UCLA DEPARTMENT OF ELECTRICAL ENGINEERING

EE102: SYSTEMS & SIGNALS

Final Examination
Date: March 19, 2020, Duration: 3 hours

INSTRUCTIONS:

- The exam has 6 problems and 16 pages.
- The exam is open book and open notes.
- Calculator are allowed.

Your name:______________________________

Student ID:______________________________

Table 1: Score Table

Problem	a	b	c	d	Score
1	5	5	5		15
2	10	5			15
3	10	5	10		25
4	5	5	5		15
5	5	5	5		15
6	5	10			15
Total					100

Problem 1 (15 pts)

State whether the following statements are TRUE or FALSE. Provide a brief explanation for each part.

(a) (5 pts) A system with the following impulse response function is time-invariant:

$$h(t,\tau) = e^{-2t-2\tau}(\sin(t)\cos(\tau) - \cos(t)\sin(\tau))u(t-\tau) \tag{1}$$

(b) (5 pts) A system with the following input-output relationship is causal:

$$y(t) = x(t-3) + \int_{t-3}^{3t} e^{-(t-\sigma)}u(t-\sigma)x(\sigma)d\sigma \tag{2}$$

(c) (5 pts) Let $z(t) = \frac{1}{3}x(t-2) + 4$, where $x(t)$ is a band-limited signal with maximum frequency 100 rad/s. The minimum sampling frequency (according to Nyquist theorem) to sample $z(t)$ is $\omega_s = 2 \times \frac{1}{3} \times 100 = \frac{200}{3}$ rad/s.

Solution:

(a) False. The impulse response function $h(t,\tau) = e^{-2(t+\tau)}\sin(t-\tau)u(t-\tau)$ can not be rewritten as $h(t-\tau)$

(b) True. The first term is a delayed version of the input so it's causal. The second term can be rewritten as

$$\begin{aligned} &\int_{t-3}^{3t} e^{-(t-\sigma)}u(t-\sigma)x(\sigma)d\sigma \\ =&\int_{t-3}^{\min\{3t,t\}} e^{-(t-\sigma)}x(\sigma)d\sigma \end{aligned}$$

Since $\min\{3t,t\} \leq t$, the second term doesn't depend on future input as well. So the system is causal.

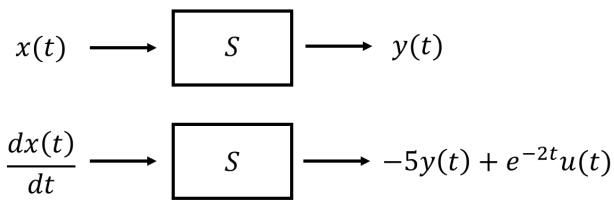
(c) False. Scaling in magnitude and delaying in time don't change the Nyquist sampling rate. $\omega_s = 2 \times 100 = 200$ rad/s.

Problem 2 (15 pts)

Consider an LTI system S with the input signal

$$x(t) = e^{-5t}u(t-1) \tag{3}$$

and corresponding output signal $y(t)$. We also know that if input $\frac{dx(t)}{dt}$ is applied to the system S, corresponding output is $-5y(t) + e^{-2t}u(t)$.



(a) (10 pts) Determine the system transfer function $H(s)$ and the impulse response function $h(t)$.

(b) (5 pts) Find the system output $y_1(t)$ if the input signal is $x_1(t) = e^{-2t}\cos(3t)u(t)$.

Solution:

(a) We have $H(s)X(s) = Y(s)$ and $sX(s)H(s) = \left[-5Y(s) + \frac{1}{s+2}\right]$. Substituting the first equation into the second, we get

$$\begin{aligned} sX(s)H(s) &= \left[-5X(s)H(s) + \frac{1}{s+2}\right] \\ H(s) &= \frac{1}{X(s)(s+5)(s+2)} \end{aligned}$$

We have $X(s) = \mathcal{L}[e^{-5t}u(t-1)] = \frac{e^{-(s+5)}}{s+5}$. Substitute in the above equation to get

