Homework 6

1) a) $\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 3y \end{cases} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\quad tr A = -3$, $\det A = 2$

$\Rightarrow \lambda = \dfrac{-3 \pm \sqrt{9 - 4.2}}{2} = \dfrac{-3 \pm \sqrt{1}}{2} = \left\langle \begin{matrix} -1 \\ -2 \end{matrix} \right.$

With $\lambda = -1$, $(A - I\lambda) v = 0$

$\Rightarrow \left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Leftrightarrow \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} v_1 + v_2 = 0 \\ -2v_1 - 2v_2 = 0 \end{cases}$

$\Rightarrow v_1 = -v_2 \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

With $\lambda = -2$, we have:

$\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

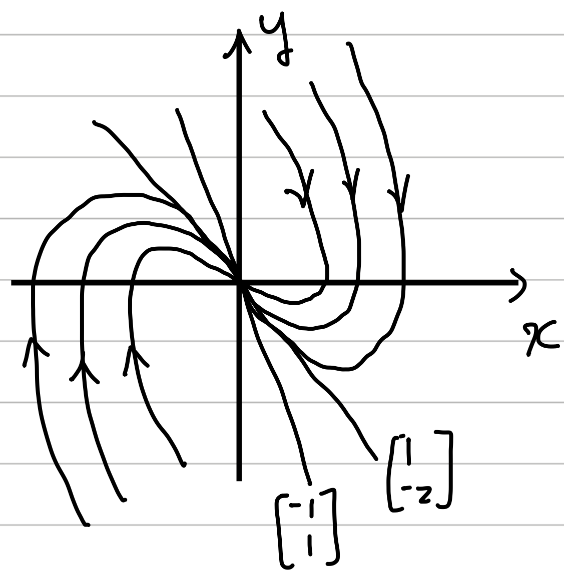
$\Rightarrow \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Leftrightarrow 2v_1 + v_2 = 0$

$\Rightarrow v_2 = -2v_1$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -2v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} v_1 \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We have $\lambda_1 = -1$, $\lambda_2 = -2 \Rightarrow \lambda_2 < \lambda_1 < 0$

$\Rightarrow$ $\boxed{\text{fixed point } (x^*, y^*) = (0,0) \text{ is a stable node}}$



b) $\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = -2x + 3y \end{cases} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \operatorname{tr} A = 0$, $\det A = -9 + 8 = -1$

$$\Rightarrow \lambda = \frac{\pm\sqrt{0 - 4.(-1)}}{2} = \pm 1$$

with $\lambda = 1$, $(A - \lambda I)\, v = 0$

$$\Rightarrow \left(\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow -v_1 + v_2 = 0$$

$$\Rightarrow v_1 = v_2$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

With $\lambda = -1$, $\left(\begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

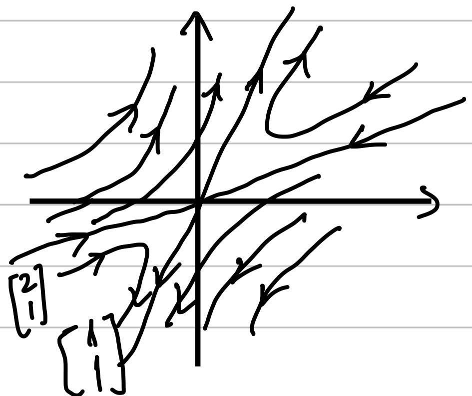
$$\Rightarrow \begin{bmatrix} -2 & 4 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow -v_1 + 2v_2 = 0$$

$$\Rightarrow v_1 = 2v_2$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_2 \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We have $\lambda_1 = 1$, $\lambda_2 = -1 \Rightarrow \lambda_2 < 0 < \lambda_1$

$$\boxed{\Rightarrow (x^*, y^*) = (0,0) \text{ is saddle point}}$$



c) $\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 8x - 6y \end{cases} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \begin{cases} \mathrm{tr} A = -2 \\ \det A = -24 + 24 = 0 \end{cases}$

$\Rightarrow \lambda = \dfrac{-2 \pm \sqrt{4 - 4.0}}{2} = \dfrac{-2 \pm 2}{2} = \left\langle \begin{matrix} 0 \\ -2 \end{matrix} \right.$

$\oplus$ $\lambda = 0 \Rightarrow A v = 0 \Rightarrow \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow 4v_1 - 3v_2 = 0 \Rightarrow v_1 = \dfrac{3v_2}{4}$

$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{3v_2}{4} \\ v_2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix} v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

