Math 134, Lec 1, Winter 2022

Lecture #1: Introduction to dynamical systems

Monday March 28$^{\text{rd}}$

Slides and lecture recording

- The lecture will be recorded and posted to the Canvas page after class. You are not allowed to store or record the lectures by any other means.

- The slides will be posted on the Canvas page after class.

TA: Jack Luong

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Learning objectives

Today we will discuss:

- Class logistics
- Class goals
- What it means to say an ODE is a first order autonomous system

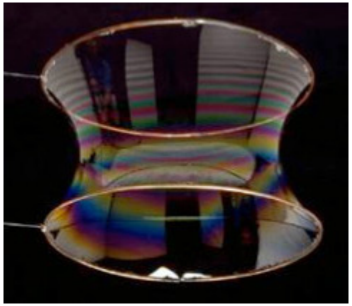
Class logistics

Introduction to dynamical systems

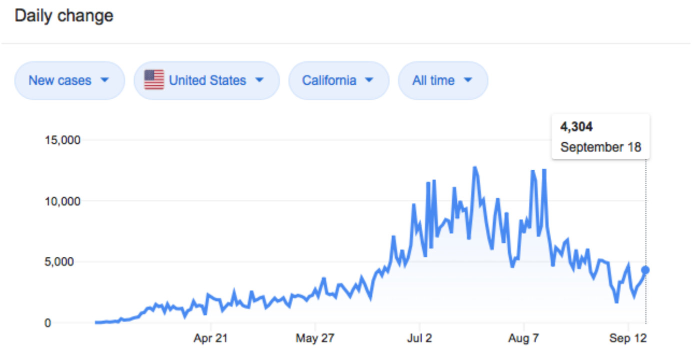
Static vs. Dynamic

Typical models of real-world phenomena can be classified as:

- **Static**



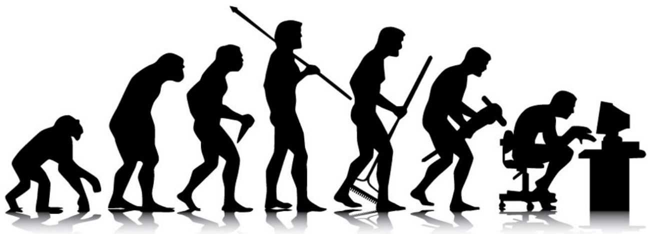
- **Dynamic**



Discrete vs. Continuous

A dynamical systems can be:

- **Discrete** $x_1, x_2, x_3, \dots$ $x_i \in \mathbb{R},\ i \geq 1$



- **Continuous**

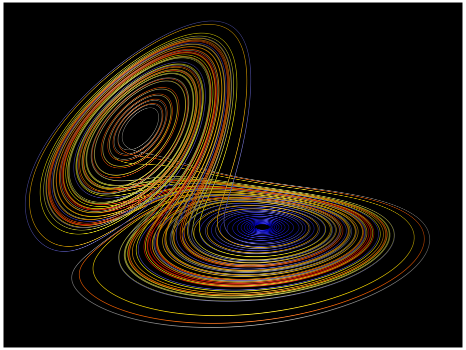
$x = x(t),\ t \geq 0$

$x \in \mathbb{R}$

$\dot{x} = f(x)$

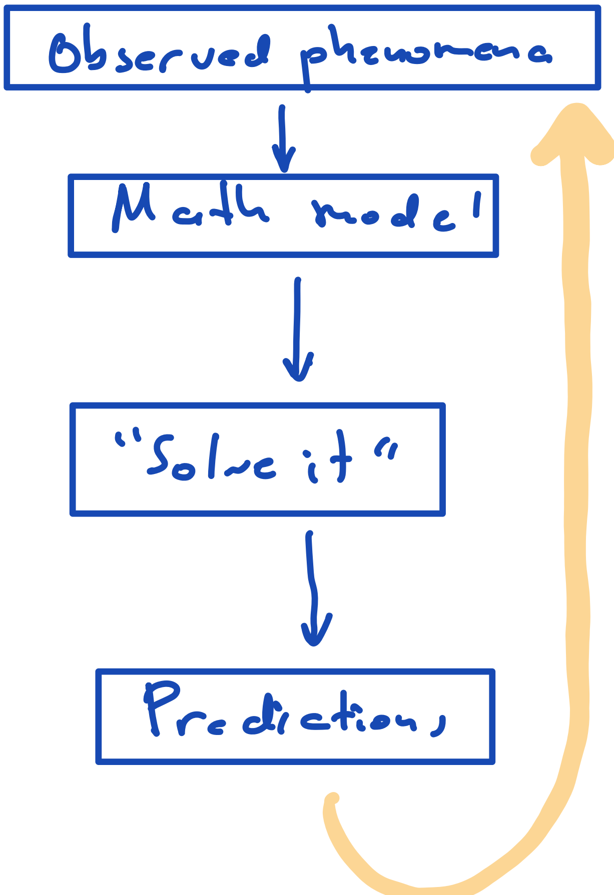
$\dot{x} = \frac{dx}{dt}$

x'



Lorenz attractor

Where do “Dynamical Systems” come from?



planetary motion

Newton ~1666

An example: the SIR model

Divide population into:

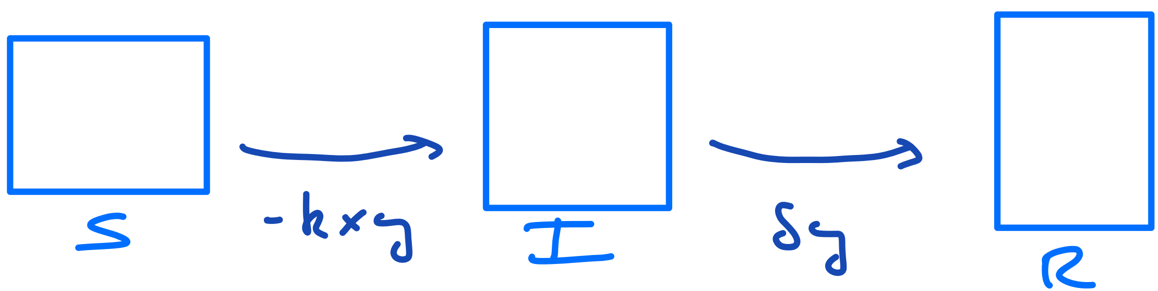
→ compartmental model

- x susceptible people
- y infectious people
- z recovered people

Model epidemic by system of ODEs

$$\begin{cases} \dot{x} = -\kappa x y \\ \dot{y} = \kappa x y - \delta y \\ \dot{z} = \delta y \end{cases}$$

where $\kappa, \delta > 0$ are constants.



$$\begin{cases} \dot{x} = -\kappa x y \\ \dot{y} = \kappa x y - \delta y \\ \dot{z} = \delta y \end{cases}$$

- What questions might we want to answer?
- How could we answer these?
- Are there other aspects of disease we might wish to model?

For the SIR model

$$\begin{cases} \dot{x} = -\kappa x y \\ \dot{y} = \kappa x y - \delta y \\ \dot{z} = \delta y \end{cases}$$

What happens to the total population $x + y + z$?

A) It decreases

B) It increases

C) It is constant

D) It oscillates

$$\frac{d}{dt}(x+y+z) = \dot{x} + \dot{y} + \dot{z} = 0$$

Consider the model

$$\begin{cases} \dot{x} = -\kappa xy + \xi z \\ \dot{y} = \kappa xy - \delta y \\ \dot{z} = \delta y - \xi z \end{cases}$$

where $\xi > 0$. What does the purple term model?

A) Loss of immunity

B) Fatalities

C) Delay between infection and symptoms

D) Birth

Consider the model

$$\begin{cases} \dot{x} = -\kappa xy \\ \dot{y} = \kappa xy - \delta y - \mu y \\ \dot{z} = \delta y \end{cases}$$

where $\mu > 0$. What does the purple term model?

A) Loss of immunity

B) Fatalities

C) Delay between infection and symptoms

D) Birth

A Modified SIR Model for the COVID-19 Contagion in Italy

Giuseppe C. Calafiore, Carlo Novara and Corrado Possieri

***Abstract*— The purpose of this work is to give a contribution to the understanding of the COVID-19 contagion in Italy. To this end, we developed a modified Susceptible-Infected-Recovered (SIR) model for the contagion, and we used official data of the pandemic up to March 30th, 2020 for identifying the parameters of this model. The non standard part of our approach resides in the fact that we considered as model parameters also the initial number of susceptible individuals, as well as the proportionality factor relating the detected number of positives with the actual (and unknown) number of infected individuals. Identifying the contagion, recovery and death rates as well as the mentioned parameters amounts to a non-convex identification problem that we solved by means of a two-dimensional grid search in the outer loop, with a standard weighted least-squares optimization problem as the inner step.**

I. Introduction

Mathematical models can offer a precious tool to public health authorities for the control of epidemics, potentially contributing to significant reductions in the number of infected people and deaths. Indeed, mathematical models can be used for obtaining short and long-term predictions, which in turn may enable decision makers optimize possible control strategies, such as containment measures, lockdowns and vaccination campaigns. Models can also be crucial in a number of other tasks, such as estimation of transmission parameters, understanding of contagion mechanisms, simulation of different epidemic scenarios, and test of various hypotheses.

Several kind of models have been proposed for describing

e.g., [7]–[13]. These models clearly provide a more detailed description of the epidemic spread than collective models but their identification is significantly harder. A first reason is that they are usually characterized by a high number of parameters and variables. A second reason, perhaps more relevant, is that the network topology is unknown in most real situations and its identification is an extremely hard task. In this paper, we focus on collective models since, thanks to their relative simplicity, they can be more suitable for non-expert operators and public health authorities, and they can provide simple but reliable models, even under scarcity of data.

Collective models are typically written in the form of differential equations or discrete-time difference equations, and are characterized by a set of parameters that are not known a-priori and have to be identified from data. However, the identification of such parameters raises several practical issues, as discussed next. An important variable in many epidemic models is the number of individuals that are infected at a given time. However, in a real epidemic scenario, only the number of infected individuals that have been detected as "positive" is available, while the actual number of infected people remains unknown. A common assumption made in the literature is that the observed cases are the actual ones. Clearly, this assumption is unrealistic and may lead to wrong epidemiological interpretations/conclusions. Other issues stem from the fact that identification of epidemic models requires in many cases to deal with non-convex

Autonomous ODEs

- The SIR model is an example of a **first order** system of **autonomous** ODEs.

the highest derivative appearing in the system

t doesn't appear on the RHS.

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n) \end{cases}$$

$\cancel{f_2(x_1, x_2, \ldots, x_n, t)}$

- We will refer to n as the **dimension** of the system.

See you next time!

Image credits:

Plateau problem: `https://faculty.math.illinois.edu/~delcour2/orpheumBubble.pdf`
COVID-19 graph: `https://www.google.com`
Generations: `https://medium.com/@pensnaku/evolution-as-a-software-developer-8829af7c126f`
Lorenz attractor: `https://sibaldiscode.com/code/the-strange-attractor/`