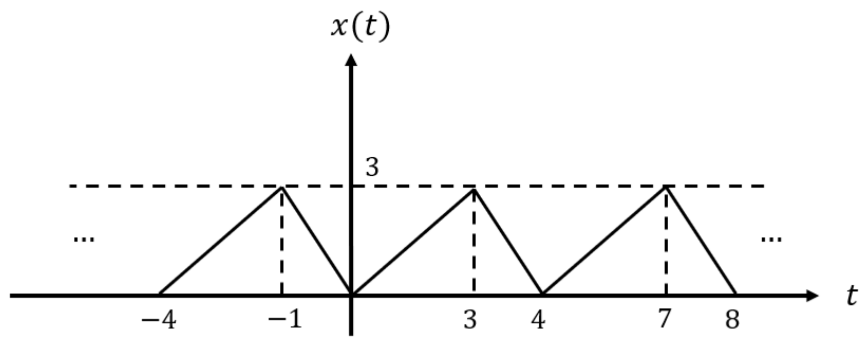
$$\begin{aligned} H(s) &= \frac{e^{s+5}}{(s+2)} \\ h(t) &= e^5 e^{-2(t+1)}u(t+1) = e^{-2t+3}u(t+1) \end{aligned}$$

(b) We have $X_1(s) = \mathcal{L}[e^{-2t}\cos(3t)u(t)] = \frac{s+2}{(s+2)^2+9}$. So $Y_1(s) = H(s)X_1(s) = \frac{e^{s+5}}{(s+2)^2+9}$. Apply inverse Laplace transform to $Y_1(s)$, the output $y_1(t)$ is

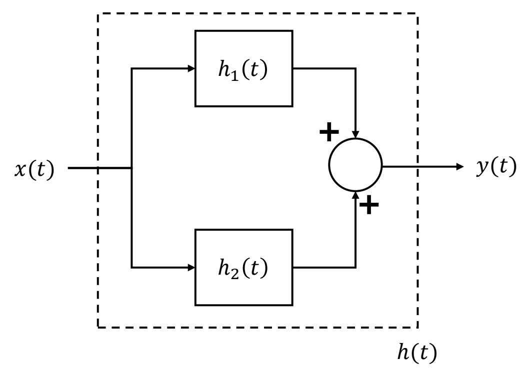
$$y_1(t) = \frac{e^5}{3}e^{-2t-2}\sin(3t+3)u(t+1)$$

Problem 3 (25 pts)

Consider a periodic signal $x(t)$



The signal is passed through a parallel system, with the following impulse response for each branch:



$$h_1(t) = \frac{\sin(3\pi t/4)}{\pi t}, \quad h_2(t) = \frac{2\sin(\pi t/4)}{t}\cos(4\pi t)$$

(a) (10 pts) Compute the Fourier series coefficients X_k of the signal $x(t)$.

(b) (5 pts) Sketch the frequency response $H(\omega)$ of the entire system.

(c) (10 pts) Compute the Fourier series coefficients Y_k of the output $y(t)$.

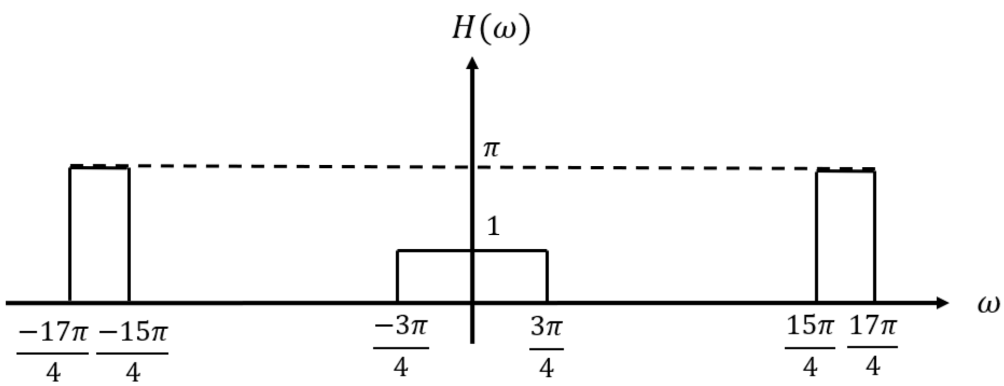
Solution:

