MATH134 - Homework 1 Suggested Solutions.

Marked: $\#1, 4$ and 6.
Suggested Solutions to Homework # 1.

Homework Problem 1. Consider a particle of mass $m > 0$ and charge $q \neq 0$ travelling in 3 dimensions under the influence of a time-dependent magnetic field of magnitude $B(t)$, pointing in the direction of $z-$axis. From Newton's second law, the position vector $\mathbf{r}(t)$ of the particle at time t satisfies the equation

$$m\ddot{\mathbf{r}}(t) = q\dot{\mathbf{r}}(t) \times \begin{bmatrix} 0 \\ 0 \\ B(t) \end{bmatrix}.$$

Show that this system can be expressed as a first order autonomous system of ODEs.

Solution. We start off by evaluating the given cross product to obtain

$$\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} \frac{q}{m}B(t)\dot{y}(t) \\ -\frac{q}{m}B(t)\dot{x}(t) \\ 0 \end{bmatrix}.$$

Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ t \end{bmatrix}$. Then, we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{q}{m}B(t)\dot{y}(t) \\ -\frac{q}{m}B(t)\dot{x}(t) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{q}{m}B(x_4(t))x_2(t) \\ -\frac{q}{m}B(x_4(t))x_1(t) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1(x_2(t), x_4(t)) \\ f_2(x_1(t), x_4(t)) \\ 0 \\ 1 \end{bmatrix}$$

where f_1 and f_2 do not depend on t explicitly and is thus a first order autonomous system of ODEs.

Alternative System:

One could consider the entire system as follows. Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \\ x_7(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ t \end{bmatrix}$. Then, the first order autonomous system is given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \\ 1 \end{bmatrix} = \begin{bmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \\ \frac{q}{m}B(x_7(t))x_5(t) \\ -\frac{q}{m}B(x_7(t))x_4(t) \\ 0 \\ 1 \end{bmatrix}.$$

Homework Problem 2. Suppose that a new short form video app, KotKit, has been launched. Write down a one-dimensional autonomous system for the growth of the app, taking $x(t)$ to be the fraction of the population that has downloaded the app at time t. Check that the fixed points and their stability match common sense.

Solution. For the growth of the app, we will use a Logistic growth model. Denote N_{pop} to be the size of the population, $r > 0$ to be the rate of growth of the app, and $K \in [0, N_{pop}]$ to be the carrying capacity. (Carrying capacity in this case might refer to the maximum reach of the app, as there could be someone in the population without the appropriate electronic device to access/download the app itself. By this definition, we must have $K < N_{pop}$.) The logistic equation describing the growth of the number of people who have downloaded the app, $N(t)$, is given by

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

We also note that $x(t) = \frac{N(t)}{N_{pop}}$ and $x'(t) = \frac{1}{N_{pop}}N'(t)$. This implies that

$$x'(t) = \frac{1}{N_{pop}}N'(t) = rx(t)\left(1 - \frac{N_{pop}}{K}x(t)\right).$$

<u>Fixed Points and Stability:</u> Set $x' = 0$. This gives $x = 0$ or $x = \frac{K}{N_{pop}}$. By plotting the graph of x' against x, a quadratic curve with negative coefficient of x^2 and $x-$intercept at $x = 0$ and $x = \frac{K}{N_{pop}}$ implies that $x = 0$ is an unstable point, while $x = \frac{K}{N_{pop}}$ is a stable point. This is consistent with common sense, as if we start from $x = 0$, any slight positive perturbation of x will incite the growth of the app. Furthermore, at $x = \frac{K}{N_{pop}}$, this implies that the population is at its carrying capacity ($N = K$). Thus, any perturbation of x from $\frac{K}{N_{pop}}$ will bring it back to the fixed point itself, which is consistent with the definition of "carrying capacity".

Note that $K < N_{pop}$ by definition and thus $\frac{K}{N_{pop}} < 1$. From our analysis above, this implies that $0 \le x(t) \le \frac{K}{N_{pop}} < 1$, which is consistent with our understanding that the "fraction of the population" is always a value between 0 and 1.

Homework Problem 3. Suppose that the one-dimensional autonomous ODE

$$\dot{x} = f(x),$$

has a fixed point x^* so that $a = f'(x^*) \neq 0$.

