Home Work 2

1) Given $f: R \to R$ be a smooth function so that $\frac{d^n f}{dx^n}$ is bounded for $n = 0, 1, 2$

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

. Using modified Euler's method, We have:

$$x_1 = x_0 + \frac{\Delta t}{2}\left[f(x_0) + f(\tilde{x}_1)\right] \text{ where } \tilde{x}_1 = x_0 + f(x_0)\Delta t$$

$$\Rightarrow x_1 = x_0 + \frac{\Delta t}{2}\left[f(x_0) + f(x_0 + f(x_0)\Delta t)\right]$$

Also, $\dot{x} = f(x) \Rightarrow \dot{x}(0) = f(x(0)) = f(x_0)$

$$\Rightarrow \ddot{x} = f'(x)\,\dot{x} \Rightarrow \ddot{x}(0) = f'(x(0)).\dot{x}(0)$$

$$\Rightarrow f(x_0 + f(x_0)\Delta t) = \dot{x}(0) + \Delta t . f'(x_0)\,\dot{x}_0$$

$$= \dot{x}(0) + \Delta t . f'(x(0))\,\dot{x}(0) = \dot{x}(0) + \Delta t\,\ddot{x}(0)$$

$$\Rightarrow x_1 = x_0 + \frac{\Delta t}{2}\left[\dot{x}(0) + \dot{x}(0) + \Delta t.\ddot{x}(0)\right]$$

$$= x_0 + \Delta t.\dot{x}(0) + \frac{1}{2}(\Delta t)^2\,\ddot{x}(0)$$

Using Taylor's theorem, then:

$$x\Delta t = x(0) + \dot{x}(0)\Delta t + \frac{\ddot{x}(0)}{2!}(\Delta t)^2 + \frac{\dddot{x}(0)}{3!}(\Delta t)^3 (z)$$

where $z \in (0, \Delta t)$

$$\Rightarrow e_1 = x\Delta t - x_1 = \frac{\dddot{x}(0)}{3!}(\Delta t)^3 (z)$$

$\Rightarrow$ for some constant $C > 0$, we have

$$|e_1| \leq C(\Delta t)^3$$

$$\boxed{\text{So the truncation error } e_1 = x(\Delta t) - x_1 \text{ satisfies } |e_1| \leq C(\Delta t)^3 \text{ for some constant } C > 0}$$

2) Given $f:(a,b) \to R$ is Lipschitz.

$\Rightarrow \forall x, y \in (a,b)$, there exist a constant $L > 0$

So that $\dfrac{|f(x) - f(y)|}{|x-y|} \leq L$ ①

Let $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{L} > 0$

for $\forall (x,y) \in (a,b)$ so that $|x-y| < \delta$,

we have ① $\Leftrightarrow$ $|f(x) - f(y)| \leq L|x-y| < \delta . L$

$\Rightarrow |f(x) - f(y)| < \dfrac{\varepsilon}{L} . L = \varepsilon$

$\Rightarrow |f(x) - f(y)| < \varepsilon$

Therefore if $f:(a,b) \to R$ is Lipschitz so f is continuous on (a,b)

3) Given $f: \mathbb{R} \to \mathbb{R}$ be Lipschitz & x^* be a fixed point of ODE: $\dot{x} = f(x)$

Since x^* is a fixed point, so $f(x^*) = 0$

Assume there exist a solution y, with $x(0) = x_0 \neq x^*$ that reaches the fixed point x^* in finite time

$f: \mathbb{R} \to \mathbb{R}$ is Lipschitz, then with x_1 & $x^* \in \mathbb{R}$ we have:

$$|f(y) - f(x^*)| \leq L\,|y - x^*|, \quad L > 0 \text{ is a constant}$$

Since $f(x^*) = 0$

$$\Rightarrow |f(y)| \leq L\,|y - x^*|$$

* Also, let function $x = F(x)$ & $y = G(x)$, both of them solve the initial value problem:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \left(\text{is same } \begin{cases} \dot{y} = f(y) \\ y(0) = y_0 \end{cases}\right)$$

