UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Midterm Examination

7^{th} February, 2022

Your name : ________________________________

Instructions

- This exam has 5 questions and 17 pages.
- The exam is closed book. One double-sided A4 sized cheat sheet is allowed. The use of calculators is permitted.
- All steps and working must be shown. No marks will be awarded for answers without math steps and/or an explanation.
- Write legibly and clearly! Any illegible work will not be graded.
- All plots must be neatly drawn and completely labelled (axes, intercepts, amplitudes) for full credit.

Good Luck!

Table 1: Score Table

Question	Total	Break up	Marks scored	Total score
1	15	3 + 3 + 3 + 3 + 3		
2	15	3 + 5 + 7		
3	20	4 + 3 + 3 + 3 + 7		
4	20	12 + 8		
5	30	4 + 4 + 4 + 10 + 8		
Total	100			

Table 3.1 One-Sided Laplace Transforms

	Function of Time	Function of s, ROC
1.	$\delta(t)$	1, whole s-plane
2.	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s] > 0$
3.	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$
4.	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$
5.	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
6.	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$
7.	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
8.	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$
9.	$2A\,e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$
10.	$\frac{1}{(N-1)!}\,t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s] > 0$
11.	$\frac{1}{(N-1)!}\,t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s] > -a$
12.	$\frac{2A}{(N-1)!}\,t^{N-1}e^{-at}\cos(\Omega_0 t + \theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s] > -a$

Table 3.2 Basic Properties of One-Sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)\ \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0+) = \lim_{s\to\infty} sF(s)$	
Final value	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$	

Simple Real Poles

If $X(s)$ is a proper rational function

$$X(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{\prod_k (s - p_k)} \tag{3.21}$$

Question 1 (15 marks)

State whether each of the following statements are true or false, with proper justification.

(a) The impulse response function $h(t)$ of a causal LTI system must be zero for all $t \leq 0$.

(b) The signal $x(t) = cos(4\omega_o t) - 4\cos(2\omega_o t)\cos^2(\omega_o t) + 2\cos(2\omega_o t)$ is periodic with period $T_o = \frac{2\pi}{\omega_o}\ \forall\, \omega_o \in (0, 2\pi)$.

(c) Function $h(t)$ is the impulse response of a causal and stable LTI system with rational transfer function $H(s)$, with $H(0) < 0$. Then, the system $\hat{h}(t) = \int_{-\infty}^{t} h(\tau)d\tau$ must be causal and stable.

(d) System S_1 yields unbounded output when an input $x(t) = e^{\sqrt{t}} \cos t u(t)$ is applied. Therefore, by the definition of BIBO stability, it must be unstable.

(e) The system given by input-output relation $y(t) = x(-t) * x(t)$ is an LTI system, where $x(t)$ is the input and $y(t)$ is the output.

Solution:

(a) False. For an LTI system to be causal, its impulse response function must satisfy $h(t) = h(t)u(t)$. The RHS term $h(t)u(t) = 0 \; \forall \; t < 0$, but $h(0)$ need not necessarily be 0 for the above condition to hold. Therfore, the statement is False. The correct statement would be : *The impulse response function $h(t)$ of a causal LTI system must be zero for all $t < 0$*

(b) False. Signal $x(t)$ evaluates to be constant for all time t. Thus, the signal is constant, and not periodic.

$$
\begin{aligned}
x(t) &= cos(4\omega_o t) - 4\cos(2\omega_o t)\cos^2(\omega_o t) + 2\cos(2\omega_o t) \\
&= cos(4\omega_o t) + 2\cos(2\omega_o t)\left(1 - 2\cos^2(2\omega_o t)\right) \\
&= cos(4\omega_o t) - 2\cos^2(2\omega_o t) \\
&= -1 \;\; \forall \, t
\end{aligned}
$$

(c) False. The LTI system is causal and stable. This means that the transfer function $H(s)$ has ROC which is a right half plane. Stability implies that the ROC must contain the $j\omega$ axis, therefore, all poles must have negative real parts. Further, since $H(0) \neq 0$, we know that there is no zero at $s = 0$. Therefore, the system $\hat{h}(t)$ has transfer function $\hat{H}(s) = \frac{H(s)}{s}$. This means that the latter system adds a pole at $s = 0$ due to the integration term. That is, we have a pole on the $j\omega$ axis, with which the system can never be stable.

