Math 134, Spring 2022

Lecture #23: Stability

May 18^{th}

Last few lectures:

~~Today we will discuss~~:

- We considered the fixed point $\boldsymbol{x}^* = 0$ of the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

- We classified the possible behaviors into:
  - Stable and unstable nodes. $0 < \lambda_1 < \lambda_2$
  - Saddle points. $\lambda_1 < 0 < \lambda_2$
  - Non-isolated fixed point(s). $\lambda_1 = 0$
  - Stable and unstable star nodes. $\lambda_1 = \lambda_2 > 0$
  - Stable and unstable degenerate nodes.
  - Centers. $\pm i\beta$
  - Stable and unstable spirals $\alpha \pm i\beta$, $\alpha > 0$

Learning objectives

Today we will discuss:

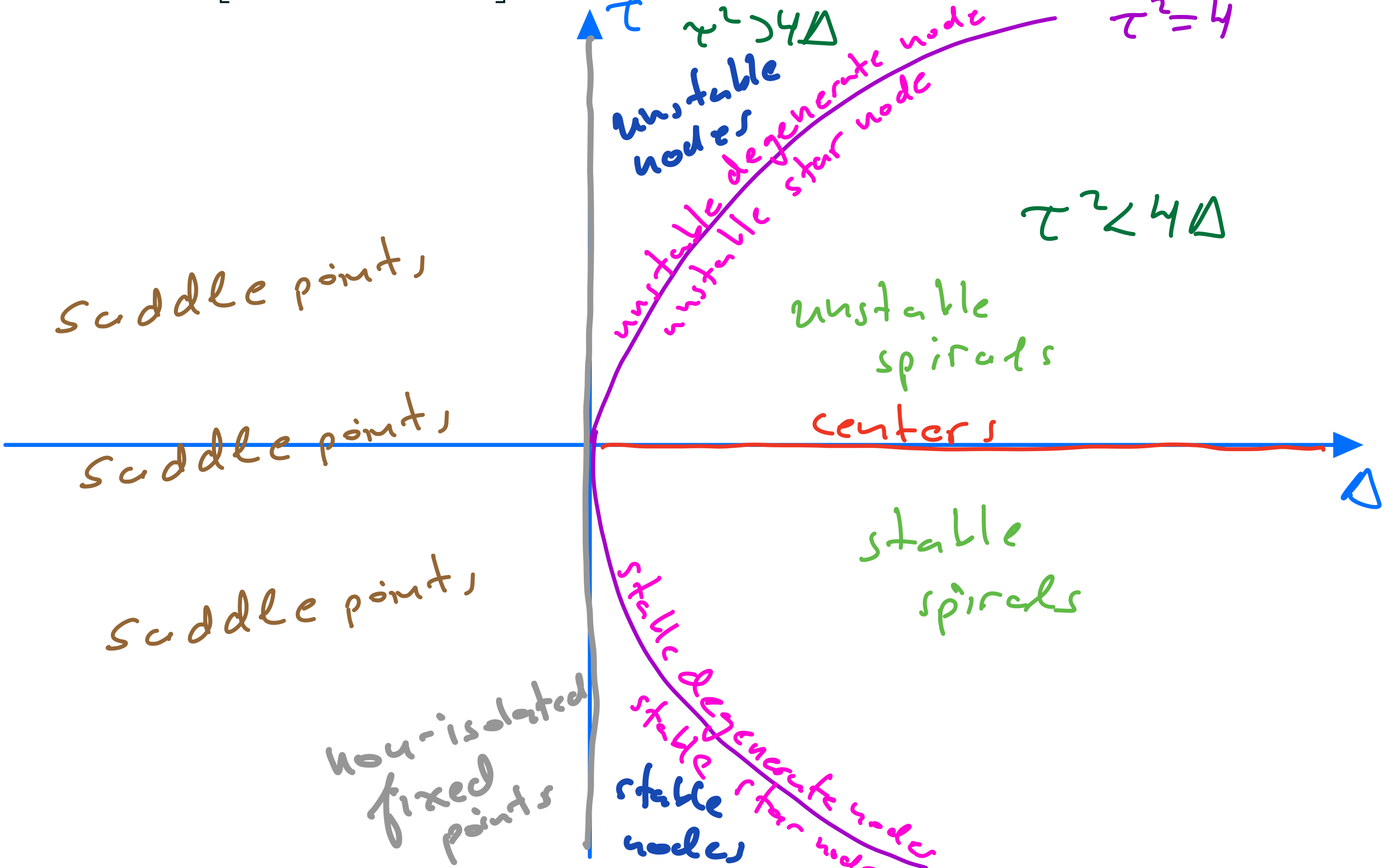
- Classification of fixed points using the trace and determinant
- What it means to say a fixed point is Lyapunov stable, neutrally stable, asymptotically stable, and unstable
- What it means to say a fixed point is hyperbolic.

