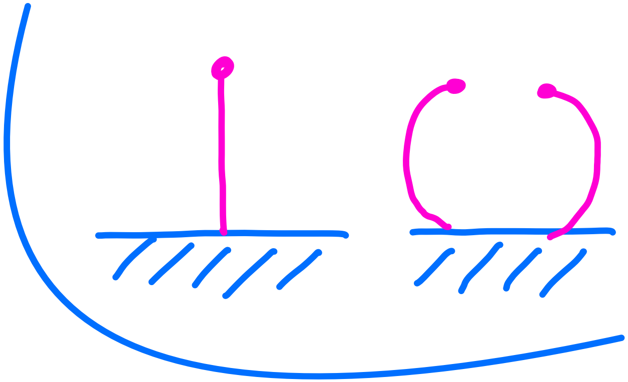
Math 134, Spring 2022

Lecture #9: Bifurcations.

Monday April 18th

Bifurcations

Symmetry



Saddle-node: $\dot{x} = r + x^2$

Transcritical: $\dot{x} = rx - x^2$

Subcritical pitchfork: $\dot{x} = rx + x^3$

$y = -x \to \dot{y} = -\dot{x} \to \dot{x} = -\dot{y}$

$\to x = -y$

SN: $-\dot{y} = r + (-y)^2 \to \dot{y} = -r - y^2$

T: $\dot{y} = ry + y^2$

SP:

$$-\dot{y} = r(-y) + (-y)^3$$

$$\dot{y} = ry + y^3$$

An example: Hysterisis!

$$\dot{x} = rx + x^3 - x^5$$
$$= x(r + x^2 - x^4)$$

$x(t) \in \mathbb{R}$
$r \in \mathbb{R}$

Fixed points: $x = 0$

$r + x^2 - x^4 = 0$

$x^2 = \frac{1}{2}[1 \pm \sqrt{1+4r}]$ if

$\frac{1}{2}[1 \pm \sqrt{1+4r}] \geq 0$

- If $r < -\frac{1}{4}$: $x = 0$ is the only root
- If $r = -\frac{1}{4}$: $x = 0$, $x = \pm\frac{1}{\sqrt{2}}$
- If $-\frac{1}{4} < r < 0$: $x = 0$, $x = \sqrt{\frac{1+\sqrt{1+4r}}{2}}$,

