Fourier Transform and its properties

$x(t)$

$$x(t) \xrightarrow{\mathcal{F}} \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$X(\omega)$

$$X(\omega) \xrightarrow{\mathcal{F}^{-1}} x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$$\delta(t) \longrightarrow 1$$

$$1 \longrightarrow 2\pi\delta(\omega)$$

$$x(t) \cdot u(t) \dashrightarrow X(\omega) = X(s)\Big|_{s=j\omega}$$

$\downarrow \mathcal{L}_s$

$X(s)$

ROC includes

$s = j\omega$

$$e^{-2t}u(t) \xrightarrow{\mathcal{F}} X(\omega) = \left.\frac{1}{s+2}\right|_{s=j\omega}$$

$\downarrow$

$$X(s) = \frac{1}{s+2}$$

$$X(\omega) = \frac{1}{j\omega+2}$$

$Re\{s\} > -2$

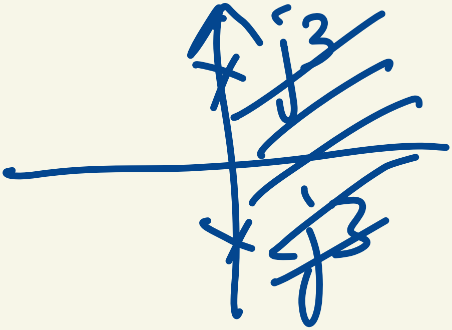
$$\cos(3t)u(t) \rightarrow X(\omega) \neq X(s)$$

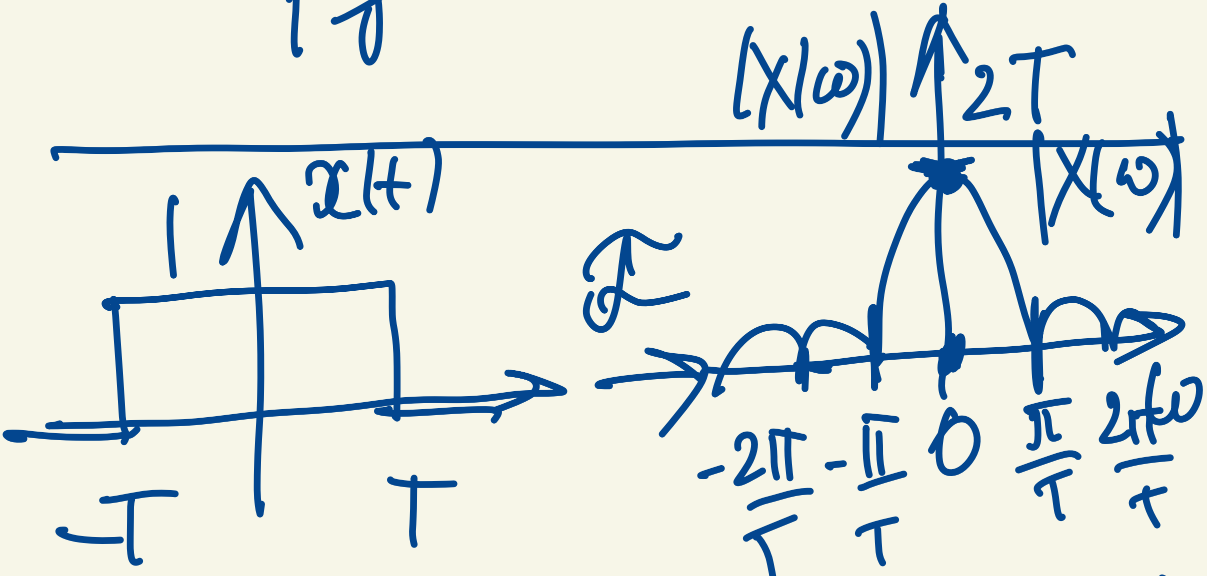
$s = j\omega$

$\downarrow$

$$X(s) = \frac{s}{s^2+9}$$

$Re\{s\} > 0 \rightarrow$ does not include $j\omega$.





rec(t, T)

time-limited.

$$X(\omega) = ?\,\text{sinc}(\omega T)$$

$$X(0) = ? = \int_{-\infty}^{+\infty} x(t)\,dt$$

$$X(0) = 2T$$

$$\text{rec}(t;T) \xrightarrow{\mathcal{F}} X(\omega) = 2T\,\text{sinc}(\omega T)$$

