Math 134, Spring 2022

Lecture #4: Gradient Flows and Numerical Methods

Monday April 2th

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

Learning objectives

Today we will discuss:

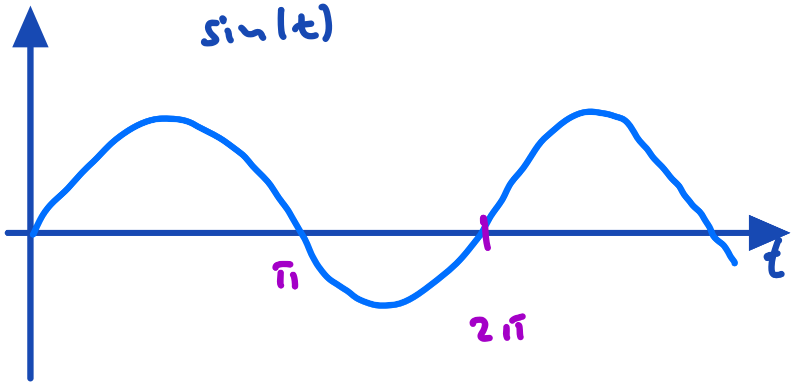
- How to show there are no non-constant periodic solutions of continuous flows on the line.
- How to write an ODE as an integral equation.
- Euler's method for numerically solving flows on the line.
- The local truncation error of a numerical method.
- The improved Euler method.
- The Runge-Kutta method.

Impossibility of oscillations

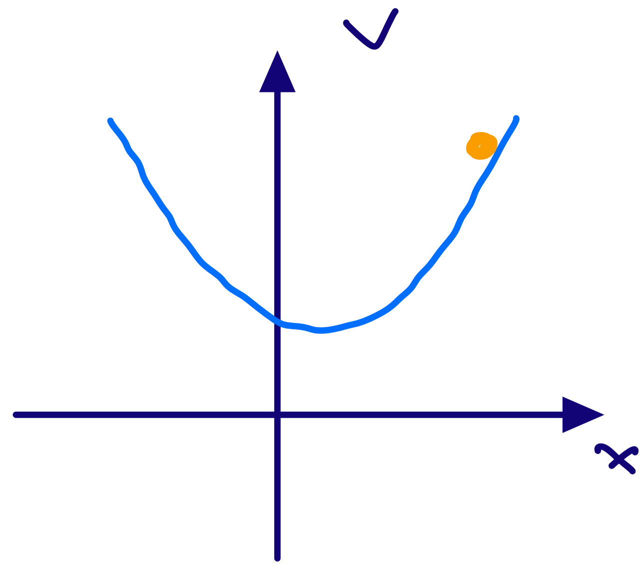
- If there exists a constant $p > 0$ so that for all t we have

$$x(t+p) = x(t)$$

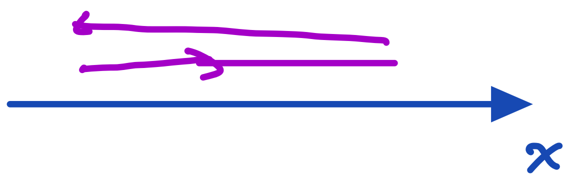
then we say the function x is **periodic**.



$$\sin(t+\underset{\substack{\uparrow\\ p}}{2\pi}) = \sin(t)$$



- All constant functions are periodic.



© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

Theorem: There are no non-constant periodic solutions of the system

$$\dot{x} = f(x).$$

Proof: Suppose that x is a periodic solution with period $p > 0$.

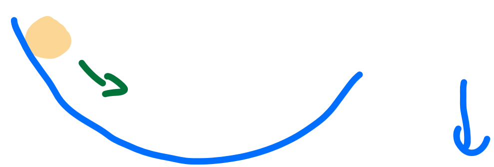
If $0 \le t \le p$ then as the potential energy is non-increasing

$$V[x(p)] \le V[x(t)] \le V[x(0)]$$

As $x(p) = x(0)$ then $V[x(t)] = \text{const}$.

$\Rightarrow$ $x(t) = \text{constant}$.

$\blacksquare$



© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

Numerical methods

Integral equations

We want to find a solution of the equation

IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

$$\frac{dx}{dt} = f(x) \rightarrow dx = f(x)\,dt$$

$$\rightarrow \int_{x_0}^{x} d\bar{x} = \int_{t_0}^{t} f(\bar{x}(\tau))\,d\tau$$

$$\rightarrow x - x_0 = \int_{t_0}^{t} f(\bar{x}(\tau))\,d\tau$$

$$\rightarrow x(t) = x_0 + \int_{t_0}^{t} f(\bar{x}(\tau))\,d\tau$$ integral eqn.

