Math 134, Spring 2022

Lecture #21: Linear systems

Monday May 16th

Last time

- We considered the 2-dimensional linear system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

- If A has distinct real eigenvalues $\lambda_1 < \lambda_2$ then the solution can be written as

$$\boldsymbol{x}(t) = C_1 e^{\lambda_1 t}\boldsymbol{v}_1 + C_2 e^{\lambda_2 t}\boldsymbol{v}_2,$$

  where $\boldsymbol{v}_1, \boldsymbol{v}_2$ are eigenvectors associated to λ_1, λ_2 and C_1, C_2 are constants.

- In this case:
  - If $\lambda_1 < \lambda_2 < 0$ we say the fixed point $\boldsymbol{x}^* = 0$ is a **stable node**.
  - If $0 < \lambda_1 < \lambda_2$ we say the fixed point $\boldsymbol{x}^* = 0$ is a **unstable node**.
  - If $\lambda_1 < 0 < \lambda_2$ we say the fixed point $\boldsymbol{x}^* = 0$ is a **saddle point**.
  - If one of λ_1, λ_2 vanishes, we have a line of fixed points: the fixed point at $\boldsymbol{x}^* = 0$ is **non-isolated**.

Learning objectives

Today we will discuss:

- Classification of fixed points for linear systems with a repeated eigenvalues.
- Classification of fixed points for linear systems with complex eigenvalues.

Linear systems

Repeated eigenvalues

Theorem: Suppose that A has a repeated (real) eigenvalue σ. Then:

- Either there exist linearly independent eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and a nonsingular matrix $P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2]$ such that

$$P^{-1}AP = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$$

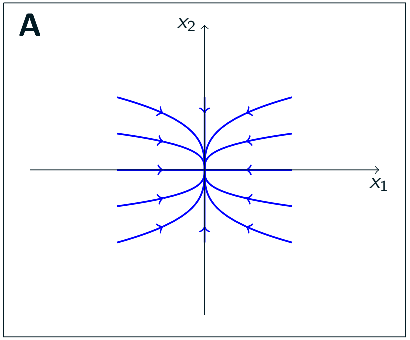
- Or there exist an eigenvector $\boldsymbol{v}$, a generalized eigenvector $\boldsymbol{w}$, and a nonsingular matrix P such that

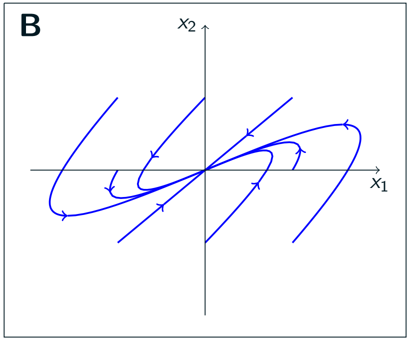
$$P^{-1}AP = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}$$

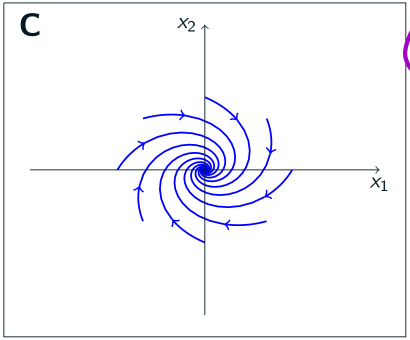
Diagonalizable case

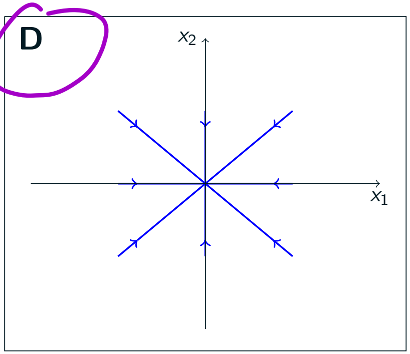
For $\sigma < 0$, which of the following phase portraits corresponds to