Because x^* & y are both solution for $\dot{x} = f(x)$

$\Rightarrow$ $F(x) = G(x)$ for every value of x

$\Rightarrow$ Following the "comparing solution" method,

x^* must be equal y to satisfy the uniqueness of solution $\Rightarrow x^* = y$

$\Rightarrow$ This is contradiction with the assumption, therefore there can not exist a solution with $x(0) = x_0 \neq x^*$ that reaches the point x^* in finite time

4) Given $\dot{x} = rx + x^3$ Where $r > 0$ is fixed

$$\Rightarrow \frac{dx}{dt} = rx + x^3 \Rightarrow \frac{dx}{rx + x^3} = dt$$

$$\Rightarrow \frac{dx}{x(r+x^2)} = dt \Rightarrow \int \frac{dx}{x(r+x^2)} = \int dt \quad (1)$$

Also, $\dfrac{1}{x(r+x^2)} = \dfrac{1}{r} \cdot \dfrac{r + x^2 - x^2}{x(r+x^2)} = \dfrac{1}{r}\left(\dfrac{1}{x} - \dfrac{x}{r+x^2}\right)$

$$= \frac{1}{r}\left[\frac{1}{x} - \frac{1}{2} \cdot \frac{2x}{r+x^2}\right]$$

$$(1) \Leftrightarrow \int \frac{1}{r}\left(\frac{1}{x} - \frac{1}{2} \cdot \frac{2x}{r+x^2}\right) dx = \int dt$$

$$= \frac{1}{r}\left[\int \frac{1}{x}\, dx - \frac{1}{2}\int \frac{2x\, dx}{r+x^2}\right] = \int dt$$

$$= \frac{1}{r}\left[\int \frac{1}{x}\, dx - \frac{1}{2}\int \frac{d(x^2+r)}{r+x^2}\right] = \int dt$$

$$= \frac{1}{r}\left[\ln x - \frac{1}{2}\ln(x^2+r)\right] = t + C$$

$$= \frac{1}{r} \ln \frac{x}{\sqrt{x^2+r}} = t + C$$

$$\Rightarrow \frac{x}{\sqrt{x^2+r}} = e^{r(t+c)} = e^{rt} \cdot e^{rc} = \underbrace{C_1}_{\text{const}} e^{rt}$$

Initially, $t=0$, $x(0) = x_0$

$$\Rightarrow \frac{x_0}{\sqrt{x_0^2+r}} = C_1 \Rightarrow \frac{x}{\sqrt{x^2+r}} = \frac{x_0}{\sqrt{x_0^2+r}} e^{rt}$$

$$\Rightarrow \frac{x^2}{x^2+r} = \frac{x_0^2}{x_0^2+r} e^{2rt}$$

$$\Rightarrow x^2(x_0^2+r) = x_0^2(x^2+r) e^{2rt}$$

$$= x_0^2 x^2 e^{2rt} + x_0^2 r e^{2rt}$$

$$\Rightarrow x^2(x_0^2 + r - x_0^2 e^{2rt}) = x_0^2 r e^{2rt}$$

$$\Rightarrow x^2 = \frac{x_0^2 r e^{2rt}}{x_0^2 - x_0^2 e^{2rt} + r} = \frac{r}{\frac{x_0^2+r}{x_0^2 e^{2rt}} - 1}$$

$$\Rightarrow |x| = \sqrt{\frac{r}{\frac{x_0^2+r}{x_0^2 e^{2rt}} - 1}}$$

We can see $|x| \to \infty$ when $\frac{x_0^2+r}{x_0^2 e^{2rt}} - 1 \to 0$

$$\Leftrightarrow \frac{x_0^2+r}{x_0^2 e^{2rt}} = 1 \Rightarrow x_0^2 + r = x_0^2 e^{2rt}$$

$$\Rightarrow e^{2rt} = \frac{x_0^2+r}{x_0^2} \Rightarrow 2rt = \ln\left(1+\frac{r}{x_0^2}\right)$$

