Math 134 - Homework 6

In problem 4 we will require the following fact: Given any 2×2 matrix A of real numbers, there exists an invertible matrix P so that $A = PMP^{-1}$ and M is one of the real canonical forms

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \qquad \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \qquad \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}$$

where $\mu \leq \lambda$, $\beta > 0$.

1. *(From Strogatz Section 5.2)* For each of the following linear systems, sketch the phase portrait, classify the fixed point $(x^*, y^*) = (0, 0)$, and indicate the directions of the eigenvectors.

   (a) $\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 3y \end{cases}$

   (b) $\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = -2x + 3y \end{cases}$

   (c) $\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 8x - 6y \end{cases}$

   (d) $\begin{cases} \dot{x} = 6x - y \\ \dot{y} = 2x + 3y \end{cases}$

2. Consider the system

$$\begin{cases} \dot{x} = x + y^2 + \frac{3}{2}y \\ \dot{y} = x + y \end{cases}$$

   (a) Find all fixed points of this system and compute the corresponding linearizations.

   (b) For each linearized system, classify the (unique) fixed point.

3. Solve problem 5.2.11 from Strogatz.

4. Given a 2×2 matrix A, let $\mathbf{u}$, $\mathbf{v}$ be the solutions of

$$\begin{cases} \dot{\mathbf{u}} = A\mathbf{u} \\ \mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{\mathbf{v}} = A\mathbf{v} \\ \mathbf{v}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

Define the (time-dependent) matrix $\Phi(t) = \begin{bmatrix} \mathbf{u}(t) & \mathbf{v}(t) \end{bmatrix}$.

(a) Show that the solution of

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

is given by

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0.$$

(b) Find $\Phi(t)$ for each of the three real canonical forms.

(c) Suppose that $B = PMP^{-1}$ for an invertible 2×2 matrix P. Show that the solution of

$$\begin{cases} \dot{\mathbf{x}} = B\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

is given by

$$\mathbf{x}(t) = P\Phi(t)P^{-1}\mathbf{x}_0.$$

(d) For each of the following matrices A, find a matrix M so that $A = PMP^{-1}$, where M is one of the real canonical forms above, and $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Then apply your answers to parts (b) and (c) to find the corresponding $\Phi(t)$.

(i) $A = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 5 \\ -2 & -5 \end{bmatrix}$

(iii) $A = \begin{bmatrix} -2 & 4 \\ -1 & -6 \end{bmatrix}$

5. For the nonlinear ODEs in (a)-(c), show that the origin is the only fixed point. What type of phase portrait does the linearization predict near the fixed point?

Use a computer program to draw the actual phase portrait. Does it look like the prediction of the linear system?

a) $\dot{x} = x^2, \quad \dot{y} = y$

b) $\dot{x} = y, \quad \dot{y} = x^2$

c) $\dot{x} = x^2 + xy, \quad \dot{y} = \frac{1}{2}y^2 + xy$