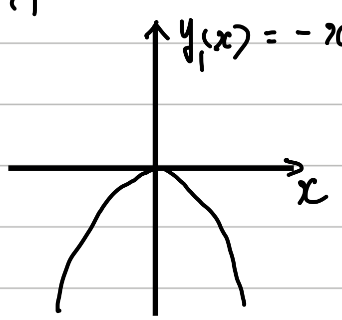
HomeWork 5

1) Given $\dot{x} = h + rx - x^2$

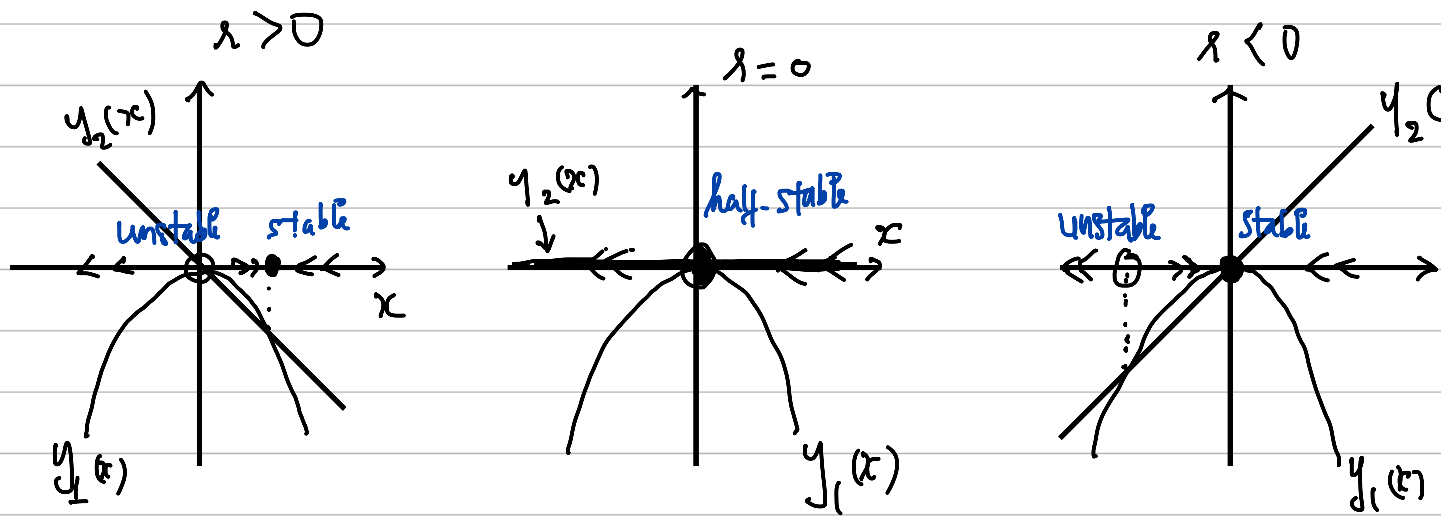
a) Let $y_1(x) = h - x^2$, $y_2(x) = -rx$, then $\dot{x} = y_1(x) - y_2(x)$

* $\boxed{\text{If } h = 0\text{, we have:}}$



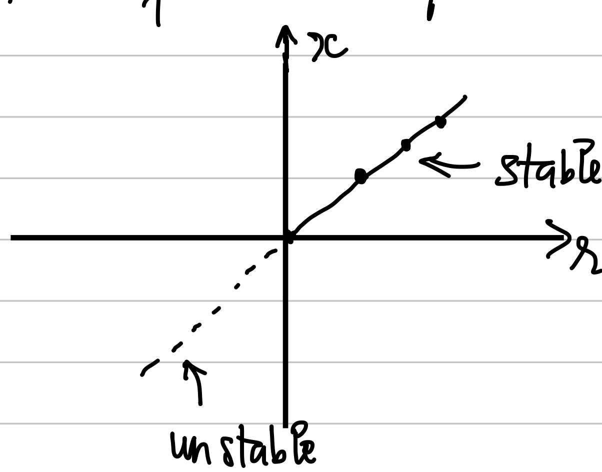
possible bifurcation point: $\begin{cases} rx - x^2 = 0 \\ r - 2x = 0 \end{cases}$

$\Leftrightarrow$ $x = 0$ & $r = 0$

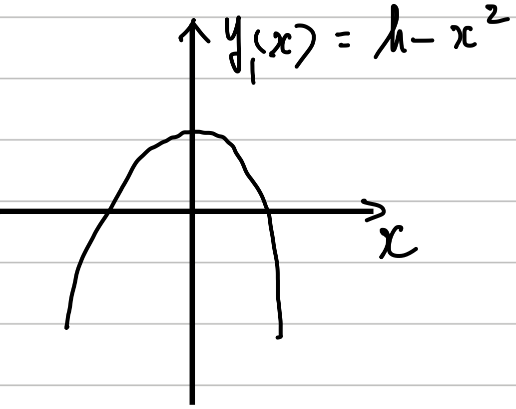


Then we have the bifurcation diagram when $h = 0$.