(a) Write down the linearization of the ODE about x^*.

(b) Show that the time required for the solution of the linearized equation found in part (a) to increase or decrease its value (depending on the sign of a) by a factor of $k > 0$ is a constant that depends only on a, k.

(c) The book defines the 'characteristic timescale' attendant to the fixed point x^* to be $\frac{1}{|a|}$. Using your answer to part (b), give an interpretation of this quantity.

Solution.
Note: The term "increase by a factor of k" is interpreted as converting x to kx rather than $(k+1)x$. Due to such ambiguity, both interpretations will be accepted and given full credit if the main gist of the solution is correct.
(a) Denote $\eta = x - x_*$. Since $f(x) \approx f(x_*) + f'(x_*)(x - x_*)$ for $|x - x_*| << 1$ (small enough), we have

$$\dot{\eta} \approx \eta f'(x^*) = a\eta.$$

(b) The question implicitly assumes that $\eta(0) = (x - x_*)(0) \neq 0$ (ie we do note start at the fixed point), and demands for the time t_* such that $\eta(t_*) = k\eta(0)$ for $k > 0$. Note that the linearized equation is separable. Solving the given ODE in (a), we have

$$\eta(t) = \eta(0)e^{at}.$$

Now, set $t = t_*$ in the equation above. We then have

$$k\eta(0) = \eta(t_*) = \eta(0)e^{at_*}$$
$$t_* = \frac{\ln(k)}{a}.$$

Note that this expression makes sense as if $a < 0$ and $\eta > 0$, we must have from (a) that $\dot{\eta} < 0$. This implies that η decreases. This implies that $k < 1$ and $\ln(k) < 0$, and thus implies that $t_* > 0$. Permuting the signs of a and η, one can check that in all cases, $t_* > 0$ and thus is consistent with common sense.
(c) Without loss of generality, we let $a > 0$. Now, we set $t_* = \frac{1}{|a|}$ and see that $\ln(k) = 1$ and thus $k = e$. Thus, in view of (b), this implies that the 'characteristic timescale' is the time taken for the distance between x and x_* to increase/decrease by a factor of e (and therefore 'significantly') if x is close enough to x^*.

Homework Problem 4. Draw a phase portrait (cf. Figure 1 on p. 37 of Strogatz) for each of the following systems, including the values and stabilities of fixed points. Overlay a sketch of a potential function on each one.

(a) $\dot{x} = x(x-1)^2$

(b) $\dot{x} = 1 - |x|$

(c) $\dot{x} = \sin(3x)$

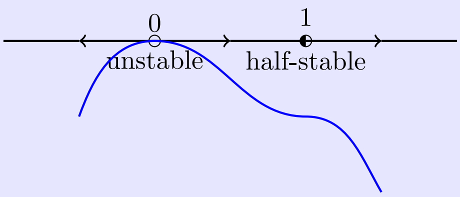
(d) $\dot{x} = \begin{cases} x \ln |x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Solution. Note: Since the question only asked us to *sketch* the potential function, I'll sketch the key features of the potential function corresponding to the nature of each fixed point without trying to solve for the potential function explicitly. You could also solve for an explicit potential function, overlay the corresponding graph of this with the phase portrait, and provide more details if a question requires you to *graph* the potential function instead.

(a) Fixed Points and Stability:

Fixed Points	Nature of Point
0	Unstable
1	Half-Stable

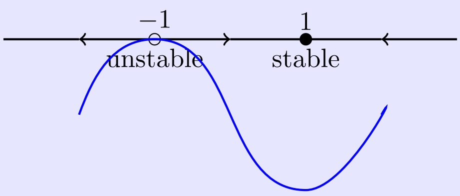
Phase Portrait and Sketch of Potential Function:



(b) Fixed Points and Stability:

Fixed Points	Nature of Point
-1	Unstable
1	Stable

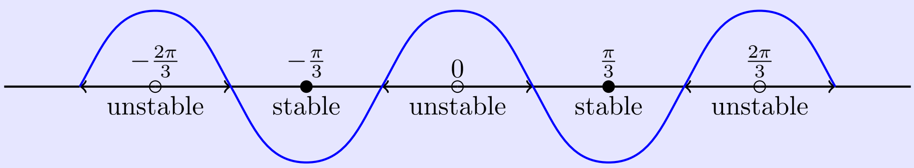
Phase Portrait and Sketch of Potential Function:



