Home Work 4

1) $\dot{x} = [\cosh(1+x) - 1][x^2 + 2x - 1]$

a) $\dot{x} = 0 \Leftrightarrow \left[\begin{array}{ll} \cosh(1+x) - 1 = 0 & (1) \\ x^2 + 2x - 1 = 0 & (2) \end{array}\right.$

(1) $\Leftrightarrow \quad \cosh(1+x) = 1 = \dfrac{e^{1+x} + e^{-(1+x)}}{2}$

$\Rightarrow \quad e^{x+1} + e^{-(x+1)} - 2 = 0$

Let $t = x+1$ & $f(t) = e^{t} + e^{-t} - 2$

$f'(t) = e^{t} - e^{-t} = 0 \Leftrightarrow e^{t} = e^{-t} \Rightarrow e^{t} = \dfrac{1}{e^{t}}$

$\Rightarrow e^{2t} = 1 = e^{0} \Leftrightarrow t = 0 \Leftrightarrow x = -1$

also, $f''(t) = e^{t} + e^{-t} > 0$

$\Rightarrow f(t)$ gets minimum at $t = 0 \Leftrightarrow x = -1$

$f_{min} = f(0) = 0$

$\Rightarrow f(x) = e^{x+1} + e^{-(x-1)} - 2$ get minimum value 0

at $x = -1$

OR $e^{x+1} + e^{-(x+1)} - 2 = 0$

has only 1 root $x = -1$

② $x^2+2x-r=0$, $\Delta = b^2-4ac = 4+4r = 4(r+1)$

$$\Rightarrow \sqrt{\Delta} = 2\sqrt{1+r}$$

If $1+r>0 \Leftrightarrow r>-1$, then there are 2 fixed point

$$\Rightarrow \begin{cases} x_1 = \dfrac{-2+\sqrt{\Delta}}{2} = \dfrac{-2+2\sqrt{1+r}}{2} \\ x_2 = \dfrac{-2-\sqrt{\Delta}}{2} = \dfrac{-2-2\sqrt{1+r}}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = -1+\sqrt{r+1} \\ x_2 = -1-\sqrt{r+1} \end{cases}$$

$\begin{cases} \text{If } r+1<0 \Rightarrow r<-1, \text{ there are no fixed point} \\ \text{If } r+1=0 \Rightarrow r=-1, \ 1 \text{ fixed point } x=-1 \end{cases}$

Combined ① & ②, We have the fixed points:

$\begin{cases} \oplus \text{ with } r \leq -1, \text{ there is one fixed point } x=-1 \\ \oplus \text{ with } r>-1, \text{ there are 3 fixed point } -1, -1\pm\sqrt{r+1} \end{cases}$

Also, $\dfrac{\partial f}{\partial x} = 0 \Leftrightarrow \Big[[\cosh(1+x)-1][x^2+2x-r]\Big]'_x = 0$

$$[\cosh(x+1)]'(x^2+2x-\ell)+\cosh(1+x)[2x+2]=0$$

We have: $[\cosh(x+1)]' = \left[\frac{e^{x+1}+e^{-(x+1)}}{2}\right]'$

$$=\frac{e^{x+1}-e^{-(x+1)}}{2}=\sinh(x+1)$$

$$\Rightarrow \frac{\partial f}{\partial x}=\sinh(x+1).(x^2+2x-\ell)+2\cosh(1+x)(x+1)=0$$

As we can see with $\ell=-1$, We have

$$\frac{\partial f}{\partial x}=\sinh(x+1)(x^2+2x+1)+2\cosh(x+1)(x+1)=0$$

$$\Leftrightarrow \sinh(x+1)(x+1)^2+(x+1).2\cosh(x+1)=0$$

$$\Leftrightarrow (x+1)[(x+1)\sinh(x+1)+2\cosh(x+1)]=0$$

$$\Leftrightarrow \left[\begin{array}{l} x+1=0 \Leftrightarrow x=-1 \\ (x+1)\sinh(x+1)+2\cosh(x+1)=0 \text{ (reject as below)} \end{array}\right.$$

