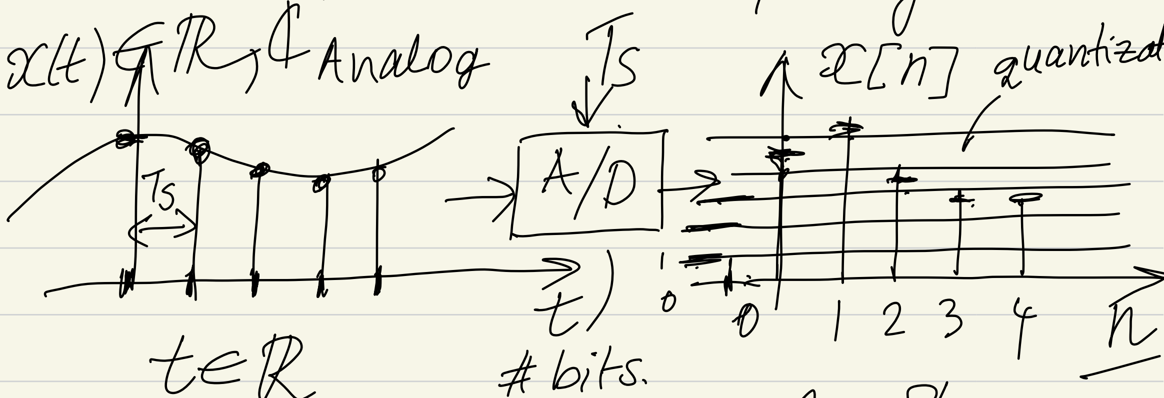
Lecture 18

Chapter 7. Sampling.



$t \in \mathbb{R}$

Continuous-Time signal

$n \in \mathbb{Z}$

sampling

Analog-to-Digital Converter.

$T_s \to$ sampling period (interval)

$f_s = \frac{1}{T_s}$ sampling frequency.

$\omega_s = 2\pi f_s$

$T = 1s$ $\qquad N = \frac{T^{=1s}}{T_{s=10^{-6}s}} = \underline{1{,}000{,}000}$

$\underline{T_s = 1\mu s}$

quantization levels
stored in bits / or bytes

1 Byte = 8 bits.

ADC : $\underline{4\text{bits}} \Rightarrow 16$ levels
k bits $\Rightarrow 2^k$ levels.

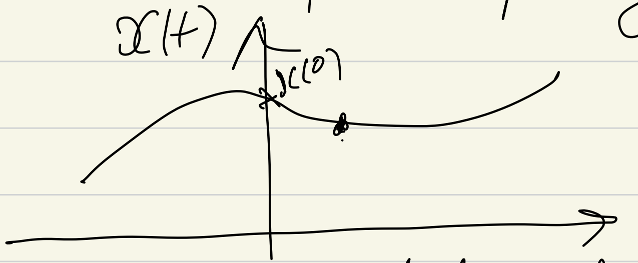
memory : $\underline{4}$ bits $\times 10^6$ samples.
= 4 MBitss
= 0.5 MBytes.

(1) What kind of signals can we sample without degradation?

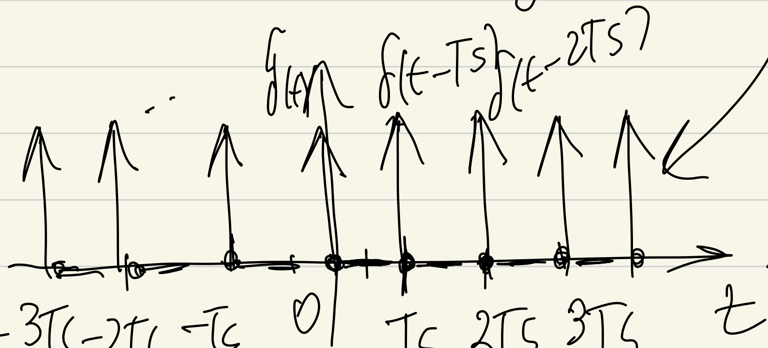
② What is the maximum T_s that ensures perfect reconstruction?

③ How do we reconstruct?

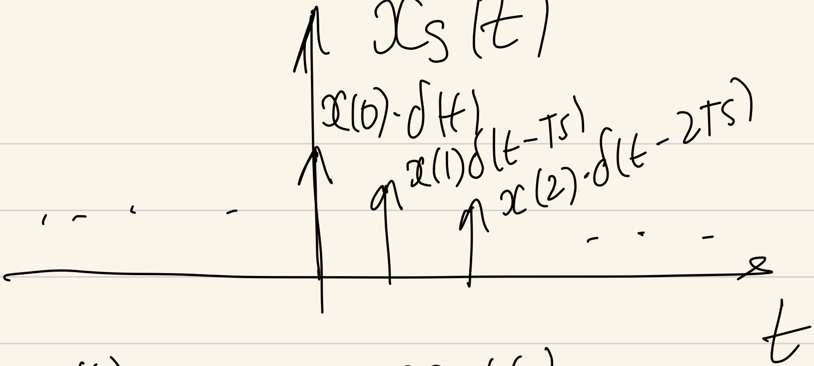
Model of sampling using multiplication with "Impulse Train"

