Math 134, Lecture 1 - Homework 1

1. In 1918, Georg Duffing introduced a nonlinear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical problems, the equations reads as follows

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)\,,$$

where δ, α, β, and ω are constants. Show that this system can be expressed as a first order autonomous system of ODEs.

2. Suppose that a new short form video app, KotKit, has been launched. Write down a one-dimensional autonomous system for the growth of the app, taking $x(t)$ to be the fraction of the population that has downloaded the app at time t. Check that the fixed points and their stability match common sense.

3. Suppose that the one-dimensional autonomous ODE

$$\dot{x} = f(x),$$

has a fixed point x^* so that $a = f'(x^*) \neq 0$.

   (a) Write down the linearization of the ODE about x^*.

   (b) Show that the time required for the solution of the linearized equation found in part (a) to increase or decrease its value (depending on the sign of a) by a factor of $k > 0$ is a constant that depends only on a, k.

   (c) The book defines the 'characteristic timescale' attendant to the fixed point x^* to be $1/|a|$. Using your answer to part (b), give an interpretation of this quantity.

4. Draw a phase portrait (cf. Figure 1 on p. 37 of Strogatz) for each of the following systems, including the values and stabilities of fixed points. Overlay a sketch of a potential function on each one.

   (a) $\dot{x} = x(x-1)^2$

   (b) $\dot{x} = 1 - |x|$

   (c) $\dot{x} = \sin(3x)$

   (d) $\dot{x} = \begin{cases} x\ln|x| & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

5. For each integer $k = 1, 2, 3, \ldots$ and each choice of $+$ or $-$, determine the stability of $x = 0$ as a fixed point of the equation

$$\dot{x} = \pm x^k$$

Restricting to the cases where $x = 0$ is stable, does making k larger result in faster or slower convergence to the fixed point? Give both a heuristic explanation and one in terms of the exact solutions. (Note that this equation is separable.)

6. (Derived from Strogatz Exercise 2.2.13) The velocity $v(t)$ of a skydiver falling to the ground is governed by

$$m\dot{v} = mg - kv^2,$$

where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

   (a) Find the exact solution for $v(t)$ when $v(0) = 0$.

   (b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.

   (c) Draw a phase portrait for this problem, and thereby re-derive a formula for the terminal velocity.