As we can see, this is transcritical bifurcation at $(x, r) = (0, 0)$



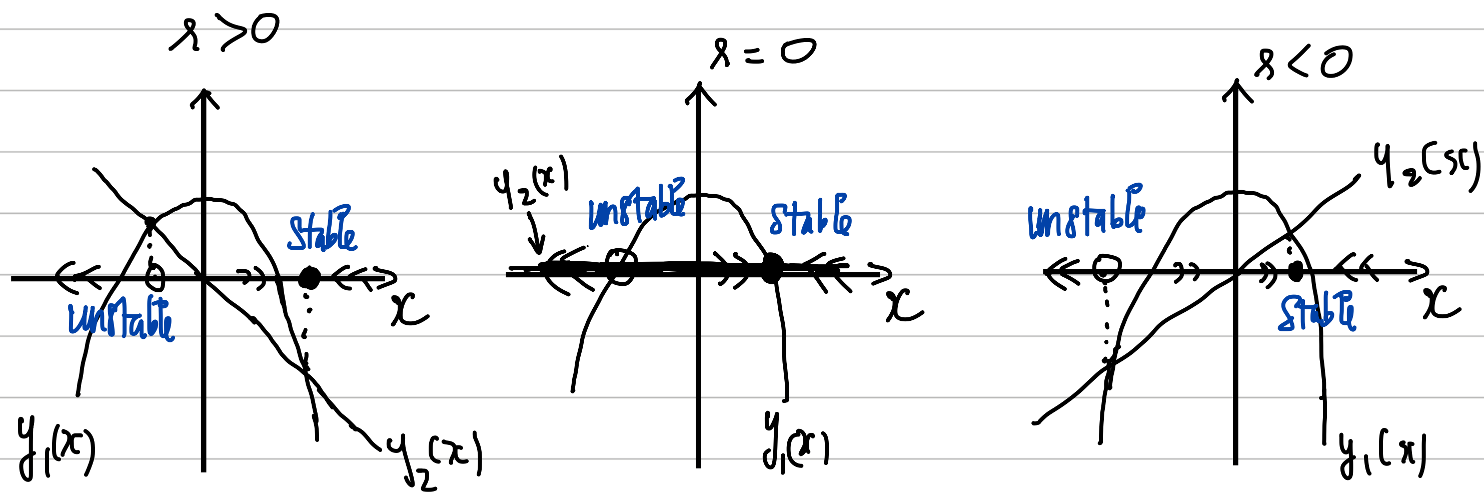
⊛ If $h>0$ we have:



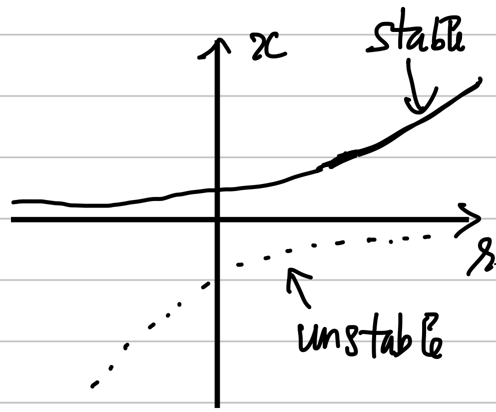
possible bifurcation point:

$$\begin{cases} h - x^2 + rx = 0 \\ -2x + r = 0 \end{cases} \Leftrightarrow \begin{cases} 2x^2 - x^2 + h = 0 \\ r = 2x \end{cases} \Leftrightarrow \begin{cases} x^2 + h = 0 \\ r = 2x \end{cases}$$

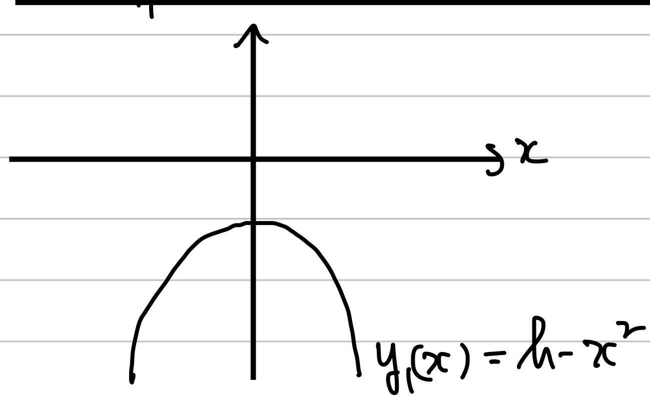
Since $h>0 \Rightarrow x^2 + h > 0 \Rightarrow$ there is no possible bifurcation poin in this case



Then, we have the bifurcation diagram.



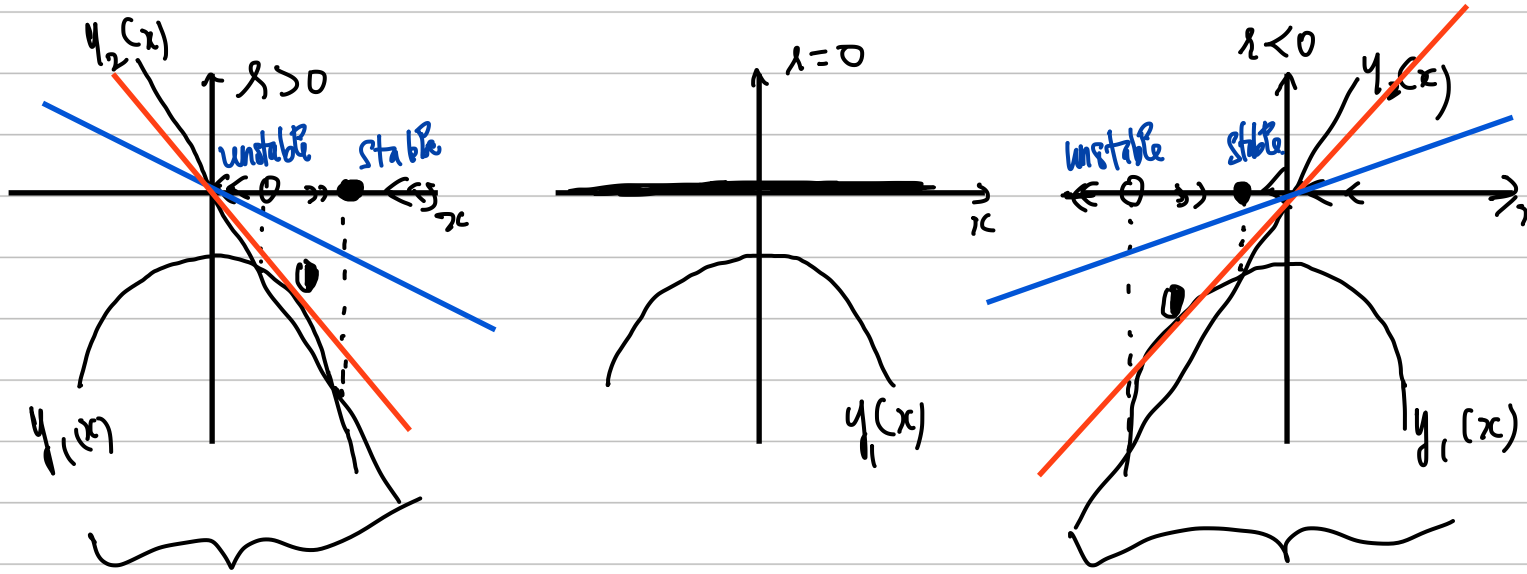
⊛ If $h<0$, we have:



The possible bifurcation point:

applying the same case of $h>0$, we have: $\begin{cases} x^2 + h = 0 \\ r = 2x \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} x = \sqrt{-h} \\ r = 2\sqrt{-h} \end{cases} \quad \text{and} \quad \begin{cases} x = -\sqrt{-h} \\ r = -2\sqrt{-h} \end{cases}$$

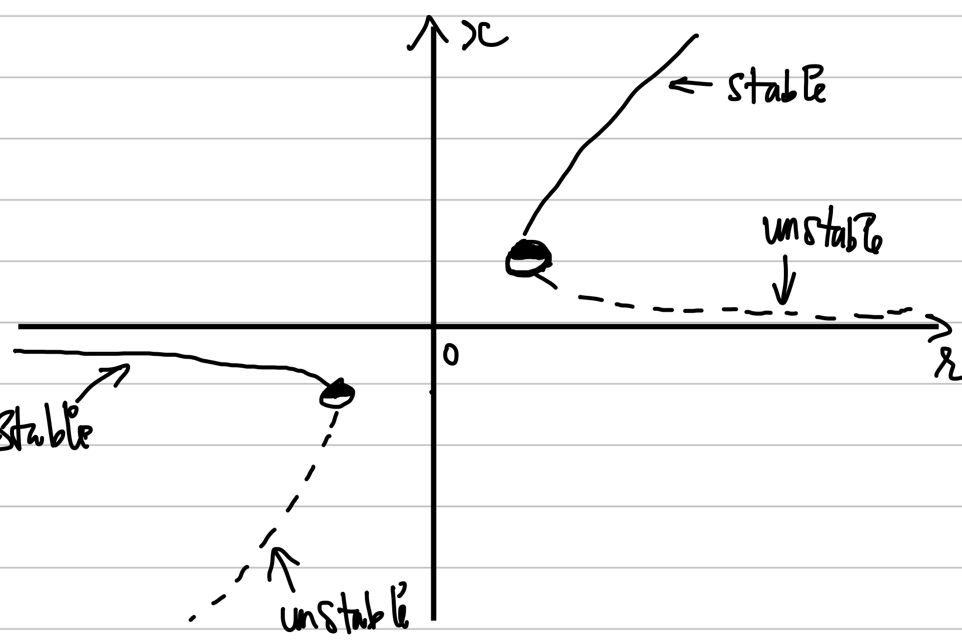


bifurcation, $\begin{cases} x = \sqrt{-h} \\ r = 2\sqrt{-h} \end{cases}$

bifurcation point at $\begin{cases} x = -\sqrt{-h} \\ r = -2\sqrt{-h} \end{cases}$

At We can see these bifurcation points are <u>saddle-point</u> since the fixed points were created & destroyed around the bifurcation points.

Then, we have a bifurcation diagram:

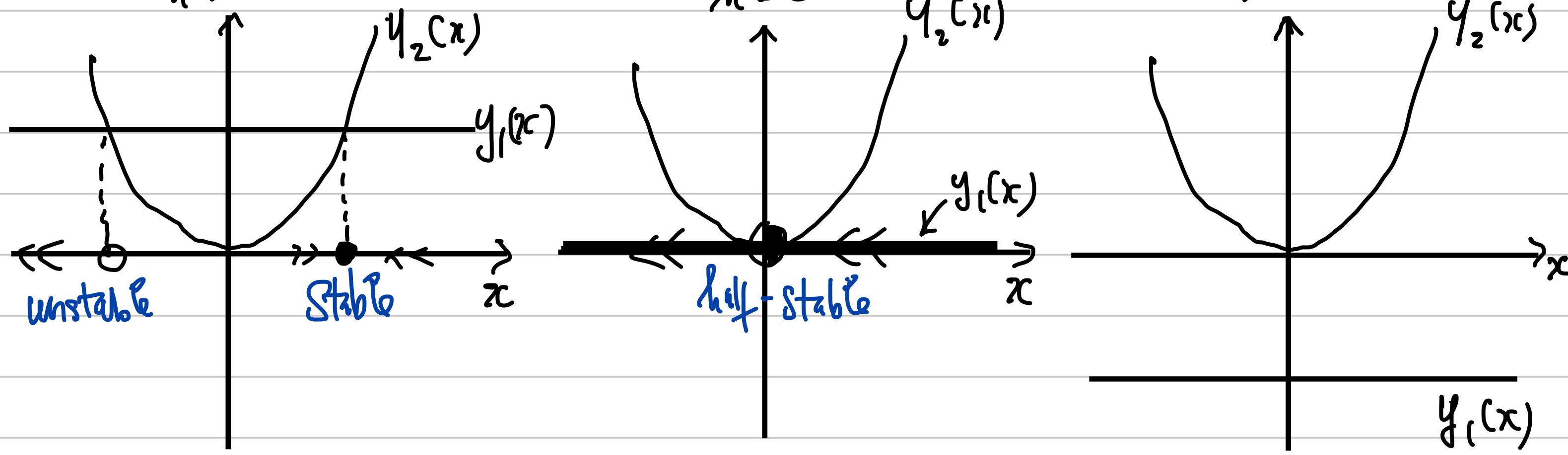


b) $\dot{x} = h + rx - x^2$

Let $y_1(x) = h$, $y_2(x) = x^2 - rx$, $\dot{x} = y_1(x) - y_2(x)$

$\boxed{\text{* If } r = 0 \text{, we have:}}$ $y_1(x) = h$, $y_2(x) = x^2$

$h > 0$ $\qquad$ $h = 0$ $\qquad$ $h < 0$



possible bifurcation point:

$$\left.\begin{aligned} f(x) = \dot{x} &= h - 2x^2 = 0 \\ \frac{\partial f}{\partial x} &= -4x = 0 \end{aligned}\right\} \Rightarrow \begin{cases} x = 0 \\ h = 0 \end{cases}$$