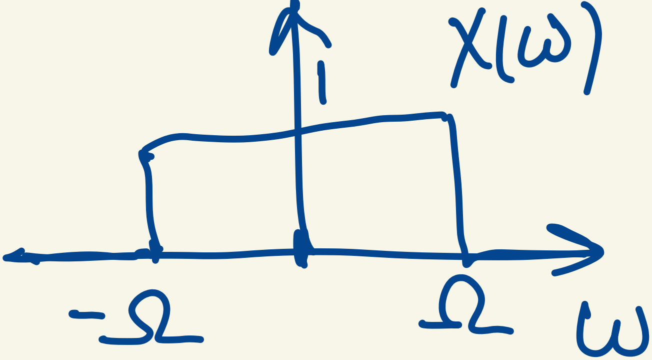
$$\text{sinc}(\omega T) = 0$$

$$\Rightarrow \sin(\omega T) = 0$$

$$\omega T = k\pi \quad k = \pm 1, \pm 2, \ldots$$

$$\omega = \frac{k\pi}{T}$$

band limited.



$\underline{\underline{\omega = 2\pi f}}$

$$X(\omega) = rec(\omega, \Omega)$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega) \cdot e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\Omega}^{\Omega} 1 \cdot e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \frac{1}{jt} e^{j\omega t} \Big|_{-\Omega}^{\Omega}$$

$$= \frac{1}{\pi t} \cdot \frac{1}{2j} \left[e^{j\Omega t} - e^{-j\Omega t} \right]$$

$$= \frac{1}{\pi t} \cdot \sin(\Omega t)$$

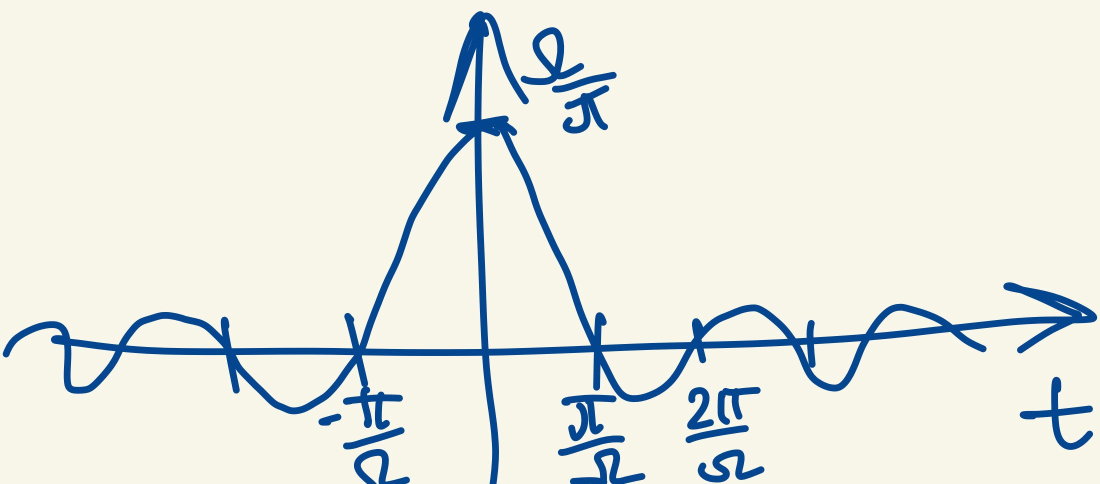
$$= \frac{\Omega}{\pi} \cdot \frac{\sin(\Omega t)}{\Omega t}$$

$$= \frac{\Omega}{\pi} \cdot \text{sinc}(\Omega t)$$

$$\frac{\Omega}{\pi}\text{sinc}(\Omega t) \longrightarrow \text{rec}(\omega, \Omega)$$

$$\text{rec}(t, T) \longrightarrow 2T\,\text{sinc}(\omega T)$$

$$x(t) = \frac{\Omega}{\pi}\text{sinc}(\Omega t)$$



$$\Omega t = k\pi \dashrightarrow t = \frac{k\pi}{\Omega}$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j\underbrace{2\pi f}_{\omega} t}\, dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Properties of Fourier Transfm.