$\lambda = -2 \Rightarrow (A - \lambda I) v = 0$

$\Leftrightarrow \left(\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

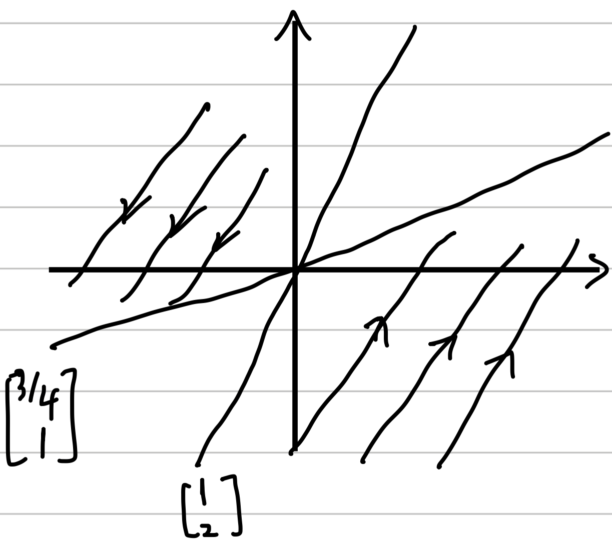
$\Rightarrow \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow 2v_1 - v_2 = 0$

$\Rightarrow v_2 = 2v_1$

$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 2v_1 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

We have $\lambda_1 = 0$ & $\lambda_2 = -2$, then the fixed point

$\boxed{(x^*, y^*) = (0,0) \text{ is a non-isolated point}}$



d) $\begin{cases} \dot{x} = 6x - y \\ \dot{y} = 2x + 3y \end{cases} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \operatorname{tr} A = 9, \ \det A = 18 + 2 = 20$

$\Rightarrow \lambda = \dfrac{9 \pm \sqrt{81 - 4.20}}{2} = \dfrac{9 \pm 1}{2} = \begin{cases} 5 \\ 4 \end{cases}$

$\lambda = 5 \Rightarrow \left(\begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Leftrightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 = v_2 \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_1$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

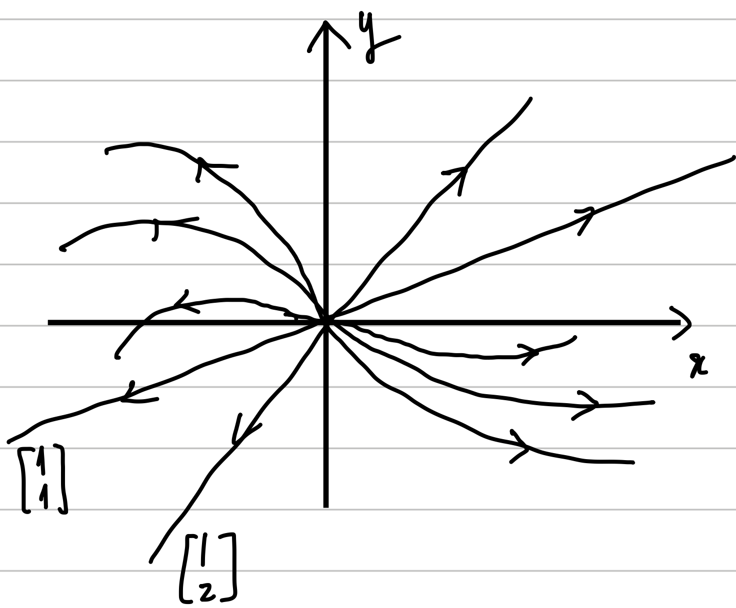
$$\lambda = 4 \Rightarrow \left(\begin{bmatrix}6 & -1\\ 2 & 3\end{bmatrix} - \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix}2 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0 \Rightarrow 2v_1 = v_2$$

$$\Rightarrow \begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}v_1\\ 2v_1\end{bmatrix} = \begin{bmatrix}1\\ 2\end{bmatrix} v_1$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix}1\\ 2\end{bmatrix}$$

We have $\lambda_1 = 5 > \lambda_2 = 4 > 0 \Rightarrow (x^*, y^*) = (0,0)$ is an unstable fixed point



2) Given $\begin{cases} \dot{x} = x + y^2 + \frac{3}{2} y = f_1(x,y) \\ \dot{y} = x + y = f_2(x,y) \end{cases}$

a) we have $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{cases} x + y^2 + \frac{3}{2} y = 0 & (1) \\ x + y = 0 \Leftrightarrow x = -y & (2) \end{cases}$