(a) For $0 \le t \le 4$, $x(t)$ is the same as $x_1(t)$, where $x_1(t) = r(t) - 4r(t-3) + 3r(t-4)$. The Laplace transform of $x_1(t)$ can help us find the Fourier series coefficients. The property used here is $X_k = \frac{1}{T_0} X_1(s = jk\omega_0)$. So we first calculate $X_1(s) = \frac{1-4e^{-3s}+3e^{-4s}}{s^2}$, $T_0 = 4$, $\omega_0 = \frac{\pi}{2}$. Then we have

$$X_k = \frac{1}{4}\frac{1 - 4e^{-3jk\pi/2} + 3e^{-j2\pi k}}{-k^2\pi^2/4} = \frac{4e^{-3jk\pi/2} - 4}{k^2\pi^2}, k \neq 0,$$

and $X_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{3}{2}$

(b) $H_1(\omega) = u(\omega + \frac{3\pi}{4}) - u(\omega - \frac{3\pi}{4})$,$H_2(\omega) = \pi[u(\omega + \frac{17\pi}{4}) - u(\omega + \frac{15\pi}{4}) + u(\omega - \frac{15\pi}{4}) - u(\omega - \frac{17\pi}{4})]$, and $H(\omega) = H_1(\omega) + H_2(\omega)$.



(c) $Y_k = H(k\omega_0)X_k = H(\frac{k\pi}{2})X_k$. Substituting the $H(\omega)$ and X_k we found in (a)(b) into the formula, we have

$$Y_k = \begin{cases} \frac{3}{2}, & k = 0 \\ \frac{4j-4}{\pi^2}, & k = 1 \\ \frac{-4j-4}{\pi^2}, & k = -1 \\ \frac{4-4}{64\pi} = 0, & k = \pm 8 \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \frac{3}{2}, & k = 0 \\ \frac{4j-4}{\pi^2}, & k = 1 \\ \frac{-4j-4}{\pi^2}, & k = -1 \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4 (15 pts)

Given the multiplication property of Fourier Transform

$$x(t)y(t) \xrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

where $X(\omega) = \mathcal{F}\{x(t)\}$ and $Y(\omega) = \mathcal{F}\{y(t)\}$, and $*$ denotes convolution.

(a) (5pts) Compute the Fourier transform of $f_1(t) = \text{sinc}^2(t)$, given that $\text{sinc}(t) = \frac{\sin(t)}{t}$.

(b) (5 pts) Compute the Fourier transform of $f_2(t) = t \times \text{sinc}^2(t)$.

(c) (5 pts) By using the result in (a) and the Parseval's theorem, calculate the following integral

$$\int_{-\infty}^{\infty} \text{sinc}^4(t) dt.$$

Solution:

(a) $F_1(\omega) = \mathcal{F}\{f_1(t)\} = \frac{1}{2\pi}\mathcal{F}\{\text{sinc}(t)\} * \mathcal{F}\{\text{sinc}(t)\} = \frac{1}{2\pi}(\pi[u(\omega+1) - u(\omega - 1)]) * (\pi[u(\omega+1) - u(\omega-1)]) = \frac{\pi}{2}(r(\omega+2) - 2r(\omega) + r(\omega-2))$.

(b) By using the frequency differentiation property in the Fourier transform table, we have $\mathcal{F}\{f_2(t)\} = j\frac{dF_1(\omega)}{d\omega} = \frac{j\pi}{2}[(u(\omega+2) - u(\omega)) - (u(\omega) - u(\omega-2))]$.

