Discussion 1

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Topics

1. Seperation of Variables
2. Chain Rule
3. Taylor's Theorem

Seperation of Variables

Seperation of Variables - If $\frac{dx}{dt} = f(x)\,g(t)$, then $\underbrace{\int \frac{dx}{f(x)}}_{\text{Function of } x} = \underbrace{\int g(t)\,dt}_{\text{Function of } t}$. Solve for $x(t)$.

Ex: $\dot{x} = x - x^2$

$$\frac{dx}{dt} = x - x^2$$

$$\frac{dx}{x - x^2} = dt$$

To integrate the LHS, use partial fraction decomposition -

$$\left(\frac{-1}{x(x-1)} = \frac{A}{x} + \frac{B}{(x-1)}\right)(x)(x-1)$$

$$0x - 1 = A(x-1) + Bx$$

- Math coefficients

$$\left.\begin{array}{ll} (x): & 0 = A + B \\ (1): & -1 = -A \end{array}\right\} \to \quad \begin{array}{l} A = 1 \\ B = -1 \end{array}$$

$$\int \frac{1}{x} - \frac{1}{x-1}\,dx = \int dt$$

$$\ln|x| - \ln|x-1| = t + C \leftarrow C \in \mathbb{R}$$

$$\ln\left(\frac{x}{x-1}\right) = t + C$$

$$\frac{x}{x-1} = e^C \cdot e^t$$

$$\frac{1}{1 - \frac{1}{x}} = D \cdot e^t \qquad D \in \mathbb{R}$$

$$\frac{1}{D} \cdot e^{-t} = 1 - \frac{1}{x}$$

$$\frac{1}{D} e^{-t} - 1 = -\frac{1}{x}$$

$$x(t) = \frac{1}{1 - Be^{-t}}$$

Initial conditions? $x(0) = x_0$

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$x(0) = x_0 = \frac{1}{1-B}$ $\rightsquigarrow$ $1-B = \frac{1}{x_0}$

$1 - \frac{1}{x_0} = B$ o If $x_0 \neq 0$

If $x_0 = 0$,

Look at $\dot{x}$ at time $t=0$:

$\dot{x} = x - x^2 = 0 - 0 = 0$. Since $\frac{dx}{dt} = 0$ at time $t=0$, $x(t) = x_0$ $\forall t$.

So, $x(t) = 0$.

Chain Rule

Single Variable Case - If $z = z(y)$, $y = y(x)$, then $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

Multivariable - If $z = z(y_1, y_2)$ and $y_1 = y_1(x_1, x_2)$, $y_2 = y_2(x_2, y_2)$, then

$$\frac{dz}{dx_1} = \frac{dz}{dy_1} \cdot \frac{dy_1}{dx_1} + \frac{dz}{dy_2} \cdot \frac{dy_2}{dx_1}$$

$$\frac{dz}{dx_2} = \frac{dz}{dy_1} \cdot \frac{dy_1}{dx_2} + \frac{dz}{dy_2} \cdot \frac{dy_2}{dx_2}$$

- Track how y_1, y_2 varies on x_1, x_2.

Ex: Suppose $\frac{dx}{dt} = f(x)$. Compute $\ddot{x}$.

Note: $\ddot{x} \neq f'(x)$, LHS is a derivative in time, RHS is a derivative in space

$$\begin{aligned}\ddot{x} &= \frac{df}{dt} \\ &= \frac{df}{dx} \cdot \frac{dx}{dt} \\ &= f'(x) \cdot \dot{x} \\ &= f'(x) \cdot f(x)\end{aligned}$$

Ex: Suppose $x(t)$ solves $\dot{x} = f(x,t)$. Compute $\ddot{x}$.

$$\frac{dx}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dt} \cdot \frac{dt}{dt}$$

$$= \frac{\partial f}{\partial x} \cdot f(x,t) + \frac{\partial f}{\partial t}$$

Taylor's Theorem

Taylor's Theorem - Let $f(x)$ be an $(n+1)$ differentiable function on $[a,b]$, and let $x_0 \in [a,b]$.

Then, for any $x \in [a,b]$, there exists a $z \in [a,b]$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x - x_0)^{n+1}$$

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{n^{th} \text{ order Taylor Polynomial}} + \underbrace{\frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x-x_0)^{n+1}}_{\text{Remainder term}}$$

- How to find Taylor Series for $f(x)$
  - Start from something known (e^x, $\sin(x)$, $\frac{1}{1-x}$, etc.)
  - Transform known functions into $f(x)$ and transform the Taylor Series as well

Ex: Find the 2^{nd} order Taylor Polynomial of $f(x) = \frac{e^{x^2}-1}{x^2}$ at $x=0$.

- Start with known: e^x.

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Substitute $x = x^2$

$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$$

Subtract 1 from both sides

$$e^{x^2}-1 = \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots\right) - 1$$

$$= \left(x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots\right)$$

Divide x^2 from both sides

$$\frac{e^{x^2}-1}{x^2} = 1 + \frac{x^2}{2!} + \frac{x^4}{3!}$$

The second degree Taylor polynomial is $1 + \frac{1}{2}x^2$

Ex: (Seperation of Variables) $\dot{x} = t \cdot e^{-x}$, $x(0) = x_0$ $\to$ $\frac{dx}{dt} = t \cdot e^{-x}$

$$\int dx\, e^x = \int dt \cdot e^t$$

$$e^x = \frac{1}{2}t^2 + C$$

Solve for Ic:

$$e^{x_0} = C$$

$$e^x = \frac{1}{2}t^2 + e^{x_0}$$

$$x = \ln\left(\frac{1}{2}t^2 + e^{x_0}\right)$$

Ex: Chain Rule Let $F(t,z) = t^{-1/4} \cdot z^2$

where $z = x \cdot t^{-1/4}$. Find $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial x}$ with respect to t, z.

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= -\frac{1}{4}\cdot t^{-5/4}\cdot z^2 + 2\cdot z\cdot t^{-1/4}\cdot -\frac{1}{4}\cdot x\cdot t^{-5/4}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$= 0 + 2\cdot z\cdot t^{-1/4}\cdot -\frac{1}{4}\cdot x\cdot t^{-5/4}\cdot t^{-1/4}$$

$$\frac{\partial f}{\partial t} = -\frac{1}{4}\cdot t^{-5/4}\cdot z^2$$

$$\frac{\partial f}{\partial z} = 2\cdot z\cdot t^{-1/4}$$

$$\frac{\partial z}{\partial t} = -\frac{1}{4}\cdot x\, t^{-5/4}$$

$$\frac{\partial z}{\partial x} = t^{-1/4}$$