$-1 < 4r < 0$
$0 < 1 + 4r < 1$

$x = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$

$x = \sqrt{\frac{1-\sqrt{1+4r}}{2}}$

$x = -\sqrt{\frac{1-\sqrt{1+4r}}{2}}$

If $r=0$: $x^3-x^5=x^3(1-x^2) \Rightarrow$ $x=0$
$x=\pm 1$

If $r>0$: $x=0$, $x=\sqrt{\frac{1+\sqrt{1+4r}}{2}}$

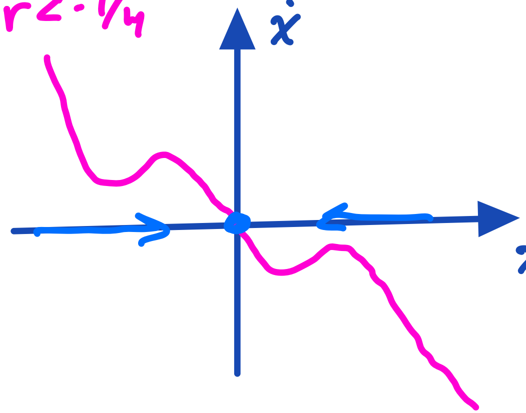
$1+4r>1$

$x=-\sqrt{\frac{1+\sqrt{1+4r}}{2}}$

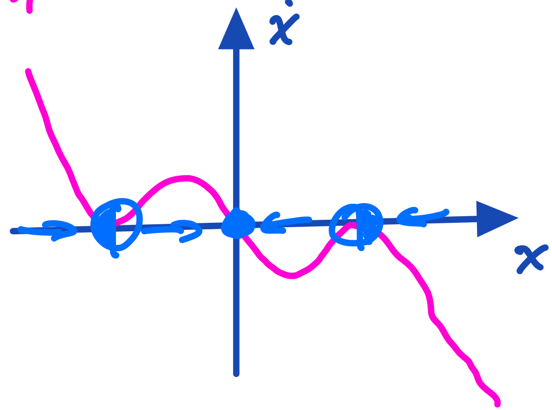
$\sqrt{1+4r}>1$

$$\dot{x} = rx + x^3 - x^5$$

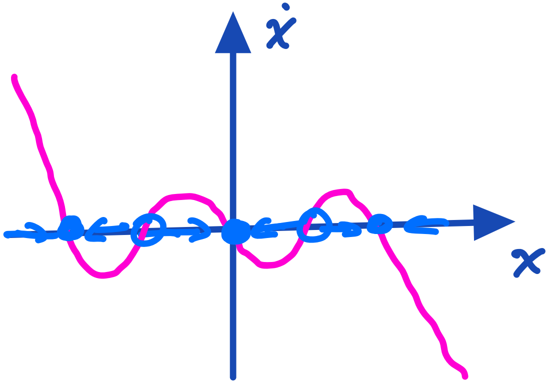
$r < -1/4$



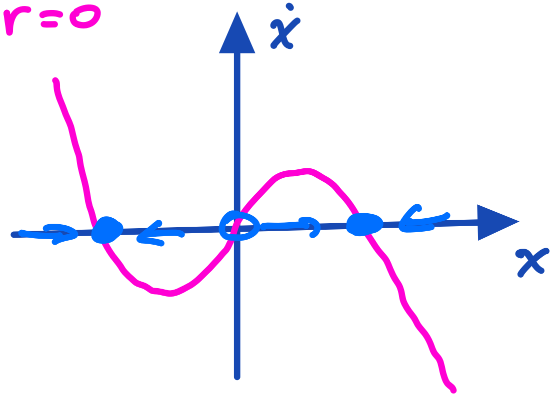
$r = -\frac{1}{4}$

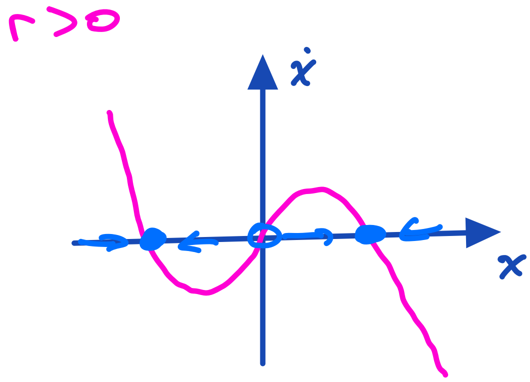


$-\frac{1}{4} < r < 0$

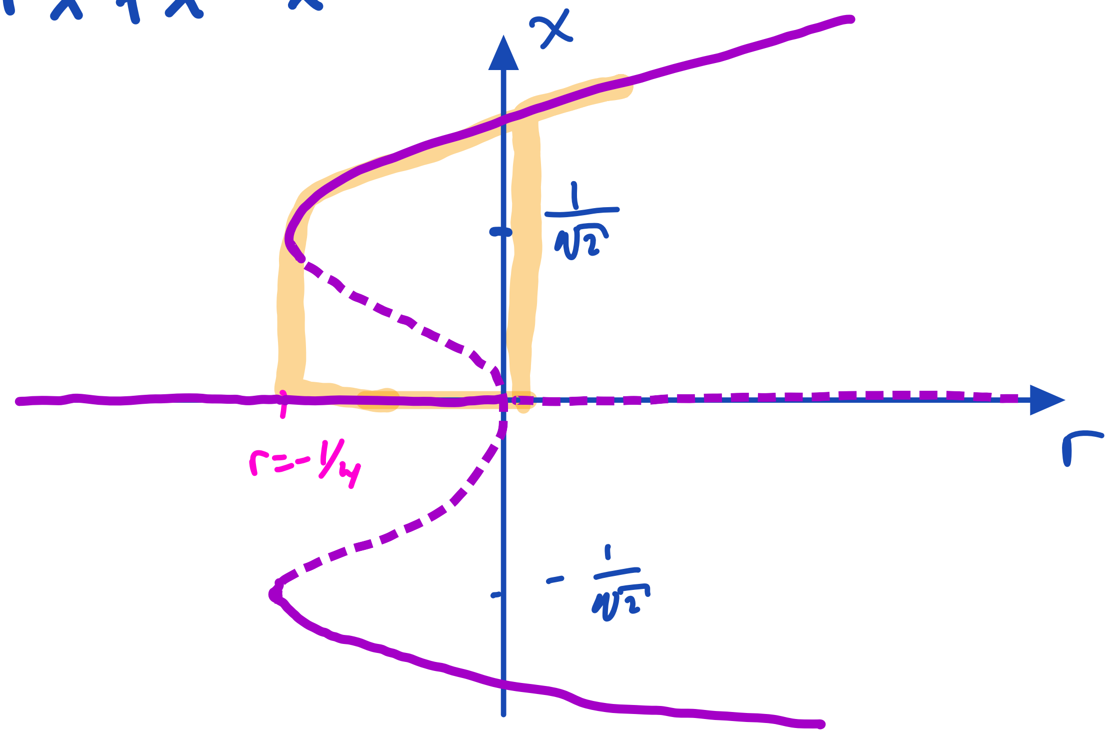


$r = 0$





$$\dot{x} = rx + x^3 - x^5$$



Hysteresis: Non-reversibility as the parameter r varies

$F:\mathbb{R}\to\mathbb{R}$

Taylor's Theorem

$F(t)$ cont. and $\frac{d^n F(t)}{dt^n}$ exists and is cont. for $1 \le n \le N+1$

