Math 134, Winter 2022

Lecture #19: Linear systems

May 11$^{\text{th}}$

Last time

- We said that $\boldsymbol{x}^*$ is a fixed point of the $2d$ system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

  if $\boldsymbol{f}(\boldsymbol{x}^*) = 0$.

- We said that the linearization of this system about the fixed point $\boldsymbol{x}^*$ is the equation

$x = x^* + \eta$

$$\dot{\eta} = \nabla \boldsymbol{f}(\boldsymbol{x}^*)\eta$$

$$\nabla f = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_1 \end{bmatrix}$$

- Started a review of linear algebra

Learning objectives

Today we will discuss:

- Review of Linear Algebra
- Reduction of linear systems to real canonical matrices.

Linear systems

Linear algebra: a review

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A\mathbb{I} = \mathbb{I}A = A$$

- We define the trace of A as

$$\operatorname{tr} A = a + d$$

- We define the determinant of A as

$$\det A = ad - bc$$

- We say that λ is an eigenvalue of A if

$$\underbrace{\det(A - \lambda \mathbb{I})}_{\text{characteristic polynomial}} = 0$$

Theorem: The eigenvalues of $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

where $\tau = \operatorname{tr} A$ and $\Delta = \det A$

Proof:

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right)$$

$$= (a-\lambda)(d-\lambda) - cb$$

$$= \lambda^2 - \underbrace{(a+d)}_{\operatorname{tr} A}\lambda + \underbrace{ad - cb}_{\det A}$$

$$= \lambda^2 - \tau\lambda + \Delta$$

using the quadratic formula we get

$$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

An example

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

A) -1 and 5

B) -4 and 8

C) 1 and 7

D) -2 and 10

$\tau = 4$ $\quad \Delta = 1 \cdot 3 - 2 \cdot 4 = -5$

$\lambda = \frac{1}{2}\left[4 \pm \sqrt{16 - 4(-5)}\right] = \frac{1}{2}[4 \pm 6]$

$= -1$ or 5

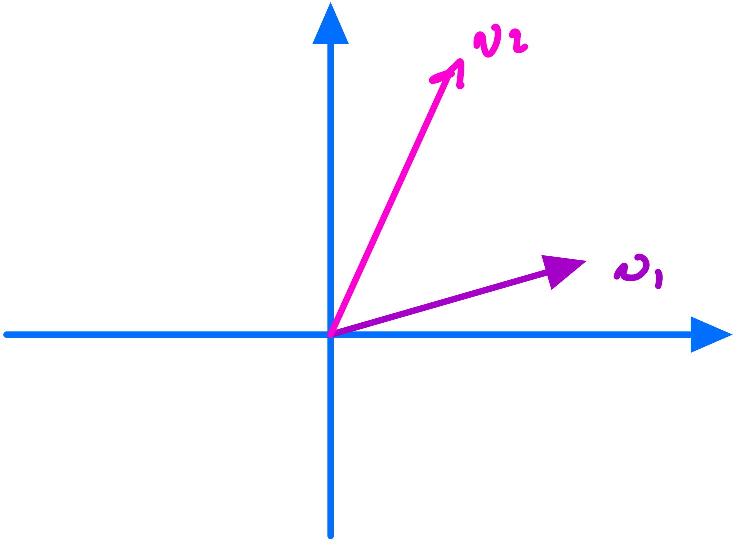
- We say that $\boldsymbol{v} \neq 0$ is an eigenvector of A with eigenvalue λ if

$$A\boldsymbol{v} = \lambda \boldsymbol{v}.$$

- We say that two vectors v_1, v_2 are lin. ind. if whenever

$$c_1 v_1 + c_2 v_2 = \underline{0}$$

we have $c_1 = c_2 = \underline{0}$



If A is a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$ then the corresponding eigenvectors are linearly independent.

$$\begin{cases} \lambda_1 \to v_1 \\ \lambda_2 \to v_2 \end{cases}$$

Find eigenvectors for

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

corresponding to each eigenvalue $\lambda = -1$ and $\lambda_2 = 5$.

$\underline{\underline{\lambda = -1}}$ //

$$Av = (-1)v$$

$$\Rightarrow Av = -Iv$$

$$\Rightarrow (A + I)v = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2v_1 + 2v_2 = 0 \\ 4v_1 + 4v_2 = 0 \end{cases}$$

$$v_1 = -v_2$$

$$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\underline{\underline{\lambda = 5}}$ //

$$Av = 5v$$

$$(A - 5I)v = 0$$

$$= \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Let A be a 2×2 real matrix. Then we are in one of three situations:

1) A has linearly independent real eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$. Taking λ_1, λ_2 to be the corresponding eigenvalues and taking $P = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2]$ we have

Always holds if $\lambda_1 \neq \lambda_2$ and are real

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

2) A has complex eigenvectors $\boldsymbol{v} \pm i\boldsymbol{w}$ with corresponding complex eigenvalues $\alpha \pm i\beta$, where $\beta \neq 0$. Taking $P = [\boldsymbol{v} \quad \boldsymbol{w}]$ we have

$i = \sqrt{-1}$

$$P^{-1}AP = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

3) A has one real eigenvector $\boldsymbol{v}$ with repeated eigenvalue σ and a generalized eigenvector $A\boldsymbol{w} = \sigma\boldsymbol{w} + \boldsymbol{v}$. Taking $P = [\boldsymbol{v} \quad \boldsymbol{w}]$ we have

$$P^{-1}AP = \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}$$

Jordan blocks

Example 1

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

$\lambda_1 = -1 \to v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 5 \to v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\to$ $P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ then

$$AP = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= P \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

Example 2: $A=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$, $\tau=2$, $\Delta=5$

$$\lambda=\frac{1}{2}\left[2\pm\sqrt{2^2-4\cdot 5}\right]=\frac{1}{2}[2\pm 4i]=1\pm 2i$$

$\lambda=1+2i$

$$[A-(1+2i)I]=\begin{bmatrix}-2i & 2\\ -2 & -2i\end{bmatrix}$$

$$A(v+iw)=\lambda(v+iw)$$

$$\Rightarrow \begin{bmatrix}1\\ i\end{bmatrix}=\underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{v}+i\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{w}$$

so $P=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$P^{-1}AP=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$$

Ex. 3: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\tau = 4$ $\Delta = 4$

$$\lambda = \frac{1}{2}\left[\tau \pm \sqrt{\tau^2 - 4\Delta}\right]$$

$$= \frac{1}{2}\left[4 \pm \sqrt{16 - 4 \cdot 4}\right] = 2$$

only one root of multiplicity 2!

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector

Look for the gen. eigenvector

$$(A - 2I)\,w = v$$

An example

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \boldsymbol{x}$$

See you next time!