$\Rightarrow$ from (1) $\Leftrightarrow \quad x + x^2 + \frac{3}{2}(-x) = 0$

$\Leftrightarrow \quad x^2 - \frac{1}{2} x = 0 \Leftrightarrow \left[\begin{array}{l} x = 0 \\ x = \frac{1}{2} \end{array}\right.$

$\Rightarrow$ fixed points: $\boxed{\begin{cases} x^* = 0 \\ y^* = 0 \end{cases} \quad \& \quad \begin{cases} x^* = \frac{1}{2} \\ y^* = -\frac{1}{2} \end{cases}}$

We have $\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 2y + \frac{3}{2} \\ 1 & 1 \end{bmatrix}$

+ $(x^*, y^*) = (0,0)$

$\Rightarrow \nabla f = \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix} \Rightarrow \boxed{\dot{\eta} = \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix} \eta}$

+ $(x^*, y^*) = \left(\frac{1}{2}, -\frac{1}{2}\right)$

$\Rightarrow \nabla f = \begin{bmatrix} 1 & -1 + 3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$

$\Rightarrow \boxed{\dot{\eta} = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix} \eta}$

b) With $\dot{\eta} = \underbrace{\begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix}}_{A} \eta$ $\Rightarrow \operatorname{tr} A = 2$, $\det = 1 - \frac{3}{2} = -\frac{1}{2}$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4 + 4 \cdot \frac{1}{2}}}{2} = \frac{2 \pm \sqrt{6}}{2}$$

we have $\lambda_1 = \frac{2 - \sqrt{6}}{2} < 0 < \lambda_2 = \frac{2 + \sqrt{6}}{2}$

$\Rightarrow$ $\boxed{\text{fixed point } (0,0) \text{ is a saddle point}}$

With $\dot{\eta} = \underbrace{\begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}}_{A} \eta$ $\Rightarrow \operatorname{tr} A = 2$, $\det A = 1 - \frac{1}{2} = \frac{1}{2}$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4 - 4 \cdot \frac{1}{2}}}{2} = \frac{2 \pm \sqrt{2}}{2}$$

Because $\lambda_1 = \frac{2 + \sqrt{2}}{2} > \lambda_2 = \frac{2 - \sqrt{2}}{2} > 0$

$\Rightarrow$ $\boxed{\text{fixed point } \left(\frac{1}{2}, -\frac{1}{2}\right) \text{ is a unstable point}}$

3) Form $A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$, $b \neq 0$

$\Rightarrow \operatorname{tr} A = 2\lambda$, $\det A = \lambda^2$

$$\Rightarrow \lambda = \frac{2\lambda \pm \sqrt{4\lambda^2 - 4\lambda^2}}{2} = \lambda$$

$\Rightarrow$ has only 1 eigenvalue λ

$$\Rightarrow (A - \lambda I)v = 0 \Leftrightarrow \left(\begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow bv_2 = 0 \Rightarrow v_2 = 0$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ Has only one dimensional eigenspace $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the eigenvalue λ.

For $\dot{x} = Ax \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow \begin{cases} \dot{x}_1 = \lambda x_1 + b x_2 & (1) \\ \dot{x}_2 = \lambda x_2 & (2) \end{cases}$$

$$(2) \Leftrightarrow \frac{dx_2}{dt} = \lambda x_2 \Rightarrow \frac{dx_2}{x_2} = \lambda dt \Rightarrow \int \frac{dx_2}{x_2} = \lambda \int dt$$

$$\Rightarrow \ln x_2 = \lambda t + C \Rightarrow x_2 = e^{\lambda t + C} = Ce^{\lambda t}$$

$$\Rightarrow (1) \Leftrightarrow \frac{dx_1}{dt} = \lambda x_1 + bce^{\lambda t}$$

$$\Rightarrow \dot{x}_1 - \lambda x_1 = bce^{\lambda t}$$

$$\begin{cases} p(t) = -\lambda \\ q(t) = bc\,e^{\lambda t} \end{cases} \Rightarrow \mu(t) = e^{\int -\lambda\, dt} = e^{-\lambda t}$$

$$\Rightarrow x_1(t) = e^{\lambda t} \int e^{-\lambda t}\, bc\, e^{\lambda t}\, dt + \frac{D}{e^{-\lambda t}}$$