(d) False. The BIBO stability condition requires that the inputs be bounded. The input $x(t) = e^{\sqrt{t}} \cos t u(t)$ is unbounded, and goes to ∞ as $t \to \infty$. The bounded-ness of the output resulting from an unbounded input cannot, therefore, be a valid criteria for checking system stability.

(e) False. The system is neither linear nor time invatiant. Thus, the system is not LTI.

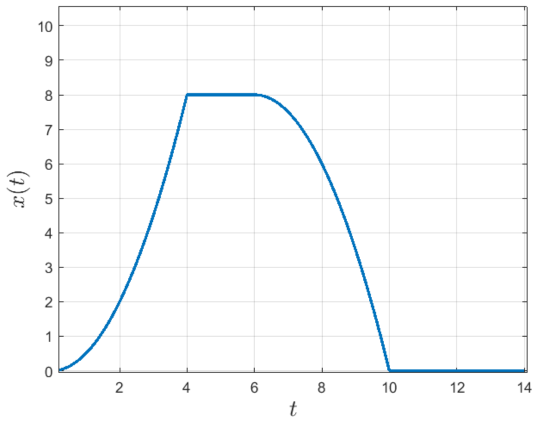
$$
\mathcal{S}\{\alpha x(t)\} = \alpha x(-t) * \alpha x(t) \neq \alpha x(-t) * x(t) = \alpha y(t)
$$

$$
\mathcal{S}\{x(t - t_o)\} = x(-t - t_o) * x(t - t_o) \neq y(t - t_o) = x(-t + t_o) * x(t - t_o)
$$

Question 2 (15 marks)

Consider the continuous time signal $x(t)$ shown below.

$$x(t) = \begin{cases} \frac{t^2}{2}, & 0 \leq t < 4 \\ 8, & 4 \leq t < 6 \\ -\frac{(t-6)^2}{2} + 8, & 6 \leq t < 10 \\ 0, & \text{otherwise} \end{cases}$$



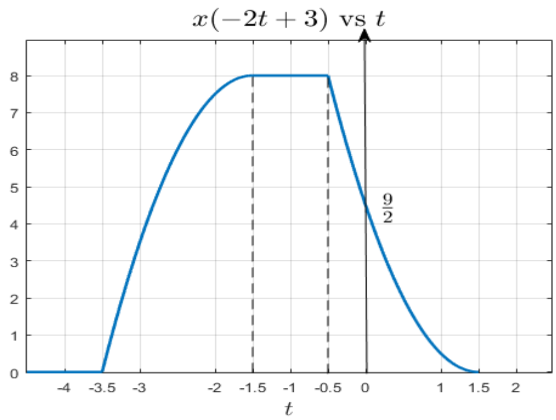
(a) Plot the signal $x(-2t+3)$ (3 marks)

(b) Compute the energy contained in $x(t)$ (5 marks)

(c) Sketch the signal $y(t) = \sum_{n=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-12n)d\tau \right\}$ and find its power, if at all it is a finite power signal. (7 marks)

Solution:

(a)

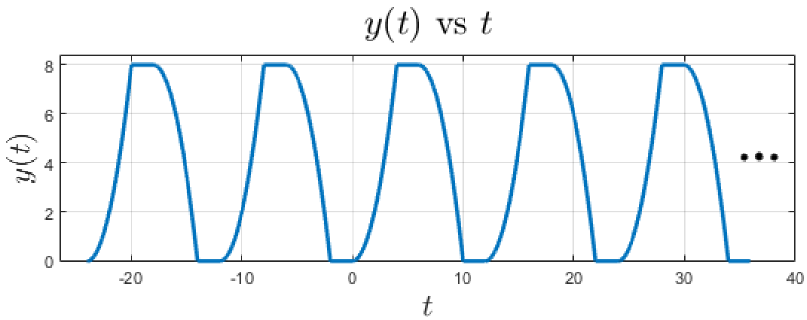


(b) Energy contained in $x(t) = E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$.