① Linearity.

$$x_1(t) \xrightarrow{\mathcal{F}} X_1(\omega)$$

$$x_2(t) \xrightarrow{\mathcal{F}} X_2(\omega)$$

$$\alpha_1 x_1(t) + \alpha_2 x_2(t)$$

$$\Big\downarrow \mathcal{F}$$

$$\alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$$

$$\mathcal{F}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} =$$

$$= \int_{-\infty}^{+\infty} (\alpha_1 x_1(t) + \alpha_2 x_2(t)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} \alpha_1 x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{+\infty} \alpha_2 x_2(t) e^{-j\omega t} dt$$

$$= \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$$

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(2) For real valued signals $x(t)$, $X(\omega)$ has complex-conjugate symmetry.

$$x(t) = x^*(t)$$

$$X(\omega) = X(-\omega)^*$$

$$|X(\omega)| = |X(-\omega)|$$ even symmetry in magnitude

$$\measuredangle X(\omega) = -\measuredangle X(-\omega)$$ spectrum odd symmetry in phase spectrum

Proof:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X^*(\omega) = \left(\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \right)^*$$

$$(a+b)^* = (a^* + b^*)$$

$$= \int_{-\infty}^{+\infty} \left(x(t) e^{-j\omega t} dt \right)^*$$

$$= \int_{-\infty}^{+\infty} x^*(t) e^{j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} x(t) e^{j\omega t} dt$$

$$= X(-\omega)$$

$$X^*(\omega) = X(-\omega)$$

③ $X(\omega) = Re\{X(\omega)\} + j\,Im\{X(\omega)\}$

real signal

$$Re\{X(\omega)\} = Re\left\{\int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt\right\}$$

$$= Re\left\{\int_{-\infty}^{+\infty} x(t)\left(\cos(\omega t) - j\sin(\omega t)\right) dt\right\}$$

$$= \int_{-\infty}^{+\infty} x(t)\cos(\omega t)\, dt$$

$$\mathrm{Im}\{X(\omega)\} = -\int_{-\infty}^{+\infty} x(t)\sin(\omega t)\,dt$$

$x(t)$ real and odd.

$$\mathrm{Re}\{X(\omega)\} = 0.$$

$$X(\omega) = j\,\mathrm{Im}\{X(\omega)\}$$

$$= -j\int_{-\infty}^{+\infty} x(t)\sin(\omega t)\,dt$$

$x(t)$ real and even

$$\mathrm{Im}\{X(\omega)\} = 0$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\cos(\omega t)\,dt$$

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④ <u>Duality</u> (any signal not necessarily real).

✓

① $x(t) \xrightarrow{\mathcal{F}} X(\omega)$

② $X(t) \xrightarrow{\mathcal{F}} 2\pi x(-\omega)$

We have seen 2 examples.

$$x(t) = \delta(t) \longrightarrow X(\omega) = 1$$

$$X(t) = 1 \longrightarrow 2\pi\delta(-\omega)$$

$x(t) =$ $X(\omega)$

$$\mathrm{rec}(t, T) \longrightarrow 2T\,\mathrm{sinc}(\omega T)$$

$X(t)$

$$\cancel{2}T\,\mathrm{sinc}(t\cdot T) \longrightarrow \cancel{2}\pi\,\mathrm{rec}(-\omega, T)$$

$T = \Omega$

$$\Omega \operatorname{sinc}(\Omega t) \rightarrow \pi \operatorname{rec}(-\omega, \Omega)$$

even symm of rec.

$$\frac{\Omega}{\pi} \operatorname{sinc}(\Omega t) \rightarrow \operatorname{rec}(\omega, \Omega)$$

Proof:

(1)
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} \, d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\rho) e^{j\rho t} \, d\rho$$

$t = -\omega$

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\rho)\, e^{-j\rho\omega}\, d\rho$$

$\rho = t$

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(t)\, e^{-j\omega t}\, dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{+\infty} X(t)\, e^{-j\omega t}\, dt$$

$$= \mathcal{F}\{X(t)\}$$

$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$

$$X(t) \xrightarrow{\mathcal{F}} 2\pi x(-\omega),$$

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⑤ Scaling Property.

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$x(at) \longrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

$a > 0$

H.W. $a < 0$

$0 < a < 1$ expand.

$1 < a < +\infty$ comp.

$X\left(\frac{1}{a}\cdot w\right)$ $\quad a<1 \quad \frac{1}{a}>1$

$a>1 \quad \frac{1}{a}<1$

$a<1$

expanded in time. → compressed in freq.

$a>1$

compressed in time → expanded in freq.

Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-j\omega t}\, dt$$

$$\boxed{\begin{aligned} at &= \tau \\ t &= \frac{\tau}{a} \\ dt &= \frac{d\tau}{a} \end{aligned}}$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\frac{\tau}{a}} \cdot \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

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6. Time-Shifting Property.

