Homework 3

1) $\dot{x} = 1 + rx + x^2$. Let $y_1(x) = x^2 + 1$, $y_2(x) = -rx$

$\Rightarrow \dot{x} = y_1(x) - y_2(x)$

We have $\dot{x} = 0 \Rightarrow 1 + rx + x^2 = 0$, $\Delta = r^2 - 4$

$$\Rightarrow x = \frac{-r \pm \sqrt{r^2 - 4}}{2}$$

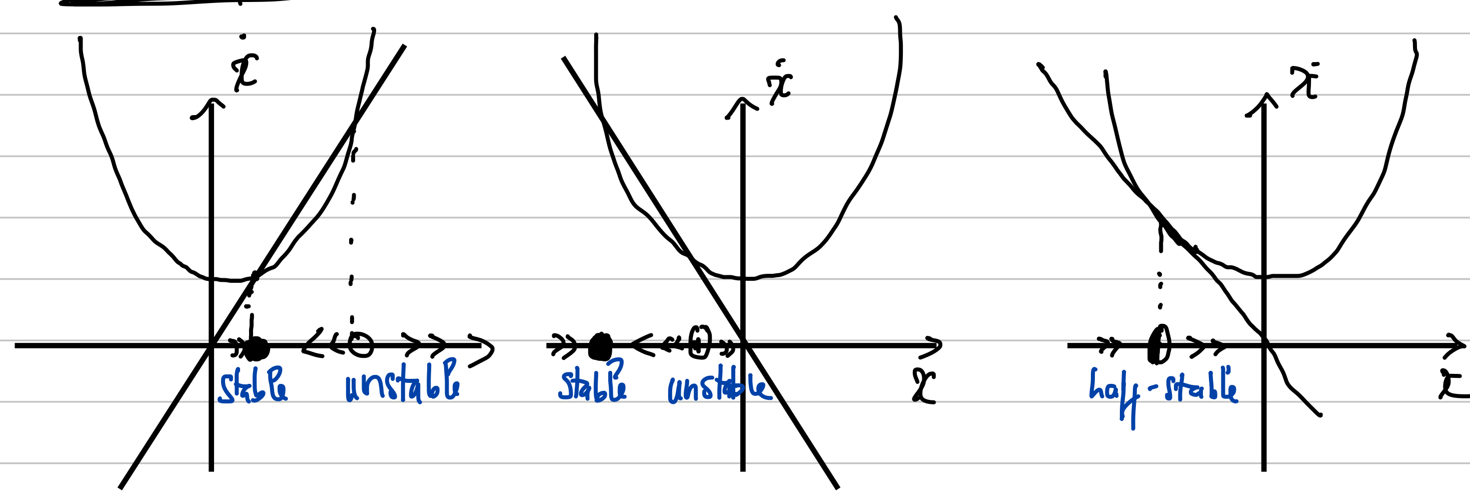
+ As we can see, if $r^2 - 4 > 0 \Leftrightarrow |r| > 2 \Rightarrow \left[\begin{array}{l} r > 2 \\ r < -2 \end{array}\right.$

$\Rightarrow$ there are 2 fixed-point

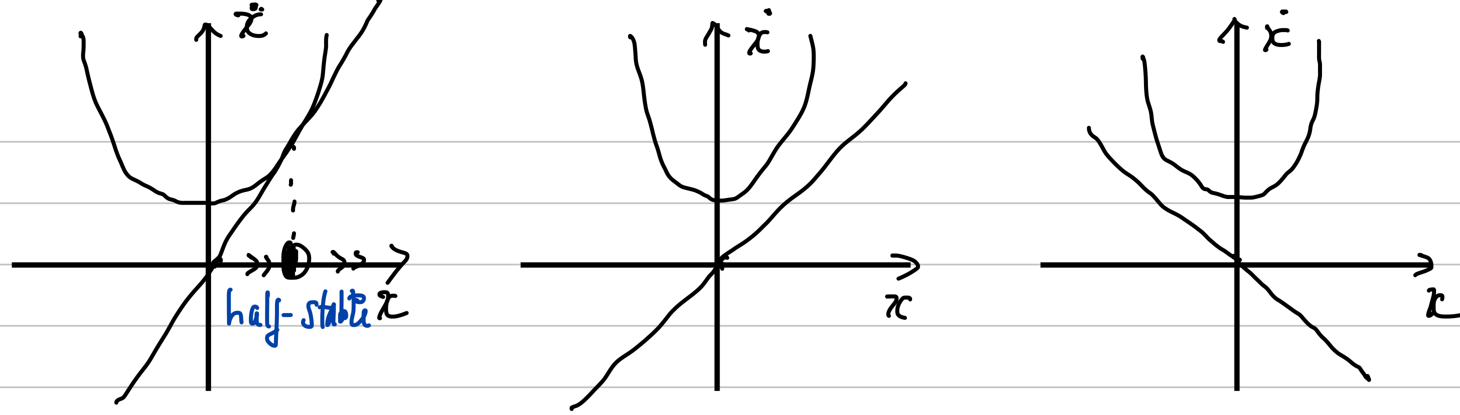
+ $r^2 - 4 = 0 \Rightarrow r = \pm 2 \Rightarrow$ 1 fixed point

+ $r^2 - 4 < 0 \Rightarrow r^2 < 4 \Leftrightarrow -2 < r < 2 \Rightarrow$ there are no fixed point, $y_1(x) = x^2 + 1$, $y_2(x) = -rx$

Vector fields:



$r < -2$ $\quad$ $r > 2$ $\quad$ $r = 2$



$r = -2$ $\qquad$ $-2 < r < 0$ $\qquad$ $0 < r < 2$

$\qquad\qquad r = -1$ $\qquad\qquad r = 1$

We have : possible bifurcation.

$$\begin{cases} \dot{x} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} x = \dfrac{-r \pm \sqrt{r^2-4}}{2} \\ 2x + r = 0 \Rightarrow x = -\dfrac{r}{2} \end{cases}$$

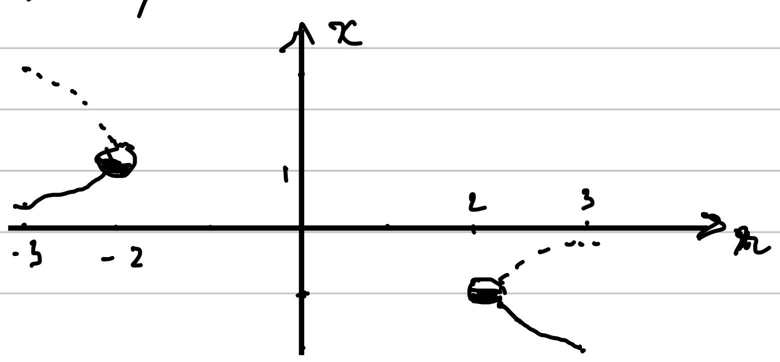
$$\Rightarrow -r \pm \sqrt{r^2-4} = -r \Rightarrow \sqrt{r^2-4} = 0$$

$$\Rightarrow \boxed{r = \pm 2}.$$

Based on the above diagram, we can realize

$$\boxed{\begin{array}{l} \text{at } r = 2 \Leftrightarrow x = -1 \\ \quad r = -2 \Leftrightarrow x = 1 \end{array} \quad \text{are } \underset{\text{bifurcation}}{\text{saddle-}} \text{ points.}}$$

Also, $\dot{x} = 0 \Leftrightarrow 1 + rx + x^2 = 0 \Leftrightarrow r = \dfrac{x^2-1}{x}$. Then, the bifurcation diagram:



b) $\dot{x} = r x - \dfrac{x}{1+x^2}$

We have $\dot{x} = 0 \Rightarrow rx = \dfrac{x}{1+x^2}$

$$\Rightarrow x\left(r - \frac{1}{1+x^2}\right) = 0 \Rightarrow \left[\begin{array}{l} x = 0 \\ r = \dfrac{1}{x^2+1} \end{array}\right.$$

$* \; r = \dfrac{1}{x^2+1} \Rightarrow x^2 + 1 = \dfrac{1}{r} \Rightarrow x^2 = \dfrac{1}{r} - 1$

$$\Rightarrow x = \pm\sqrt{\frac{1}{r} - 1}$$

$*$ Also $\dfrac{\partial f}{\partial x} = 0 \Leftrightarrow \left(rx - \dfrac{x}{1+x^2}\right)' = 0$

$$\Rightarrow r - \frac{(1+x^2) - x\,2x}{(1+x^2)^2} = 0$$

$$\Rightarrow r - \frac{1-x^2}{(1+x^2)^2} = 0 \Leftrightarrow r = \frac{1-x^2}{(1+x^2)^2}$$

$*$ When $x = 0 \Rightarrow r = 1$

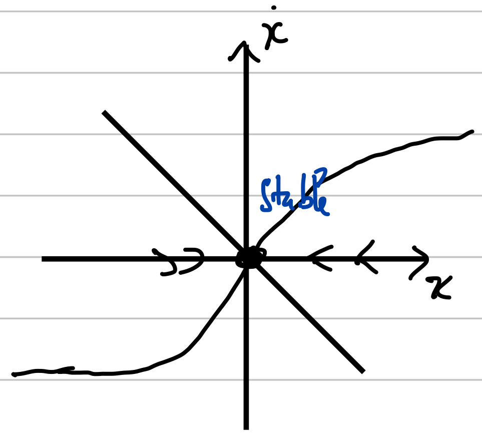
$*$ When $r = \dfrac{1}{x^2+1} \Rightarrow \dfrac{1}{x^2+1} = \dfrac{1-x^2}{(1+x^2)^2}$

$\Rightarrow 1 = \dfrac{1-x^2}{1+x^2} \Rightarrow x = 0 \Rightarrow r = 1$ (Same as above)

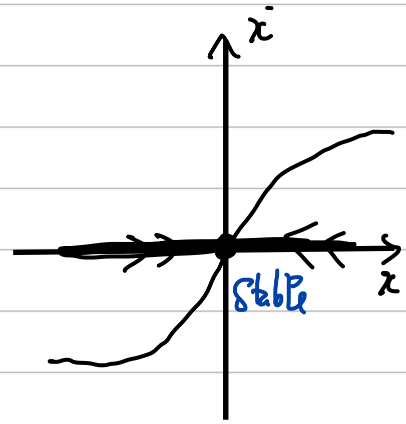
⇒ $\boxed{\text{there is one posible bifurcation at } x = 0 \text{ when } r = 1}$