Check $(x+1)\sinh(x+1)+2\cosh(x+1)=0$

Let $t=x+1 \Rightarrow t\sinh t+2\cosh t=0$

Let $f(t)=t\sinh t+2\cosh t$

$$= t.\frac{e^t - e^{-t}}{2} + 2\frac{e^t + e^{-t}}{2}$$

$$= \frac{1}{2}t(e^t - e^{-t}) + e^t + e^{-t}$$

+ Since $t \geq 0 \Rightarrow e^t - e^{-t} \geq 0$

$$\Rightarrow \frac{1}{2}t(e^t - e^{-t}) + e^t + e^{-t} > 0.$$

+ $t < 0 \Rightarrow e^t - e^{-t} < 0$

$$\Rightarrow t(e^t - e^{-t}) > 0$$

$$\Rightarrow \frac{1}{2}t(e^t - e^{-t}) + e^t + e^{-t} > 0$$

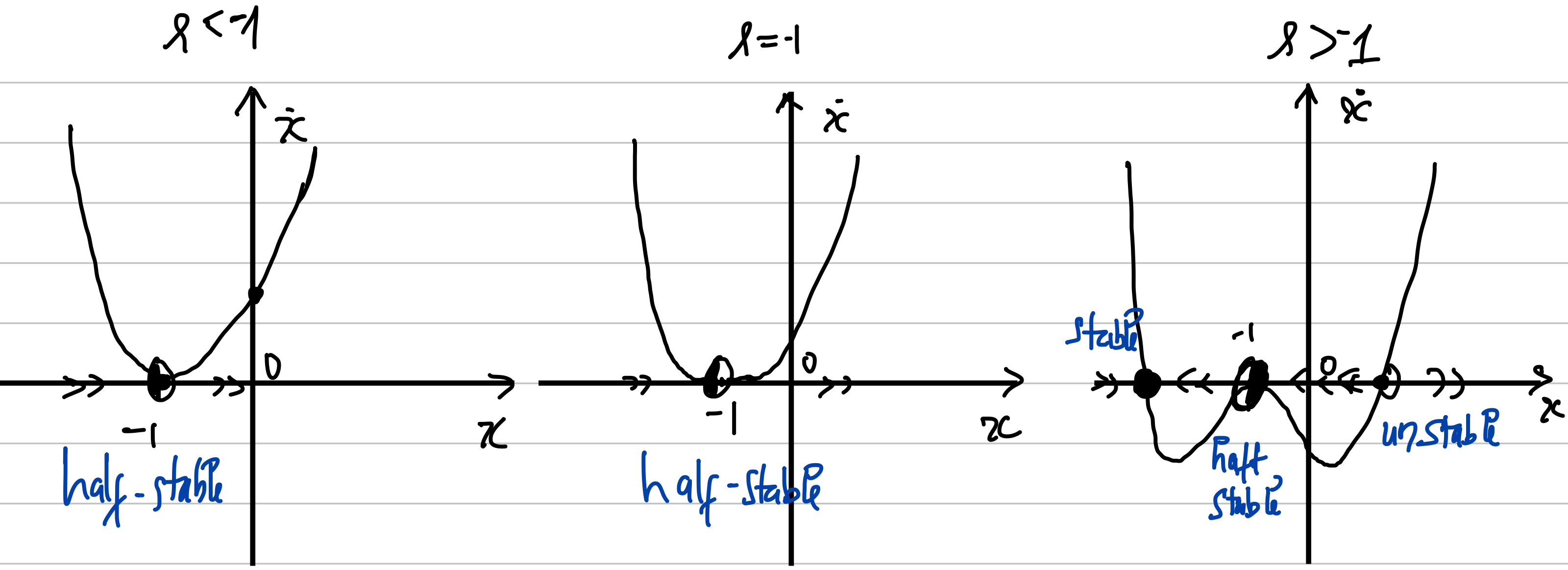
$\Rightarrow$ therefore $t\sinh t + 2\cosh t$ or $(x+1)\sinh(x+1) + 2\cosh(x+1)$ always greater than 0

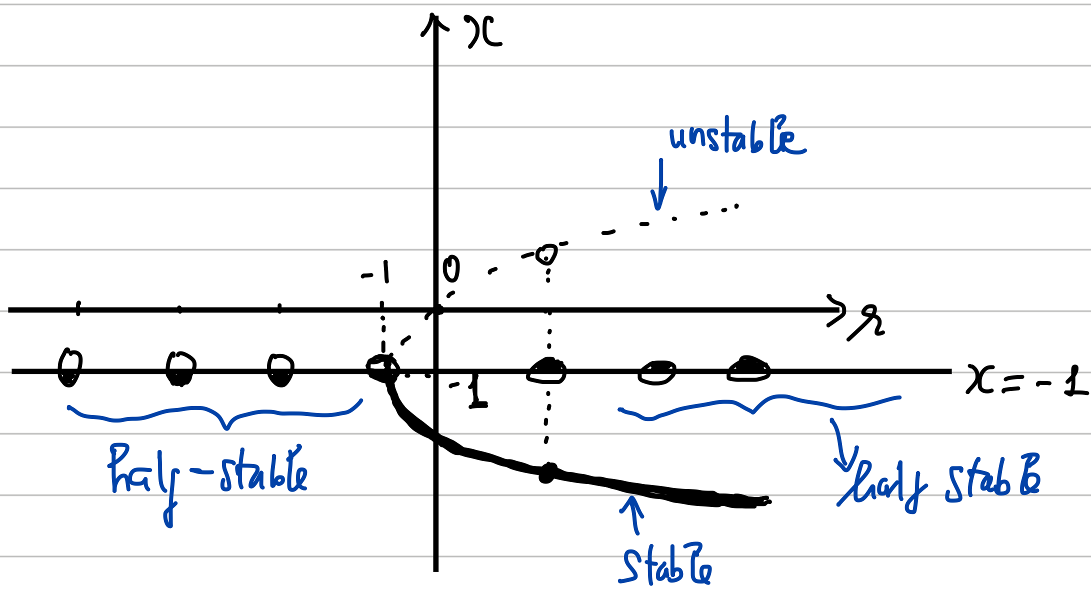
$$\boxed{\text{Finally, the posible bifurcation is at } (r, x) = (-1, -1)}$$

b) $\dot{x} = [\cosh(1+x) - 1](x^2 + 2x - r)$

Based on part a, we can have the diagram:

$$\begin{cases} r = x^2 + 2x \\ \cosh(x+1) = 1 \Leftrightarrow x = -1 \end{cases}$$





