Math 134, Spring 2022

Lecture #3: Linear stability analysis & potentials

Friday April 1^{st}

Slides and lecture recording

- The lecture will be recorded and posted to the Canvas page after class. You are not allowed to store or record the lectures by any other means.

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Friday April 1^{st}

Last time

- We considered continuous flows on the line given by the ODE
$$\dot{x} = f(x).$$
- We said that x^* is a fixed point if $f(x^*) = 0$.
- We discussed what it means to say a fixed point is stable, unstable, and half-stable.

Learning objectives

Today we will discuss:

- How to compute the linearization about a fixed point.
- How to use the linearization to determine the stability of a fixed point.
- How to find the potential function for a flow on the line.
- How to show there are no non-constant periodic solutions of continuous flows on the line (time permitting).

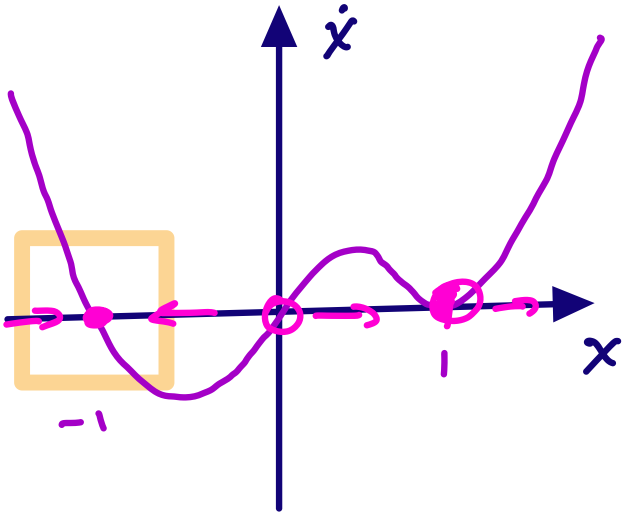
Linear stability analysis

Question: Can we say more about what happens close to fixed points?

$$\dot{x} = x(x+1)(x-1)^2$$

(Local method)