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$x(t-d) \longrightarrow X(\omega)e^{-j\omega d}$$

Proof:

$$\mathcal{F}\{x(t-d)\} =$$

$$= \int_{-\infty}^{+\infty} x(t-d)\, e^{-j\omega t}\, dt$$

$$t-d = \tau \Rightarrow t = \tau + d$$

$$dt = d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega(\tau+d)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau} \cdot e^{-j\omega d}\, d\tau$$

$$= e^{-j\omega d} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= e^{-j\omega d}\, X(\omega)$$ end of proof.

$$x(t) \longrightarrow X(\omega)$$

$$x(t-d) \longrightarrow X(\omega) \cdot e^{-j\omega d}$$

$$\left| X(\omega) e^{-j\omega d} \right| = |X(\omega)|$$

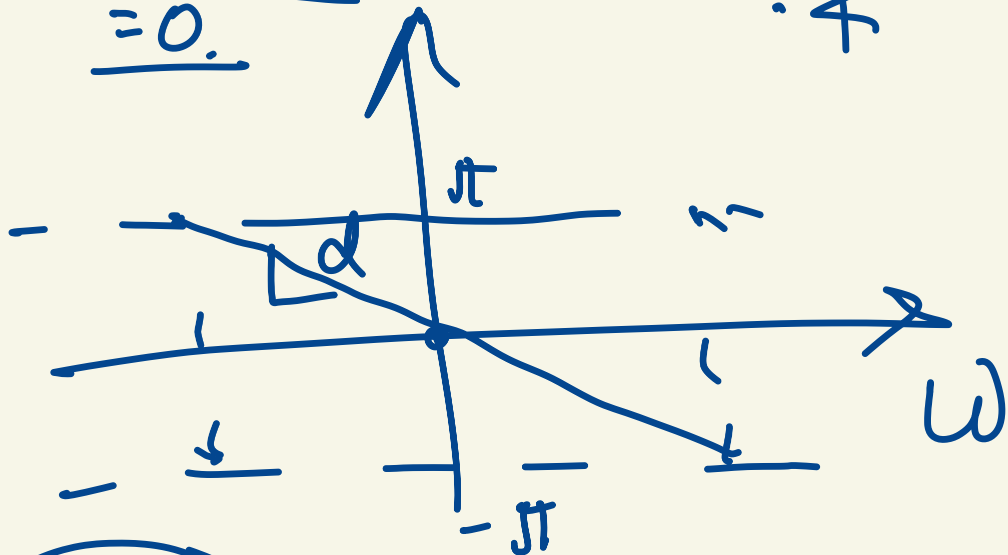
$$\measuredangle \left\{ X(\omega) \cdot e^{-j\omega d} \right\} =$$

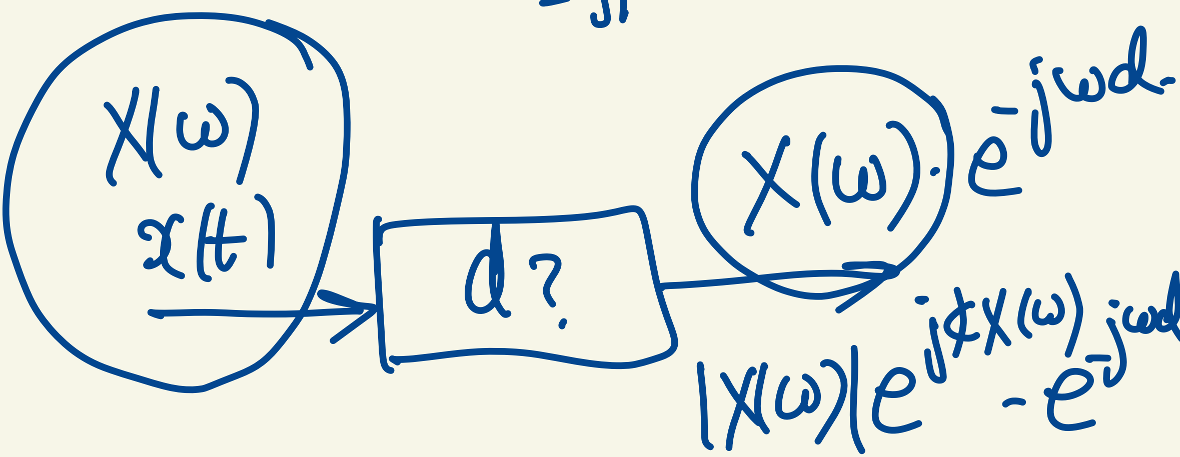
$$= \measuredangle \left\{ |X(\omega)| e^{j \measuredangle X(\omega)} \cdot e^{-j\omega d} \right\}$$

$$= \measuredangle \left\{ |X(\omega)| \cdot e^{j \underbrace{(\measuredangle X(\omega) - \omega d)}} \right\}$$