Linear systems

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \operatorname{tr} A = a + d \quad \det A = ad - cb$

Classification using trace and determinant

$$\lambda = \frac{1}{2}\left[\tau \pm \sqrt{\tau^2 - 4\Delta}\right] \quad \text{where} \quad \tau = \operatorname{tr} A \quad \text{and} \quad \Delta = \det A$$

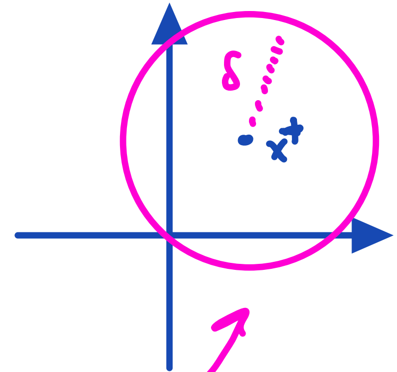


Stability

We now want to understand the stability of fixed points $\boldsymbol{x}^*$ of the 2d system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

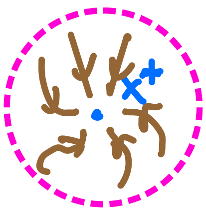
To do this, we require some definitions...



- Define the **open ball** of radius $\delta > 0$ about $\boldsymbol{x}^*$ to be

$$B(\boldsymbol{x}^*, \delta) = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x} - \boldsymbol{x}^*| < \delta\}$$

- We say that $\boldsymbol{x}^*$ is **attracting** if there exists $\delta > 0$ so that given any $\boldsymbol{x}(0) \in B(\boldsymbol{x}^*, \delta)$ we have



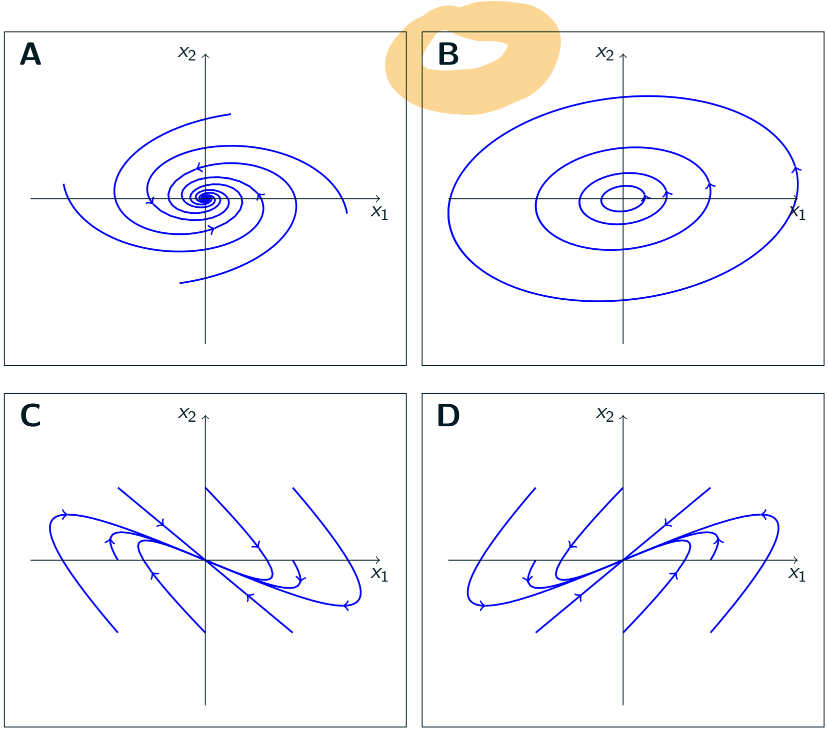
$$\boldsymbol{x}(t) \to \boldsymbol{x}^* \quad \text{as} \quad t \to \infty$$

- We say $\boldsymbol{x}^*$ is **globally attracting** if we can replace $B(\boldsymbol{x}^*, \delta)$ by $\mathbb{R}^2$.