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \boldsymbol{x}$$







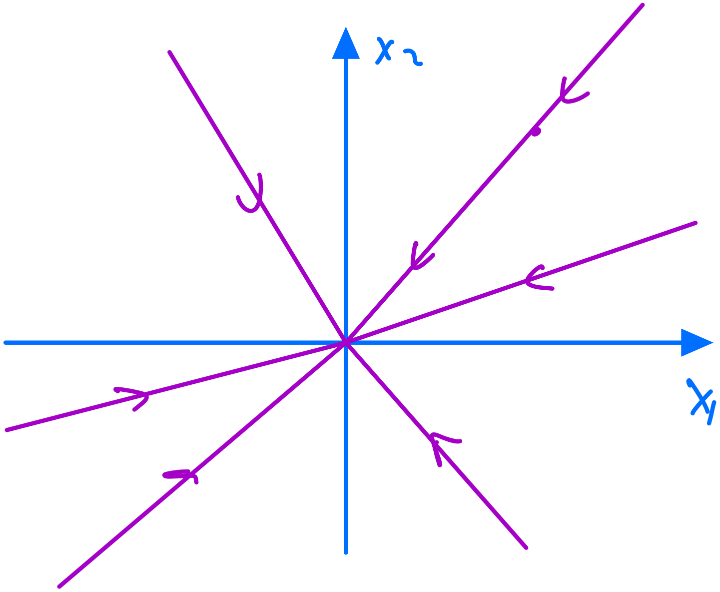


$$\begin{cases} \dot{x}_1 = \sigma x_1 \\ \dot{x}_2 = \sigma x_2 \end{cases} \Rightarrow \begin{cases} x_1(t) = x_1(0)e^{\sigma t} \\ x_2(t) = x_2(0)e^{\sigma t} \end{cases}$$

$$\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

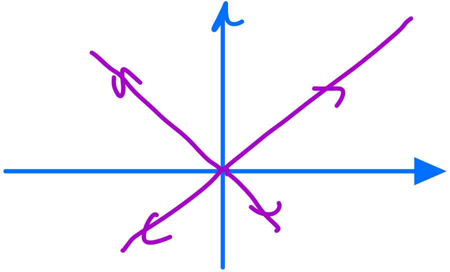
so the only critical point is

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an stable star

$\sigma > 0$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an unstable star

Nondiagonalizable case

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix} \boldsymbol{x}$$

$\lambda = \sigma$

We have one eigenvalue of multiplicity 2 and only one eigen-vector

$$\begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sigma \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general solution is

$$x(t) = (A + Bt) e^{\sigma t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + B e^{\sigma t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\sigma < 0$

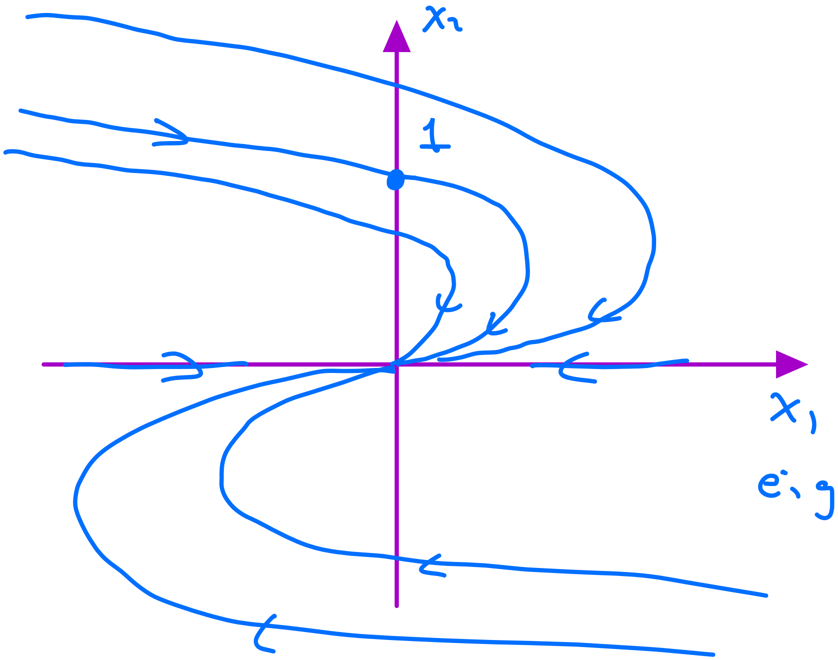
$$x(t) = \begin{bmatrix} A+Bt \\ B \end{bmatrix} e^{\sigma t}$$