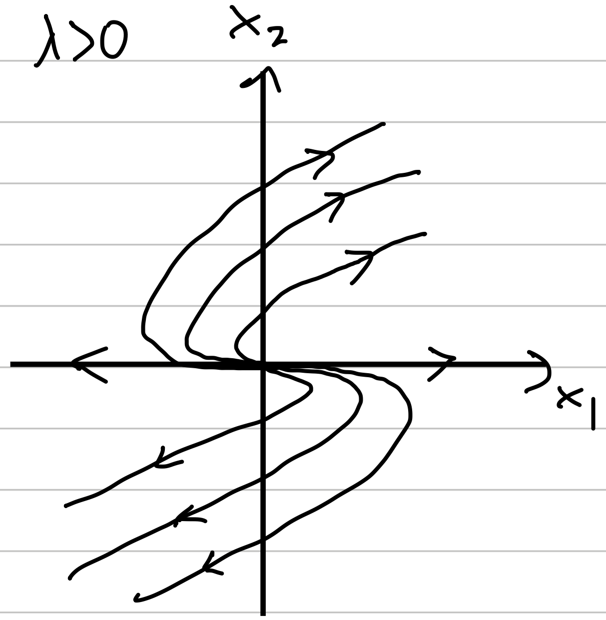
$$\Rightarrow x_1(t) = e^{\lambda t} \int bc\, dt + De^{\lambda t}$$

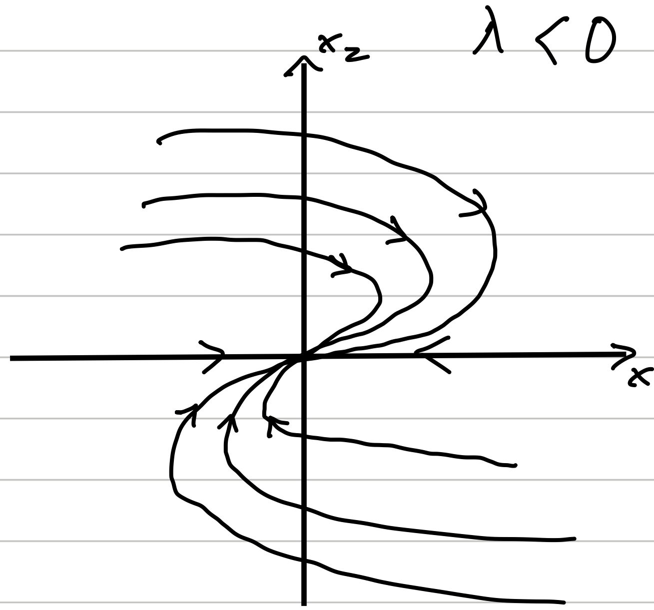
$$= e^{\lambda t} . bct + De^{\lambda t}$$

$$\Rightarrow x_1(t) = e^{\lambda t} [bct + D]$$

$$\boxed{\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda t}[bct + D] \\ ce^{\lambda t} \end{bmatrix} = \begin{bmatrix} e^{\lambda t} t\, bc + De^{\lambda t} \\ ce^{\lambda t} \end{bmatrix} = \begin{bmatrix} bc & D \\ D & c \end{bmatrix} \begin{bmatrix} e^{\lambda t} t \\ e^{\lambda t} \end{bmatrix}}$$

Phase potrait:


λ>0
x2
x1


x2
λ < 0
x1

$\boxed{\text{4) Given } A \text{ is } 2\times 2 \text{ matrix}}$

$$\begin{cases} \dot{u} = Au \\ u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \qquad \begin{cases} \dot{v} = Av \\ v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases} \qquad \phi(t) = [u(t) \;\; v(t)]$$

a) With $x(t) = \phi(t)\, x_0 = [u(t) \;\; v(t)]\, x_0$

$$\Rightarrow \dot{x} = [\dot{u} \;\; \dot{v}]\, x_0 = [Au \;\; Av]\, x_0$$
$$= A[u \;\; v]\, x_0 = A\phi(t)\, x_0$$

$$\Rightarrow \dot{x} = A\,\phi(t).x_0 = Ax(t) \Leftrightarrow \dot{x} = Ax(t) \quad (1)$$

Also, $x(t) = \phi(t)\, x_0$

$$\Rightarrow x(0) = \phi(0)\, x_0 = [u(0) \;\; v(0)]\, x_0$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_0$$

$$\Rightarrow x(0) = x_0 \quad (2)$$

From (1) & (2), $x(t) = \phi(t)\, x_0$ sastifies the equation:

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

$\Rightarrow$ $x(t) = \phi(t)\, x_0$ is the solution of above equation

b) We have:

$$\begin{cases} \dot{u} = Au \\ \dot{x} = Ax \end{cases} \Rightarrow \frac{\dot{u}}{\dot{x}} = \frac{u}{x} \Leftrightarrow \frac{du}{dx} = \frac{u}{x}$$

$$\Rightarrow \frac{du}{u} = \frac{dx}{x} \Rightarrow \begin{cases} \frac{du_1}{u_1} = \frac{dx_1}{x_1} \\ \frac{du_2}{u_2} = \frac{dx_2}{x_2} \end{cases}$$

$$\Rightarrow \begin{cases} \ln u_1 = \ln x_1 + C_1 \\ \ln u_2 = \ln x_2 + C_2 \end{cases} \Leftrightarrow \begin{cases} u_1(t) = C_1 x_1(t) \\ u_2(t) = C_2 x_2(t) \end{cases}$$

Since $u(0) = \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$

$$\Rightarrow u(0) = \begin{bmatrix} C_1 x_1(0) \\ C_2 x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} C_1 = 1/x_1(0) \\ C_2 = 0 \end{cases}$$

$$\Rightarrow u(t) = \begin{bmatrix} \frac{1}{x_{01}} x_1(t) \\ 0 \end{bmatrix} = x_1(t) \begin{bmatrix} \frac{1}{x_{01}} \\ 0 \end{bmatrix}$$

Similarly, we also have:

$$\begin{cases} \ln v_1 = \ln x_1 + D_1 \\ \ln v_2 = \ln x_2 + D_2 \end{cases} \Rightarrow \begin{cases} v_1(t) = D_1 x_1(t) \\ v_2(t) = D_2 x_2(t) \end{cases}$$

$$\Rightarrow v(0) = \begin{bmatrix} V_1(0) \\ V_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} D_1 x_1(0) \\ D_2\, x_2(0) \end{bmatrix} \Rightarrow \begin{cases} D_1 = 0 \\ D_2 = \dfrac{1}{x_2(0)} \end{cases}$$

$$\Leftrightarrow \begin{cases} D_1 = 0 \\ D_2 = \dfrac{1}{x_{02}} \end{cases} \Rightarrow v(t) = \begin{bmatrix} 0 \\ \dfrac{1}{x_{02}} x_2(t) \end{bmatrix}$$

$$\Rightarrow v(t) = x_2(t) \begin{bmatrix} 0 \\ \dfrac{1}{x_{02}} \end{bmatrix}$$

$$\Rightarrow \phi(t) = \begin{bmatrix} \dfrac{1}{x_{01}} x_1(t) & 0 \\ 0 & \dfrac{1}{x_{02}} x_2(t) \end{bmatrix}$$

$$\Rightarrow \phi(t) = \begin{bmatrix} \dfrac{1}{x_{01}} & 0 \\ 0 & \dfrac{1}{x_{02}} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\Rightarrow \phi(t) = \underbrace{\begin{bmatrix} \dfrac{1}{x_{01}} & 0 \\ 0 & \dfrac{1}{x_{02}} \end{bmatrix}}_{E} x(t)$$

$$\Rightarrow \phi(t) = E\, x(t) = E \,.\, \phi(t) \,.\, x_0$$

$$\Rightarrow \phi^{-1}(t)\, \phi(t) = \phi^{-1}(t)\, E\, \phi(t)\, x_0$$

$$\Rightarrow \quad \phi^{-1}(t)\, E\, \phi(t)\, x_0 = \mathbb{I}$$

$$\boxed{\begin{array}{l} \text{Therefore } \phi(t) = \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} x(t) \text{ with } x(t) \text{ is the} \\ \text{solution of } \begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \text{ OR } \phi(t) \text{ is a matrix satisfies} \\ \qquad \phi^{-1}(t) \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} \phi(t)\, x_0 = \mathbb{I} \end{array}}$$

c) P is a matrix 2×2 and $B = PMP^{-1}$

we have $x(t) = P\,\phi(t)\,P^{-1}x_0$

$$\Rightarrow \quad \dot{x} = P.\,\dot{\phi}(t)\,P^{-1}x_0$$

As the beginning, we have $\phi(t) = [u(t) \quad v(t)]$

$$\Rightarrow \quad \dot{x} = P[\dot{u}(t) \quad \dot{v}(t)]\,P^{-1}x_0 = P[Au \quad Av]\,P^{-1}x_0$$

$$= PA[u \quad v]\,P^{-1}x_0 \quad = PA\,\phi(t)\,P^{-1}x_0$$

$$= PA\underbrace{P^{-1}.P}_{\mathbb{I}}\,\phi(t)\,P^{-1}x_0$$

$$= PAP^{-1}.\ P\,\phi(t)\,P^{-1}x_0$$

$$= \underbrace{PAP^{-1}}_{B}.\ x(t)$$

$\Rightarrow \dot{x} = PAP^{-1} x(t)$, Let $M = A$ & $B = PAP^{-1}$

$\Leftrightarrow B = PMP^{-1}$

$\Rightarrow \dot{x}(t) = B.x(t)$ (1)