(c) Fixed Points and Stability:

Fixed Points	Nature of Point
$\frac{n\pi}{3}$, n is odd	Stable
$\frac{n\pi}{3}$, n is even	Unstable

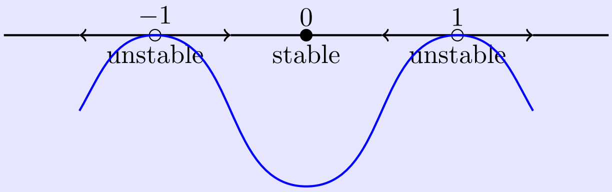
Phase Portrait and Sketch of Potential Function:



(d) Fixed Points and Stability:

Fixed Points	Nature of Point
± 1	Unstable
0	Stable

Phase Portrait and Sketch of Potential Function:



Note for (d): The gradient of V at $x = 0$ must be 0. This is because $-\frac{\mathrm{d}V}{\mathrm{d}x} = f(x)$ implies that at a fixed point, $f(x_*) = 0$ and thus $-\frac{\mathrm{d}V}{\mathrm{d}x}(x_*) = 0$. It is not that your graphing software lied to you, it is just that you did not zoom in to your diagram close enough! This comment holds for all parts of this question - if you have a fixed point, the potential function must have a gradient of 0 at that point (including in (a) at the half-stable point).

Homework Problem 5. For each integer $k = 1, 2, 3...$ and each choice of $+$ or $-$, determine the stability of $x = 0$ as a fixed point of the equation

$$\dot{x} = \pm x^k.$$

Restricting to the cases where $x = 0$ is stable, does making k larger result in faster or slower convergence to the fixed point? Give both a heuristic explanation and one in terms of the exact solutions.

Solution. If k is even, the fixed point $x = 0$ is always half-stable regardless of the choice of sign. This is because the value of $\dot{x}$ at 0^+ and 0^- takes the same sign. If k is odd and if we pick the $+$ sign, this implies that the sign of x is equals to the sign of $\dot{x}$. This implies that $\dot{x}(0^+) > 0$ and $\dot{x}(0^-) < 0$, implying that $x = 0$ is an unstable point. Using an analogous argument, we can deduce that if k is odd and if we pick the $-$ sign, then $x = 0$ is a stable point.

Note that restricting to the cases where $x = 0$ is stable is equivalent to looking at

$$\dot{x} = -x^k$$

where k is an odd integer.
Heuristic Argument: For x close to 0 (and thus less than 1), $|x^{k_1}| < |x^{k_2}|$ for $k_1 > k_2$. This implies that for a larger k, the magnitude of $\dot{x}$ is smaller and thus implies a slower convergence to the fixed point.
Using Exact Solutions: The equation is separable. Thus, one can check that the solution to the ODE above is given by $x(t) = \begin{cases} \left(\frac{1}{(k-1)\left(t+\frac{x(0)^{1-k}}{k-1}\right)}\right)^{\frac{1}{k-1}} & \text{for } k > 1 \\ x(0)e^{-t} & \text{for } k = 1 \end{cases}$.
For $k \neq 1$ and for large values of t, we have $x(t) \approx \left(\frac{1}{k-1}\right)^{\frac{1}{k-1}} \left(\frac{1}{t}\right)^{\frac{1}{k-1}}$ and $\frac{1}{t} \to 0$. Thus, a larger value of k implies that $\frac{1}{k-1}$ is smaller, and thus $x(t)$ decays to 0 at a slower rate. For any $k \neq 1$, the polynomial decay rate will be slower than that in $k = 1$ where the decay rate is exponential.

Note that in the solution of the ODE given above, since k is odd, then $1 - k$ is necessarily even. This implies that $x(0)$ can take both signs $\pm$ without giving rise to imaginary solutions.

Homework Problem 6. The velocity $v(t)$ of a skydiver falling to the ground is governed by

$$m\dot{v} = mg - kv^2,$$

where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

(a) Find the exact solution for $v(t)$ when $v(0) = 0$.

(b) Find the limit for $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.

(c) Draw a phase portrait for this problem, and thereby re-derive a formula for the terminal velocity.

