Math 134, Spring 2022

Lecture #21: Linear systems

Monday May 16$^{\text{th}}$

Last time

- We considered the 2-dimensional linear system
$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$
- If A has distinct real eigenvalues $\lambda_1 < \lambda_2$ then the solution can be written as
$$\boldsymbol{x}(t) = C_1 e^{\lambda_1 t}\boldsymbol{v}_1 + C_2 e^{\lambda_2 t}\boldsymbol{v}_2,$$
where $\boldsymbol{v}_1, \boldsymbol{v}_2$ are eigenvectors associated to λ_1, λ_2 and C_1, C_2 are constants.
- In this case:
  - If $\lambda_1 < \lambda_2 < 0$ we say the fixed point $\boldsymbol{x}^* = 0$ is a **stable node**.
  - If $0 < \lambda_1 < \lambda_2$ we say the fixed point $\boldsymbol{x}^* = 0$ is a **unstable node**.
  - If $\lambda_1 < 0 < \lambda_2$ we say the fixed point $\boldsymbol{x}^* = 0$ is a **saddle point**.
  - If one of λ_1, λ_2 vanishes, we have a line of fixed points: the fixed point at $\boldsymbol{x}^* = 0$ is **non-isolated**.

Learning objectives

Today we will discuss:

- Classification of fixed points for linear systems with a repeated eigenvalues.
- Classification of fixed points for linear systems with complex eigenvalues.

Linear systems

Repeated eigenvalues

Theorem: Suppose that A has a repeated (real) eigenvalue σ. Then:

- Either there exist linearly independent eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and a nonsingular matrix $P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2]$ such that

$$P^{-1}AP = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

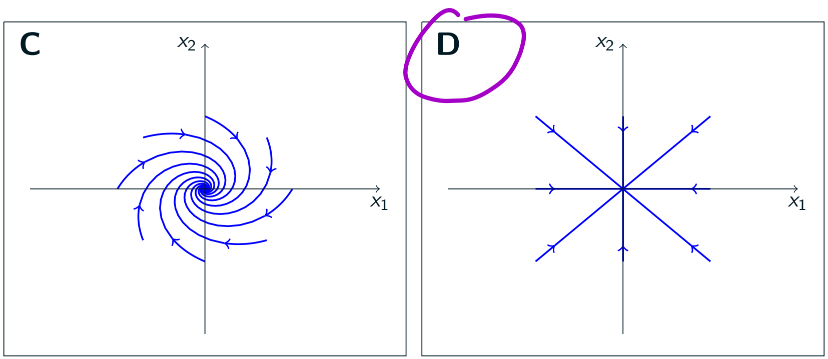
- Or there exist an eigenvector $\boldsymbol{v}$, a generalized eigenvector $\boldsymbol{w}$, and a nonsingular matrix P such that

$$P^{-1}AP = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}$$

Diagonalizable case

For $\sigma < 0$, which of the following phase portraits corresponds to

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \boldsymbol{x}$$

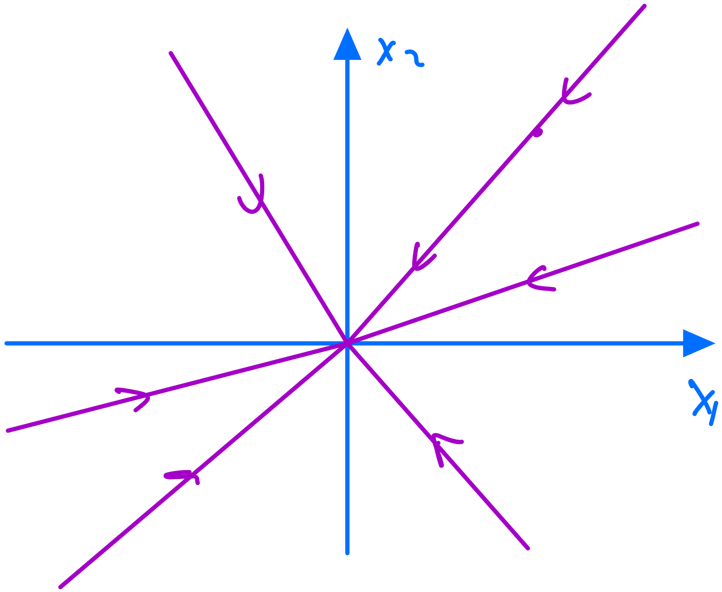


$$\begin{cases}\dot{x}_1 = \sigma x_1 \\ \dot{x}_2 = \sigma x_2\end{cases} \Rightarrow \begin{cases} x_1(t) = x_1(0)e^{\sigma t} \\ x_2(t) = x_2(0)e^{\sigma t}\end{cases}$$

$$\begin{bmatrix}\sigma & 0 \\ 0 & \sigma\end{bmatrix}\begin{bmatrix}0 \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

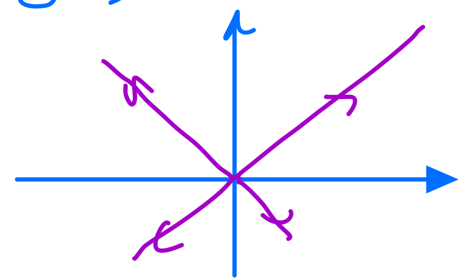
so the only critical point is

$$x^* = \begin{bmatrix}0 \\ 0\end{bmatrix}$$



$\begin{bmatrix}0 \\ 0\end{bmatrix}$ is an stable star

$\sigma > 0$



$\begin{bmatrix}0 \\ 0\end{bmatrix}$ is an unstable star

Nondiagonalizable case

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix} \boldsymbol{x}$$

$\lambda = \sigma$

we have one eigenvalue of multiplicity 2 and only one eigenvector

$$\begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sigma \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general sol is