Example: stable node at $x^*=0$ $\quad \dot{x} = Ax$

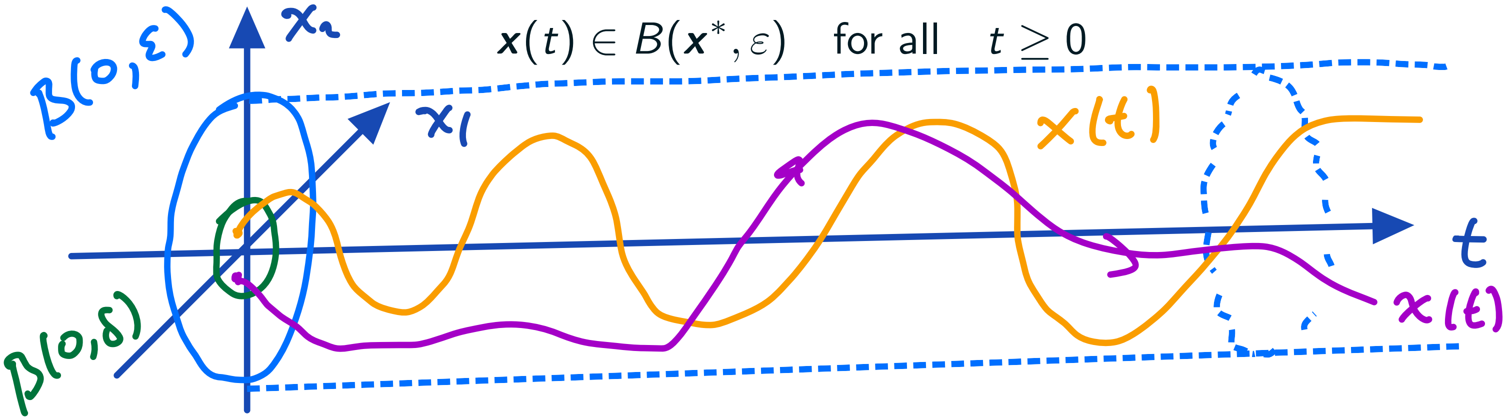
An example

In which of the following phase portraits is $\boldsymbol{x}^* = 0$ **not** attracting?



- We say that $\boldsymbol{x}^*$ is **Lyapunov stable** if given any $\varepsilon > 0$ there exists $\delta > 0$ so that for every $\boldsymbol{x}(0) \in B(\boldsymbol{x}^*, \delta)$ we have

$$\boldsymbol{x}(t) \in B(\boldsymbol{x}^*, \varepsilon) \quad \text{for all} \quad t \geq 0$$



- If a fixed point is Lyapunov stable but not attracting, we say it is **neutrally stable**.

- If a fixed point is Lyapunov stable and attracting, we sat it is **asymptotically stable**.

unstable
- everything else

- A fixed point that it neither attracting nor Lyapunov stable is **unstable**.