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^4 \frac{t^2}{4} dt + \int_4^6 64 dt + \int_6^{10} \left(-\frac{(t-6)^2}{2} + 8\right)^2 dt = 315.733$$

(c) We can simplify the expression for $y(t)$ using the sifting property of $\delta(t)$.

$$y(t) = \sum_{n=-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-12n)d\tau \right\} = \sum_{n=-\infty}^{\infty} x(t-12n)$$



Signal $y(t)$ is periodic with period $T_o = 12$. Thus, the power of $y(t)$, given by P_y, is calculated as follows:

$$P_y = \frac{1}{12} \int_0^{12} |x(t)|^2 dt = \frac{E_x}{12} = 26.311$$

Question 3 (20 marks)

Consider a system S with input-output relation as shown below, where b is a real constant.

$$y(t) = x(t) - \int_{-\infty}^{t} x(\tau)e^{-(t+b\tau)}d\tau$$

(a) Find the impulse response function $h(t, \tau)$ as a function of b. (4 marks)

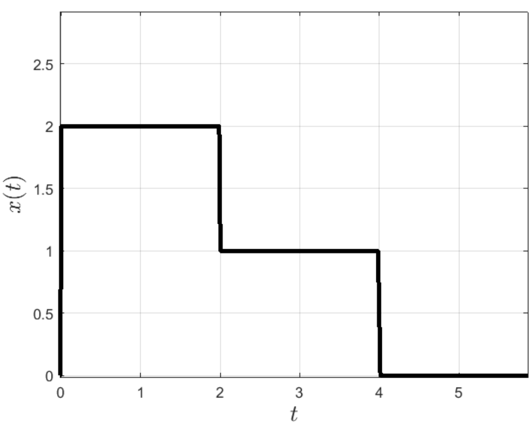
(b) What should be the value of b for system S to be LTI? (3 marks)

(c) Is the LTI system S with impulse response $h(t)$ causal? Justify your answer. (3 marks)

(d) Is the LTI system S with impulse response $h(t)$ stable? Justify your answer. (3 marks)

(e) Compute and sketch the output $y(t)$ when the input $x(t)$ shown below is applied to the LTI system S. (7 marks)

$x(t) = 0 \ \ \forall \ t < 0, t \geq 4$. $x(0) = 2$, $x(2) = 1$.



Hint: *First compute unit step response of system S. Then use properties of LTI systems to find the output.*

Solution:

(a) The impulse response function can be computed by applying an input of $\delta(t-\tau)$ to the system.

$$\begin{aligned}
h(t,\tau) &= \delta(t-\tau) - \int_{-\infty}^{t} \delta(\sigma-\tau)e^{-(t+b\sigma)}d\sigma \\
&= \delta(t-\tau) - \int_{-\infty}^{\infty} \delta(\sigma-\tau)e^{-(t+b\sigma)}u(t-\sigma)d\sigma \\
&= \delta(t-\tau) - e^{-(t+b\tau)}u(t-\tau)
\end{aligned}$$

(b) For system S to be LTI, the impulse response function computed in part (a) must satisfy $h(t,\tau) = h(t-\tau, 0)$. In other words, $h(t,\tau)$ must only be a function of $\tilde{\tau} = (t-\tau)$. This is only possible if $b = -1$. Thus, the LTI system has impulse response $h(t) = \delta(t) - e^{-t}u(t)$ when $b = -1$.