$$\Rightarrow t = \underbrace{\frac{1}{2r}\ln\left(1+\frac{r}{x_0^2}\right)}_{\text{constant} \Leftrightarrow \text{finite time}} \quad \text{with } x_0 \neq 0,\ r>0 \text{ fixed}$$

$$\boxed{\begin{array}{l}\text{Therefore } |x(t)| \to \infty \text{ in finite time} \\ \text{that } t = \frac{1}{2r}\ln\left(1+\frac{r}{x_0^2}\right),\ x_0 \neq 0,\ r>0 \text{ fixed}\end{array}}$$

5) Given $\dot{x} = 1 + x^{10}$.

Check the system $\begin{cases} \dot{y} = g(y) = 1 + y^2 \\ y(0) = y_0 \end{cases}$

$$\dot{y} = 1 + y^2 \Rightarrow \frac{dy}{dt} = 1 + y^2 \rightarrow \frac{dy}{1+y^2} = dt$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int dt \Rightarrow \tan^{-1} y = t + C$$

$$\Rightarrow y = \tan(t + c)$$

with $t = 0 \Rightarrow y(0) = y_0 = \tan C \Rightarrow C = \tan^{-1} y_0$

$$\Rightarrow y = \tan(t + \tan^{-1} y_0)$$

with $t = \underbrace{\frac{\pi}{2} - \tan^{-1} y_0}_{\text{finite time}} \Rightarrow y\left(\frac{\pi}{2} - \tan^{-1} y_0\right) = \tan\left(\frac{\pi}{2}\right) \rightarrow \infty$

$\Rightarrow$ the solution $y(t)$ of $\begin{cases} \dot{y} = 1 + y^2 \\ y(0) = y_0 \end{cases}$ blows up in finite time

Back to the system $\begin{cases} \dot{x} = f(x) = 1 + x^{10} \\ x(0) = x_0 \end{cases}$

Let $|x| > 1 \Rightarrow f(x) = 1 + x^{10} \geq g(x) = 1 + x^2$

if $x(t)$ is the solution of $\begin{cases} \dot{x} = f(x) = 1 + x^{10} \\ x(0) = x_0 \end{cases}$ with $x_0 \geq y_0$

following the comparison solution, we have:

$y(t) \leq x(t)$ for all $t \in$ interval $[0, T]$ with

T is arbitiary number $\in R$

As we proved before that $y(t)$ blows up in a finite time

$$\boxed{\text{therefore the solution } x(t) \text{ of ODE: } \begin{cases} \dot{x} = f(x) = 1 + x^{10} \\ x(0) = x_0 \end{cases} \text{ is also blows up in a finite time}}$$

6) Given $\dot{x} = r + \frac{1}{4}x - \frac{x}{1+x}$.

We have $\dot{x} = 0 \Leftrightarrow r + \frac{1}{4}x - \frac{x}{1+x} = 0$

$\Leftrightarrow r + \frac{1}{4}x = \frac{x}{1+x} \Rightarrow$ the intersection of the line $r + \frac{1}{4}x$ & the graph $\frac{x}{1+x}$ correspond to fixed points

Firstly, we need to find the bifurcation point, we impose the condition that the graph of $r + \frac{1}{4}x$ & $\frac{x}{1+x}$ intersect tangentially. Thus, we demand equality of the functions & their derivative. Then, we check:

$$\begin{cases} \underbrace{r + \frac{1}{4}x}_{y_1(x)} = \underbrace{\frac{x}{x+1}}_{y_2(x)} \\ \frac{d}{dx}\left(r + \frac{1}{4}x\right) = \frac{d}{dx}\left(\frac{x}{x+1}\right) \end{cases} \Leftrightarrow \begin{cases} r + \frac{1}{4}x = \frac{x}{x+1} \quad (1) \\ \frac{1}{4} = \frac{1}{(x+1)^2} \quad (2) \end{cases}$$