Neutrally stable
- centers
- "stable" non-isolated fixed points

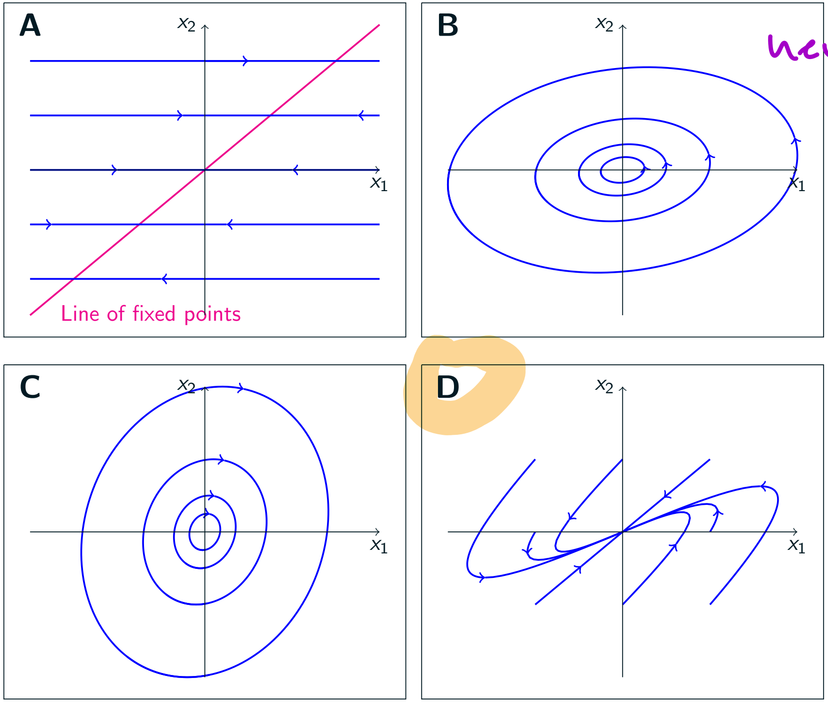
Asymp. stable
- stable nodes
- stable spirals
- " star nodes
- stable deg. nodes

An example

In which of the following phase portraits is $\boldsymbol{x}^* = 0$ **not** neutrally stable?

neut stable

neut stable



neut. stable

asymp. stable

Fixed points of nonlinear systems

Recall that for $1d$ systems

$$\dot{x} = f(x)$$

the linearization

$$\dot{\eta} = f'(x^*)\eta$$

determines the stability of a fixed point x^* whenever $f'(x^*) \neq 0$.

Question: For the $2d$ system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

when does the linearization

$$\dot{\boldsymbol{\eta}} = \nabla \boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{\eta},$$

determine the stability of the fixed point $\boldsymbol{x}^*$?

A cautionary tale

2D system

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) \\ \dot{y} = x + ay(x^2 + y^2) \end{cases}$$

$a \in \mathbb{R}$ is a given const.

$(x^*, y^*) = (0,0)$ is a critical point

Linearization: $f(x,y) = \begin{bmatrix} -y + ax(x^2+y^2) \\ x + ay(x^2+y^2) \end{bmatrix}$

$$\nabla f(x,y) = \begin{bmatrix} a(x^2+y^2) + 2ax^2 & -1 + 2axy \\ 1 + 2ayx & a(x^2+y^2) + 2ay^2 \end{bmatrix}$$

so $\nabla f(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

So the lin. at $(0,0)$ is

$\dot{\eta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \eta$ eig. $\lambda = \pm i$

so we get that the origin is a center

When can we neglect higher order terms?

Hyperbolic fixed points

We say a fixed point $\boldsymbol{x}^*$ is **hyperbolic** if all eigenvalues of $\nabla \boldsymbol{f}(\boldsymbol{x}^*)$ satisfy

$$\operatorname{Re}\lambda \neq 0$$

An example

For which of the following systems is the fixed point at $\boldsymbol{x}^* = (1, 1)$ **not** hyperbolic?

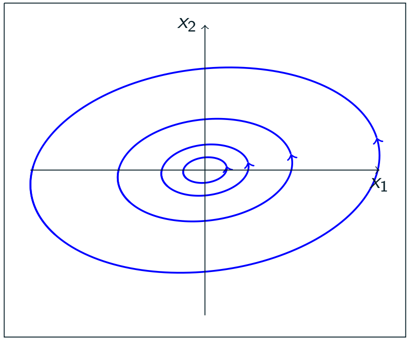
A) $\begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = 1 - xy \end{cases}$