Also, with $x(t) = P.\phi(t)P^{-1} x_0$

$$\Rightarrow x(0) = P\begin{bmatrix} u(0) & v(0) \end{bmatrix} P^{-1} x_0$$

$$= P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} x_0 = PP^{-1}x_0 = \mathbb{I}x_0$$

$$= x_0$$

$\Rightarrow x(0) = x_0$ (2)

Based on (1) & (2), $x(t) = P\phi(t)P^{-1}x_0$ sastifies the equation

$$\begin{cases} \dot{x} = Bx \text{ with } B = PAP^{-1} \\ x(0) = x_0 \end{cases} \quad (\mathrm{I})$$

$\Rightarrow$ $\boxed{x(t) = P\phi(t)P^{-1}x_0 \text{ is the solution of } (\mathrm{I})}$

d) $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(i) $A = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \quad A = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} \quad \text{or} \quad \begin{cases} \dot{x} = Bx \\ x(0) = x_0 \end{cases} \quad B = PMP^{-1} = A$$

$$\operatorname{tr} A = 4, \quad \det A = -5 + 8 = 3$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \begin{matrix} \nearrow 3 \\ \searrow 1 \end{matrix}$$

$\Rightarrow \boxed{M = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}}$ is one of the real canonical form

* Applying for b & c

With $\lambda = 3$, $(A - \lambda I)v = 0$

$$\Rightarrow \left(\begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Leftrightarrow v_1 + 2v_2 = 0$$

$$\Leftrightarrow v_1 = -2v_2$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} v_2 \Rightarrow \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

With $\lambda = 1$ $\Rightarrow \left(\begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 = -v_2$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ general solution :

$$x(t) = C_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow x(0) = C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2C_1 - C_2 = x_{01} \\ C_1 + C_2 = x_{02} \end{cases} \Rightarrow \begin{cases} C_1 = -x_{01} - x_{02} = a_1 \\ C_2 = 2x_{02} + x_{01} = a_2 \end{cases}$$

$$\Rightarrow x(t) = a_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + a_2 e^{t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2a_1 e^{3t} - a_2 e^{t} \\ a_1 e^{3t} + a_2 e^{t} \end{bmatrix} = \begin{bmatrix} -2a_1 & -a_2 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{t} \end{bmatrix}$$

$$\Rightarrow \phi(t) = \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} x(t)$$

$$\Rightarrow \boxed{\phi(t) = \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} \begin{bmatrix} -2a_1 & -a_2 \\ a_1 & a_2 \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{t} \end{bmatrix}}$$

with $a_1 = -x_{01} - x_{02}$ & $a_2 = 2x_{02} + x_{01}$

ii) $A = \begin{bmatrix} 1 & 5 \\ -2 & -5 \end{bmatrix}$

$\operatorname{tr} A = -4$, $\det A = -5 + 10 = 5$

$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2j}{2} = \begin{cases} -2 + j \\ -2 - j \end{cases}$$

$\Rightarrow \alpha = -2, \beta = 1$

$\Rightarrow \boxed{M = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}}$ is one of the real canonical form

Applying for part b & c, we have

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$ has general solution:

$$x(t) = \begin{bmatrix} Ae^{\alpha t} \sin(\beta t + B) \\ Ae^{t} \cos(\beta t + B) \end{bmatrix} \Rightarrow x(0) = \begin{bmatrix} A \sin B \\ A \cos B \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

$$\Rightarrow \operatorname{tg} B = \frac{x_{01}}{x_{02}} \Rightarrow B = \operatorname{tg}^{-1} \frac{x_{01}}{x_{02}}$$

$$\Rightarrow A = \frac{x_{02}}{\cos\left(\operatorname{tg}^{-1} \frac{x_{01}}{x_{02}}\right)}$$

$$\Rightarrow \phi(t) = \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} x(t) = \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} \begin{bmatrix} Ae^{2t} \sin(\beta t + B) \\ Ae^{t} \cos(\beta t + B) \end{bmatrix}$$

with $x_{01} = x_1(0)$, $x_{02} = x_2(0)$, $A = \dfrac{x_{02}}{\cos\left(\operatorname{tg}^{-1} \frac{x_{01}}{x_{02}}\right)}$, $B = \operatorname{tg}^{-1} \dfrac{x_{01}}{x_{02}}$

$x_{02} \neq 0$

iii) $A = \begin{bmatrix} -2 & 4 \\ -1 & -6 \end{bmatrix}$ $trA = -8$, $\det A = 12 + 4 = 16$