$(2) \Leftrightarrow (x+1)^2 = 4 \Rightarrow \left[\begin{array}{l} x+1 = 2 \\ x+1 = -2 \end{array}\right. \Rightarrow \left[\begin{array}{l} x = 1 \\ x = -3 \end{array}\right.$

* With $x = 1 \Rightarrow (1) \Leftrightarrow \lambda + \frac{1}{4} = \frac{1}{2}$

$\Rightarrow \lambda = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

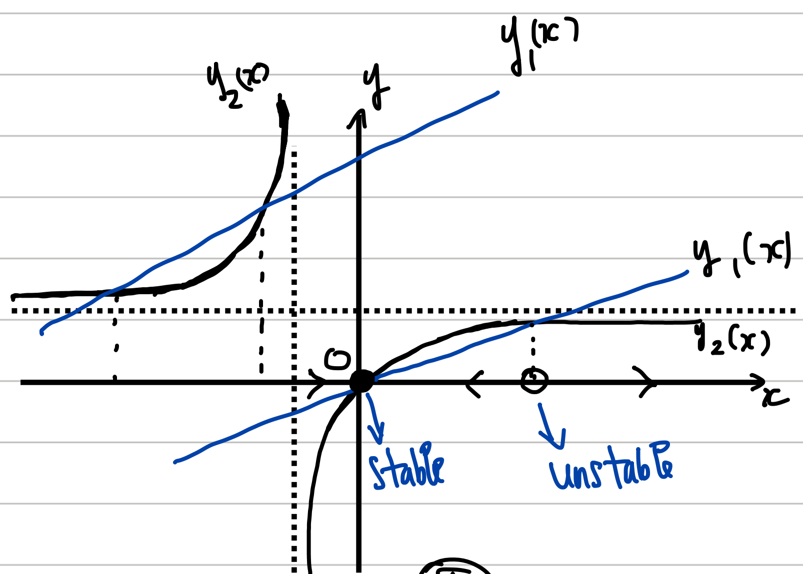
$\Rightarrow$ the first bifurcation point is $\lambda_1 = \frac{1}{4}$, occurs at $x = 1$

* With $x = -3 \Rightarrow (1) \Leftrightarrow \lambda - \frac{3}{4} = \frac{-3}{-2} = \frac{3}{2}$

$\Rightarrow \lambda = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$

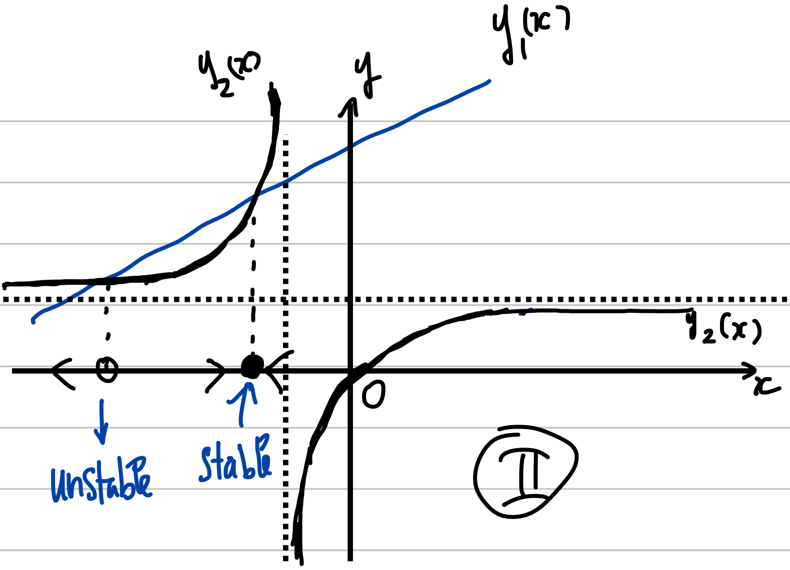
$\Rightarrow$ the second bifurcation point is $\lambda_2 = \frac{9}{4}$, occurs at $x = -3$