Let $y_1(x) = rx$, $y_2(x) = \frac{x}{1+x^2}$, $\dot{x} = y_1(x) - y_2(x)$

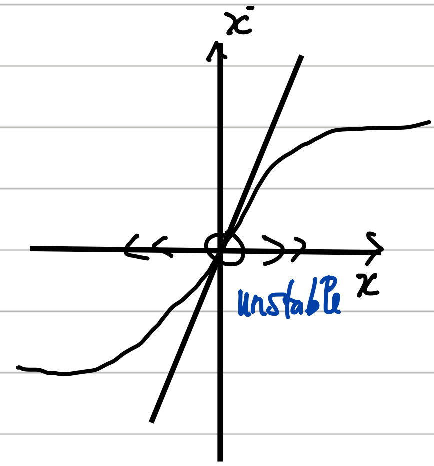
$r < 0$



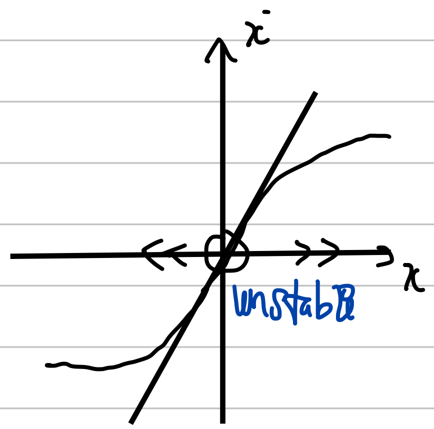
$r = 0$



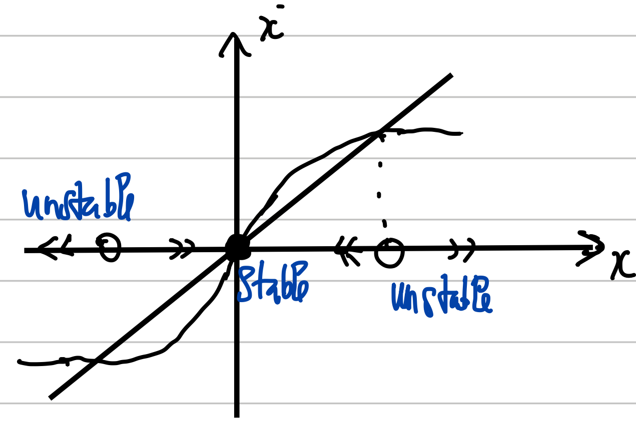
$r > 0$



$r = 1$

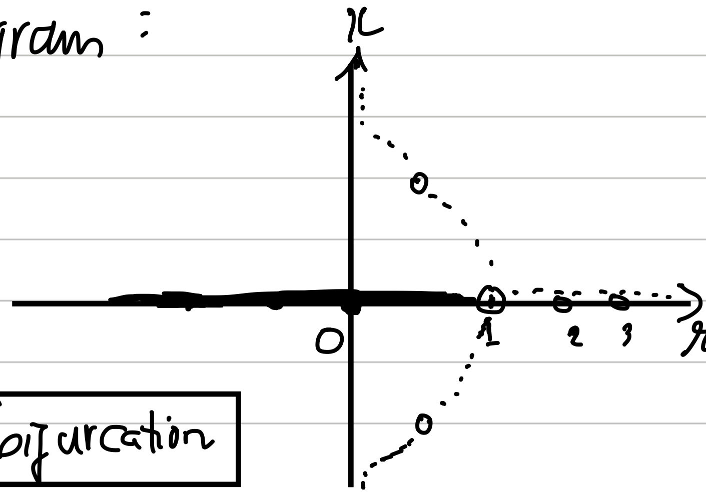


$0 < r < 1$



We have the bifurcation diagram:

$$\dot{x} = 0 \iff r = \frac{1}{x^2+1}$$



Based on the diagram with $\begin{cases} r = 1, \\ x = 0 \end{cases}$ the bifurcation point is a $\boxed{\text{subcritical pitchfork bifurcation}}$

c) $\dot{x} = x - rx(1-x) = x[1 - r(1-x)] = f(x)$

We have $\dot{x} = 0 \Rightarrow \begin{cases} x = 0 \\ 1 - r + rx = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ rx = r - 1 \end{cases}$

⊕ $\dfrac{\partial f}{\partial x} = 0 \Rightarrow 1 - r + 2xr = 0$

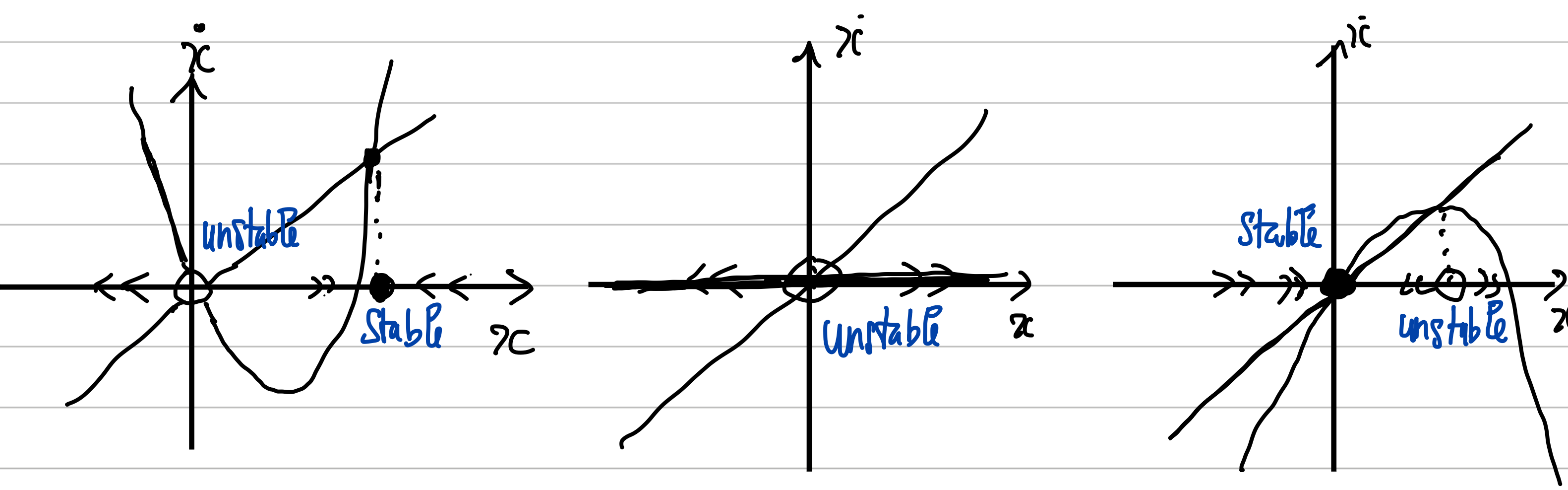
$\Rightarrow \quad 1 - r + 2(r-1) = 0$

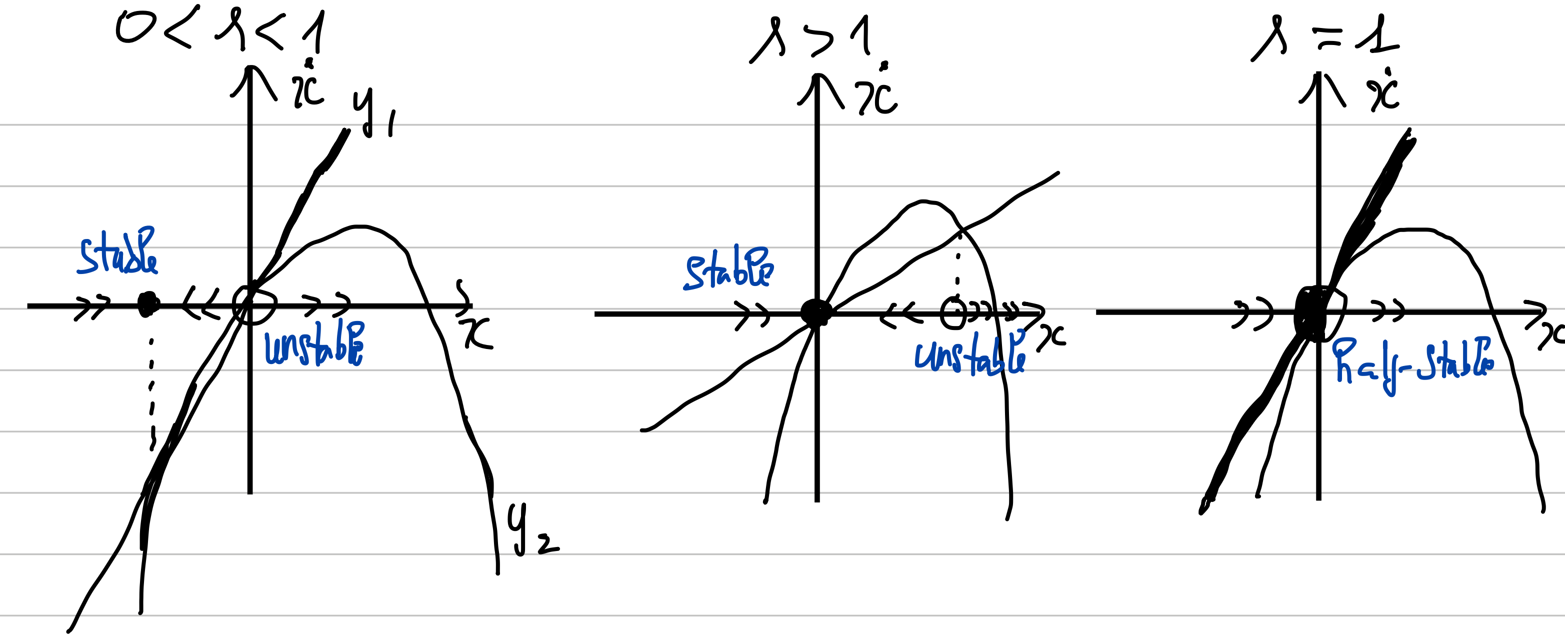
$\Rightarrow \quad 1 - r + 2r - 2 = 0 \Rightarrow r = 1 \Rightarrow x = 0$

$\Rightarrow$ there is a possible bifurcation at $\boxed{x = 0 \text{ when } r = 1}$