Note:

$$\boxed{\text{Every points on } x = -1 \text{ are half-stable points}}$$

c) let $\begin{cases} y = x - x_c \\ s = r - r_c \end{cases} \Rightarrow \begin{cases} y = x + 1 \\ s = r + 1 \Leftrightarrow r = s - 1 \end{cases}$

$$\Rightarrow f(y, s) = \dot{x} = (\cosh y - 1)(x^2 + 2x - s + 1)$$

we have $x^2 + 2x - s + 1 = (x+1)^2 - s$

$$= y^2 - s$$

$$\Rightarrow f(y,s) = (\cosh y - 1)(y^2 - s)$$

$$\Rightarrow \boxed{\dot{y} = f(y,s) = (\cosh y - 1)(y^2 - s)}$$

d) With $(y,s) = (0,0)$

$$\Rightarrow \dot{y} = (\cosh 0 - 1)(0 - 0) = 0$$

$\Rightarrow (y,s) = (0,0)$ is possible bifurcation

Applying Taylor series, we have:

$$f(y,s) = \sum_{n=0}^{N} \sum_{j=0}^{n} \frac{1}{(n-j)!\, j!} \cdot \frac{\partial^n f}{\partial y^{n-j} \partial s^j}(0,0)\, y^{n-j} s^j$$

$$= f(0,0) + \frac{\partial f}{\partial y}\Big|_{(0,0)} y + \frac{\partial f}{\partial s}\Big|_{(0,0)} s$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)} y^2 + \frac{\partial^2 f}{\partial y \partial s}\Big|_{(0,0)} y.s + \frac{1}{2} \frac{\partial^2 f}{\partial s^2}\Big|_{(0,0)} s^2$$

$$+ \frac{1}{3!} \frac{\partial^3 f}{\partial y^3}\Big|_{(0,0)} y^3 + \frac{1}{2!} \frac{\partial^3 f}{\partial y^2 \partial s}\Big|_{(0,0)} y^2.s$$

$$+ \frac{1}{2!} \frac{\partial^3 f}{\partial y \partial s^2}\Big|_{(0,0)} y s^2 + \frac{1}{3!} \cdot \frac{\partial^3 f}{\partial s^3}\Big|_{(0,0)} . s^3$$

$$+ \frac{1}{4!} \frac{\partial^4 f}{\partial y^4}\Big|_{(0,0)} y^4 + \frac{1}{3!} \frac{\partial^4 f}{\partial y^3 \partial s}\Big|_{(0,0)} y^3 s$$

$$+ \frac{1}{2!\,2!} \cdot \frac{\partial^4 f}{\partial y^2 \partial s^2} y^2 s^2 + \frac{1}{3!} \frac{\partial^4 f}{\partial y \partial s^3}\Big|_{(0,0)} y . s^3$$

$$+ \frac{1}{4!} \frac{\partial^4 f}{\partial s^4}\Big|_{(0,0)} s^4$$

We have : $\cosh(y) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} y^{2n}$. Need to check it

at fourth order : $\cosh(y) \simeq 1 + \frac{1}{2} y^2 + \frac{1}{4!} y^4$

$$= 1 + \frac{1}{2} y^2 + \frac{1}{24} y^4$$

$$\Rightarrow f(y,s) = \left(\frac{1}{2} y^2 + \frac{1}{24} y^4\right)(y^2 - s)$$

- $\frac{\partial f}{\partial y}\Big|_{(0,0)} = \left(y + \frac{1}{6} y^3\right)(y^2 - s) + 2y\left(\frac{1}{2} y^2 + \frac{1}{24} y^4\right)$
  $= 0$

- $\frac{\partial f}{\partial s}\Big|_{(0,0)} = -\left(\frac{1}{2} y^2 + \frac{1}{24} y^4\right) = 0 \Rightarrow \frac{\partial^2 f}{\partial s^2}\Big|_{(0,0)} = 0$

- $\frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)} = \left(1 + \frac{1}{2} y\right)(y^2 - s) + \left(y + \frac{1}{6} y^3\right)(2y) + 3y^2 + \frac{5}{12} y^4$
  $= 0$

$\oplus \left.\dfrac{\partial^2 f}{\partial y\, \partial s}\right|_{(0,0)} = -\left[y + \frac{1}{6}y^3\right] = 0.$

$\oplus \left.\dfrac{\partial^3 f}{\partial y^3}\right|_{(0,0)} = \frac{1}{2}(y^2 - s) + (1+\frac{1}{2}y)2y + 4y + \frac{4}{3}y^3$
$+ 6y + \frac{5}{3}y^3 = 0$

$\oplus \left.\dfrac{\partial^3 f}{\partial y^2 \partial s}\right|_{(0,0)} = \dfrac{\partial f}{\partial y}\left(\dfrac{\partial^2 f}{\partial y\, \partial s}\right)\Big|_{(0,0)} = -\left[1 + \frac{1}{2}y^2\right]\Big|_{(0,0)} = -1$