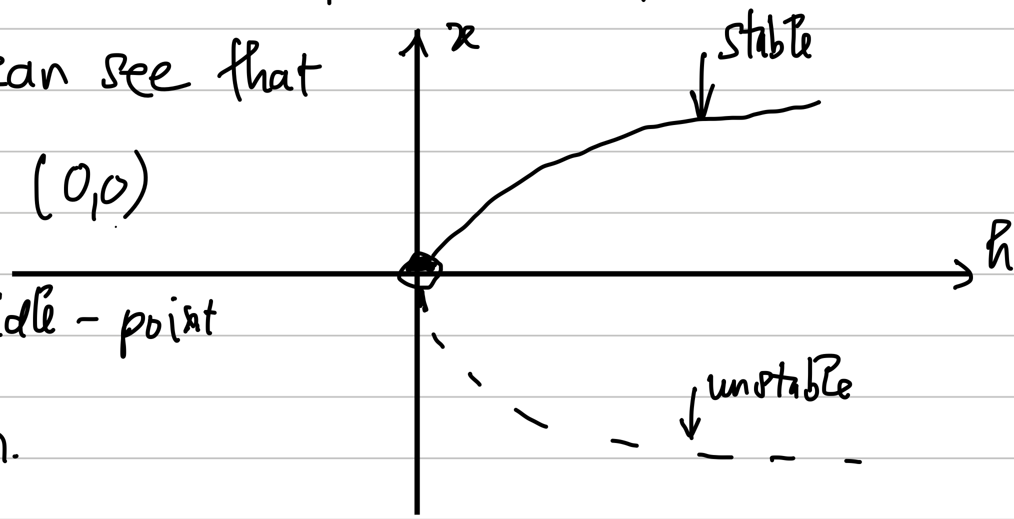
Then we have a bifurcation diagram:

As we can see that

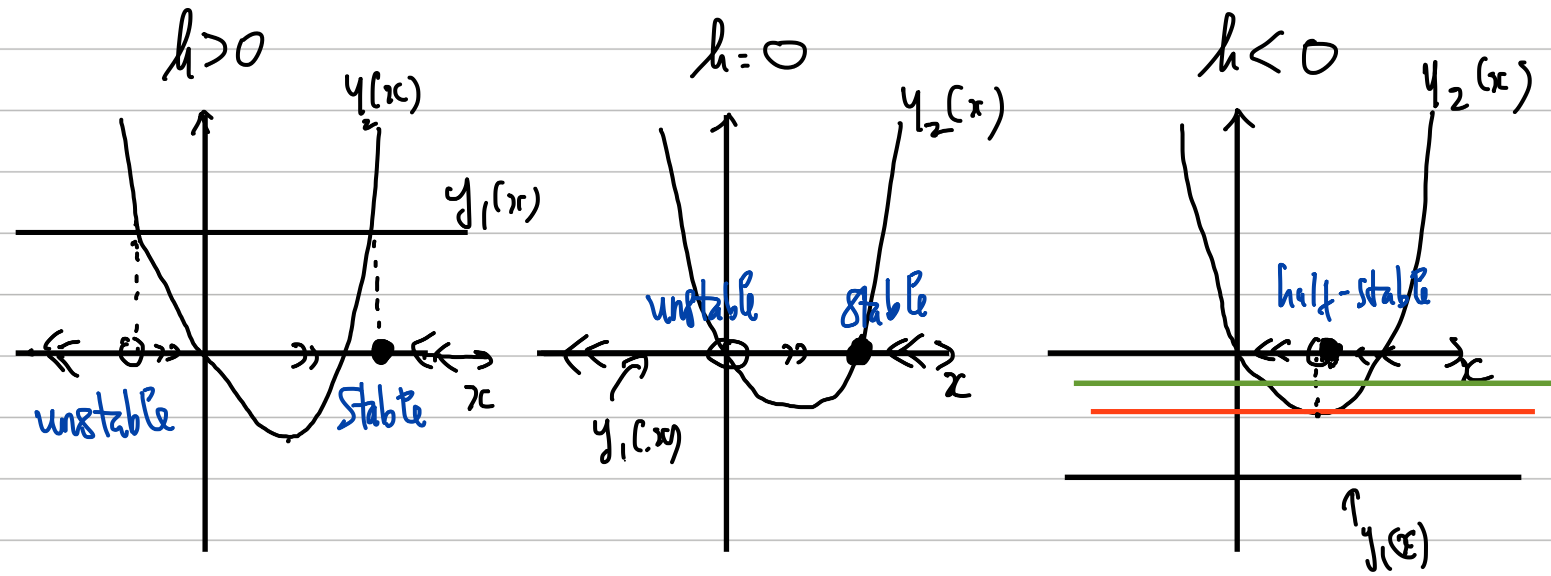
this point $(0,0)$

is a saddle-point

bifurcation.



$\boxed{\text{*If } r>0, \text{ we have:}}$ $y_1(x)=h,\ y_2=x^2-rx$



possible bifurcation points:

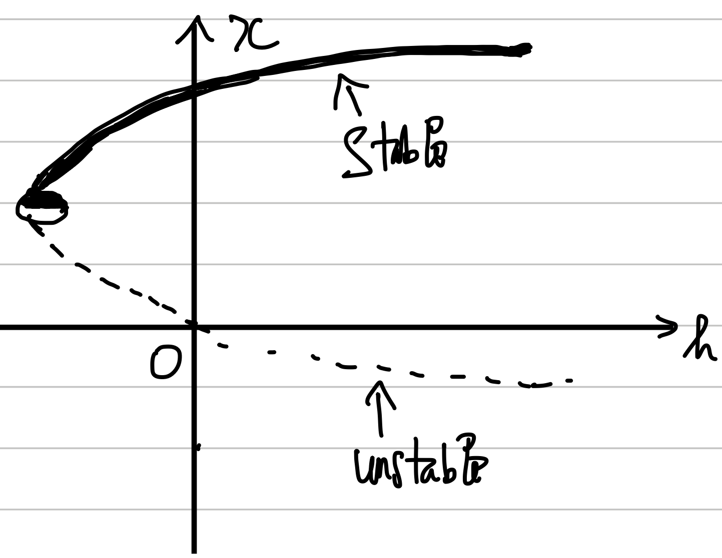
$$\begin{cases} h+rx-x^2=0 \\ r=2x \end{cases} \Leftrightarrow \begin{cases} h+2x^2-x^2=0 \Leftrightarrow h+x^2=0 \\ r=2x>0 \Leftrightarrow x>0 \text{ Since } r>0 \end{cases}$$