Let $y_1(x) = x$, $y_2(x) = rx(1-x)$

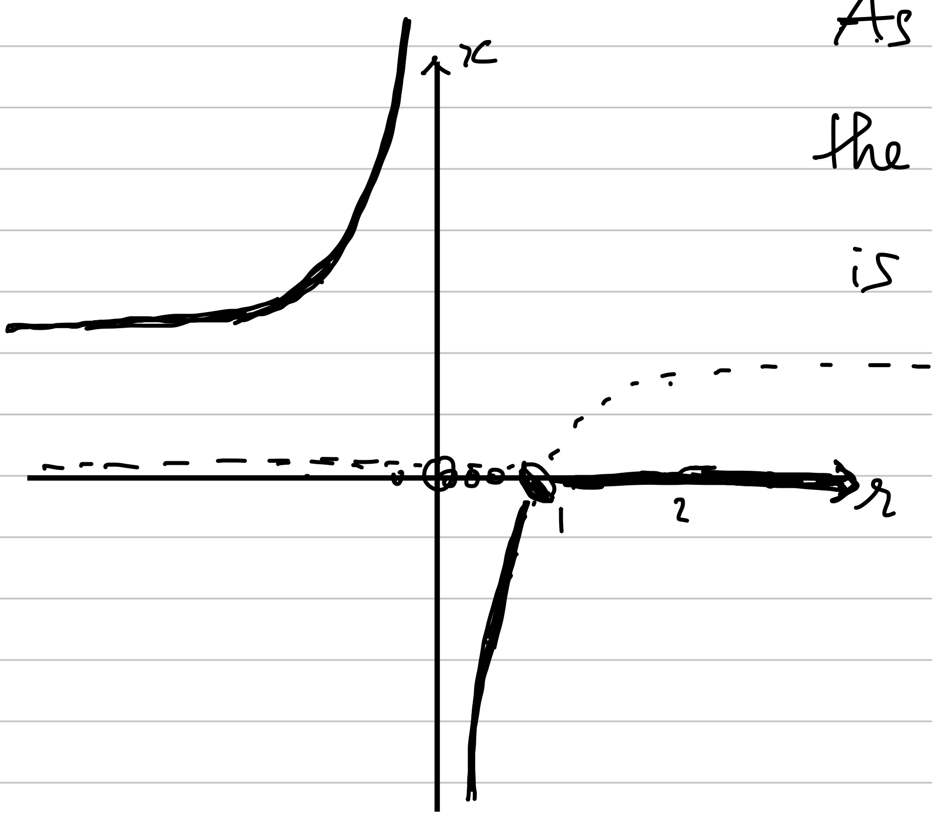
$r < 0$ $\qquad\qquad$ $r = 0$ $\qquad\qquad$ $r > 0$





Also, $\dot{x} = 0 \Leftrightarrow x = rx(1-x) \Rightarrow r = \frac{1}{1-x}$

we have bifucation diagram.



As we can see the diagram the point $\boxed{(x,r) = (0,1)}$ is a $\boxed{\text{transcritical bifurcation point}}$

d) $\dot{x} = r - \cosh x = f(x)$

we have:

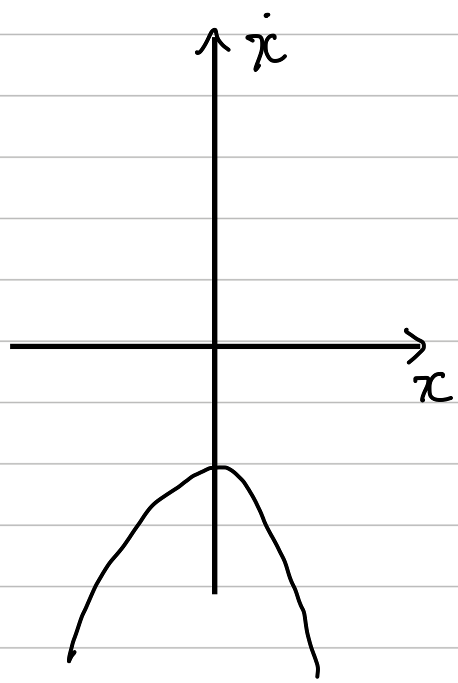
$$\begin{cases} \dot{x} = 0 \iff \lambda = \cosh x \\ \dfrac{\partial f}{\partial x} = 0 \iff \sinh x = 0 \iff x = 0 \end{cases}$$

with $x = 0 \Rightarrow \cosh x = \frac{1}{2}\left[e^{x} + e^{-x}\right] = 1$

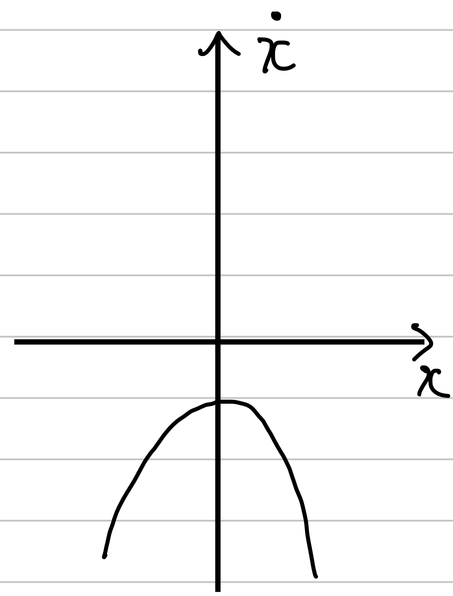
$\Rightarrow \lambda = 1 \rightarrow$ posible bifurcation at $(x, \lambda) = (0, 1)$

<u>We have:</u>

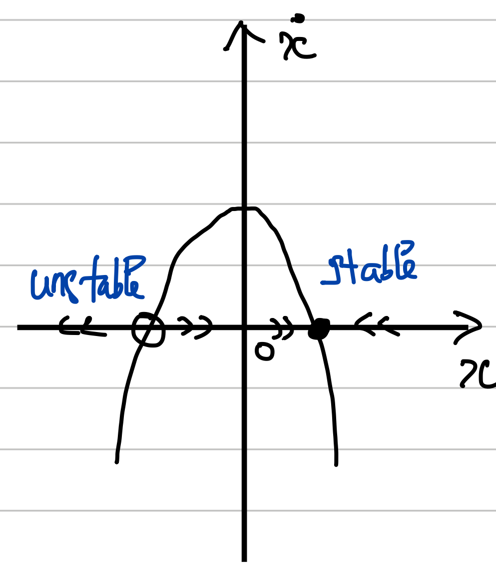
$\lambda < 0$



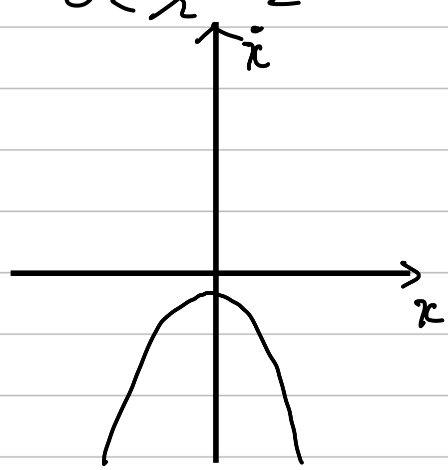
$\lambda = 0$



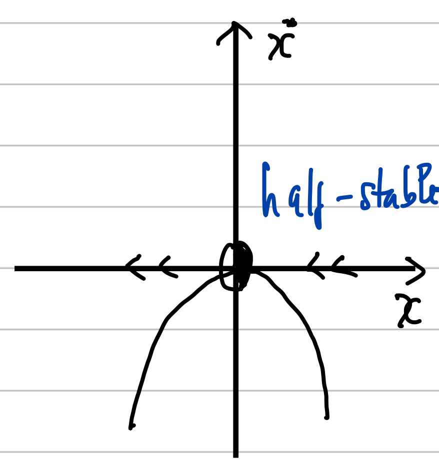
$\lambda > 0$



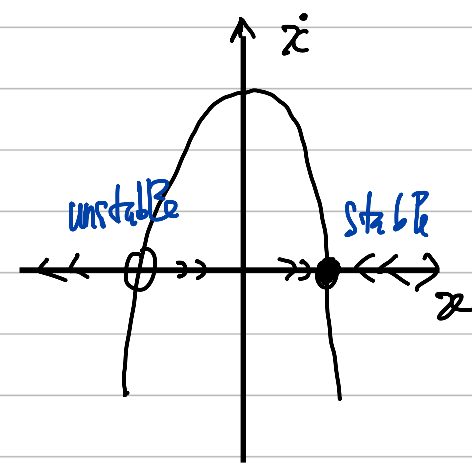
$0 < \lambda < 1$



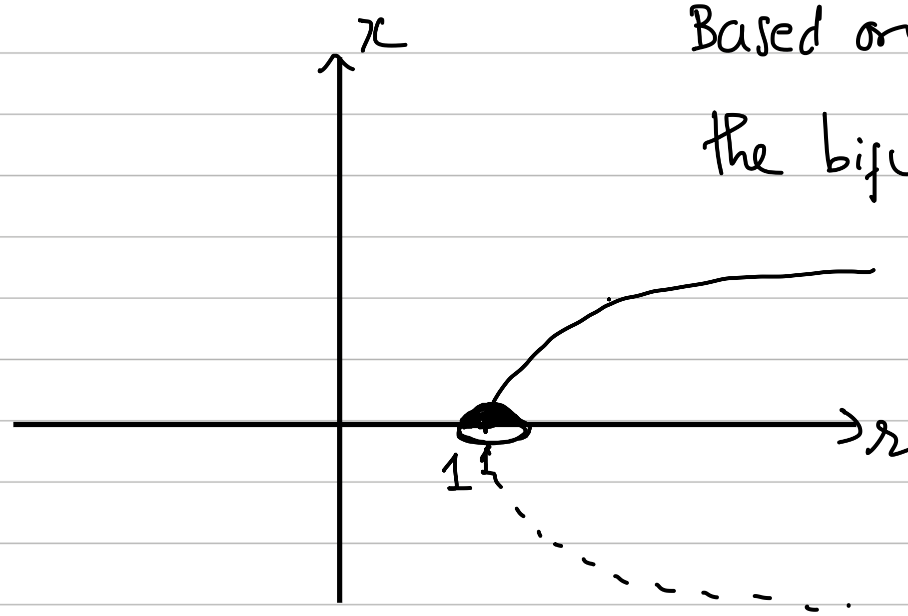
$\lambda = 1$



$\lambda > 1$



We have the bifurcation diagram:



Based on diagram, we can see the bifucation point $\boxed{(x,r) = (0,1)}$ $\boxed{\text{is a Saddle bifurcation point}}$

2) Given $\dot{x} = -V'(x)$

a) $\dot{x} = r - x^2$

We have: $\begin{cases} \dot{x} = 0 & \Rightarrow r = x^2 \\ \dfrac{\partial f}{\partial x} = 0 & \Rightarrow -2x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ r = 0 \end{cases}$