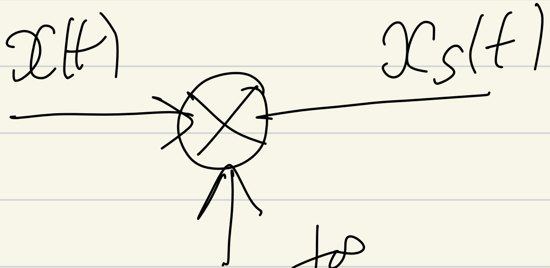


$\times$ multiply



$P_{T_s}(t)$





$$P_{T_s}(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT_s)$$

Impulse Train

$$x_s(t) = x(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t-kT_s)$$

get

$$= \sum_{k=-\infty}^{+\infty} x(t)\cdot\delta(t-kT_s)$$

$x(t)=$

$$= \sum_{k=-\infty}^{+\infty} \underline{x(kT_s)}\cdot\delta(t-kT_s)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)$$

true for all signals.

$$x(t) \neq \sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t-kT_s)$$

$$x_s(t) = \sum_{k=-\infty}^{+\infty} x(kT_s)\,\delta(t-kT_s)$$

?

$x(t)$

$$x(t) \neq x_s(t)$$

How is $X_s(\omega)$ related to $X(\omega)$

$$x_s(t) = x(t) \cdot p_{T_s}(t) \quad \text{periodic}$$

$$p_{T_s}(t) = \sum_{k=-\infty}^{+\infty} P_k\, e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$P_K = \frac{1}{T_s} \int_{T_s} \delta(t) \cdot e^{-jK\omega_s t} \, dt$$

$$= \frac{1}{T_s} \int_{T_s} \delta(t) \cdot \cancelto{1}{e^{-jK\omega_s \cdot 0}} \, dt$$

$$= \frac{1}{T_s}.$$

$$P_K = \frac{1}{T_s} \qquad K = 0, \pm 1, \ldots$$

$$P_{T_s}(t) = \frac{1}{T_s} \sum_{K=-\infty}^{+\infty} e^{jK\omega_s t}$$

$$x_s(t) = x(t) \cdot \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} e^{jk\omega_s t}$$

$$\rightarrow = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x(t) e^{jk\omega_s t}$$

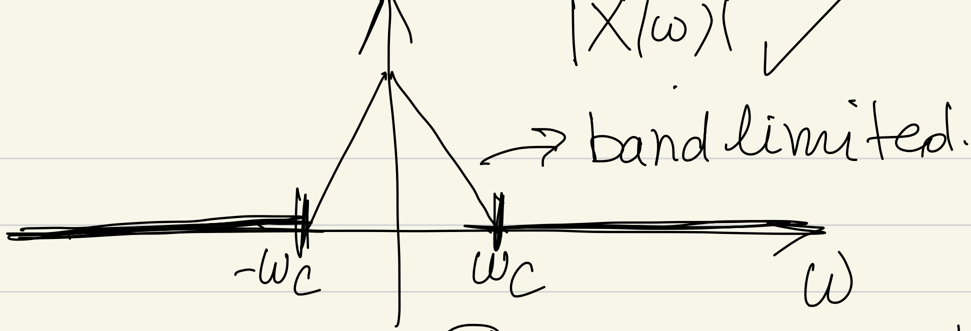
$$X_s(\omega) = \mathcal{F}\left\{ \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x(t) e^{jk\omega_s t} \right\}$$

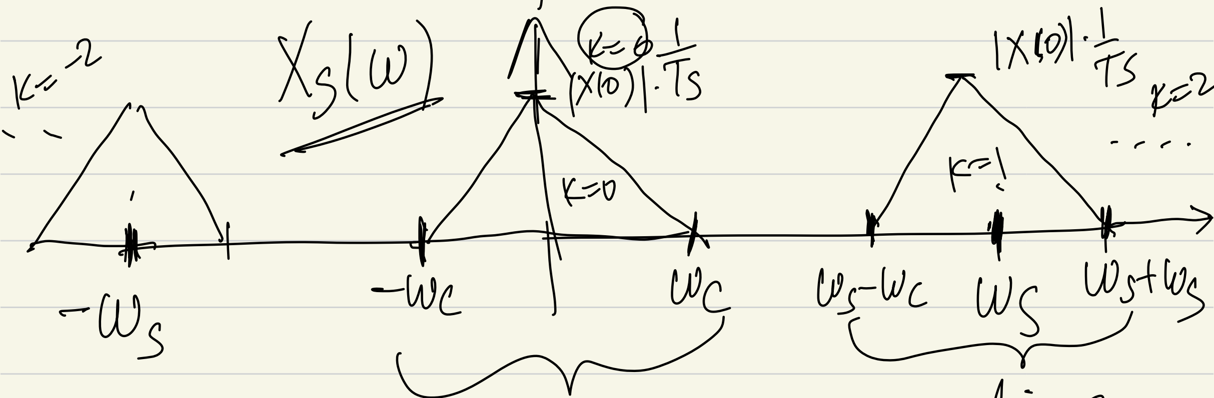
$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \mathcal{F}\left\{ x(t) e^{jk\omega_s t} \right\}$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X(\omega - k\omega_s)$$