$\oplus \left.\dfrac{\partial^3 f}{\partial y\, \partial s^2}\right|_{(0,0)} = \dfrac{\partial f}{\partial y}\left(\dfrac{\partial^2 f}{\partial s^2}\right)\Big|_{(0,0)} = 0$

$\oplus \left.\dfrac{\partial^3 f}{\partial y\, \partial s^2}\right|_{(0,0)} = \dfrac{\partial f}{\partial y}\left(\dfrac{\partial^2 f}{\partial s^2}\right)\Big|_{(0,0)} = 0$

$\oplus \left.\dfrac{\partial^3 f}{\partial s^3}\right|_{(0,0)} = 0$

$\oplus \left.\dfrac{\partial^4 f}{\partial y^4}\right|_{(0,0)} = y + 2 + 2y + 4 + 4y^2 + 6 + 5y^2 = 12$

$\oplus \left.\dfrac{\partial^4 f}{\partial y^3 \partial s}\right|_{(0,0)} = \dfrac{\partial f}{\partial y}\left(\dfrac{\partial^3 f}{\partial y^2 \partial s}\right) = -y = 0$

$\oplus$ $\left.\dfrac{\partial^4 f}{\partial y^2 \partial s^2}\right|_{(0,0)} = \dfrac{\partial f}{\partial y}\left(\dfrac{\partial^3 f}{\partial y \partial s^2}\right)\Big|_{(0,0)} = 0$

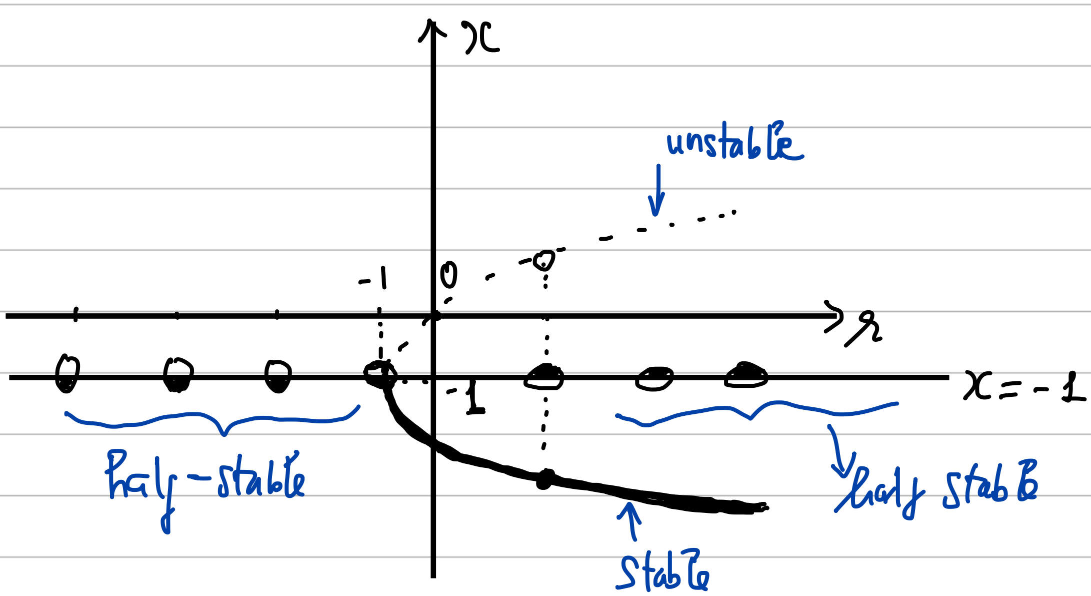
$\oplus$ $\left.\dfrac{\partial^4 f}{\partial y \partial s^3}\right|_{(0,0)} = \dfrac{\partial f}{\partial y}\left(\dfrac{\partial^3 f}{\partial s^3}\right)\Big|_{(0,0)} = 0$

$\oplus$ $\left.\dfrac{\partial^4 f}{\partial s^4}\right|_{(0,0)} = 0$

$\Rightarrow f(y,s) = 0 + \frac{1}{2}(-1)\ y^2 s + \frac{1}{4!}\cdot 12 y^4 + 0$

$\Rightarrow \boxed{f(y,s) = -\frac{1}{2}y^2 s + \frac{1}{2}y^4 + O(H.O.T)}$

e) At part b, we have the bifurcation diagram below.



As we can see that at bifurcation point $(x,r) = (-1,-1)$, there is existed a transition from stable to unstable

However, at $x = -1$, every point $(x, \lambda) = (-1, \lambda)$ are always half-stable, not stable or unstable. Hence, it must not be saddle, transcritical, pitchfork, or any type of bifurcation we have learned so far.

Furthermore, checking the Taylor series we did in part d,

we can see that with $f(y, s)$, even $\frac{\partial^n f}{\partial s^n}(0,0) = 0$ for all n,

however, $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 0$ $\left.\frac{\partial^2 f}{\partial y \partial s}\right|_{(0,0)} = 0$, means

both $\frac{\partial^2 f}{\partial y^2}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial s}(0,0)$ are not different 0

$\Rightarrow$ it can not be transcritical bifurcation

Also, $\frac{\partial f}{\partial s}(0,0)$ is also equal to 0

$\Rightarrow$ it can not be saddle-node bifurcation also.