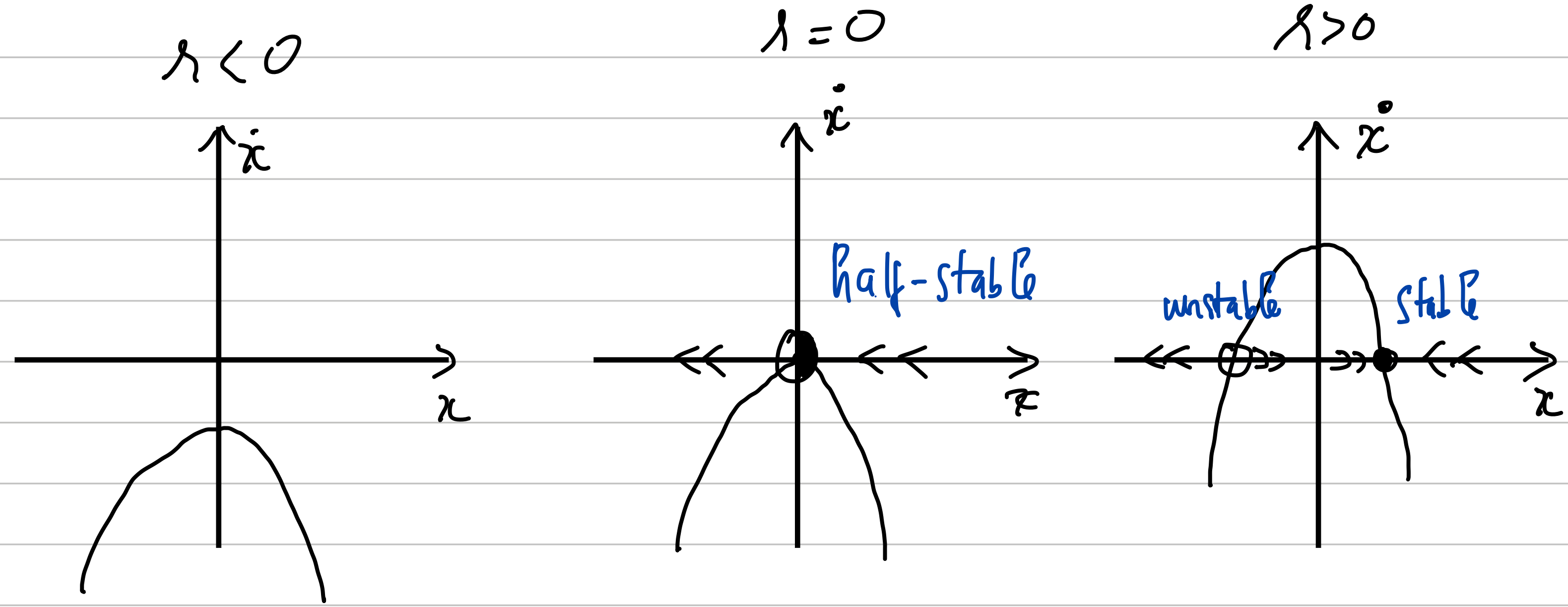
$\Rightarrow$ the posible bifurcation point $(x, r) = (0, 0)$.

Also: $r - x^2 = -V'(x)$

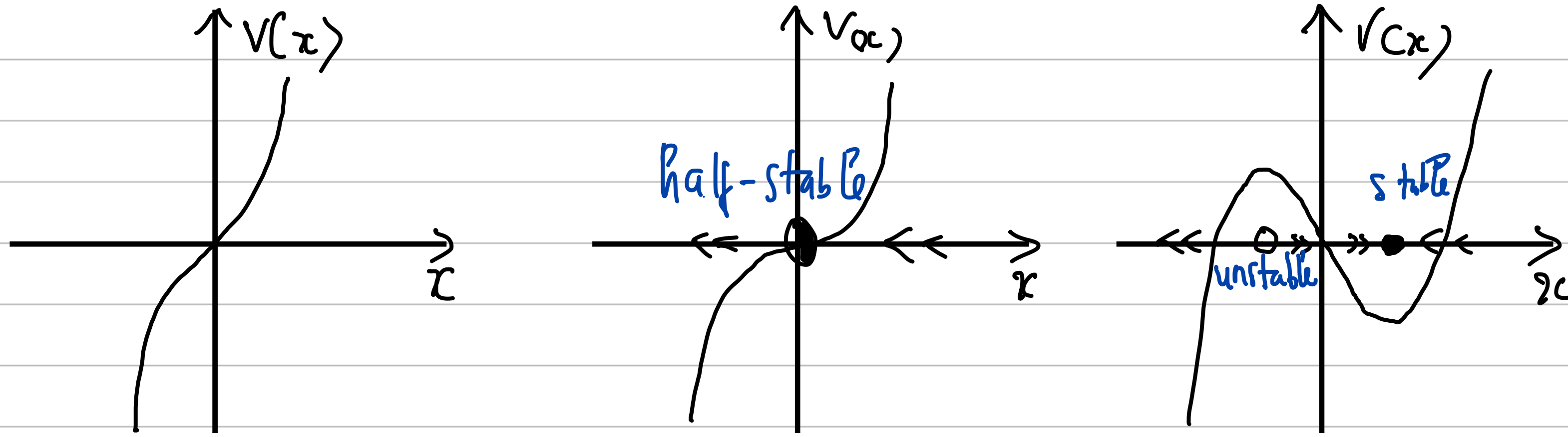
$$\Rightarrow V(x) = -\int (r - x^2)\,dx = \int (x^2 - r)\,dx.$$

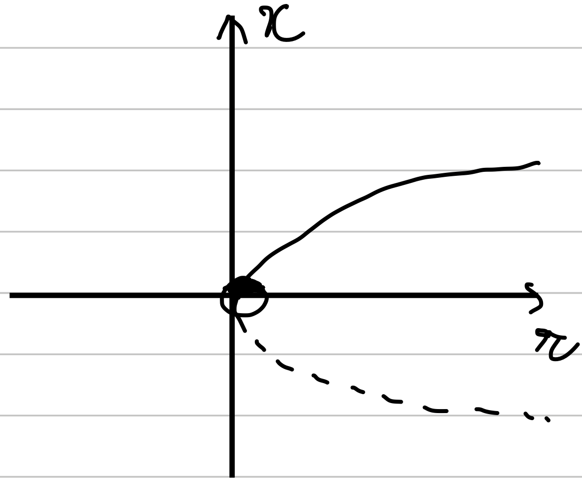
$$\Rightarrow V(x) = \frac{1}{3}x^3 - rx$$

We have:



$r<0$ $\qquad$ $r=0$ $\qquad$ $r>0$





$V(x)=0 \iff \frac{1}{3}x^3 = rx$

$\Rightarrow \quad \frac{1}{2}x^2 = r$

$\boxed{\text{Based on bifurcation, we can see the point } (x,r)=(0,0) \text{ is a saddle point}}$

b) $\dot{x} = rx - x^2 = f(x,r)$

We have:

$$\begin{cases} \dot{x} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \iff \begin{cases} x(r-x)=0 \\ r-2x=0 \end{cases} \iff \begin{cases} \left[\begin{array}{l} x=0 \\ r=x \end{array}\right. \\ r=2x \end{cases}$$

$\Rightarrow \begin{cases} x=0 \\ r=0 \end{cases}$ is posible bifurcation point

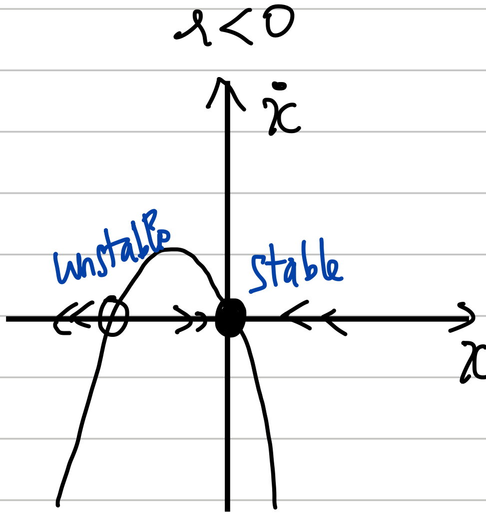
Also, $\dot{x} = -V'(x)$

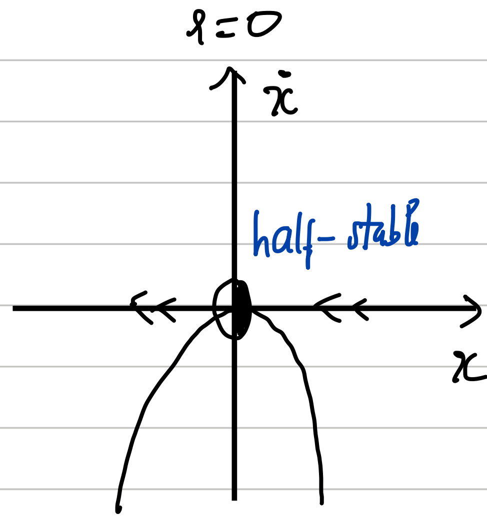
$$\Rightarrow \quad V(x) = -\int (rx - x^2)\,dx = \int (x^2 - rx)\,dx$$

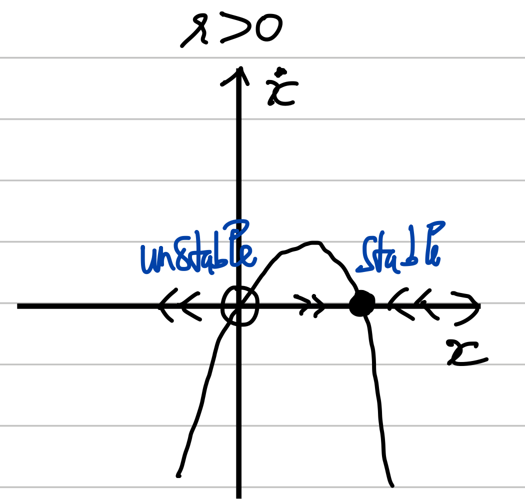
$$\Rightarrow \quad V(x) = -\frac{1}{2}rx^2 + \frac{1}{3}x^3 = \frac{1}{3}x^3 - \frac{1}{2}rx^2$$

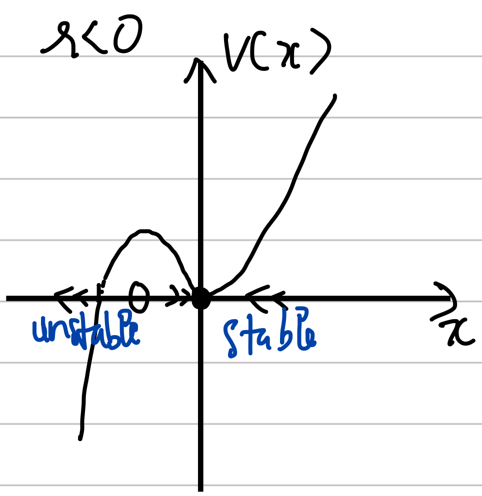
We have the diagram:

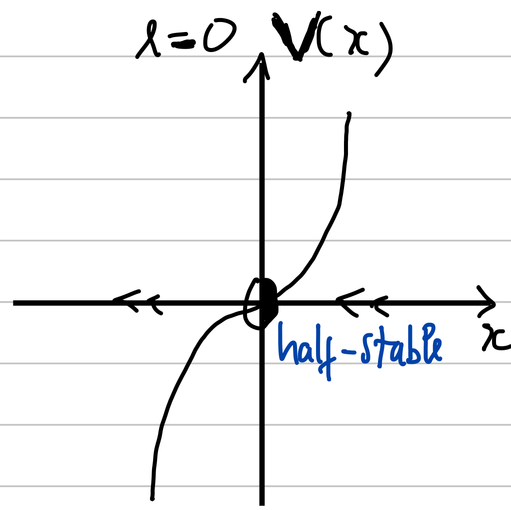
$$\dot{x} = rx - x^2$$

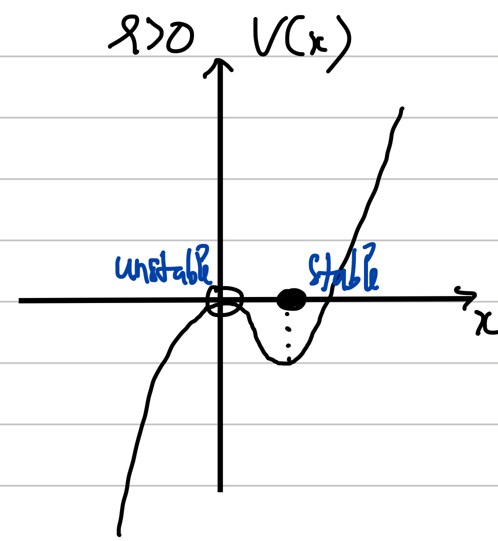






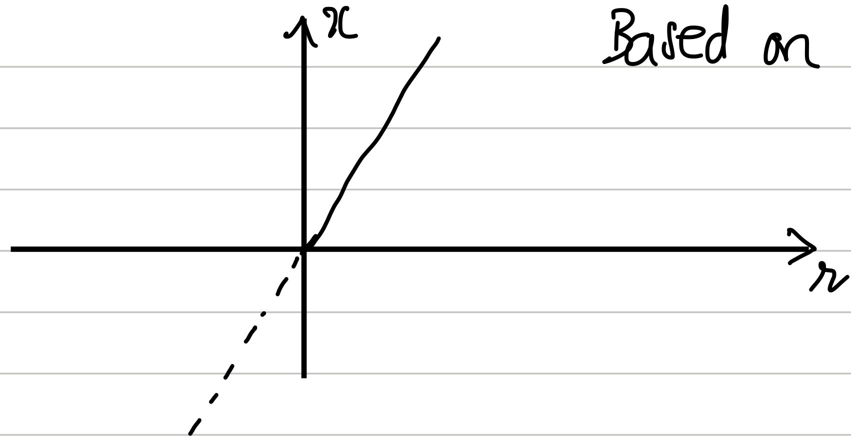






$$V(x) = 0 \Leftrightarrow \frac{1}{3}x^3 - \frac{1}{2}rx^2 = 0 \Rightarrow 2x^3 - 3rx^2 = 0$$

$$\Leftrightarrow 2x - 3r = 0 \Rightarrow r = \frac{2x}{3} \Leftrightarrow x = \frac{3r}{2}$$



Based on the diagram, we can see at $(x, r) = (0,0)$ is a $\boxed{\text{transcritical bifurcation point}}$

c) $\dot{x} = rx - x^3$

let $\begin{cases} \dot{x} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \Leftrightarrow \begin{cases} x(r - x^2) = 0 \\ r - 3x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x = 0 \\ r = x^2 \end{array}\right. \\ r = 3x^2 \end{cases}$

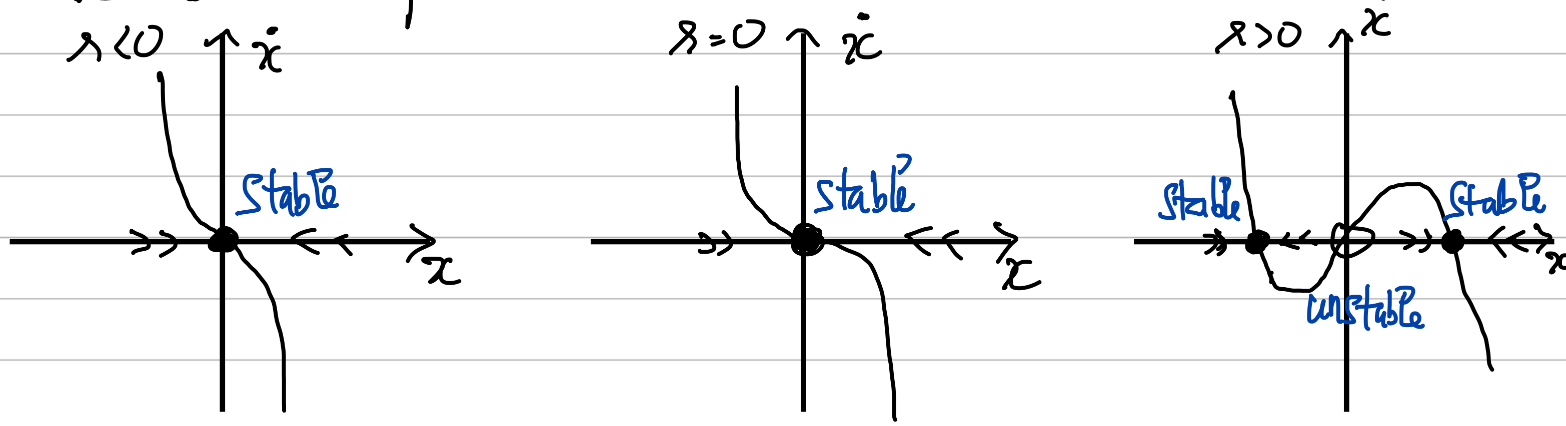
$\Leftrightarrow \begin{cases} x = 0 \\ r = 0 \end{cases}$ is a possible bifurcation point.

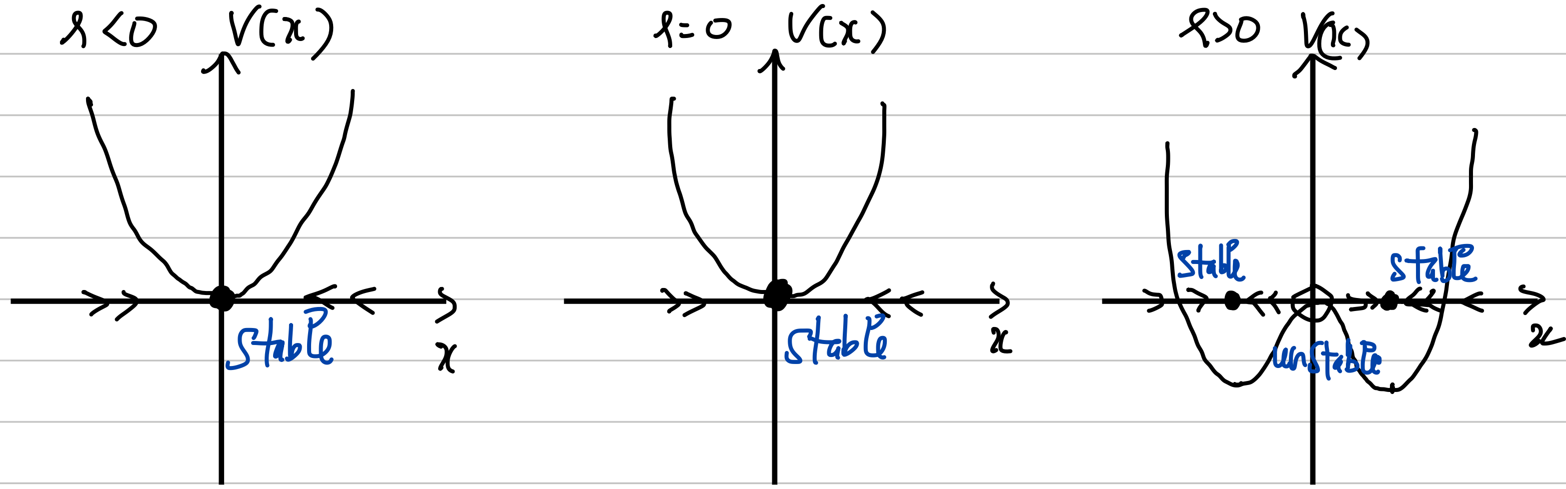
Also, $\dot{x} = rx - x^3 = -V'(x)$

$\Rightarrow V(x) = -\int (rx - x^3)\,dx = \int (x^3 - rx)\,dx$

$\Rightarrow V(x) = \dfrac{1}{4}x^4 - \dfrac{1}{2}rx^2$

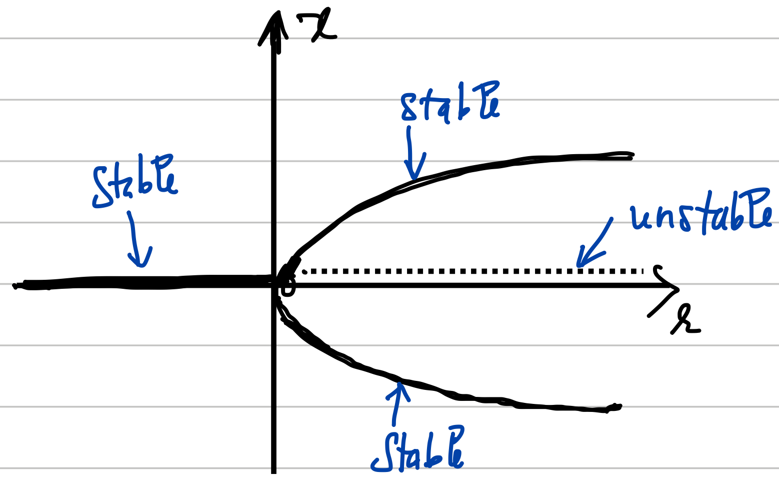
We have diagram:





We have: $V(x) = 0 \iff x^4 - 2rx^2 = 0 \Rightarrow x^2 = 2r$

$\iff r = \dfrac{x^2}{2}$



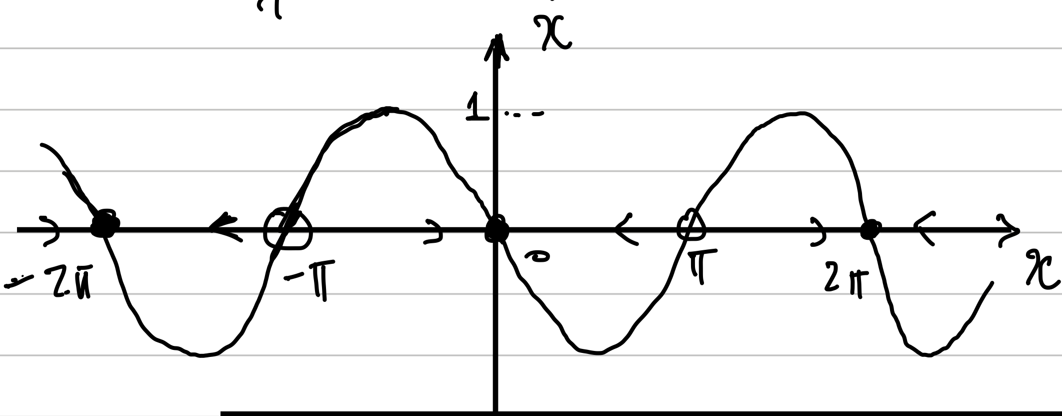
Based on bifurcation, we have point $(x, r) = (0, 0)$ is a supercritical pitchfork bifurcation point

3) $\dot{x} = rx - \sin x$

a) When $r = 0 \Rightarrow \dot{x} = -\sin x = f(r, x)$.

$\Rightarrow \dot{x} = 0 \Rightarrow \sin x = 0 \Rightarrow x = k\pi,\ k \in \mathbb{Z}$

Then the vector field:



As we can see, at $x = 2n\pi$ are stable points

at $x = (2n+1)\pi$ are unstable points

with $n \in \mathbb{Z}$.

b) When $r = 1 \Rightarrow \dot{x} = x - \sin x = f(x)$

We have: $\begin{cases} f(x) = x - \sin x = 0 \\ \dfrac{\partial f}{\partial x} = 1 - \cos x = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ x = \pi + 2k\pi \end{cases}$ $(k \in \mathbb{Z})$

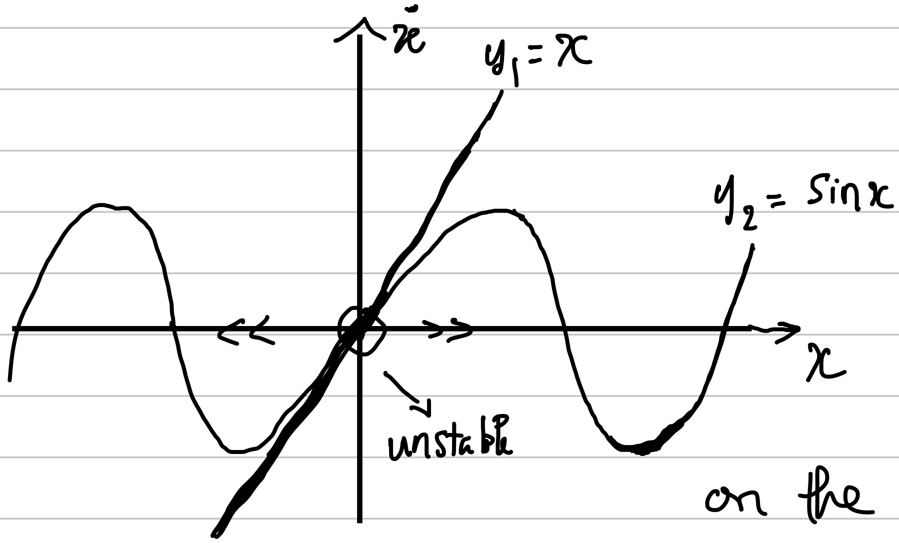
$\Rightarrow x = 0 \Leftrightarrow y_1(x) = x$ is tangible of $y_2(x) = \sin x$.

Besides, $\cos x \leq 1 \Rightarrow \dfrac{\partial f}{\partial x} = 1 - \cos x \geq 0 \Rightarrow f(x)$ is strictly increasing.

Also, both $f(x)$ & $\frac{\partial f}{\partial x}$ are continuous with $(x_0, x_0) \in \mathbb{R}^2 \Rightarrow x_0 = 0$ is a only solution for the initial value: $\dot{x} = x - \sin x$

Or there is only one fixed point $x = 0$

We also have:

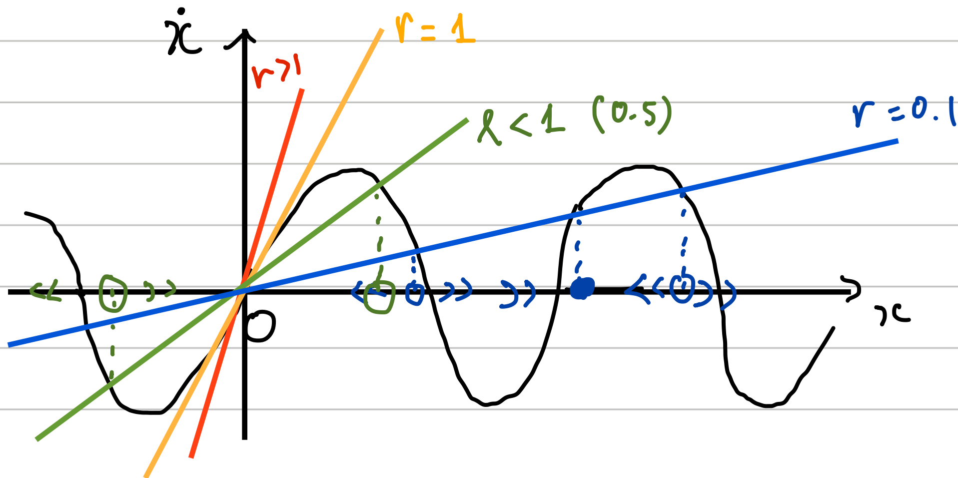


$\dot{x} = x - \sin x = y_1 - y_2$

as we can see, If y_1 lies above $y_2 \Leftrightarrow y_1 > y_2$ means $\dot{x} > 0$. Otherwise, $\dot{x} < 0$. Then, as showed on the diagram, we realize this fixed point is $\boxed{\text{unstable}}$

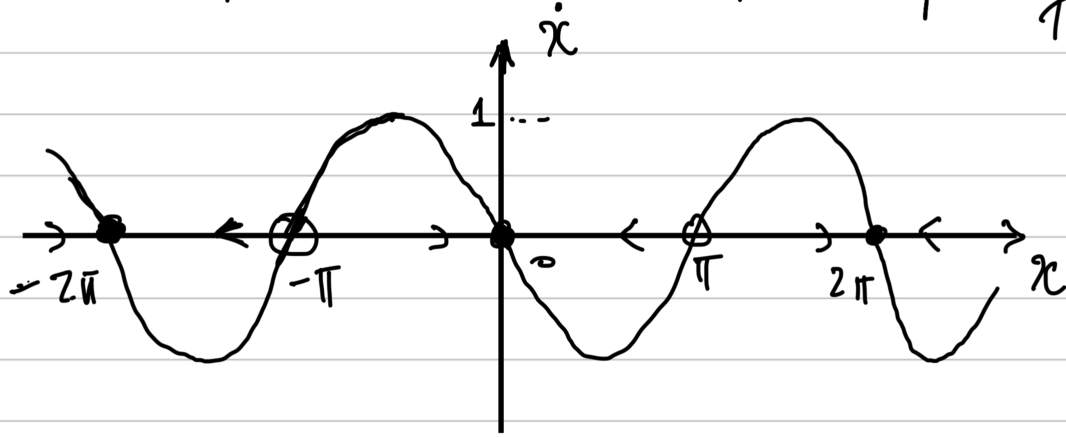
c) As r decrease from ∞ to 0, $\dot{x} = rx - \sin x$

$\dot{x} = rx - \sin x$



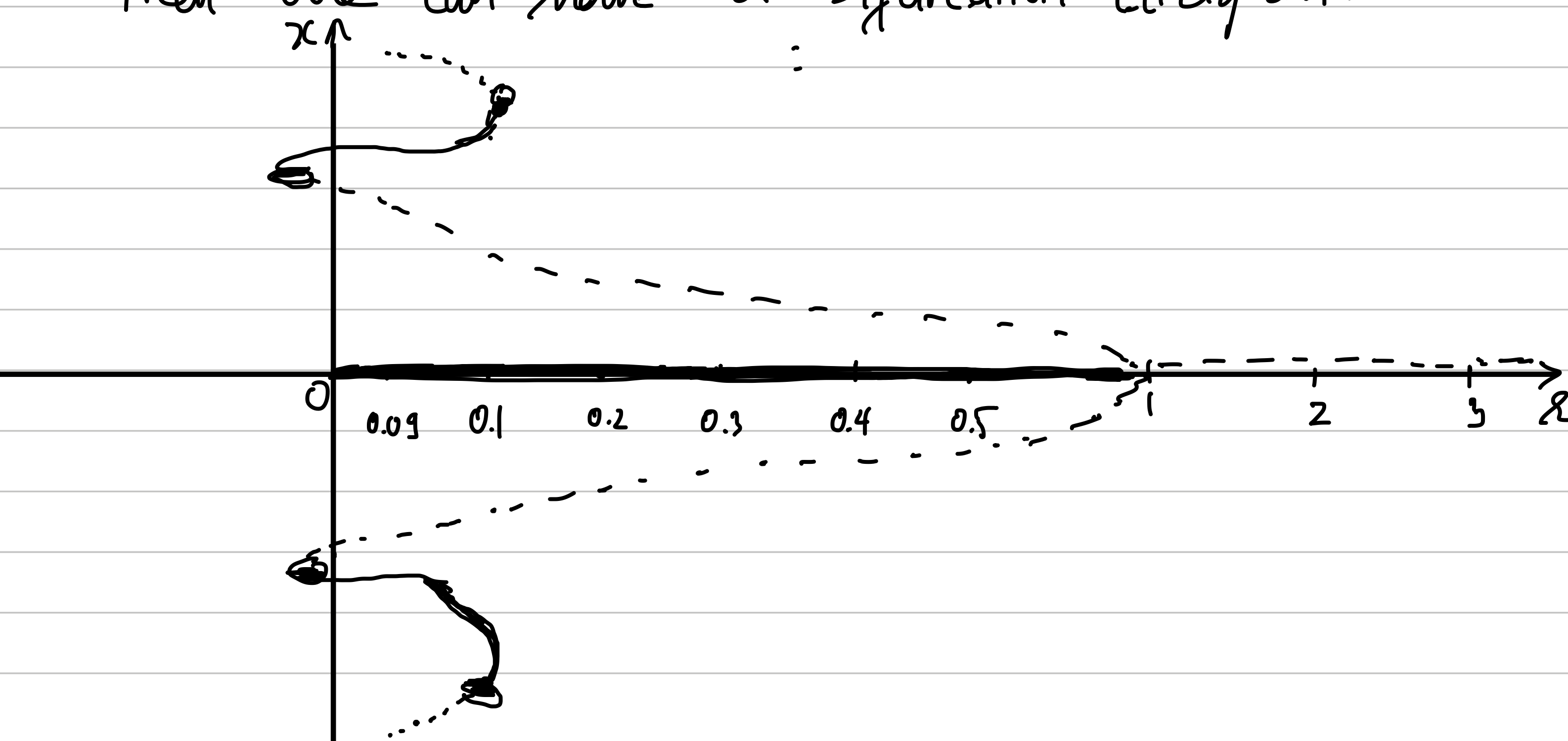
Based on the diagram, we we decrease the value of r, there are more intersection between $y_1 = r x$ and $y_2 = \sin x$ or the $f(x)$ will have more fixed points.