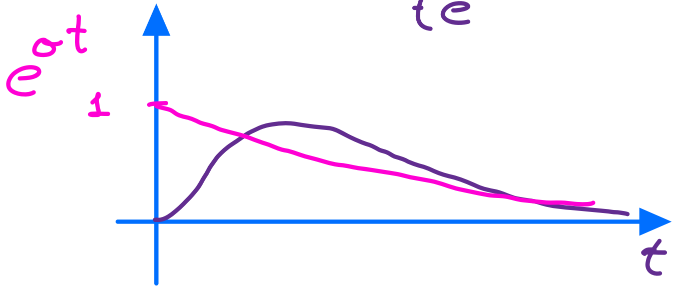
$$x(0) = \begin{bmatrix} A \\ B \end{bmatrix}$$

$A=0$ and $B=1$

$$\begin{cases} A = 0 \\ B = 1 \end{cases}$$

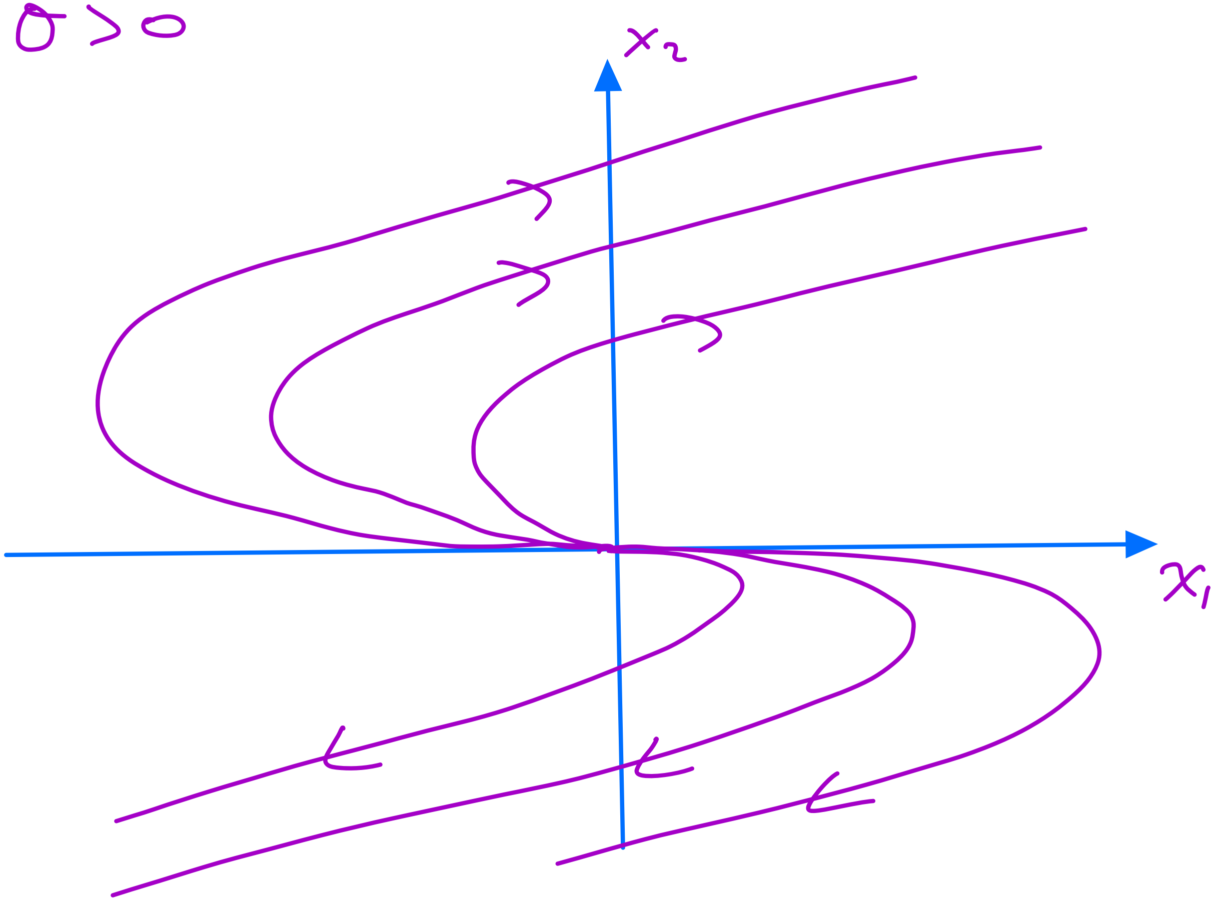
$$x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} e^{\sigma t}$$





eigendirection $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\sigma > 0$



An example

Sketch the phase portrait for the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \boldsymbol{x}$$

$\tau = -2$

$\Delta = (-3)(1) - (4)(-1)$

$= 1$

Eigenvalues: $\tau = \frac{1}{2}\left[-2 \pm \sqrt{\underbrace{4-4}_{=0}}\right] = -1$

Eigenvectors:

$$\left(\begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \underline{v} = \underline{0}$$

here $\underline{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ will do.