pick ω_s $\Rightarrow$ pick T_s ✓

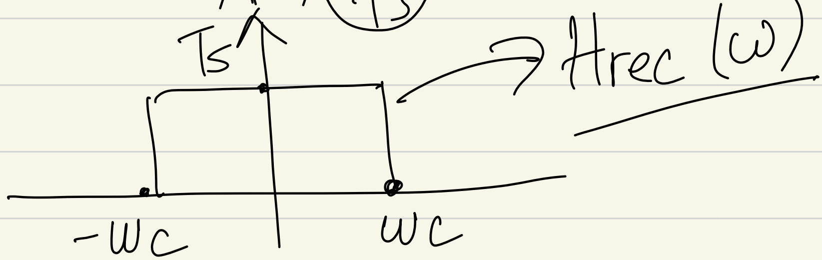
$$\omega_s = \frac{2\pi}{T_s}.$$





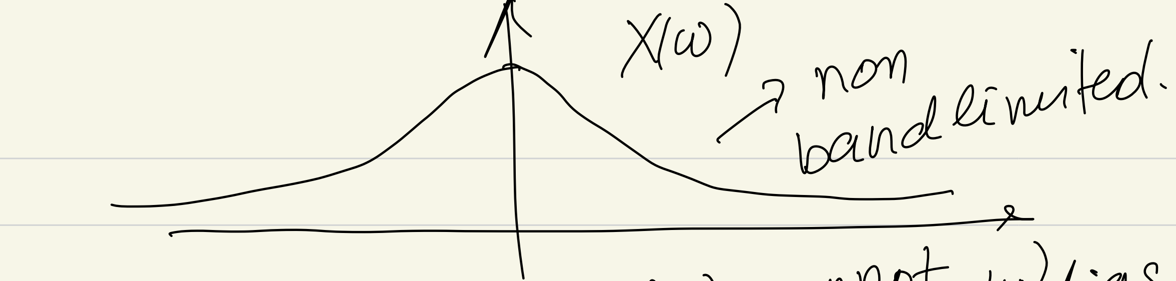
contains $X(\omega)\cdot\frac{1}{T_s}$

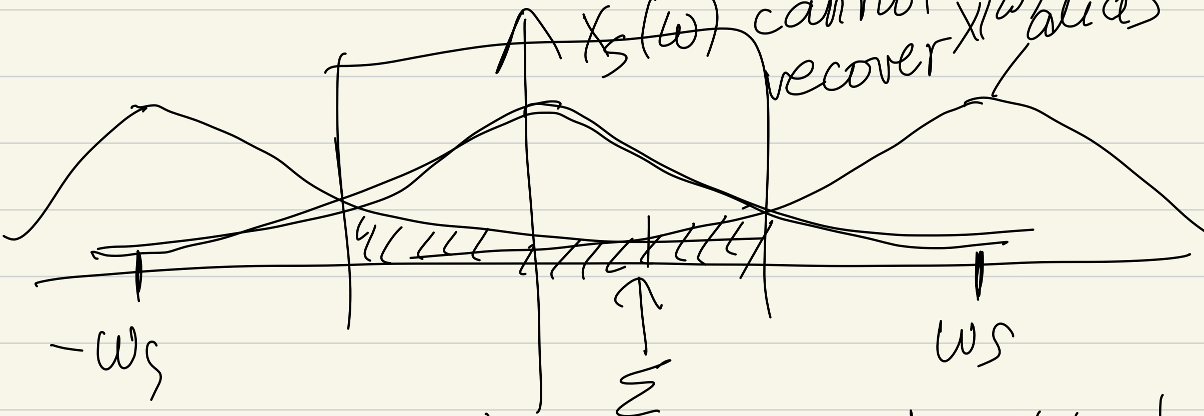
alias

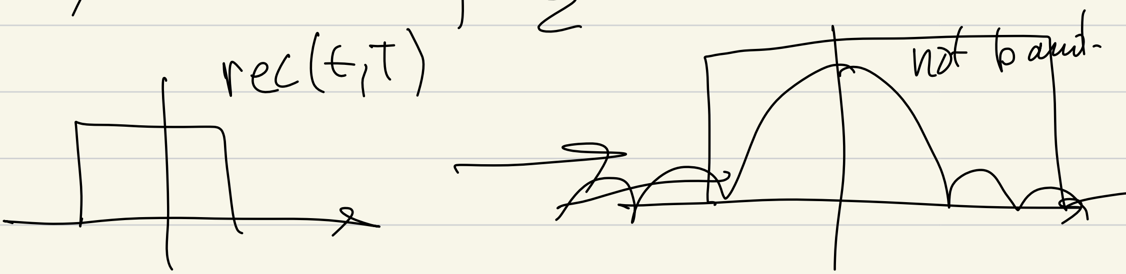


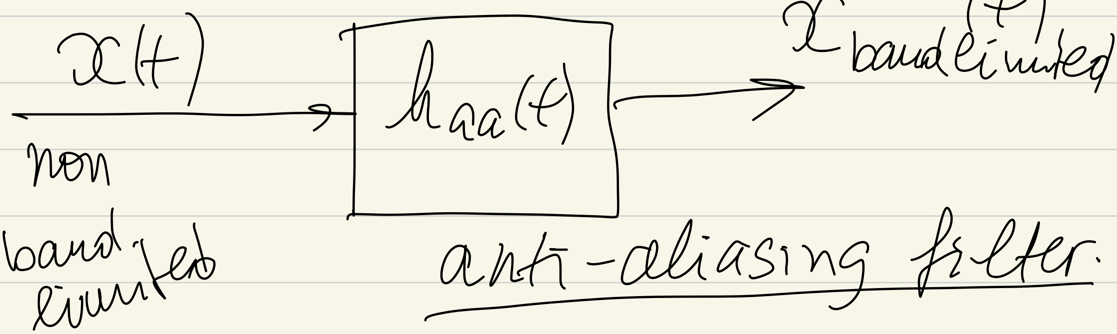
$$\underline{X_s(\omega)}\cdot\underline{H_{rec}(\omega)} = \underline{X(\omega)}$$

$$x(kT_s) \xrightarrow{?} x(t)$$



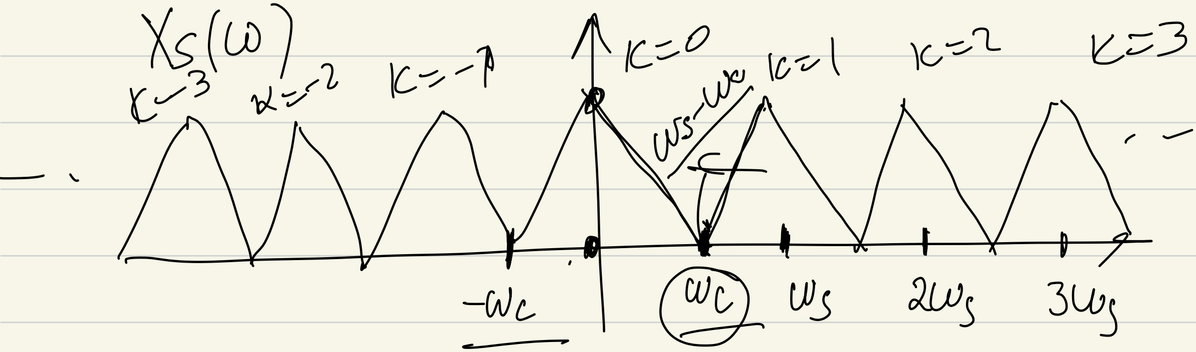






anti-aliasing filter.

Critical T_s (or equiv. ω_s)



$$\omega_s - \omega_c > \omega_c$$

