Math 134, Spring 2022

Lecture #20: Linear systems

Friday May 13th

Learning objectives

Today we will discuss:

- What it means to say $\boldsymbol{x}^* = 0$ is a stable or unstable node of an uncoupled system.
- What it means to say $\boldsymbol{x}^* = 0$ is a saddle point of an uncoupled system.
- The stable and unstable manifolds associated to a saddle point of an uncoupled system.
- Classification of fixed points for linear systems with distinct real eigenvalues.

Linear systems

Uncoupled linear systems

We say that the linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

is **uncoupled** if

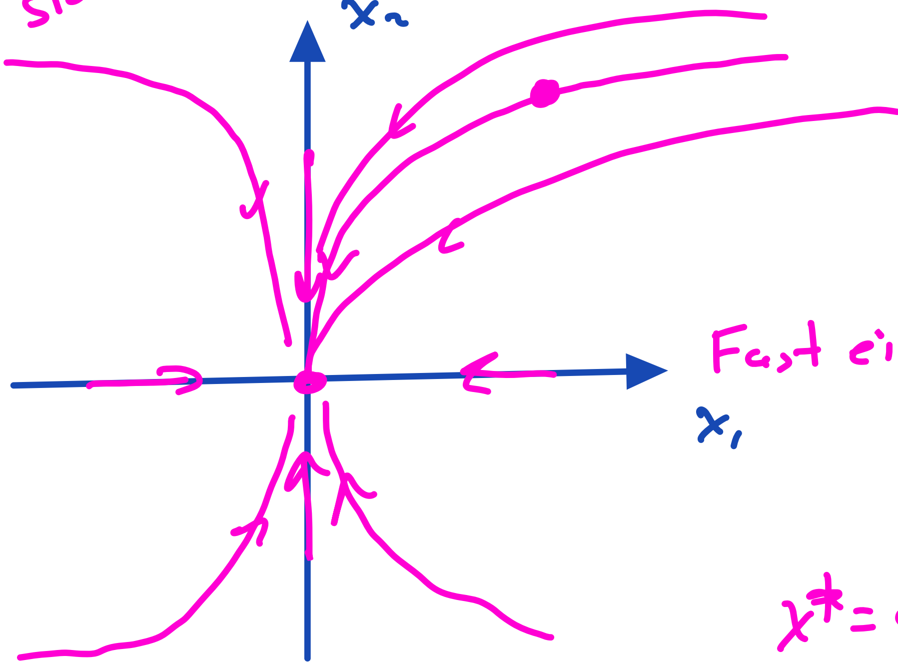
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Assume that $\lambda_1 \neq \lambda_2$

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 \\ \dot{x}_2 = \lambda_2 x_2 \end{cases} \Rightarrow \begin{cases} x_1(t) = x_1(0) e^{\lambda_1 t} \\ x_2(t) = x_2(0) e^{\lambda_2 t} \end{cases}$$

Case 1: $\lambda_1 < \lambda_2 < 0$

$$\begin{cases} x_1(t) = x_1(0)\, e^{\lambda_1 t} \\ x_2(t) = x_2(0)\, e^{\lambda_2 t} \end{cases}$$

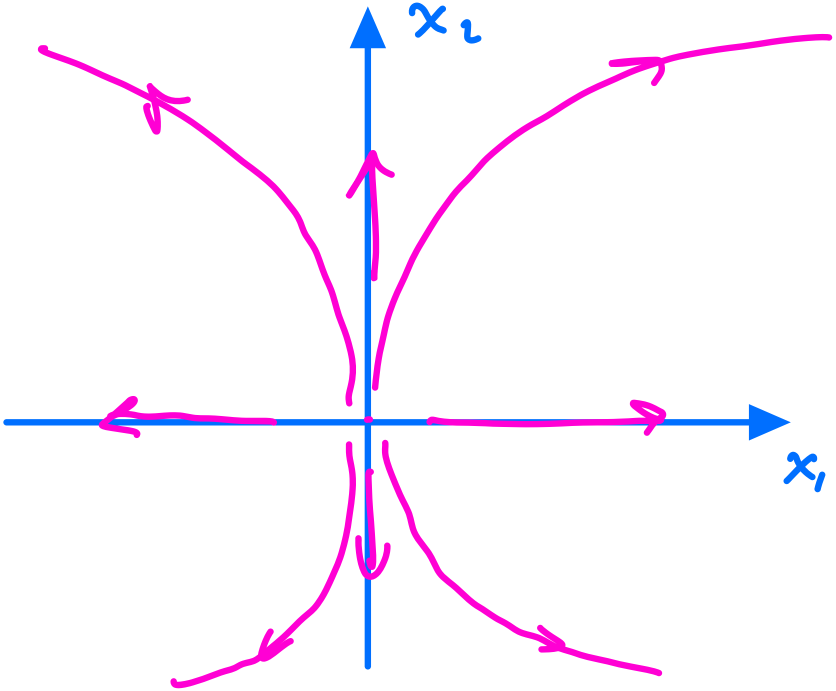


$x^* = 0$ is a stable node

slow eigendirection

Case 2: $\lambda_1 > \lambda_2 > 0$

$$\begin{cases} x_1(t) = x_1(0)\, e^{\lambda_1 t} \\ x_2(t) = x_2(0)\, e^{\lambda_2 t} \end{cases}$$