$$\Rightarrow h=-x^2 \Rightarrow (x,h)=(x,-x^2)$$

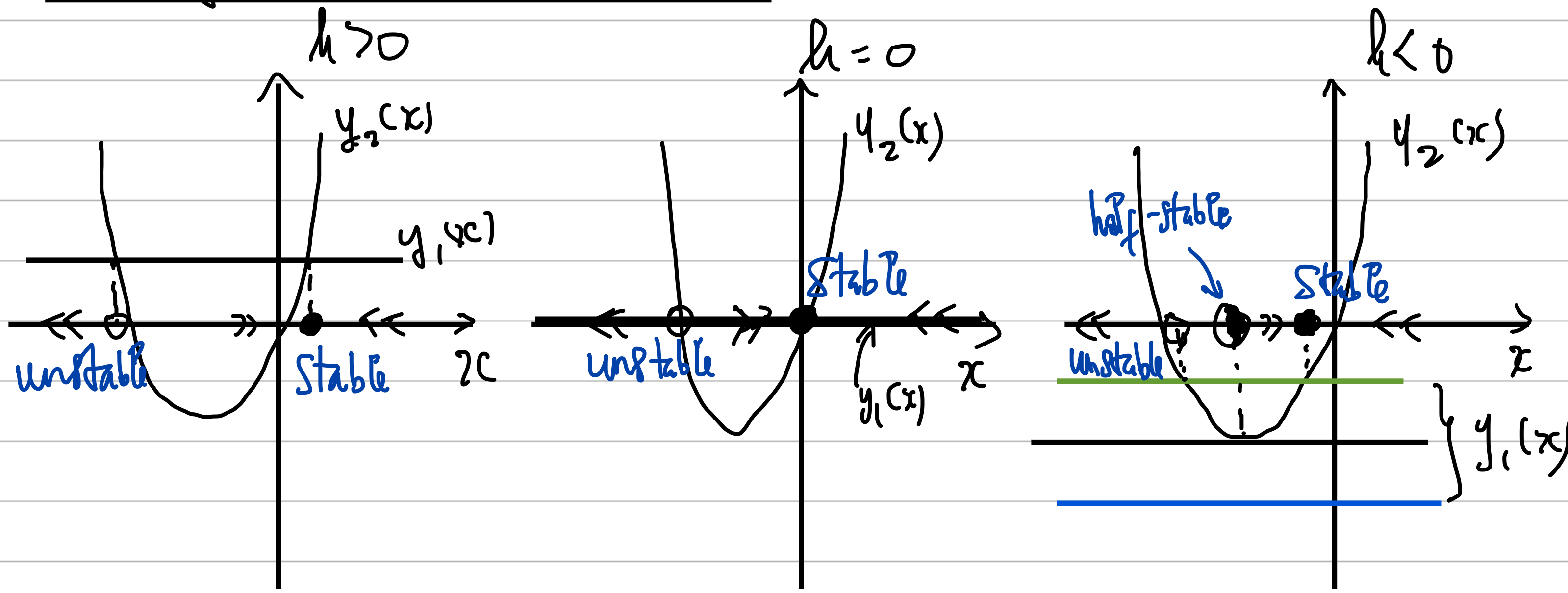
are possible bifurcation points.

Then we have a bifurcation diagram:

We realize this bifurcation point is saddle point

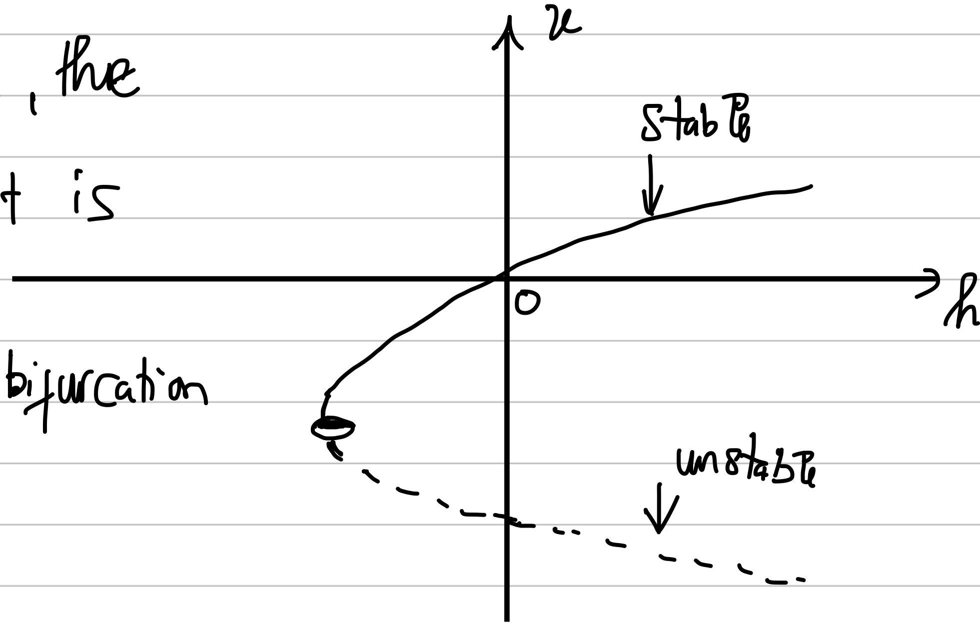


$\boxed{\text{* If } r<0, \text{ we have:}}$ $y_1(x)=h,\ y_2=x^2-rx$



Applying the same process as previous case, we have $(x,h)=(x,-x^2)$ are possible bifurcation points. $(x<0)$