Based on the diagram, with $r \geq 1$, at $x=0$ is allways an unstable point. However if $r<1$ at $x=0$ will be changed to stable point Until $r=0$, we will have the diagram from part a



$(r=0)$

Then we can have a bifurcation diagram:



Based on the diagram, we can see when r is decreased from $\infty \to 0$, there is a subcritical pitchfork bifurcation at $r = 1$ and infinite saddle point bifurcation when $0 < r < 1$.

d) Given $0 < r \ll 1$

As we showed in part a, in $0 < r \ll 1$, there are saddle point bifurcation, so it satisfies:

$$\begin{cases} \dot{x} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} rx = \sin x \Rightarrow r = \dfrac{\sin x}{x} \\ r = \cos x \end{cases}$$

Also, $r \ll 1 \Rightarrow \cos x \ll 1 \Rightarrow \cos x \approx 0$

$$\Rightarrow x \approx \frac{\pi}{2} + 2n\pi \quad \text{with } n \geq 0$$

Because $r > 0 \Rightarrow \begin{cases} \cos x > 0 \\ \dfrac{\sin x}{x} > 0 \end{cases} \Rightarrow x$ has to be positive

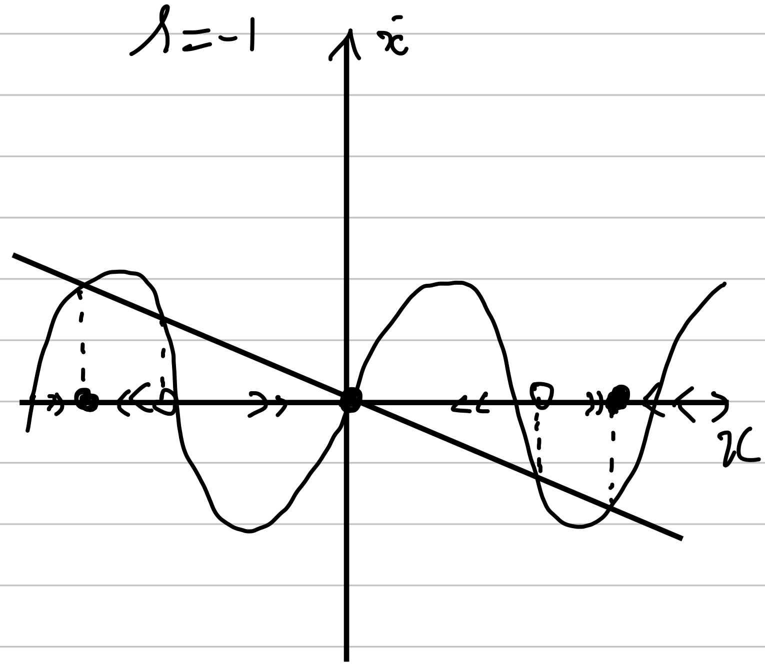
$$\Rightarrow r = \frac{\sin\left(\frac{\pi}{2} + 2n\pi\right)}{\frac{\pi}{2} + 2n\pi} = \frac{1}{\frac{\pi}{2} + 2n\pi} = \frac{1}{\frac{\pi}{2} + \frac{4n\pi}{2}}$$

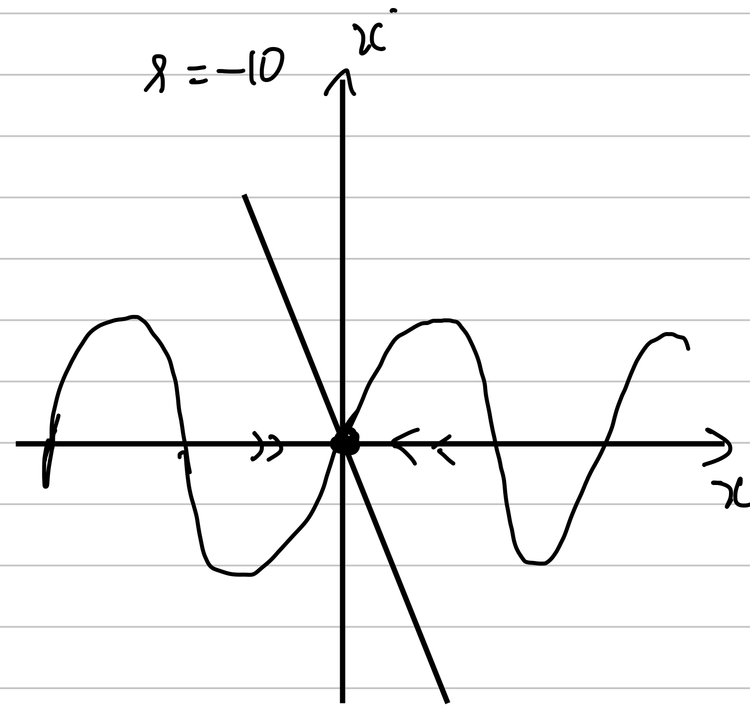
$$\Rightarrow \boxed{r = \frac{2}{\pi(4n+1)} = \left[\frac{(4n+1)\pi}{2}\right]^{-1}}$$

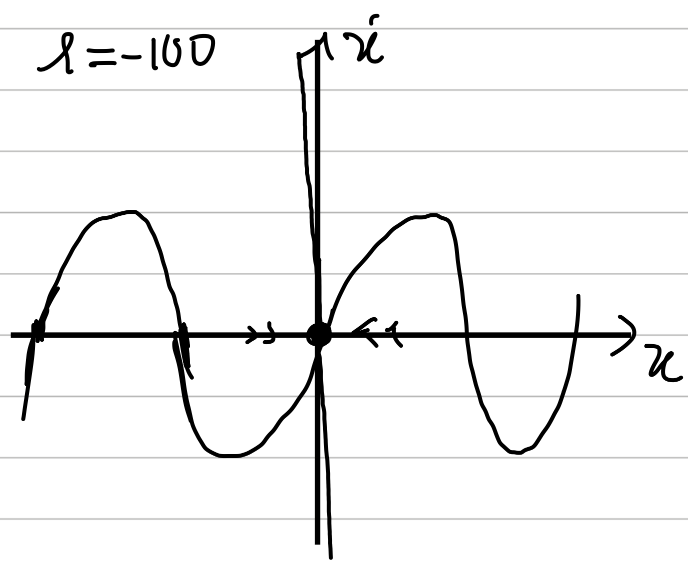
e) With r decrease from 0 → ∞

we can take some example of r like below:

$$\dot{x} = rx - \sin x$$







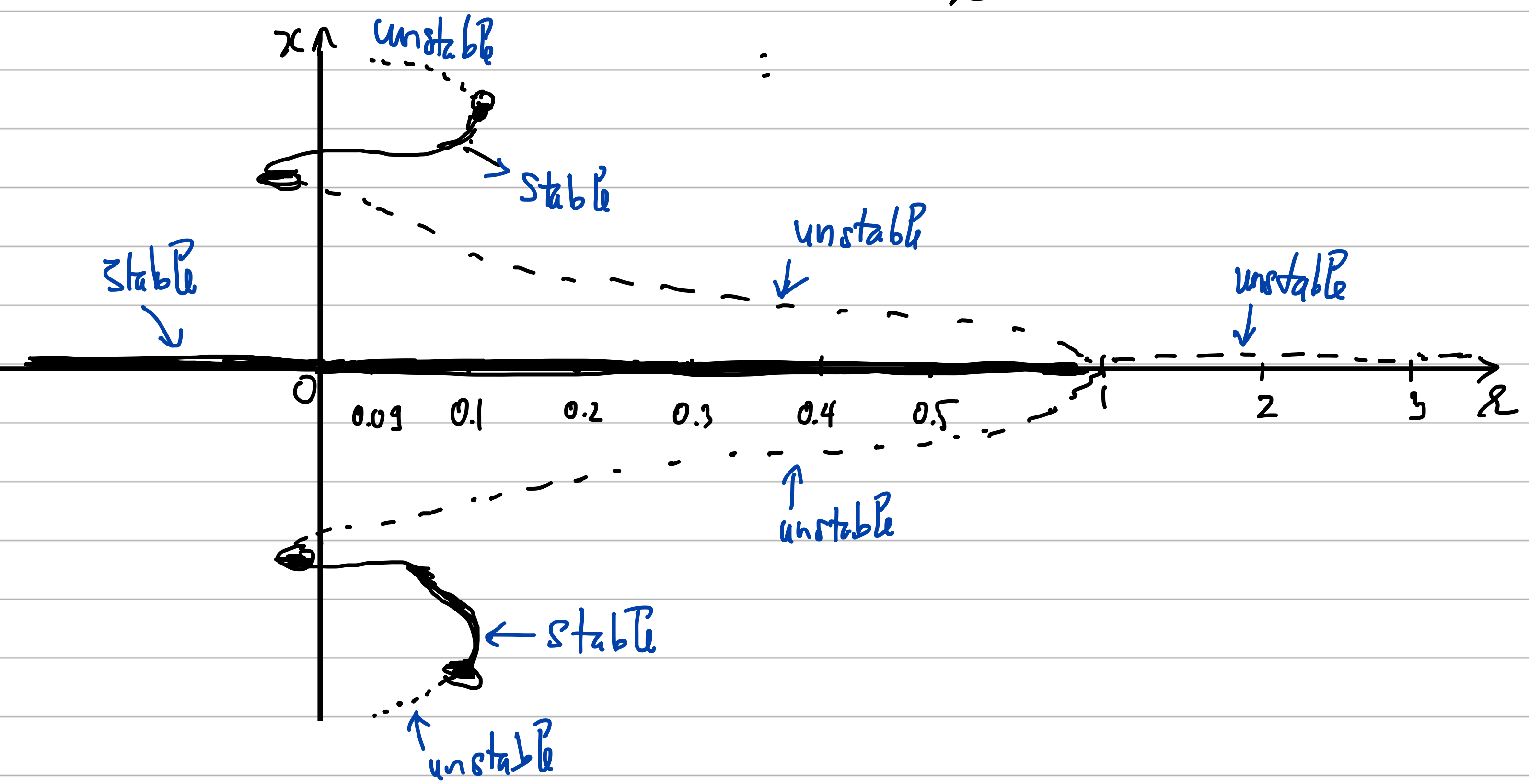
We can see when r is very low, approaches to −∞, there is only 1 stable fixed point

means the saddle node

bifurcation has been dissapear and having the stable point

f) Based on previous part, we can have the bifurcation diagram: $(-\infty < r < \infty)$