(c) We can rewrite the integral as

$$\int_{-\infty}^{\infty} \text{sinc}^4(t) dt = \int_{-\infty}^{\infty} |\text{sinc}^2(t)|^2 dt.$$

By using the Parseval's theorem, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |\text{sinc}^2(t)|^2 dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\frac{\pi}{2}(r(\omega+2) - 2r(\omega) + r(\omega-2))|^2 d\omega \\
&= \frac{\pi}{8}\int_{-2}^{0}(\omega+2)^2 d\omega + \frac{\pi}{8}\int_{0}^{2}(-\omega+2)^2 d\omega \\
&= \frac{\pi}{4}\int_{0}^{2}(-\omega+2)^2 d\omega = \frac{\pi}{4} \times \frac{8}{3} = \frac{2\pi}{3}
\end{aligned}$$

Problem 5 (15 pts)

Let S be the Linear system

$$x(t) \to [S] \to y(t),$$

described by the differential equation

$$\frac{1}{a}\frac{dy(t)}{dt} + y(t) = \frac{1}{a}\frac{dx(t)}{dt} - x(t), \quad \text{and } y(0) = x(0) = 0$$

where $a > 0$.

(a) (5 pts) Find the transfer function $H(s)$ and the frequency response $H(\omega)$.

(b) (5 pts) Find the system output $y(t)$ if $x(t) = e^{-at}cos(3t)u(t)$.

(c) (5 pts) Show that the magnitude of the frequency response, $|H(\omega)|$, satisfies

$$|H(\omega)| = \text{constant, for all } \omega.$$

System S is the "all pass" filter since it passes all frequencies of any $X(\omega) = \mathcal{F}\{x(t)\}$.

Solution:

(a) Applying the Laplace transform to the differential equation, we have $\frac{1}{a}sY(s) + Y(s) = \frac{1}{a}sX(s) - X(s)$. So the transfer functions are

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-a}{s+a}, \quad H(\omega) = \frac{j\omega - a}{j\omega + a}.$$

(b) $X(s) = \mathcal{L}[e^{-at}\cos(3t)u(t)] = \frac{s+a}{(s+a)^2+9}$. $Y(s) = H(s)X(s) = \frac{s-a}{(s+a)^2+9} = \frac{s+a}{(s+a)^2+9} - \frac{2a}{3}\frac{3}{(s+a)^2+9}$. Apply inverse Laplace transform to $Y(s)$, the output $y(t)$ is

$$y(t) = e^{-at}\cos(3t)u(t) - \frac{2a}{3}e^{-at}\sin(3t)u(t)$$

(c) The magnitude of the system frequency response $|H(\omega)| = \frac{|j\omega-a|}{|j\omega+a|} = \frac{\sqrt{a^2+\omega^2}}{\sqrt{a^2+\omega^2}} = 1$. So we proved it's a constant and the system is a all pass filter.

Problem 6 (15 pts)

For a continuous-time LTI system with a real, causal impulse response $h(t)$, the frequency response $H(\omega)$ and $h(t)$ can be completely specified by the real part of its frequency response, $\mathfrak{Re}\{H(\omega)\}$. The property is generally referred as *real-part sufficiency*. In the following, we want to show this property by examining the even part of $h(t)$, denoted as $h_e(t)$.

(a) (5 pts) Given that $h(t)$ is a real and causal impulse response, express the Fourier transform of $h_e(t)$ in terms of $H(\omega)$.

(b) (10 pts) Using what we observed in (a), specify $h(t)$ if the real part of the frequency response of this causal system is

$$\mathfrak{Re}\{H(\omega)\} = \cos(\omega)$$

Solution:

(a) $h_e(t) = \frac{h(t)+h(-t)}{2}$, $H_e(\omega) = \mathcal{F}\{h_e(t)\} = \frac{1}{2}(H(\omega) + H(-\omega))$. Since the impulse response is real, we have $H(-\omega) = H(\omega)^*$. Substitute it into the expression of $H_e(\omega)$ to get

$$H_e(\omega) = \frac{1}{2}(H(\omega) + H(\omega)^*) = \mathfrak{Re}\{H(\omega)\}$$

(b) From (a), we know that $h_e(t) = \frac{1}{2}[\delta(t+1) + \delta(t-1)]$. Since $h_e(t) = \frac{1}{2}(h(t) + h(-t))$ and $h(t)$ is causal, the $h_e(t)$ where $t > 0$ corresponds to $\frac{1}{2}h(t)$, and the $h_e(t)$ where $t < 0$ corresponds to $\frac{1}{2}h(-t)$. Therefore, we have

$$h(t) = \delta(t-1).$$