Solution. (a) Rearranging, we have

$$\frac{1}{1 - \frac{k}{mg}v^2}\dot{v} = g.$$

Note that

$$\begin{aligned}
\int \frac{\mathrm{d}v}{1 - \frac{k}{mg}v^2} &= \int \frac{\mathrm{d}v}{(1 - \sqrt{\frac{k}{mg}}v)(1 + \sqrt{\frac{k}{mg}}v)} \\
&= \int \left(\frac{1}{2}\frac{1}{(1 + \sqrt{\frac{k}{mg}}v)} + \frac{1}{2}\frac{1}{(1 - \sqrt{\frac{k}{mg}}v)} \right) \mathrm{d}v \\
&= \frac{1}{2}\sqrt{\frac{mg}{k}} \ln \left| \frac{1 + \sqrt{\frac{k}{mg}}v}{1 - \sqrt{\frac{k}{mg}}v} \right|.
\end{aligned}$$

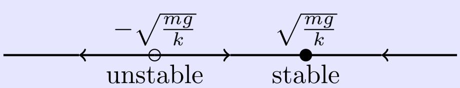
By separation of variables, this implies that

$$\frac{1}{2}\sqrt{\frac{mg}{k}} \ln \left| \frac{1 + \sqrt{\frac{k}{mg}}v}{1 - \sqrt{\frac{k}{mg}}v} \right| = gt + C,$$

with $C = 0$ if $v(0) = 0$ is used. Rearranging, we then have

$$v(t) = \frac{e^{2\sqrt{\frac{gk}{m}}t} - 1}{e^{2\sqrt{\frac{gk}{m}}t} + 1}\sqrt{\frac{mg}{k}}.$$

(b) As $t \to \infty$, we have $\frac{e^{2\sqrt{\frac{gk}{m}}t}-1}{e^{2\sqrt{\frac{gk}{m}}t}+1} \to 1$ (this can be verified using L' Hopital Rule or observing that the exponential term will dominates constants) and thus $v \to \sqrt{\frac{mg}{k}}$.
(c) Solving $\dot{v} = 0$ yields $v = \pm\sqrt{\frac{mg}{k}}$, with $v = -\sqrt{\frac{mg}{k}}$ being an unstable point and $v = \sqrt{\frac{mg}{k}}$ being a stable point. The phase portrait is shown below.



Since $v(0) = 0$ and $\dot{v}(0) > 0$, this means that v will approach the stable fixed point as $t \to \infty$. This implies that the terminal velocity is given by the value obtained at the stable equilibrium point, $v = \sqrt{\frac{mg}{k}}$.

Homework Problem 7. Consider the equation

$$\begin{cases} \dot{x} = \sin(x) \\ x(0) = \frac{\pi}{2}. \end{cases}$$

(a) Use Euler's method with step size $\Delta t = 0.2$ to approximate $x(1)$. Tabulate your results, keeping at least six decimal places.

(b) Find the true value of $x(1)$ and compute the error of your approximation in part (a).

Solution. (a) The Euler's method gives the following scheme

$$x_{n+1} = x_n + f(x_n)\Delta t$$

with $f(x) = \dot{(x)} = \sin(x)$, $x_0 = \frac{\pi}{2}$ and $\Delta t = 0.2$. The value of x at $t = 1$ corresponds to x_5. The value at each iteration is shown below.

n	x_n
1	1.770796
2	1.966810
3	2.151331
4	2.318565
5	$\boxed{2.465206}$

(b) Note that the ODE is separable. This implies that $\int \operatorname{cosec} x \mathrm{d}x = \int \mathrm{d}t$. Since $\int \operatorname{cosec}(x)\mathrm{d}x = -\ln|\cot(x) + \operatorname{cosec}(x)|$, by rearranging the terms, we have

$$\frac{\cos(x) + 1}{\sin(x)} = Ae^{-t}$$

for some arbitrary constant A. Using $x(0) = \frac{\pi}{2}$ yields $A = 1$. Furthermore, by the double angle formula $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ and $\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1$, we have

$$x(t) = 2\operatorname{arccot}(e^{-t}).$$

Thus, $x(1) \approx 2.436566$. The error (to 3 s.f) is given by

$$|2.436566 - 2.465206| = \boxed{0.0286}.$$