$\dot{\underline{x}} = A \underline{x}$

Fast eigendirection

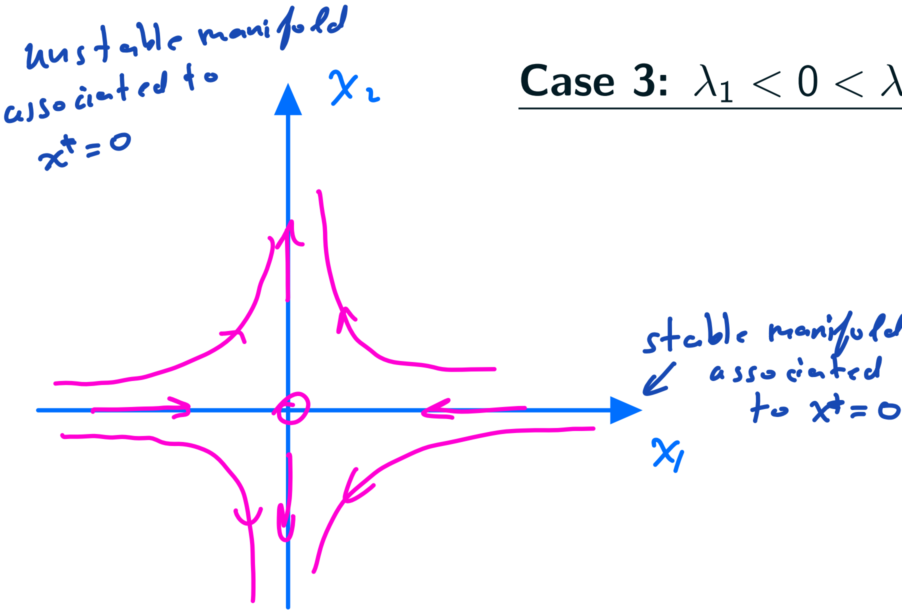
$x^* = 0$ is an unstable node

Case 3: $\lambda_1 < 0 < \lambda_2$

$$\underline{\dot{x}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underline{x}$$

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 \\ \dot{x}_2 = \lambda_2 x_2 \end{cases}$$

$$\begin{cases} x_1(t) = x_1(0) e^{\lambda_1 t} \\ x_2(t) = x_2(0) e^{\lambda_2 t} \end{cases}$$



$x^* = 0$ is a saddle point

Stable manifold:

$$W^s(\underline{0}) = \{ \underline{x}_0 \in \mathbb{R}^2 \mid x(t) \to 0, \text{ as } t \to \infty \}$$

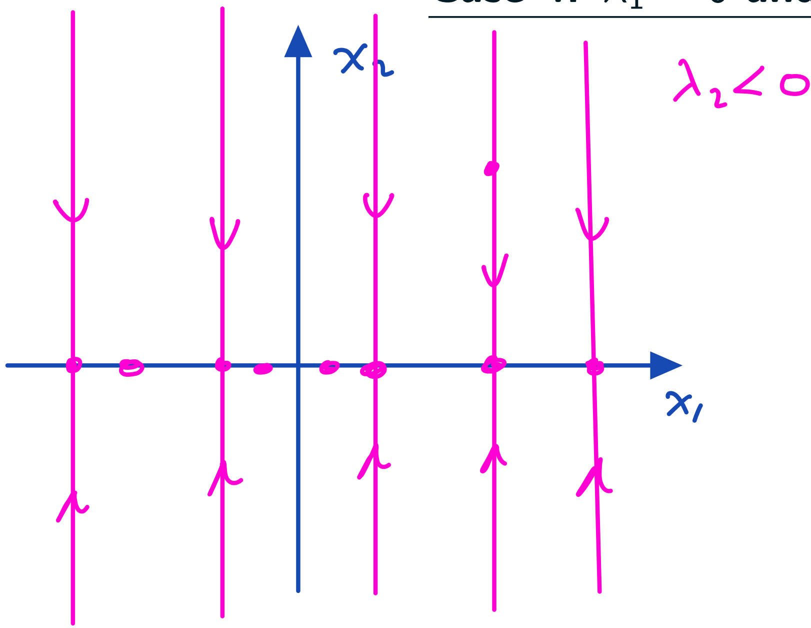
Unstable manifold:

$$W^u(\underline{0}) = \{ \underline{x}_0 \in \mathbb{R}^2 \mid x(t) \to 0, \text{ as } t \to -\infty \}$$

