UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Final Examination
Date: March 19, 2020, Duration: 3 hours

INSTRUCTIONS:

- The exam has 6 problems and 16 pages.
- The exam is open book and open notes.
- Calculator are allowed.

Your name:____

Student ID:____

Table 1: Score Table

Problem	a	b	c	d	Score
1	5	5	5		15
2	10	5			15
3	10	5	10		25
4	5	5	5		15
5	5	5	5		15
6	5	10			15
Total					100

Table 3.1 Basic Properties of One-sided Laplace Transforms

Causal functions and constants	$\alpha f(t), \beta g(t)$	$\alpha F(s), \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$tf(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt'$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0-) = \lim_{s\to\infty} sF(s)$	

Table 3.2 One-sided Laplace Transforms

	Function of time	**Function of s, ROC**
(1)	$\delta(t)$	1, whole $s-$ plane
(2)	$u(t)$	$\frac{1}{s}, \mathcal{R}e[s] > 0$
(3)	$r(t)$	$\frac{1}{s^2}, \mathcal{R}e[s] > 0$
(4)	$e^{-at}u(t), a > 0$	$\frac{1}{s+a}, \mathcal{R}e[s] > -a$
(5)	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}, \mathcal{R}e[s] > 0$
(6)	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}, \mathcal{R}e[s] > 0$
(7)	$e^{-at}\cos(\Omega_0 t)u(t), a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}, \mathcal{R}e[s] > -a$
(8)	$e^{-at}\sin(\Omega_0 t)u(t), a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}, \mathcal{R}e[s] > -a$
(9)	$2Ae^{-at}\cos(\Omega_0 t + \theta)u(t), a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}, \mathcal{R}e[s] > -a$
(10)	$\frac{1}{(N-1)!}t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
(11)	$\frac{1}{(N-1)!}t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
(12)	$\frac{2A}{(N-1)!}t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}, \mathcal{R}e[s] > -a$

Table 5.1 Basic Properties of the Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert} X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n},\ n \geq 1,$ integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\Omega} X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$
		$\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt,$ real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt,$ imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
1	$\delta(t)$	1
2	$\delta(t-\tau)$	$e^{-j\Omega\tau}$
3	$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
4	$u(-t)$	$\frac{-1}{j\Omega} + \pi\delta(\Omega)$
5	$\text{sgn}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
6	$A, \quad -\infty < t < \infty$	$2\pi A\delta(\Omega)$
7	$Ae^{-at}u(t),\ a > 0$	$\frac{A}{j\Omega + a}$
8	$Ate^{-at}u(t),\ a > 0$	$\frac{A}{(j\Omega + a)^2}$
9	$e^{-a\lvert t\rvert},\ a > 0$	$\frac{2a}{a^2 + \Omega^2}$
10	$\cos(\Omega_0 t), \quad -\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
11	$\sin(\Omega_0 t), \quad -\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
12	$A[u(t+\tau) - u(t-\tau)],\ \tau > 0$	$2A\tau \frac{\sin(\Omega\tau)}{\Omega\tau}$
13	$\frac{\sin(\Omega_0 t)}{\pi t}$	$u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$
14	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$

Problem 1 (15 pts)

State whether the following statements are TRUE or FALSE. Provide a brief explanation for each part.

(a) (5 pts) A system with the following impulse response function is time-invariant:

$$h(t, \tau) = e^{-2t-2\tau}(\sin(t)\cos(\tau) - \cos(t)\sin(\tau))u(t - \tau) \tag{1}$$

(b) (5 pts) A system with the following input-output relationship is causal:

$$y(t) = x(t - 3) + \int_{t-3}^{3t} e^{-(t-\sigma)}u(t - \sigma)x(\sigma)d\sigma \tag{2}$$

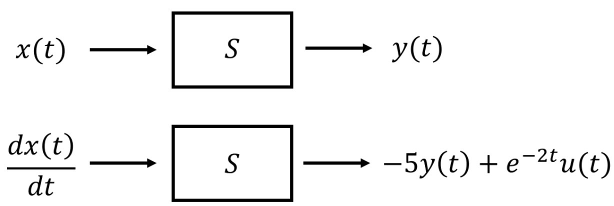
(c) (5 pts) Let $z(t) = \frac{1}{3}x(t - 2) + 4$, where $x(t)$ is a band-limited signal with maximum frequency 100 rad/s. The minimum sampling frequency (according to Nyquist theorem) to sample $z(t)$ is $\omega_s = 2 \times \frac{1}{3} \times 100 = \frac{200}{3}$ rad/s.

Problem 2 (15 pts)

Consider an LTI system S with the input signal

$$x(t) = e^{-5t}u(t-1) \tag{3}$$

and corresponding output signal $y(t)$. We also know that if input $\frac{dx(t)}{dt}$ is applied to the system S, corresponding output is $-5y(t) + e^{-2t}u(t)$.

