Discussion 5

Wednesday, April 27, 2022 6:05 PM

Topics

1. Bifurcation Normal Forms + Taylor Series
2. Nondimensionalization

Logistics: Extra review OH: http://whenisgood.net/ww3qwnm next week

Bifurcation Normal Forms + Taylor Series

Bifurcation Normal Form - Polynomial examples of different bifurcation types. All examples of a specific bifurcation "looks like" the corresponding prototype

Normal Forms:

Saddle Node - $\dot{x} = r - x^2$
$\dot{x} = r + x^2$

Transcritical - $\dot{x} = rx - x^2$
$\dot{x} = rx + x^2$

Supercritical Pitchfork - $\dot{x} = rx - x^3$

Subcritical Pitchfork - $\dot{x} = rx + x^3$

- What do we mean when we say f(x) looks like one of these polynomials?
  - we mean the Taylor Series of f(x) looks like one of these polynomials
  - Conversely, if the Taylor Series does not look like a normal form, then we have a new bifurcation.

2D Taylor's Theorem: [WHERE $(x^*, r^*) = (0,0)$ ...]

Taylor's Theorem

Theorem: Suppose that all partial derivatives of $f(x,r)$ up to order $N+1$ are continuous. Then,

$$f(x,r) = \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{1}{(n-j)!j!} \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(0,0) x^{n-j} r^j + R_N(x,r),$$

where the remainder term can be written as

$$R_N(x,r) = \sum_{j=0}^{N+1} \frac{1}{(N+1-j)!j!} \frac{\partial^{N+1} f}{\partial x^{N+1-j} \partial r^j}(tx, tr) x^{N+1-j} r^j,$$

for some $0 < t < 1$.

Ex: "Find the 2^{nd} order Taylor Polynomial of $f(x,r) = r + 2x - 1 - 2e^{x-1}$ around the possible bifurcation point."

- We do this via Taylor Series
- First look for possible bifurcation points:

$$0 = f(x,r) = r + 2x - 1 - 2e^{x-1} \rightarrow r = 1$$

$$0 = \frac{\partial f}{\partial x}(x,r) = 2 - 2e^{x-1} \rightsquigarrow 1 = e^{x-1},\ x = 1$$

$$f(x,r) = \sum_{n=0}^{2} \sum_{j=0}^{n} \frac{1}{(n-j)!\,j!} \cdot \frac{\partial^n f}{\partial x^{n-j} \partial r^j}(1,1)\ (x-1)^{n-j} \cdot (r-1)^j$$

- How to "unwind" this?
- Double for loop
for n=0
for j=0 ...

n=0: j=0
n=1: j=0, j=1
n=2: j=0, j=1, j=2

$$= \overset{0}{\cancel{f(1,1)}} + \overset{0}{\cancel{\frac{\partial f}{\partial x}}}\Big|_{(1,1)} (x-1)^1 + \frac{\partial f}{\partial r}\Big|_{(1,1)} (r-1)^1$$

$$+ \frac{1}{2}\frac{\partial f}{\partial x^2}\Big|_{(1,1)} (x-1)^2 + \underbrace{\frac{\partial f}{\partial x \partial r}\Big|_{(1,1)}}_{0} (x-1)(r-1) + \underbrace{\frac{\partial f}{\partial r^2}\Big|_{(1,1)}}_{0} (r-1)^2$$

$$= (r-1) + \frac{1}{2}(-2)(x-1)^2$$

$$= (r-1) - (x-1)^2$$

Saddle Node Theorem -

Theorem: Suppose that

$$\dot{x} = f(x,r)$$

has a bifurcation at $(x,r) = (x^*, r^*)$. If

$$\frac{\partial f}{\partial r}(x^*, r^*) \neq 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x^*, r^*) \neq 0.$$

Then (x^*, r^*) is a saddle-node bifurcation.

Intuition:

$$f(x,r) = \overset{0}{\cancel{f(x^*,r^*)}} + \frac{\partial f}{\partial x}\overset{0}{\cancel{(x^*,r^*)}}(x-x^*) + \frac{\partial f}{\partial r}(x^*,r^*)(r-r^*)$$

$$+ \frac{1}{2}\frac{\partial f}{\partial x^2}(x^*,r^*)(x-x^*) + \underbrace{\frac{\partial f}{\partial x \partial r}(x^*,r^*)(r-r^*)(x-x^*) + \frac{1}{2}\frac{\partial f}{\partial r^2}(r-r^*)^2 + H.O.T}_{\text{"Sufficiently Small"}}$$

Transcritical Theorem -

Theorem: Suppose that

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Transcritical Theorem-

Theorem: Suppose that

$$\dot{x} = f(x, r)$$

has a bifurcation at $(x, r) = (x^*, r^*)$. If

$$\frac{\partial^n f}{\partial r^n}(x^*, r^*) = 0 \text{ for all } n, \quad \frac{\partial^2 f}{\partial x \partial r}(x^*, r^*) \neq 0, \quad \frac{\partial^2 f}{\partial x^2}(x^*, r^*) \neq 0.$$

Then (x^*, r^*) is a transcritical bifurcation.

