MATH134 - Homework 2 Suggested Solutions.

Suggested Solutions to Homework # 2.

Homework Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function so that $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f$ is bounded for $n = 0, 1, 2$ and consider the ODE:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Let x_1 be the approximation to $x(\Delta t)$ obtained from the improved Euler method. using Taylor's Theorem, show that the local truncation error $e_1 = x(\Delta t) - x_1$ satisfies

$$|e_1| \leq C(\Delta t)^3$$

for some $C > 0$.

Solution. Recall that the Improved Euler Method is given by

$$\begin{cases} \widetilde{x}_{n+1} = x_n + f(x_n)(\Delta t) \\ x_{n+1} = x_n + \frac{1}{2}(f(x_n) + f(\widetilde{x}_{n+1}))(\Delta t). \end{cases}$$

Starting from $\dot{x} = f(x)$, we take an extra time derivative and apply the chain rule to obtain (here, we show the explicit time dependence for once to see the 'hidden' chain rule)

$$\ddot{x} = f'(x(t))x'(t) = f'(x(t))\dot{x}(t) = f'(x)f(x).$$

Furthermore, we have

$$\dddot{x} = f''(x)\dot{x}f(x) + f'(x)f'(x)\dot{x} = f''(x)f'(x)f(x) + (f'(x))^2 f(x).$$

Thus, using the Taylor's Remainder theorem for $x(\Delta t)$ about $x = 0$ up to the third order, we have for some $d_1 \in (0, \Delta t)$,

$$\begin{aligned} x(\Delta t) &= x(0) + \dot{x}(0)\Delta t + \frac{\ddot{x}(0)}{2!}(\Delta t)^2 + \frac{\dddot{x}(d)}{3!}(\Delta t)^3 \\ &= x_0 + f(x(0))\Delta t + \frac{f'(x(0))f(x(0))}{2!}(\Delta t)^2 \\ &+ \frac{f''(x(d_1))f'(x(d_1))f(x(d_1)) + (f'(x(d_1)))^2 f(x(d_1))}{3!}(\Delta t)^3 \end{aligned} \tag{1}$$

Next, from the scheme, we substitute the equation for $\widetilde{x}_{n+1}$ into that for x_{n+1} and set $n = 0$, we obtain

$$\begin{aligned} x_1 &= x_0 + \frac{1}{2}(f(x_0) + f(x_0 + f(x_0)\Delta t))\Delta t \\ &= x_0 + \frac{1}{2}f(x(0))(\Delta t) + \frac{1}{2}f(x_0 + f(x_0)\Delta t))\Delta t \\ &= x_0 + \frac{1}{2}f(x_0)(\Delta t) + \frac{1}{2}(f(x_0) + f'(x_0)[f(x_0)\Delta t] + \frac{1}{2!}f''(x_0)[f(d_2)\Delta t]^2)\Delta t \\ &= x_0 + f(x_0)(\Delta t) + \frac{f'(x_0)f(x_0)}{2!}(\Delta t)^2 + \frac{1}{4}f''(x_0)(f(d_2))^2(\Delta t)^3 \end{aligned} \tag{2}$$

where we have used a Taylor expansion for $f(x_0 + f(x_0)\Delta t)$ about $x = x_0$ up to the second order, and $d_2 \in I$, an open interval with endpoints x_0 and $f(x_0)\Delta t$.

From (1) and (2), we have

$$\begin{aligned} |e_1| &= |x(\Delta t) - x_1| \\ &= |\frac{f''(x(d_1))f'(x(d_1))f(x(d_1)) + (f'(x(d_1)))^2 f(x(d_1))}{3!} - \frac{1}{4}f''(x_0)(f(d_2))^2|(\Delta t)^3 \\ &\leq C(\Delta t)^3 \end{aligned}$$

since $x_0 = x(0)$ and the coefficient of $(\Delta t)^3$ contains terms in f, f' and f'' which are bounded as assumed in the question.

Homework Problem 2. Suppose $f : (a, b) \to \mathbb{R}$ is Lipschitz. Show that f is continuous on (a, b).

Solution. Note that the interval I is fixed, and so is the Lipschitz constant L. To show that f is continuous on (a, b), we have to show that it is continuous on every point $x \in (a, b)$. Now, fix an $x \in (a, b)$. To show continuity at x, this is equivalent to showing that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $y \in (a, b)$ and $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. In addition, we recall that since f is Lipschitz, then we have for all $x, y \in (a, b)$,

$$|f(x) - f(y)| \le L|x - y|.$$

Proof: Let $\varepsilon > 0$ be given. Pick $\delta = \frac{\varepsilon}{L}$. Since f is Lipschitz, we have for all $y \in (a, b)$ and $|x - y| < \delta$,

$$|f(x) - f(y)| \le L|x - y| < L\left(\frac{\varepsilon}{L}\right) = \varepsilon.$$

Since this is true for all $x \in (a, b)$, then f is continuous on (a, b).