Generalized eigenvector: $A\underline{w} = \lambda \underline{w} + \underline{v}$

solving this system we get

$$\underline{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

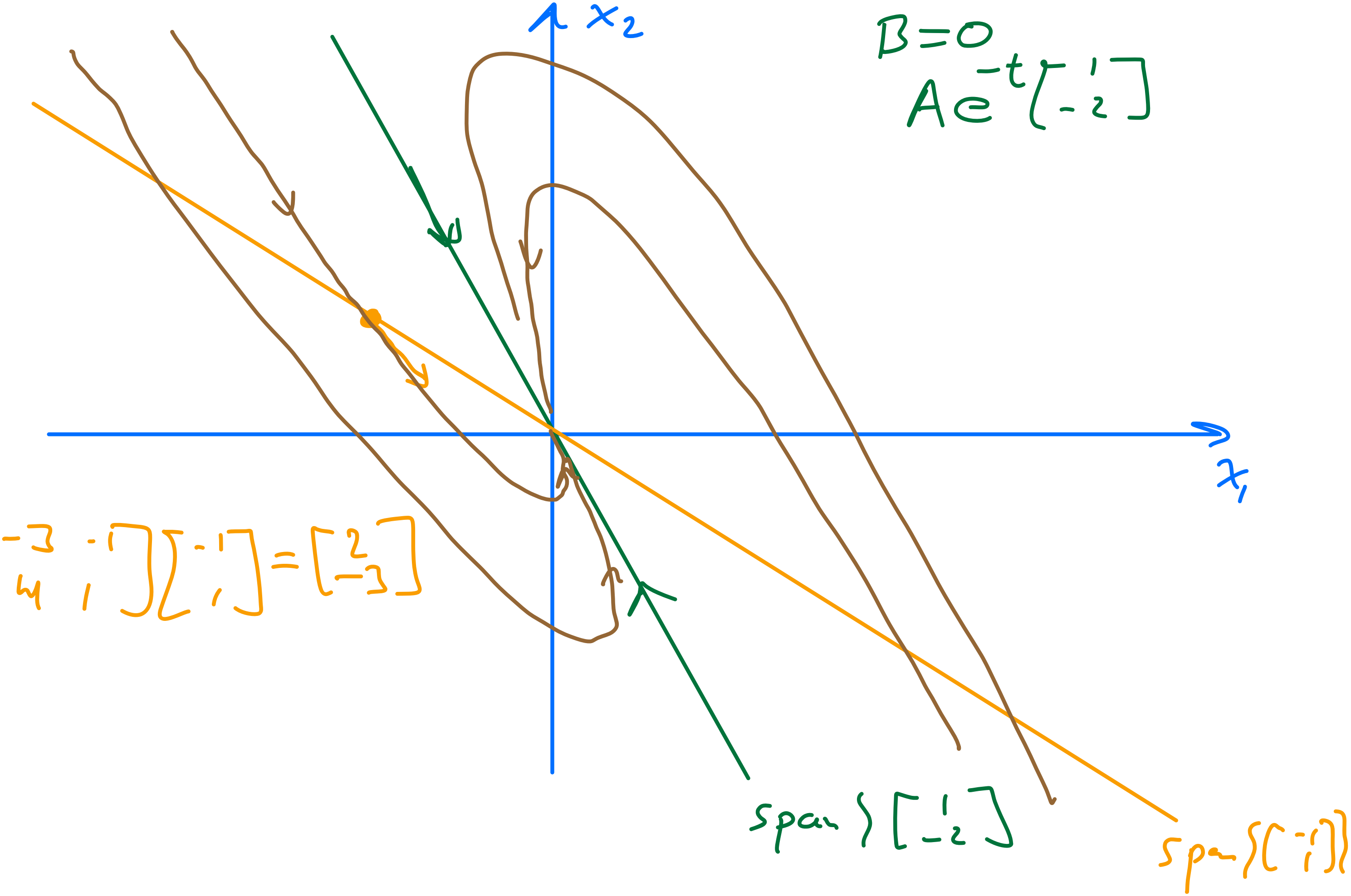
So, the general sol. is

$$x(t) = (A+Bt)e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + Be^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x(t) = (A+Bt)e^{-t}\begin{bmatrix}1\\-2\end{bmatrix} + Be^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$$