Taylor's Theorem:

$$F(t) = \sum_{n=0}^{N} \frac{1}{n!} \frac{d^n F(0)}{dt^n} t^n + R_N(t)$$

where $R_N(t) = \underbrace{\frac{1}{(N+1)!} \frac{d^{N+1}}{dt^{N+1}} F(\tilde{t})\, t^{n+1}}_{\text{residue in Lagrange form}}$ for some $\tilde{t} \in (0,t)$.

Ex: $F(t) = e^t$

$$F(t) = \sum_{n=0}^{N} \frac{1}{n!} t^n + \underbrace{R_N(t)}_{\frac{1}{(N+1)!} e^{\tilde{t}} t^{N+1}} \qquad \tilde{t} \in (0,t)$$

Let $f(x,r)$ be smooth (i.e. $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial r^m} f$ is cont. for all $m,n \geq 1$)

Let $F(t) = f(tx, tr)$ and let, note the follow

$$\frac{d^n F}{dt^n}(t) = \sum_{j=0}^{n} \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(tx, tr)\, x^{n-j} r^j$$

$$\binom{n}{j} = \frac{n!}{(n-j)!\, j!}$$

chain rule

$$f(x,r) = F(1) = \sum_{n=0}^{N} \frac{1}{n!} \frac{d^n F}{dt^n}(0) + R_N(1)$$

$$= \sum_{j=0}^{N} \sum_{j=0}^{n} \frac{1}{(n-j)!\, j!} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(0,0)\, x^{n-j} r^j$$

$$+ R_N(1)$$

Taylor's Theorem

Theorem: Suppose that all partial derivatives of $f(x, r)$ up to order $N+1$ are continuous.

Then,

$$f(x,r) = \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{1}{(n-j)!j!} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(0,0) x^{n-j} r^j + R_N(x,r),$$

where the remainder term can be written as

$$R_N(x,r) = \sum_{j=0}^{N+1} \frac{1}{(N+1-j)!j!} \frac{\partial^{N+1} f}{\partial x^{N+1-j} \partial r^j}(\tilde{t}x, \tilde{t}r) x^{N+1-j} r^j,$$

for some $0 < \tilde{t} < 1$.

A special case

$N=2$

$$f(x,r) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial r}(0,0)r$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{\partial^2 f}{\partial x \partial r}(0,0)xr$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial r^2}(0,0)r^2 + R_2(x,r)$$

An example

$\frac{\partial^{n+m} f}{\partial x^n \partial r^m}$ exist for all $n,m \geq 1$.

Consider the function

$$f(x,r) = (r^2 + x)e^x$$

Which of the following is the correct Taylor series expansion to quadratic order at $x = 0,\ r = 0$?

A) $x + \frac{1}{2}x^2 + \frac{1}{2}r^2 + \ldots$

B) $x - x^2 + r^2 + \ldots$

C) $x + r - xr + \ldots$

D) $x + x^2 + r^2 + \ldots$

$$(r^2+x)e^x = (r^2+x)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{h.o.t.}\right)$$
$$= r^2 + r^2 x + \text{h.o.t.}$$
$$+ x + x^2 + \frac{x^3}{2!} + \text{h.o.t.}$$
$$= x + x^2 + r^2 + \text{h.o.t.}$$

$$f(x,r) = f(0,0) + \frac{\partial f}{\partial x}(0,0)\,x + \frac{\partial f}{\partial r}(0,0)\,r$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)\,x^2 + \frac{\partial^2 f}{\partial x \partial r}(0,0)\,xr$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial r^2}(0,0)\,r^2 + R_2(x,r)$$

$$f(0,0) = (r^2 + x)e^x \Big|_{\substack{x=0\\ r=0}} = 0$$

$$\frac{\partial f}{\partial x}(0,0) = \left(e^x + (r^2+x)e^x\right)\Big|_{\substack{x=0\\ r=0}} = 1$$

$$\frac{\partial f}{\partial r}(0,0) = 2re^x \Big|_{\substack{x=0\\ r=0}} = 0$$

$$\frac{\partial^2 f}{\partial r^2}(0,0) = 2 \qquad \frac{\partial^2 f}{\partial x \partial r}(0,0) = 0 \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = 2$$

$$f(x,r) = 0 + x + 0\cdot r + x^2 + \tfrac{1}{2}\cdot 0 \cdot xr + 2\cdot r^2/2 + h.o.t.$$
$$= x + x^2 + r^2 + h.o.t.$$

See you next time!