(c) The LTI system S is causal if $h(t) = h(t)u(t)$. Evaluating the RHS, we get:

$$h(t)u(t) = \delta(t)u(t) - e^{-t}u(t)u(t) = \delta(t) - e^{-t}u(t) = h(t)$$

Therefore, the system is causal.

(d) To check for BIBO stability, we need to check whether $\int_{-\infty}^{\infty} |h(t)|dt < \infty$. In other words, we need to show that the integral $\int_{-\infty}^{\infty} |h(t)|dt$ has a finite upper bound.

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} |\delta(t) - e^{-t}u(t)|dt \\
&\leq \int_{-\infty}^{\infty} |\delta(t)|dt + \int_{-\infty}^{\infty} |e^{-t}u(t)|dt = \int_{-\infty}^{\infty} \delta(t)dt + \int_{-\infty}^{\infty} e^{-t}u(t)dt \\
&= 1 + \int_{0}^{\infty} e^{-t}dt \\
&= 2
\end{aligned}$$

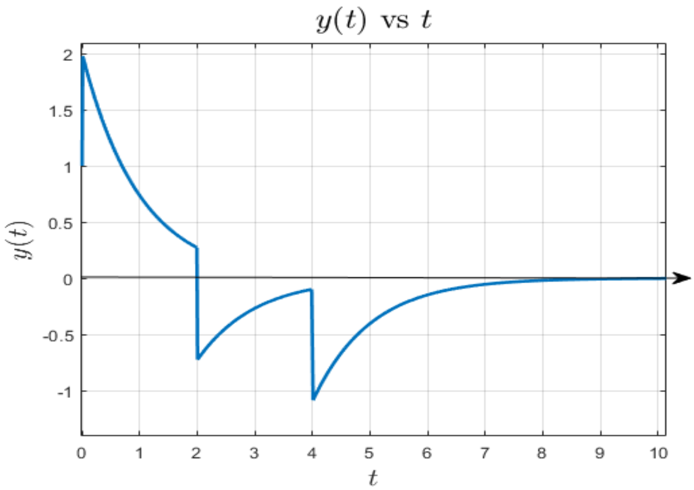
The integral $\int_{-\infty}^{\infty} |h(t)|dt$ has an upper bound of 2. Therefore, $h(t)$ is absolutely integrable, making the system BIBO stable.

(e) The unit step response $\mu(t)$ is calculated as follows:

$$\begin{aligned}
\mu(t) &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} \left[\delta(\tau) - e^{-\tau}u(\tau)\right] u(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau)u(t-\tau)d\tau - \int_{-\infty}^{\infty} e^{-\tau}u(\tau)u(t-\tau)d\tau \\
&= u(t) - \int_{0}^{t} e^{-\tau}d\tau \\
&= \begin{cases} u(t) - \int_{0}^{t} e^{-\tau}d\tau & if t \geq 0 \\ 0 & if t < 0 \end{cases} \\
&= u(t) - \left(\int_{0}^{t} e^{-\tau}d\tau\right) u(t) \\
&= u(t) - (1 - e^{-t})u(t) \quad = \quad e^{-t}u(t)
\end{aligned}$$

Since $x(t) = 2u(t) - u(t-2) - u(t-4)$, we can find the output $y(t)$ by exploiting properties of linearity and time invariance in the following manner.

$$\begin{aligned}
y(t) &= 2\mu(t) - \mu(t-2) - \mu(t-4) \\
&= 2e^{-t}u(t) - e^{-(t-2)}u(t-2) - e^{-(t-4)}u(t-4)
\end{aligned}$$



Question 4 (20 marks)

Find the Laplace Transforms of the following time domain signals. (12 + 8 marks)

(a) $x(t) = \int_{-\infty}^{2t} e^{-3\tau}\tau\sin(2\pi\tau)u(\tau-3)d\tau$

Hint: *Use property* $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \implies x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|}X(\frac{s}{a})$, *where* a *is a real constant.*