$B=0$

$$Ae^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$$



$$\begin{bmatrix}-3 & -1\\4 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}$$

Complex eigenvalues: A special case

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \boldsymbol{x}$$

$$x(t) = \begin{bmatrix} A e^{\alpha t} \sin(\beta t + B) \\ A e^{\alpha t} \cos(\beta t + B) \end{bmatrix}$$

where

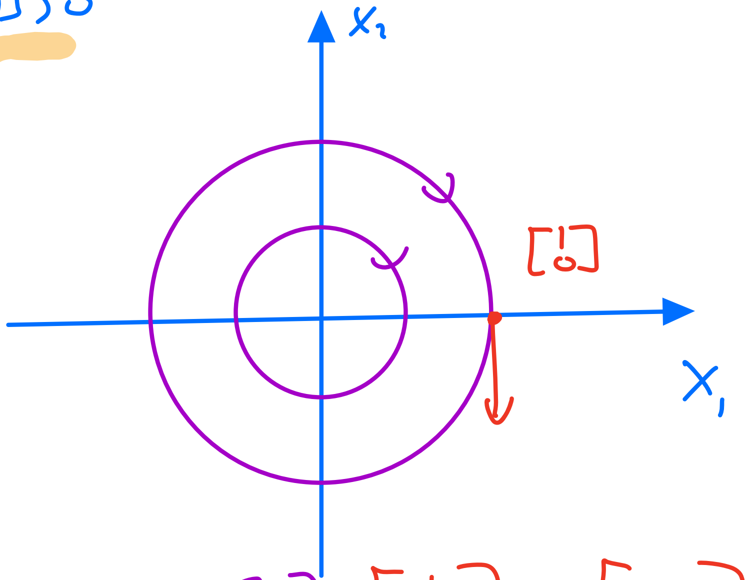
$$x(0) = \begin{bmatrix} A \sin(B) \\ A \cos(B) \end{bmatrix}$$

Case 1: $\alpha = 0$

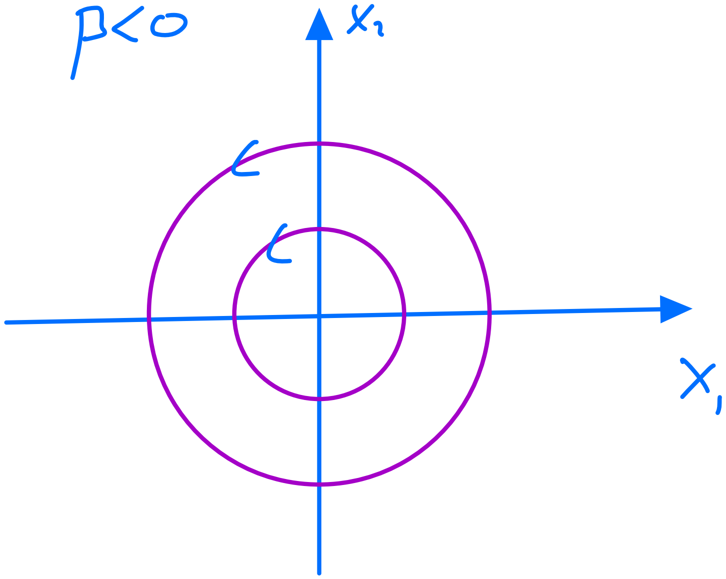
$$x(t) = \begin{bmatrix} A\sin(\beta t + B) \\ A\cos(\beta t + B) \end{bmatrix}$$

