Math 134, Summer 2022

Lecture #17: Linear systems

May 4^{th}

Learning objectives

Today we will discuss:

- The phase space and trajectories of a $2d$ system.
- Fixed points of $2d$ systems.
- The linearization of a $2d$ system about a fixed point.
- Fixed points of linear systems.
- Eigenvalues and eigenvectors of 2×2 matrices.

Linear systems

An example

Recall the bead in a hoop model

$$\varepsilon\ddot{\phi} + \dot{\phi} = \sin\phi\big[\gamma\cos\phi - 1\big].$$

$2d$ systems

A solution of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

has the form

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Fixed points

We say that $\boldsymbol{x}^* \in \mathbb{R}^2$ is a **fixed point** of the $2d$ system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

if $f(\boldsymbol{x}^*) = 0$.

The linearized system

Suppose that $\boldsymbol{x}^*$ is a fixed point of the $2d$ system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}).$$

The **linearization** of this system at $\boldsymbol{x}^*$ is the equation

$$\dot{\boldsymbol{\eta}} = \nabla \boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{\eta}$$

where the **gradient**

$$\nabla \boldsymbol{f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

An example

Recall the pendulum with $\varepsilon = 1$ and $\gamma = 0$:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} x_2 \\ -x_2 - \sin(x_1) \end{bmatrix}.$$

An example

Consider the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} x_1 x_2 - 1 \\ x_1^2 - x_2^2 \end{bmatrix}$$

Which of the following is the linearization at the fixed point $(1, 1)$?

A) $\dot{\boldsymbol{\eta}} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \boldsymbol{\eta}$

B) $\dot{\boldsymbol{\eta}} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \boldsymbol{\eta}$

C) $\dot{\boldsymbol{\eta}} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \boldsymbol{\eta}$

D) $\dot{\boldsymbol{\eta}} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \boldsymbol{\eta}$

Linear systems

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \quad \text{where} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

See you next time!