$$\omega_s > 2\omega_c$$

$\omega_s = 2\omega_c \rightarrow$ Nyquist frequency.

Sampling Theorem
(Nyquist Theorem)
(Shannon-Nyquist Theorem)

Let $x(t)$ be a band-limited signal, such that $X(\omega)=0$ for $|\omega| > \omega_c$. Then, $x(t)$ is uniquely determined by its samples $x(nT_s)$ if

$$\omega_s = \frac{2\pi}{T_s} > 2\omega_c$$

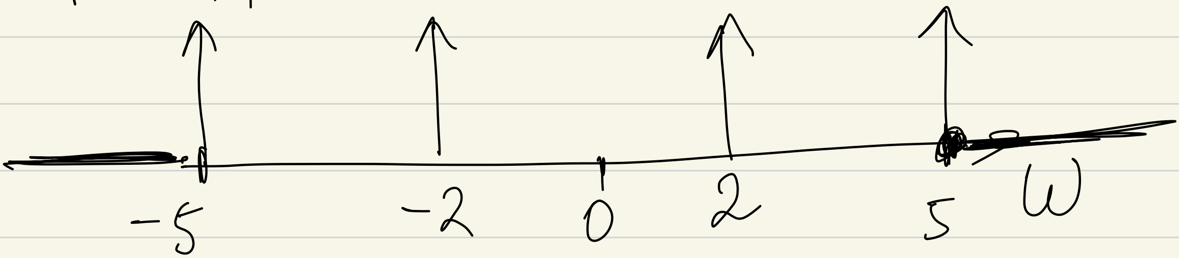
Example:

$T_1 = \pi$ $\qquad\qquad$ $T_2 = \frac{2\pi}{5}$

$$x(t) = 3\cos\left(2t + \frac{\pi}{4}\right) + 2\sin\left(5t - \frac{\pi}{8}\right)$$

What is max T_s for $x(t)$.