$= \measuredangle X(\omega) - \omega d.$

$\underline{\underline{=0.}}$





measure phase.

$$\angle Y(\omega) = \underbrace{\angle X(\omega)}_{\text{known}} - \omega d.$$

subtract it out.

$$\angle Y(\omega) - \angle X(\omega) = \underline{\underline{-\omega d}}.$$

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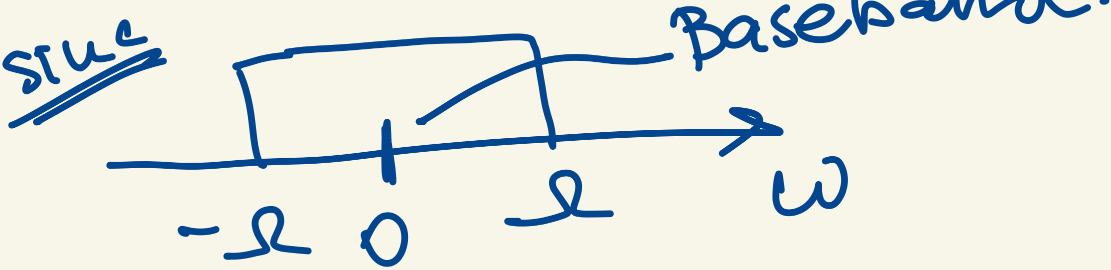
7 Frequency shifting property.

$$x(t) \longrightarrow X(\omega)$$

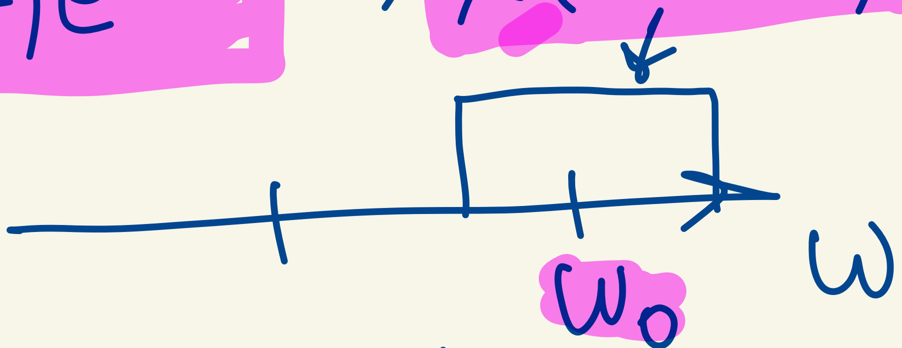
$$x(t) \cdot e^{j\omega_0 t} \longrightarrow X(\omega - \omega_0)$$