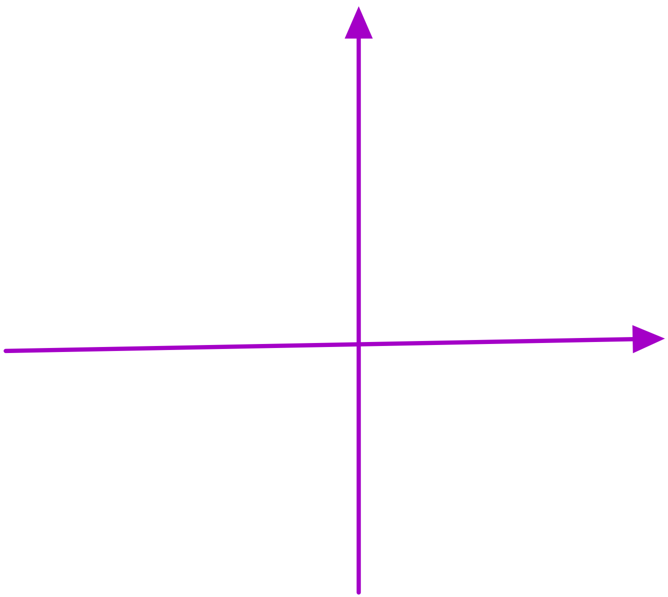
$$x(t) = (A + Bt) e^{\sigma t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + B e^{\sigma t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\sigma < 0$

$$x(t) = \begin{bmatrix} A + Bt \\ B \end{bmatrix} e^{\sigma t}$$

$$x(0) = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\begin{cases} A = 0 \\ B = 1 \end{cases}$$



An example

Sketch the phase portrait for the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}$$

Complex eigenvalues: A special case

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \boldsymbol{x}$$

Case 1: $\alpha = 0$

Case 2: $\alpha > 0$

Case 3: $\alpha < 0$

Complex eigenvalues

Theorem: Suppose that A has complex eigenvalues $\alpha \pm i\beta$. Then there exist linearly independent (real) vectors $\boldsymbol{v}$, $\boldsymbol{w}$ so that

$$A\boldsymbol{v} = \alpha\boldsymbol{v} - \beta\boldsymbol{w}$$

$$A\boldsymbol{w} = \beta\boldsymbol{v} + \alpha\boldsymbol{w}.$$

and there exists a matrix $P = [\boldsymbol{v} \quad \boldsymbol{w}]$ such that

$$P^{-1}AP =$$

and the general solution of

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

is given by

$$\boldsymbol{x}(t) = C_1 e^{\alpha t}\sin(\beta t + C_2)\boldsymbol{v} + C_1 e^{\alpha t}\sin(\beta t + C_2)\boldsymbol{w}$$

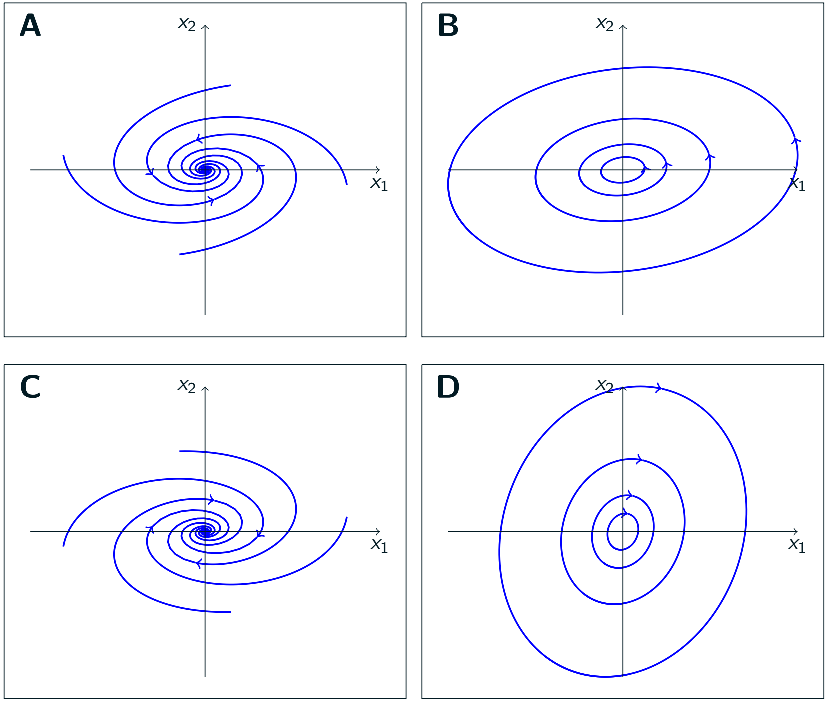
An example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 4 & 2 \\ -5 & -2 \end{bmatrix} \boldsymbol{x}$$

An example

Which of the following phase portraits corresponds to the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & -10 \\ 5 & -1 \end{bmatrix} \boldsymbol{x}$$



See you next time!