$$(A\sin(\beta t + B))^2 + (A\cos(\beta t + B))^2 = A^2$$

$\beta > 0$



$\beta < 0$

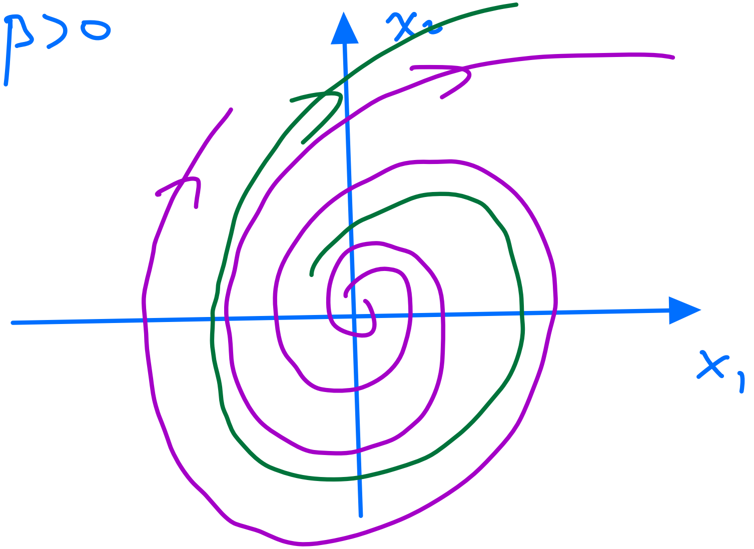


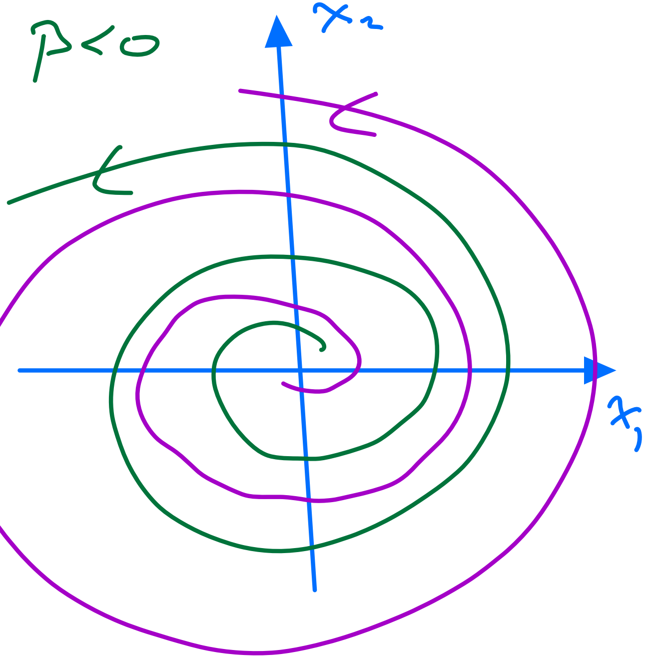
$$\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\beta \end{bmatrix}$$

the origin is called a center

Case 2: $\alpha > 0$

$$x(t) = \begin{bmatrix} A\, e^{\alpha t} \sin(\beta t + B) \\ A\, e^{\alpha t} \cos(\beta t + B) \end{bmatrix}$$