Linearization



$$\eta(t) = x(t) - (-1)$$

$$x(t) = \eta(t) - 1$$

$$\dot{x}(t) = \dot{\eta}(t)$$

$$x(x+1)(x-1)^2 = (\eta-1)(\eta)(\eta-2)^2$$

$$= -4\eta + O(\eta^2)$$

$$\dot{\eta} \simeq -4\eta$$

$$\rightarrow \quad \dot{\eta} = -4\eta$$

$$x(t) = \eta(t) - 1$$

$$\rightarrow \quad g(\eta) = O(\eta^2) \text{ as } \eta \to 0$$

$$|g(\eta)| \leq C\eta^2$$

$$\eta^2, \eta^3$$

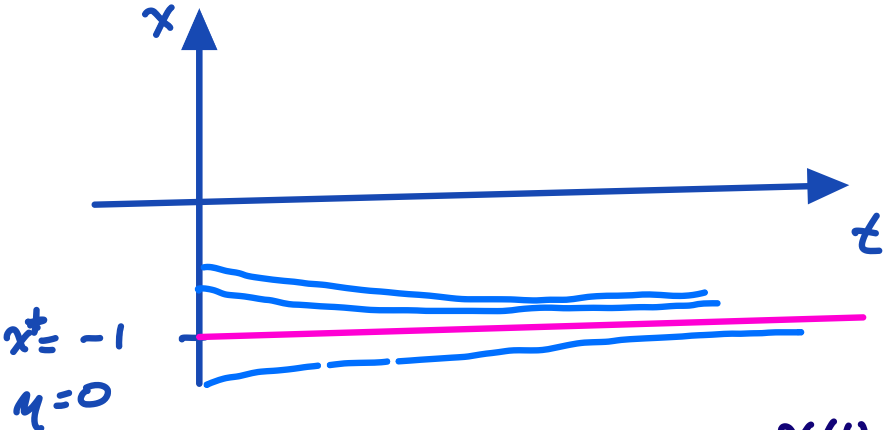
$$\dot{x} = f(x)$$

Near $x=-1$, $y=x+1$ and it satisfies

$$\dot{y} = -4y \qquad \int \frac{dy}{y} = -\int 4\,dt$$

$$y(t) = Ce^{-4t} \qquad \ln|y| = -4t + A$$

$$y = Ce^{-4t}$$



$$x(t) = y(t) - 1 \xrightarrow[t\to\infty]{} -1$$

The linearization

Suppose that x^* is a fixed point of the system

$$\dot{x} = f(x).$$

The **linearization** of this equation about x^* is the equation

$$\dot{\eta} = f'(x^*)\eta.$$

Assume that x^* is a fixed point, i.e. $f(x^*)=0$

Let $\eta = x - x^*$. Then

$$\dot{\eta} = \dot{x} = f(x)$$

Taylor's Theorem $\rightarrow$

$$= \cancel{f(x^*)}^{\,0} + f'(x^*)\eta + \tfrac{1}{2}f''(x^*)\eta^2 + \cdots$$

$$= f'(x^*)\eta + O(\eta^2), \text{ as } \eta \to 0$$

The eqn. $\dot{\eta} = f'(x^*)\eta$ is the linearization at $x = x^*$

$$\eta(t) = Ce^{f'(x^*)t} \xrightarrow[t\to\infty]{} \begin{cases} 0, & f'(x^*)<0 \\ \pm\infty, & f'(x^*)>0 \end{cases}$$

$$x(t) = x^* + \eta$$

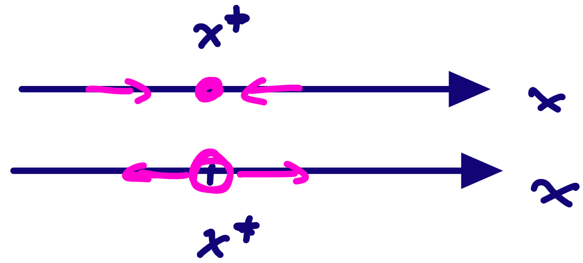
$$\begin{cases} \dot{\eta} = f'(x^*)\eta \\ \eta(0) = \eta \end{cases}$$

Theorem: Suppose that x^* is a fixed point of the system

$$\dot{x} = f(x).$$

Then, if

- $f'(x^*) < 0$, the fixed point x^* is stable.
- $f'(x^*) > 0$, the fixed point x^* is unstable.



Proof (sketch)

Look at the linearization, $\eta = x - x^*$

$$\dot{\eta} = f'(x^*)\,\eta \quad \rightarrow \quad x(t) = x^* + \eta(t)$$

An example

Determine the stability of the fixed point $x^* = 0$ of

$$\dot{x} = \underbrace{\int_0^x e^{-\frac{1}{2}y^2}\, dy - 2x.}_{f(x)}$$

A) Stable

B) Unstable

C) Half-stable

D) None of the above

$$f'(x) = e^{-\frac{1}{2}x^2} - 2$$

FTC

$$f'(0) = 1 - 2 < 0$$

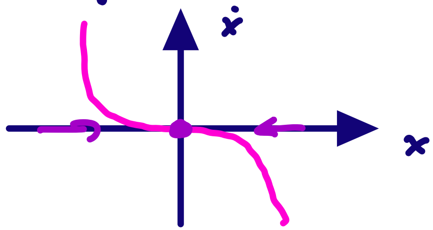
$$\dot{\eta} = -\eta$$

What happens if $f'(x^*)=0$?

Answer: Anything!!