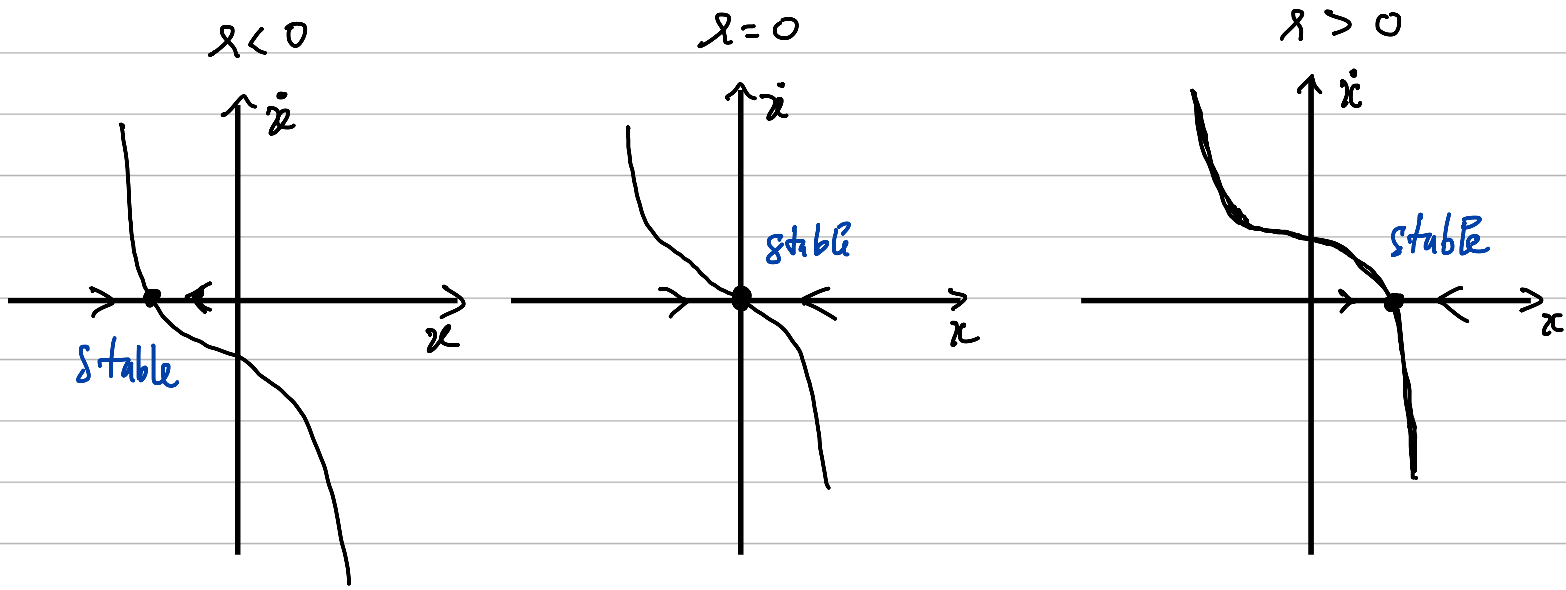
$$r = \frac{\sin x}{x}$$



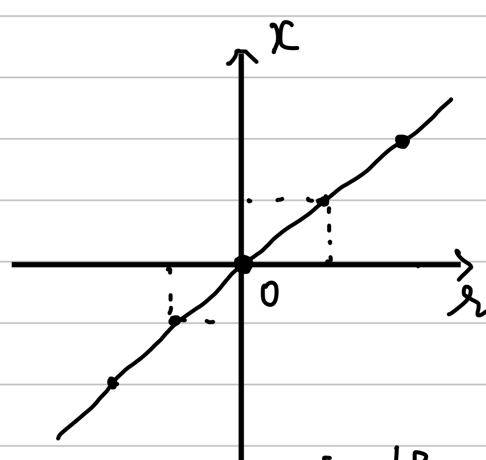
4) a) $\dot{x} = r^3 - x^3 = f(x,r)$

Firstly, we will find the possible bifurcation points by using

$$\begin{cases} \dot{x} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \Leftrightarrow \begin{cases} x = r \\ x = 0 \end{cases} \Leftrightarrow x = r = 0$$



Then we can have the bifurcation diagrams:



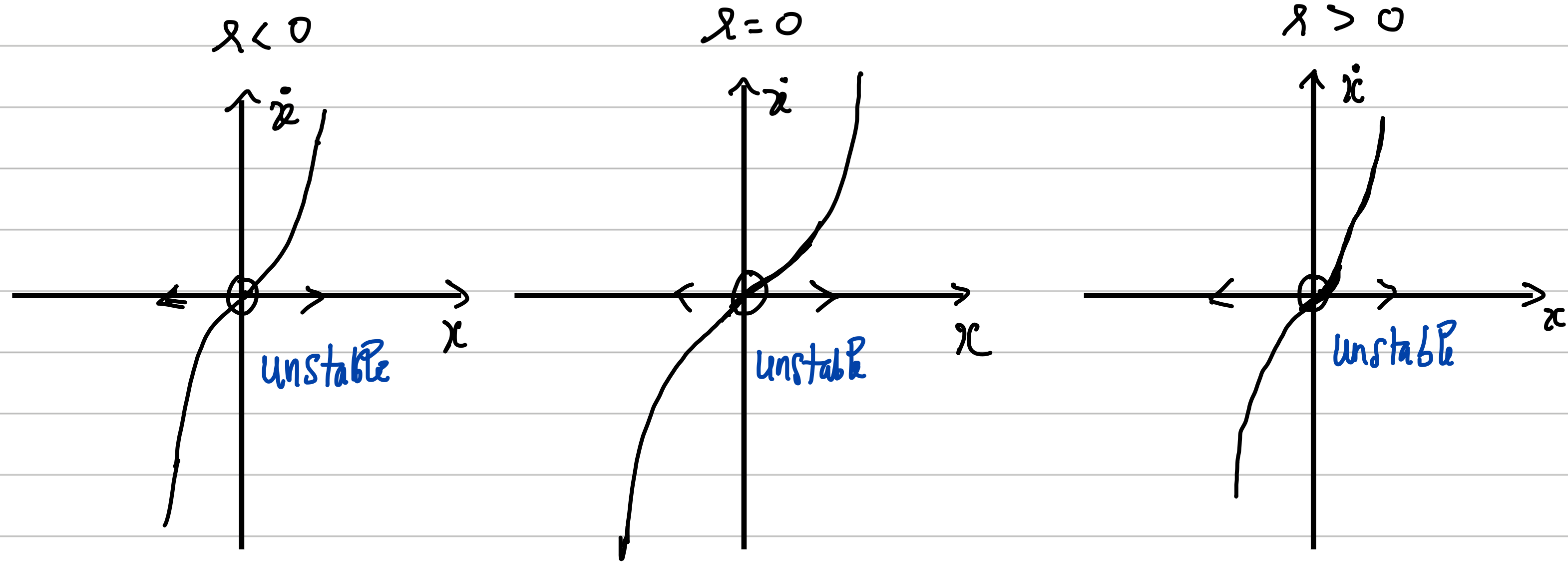
Based on the diagram, we can see it is allways stable at all point of x & r included $(x,r) = (0,0)$ which is fixed point. Hence, there is no change in the fixed point structure and alway stable.

$\Rightarrow$ $\boxed{\text{There is no actual bifurcation point}}$

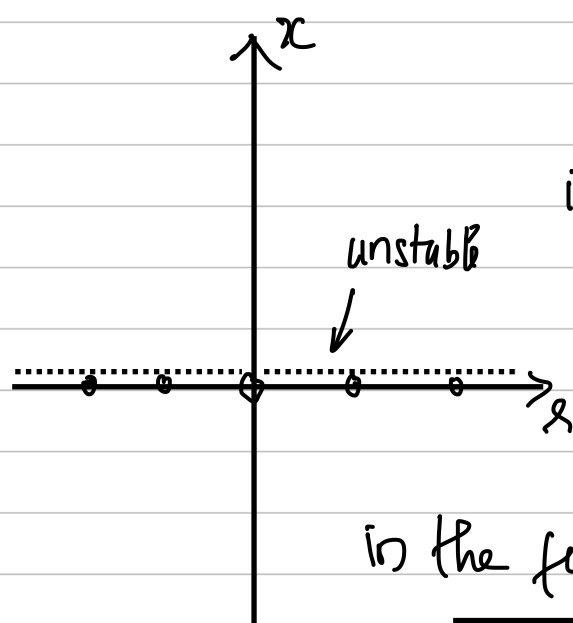
b) $\dot{x} = r^2 x + x^3 = f(x, r)$

The possible bifurcation points will satisfy :

$$\begin{cases} \dot{x} = 0 \\ \dfrac{\partial f}{\partial x} = 0 \end{cases} \Leftrightarrow \begin{cases} x(r^2 + x^2) = 0 \\ r^2 + 3x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ r = 0 \end{cases}$$



Then we can have a bifurcation diagram.



Based on the diagram, we can see it is always unstable at all point of x & r included $(x, r) = (0, 0)$ which is fixed point. Hence, there is no change in the fixed point structure and always unstable

$\Rightarrow$ $\boxed{\text{There is no actual bifurcation point}}$