Math 134, Spring 2022

Lecture #6: Existence and uniqueness & an Intro to bifurcations.

Friday April 7$^{\text{th}}$

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Learning objectives

Today we will discuss:

- What it means to say a function is (globally) Lipschitz continuous.
- Properties of Lipschitz functions.
- Global solutions.
- Comparing solutions of ODEs.
- Intro to bifurcation theory

Existence and uniqueness

Definition: (locally Lipschitz cont.)

Let $f:(a,b)\to\mathbb{R}$. f is called locally Lipschitz cont. if for every $[d,c]\subset(a,b)$ there exists $K>0$ such that

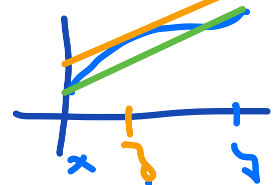
$$|f(x)-f(y)|\leq K|x-y| \quad \forall x,y\in[d,c]$$

Definition (~~Globally~~ Lipschitz cont.)

A function $f:(a,b)\to\mathbb{R}$ is said to be globally Lipschitz cont. if there exists $L>0$ s.t.

$$|f(x)-f(y)|\leq L|x-y| \quad \forall x,y\in(a,b)$$

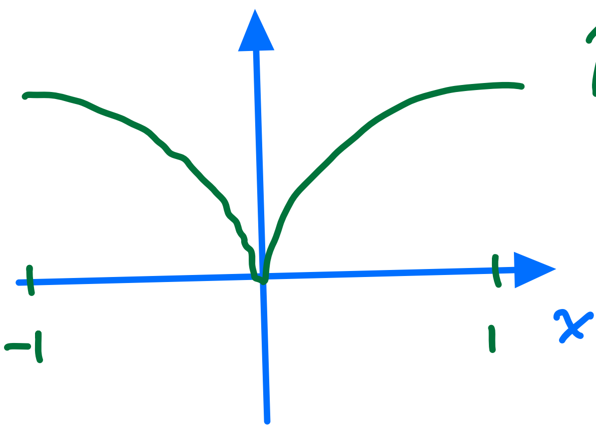
MVT $|f(x)-f(y)| = |f'(\xi)||(x-y)|$



Global and local Lipschitz functions

Fact: Cont. diff. on $[a,b]$ $\subset$ globally Lipschitz cont. on $[a,b]$ $\subset$ cont. on $[a,b]$

Example 1: $f(x)=\sqrt{|x|}$

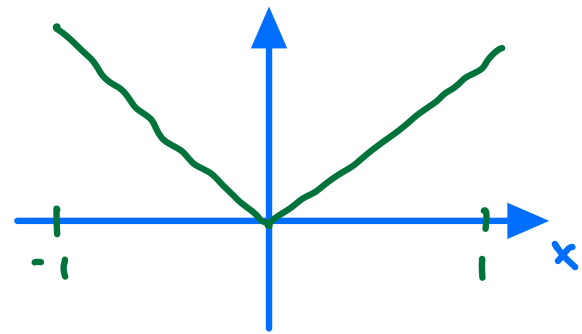


$f'(x)=\frac{1}{2\sqrt{x}}$

$x>0$

$[-1,1]$

Example 2:
$g(x)=|x|$ is not diff. but it is Glob. Lipschitz on $[-1,1]$



Finite time blowup: An example

Does the solution of

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 = x_0 \end{cases}$$

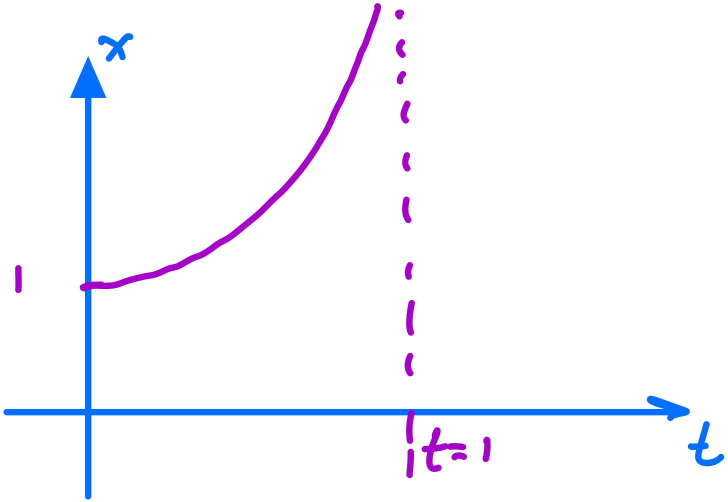
blow up in finite time?

$$\frac{dx}{dt} = x^2$$

$$\int_{x_0}^{x} \frac{dx}{x^2} = \int_{t_0}^{t} d\bar{t}$$

$$\dots$$

$$\to \quad x(t) = \frac{1}{1-t}$$



Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous.

Then there exists a unique **global** solution $x\colon \mathbb{R} \to \mathbb{R}$ of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Proof: We will omit it.

An example

Is $f(x) = x^2$ Lipschitz continuous on $\mathbb{R}$? No

Let $x>0$, and $y=0$

$$|f(x)-f(y)| = x^2 = x\,|x-y|$$

$f(y) = 0$

$$|f(x)-f(y)| \le k|x-y|$$

Given $L>0$, if $x>L$ then

$$|f(x)-f(y)| > L\,|x-y|$$

so f is not Lipschitz on $\mathbb{R}$.