$\dot{x} = -x^3 \quad f'(x) = -3x^2$

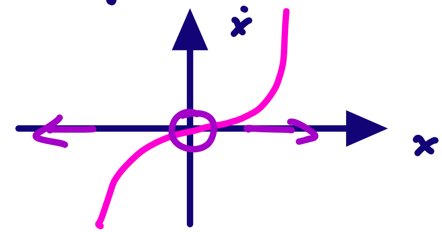
$f'(0)=0$



$x^*=0$ is stable

$\dot{x} = x^3 \quad f'(x) = 3x^2$

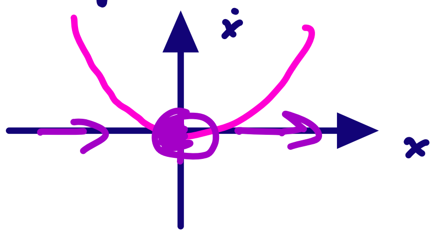
$f'(0)=0$



$x^*=0$ is unstable

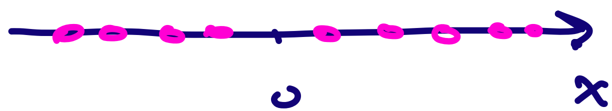
$\dot{x} = x^2 \quad f'(x) = 2x$

$f'(0)=0$



$x^*=0$ is half-stable point

$\dot{x} = 0$



~~**Question.** What happens when $f'(x^*) = 0$?~~

Potentials

Potentials

- Let $f : \mathbb{R} \to \mathbb{R}$ be smooth ~~continuous~~ and consider the system

$$\dot{x} = f(x).$$

- A function $V : \mathbb{R} \to \mathbb{R}$ so that

$$f(x) = -V'(x)$$

  is called a **potential** for f.

$$V(x) = -\int_0^x f(y)\,dy$$

- Our system can be written as a **gradient flow**

$$\dot{x} = -V'(x).$$

An example

Recall: a potential V satisfies $f(x) = -V'(x)$.

Which of the following is a potential function for

$$\dot{x} = x - x^3$$

A) $1 - 3x^2 \xrightarrow{-\frac{d}{dx}} 6x$

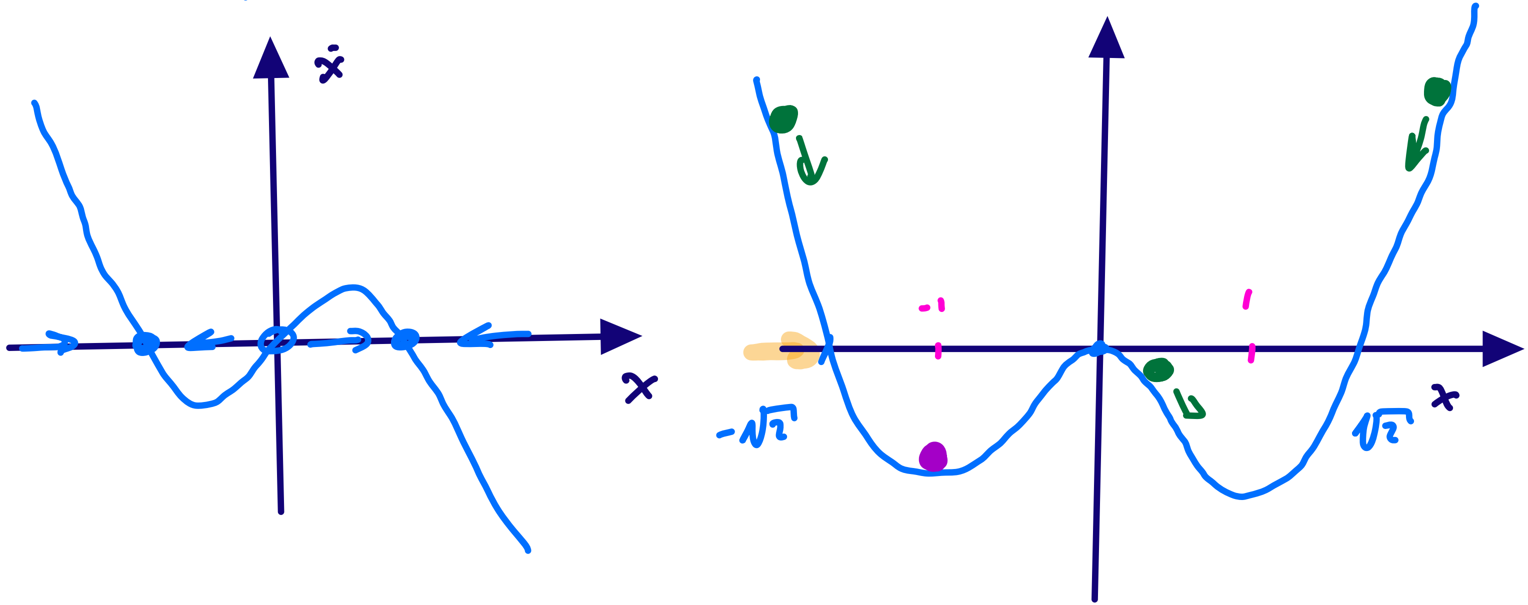
B) $\frac{1}{2}x^2 - \frac{1}{4}x^4 \longrightarrow -(x - x^3) = -x + x^3$