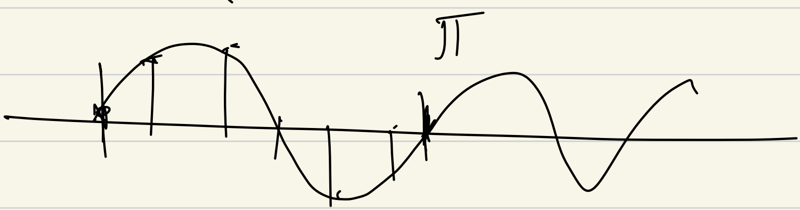
$|X(\omega)|$



$\omega_s > 2\omega_c$ $\qquad \omega_c = 5$

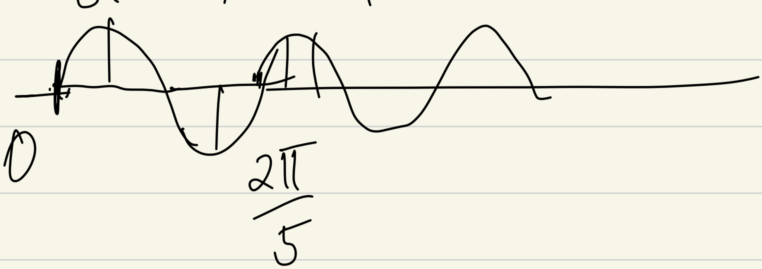
$\omega_s > 10$ $\qquad \boxed{T_s < \frac{2\pi}{10}}$

cos(2t)

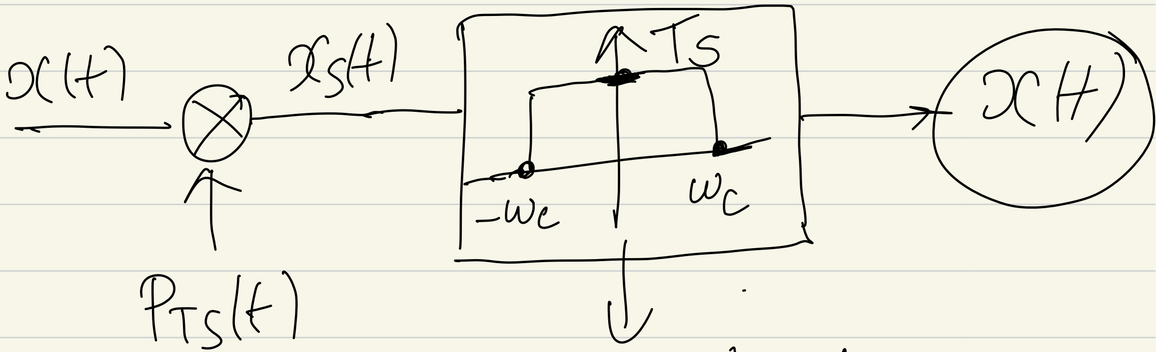


$\frac{2\pi}{10} = 0.2\pi$

sin(2t) +



Reconstruction.



reconstruction filter.

$$H_{rec}(\omega) = T_s \, rec(\omega, \omega_c)$$

$$\omega_s = 2\omega_c \rightarrow \omega_c = \frac{\omega_s}{2}$$

$$H_{rec}(\omega) = \underline{T_s} \, rec\left(\omega, \underline{\frac{\omega_s}{2}}\right)$$

$$x(t) = x_s(t) * \underline{h_r(t)}.$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} \underbrace{x(nT_s)}_{\text{input}} \delta(t - nT_s)$$

$$h_r(t) = ?\ \frac{\Omega}{\pi}\operatorname{sinc}(\Omega t) \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{rec}(\omega, \Omega)$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\cancel{\frac{T_s \omega_s}{2\pi}} \operatorname{sinc}\left(\frac{\omega_s}{2} t\right) \overset{\mathcal{F}}{\longrightarrow} T_s \operatorname{rec}\left(\omega, \frac{\omega_s}{2}\right)$$

$$h_r(t) = \operatorname{sinc}\left(\frac{\omega_s}{2} t\right)$$

$$x(t) = h_r(t) * \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s)$$

$$\underline{\underline{x(t)}} = \text{sinc}\left(\frac{\omega_s}{2}t\right) * \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s) =$$

$$= \sum_{n=-\infty}^{+\infty} \text{sinc}\left(\frac{\omega_s}{2}t\right) * \left[\underline{\underline{x(nT_s)}} \cdot \delta(t - nT_s)\right] =$$

$$\sum_{n=-\infty}^{+\infty} x(nT_s)\left[\underbrace{\text{sinc}\left(\frac{\omega_s}{2}t\right)}_{f(t)} * \underbrace{\delta(t-nT_s)}_{\delta(t-\tau)}\right]$$