Also, if we apply checking Taylor series for

$$f(y,s) \approx -\frac{1}{2}y^2 s + \frac{1}{2}y^4$$

We also have:

$$\frac{\partial f}{\partial s}(0,0) = -\frac{1}{2}y^2\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y \partial s}(0,0) = -y\Big|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y} = -ys + 2y^3 \Rightarrow \frac{\partial^2 f}{\partial y^2}(0,0) = -s + 6y^2\Big|_{(0,0)}$$

$$= 0$$

$\Rightarrow$ It could not be either saddle-node or transcritical bifurcation point at $(x,\lambda) = (0,0)$

$$\boxed{\text{Therefore, based on using both bifurcation diagram and Taylor series, this is a new type of bifurcation}}$$

2) Given $\dot{N} = rN\left(1 - \frac{N}{K}\right)$, initial condition N_0

$\Leftrightarrow N(0) = N_0$

a) We have:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = rN - \frac{rN^2}{K} = \frac{rNK - rN^2}{K}$$

$$\Rightarrow \frac{dN}{rNK - rN^2} = \frac{dt}{K} \Rightarrow \frac{1}{r} \cdot \frac{1}{N(K-N)} dN = \frac{dt}{K}$$

$$\Rightarrow \frac{1}{r} \int \frac{1}{N(K-N)} dN = \int \frac{dt}{K}$$

$$\Rightarrow \frac{1}{rK} \int \left(\frac{1}{N} + \frac{1}{K-N}\right) dN = \frac{1}{K} \int dt$$

$$\Rightarrow \frac{1}{r} \int \left(\frac{1}{N} - \frac{1}{N-K}\right) dN = \int dt$$

$$\Rightarrow \frac{1}{r}\left[\ln N - \ln(N-K)\right] = t + C$$

$$\Rightarrow \frac{1}{r} \ln \frac{N}{N-K} = t + C \qquad \text{Since } N(0) = N_0$$

$$\Rightarrow \frac{1}{r} \ln \frac{N_0}{N_0 - K} = C$$

$$\Rightarrow \frac{1}{r} \ln \frac{N}{N-K} = t + \frac{1}{r} \ln \frac{N_0}{N_0-K}$$

$$\Rightarrow \frac{1}{r}\left[\ln \frac{N}{N-K} - \ln \frac{N_0}{N_0-K}\right] = t$$

$$\Rightarrow \frac{1}{r} \ln\left[\frac{N(N_0-K)}{N_0(N-K)}\right] = t$$

$$\Rightarrow r = \frac{1}{t} \ln \frac{N(N_0-K)}{N_0(N-K)} = \frac{1}{t} \ln\left[\frac{\left(1-\frac{K}{N_0}\right)}{\left(1-\frac{K}{N}\right)}\right]$$

$\Rightarrow$ the dimension of r is $\frac{1}{t}$ or $\frac{1}{\text{time}}$

We also see that K & N_0 should has the same dimension as N, so $\frac{K}{N_0}$ or $\frac{K}{N}$ could serve as a dimensionless population level OR $\frac{N_0}{K}$ or $\frac{N}{K}$ are also as a dimensionless

b) As we have: $\dot{N} = rN\left(1-\frac{N}{K}\right)$ (1)

Let $x = \frac{N}{K}$ is dimensionless time $\Rightarrow \dot{x} = \frac{1}{K}.\dot{N}$ & $x(0) = \frac{N_0}{K}$ is also dimensionless time

$$\Rightarrow (1) \Leftrightarrow \frac{\dot{N}}{K} = \frac{rN}{K}\left(1 - \frac{N}{K}\right)$$

$$\Leftrightarrow \begin{cases} \dot{x} = rx(1-x) \\ x(0) = \frac{N_0}{K} = x_0 \end{cases}$$

let $\tau = rt$ is a dimension less time

$$\dot{x} = rx(1-x) \Leftrightarrow \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} = rx(1-x)$$

$$\Leftrightarrow \frac{dx}{d\tau} \cdot \frac{r\,dt}{dt} = rx(1-x)$$

$$\Leftrightarrow \begin{cases} \frac{dx}{d\tau} = x(1-x) \\ x(0) = x_0 = \frac{N_0}{K} \end{cases}$$

Therefore, the original system can be rewritten in dimension form:

$$\boxed{\begin{cases} \frac{dx}{d\tau} = x(1-x), \text{ with dimensionless variable:} \\ x(0) = x_0 \quad , x = \frac{N}{K}, \; x_0 = \frac{N_0}{K}, \; \& \; \tau = rt = \ln\left[\frac{\left(1 - \frac{K}{N_0}\right)}{\left(1 - \frac{K}{N}\right)}\right] \end{cases}}$$

c) $\dot{N} = rN\left(1-\frac{N}{K}\right)$ ① $N(0)=N_0$

let $u = \frac{N}{N_0} \Rightarrow u(0) = \frac{N(0)}{N_0} = \frac{N_0}{N_0} = 1$

$\Rightarrow \dot{u} = \frac{1}{N_0}\dot{N}$ & $uN_0 = N \Rightarrow \frac{N}{K} = \frac{uN_0}{K}$

