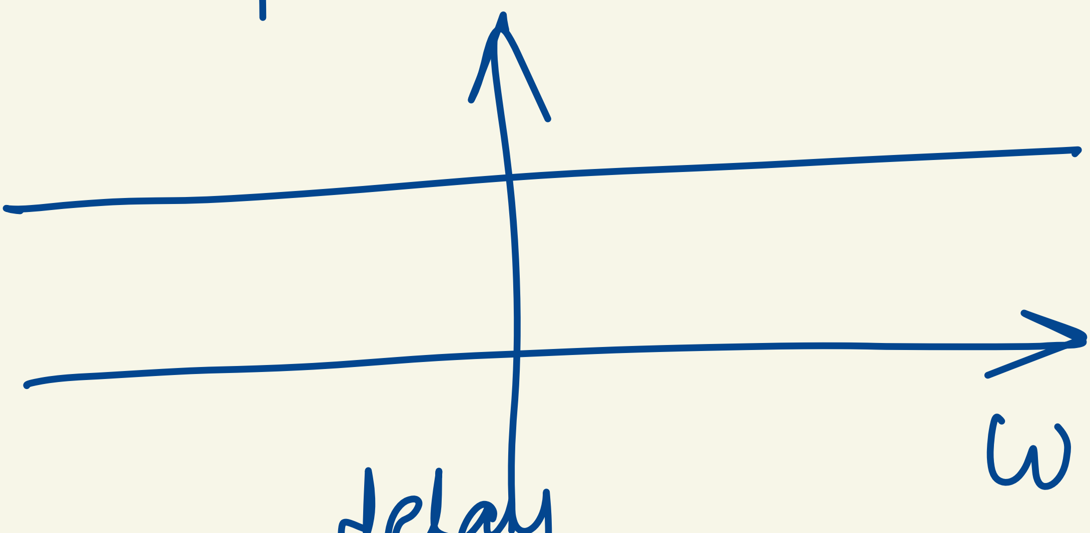


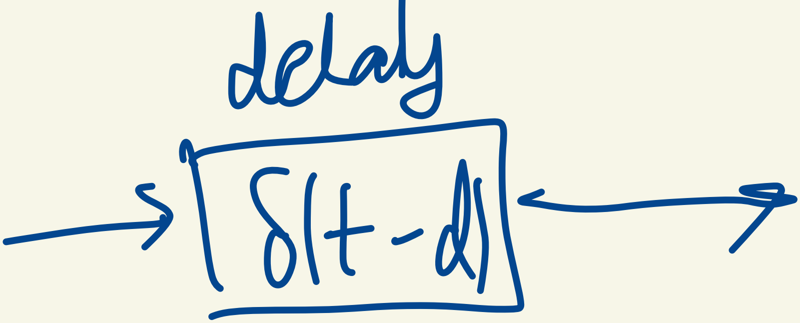
Band Stop Filter

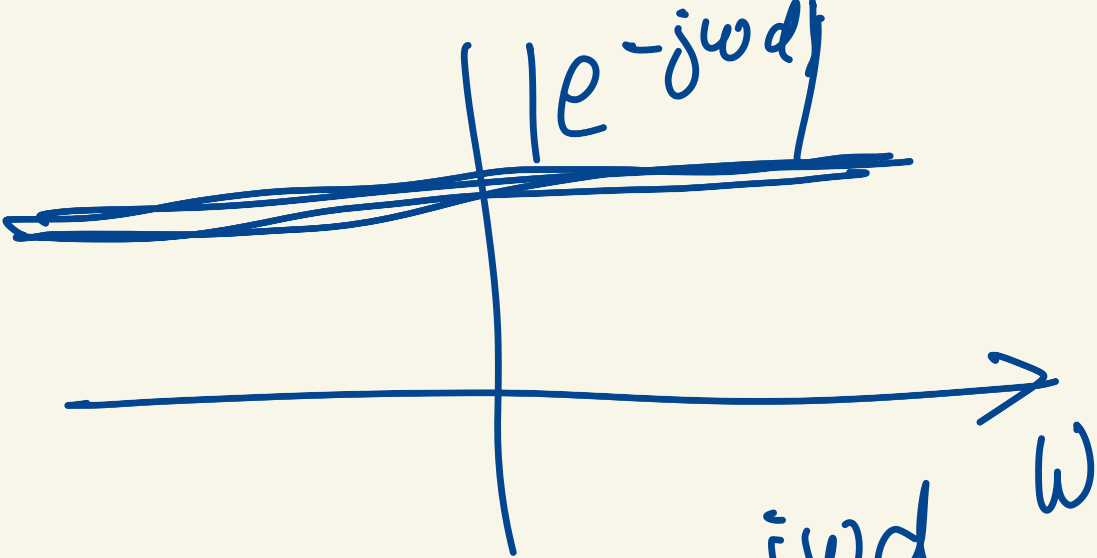




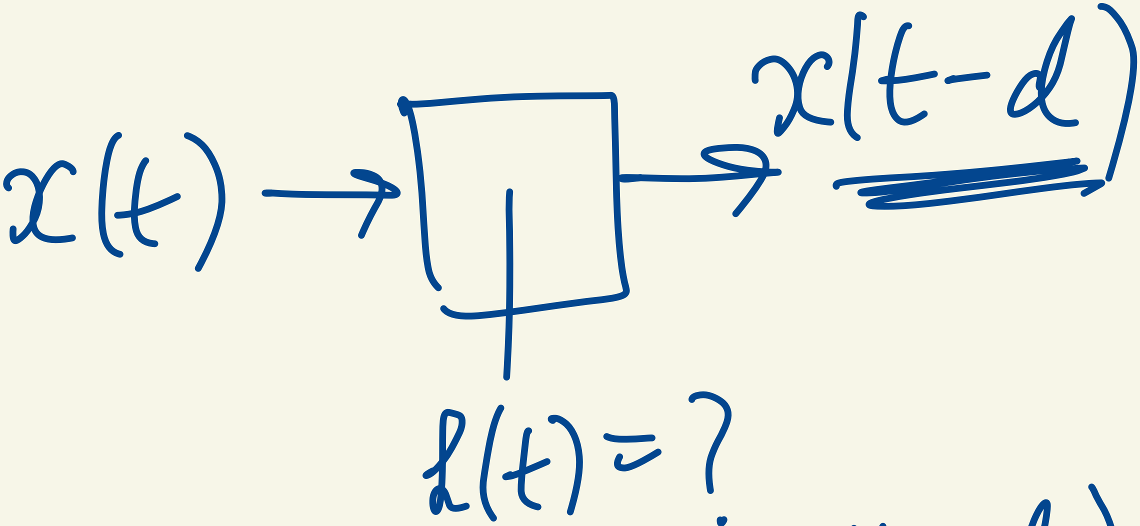
All pass Filter.







$$\mathcal{F}\{\delta(t-d)\} = e^{-j\omega d}$$



$$x(t) * \underbrace{\delta(t-d)}_{h(t)} = x(t-d)$$

$$x(t) \longrightarrow \boxed{?} \longrightarrow 0.5x(t-1) + 0.2x(t-3) + 0.2x(t-5)$$

wireless channel.