“Sufficiently Small”

If saddle node case:

$$f(x,r) = \frac{\partial f}{\partial r}(x^*, r^*) \cdot (r - r^*) + \frac{1}{2} \frac{\partial f}{\partial x^2}(x^*, r^*)(x - x^*)^2$$

If transcritical case:

$$f(x,r) = \frac{1}{2} \frac{\partial f}{\partial x^2}(x^*, r^*)(x - x^*)^2 + \frac{\partial f}{\partial x \cdot \partial r}(x^*, r^*)(x - x^*)(r - r^*)$$

Let $X = x - x^*$, $R = r - r^*$,

Saddle node:

$$f(x,r) = A \cdot R + B \cdot X^2$$

Transcritical:

$$f(x,r) = C \cdot R \cdot X + B X^2$$

Ex: “Classify the bifurcation in $f(x,r) = r + 2x - 1 - 2xe^{x-1}$.”

Recall

$$f(x,r) = \underbrace{(r-1)}_{R} - \underbrace{(x-1)}_{X}{}^2 + H.O.T$$

$= R - X^2$, so the bifurcation at $r = 1$, $x = 1$ is a saddle node.

Nondimensionalization

Dimensionless Analysis- Removing dimension from ODE

1. Rescales ODE and makes it easier to understand scale

Ex: $y(t) = e^t$ ← seconds

y: Population in 1,000,000

- If t is large ($t \approx 10$)
- Population in $1{,}000{,}000 \approx e^{10}$

Ex: $Re = \frac{\rho u L}{\mu}$ Reynolds Number

ρ = density (kg/m^3)
u = flow speed (m/s)
L = length (m)
μ = Dynamic viscosity ($\frac{kg}{m \cdot s}$)

2. Reduces # of parameters by meaning them into dimensionless groups

Ex: “$\dot{q} = k_1 s_0 - k_2 \cdot q + \frac{k_3 q^2}{k_4^2 + q^2}$. Here, $q(t)$, s_0 are chemical concentrations. Show this system can be put into dimensionless form $\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$ where $r \geq 0$, $s \geq 0$ are dimensionless groups.”

$k > 0$, s_0) constants

1. Let $x = \frac{q}{M}$, $\tau = \frac{t}{T}$ where M and T are TBD. We also get the relations in derivatives:

$$dx = \frac{dq}{M}$$

$$\frac{d}{dt} = \left(\frac{d\tau}{dt}\right) \cdot \frac{d}{d\tau} \rightarrow \frac{d}{dt} = \frac{1}{T} \cdot \frac{d}{d\tau}$$

$$\frac{dx}{dt} = \frac{1}{M \cdot T} \cdot \frac{dq}{d\tau}$$

2. Substitute and plug into ODE:

$$\frac{M}{T} \cdot \frac{dx}{d\tau} = k_1 \cdot s_0 - k_2 \cdot (Mx) + \frac{k_3 \cdot (Mx)^2}{k_4^2 + (Mx)^2}$$

$$\frac{dx}{d\tau} = \frac{T}{M} \cdot k_1 \cdot s_0 - \frac{k_2}{T} \cdot x + \frac{T}{M} \cdot \frac{k_3 \cdot M^2 \cdot x^2}{k_4^2 + M^2 x^2}$$

3. Force M, T to be what you want

Let $s_0 = \frac{T}{M} \cdot k_1 \cdot s_0$

$r = k_2 / T$

Now look at the fraction:

$$\frac{T\, k_3 \cdot M \cdot x^2}{k_4^2 + M^2 x^2} = \frac{T \cdot k_3 \cdot M \cdot x^2}{k_4^2 \left(1 + \frac{M^2}{k_4^2} x^2\right)}$$

Let $\frac{M^2}{k_4^2} = 1$ ($M = k_4$)

Let $\frac{T \cdot k_3 \cdot M}{k_4^2} = 1$ ($T = \frac{k_4}{k_3}$)

So, $\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$