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

An example

Write the equation

$$\begin{cases} \dot{x} = \sin x \\ x(0) = 1 \end{cases}$$

as an integral equation

Numerical approximation

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

Euler's method

- Want to approximate the solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

- Given a timestep Δt, for $n \geq 0$ define

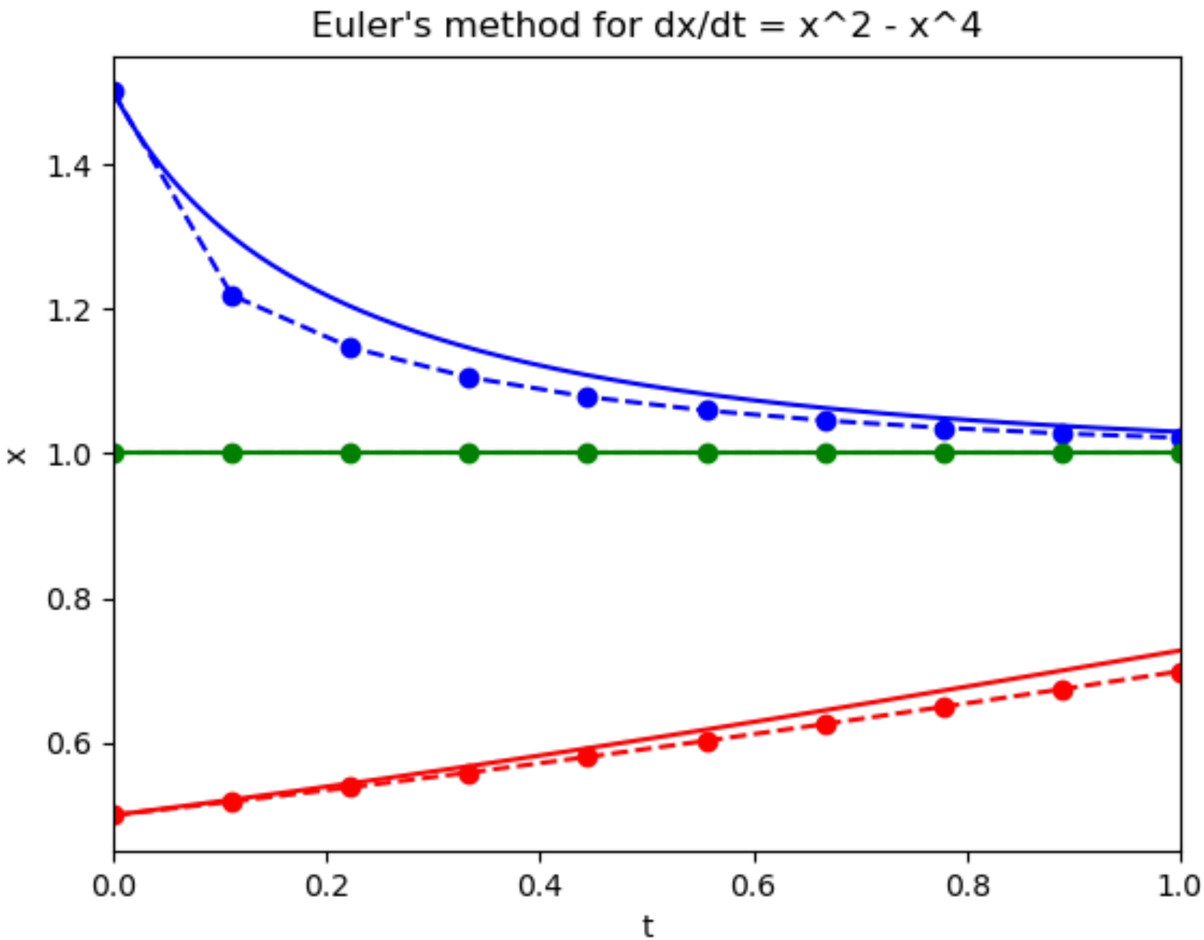
$$x_{n+1} = x_n + f(x_n)\Delta t$$

- Take x_n to be our approximation to $x(n\Delta t)$

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

An example

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.



© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.

See you next time!

© 2022 Pedro Aceves Sanchez. These slides may not be distributed without written permission from the author.