Discussion 3

Wednesday, April 13, 2022 6:06 PM

Topics
1. Numerical Methods and Local Truncation Errors
2. Blowup Time and the Comparison Principle
3. Saddle Node Bifurcations
4. Hw (2 + 3)

Numerical Methods and LTE

- Setup: $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$

Local Truncation Error - $e_1 = x(\Delta t) - x_1$

- $x(\Delta t)$ = exact solution of ODE
- x_1 = approximation of $x(\Delta t)$ through the numerical method
- How to find LTE? Taylor Expand (usually around x_0)

Ex: "Show the Trapezoidal Method has an LTE which is $\mathcal{O}(\Delta t^3)$"

Ex: Trapezoidal method

$$y_{n+1} = y_n + \frac{h}{2}\left(f(y_n) + f(y_{n+1})\right)$$ Trapezoidal Method

$$e_1 = y(\Delta t) - y_1$$

$$y(\Delta t) = y(0) + y'(0)\Delta t + \frac{y''(0)}{2!}\Delta t^2 + \frac{y'''(z)}{3!}\Delta t^3 \quad z \in (0, \Delta t)$$

Want to turn $f(y_1)$ into an expression involving only y_0 and Δt

$$f(y_1) = f(y(0)) + f'(y(0))\cdot(y_1 - y(0)) + \frac{f''(w)}{2!}\cdot(y_1 - y(0))^2 \quad w \in (y(0), y_1)$$

$y_1 \approx y(\Delta t)$

$$e_1 = y(0) + y'(0)\Delta t + \frac{y''(0)}{2!}\Delta t^2 + \frac{y'''(z)}{3!}\Delta t^3$$
$$- \left(y(0) + \frac{\Delta t}{2}\left(f(y(0)) + f(y(0)) + f'(y(0))\cdot(y_1 - y(0)) + \frac{f''(w)}{2!}(y_1 - y(0))^2\right)\right)$$

$$y(\Delta t) = y(0) + \underset{+f(y(0))\Delta t}{y'(0)\Delta t} + \frac{y''(p)}{2!}\Delta t^2 \quad p \in (0, \Delta t) \qquad f(y(0))\Delta t + \frac{y''(p)}{2!}\Delta t^2$$

$$e_1 = \frac{y'''(z)}{3!}\Delta t^3 - \frac{y''(p)}{2!\cdot 2}\Delta t^3 + \frac{f''(w)}{2!}\Delta t\cdot\left(f(y(0))\Delta t^2 + \dots\right)$$
$$= \Delta t^3(\sim)$$
$$\leq C\cdot\Delta t^3$$

Blowup Time and the Comparison Principle

- How do we show a solution reaches ∞ in finite time?
  - Analytic Solution
    - Complicated
    - Not always possible
  - Comparison Principle

Theorem: Let $f \le g$ be smooth and let $x_0 \le y_0$. Suppose that x, y are solutions of the ODEs

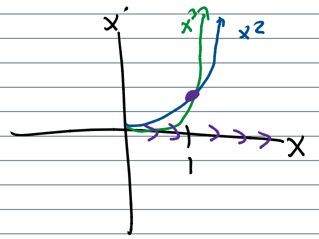
$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y) \\ y(0) = y_0 \end{cases}$$

on a time interval $[0, T]$. Then $x(t) \le y(t)$ for all $t \in [0, T]$.

- When to use:
  - $f(x)$ looks close to something I know how to deal with.
  - I can't integrate / it's hard to integrate $f(x)$ directly.