Homework Problem 3. Recall that the Picard iterates for the ODE

$$\dot{x} = f(x)$$

are defined by taking $x_0(t) = x(0)$ and

$$x_{n+1}(t) = x(0) + \int_0^t f(x_n(s))\mathrm{d}s.$$

Compute the iterates $x_n(t)$ for $n = 1, 2, 3, 4$ in the case that $f(x) = 2 - x$ and $x(0) = 1$.

Solution. We will start off by simplifying the iterates as follows:

$$\begin{aligned} x_{n+1}(t) &= 1 + \int_0^t 2 - x_n(s)\mathrm{d}s \\ &= 1 + 2t - \int_0^t x_n(s)\mathrm{d}s. \end{aligned}$$

One can then iterate using this formula to obtain:

n	$x_n(t)$
1	$1+t$
2	$1+t-\frac{t^2}{2}$
3	$1+t-\frac{t^2}{2}+\frac{t^3}{6}$
4	$1+t-\frac{t^2}{2}+\frac{t^3}{6}-\frac{t^4}{24}$

.

Homework Problem 4. (Exercise 2.5.1 in Strogatz) A particle travels on the half-line $[0, \infty)$ with velocity given by $\dot{x} = -x^c$, where c is real and constant.

(a) Find all values of c such that the origin $x = 0$ is a stable fixed point.

(b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in a finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

Solution.
(a) Note that $c \neq 0$. If $c = 0$ then $\dot{x} = -1$, and this system does not have a fixed point. Furthermore, if $c < 0$, then the equation $x^c = 0$ does not admit $x = 0$ as a fixed point.
For any $c > 0$, solving for the fixed point yields $x^c = 0$ and thus $x = 0$ for all $c > 0$.
We will then show that $x = 0$ is a stable fixed point (from the right, as the trajectory is limited to the right half-line). For $x > 0$, $x^c > 0$ and $\dot{x} = -x^c < 0$, implying that the flow is to the left for points to the right of $x = 0$. This implies that $x = 0$ is a stable point.
(b) Here, we note that this equation is separable (see Homework 1 Question 5, replacing odd integers k with a non-zero real positive number c). The solution is given by

$$x(t) = \begin{cases} ((c-1)t + x(0)^{1-c})^{\frac{1}{1-c}} & \text{for } c \neq 1, c > 0 \\ x(0)e^{-t} & \text{for } c = 1 \end{cases}.$$

If $x(0) = 0$, then the particle will reach origin in finite time (ie no time). Thus, we shall consider the case for which $x(0) \neq 0$.
For $c = 1$, we see that $x(t) \to 0$ only if $t \to \infty$.
For $c > 1$, since $(c-1)t > 0$ and $x(0) > 0$ implies $x(0)^{1-c} > 0$, and thus, the expression for $x(t) > 0$ for all time $t > 0$. Thus, $x(t) = \left(\frac{1}{(c-1)t + x(0)^{1-c}}\right)^{\frac{1}{c-1}} \to 0$ only as $t \to \infty$.
For $0 < c < 1$, since the exponent is now $\frac{1}{1-c} > 0$, we can set $(c-1)t + x(0)^{1-c} = 0$ to obtain $t = \frac{x(0)^{1-c}}{1-c} > 0$ such that $x(t) = 0$. Thus, the particle does reach origin in finite time.
For $x(0) = 1$, the time taken for the particle to travel from $x = 1$ to $x = 0$, denoted by t_p, is given by

$$t_p(c) = \begin{cases} \frac{1}{1-c} & \text{if } 0 < c < 1 \\ \infty & \text{if } c \geq 1 \end{cases}.$$

Homework Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz and let x^* be a fixed point of the ODE

$$\dot{x} = f(x).$$

Show that there cannot exist a solution $x(0) = x_0 \neq x^*$ that reaches the fixed point x^* in finite time.

Solution. Suppose such a solution exists, in which we denote it by x_1 with $x_1(0) = x_0 \neq x^*$. If the solution reaches x^* in finite time, then there exists $T \in \mathbb{R}$ such that $x_1(T) = x^*$.
Now, we consider $x_2(t) = x^*$ for all time t. We claim that this is a solution to the given ODE. Since $x_2(t)$ is a constant function, then $\dot{x_2}(t) = 0$ for all t. On the right hand side of the ODE, we have $f(x(t)) = f(x^*) = 0$ by the definition that a fixed point x^* is the solution to $f(x) = 0$. This implies that $x_2(T) = x^*$ too.