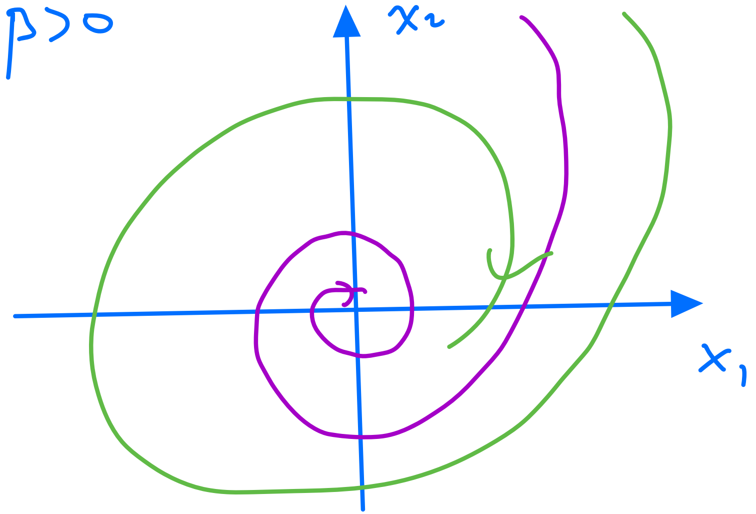


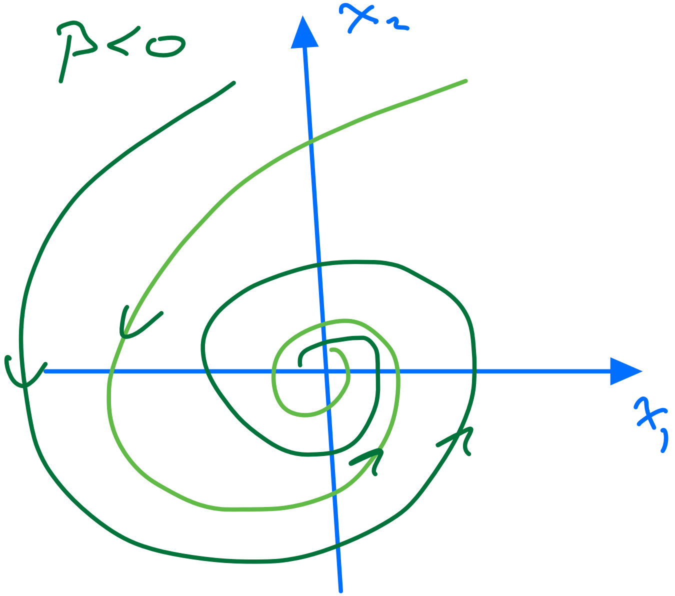


the origin is called an unstable spiral

Case 3: $\alpha < 0$

$$x(t) = \begin{bmatrix} A e^{\alpha t} \sin(\beta t + B) \\ A e^{\alpha t} \cos(\beta t + B) \end{bmatrix}$$





the origin is called an stable spiral

Complex eigenvalues

Theorem: Suppose that A has complex eigenvalues $\alpha \pm i\beta$. Then there exist linearly independent (real) vectors $\mathbf{v}$, $\mathbf{w}$ so that

$$A\mathbf{v} = \alpha\mathbf{v} - \beta\mathbf{w}$$

$$A\mathbf{w} = \beta\mathbf{v} + \alpha\mathbf{w}.$$

$$A[v+iw]=[\alpha+i\beta][v+iw]$$

and there exists a matrix $P = [\mathbf{v} \quad \mathbf{w}]$ such that

$$P^{-1}AP = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

and the general solution of

$$\dot{\mathbf{x}} = A\mathbf{x}$$

is given by

$$\mathbf{x}(t) = C_1 e^{\alpha t} \sin(\beta t + C_2)\mathbf{v} + C_1 e^{\alpha t} \sin(\beta t + C_2)\mathbf{w}$$

An example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 4 & 2 \\ -5 & -2 \end{bmatrix} \boldsymbol{x}$$

$\tau = 2$

$\Delta = 2$

Eigenvalues:

$$\lambda = \frac{1}{2}\left[2 \pm \sqrt{4-8}\right] = 1 \pm i$$

Eigenvectors:

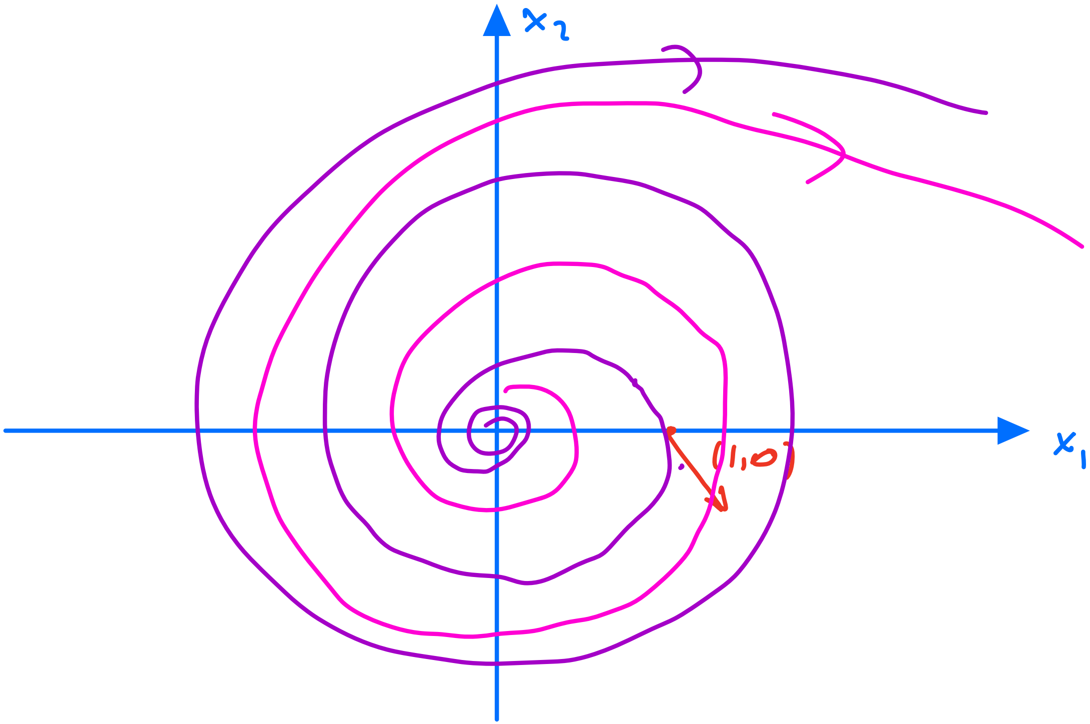
$$\left(\begin{bmatrix} 4 & 2 \\ -5 & -2 \end{bmatrix} - (1+i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\underline{z} = \underline{0}$$