$$\underline{f(t)} * \delta(t-\underline{\tau}) =$$

$$= f(t-\underline{\tau})$$

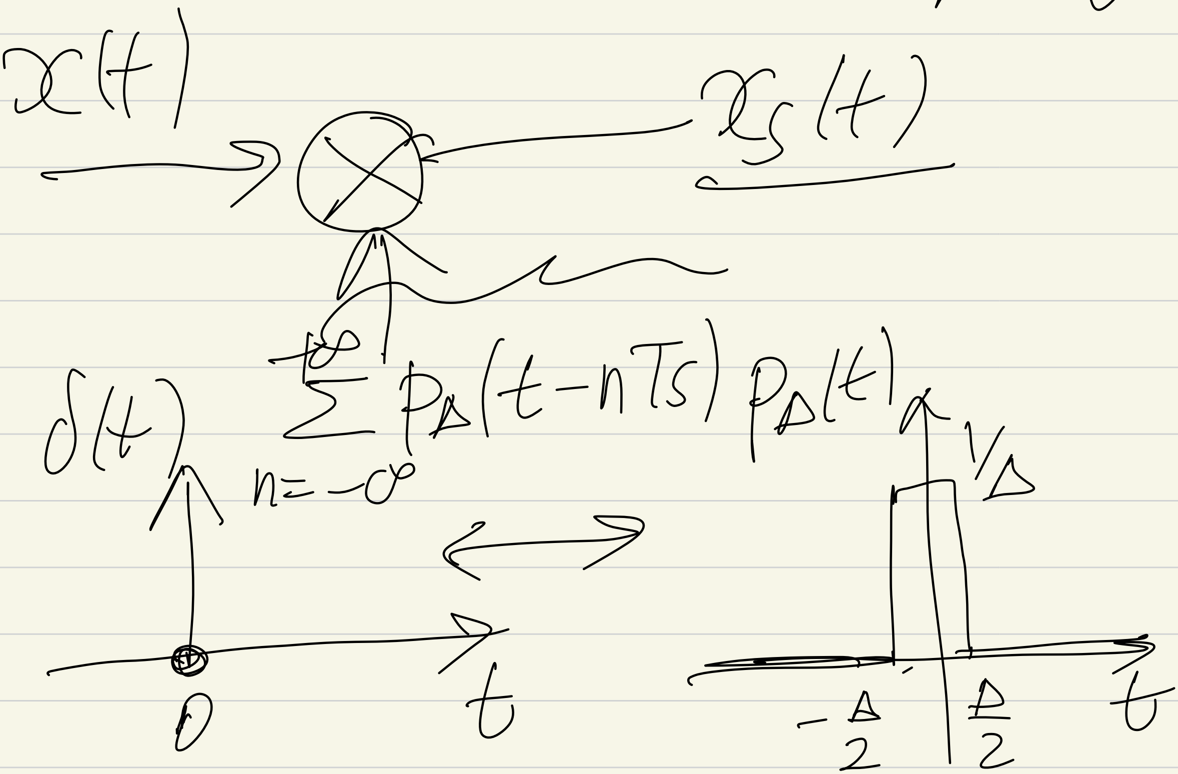
review Midterm
or derive!

$$= \sum_{n=-\infty}^{+\infty} x(nT_s) \cdot \operatorname{sinc}\left(\frac{\omega_s}{2}(t-nT_s)\right)$$

$$\underline{\underline{x(t)}} = \sum_{n=-\infty}^{+\infty} \underline{x(nT_s)} \operatorname{sinc}\left(\frac{\omega_s}{2}(t-nT_s)\right)$$

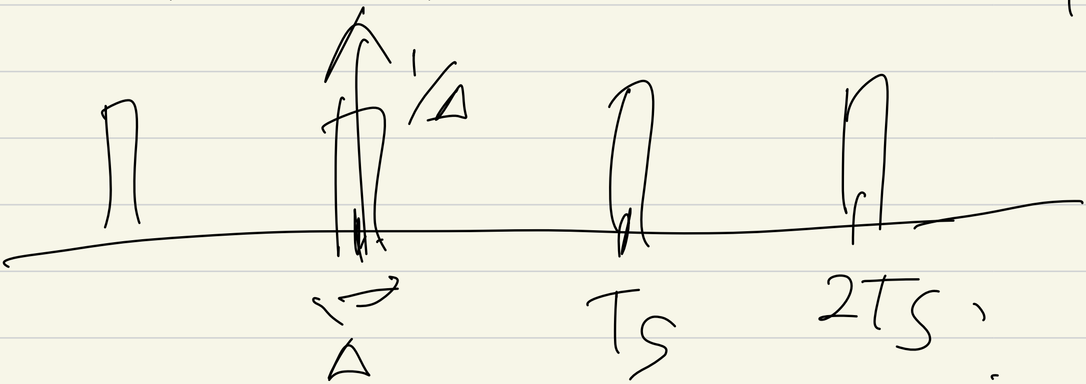
"sinc" interporation

More realistic view of sampling



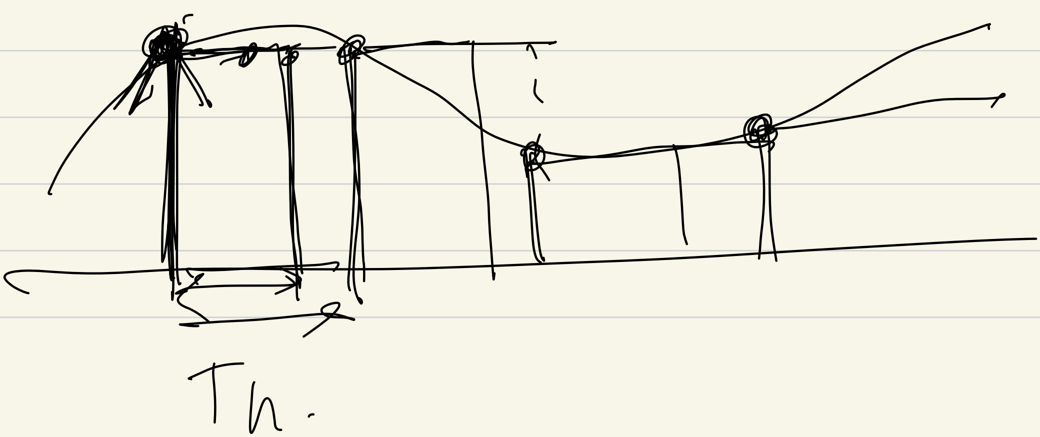
$$\lim_{\Delta \to 0} p_\Delta(t) = \delta(t)$$