$$x(t)e^{-j\omega_0 t} \rightarrow X(\omega+\omega_0)$$

$$x(t) \rightarrow X(\omega)$$

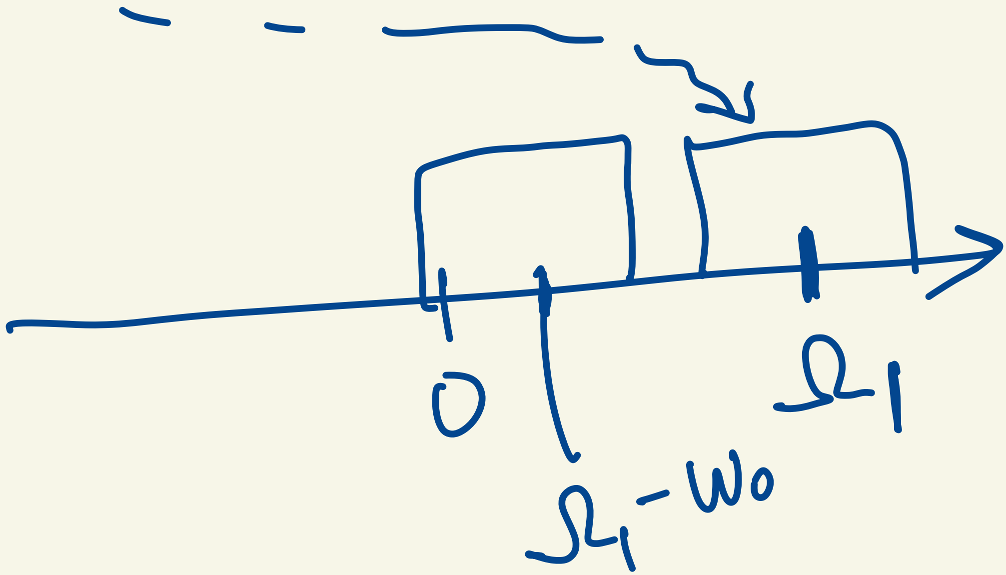


$$x(t)e^{+j\omega_0 t} \rightarrow X(\omega-\omega_0)$$



upconversion.

$$x(t)e^{-j\omega_0 t} \rightarrow X(\omega+\omega_0)$$



down conversion

$$\mathcal{F}\left\{x(t)\,e^{\pm j\omega_0 t}\right\}$$

$$= \int_{-\infty}^{+\infty} x(t)\,e^{\pm j\omega_0 t} \cdot e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j(\omega \mp \omega_0)t}\, dt$$

$$= X(\omega \mp \omega_0)$$

$$x(t) \cdot \cos(\omega_0 t)$$

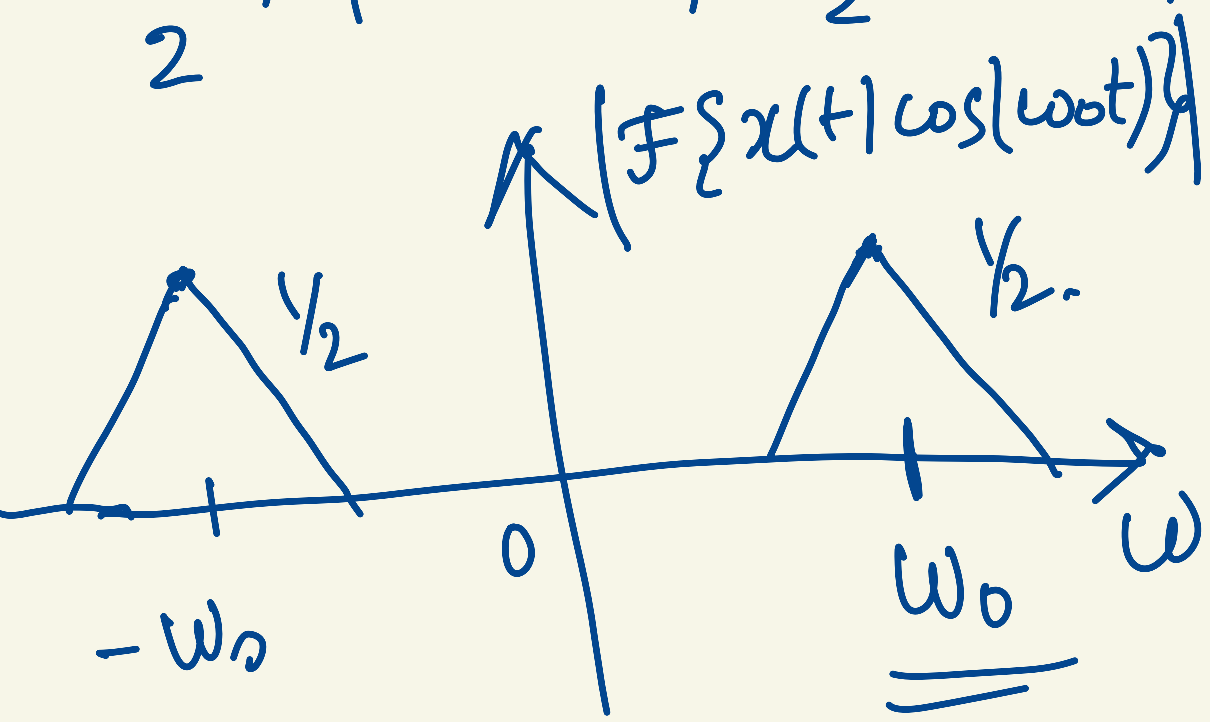
$$\mathcal{F}\{x(t)\cos(\omega_0 t)\} =$$

$$= \mathcal{F}\left\{x(t)\, \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{F}\left\{x(t)e^{j\omega_0 t}\right\}$$

$$+ \frac{1}{2}\mathcal{F}\left\{x(t)e^{-j\omega_0 t}\right\}$$

$$= \frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$



$\Rightarrow$ more in 132A