① $\Leftrightarrow \frac{\dot{N}}{N_0} = \frac{rN}{N_0}\left(1-\frac{N}{K}\right)$

$$\Rightarrow \begin{cases} \dot{u} = ru\left(1-\frac{uN_0}{k}\right) \\ u(0)=1 \end{cases}$$

let $\tau = rt$ is dimesionless time

$= \ln \frac{N(N_0-K)}{N_0(N-K)}$ & $d\tau = r\,dt$

$$\Rightarrow \begin{cases} \frac{du}{d\tau}\cdot\frac{d\tau}{dt} = ru\left(1-\frac{N_0}{k}u\right) \\ u(0)=1 \end{cases}$$

$$\Rightarrow \frac{du}{d\tau}\, r\,\frac{dt}{dt} = ru\left(1-\frac{N_0}{k}u\right)$$

$\left\{ u(0) = 1 \right.$

$$\Rightarrow \begin{cases} \dfrac{du}{d\tau} = u\left(1 - \dfrac{N_0}{K} u\right) \\ u(0) = 1 \end{cases}$$

let $\alpha = \dfrac{N_0}{K}$ is a dimensionless variable

Therefore, the system can also be written in the dimension less:

$$\begin{cases} \dfrac{du}{d\tau} = u(1 - \alpha u), \text{ with } u = \dfrac{N}{N_0}, \ \tau = \lambda t = \ln . \dfrac{N(N_0 - K)}{N_0(N - K)} \\ u(0) = 1 \end{cases}$$

, and $\alpha = \dfrac{N_0}{K}$, $v(0) = 1$ are dimensionless variable.

3) Given $\dot{x} = \ln(1+x) - r x$

We need to find posible bifurcation point.

$$\begin{cases} \dot{x} = 0 \\ \frac{\partial f}{\partial x} = 0 \end{cases} \iff \begin{cases} \ln(1+x) = r x & (1) \\ \frac{1}{1+x} - r = 0 & (2) \end{cases}$$

at $(x^*, r^*) = (0,1)$, we have:

(1) $\iff \ln(1+x^*) - r^* x^* = \ln(1+0) - 1\times 0$

$= 0$ (corrected)

(2) $\iff \frac{1}{1+x^*} - r^* = \frac{1}{1+0} - 1 = 0$ (corrected)

$\Rightarrow (x^*, r^*) = (0,1)$ is a posible bifurcation.

We have, $\frac{\partial f}{\partial x} = \frac{1}{1+x} - r$

$\Rightarrow \frac{\partial^2 f}{\partial x \partial r}(0,1) = -1 \neq 0$ (I)

$\Rightarrow \frac{\partial^2 f}{\partial x^2}(0,1) = -\frac{1}{(1+x^*)^2}\Big|_{(0,1)} = -1 \neq 0$ (II)

Also, $f(x,r) = \ln(1+x) - rx$

$$\Rightarrow \frac{\partial f}{\partial r}(0,1) = -x\Big|_{(0,1)} = 0$$

$$\Rightarrow \frac{\partial^n f}{\partial r^n}(0,1) = 0$$

$$\Rightarrow \frac{\partial^n f}{\partial r^n}(0,1) = 0 \text{ for all } n \quad \text{III}$$

$$\boxed{\text{Combining I, II, \& III, we can conclude that } (x^*, r^*) = (0,1) \text{ is a transcritical bifurcation by using the Trancritical Bifurcation Theorem.}}$$

4) Given $\varepsilon\ddot{x}+\dot{x}+x=0$, $x(0)=1$, $\dot{x}(0)=0$

a) For all $\varepsilon>0$, let $x=e^{\lambda t}$, then we have the equation:

$$\varepsilon\lambda^2+\lambda+1=0 \quad , \Delta = 1-4\varepsilon$$

⊛ If $\Delta>0 \Leftrightarrow 1-4\varepsilon>0 \Leftrightarrow 4\varepsilon<1 \Leftrightarrow \varepsilon<\frac{1}{4}$

or $0<\varepsilon<\frac{1}{4}$, $\Rightarrow \lambda_{1,2}=\dfrac{-1\pm\sqrt{1-4\varepsilon}}{2\varepsilon}$

$\Rightarrow$ 2 distinct real root $\lambda_1=\dfrac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}$ & $\lambda_2=\dfrac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}$

$$\Rightarrow x(t)=C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}$$

$$\Rightarrow \dot{x}(t)=C_1\lambda_1e^{\lambda_1 t}+C_2\lambda_2e^{\lambda_2 t}$$

$$\begin{cases} x(0)=1 \\ \dot{x}(0)=0 \end{cases} \Leftrightarrow \begin{cases} C_1+C_2=1 \\ C_1\lambda_1+C_2\lambda_2=0 \end{cases} \Leftrightarrow \begin{cases} C_1=1-C_2 \\ (1-C_2)\lambda_1+C_2\lambda_2=0 \end{cases}$$