$$\rightarrow \begin{bmatrix} 3-i & 2 \\ -5 & -3-i \end{bmatrix}\underline{z} = \underline{0}$$

$$z = \begin{bmatrix} 2 \\ i-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & \overset{\underline{v}}{0} \\ -3 & 1 \end{bmatrix} \qquad \overset{\underline{w}}{P}{}^{-1}AP = \begin{bmatrix} \overset{\overset{\alpha}{\downarrow}}{1} & \overset{\overset{\beta}{\downarrow}}{1} \\ -1 & 1 \end{bmatrix}$$

$$x(t) = Ae^{t}\sin(t+B)\begin{bmatrix}2\\-3\end{bmatrix} + Ae^{t}\cos(t+B)\begin{bmatrix}0\\1\end{bmatrix}$$

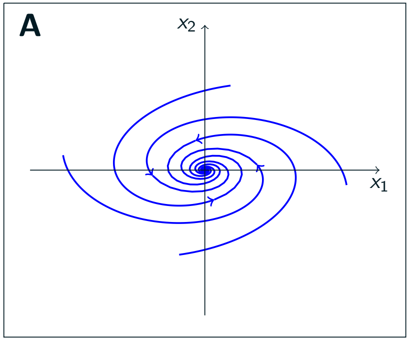


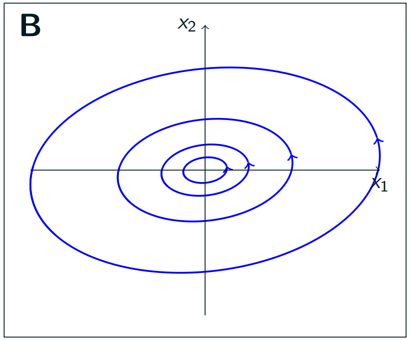
$$\begin{bmatrix}4 & 2\\-5 & -2\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\-5\end{bmatrix}$$

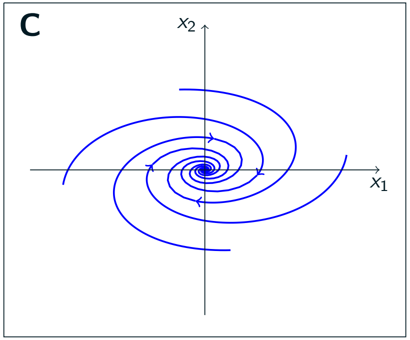
An example

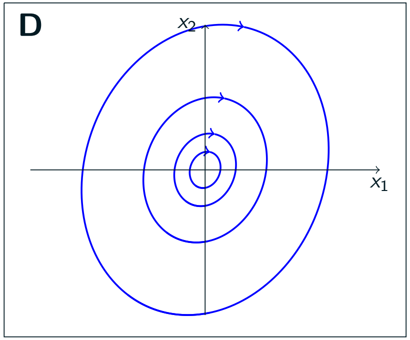
Which of the following phase portraits corresponds to the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & -10 \\ 5 & -1 \end{bmatrix} \boldsymbol{x}$$









See you next time!