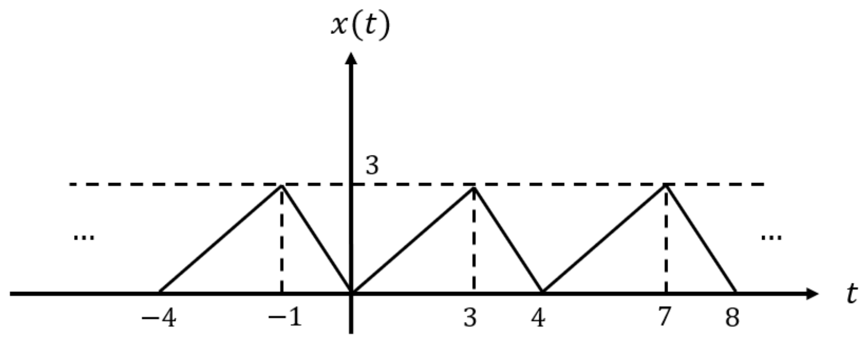


(a) (10 pts) Determine the system transfer function $H(s)$ and the impulse response function $h(t)$.

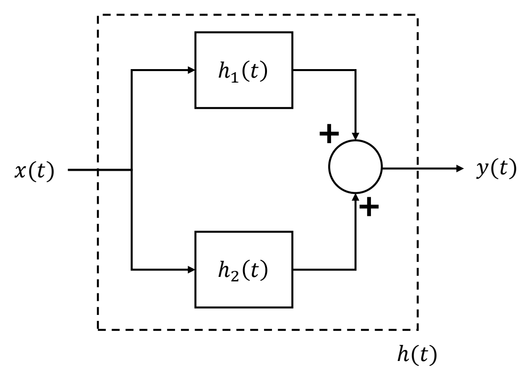
(b) (5 pts) Find the system output $y_1(t)$ if the input signal is $x_1(t) = e^{-2t}\cos(3t)u(t)$.

Problem 3 (25 pts)

Consider a periodic signal $x(t)$



The signal is passed through a parallel system, with the following impulse response for each branch:



$$h_1(t) = \frac{\sin(3\pi t/4)}{\pi t}, \quad h_2(t) = \frac{2\sin(\pi t/4)}{t}\cos(4\pi t)$$

(a) (10 pts) Compute the Fourier series coefficients X_k of the signal $x(t)$.

(b) (5 pts) Sketch the frequency response $H(\omega)$ of the entire system.

(c) (10 pts) Compute the Fourier series coefficients Y_k of the output $y(t)$.

Problem 4 (15 pts)

Given the multiplication property of Fourier Transform

$$x(t)y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

where $X(\omega) = \mathcal{F}\{x(t)\}$ and $Y(\omega) = \mathcal{F}\{y(t)\}$, and $*$ denotes convolution.

(a) (5pts) Compute the Fourier transform of $f_1(t) = \text{sinc}^2(t)$, given that $\text{sinc}(t) = \frac{\sin(t)}{t}$.

(b) (5 pts) Compute the Fourier transform of $f_2(t) = t \times \text{sinc}^2(t)$.

(c) (5 pts) By using the result in (a) and the Parseval's theorem, calculate the following integral

$$\int_{-\infty}^{\infty} \text{sinc}^4(t) dt.$$

Problem 5 (15 pts)

Let S be the Linear system

$$x(t) \rightarrow [S] \rightarrow y(t),$$

described by the differential equation

$$\frac{1}{a}\frac{dy(t)}{dt} + y(t) = \frac{1}{a}\frac{dx(t)}{dt} - x(t), \quad \text{and } y(0) = x(0) = 0$$

where $a > 0$.

(a) (5 pts) Find the transfer function $H(s)$ and the frequency response $H(\omega)$.

(b) (5 pts) Find the system output $y(t)$ if $x(t) = e^{-at}cos(3t)u(t)$.

(c) (5 pts) Show that the magnitude of the frequency response, $|H(\omega)|$, satisfies

$$|H(\omega)| = \text{constant, for all } \omega.$$

System S is the "all pass" filter since it passes all frequencies of any $X(\omega) = \mathcal{F}\{x(t)\}$.

Problem 6 (15 pts)

For a continuous-time LTI system with a real, causal impulse response $h(t)$, the frequency response $H(\omega)$ and $h(t)$ can be completely specified by the real part of its frequency response, $\mathfrak{Re}\{H(\omega)\}$. The property is generally referred as *real-part sufficiency*. In the following, we want to show this property by examining the even part of $h(t)$, denoted as $h_e(t)$.

(a) (5 pts) Given that $h(t)$ is a real and causal impulse response, express the Fourier transform of $h_e(t)$ in terms of $H(\omega)$.

(b) (10 pts) Using what we observed in (a), specify $h(t)$ if the real part of the frequency response of this causal system is

$$\mathfrak{Re}\{H(\omega)\} = \cos(\omega)$$