(b) $y(t) = \int_{-\infty}^{t} \tau^2 e^{-(4t+\tau)}u(\tau)d\tau$

Hint: *Express* $y(t)$ *as a convolution integral and use the convolution property of Laplace Transform.*

Solution:

(a) $x(t) = \int_{-\infty}^{2t} e^{-3\tau}\tau\sin(2\pi\tau)u(\tau-3)d\tau$

Let $x(t) = x_o(2t)$, where $x_o(t) = \int_{-\infty}^{t} e^{-3\tau}\tau\sin(2\pi\tau)u(\tau-3)d\tau$.

$$\boxed{X(s) = \frac{1}{2}X_o\left(\frac{s}{2}\right)} \tag{1}$$

Further, Let $x_o(t) = \int_{-\infty}^{t} x_1(\tau)d\tau$, where $x_1(t) = e^{-3t}t\sin(2\pi t)u(t-3)$.

$$\boxed{X_o(s) = \frac{X_1(s)}{s}} \tag{2}$$

$$\begin{aligned} \text{Let } x_2(t) &= t\sin(2\pi t)u(t-3) \\ &= (t-3)\sin(2\pi t - 6\pi)u(t-3) + 3\sin(2\pi t - 6\pi)u(t-3) \end{aligned}$$

$$\begin{aligned} \text{Thus, } X_2(s) &= -e^{-3s}\frac{d}{ds}\left[\frac{2\pi}{s^2+4\pi^2}\right] + e^{-3s}\frac{6\pi}{s^2+4\pi^2} \\ &= 2\pi e^{-3s}\left[\frac{2}{(s^2+4\pi^2)^2} + \frac{3}{s^2+4\pi^2}\right] \end{aligned}$$

Since $x_1(t) = e^{-3t}x_2(t)$, $X_{(s)}$ can be obtained as $X_1(s) = X_3(s+3)$

$$X_1(s) = 2\pi e^{-3(s+3)}\left[\frac{2}{((s+3)^2+4\pi^2)^2} + \frac{3}{(s+3)^2+4\pi^2}\right] \tag{3}$$

From (1) and (2), we get $X(s) = \frac{1}{2}\left(\frac{2}{s}X_1\left(\frac{s}{2}\right)\right) = \frac{1}{s}X_1\left(\frac{s}{2}\right)$.

Thus, $X(s)$ can be obtained as:

$$\boxed{X(s) = \frac{2\pi e^{-3\left(\frac{s}{2}+3\right)}}{s((\frac{s}{2}+3)^2+4\pi^2)}\left[\frac{2(\frac{s}{2}+3)}{(\frac{s}{2}+3)^2+4\pi^2} + 3\right] \quad \text{ROC: } \mathcal{R}e\{s\} > 0}$$

(b) We express $y(t)$ as a convolution integral of the form $y(t) = \tilde{x}(t) * h(t)$. Having identified $\tilde{x}(t)$ and $h(t)$, we can write $Y(s) = \tilde{X}(s)H(s)$.

$$\begin{aligned}
y(t) &= \int_{-\infty}^{t} \tau^2 e^{-(4t+\tau)} u(\tau) d\tau \\
&= \int_{-\infty}^{\infty} \tau^2 e^{-(4t+\tau)} u(\tau) u(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} \tau^2 e^{-(4t-4\tau)} e^{-5\tau} u(\tau) u(t-\tau) d\tau \\
&= \int_{-\infty}^{\infty} \underbrace{\tau^2 e^{-5\tau} u(\tau)}_{\tilde{x}(\tau)} \underbrace{e^{-(4t-4\tau)} u(t-\tau)}_{h(t-\tau)} d\tau
\end{aligned}$$

Thus, $\tilde{x}(t) = t^2 e^{-5t} u(t) \implies \tilde{X}(s) = \frac{2}{(s+5)^3}$, $\mathcal{R}e\{s\} > -5$
And $h(t) = e^{-4t} u(t) \implies H(s) = \frac{1}{(s+4)}$, $\mathcal{R}e\{s\} > -4$