Case 4: $\lambda_1 = 0$ and $\lambda_2 \neq 0$

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underline{x}$$

$$\begin{cases} x_1(t) = x_1(0) \\ x_2(t) = x_2(0)\, e^{\lambda_2 t} \end{cases}$$



$$\begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

non-isolated critical points

An example

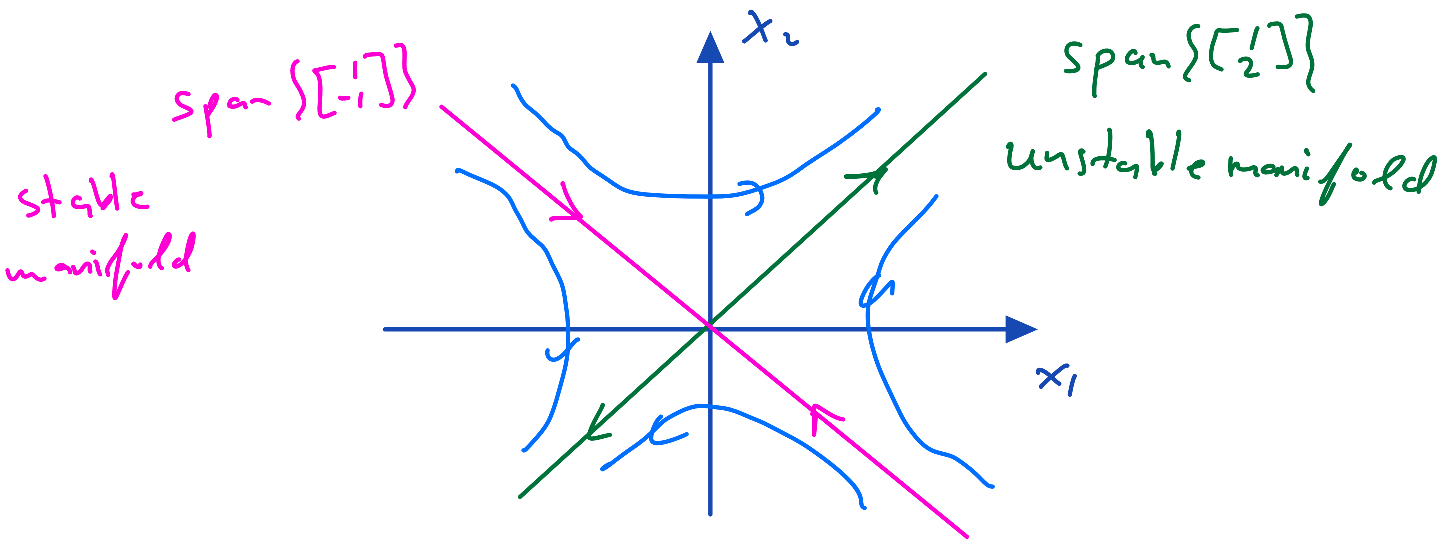
$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} x$$

$\text{span}\{u\} = \{\lambda u \mid \lambda \in \mathbb{R}\}$

$\lambda_1 = -1 \rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 5 \rightarrow v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