We have: $(1-C_2)\lambda_1+C_2\lambda_2=0$

$$\Rightarrow \lambda_1-C_2\lambda_1+C_2\lambda_2=0 \Rightarrow \lambda_1=C_2(\lambda_1-\lambda_2)$$

$$\Rightarrow C_2=\frac{\lambda_1}{\lambda_1-\lambda_2} \Rightarrow C_1=1-C_2=1-\frac{\lambda_1}{\lambda_1-\lambda_2}=\frac{-\lambda_2}{\lambda_1-\lambda_2}$$

$$\Rightarrow \boxed{x(t)=\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1 t}+\frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2 t}} \quad \text{with } \lambda_{1,2}=\frac{-1\pm\sqrt{1-4\varepsilon}}{2\varepsilon}$$

* If $\Delta = 0 \iff 1 - 4\varepsilon = 0 \iff \varepsilon = \frac{1}{4}$

$\Rightarrow$ 1 root $\lambda = -\frac{1}{2\varepsilon} = -2 \Rightarrow$ the only one solution

$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$$

$$\Rightarrow \dot{x}(t) = C_1 \lambda e^{\lambda t} + C_2 \left[e^{\lambda t} + t \lambda e^{\lambda t} \right]$$

$$\Rightarrow \begin{cases} x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} C_1 + 0 = 1 \Rightarrow C_1 = 1 \\ C_1 \lambda + C_2 [1 + 0] = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = -\lambda \end{cases}$$

$$\Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 2 \end{cases} \Rightarrow x(t) = e^{\lambda t} + \frac{1}{2} t e^{\lambda t}$$

or $\boxed{x(t) = e^{-2t} + 2t e^{-2t}}$

* If $\Delta < 0 \Rightarrow 1 - 4\varepsilon < 0 \iff 1 < 4\varepsilon \iff \varepsilon > \frac{1}{4}$

$$\Rightarrow \lambda_{1,2} = \frac{-1 \pm j\sqrt{4\varepsilon - 1}}{2\varepsilon} = \alpha \pm j\beta$$

$$\Rightarrow \alpha = \frac{-1}{2\varepsilon}, \quad \beta = \frac{\sqrt{4\varepsilon - 1}}{2\varepsilon}$$

$\Rightarrow$ solution : $x(t) = e \left[A_1 \cos \beta t + A_2 \sin \beta t \right]$

or $x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

With λ_1 & λ_2 are complex

$$\begin{cases} x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \Rightarrow \begin{cases} A_1 + A_2 = 1 \\ A_1 \lambda_1 e^{\lambda_1 t} + A_2 \lambda_2 e^{\lambda_2 t} \Big|_0 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} A_1 + A_2 = 1 \\ A_1 \lambda_1 + A_2 \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} A_1 = \dfrac{\lambda_2}{\lambda_2 - \lambda_1} \\ A_2 = \dfrac{\lambda_1}{\lambda_1 - \lambda_2} \end{cases}$$

With $\lambda_1 = \alpha + j\beta$ & $\lambda_2 = \alpha - j\beta$

$$\Rightarrow \lambda_2 - \lambda_1 = \alpha - j\beta - \lambda - j\beta = -2j\beta$$

$$\Rightarrow \lambda_1 - \lambda_2 = 2j\beta$$

$$\Rightarrow A_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{\alpha - j\beta}{-2j\beta} = \frac{j\beta - \alpha}{2j\beta}$$

$$A_2 = \frac{\alpha + j\beta}{2j\beta}$$

$\Rightarrow$ <u>Solution:</u> $x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

$$\Rightarrow \boxed{x(t) = \frac{j\beta - \alpha}{2j\beta} e^{\lambda_1 t} + \frac{\alpha + j\beta}{2j\beta} e^{\lambda_2 t} \quad \text{with } \lambda_1 = \frac{-1}{2\varepsilon} + \frac{j\sqrt{4\varepsilon - 1}}{2\varepsilon} \ \& \ \lambda_2 = \frac{-1}{2\varepsilon} - \frac{j\sqrt{4\varepsilon-1}}{2\varepsilon}}$$

b) $0 < \varepsilon << 1 \Rightarrow 1 - 4\varepsilon \geq 0 \Rightarrow$ there are 2 cases

* Firstly, $\varepsilon = \frac{1}{4} \Rightarrow 1 - 4\varepsilon = 0$, based on part a, we have the only 1 solution (repeated root)

$$\boxed{x(t) = e^{-2t} + 2te^{-2t}}$$

* Secondly, $1 - 4\varepsilon > 0 \Rightarrow$ there are 2 distinct root for λ_1 & λ_2, based on part a, we have:

$$x(t) = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 t} \quad \text{with } \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$$

Also, $\lambda_2 - \lambda_1 = \frac{-1 - \sqrt{1-4\varepsilon}}{2\varepsilon} - \frac{-1 + \sqrt{1-4\varepsilon}}{2\varepsilon}$

$$= \frac{-1 - \sqrt{1-4\varepsilon} + 1 - \sqrt{1-4\varepsilon}}{2\varepsilon} = -\frac{\sqrt{1-4\varepsilon}}{\varepsilon}$$

$$\Rightarrow \frac{\lambda_2}{\lambda_2 - \lambda_1} = \frac{-1 - \sqrt{1-4\varepsilon}}{2\varepsilon} \cdot \frac{(-\varepsilon)}{\sqrt{1-4\varepsilon}} = \frac{1 + \sqrt{1-4\varepsilon}}{2\sqrt{1-4\varepsilon}}$$