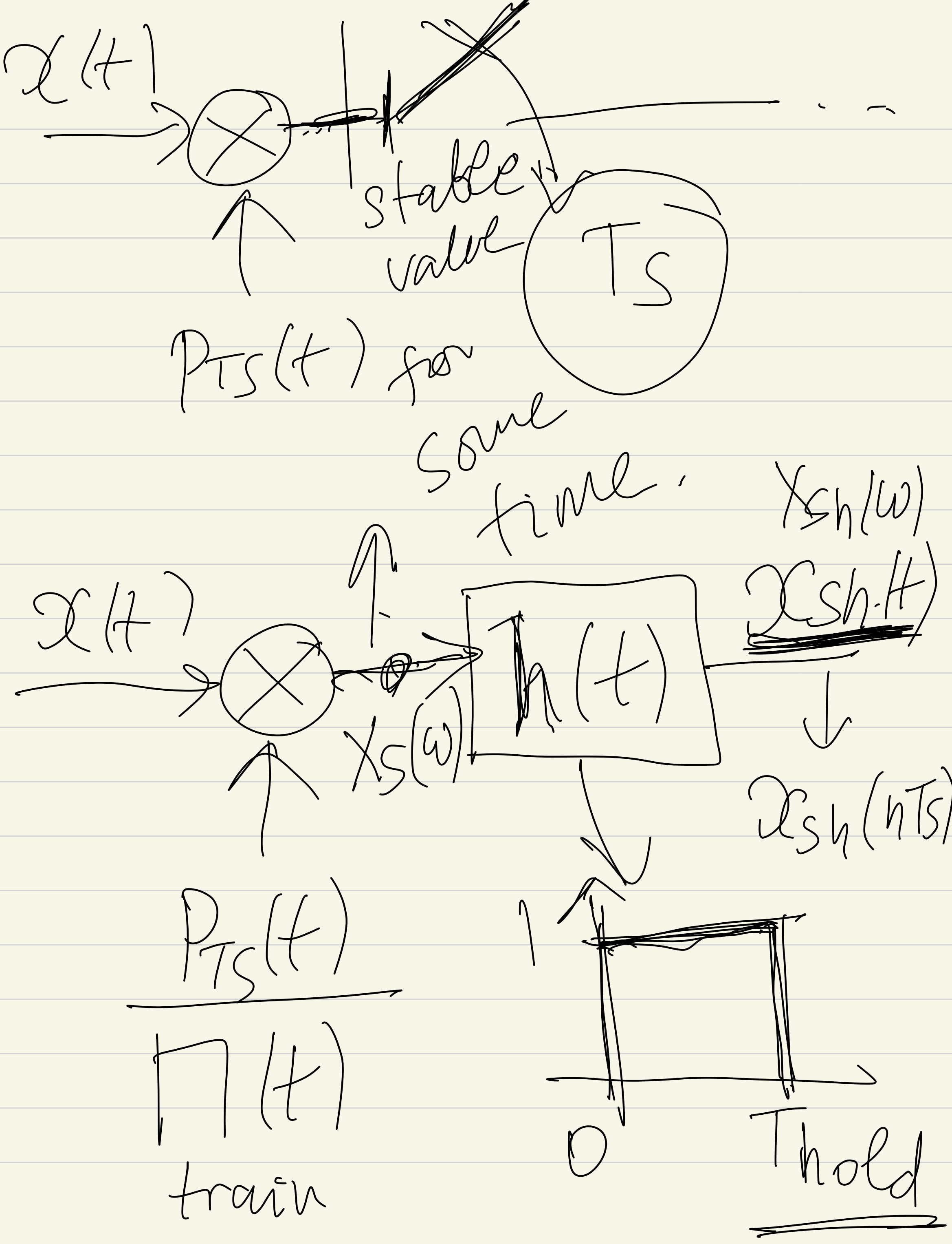
Fourier series of



$$\Pi_{T_S}(t) = \sum \Pi_k e^{jk\omega_s t}$$

"Sample and Hold"



