Analysis of Signals through Decomposition into Elementary Signals

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \phi_n(t)$$

elementary signals

$$\{\phi_n(t)\}_{n=-\infty}^{+\infty}$$

It will helpful in Linear Systems Analysis

$$\alpha x_1(t)+\beta x_2(t) \longrightarrow \boxed{S} \longrightarrow \alpha\cdot y_1(t)+\beta y_2(t)$$

$$\sum_{n=-\infty}^{+\infty} X_n \phi_n(t) \longrightarrow \boxed{S} \longrightarrow \sum_{n=-\infty}^{+\infty} X_n S\{\phi_n(t)\}$$

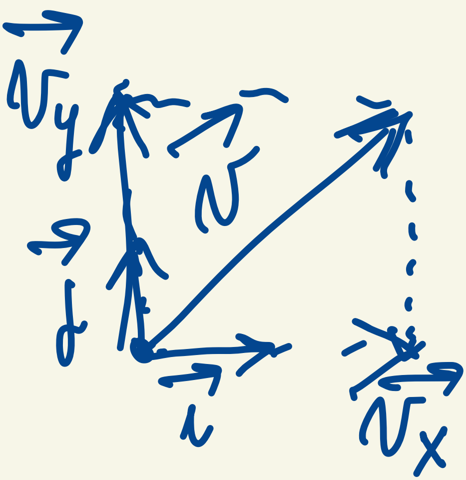
$$\{\phi_n(t)\}_{n=-\infty}^{+\infty}$$

1. What are good elementary signals?

2. What are their properties?

Building some intuition about Basis for represent.

Vectors



2-D vectors

$\vec{i}, \vec{j}$

Properties:

1) orthogonal
2) unit norm

We defined "dot" product

$$\langle \vec{v}, \vec{u} \rangle = \vec{v} \cdot \vec{u} \triangleq |\vec{v}| \cdot |\vec{u}| \cdot \cos(\measuredangle(\vec{u}, \vec{v}))$$

$$\langle \vec{i}, \vec{j} \rangle = |\vec{i}|\,|\vec{j}| \cos(90) = 0$$

orthogonality means

$$\langle \vec{u}, \vec{v} \rangle = 0$$

$$\langle \vec{i}, \vec{i} \rangle = |\vec{i}| \cdot |\vec{i}| \cdot \cos 0$$

$$= |\vec{i}|^2 = 1$$

$$\langle \vec{j}, \vec{j} \rangle = |\vec{j}|\,|\vec{j}| \cos 0 = 1$$

$$\vec{v} = \vec{v_x} + \vec{v_y}$$

$$\vec{v}_x = \langle \vec{v}, \vec{i} \rangle \vec{i} = v_x \cdot \vec{i}$$

$$\vec{v}_y = \underline{\langle \vec{v}, \vec{j} \rangle} \cdot \vec{j} = \underline{v_y} \cdot \vec{j}$$

inner product of vector and basis is projection of that vector onto the basis!

proof:

$$\langle \vec{v}, \vec{i} \rangle = (v_x \cdot \vec{i} + v_y \cdot \vec{j}) \cdot \vec{i}$$

$$= v_x \cdot \vec{i} \cdot \vec{i} + \cancelto{0}{v_y \cdot \vec{j} \cdot \vec{i}}$$

$$= v_x$$

$$\vec{v} = \underbrace{\langle \vec{v}, \vec{i} \rangle}\cdot \vec{i} + \underbrace{\langle \vec{v}, \vec{j} \rangle}\cdot \vec{j}$$

Lets come back to signals.

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \underline{\underline{\phi_n(t)}}.$$

basis elem. sign.

$t \in [a, b]$

inner product definition for signals:

$$\langle \phi_n(t), \phi_m(t) \rangle \triangleq$$

$$\triangleq \int_a^b \phi_n(t) \cdot \phi_m^*(t)\, dt$$

orthogonal basis has

$$\int_a^b \phi_n(t) \cdot \phi_m^*(t)\, dt = 0 \text{ for } n \neq m \ \forall m, n$$

$$\phi_n(t) \perp \phi_m(t) \qquad a \cdot a^* = |a|^2$$

unit norm

$$\int_a^b \phi_n(t) \cdot \phi_n^*(t)\, dt = \int_a^b |\phi_n(t)|^2 dt = 1$$

Periodic Signals decomposition

$x(t) \quad -\infty < t < +\infty$

$\exists T_0$ such that

$$x(t) = x(t+T_0) = x(t+2T_0) = \dots = x(t+kT_0)$$

Periodic signals can be decomposed

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \phi_n(t)$$

such that

$$\boxed{\phi_n(t) = \frac{1}{\sqrt{T_0}} e^{jn\omega_0 t}}$$

$$\omega_0 = \frac{2\pi}{T_0}$$ fundamental "angular" frequency

$T_0 \rightarrow$ fundamental period

$$f_0 = \frac{1}{T_0} \rightarrow \phi_n(t) = \frac{1}{\sqrt{T_0}} e^{jn \cdot 2\pi f_0 t}$$

$$\left\{ \frac{1}{\sqrt{T_0}} e^{jn\omega_0 t} \right\}_{n=-\infty}^{+\infty} \rightarrow$$ Fourier basis.