C) $3x^2 - 1 \longrightarrow -(6x) = -6x$

$\longrightarrow$ D) $\frac{1}{4}x^4 - \frac{1}{2}x^2 \longrightarrow x - x^3$

$$\dot{x} = x - x^3 = x(1-x)(1+x)$$

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 = \frac{1}{4}x^2(x^2-2) = \frac{1}{4}x^2(x+\sqrt{2})(x-\sqrt{2})$$



Theorem: Let $V\colon \mathbb{R} \to \mathbb{R}$ be smooth and consider the system

$$\dot{x} = -V'(x).$$

Then the **potential energy** $V(x(t))$ is non-increasing (as a function of time). Further, if $x(t)$ is not a fixed point for all $t \in (T_1, T_2)$ then the potential energy is strictly decreasing on (T_1, T_2).

Proof: Compute:

$$\frac{d}{dt}\left[V(x(t))\right] = \frac{dV}{dx}(x(t))\,\dot{x}(t) = -\left[\frac{dV}{dx}(x(t))\right]^2 \leq 0$$

with equality iff $V'(x(t)) = 0$, so $x(t)$ is a fixed point

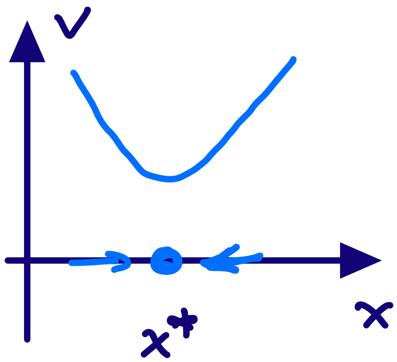
Corollary: Let $V : \mathbb{R} \to \mathbb{R}$ be smooth and consider the system

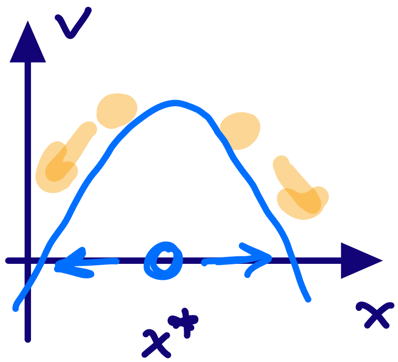
$$\dot{x} = -V'(x).$$

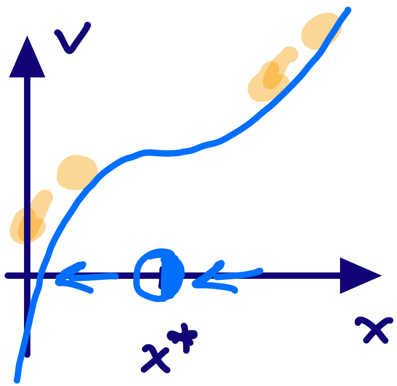
If x^* is an isolated critical point of V then

- If it is a local minima of V, it is a stable fixed point.
- If it is a local maxima of V, it is an unstable fixed point.
- If it is an inflection point of V, it is a half-stable fixed point.

Proof:







An example (Exercise!)

Consider the system

$$\dot{x} = -V'(x)$$

Suppose that:

- V is smooth.
- The only solutions of $V(x) = 0$ are $x = \pm 1$.
- $V(0) = -1$.

Which of the following is a true statement?

A) 0 is a stable fixed point

B) There are no unstable fixed points

C) There is a stable fixed point in $(-1, 1)$

D) There is at least one unstable fixed point

- V is smooth.
- The only solutions of $V(x) = 0$ are $x = \pm 1$.
- $V(0) = -1$.

See you next time!