Then we have the bifurcation diagram:



In this case, the bifurcation point is also a saddle bifurcation point

2) Problem 3.5.8

Given $\dot{u} = au + bu^3 - cu^5$ ①

Where $x = \frac{u}{U}$, $\tau = \frac{t}{T}$ $\Rightarrow u = xU$

We have $\dot{u} = \frac{du}{dt} = \frac{d(x.U)}{d(\tau T)} = \frac{U\,dx}{T\,d\tau}$

$\Rightarrow$ ① $\Leftrightarrow$ $\frac{U}{T}\frac{dx}{d\tau} = a.\,xU + bx^3U^3 - cx^5U^5$

$$\Rightarrow \frac{dx}{d\tau} = aTx + bTx^3U^2 - cTx^5U^4$$

$$= aTx + bTU^2x^3 - cTU^4x^5$$

Then let $aT = r$, $bTU^2 = 1$ & $cTU^4 = 1$

$\Rightarrow T = \frac{r}{a}$ & $bTU^2 = cTU^4 \Rightarrow bU^2 = cU^4$

$\Rightarrow$ $b = cU^2 \Rightarrow U^2 = \frac{b}{c} \Rightarrow U = \pm\sqrt{\frac{b}{c}}$ (since U, T, r are to be determined in terms of a, b & c)

$\Rightarrow$ $bTU^2 = 1 \Leftrightarrow bT.\frac{b}{c} = 1 \Rightarrow T = \frac{c}{b^2}$

Also, $T = \frac{r}{a} \Rightarrow r = aT = a.\frac{c}{b^2} = \frac{ac}{b^2}$

Therefore, with $r = \frac{ac}{b^2}$, $T = \frac{c}{b^2}$, $U = \pm\sqrt{\frac{b}{c}}$,

the first order system $\dot{u} = au + bu^3 - cu^5$ can be rewritten as:

$$\frac{dx}{d\tau} = rx + x^3 - x^5.$$

3)

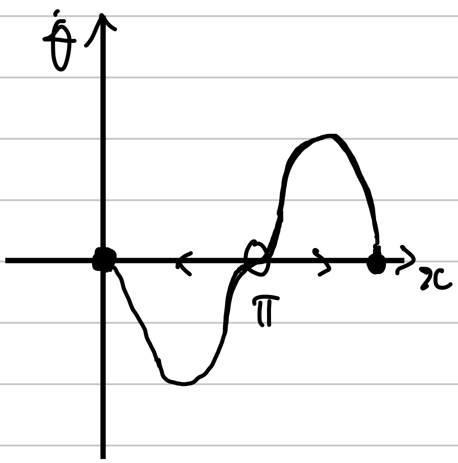
$\boxed{\text{(*) } 4.3.3.}$

Given $\dot{\theta} = \mu \sin\theta - \sin 2\theta$

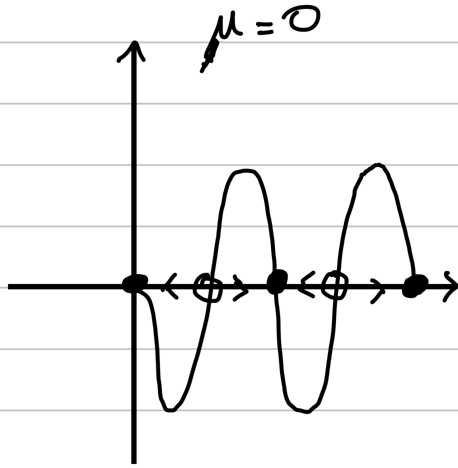
We have $\dot{\theta} = 0 \Leftrightarrow \mu \sin\theta = \sin 2\theta = 2\sin\theta\cos\theta$

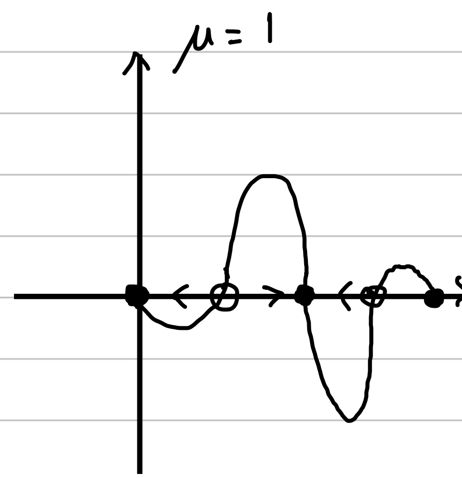
$\Rightarrow \mu = 2\cos\theta$

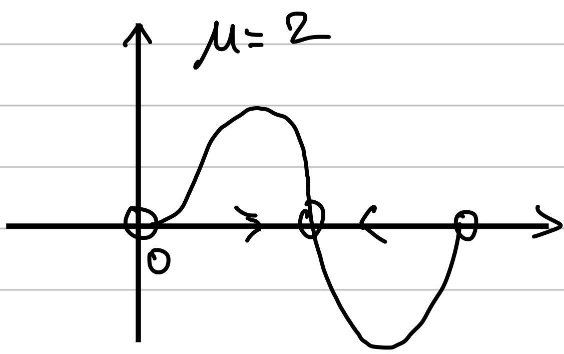
If $\mu = -2 \Rightarrow \cos\theta = -1$ & $\dot{\theta} = -2\sin\theta - \sin 2\theta$



Then do same for varies μ, we also have some phase portait







Based on the diagrams, we can conclude that:

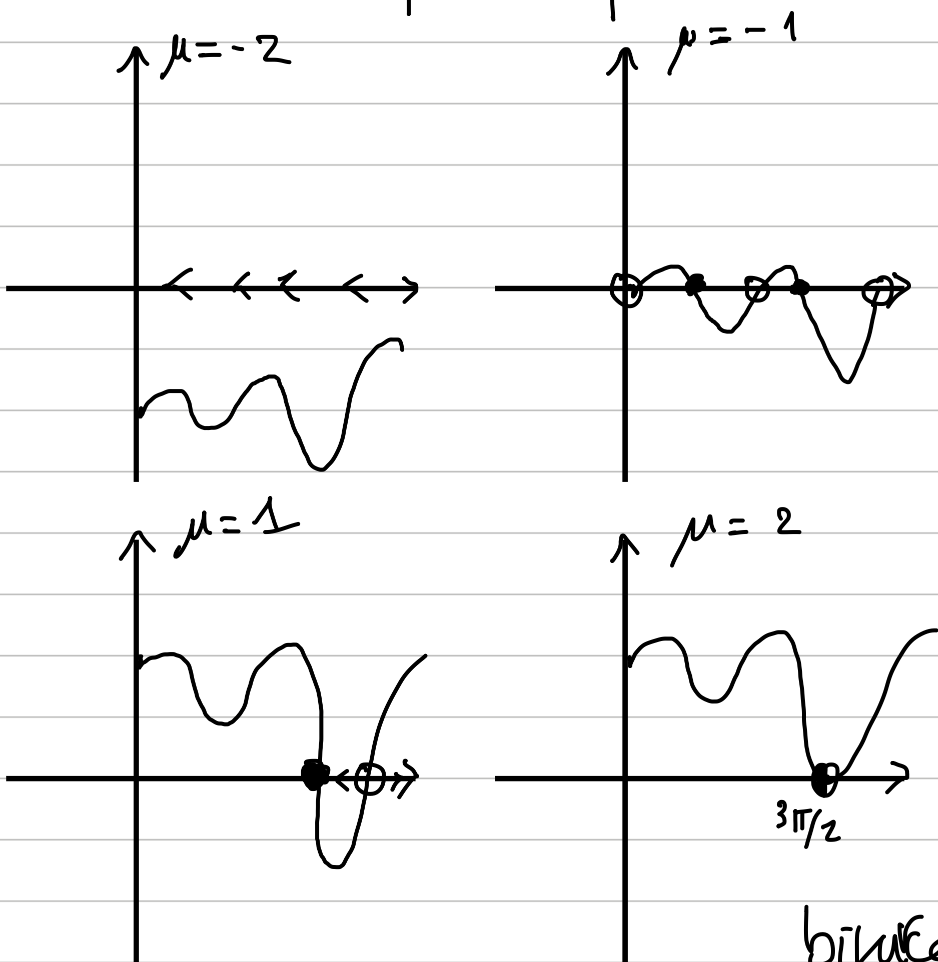
* at $\theta = \pi$ & $\mu = -2$, there is a subcritical pitchfork bifurcation point

* at $\theta = 0$ & $\mu = 2$, there is a subcritical pitchfork bifurcation point.

$\boxed{4.3.6}$, Given $\dot{\theta} = \mu + \sin\theta + \cos 2\theta$

$\Rightarrow \dot{\theta} = 0 \Leftrightarrow \mu = -\sin\theta - \cos 2\theta$

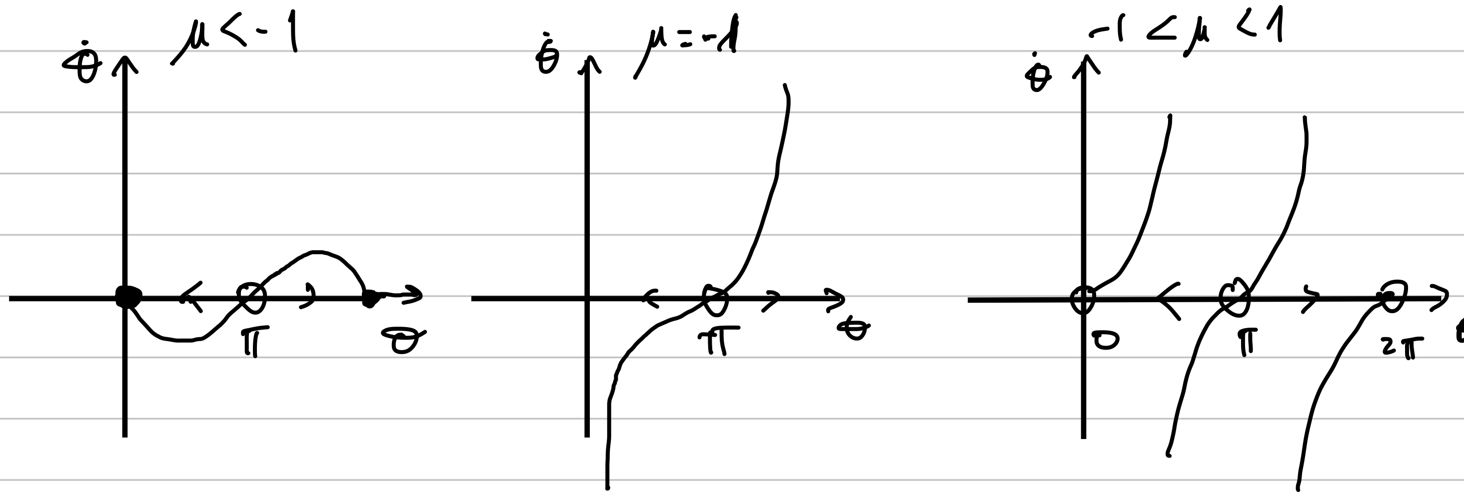
We have phase potraits.