Basis is periodic w/ T_0!

$$\phi_n(t) = \phi_n(t+T_0)\ ?$$

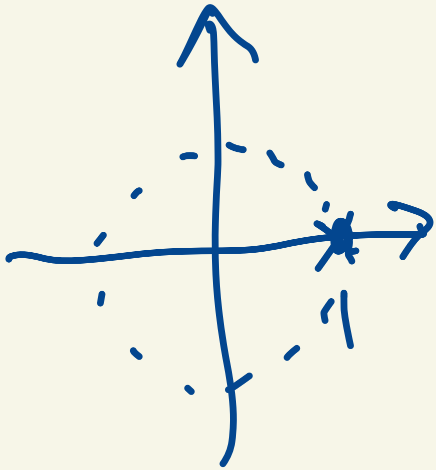
$$\frac{1}{\cancel{\sqrt{T_0}}} e^{jn\omega_0 t} = \frac{1}{\cancel{\sqrt{T_0}}} e^{jn\omega_0 (t+T_0)}$$

$$e^{jn\omega_0 t} = e^{jn\omega_0 t} \cdot e^{jn\omega_0 T_0}$$

$$e^{jn\omega_0 T_0} = 1$$

$$e^{jn\frac{2\pi}{\cancel{T_0}} \cdot \cancel{T_0}} = 1$$

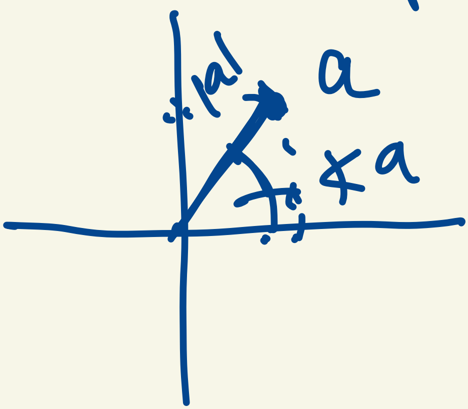
$$\underline{e^{jn\cdot 2\pi}} = 1$$



hint

$$a = \underline{|a| \cdot e^{j\measuredangle a}}$$

$$\underline{\underline{e^{j\measuredangle a}}} = \cos(\measuredangle a) + j\sin(\measuredangle a)$$



equivalent

$$a = \underline{\mathrm{Re}\{a\}} + j\,\mathrm{Im}\{a\}$$

Orthogonality check:

$$\langle \phi_n(t), \phi_m(t) \rangle \stackrel{?}{=} 0$$

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{1}{\sqrt{T_0}} e^{jn\omega_0 t} \cdot \left(\frac{1}{\sqrt{T_0}} e^{jm\omega_0 t} \right)^* dt$$

$$= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{1}{T_0} e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t} dt$$

$$a = |a| \cdot e^{j\measuredangle a} \qquad = \qquad a = Re\{a\} + jIm\{a\}$$

$$a^* = |a| e^{-j\measuredangle a} \qquad a^* = Re\{a\} - jIm\{a\}$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} e^{j(n-m)\omega_0 t} \, dt$$

$$\int e^{at} \, dt = \frac{1}{a} e^{at}$$

$$= \frac{1}{T_0} \frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0} \Bigg|_{-\frac{T_0}{2}}^{\frac{T_0}{2}}$$

$$= \frac{1}{j(n-m)\omega_0 \cdot T_0} \cdot \left[e^{j(n-m)\omega_0 \frac{T_0}{2}} - e^{-j(n-m)\omega_0 \frac{T_0}{2}} \right]$$

$$= \frac{1}{j(n-m)2\pi} \left[e^{j(n-m)\pi} - e^{-j(n-m)\pi} \right]$$

$$= \frac{1}{(n-m)\pi} \frac{e^{j(n-m)\pi} - e^{-j(n-m)\pi}}{2j}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$= \frac{1}{(n-m)\pi} \cdot \sin(n-m)\pi$$

$$\langle \phi_n(t), \phi_m(t) \rangle = \frac{\sin(n-m)\pi}{(n-m)\pi}$$

$n \neq m$ $\qquad \frac{\sin(n-m)\pi}{(n-m)\pi} = 0$

basis are orthogonal.

$n = m$ $\qquad \frac{\sin(n-m)\pi}{(n-m)\pi} = 1$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$

basis are unit norm.