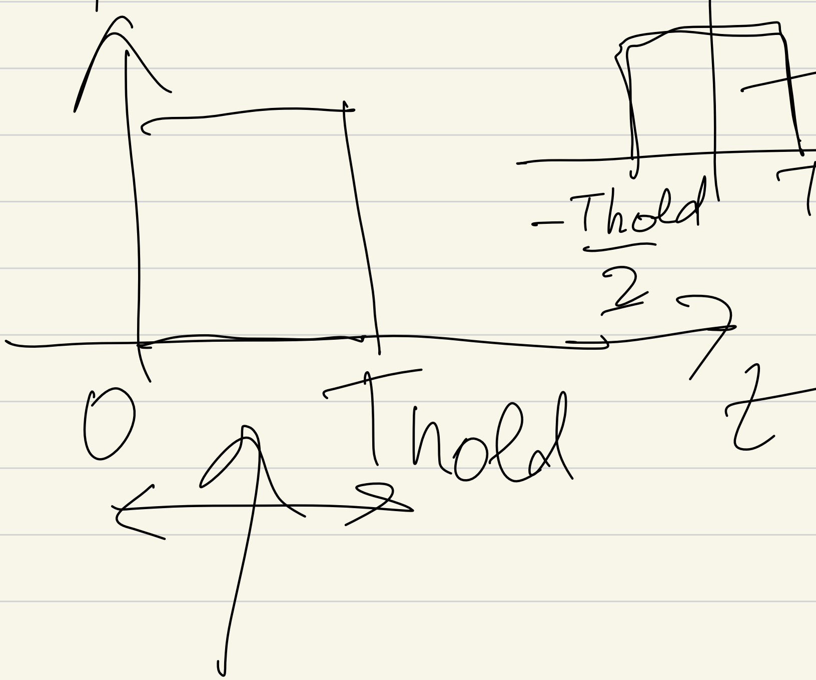


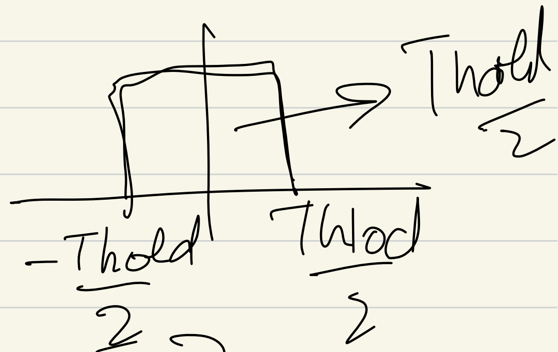
$x(t)$
stable value
T_s
$P_{T_S}(t)$ for some time.
$x(t)$
$X_s(\omega)$
$h(t)$
$X_{sh}(\omega)$
$x_{sh}(t)$
$x_{sh}(nT_s)$
$\frac{P_{T_S}(t)}{\sqcap(t)}$
train
1
0
T_{hold}

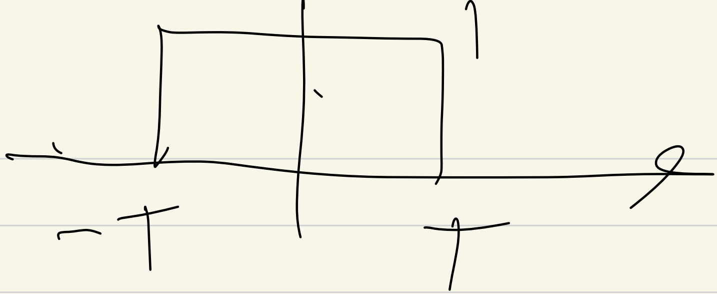
$X_s(\omega)$ know!

$$X_{sh}(\omega) = \underbrace{X_s(\omega)}_{\text{aliased spectrum}} \cdot \underline{H(\omega)}$$

$h(t)$







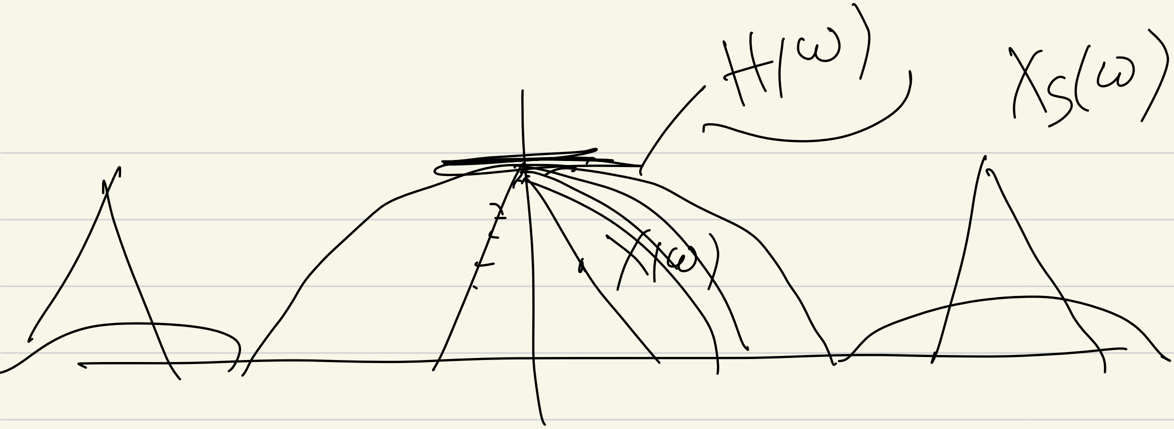
$$rec(t, T) \rightarrow 2T\,sinc(\omega T)$$

$$rec\left(t, \frac{Thold}{2}\right) \rightarrow Thold\,sinc\left(\omega \frac{Thold}{2}\right)$$

$$rec\left(t - \frac{Thold}{2}, \frac{Thold}{2}\right) \rightarrow$$

$$e^{-j\omega \frac{Thold}{2}} \cdot Thold\left(sinc\left(\omega \frac{Thold}{2}\right)\right)$$

$$\underline{\underline{H(\omega)}}$$



$$X_{sh}(\omega) = X(\omega) \cdot H(\omega)$$