Therefore $$\boxed{Y(s) = \frac{2}{(s+5)^3(s+4)} \text{ , } \mathcal{R}e\{s\} > -4}$$

Question 5 (30 marks)

Consider a causal LTI system S_o with transfer function $H_\text{o}(s)$ as given below.

$$H_\text{o}(s) = \frac{e^{-2s}(s-2)}{(s+2)(s^2-1)}$$

(a) Plot the pole-zero constellation of $H_\text{o}(s)$ and indicate its ROC (region of convergence). Is system S_o BIBO stable? Explain why. (4 marks)

(b) Consider system S_1 with transfer function $H_1(s)$ as shown below. (4 marks)

$$H_1(s) = \frac{(s-1)}{(s^2-4)(s+3)}$$

Comment on the BIBO stability of S_1 with the help of the pole-zero constellation of $H_1(s)$.

(c) Consider the cascaded system $S_\text{o}S_1$ with transfer function $H_2(s)$. (4 marks)

$$\rightarrow \boxed{H_2(s)} \rightarrow \quad \equiv \quad \rightarrow \boxed{H_\text{o}(s)} \rightarrow \boxed{H_1(s)} \rightarrow$$

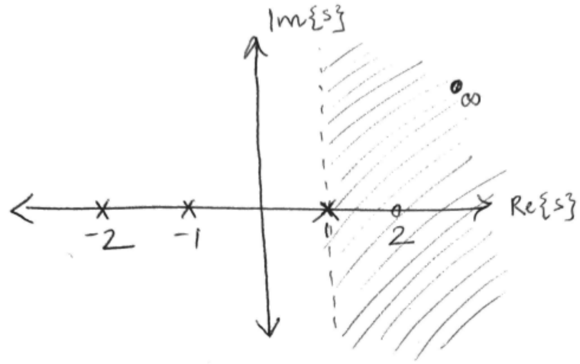
Plot the pole-zero constellation of $H_2(s)$ and indicate its ROC. Is the system $S_\text{o}S_1$ BIBO stable?

(d) Find the Inverse Laplace transform $h_2(t)$ of the cascaded system $S_\text{o}S_1$, given that $S_\text{o}S_1$ is causal. (10 marks)

(e) Find the output $y(t)$ when the input $x(t) = \frac{d}{dt}\delta(t) + 3\delta(t) + 2u(t)$ is applied to system $h_3(t)$, where its transfer function $H_3(s)$ has the same poles and zeros as $H_2(s)$ plus one additional zero at $s = 0$. (8 marks)

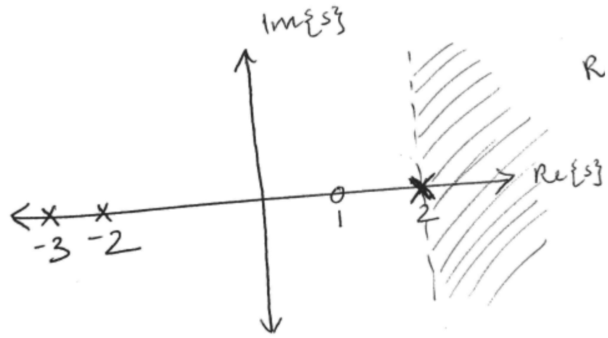
Solution:

(a) $H_o(s) = \frac{e^{-2s}(s-2)}{(s+2)(s^2-1)}$. Poles: $s = -2, -1, 1$. Zeros: $s = 2, \infty$. ROC: $\mathcal{R}e\{s\} > 1$.



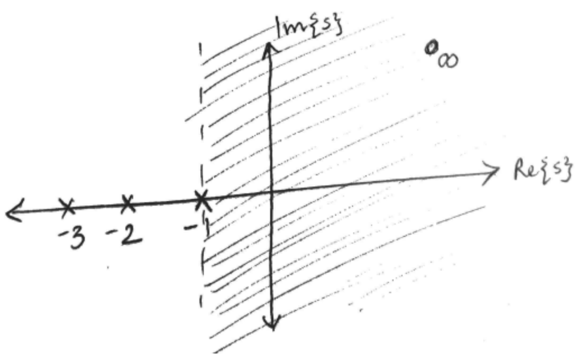
The system is not BIBO stable, because the ROC does not contain the $j\omega$ axis.