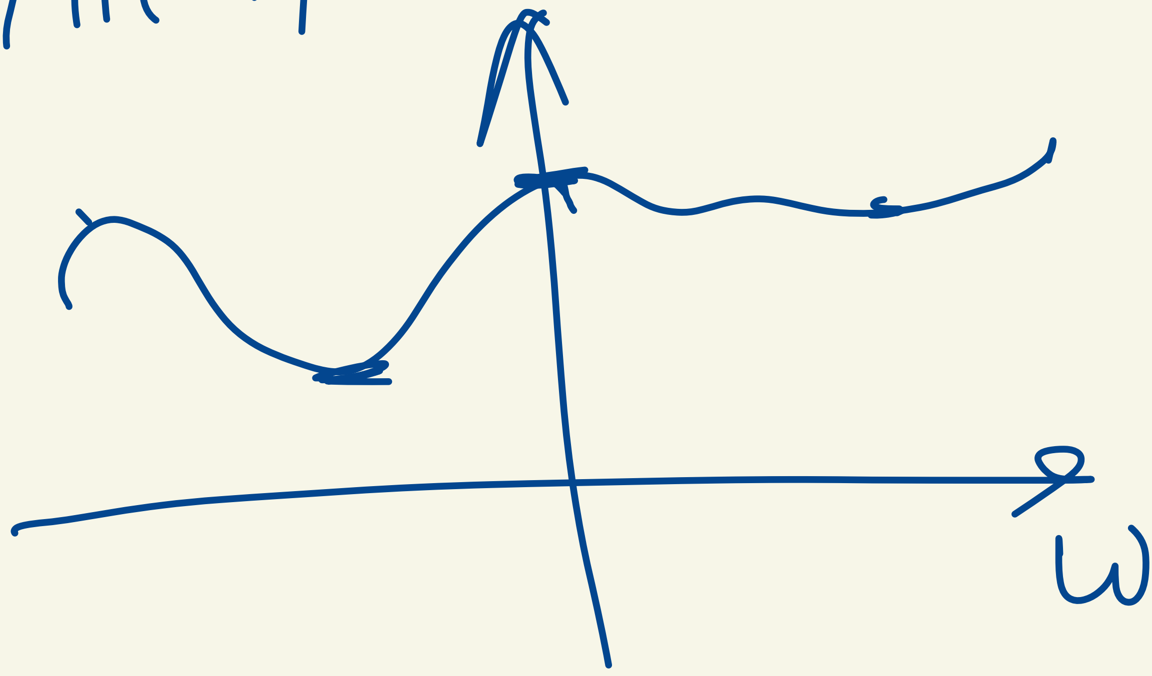
$$h(t) = 0.5\delta(t-1) + 0.2\delta(t-3)\ 0.2\delta(t-5)$$

$$H(\omega) = 0.5e^{-j\omega} +$$

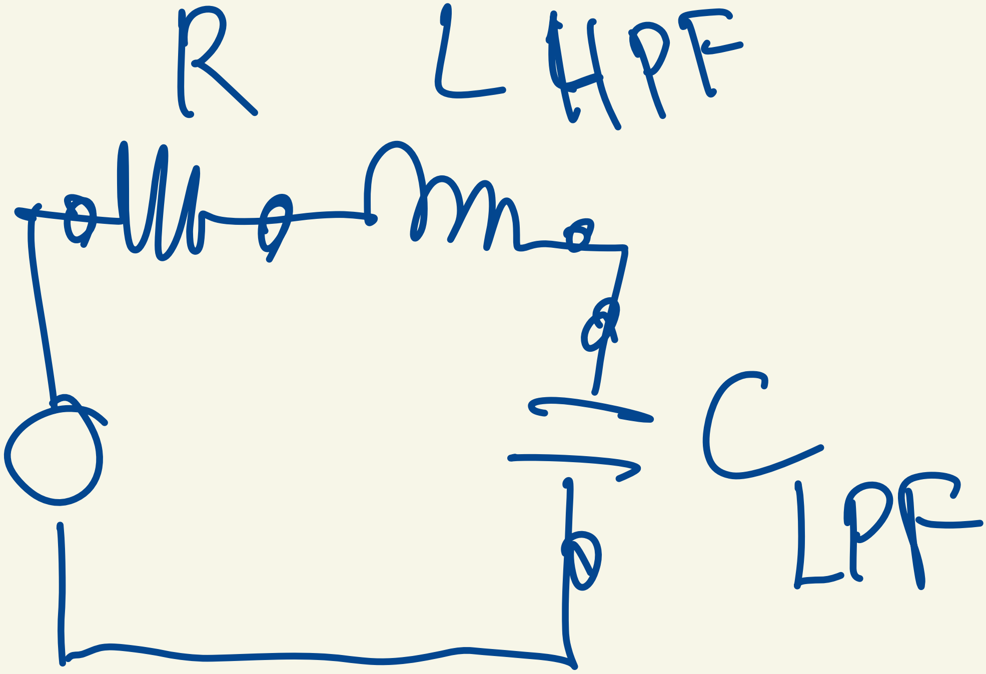
$+0.2e^{-j3\omega}$

$+0.2e^{-j5\omega}$

$|H(\omega)|$



BPF



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