$\Rightarrow \lambda = \dfrac{-8 \pm \sqrt{64-64}}{2} = -4 \Rightarrow$ repeated eigence value

$\Rightarrow \boxed{M = \begin{bmatrix} -4 & 1 \\ 0 & -4 \end{bmatrix}}$ is one of the real canonical form

$\oplus$ $(A - \lambda I)v = 0 \Rightarrow \left(\begin{bmatrix} -2 & 4 \\ -1 & -6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 + 2v_2 = 0$

$\Rightarrow v_1 = -2v_2$

$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} v_2 \Rightarrow \vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

We have:

$$A - \sigma I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$$

$\Rightarrow$ $x = 2,\ y = 4,\ z = -1$

$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} q \\ p \end{bmatrix} \Rightarrow q = -2$

$\Rightarrow W = \begin{bmatrix} y \\ -x + \frac{q}{y} \end{bmatrix} = \begin{bmatrix} 4 \\ -2 + \frac{-2}{4} \end{bmatrix} = \begin{bmatrix} 4 \\ -5/2 \end{bmatrix}$

$$\Rightarrow \vec{w} = \begin{bmatrix} 4 \\ -5/2 \end{bmatrix}$$

$\Rightarrow$ general solution:

$$x(t) = (A+Bt)e^{-4t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + Be^{-4t}\begin{bmatrix} 4 \\ -5/2 \end{bmatrix}$$

$$\Rightarrow x(0) = A\begin{bmatrix} -2 \\ 1 \end{bmatrix} + B\begin{bmatrix} 4 \\ -5/2 \end{bmatrix}$$

$$= \begin{bmatrix} -2A + 4B \\ A - \frac{5B}{2} \end{bmatrix} = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2A + 4B = x_{01} \\ 2A - 5B = 2x_{02} \end{cases} \Rightarrow \begin{cases} B = -x_{01} - 2x_{02} \\ A = -4x_{02} - \frac{5}{2}x_{01} \end{cases}$$

$$\boxed{\begin{array}{l} \Rightarrow \phi(t) = \begin{bmatrix} \frac{1}{x_{01}} & 0 \\ 0 & \frac{1}{x_{02}} \end{bmatrix} x(t) \\ \\ \text{with } x(t) = (A+Bt)e^{-4t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + Be^{-4t}\begin{bmatrix} 4 \\ -5/2 \end{bmatrix} \\ \\ A = -4x_{02} - 2.5x_{01} \ \& \ B = -x_{01} - 2.x_{02} \end{array}}$$

5) a) $\begin{cases} \dot{x} = x^2 = 0 \\ \dot{y} = y = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\Rightarrow$ the origin is the only fixed point.

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ <u>linearization</u> : $\dot{\eta} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \eta$

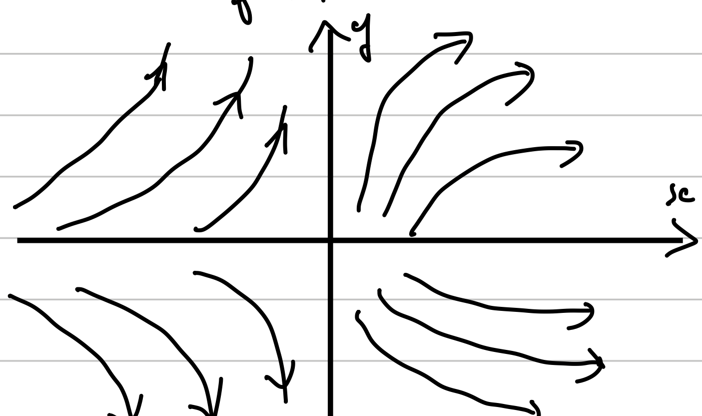
$\Rightarrow \operatorname{tr} A = 1$
$\Rightarrow \det A = 0$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1-0}}{2} = \begin{cases} 1 \\ 0 \end{cases}$$

We have $\lambda_1 = 0$ & $\lambda_2 = 1 \neq 0$

$\Rightarrow$ the fixed point $(x^*, y^*) = (0, 0)$ is a non isolated critical point

<u>Phase potrait</u> :



As we can see this is matched with the prediction of the linear system

b) $\begin{cases} \dot{x} = y = 0 \\ \dot{y} = x^2 = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \\ x = 0 \end{cases}$

$\Rightarrow$ there is only one fixed point as original point

$(x^*, y^*) = (0,0)$

We have: $\nabla f = \begin{bmatrix} 0 & 1 \\ 2x & 0 \end{bmatrix} \Rightarrow \nabla f = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