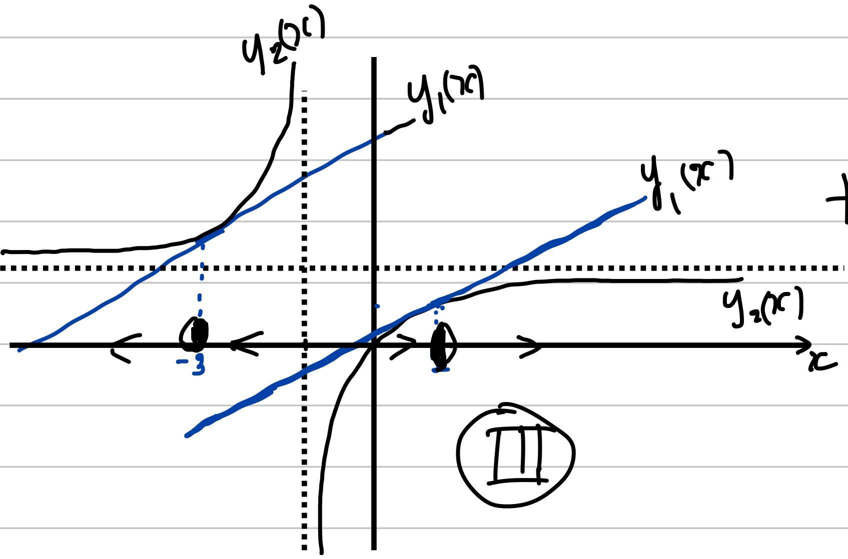
Secondly, we check the graph:



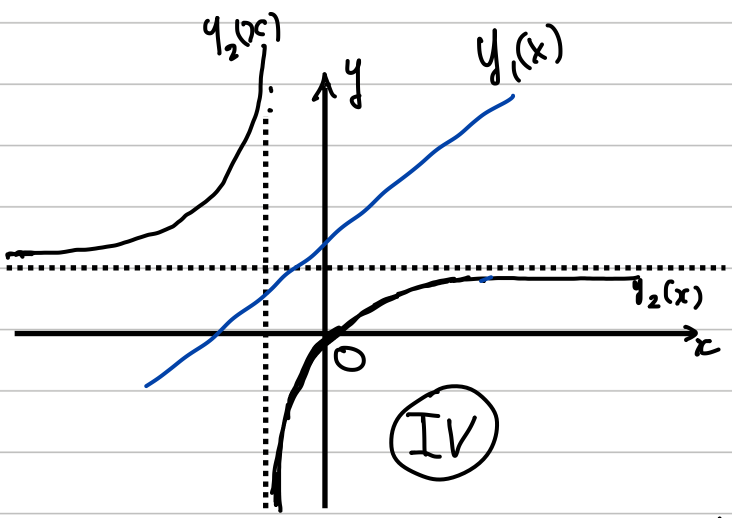
(I)

If the line lies above the curve, $y_1(x) > y_2(x)$ $\Leftrightarrow \dot{x} > 0$. Otherwise, $\dot{x} < 0$ if line lies below the curve. So, we see that the stable & unstable showed on the graph (I) & (II)





From the graph III, as we can see that if the line moved to become tangent to the curve at $r = \frac{1}{4}, \frac{9}{4}$ the fixed points are the half-stable fixed points.



Besides, check the graph IV, if we move the line as graph IV, there are no fixed-point

Finally, we can realize that, at $(x, r) = (1, \frac{1}{4})$ and $(-3, \frac{9}{4})$, the fixed-point are created & destroyed

$\Rightarrow (x,\lambda) = (1, \frac{1}{4})$ & $(-3, \frac{9}{4})$ are saddle-node bifurcation.

$$\boxed{\begin{array}{l}\text{Therefore, with } \lambda = \frac{1}{4} \text{ \& } \frac{9}{4} \text{ we will have} \\ \text{2 saddle-node bifurcation, they are} \\ (x,\lambda) = (1, \frac{1}{4}) \text{ \& } (-3, \frac{9}{4})\end{array}}$$