Ex: "Show $\dot{x} = x^3$, $x(0) = 1$ blows up in finite time"

- Can solve by seperation of variables (solution is $x(t) = \sqrt{\frac{1}{1-2t}}$)
- How can we use comparison principle?
  - Comparison candidate: $\dot{x} = x^2$, $x(0) = 1$
  - Inequality? $x^2 \le x^3$ if $|x| > 1$
    - We know $|x| > 1$ because of the ODE and initial condition



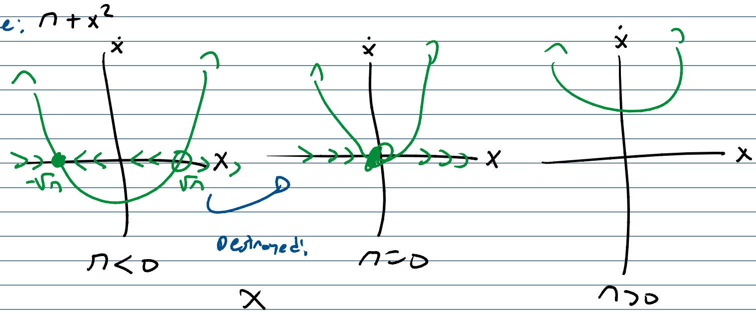
- So, $\begin{cases} \dot{y} = y^2 \\ y(0) = 1 \end{cases}$ can be used for comparison. Solve this ODE $\to$ $y(t) = \frac{1}{1-t}$. Since $y(t) \to \infty$ as $t \to 1$ and $y(t) \le x(t)$, $x(t)$ also has a finite blowup time.

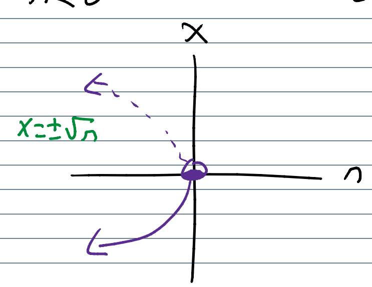
Saddle Node Bifurcations

Bifurcation - If we have an ODE $\dot{x} = f(x, \lambda)$, then we get a bifurcation at $\lambda = \lambda_0$ when the fixed point behavior radically changes

Saddle Node: Fixed points destroyed and created. Two fixed points collide and destroy each other

Prototype: $r + x^2$

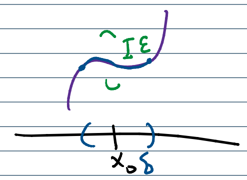




- As r varies, track where the fixed points are
- Dashed line for unstable, solid line for stable

Homework

2. Continous (at a point x_0) - A function f such that $\forall \varepsilon > 0, \exists \delta > 0$, if $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$



Ex: "Show $f(x)=x^2$ is continuous on $[1,2]$."

Let x_0 be a point in $[1,2]$, and let $\varepsilon>0$ be given.

Goal: Get from $|x^2-x_0^2|<\varepsilon \rightarrow |x-x_0|<\delta$.

Scratchwork

$$|x^2-x_0^2|<\varepsilon$$
$$|x-x_0|\cdot|x+x_0|<\varepsilon$$
$$|x-x_0|<\frac{\varepsilon}{|x+x_0|}$$

, How small can $\frac{\varepsilon}{|x+x_0|}$ be?

Since $x, x_0 \in [1,2]$, $|x+x_0|<|2+2|<4$. So,

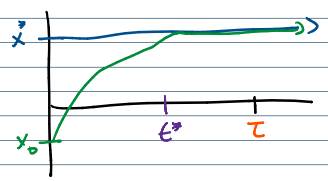
$$|x-x_0|<\frac{\varepsilon}{4}. \text{ So, } \delta=\frac{\varepsilon}{4}.$$

Let $\delta=\frac{\varepsilon}{2}$. Then, if $|x-x_0|<\delta=\varepsilon/4$,

$$|x-x_0|\cdot|x+x_0|<\frac{\varepsilon}{4}\cdot|x+x_0|$$
$$|x^2-x_0^2|<\frac{\varepsilon}{4}\cdot|4|<\varepsilon.$$

So, $f(x)$ is continuous at x_0, for any $x_0\in[1,2]$.

3. What if there was a trajectory $x(t)$ that got to fixed point x^* in finite time?



- What if we reverse the ODE? Start from time τ and go to 0?

Is there anything about uniqueness we can use?