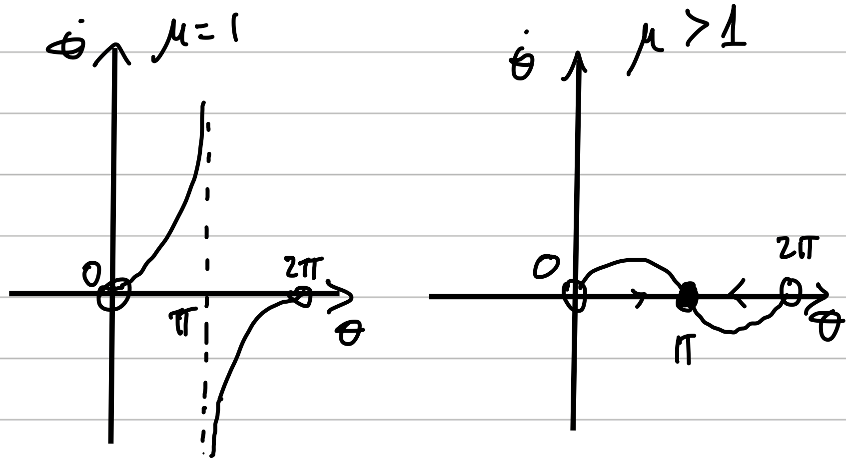


As we can see, at $\theta = \frac{\pi}{2}$ & $\mu = 0$ there is a saddle-point bifurcation. And also having a saddle-point bifurcation at $\theta = \frac{3\pi}{2}$ & $\mu = 2$

$\boxed{\circledast 4.3.4}$, Given $\dot{\theta} = \dfrac{\sin\theta}{\mu + \cos\theta}$

We have phase potraits



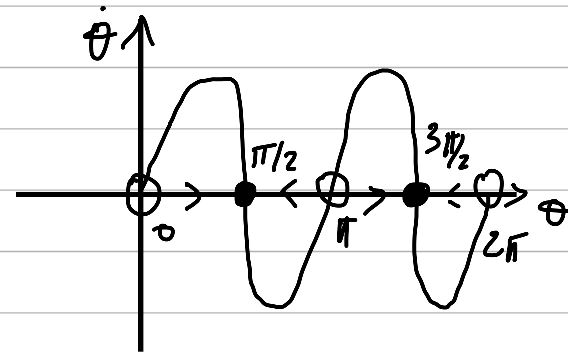


Then, at $\theta = \pi$ & $\mu = -1$ there has a transcritical bifurcation point.

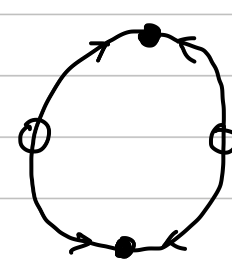
$\boxed{\circledast\ 4.1.3:}$ $\dot{\theta} = \sin 2\theta = 0$

$\iff 2\theta = k\pi \quad (k \in \mathbb{N})$

$\iff \theta = \dfrac{k\pi}{2}$



We have phase potrait



$\Rightarrow$ at $\theta = \frac{k\pi}{2}$ with k is even, points are unstable fixed point, ex: $0, \pi, 2\pi, 3\pi, \ldots$

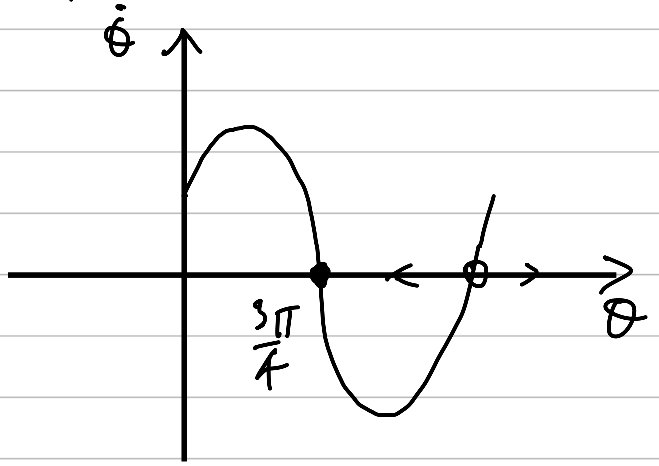
at $\theta = \frac{k\pi}{2}$ with k is odd, points are stable fixed points., ex: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

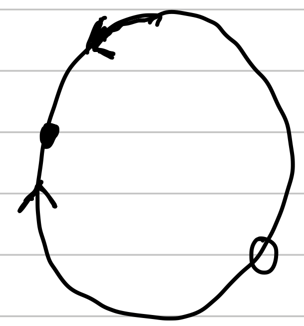
$\boxed{(*)\ 4.15,}$ Given $\dot{\theta} = \sin\theta + \cos\theta = 0$

$\Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1$

$\Rightarrow \theta = -\frac{\pi}{4} + k\pi$

Phase potrait:





Then we can conclude:

at $\theta = -\frac{\pi}{4} + k\pi$, k is odd, there is a stable fixed point
ex: $\theta = 3\pi/4, 11\pi/4, 19\pi/4, \ldots$

at $\theta = -\pi/4 + k\pi$, k is even, there is a unstable fixed point ex: $\frac{7\pi}{4}, \frac{15\pi}{4}, \frac{23\pi}{4}, \dots$

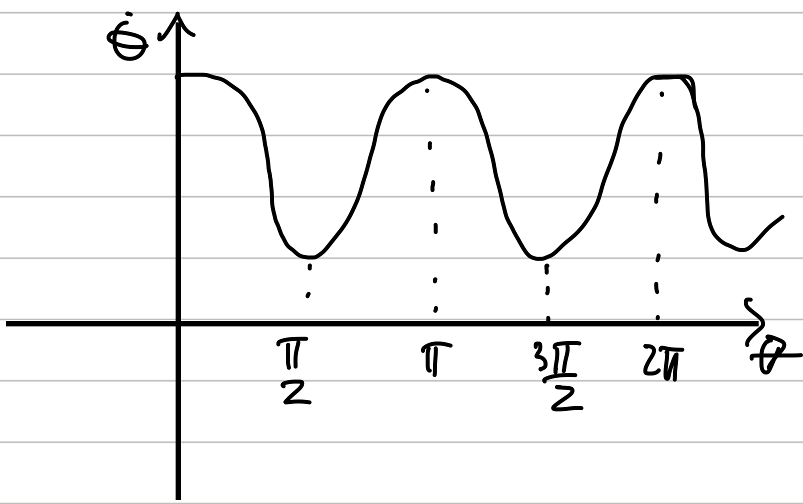
$\boxed{4.1.6}$ given $\dot{\theta} = 3 + \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = -3$

However $-1 \le \cos 2\theta \le 1 \Rightarrow \cos 2\theta + 3 \ge 2 > 0$

$\Rightarrow \dot{\theta}$ can not be 0 in this case

Phase potrait:



Therefore, the phase potrait could not be drawn on the circle.