And $\frac{\lambda_1}{\lambda_1 - \lambda_2} = \frac{-1 + \sqrt{1-4\varepsilon}}{2\varepsilon} \cdot \frac{\varepsilon}{\sqrt{1-4\varepsilon}} = \frac{-1 + \sqrt{1-4\varepsilon}}{2\sqrt{1-4\varepsilon}}$

$$\Rightarrow x(t) = \frac{1+\sqrt{1-4\varepsilon}}{2\sqrt{1-4\varepsilon}} e^{\lambda_1 t} + \frac{\sqrt{1-4\varepsilon}-1}{2\sqrt{1-4\varepsilon}} e^{\lambda_2 t}$$

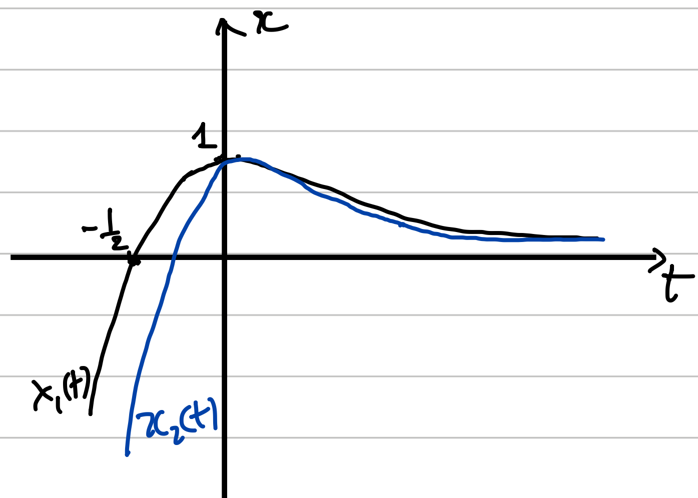
with $\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

$\Rightarrow$ Therefore, when $\varepsilon \ll 1$, there are two widely separate time scale in the problem.

$$\begin{cases} x_1(t) = e^{-2t} + 2t\,e^{-2t} \\ x_2(t) = \dfrac{1+\sqrt{1-4\varepsilon}}{2\sqrt{1-4\varepsilon}} e^{\lambda_1 t} + \dfrac{\sqrt{1-4\varepsilon}-1}{2\sqrt{1-4\varepsilon}} e^{\lambda_2 t} \end{cases}$$

(with $\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$)

c)



d) With $\dot{x} + x = 0 \Leftrightarrow \dfrac{dx}{dt} = -x$

$$\Rightarrow -\frac{dx}{x} = dt \Leftrightarrow -\int \frac{dx}{x} = \int dt$$

$$\Rightarrow \quad -\ln x = t + C$$

$$\Rightarrow \quad x = e^{-(t+c)} = C_1 e^{-t}$$

$$\Rightarrow \quad x(t) = C_1 e^{-t} \quad \Rightarrow \quad \dot{x}(t) = -C_1 e^{-t}$$

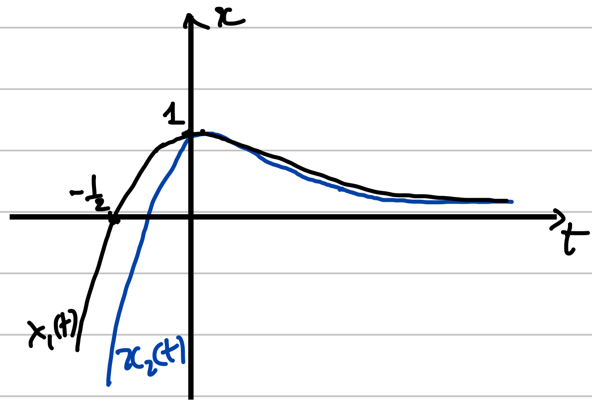
Since $x(0) = 1 \Rightarrow C_1 = 1$

$$\Rightarrow \quad x(t) = e^{-t}$$

So we compare:

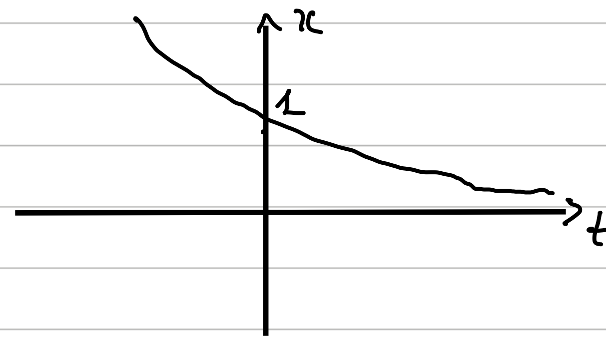
$\varepsilon \ddot{x} + \dot{x} + x = 0$ (I)

$\varepsilon << 1$



$\dot{x} + x = 0$ (II)

$x(t) = e^{-t}$



Based on 2 diagrams, we can see when $t \to \infty$, (I) become to be similar to II and gradually be zero

Therefore, we can replace $\varepsilon \ddot{x} + \dot{x} + x = 0$ by singular limit $\dot{x} + x = 0$ when $t \to \infty$