Normal forms:

Saddle-node bif.

$$\dot{x} = f(x,r)$$

$$= \cancel{f(x^*,r^*)}^{\,0} + \underbrace{\partial_x \cancel{f(x^*,r^*)}^{\,0}}_{q_1}(x-x^*)$$

$$+ \underbrace{\partial_r f(x^*,r^*)}_{p_1}(r-r^*)$$

$$+ \underbrace{\frac{1}{2}\partial^2_{xx} f(x^*,r^*)}_{q_2}(x-x^*)^2$$

$$+ \underbrace{\partial^2_{xr} f(x^*,r^*)}_{p_2}(x-x^*)(r-r^*)$$

$$+ \underbrace{\frac{1}{2}\partial^2_{rr} f(x^*,r^*)}_{Q}(r-r^*)^2$$

$$+ h.o.t.$$

$f(x^*,r^*)=0$

$\frac{\partial f}{\partial x}(x^*,r^*)=0$

Theorem: Suppose that $f(x^*, r^*) = 0$, $q_1 = 0$, $p_1 \neq 0$, $q_2 \neq 0$, then

$$\dot{x} = f(x, r)$$

undergoes a saddle-node bif. at (x^*, r^*) and

$$\dot{x} = \frac{\partial f}{\partial r}(x^*, r^*)(r - r^*) + \frac{1}{2}\partial^2_{xx} f(x^*, r^*)(x - x^*)^2 + \mathcal{O}(\varepsilon^3)$$

for $|r - r^*| < \varepsilon^2$ and $|x - x^*| < \varepsilon$

Moreover, there exists a change of variables

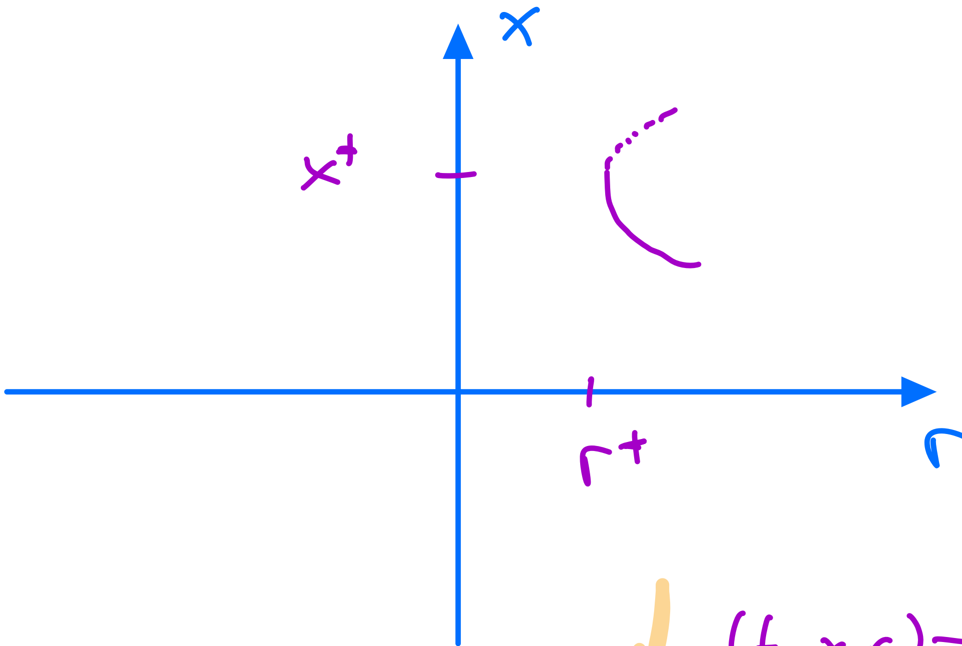
$$(t, x, r) \longmapsto (s, y, R) \text{ such that}$$

$$\dot{x} = p_1(r - r^*) + q_2(x - x^*)^2 + \text{h.o.t.}$$

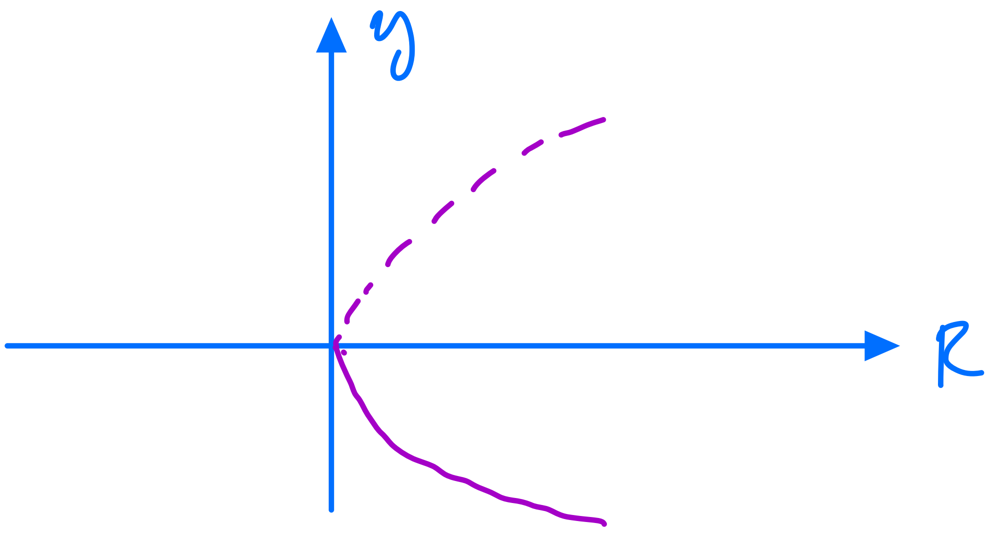
takes the form

$$\frac{dy}{ds} = R + y^2 \quad \left(\begin{array}{c}\text{saddle-node}\\ \text{bif.}\end{array}\right)$$

near $(0, 0) = (y(x^*), R(r^*))$



$(t, x, r) \to (s, y, R)$



The proof is in Wiggins

Example: $\dot{x} = r + x - 1 - e^{x-1}$

Possible bif. points?

$$\begin{cases} r + x - 1 - e^{x-1} = 0 \\ 1 - e^{x-1} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} r + x - 2 = 0 \\ e^{x-1} = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ r = 1 \end{cases}$$

$$f(x,r) = r + \cancel{x-1} - \left[1 + \cancel{(x-1)} + \frac{1}{2}(x-1)^2 + h.o.t \right]$$
$$= (r-1) - \frac{1}{2}(x-1)^2 + h.o.t.$$