We now define: (Note - This technique is the rigorous way in which we run the ODE backwards)

$$y(t) = x(T - t)$$

and note that

$$\dot{y}(t) = -\dot{x}(T - t) = -f(x(T - t)) = -f(y(t)).$$

Then, we have the corresponding $y_1(t) = x_1(T - t)$ and $y_2(t) = x_2(T - t)$ with $y_1(0) = x_*$ and $y_2(0) = x_*$. One can also show that if f is Lipschitz, so is $-f$.[a] Since $-f$ is Lipschitz, by **global** uniqueness of solution on $\mathbb{R}$, we must have $y_1(t) = y_2(t)$ for all t. In particular, we have

$$x_0 = x_1(0) = y_1(T) = y_2(T) = x_2(0) = x_*,$$

contradicting $x_0 \neq x^*$.
Remark: We do not have to do the trick of running the ODE backwards if we have complete understanding of the proof of global uniqueness. One can show that we have a global unique solution for any initial/boundary condition (ie the given condition could be in the form of $x(t_0) = C$ for some C and t_0) as long as f is Lipschitz on $\mathbb{R}$. This is because the proof corresponds to using $x(t) = x_0 + \int_0^t \text{“something”} \mathrm{d}t$, and that to prove uniqueness, we take $x_1(t) - x_2(t) = \int_0^t \text{“something”}_1 - \text{“something”}_2 \mathrm{d}t$. Thus, we could have used $x(t) = x_{t_0} + \int_0^{t_0} \text{“something”} \mathrm{d}t$ instead and that x_{t_0} for both x_1 and x_2 cancels, leaving the bulk of the proof unchanged posterior to taking $x_1(t) - x_2(t)$.

Remark: One can do a similar argument using time translation too instead of time reversal.
Remark: One could also apply the modified global uniqueness theorem stating that we could instead impose a boundary condition of $x(t_0) = x_0$ instead of say at $t = 0$.

[a] $|-f(x) - (-f(y))| = |f(x) - f(y)|$ and we can use the same Lipschitz constant L to bound this from above by $L|x - y|$. This implies that $-f$ is Lipschitz with Lipschitz constant L.

Homework Problem 6. (Exercise 2.5.3 in Strogatz.) Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $|x(t)| \to \infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

Solution. We note that the given ODE is separable. This involves integrating the expression:

$$\int \frac{\mathrm{d}x}{rx + x^3},$$

in which one can use the partial fraction decomposition

$$\frac{1}{rx + x^3} = \frac{1}{r}\frac{1}{x} - \frac{1}{r}\frac{x}{x^2 + r}$$

and the fact that $\int \frac{x}{x^2+r}\mathrm{d}x = \frac{1}{2}\ln(x^2 + r)$ to solve the equation by separation of variables. Upon further simplification, one can then show that

$$x(t) = \frac{x_0 e^{rt}\sqrt{r}}{\sqrt{x_0^2(1 - e^{2rt}) + r}}.$$

From the expression above, we set the denominator to be 0. This implies that $x_0^2(1 - e^{2rt}) + r = 0$ and thus $t = \frac{1}{2r}\ln\left(1 + \frac{r}{x_0^2}\right)$. Indeed, given any initial condition $x_0 \neq 0$, we have a finite time $t_* = \frac{1}{2r}\ln\left(1 + \frac{r}{x_0^2}\right) > 0$ such that $|x(t)| \to \infty$ as $t \to t_*$.

Alternative: Note that if $x_0 = x(0) > 0$, then $\dot{x}(0) = rx(0) + x(0)^3 > 0$ and by continuity of $rx + x^3$, x is increasing in an interval containing 0. We can repeat this argument to show that x is increasing for all $t \geq 0$ and thus $x(t) \geq x(0) > 0$. Thus, we can show that $rx + x^3 \geq x^3$. By Discussion Supplement 3 Exercise 1, since $\alpha = 3 > 1$ and $x(0) > 0$, then the solution corresponding to $\dot{y} = y^3$ and $y(0) = x(0) > 0$ blows up in finite time with $x(t) \geq y(t)$. This implies that $x(t)$ must blow up at the same time in which $y(t)$ does.

For $x_0 = x(0) < 0$, we note that $\dot{x}(0) = rx(0) + x(0)^3 < 0$ since $r > 0$. Define $z(t) = -x(t)$ (Intuition: blows up in the negative direction). we have $\dot{z} = -\dot{x} = -rx - rx^3 = r(-z) + r(-z)^3 = rz + z^3$ with $y(0) = -x(0) > 0$. We repeat the exact same argument in the previous part to show that $z(t) \to \infty$ in finite time. Since $z(t) = -x(t)$, this implies that $x(t) \to -\infty$ in finite time (at the same time as z does), which regardless of the sign, still represents a solution blowing up in finite time.

Remark: The first method is suitable for anyone who knows how to do the corresponding integration, and thus will end up to be a "brainless" method (the effort comes from actually performing the integration). The second method is suitable for anyone who wants to practice some skills in Analysis and/or we are dealing with a function that is challenging to perform an integration on.