Is $h(x)=x^2$ Lipschitz on $[-1,1]$? Yes

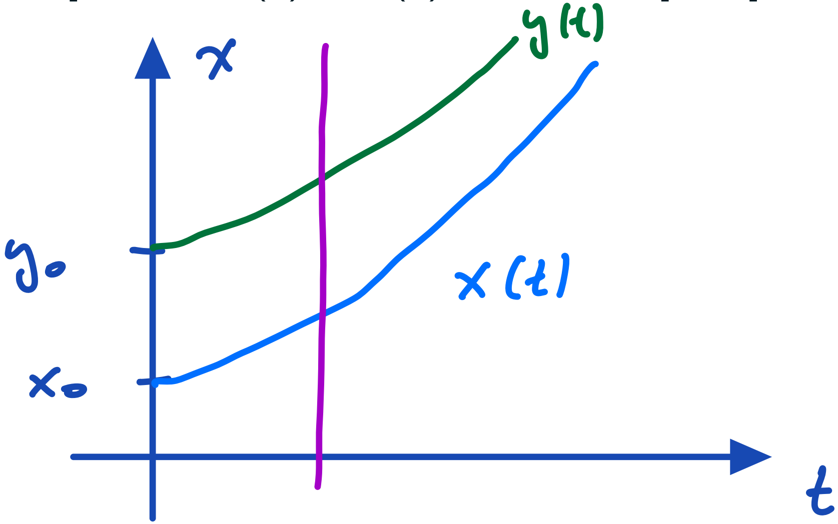
Comparing solutions

Let $f \leq g$ be smooth and let $x_0 \leq y_0$. Suppose that x and y are solutions of the ODEs

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y) \\ y(0) = y_0 \end{cases}$$

on an interval $[0, T]$. Then $x(t) \leq y(t)$ for all $t \in [0, T]$.

Proof(Sketch)



Comparing solutions: Example

Show that the solution of

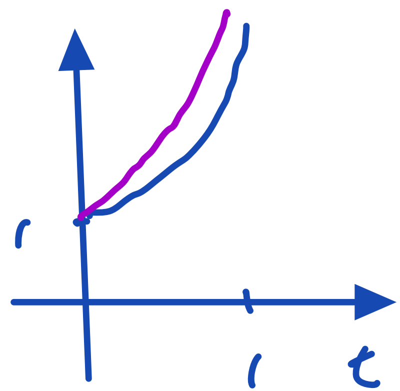
$$\begin{cases} \dot{x} = 1 + \sin(x) + x^2 \\ x(0) = 1 \end{cases}$$

blows up in finite time.

Sol. $\begin{cases} \dot{y} = y^2 \\ y(0) = 1 \end{cases} \to y(t) = \dfrac{1}{1-t}$

- As $1 + \sin(x) \geq 0 \to 1 + \sin(x) + x^2 \geq x^2$
- Thanks to the previous result

$$x(t) \geq \frac{1}{1-t} \quad \text{so as} \quad t \to 1^-, \; x(t) \to \infty$$



An Intro to bifucations

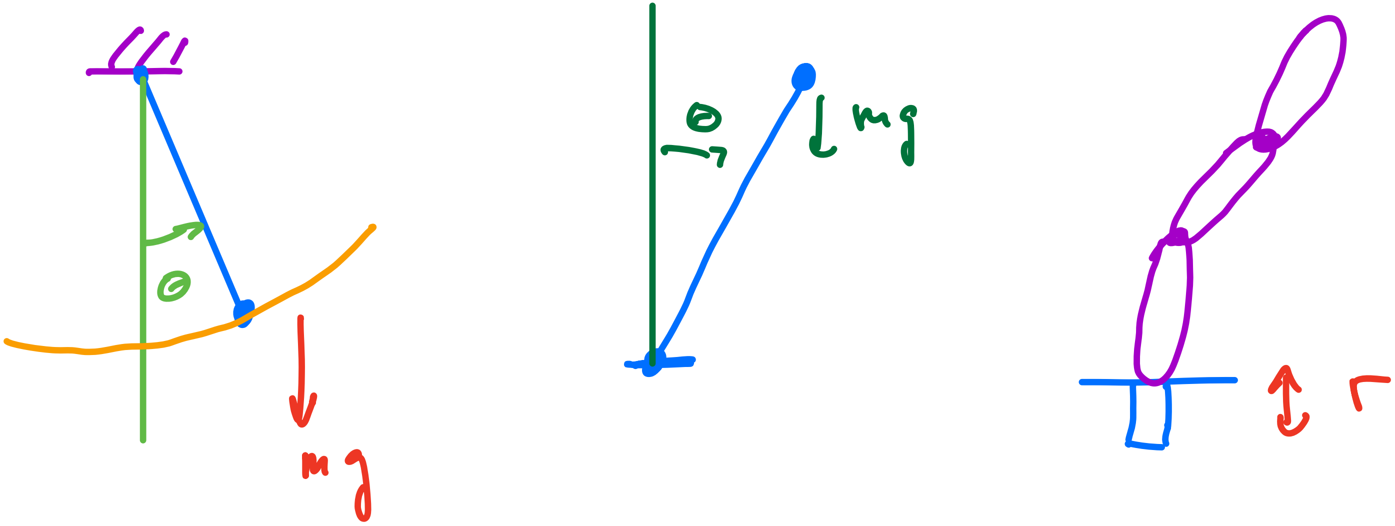
External parameters

- Consider the ODE

$$\dot{x} = f(x, r)$$

where r is a parameter of the model.

- **Question:** How do the dynamics vary as we vary r?



An example

$$\dot{x} = r + x^2$$

Definition!

Consider the following autonomous system

$$\dot{x} = f(x, \lambda)$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. A **bifurcation** occurs at parameter $\lambda = \lambda_0$ if there are parameter values λ_1 arbitrarily close to λ_0 with dynamics topologically inequivalent from those at λ_0.

See you next time!