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \underline{\underline{\Phi_n(t)}}$$

$\searrow$ Fourier basis

We will use

$$\phi_n(t) = e^{jn\omega_0 t}$$ (dropped normality)

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle = \underline{0} \quad m \neq n$$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$$

$$\langle x(t), e^{jn\omega_0 t} \rangle$$

$$= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-jn\omega_0 t} \, dt$$

$$= \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left(\sum_{m=-\infty}^{+\infty} X_m e^{jm\omega_0 t} \right) e^{-jn\omega_0 t} \, dt$$

$$= \int_{-\frac{T_0}{2}}^{T_0/2} \left(\sum_{m=-\infty}^{+\infty} X_m e^{j(m-n)\omega_0 t} \right) dt$$

$$= \sum_{m=-\infty}^{+\infty} \int_{-T_0/2}^{T_0/2} X_m e^{j(m-n)\omega_0 t} \, dt$$

$$= \sum_{m=-\infty}^{+\infty} X_m \underbrace{\int_{-\frac{T_0}{2}}^{T_0/2} e^{j(n-m)\omega_0 t}\, dt}_{=\begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases}}$$

$$= X_n \cdot T_0$$

$$\langle x(t), e^{jn\omega_0 t} \rangle = \underline{\underline{X_n \cdot T_0}}$$

$$X_n = \frac{1}{T_0} \langle x(t), e^{jn\omega_0 t} \rangle$$

$$\boxed{X_n = \frac{1}{T_0} \int_{T_0} x(t) \cdot e^{-jn\omega_0 t}\, dt}$$

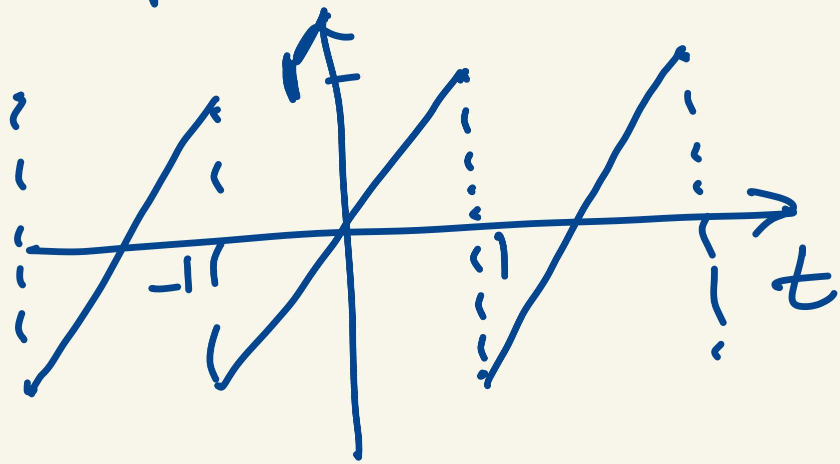
Fourrier Series coeff.

Fourier series for periodic signals.

1. $x(t)$ periodic w/ T_0
2. $\omega_0 = \frac{2\pi}{T_0}$
3. $\left\{ e^{jn\omega_0 t} \right\}_{n=-\infty}^{+\infty}$
4. $X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$
5. $x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_0 t}$

Fourier series representat. of periodic signals.

Example: $x(t)$



$x(t) = t \qquad -1 \le t < 1$

① $T_0 = 2$

② $\omega_0 = \frac{2\pi}{2} = \pi$

③ $\left\{ e^{jn\pi t} \right\}_{n=-\infty}^{+\infty}$

④ $X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt$

$= \frac{1}{2}\int_{-1}^{1} t\,e^{-jn\pi t}\,dt$

cheat sheet

$\int t e^{at}\,dt = \left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at}$

$a = -jn\pi$

$= \frac{1}{2}\left[\frac{t}{-jn\pi} - \frac{1}{(-jn\pi)^2}\right]e^{-jn\pi t}\Big|_{-1}^{1}$

$(-j)^2 = -1$

$$= \frac{1}{2}\left[\frac{jt}{n\pi} e^{-jn\pi t}\right]_{-1}^{1} +$$

$$\frac{1}{2}\left[\frac{1}{n^2\pi^2} e^{-jn\pi t}\right]_{-1}^{1}$$

$$= \frac{\overset{0}{\cancel{1}}}{2}\left[\frac{1}{n\pi} e^{-jn\pi\cdot 1} - \frac{-1}{n\pi} e^{-jn\pi(-1)}\right]$$

$$+ \frac{1}{2}\left[\frac{1}{n^2\pi^2} e^{-jn\pi\cdot 1} - \frac{1}{n^2\pi^2} e^{-jn\pi(-1)}\right] \rightarrow 0$$

$$e^{-jn\pi} = e^{jn\pi}$$

$$= \frac{-j}{2n\pi}\left[e^{-jn\pi} + e^{jn\pi}\right]$$

$$= \frac{\cancel{2} j e^{jn\pi}}{\cancel{2} n\pi}$$

$$= \frac{-j}{n\pi}(-1)^n \qquad n = \pm 1, \pm 2, \ldots$$

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\cdot \cancel{e^{-j\omega_0 t}}\, dt$$

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

DC ("Direct Current") coefficient.

$$X_0 = \frac{1}{2}\int_{2} t\,dt = 0$$

$$x(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{j(-1)^n}{n\pi} e^{jn\pi t}$$