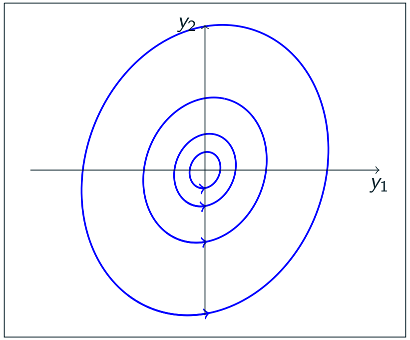
B) $\begin{cases} \dot{x} = x - 1 \\ \dot{y} = y^2 - 1 \end{cases}$

C) $\begin{cases} \dot{x} = y^2 - 1 \\ \dot{y} = 1 - x \end{cases}$

D) $\begin{cases} \dot{x} = 1 - x^2 \\ \dot{y} = 2 - y - x \end{cases}$

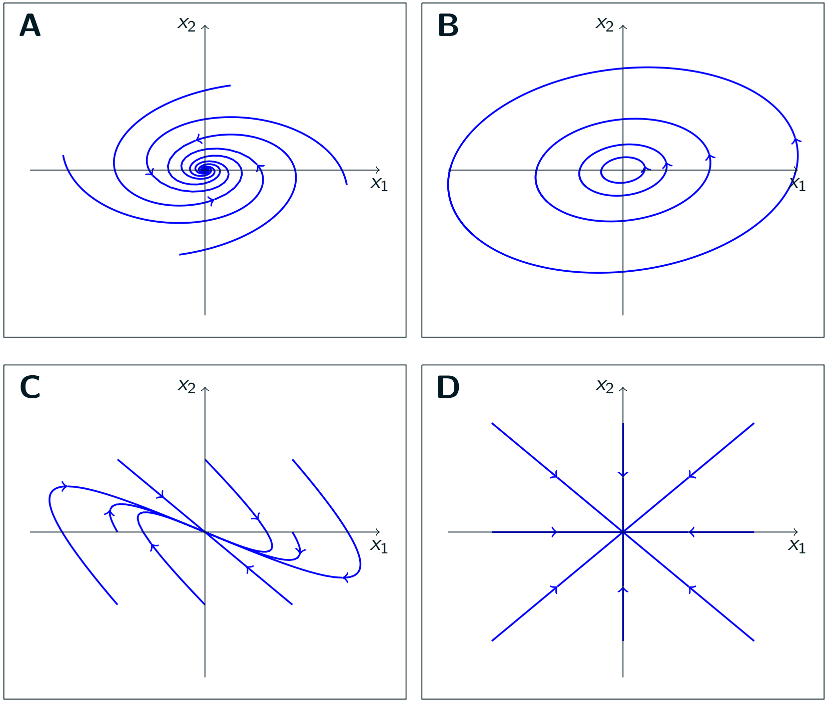
Continuous changes of variable





An example

Which of the following phase portraits is **not** topologically equivalent to the other three?



The Hartman–Grobman Theorem

Theorem: Let $\boldsymbol{f}$ be smooth and $\boldsymbol{x}^*$ be a hyperbolic fixed point of

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Then there is a continuous change of variables from a neighborhood of the fixed point $\boldsymbol{x}^*$ to a neighborhood of the fixed point $\boldsymbol{\eta}^* = 0$ of the linearized flow

$$\dot{\boldsymbol{\eta}} = \nabla \boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{\eta}.$$

An example

Consider the system

$$\begin{cases} \dot{x} = x^2 - 4y \\ \dot{y} = y - 1 \end{cases}$$

Is the fixed point at $(2, 1)$ asymptotically stable?

The (improved) Hartman–Grobman Theorem

Theorem: Let $\boldsymbol{f}$ be smooth and $\boldsymbol{x}^*$ be a hyperbolic fixed point of

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

Then there is a **continuously differentiable** change of variables from a neighborhood of the fixed point $\boldsymbol{x}^*$ to a neighborhood of the fixed point $\boldsymbol{\eta}^* = 0$ of the linearized flow

$$\dot{\boldsymbol{\eta}} = \nabla \boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{\eta}.$$

Moreover, **the derivative of the change of variables at the fixed point is the identity**.

The Lotka–Volterra model

$$\begin{cases} \dot{x} = x(3 - x - 2y) \\ \dot{y} = y(2 - x - y) \end{cases}$$

See you next time!