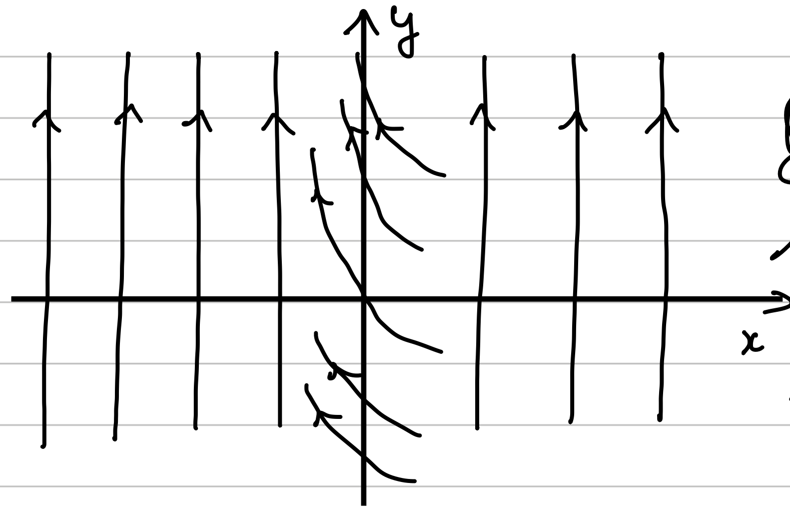
$\Rightarrow$ the corresponding linearization:

$\dot{\eta} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \eta \qquad \begin{matrix} \mathrm{tr} A = 0 \\ \det A = 0 \end{matrix}$

$\Rightarrow \lambda = \dfrac{0 \pm \sqrt{0}}{2} = 0$

$\Rightarrow$ $\boxed{\text{the linearization predict the fixed point as a non-isolated fixed point.}}$

Phase potrait:



Based on the phase potrait, we can see it looks like the prediction of the linear system

c) $\begin{cases} \dot{x} = x^2 + xy = 0 \Rightarrow x(x+y) = 0 \\ \dot{y} = \frac{1}{2}y^2 + xy = 0 \Rightarrow y(\frac{1}{2}y + x) = 0 \end{cases}$

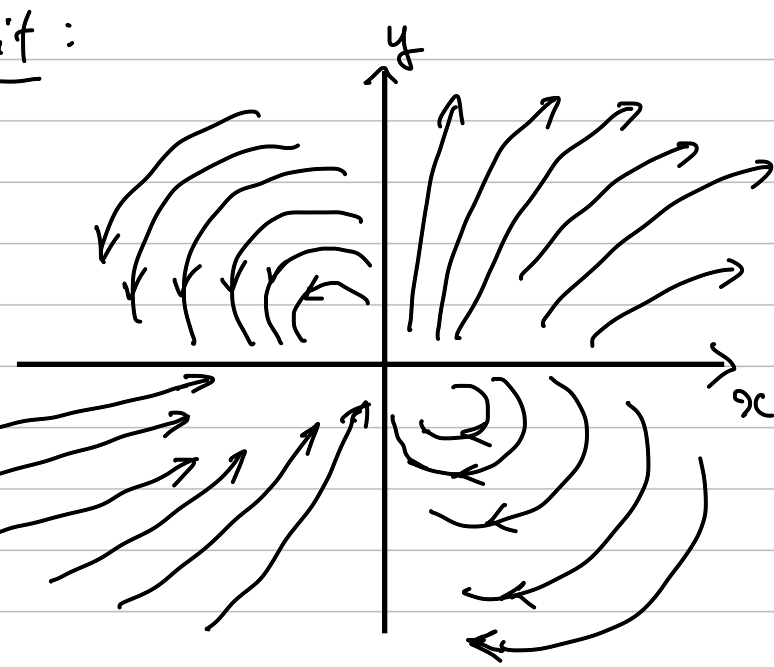
$\Leftrightarrow \begin{cases} \left[\begin{array}{l} x = 0 \\ x = -y \end{array}\right. \\ \left[\begin{array}{l} y = 0 \\ \frac{1}{2}y = -x \end{array}\right. \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$ $\Rightarrow$ the origin is only the fixed point

We have: $\nabla f = \begin{bmatrix} 2x + y & x \\ y & y + x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \operatorname{tr} A = 0, \det A = 0$

$\Rightarrow \lambda = 0 \Rightarrow$ $\boxed{\text{This is a non-isolated fixed point}}$

Phase potrait:



It looks like the same as the prediction of the linear system.