Math 134, Winter 2022

Lecture #5: Numerical methods & Existence and Uniqueness.

Wednesday April 5$^{\text{th}}$

Learning objectives

Today we will discuss:

- The local truncation error of a numerical method.
- The improved Euler method.
- The Runge–Kutta method.
- The Cauchy–Peano Existence Theorem.
- Picard-Lindelöf Existence and Uniquness Theorem

Numerical methods

The local truncation error

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

- Let $x_n \approx x(n\Delta t)$
- Define the **local truncation error**

$$e_1 = x(\Delta t) - x_1$$

Euler's method: $x_1 = x_0 + f(x_0)\Delta t$

Taylor's theorem with Lagrange residue

$$x(\Delta t) = \underbrace{x(0)}_{= x_0} + x'(0)\Delta t + \frac{x''}{2}(\xi)\Delta t^2 \qquad \xi \in (0, \Delta t)$$

Since $\dot{x} = f(x) \Rightarrow \dot{x}(0) = f(x(0)) = f(x_0)$

$\ddot{x} = f'(x)\, x' = f'(x) f(x)$

So if f and f' are bounded and cont.

then $|\text{Remainder}| \leq C(\Delta t)^2$

$$e_1 = x(\Delta t) - x_1$$
$$= x(\Delta t) - (x_0 + f(x_0)\Delta t)$$
$$= \frac{x''(\xi)}{2}(\Delta t)^2$$

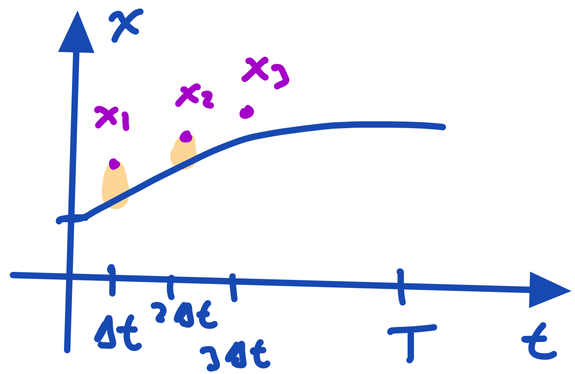
$$c = \max_{y \in [0,T]} |f(y) f'(y)|$$

$$\rightarrow \quad |e_1| \leq c(\Delta t)^2$$

$\Rightarrow$ Euler's method is 1st order because
$|e_1| \leq c(\Delta t)^2 \leftarrow$ local error

$$\underbrace{|e_1| + |e_2| + \cdots + |e_n|}_{\text{global error}} \leq \underbrace{c\,n(\Delta t)^2}_{c\,n\left(\frac{T}{n}\right)^2} = cT\Delta t$$