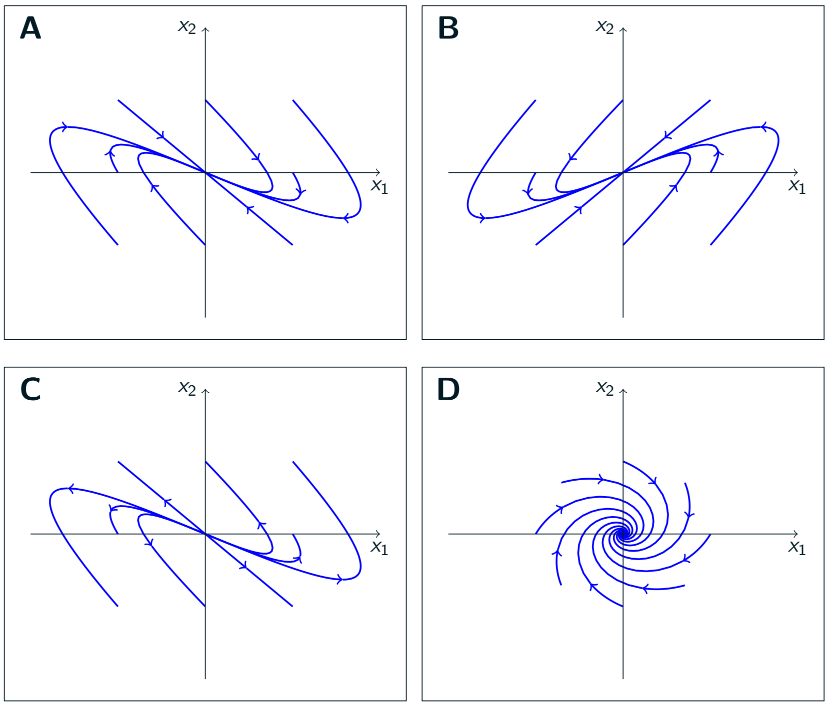
An example

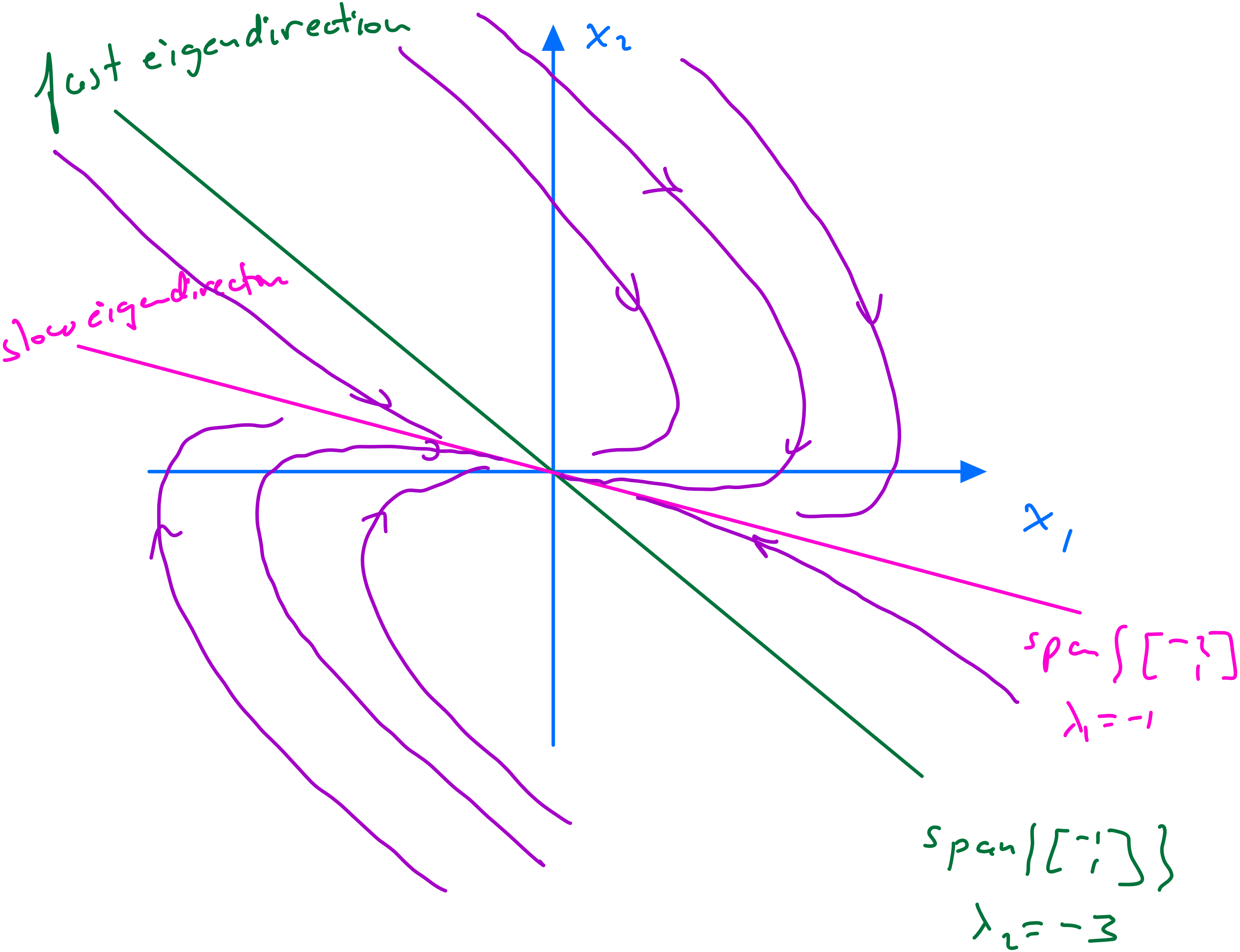
Which of the following phase portraits corresponds to the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 4 \\ -2 & -5 \end{bmatrix} \boldsymbol{x}$$

$\lambda_1 = -3 \rightarrow v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \rightarrow v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$




fast eigendirection
slow eigendirection
x_2
x_1
span{[-3; 1]}
$\lambda_1 = -1$
span{[-1; 1]}
$\lambda_2 = -3$

See you next time!