(b) Poles: $s = -3, -2, 2$. Zeros: $s = 1$. ROC: $\mathcal{R}e\{s\} > 2$



The system is not BIBO stable, because the ROC does not contain the $j\omega$ axis.

(c) $H_2(s) = \frac{e^{-2s}(s-2)}{(s+2)(s^2-1)} \frac{(s-1)}{(s^2-4)(s+3)} = \frac{e^{-2s}}{(s+2)^2(s+1)(s+3)}$
Poles: $s = -2, -2, -1, -3$. Zeros: $s = \infty$. ROC: $\mathcal{R}e\{s\} > -1$



The system is BIBO stable, because the entire $j\omega$ axis is contained in the ROC.

(d) We want to find the inverse laplace transform of $H_2(s) = \frac{e^{-2s}}{(s+2)^2(s+1)(s+3)}$.
Let $\tilde{H}_2(s) = \frac{1}{(s+2)^2(s+1)(s+3)}$. We express $\tilde{H}_2(s)$ as a sum of single-pole fractional parts, and find the inverse laplace transform $\tilde{h}_2(t)$. Then, we find $h_2(t) = \tilde{h}_2(t-2)$.

$$\tilde{H}_2(s) = \frac{1}{(s+2)^2(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)} + \frac{C}{(s+2)} + \frac{D}{(s+2)^2}$$

$$\begin{aligned}
A &= \tilde{H}_2(s)(s+1)\Big|_{s=-1} = \frac{1}{2} \\
B &= \tilde{H}_2(s)(s+3)\Big|_{s=-3} = -\frac{1}{2} \\
D &= \tilde{H}_2(s)(s+2)^2\Big|_{s=-2} = -1 \\
C &= \frac{d}{ds}\left(\tilde{H}_2(s)(s+2)^2\right)\Big|_{s=-2} = 0
\end{aligned}$$

Thus,

$$\tilde{H}_2(s) = \frac{1}{2}\frac{1}{(s+1)} - \frac{1}{2}\frac{1}{(s+3)} - \frac{1}{(s+2)^2}$$

Therefore, the inverse laplace transform can be computed as

$$\tilde{h}_2(t) = \left[\frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} - te^{-2t}\right]u(t)$$

We then obtain $h_2(t)$ as $h_2(t) = \tilde{h}_2(t-2)$.

$$h_2(t) = \left[\frac{1}{2}e^{-(t-2)} - \frac{1}{2}e^{-3(t-2)} - (t-2)e^{-2(t-2)}\right]u(t-2)$$

(e) $\tilde{H}_3(s) = \frac{e^{-2s}.s}{(s+2)^2(s+1)(s+3)}$. The input $x(t) = \frac{d}{dt}\delta(t) + 3\delta(t) + 2u(t)$.

Thus, $X(s) = s - \delta(0^-) + 3 + \frac{2}{s} = \frac{s^2+3s+2}{s} = \frac{(s+1)(s+2)}{s}$.

The output $y(t)$ can be computed as follows:

$$\begin{aligned}
Y(s) &= \frac{e^{-2s}.s}{(s+2)^2(s+1)(s+3)}\frac{(s+1)(s+2)}{s} \\
&= e^{-2s}\frac{1}{(s+2)(s+3)} \\
&= e^{-2s}\left[\frac{1}{s+2} - \frac{1}{s+3}\right]
\end{aligned}$$

Thus, the inverse laplace transform $y(t)$ can be obtained as:

$$y(t) = e^{-2(t-2)}u(t-2) - e^{-3(t-2)}u(t-2)$$