$$c \cancel{2} T \frac{\Delta t}{\cancel{2}}$$



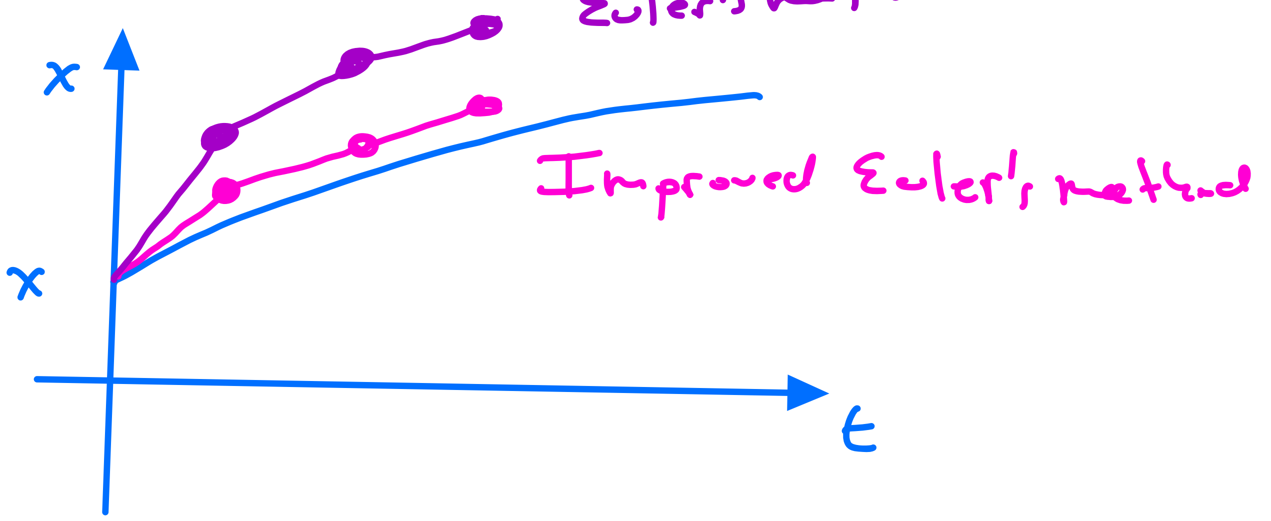
$\Delta t = \frac{T}{n}$

$T = n\Delta t$

Question: How could we improve our approximation?

- Take smaller timesteps (Δt)
  ↳ more computations
  ↳ "Round-off errors"

- Improve the approximation



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Improved Euler's method

- Want to approximate the solution of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

- For $n \geq 0$:
    - Make a first approximation:

$$\widetilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

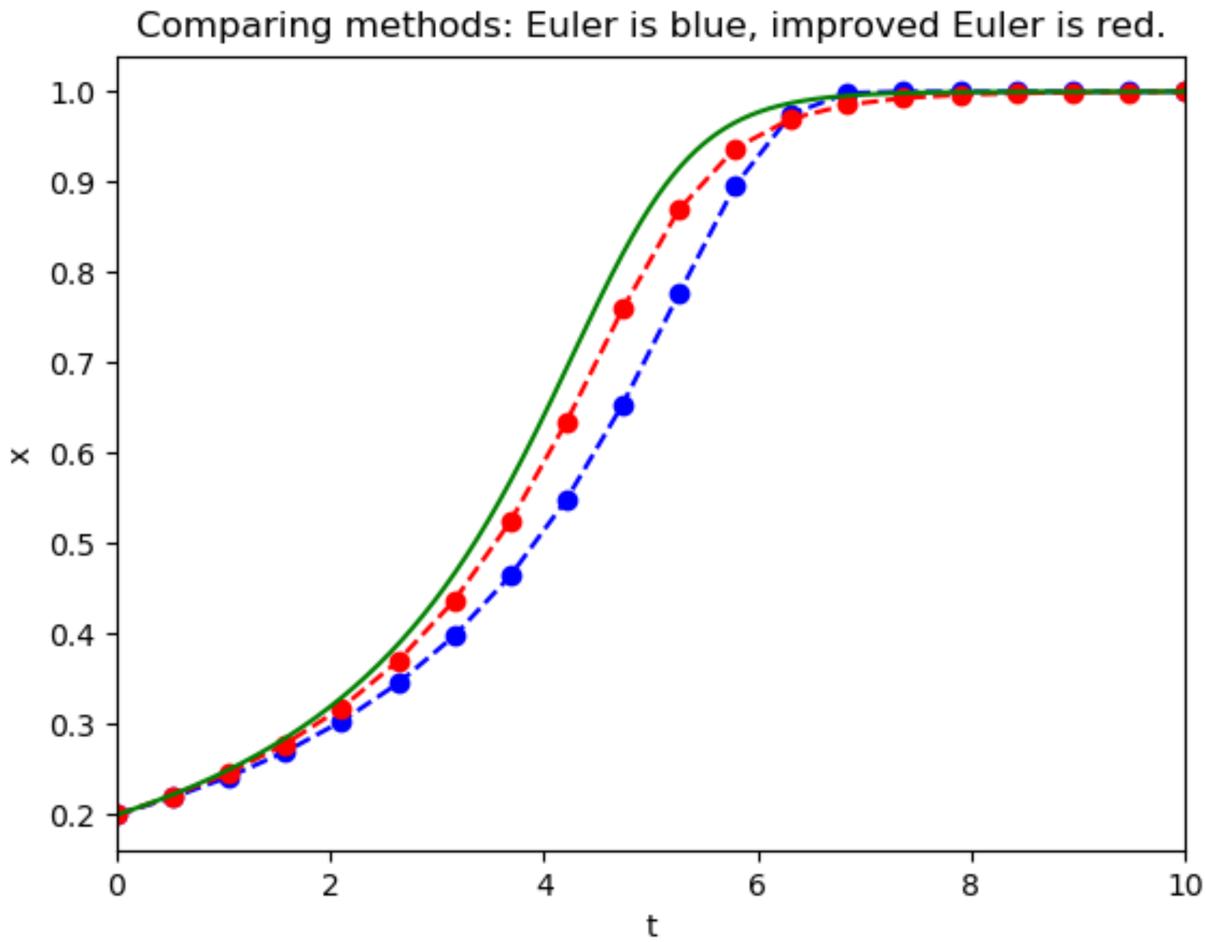
    - Use this to make a better approximation:

$$x_{n+1} = x_n + \frac{1}{2}\Big[f(x_n) + f(\widetilde{x}_{n+1})\Big]\Delta t$$

- Take x_n to be our approximation to $x(n\Delta t)$.

An example

$$\dot{x} = x^2 - x^4$$


Comparing methods: Euler is blue, improved Euler is red.
x
t
0
2
4
6
8
10
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

Runge–Kutta 4th order method

- For $n \geq 0$:
  - Take:

$$\begin{aligned} k_n^{(1)} &= f(x_n)\Delta t, \\ k_n^{(2)} &= f(x_n + \tfrac{1}{2}k_n^{(1)})\Delta t, \\ k_n^{(3)} &= f(x_n + \tfrac{1}{2}k_n^{(2)})\Delta t, \\ k_n^{(4)} &= f(x_n + k_n^{(3)})\Delta t, \end{aligned}$$

  - Set

$$x_{n+1} = x_n + \tfrac{1}{6}\left[k_n^{(1)} + 2k_n^{(2)} + 2k_n^{(3)} + k_n^{(4)}\right]$$

- Take x_n to be our approximation to $x(n\Delta t)$
- The local truncation error satisfies

$$|e_1| \leq C(\Delta t)^5$$

Existence & Uniqueness

An example

Consider the ODE

$$\begin{cases} \dot{x} = \sqrt{|x|} \\ x(0) = 0 \end{cases}$$

Which of the following is the solution?

A) $x(t) = 0$

B) $x(t) = \frac{1}{4}t^2$

C) Both of these

D) Neither of these

$$\begin{cases} \dot{x} = \sqrt{|x|} \\ x(0) = 0 \end{cases}$$

An existential crisis

- If solutions might not be unique, how can we even talk about **the** solution?
- Wait a minute ... does a solution necessarily exist?

The Cauchy–Peano Existence Theorem Let $f\colon (a,b) \to \mathbb{R}$ be continuous and $x_0 \in (a,b)$. Then there **exists** some $\delta > 0$ and a solution $x\colon [-\delta, \delta] \to \mathbb{R}$ of the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Picard–Lindelöf Existence and Uniqueness Theorem. Let $f\colon (a,b) \to \mathbb{R}$ be continuous and $x_0 \in (a,b)$. If f is locally Lipschitz continuous, then there **exists a unique local solution** $\bar{x}(t) \in C^1(I, \mathbb{R})$ of the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

where I is some interval around t_0.

Proof: We will not cover it. If you are interested, you could consult the book *Differential Equations: A Dynamical Systems Approach, Ordinary Differential Equations* by J. H. Hubbard and B. H. West, published by Springer Verlag. You can download this book for free through the UCLA subscription, just look for the book on the Springer Verlag webpage when connected on campus.

See you next time!