Math 134 - Lecture 2
Fall 2020
Midterm 1
Due 10/28/2020 before 10am

This exam contains 11 pages (including this cover page) and 6 problems. There are a total of 50 points available.

- Attempt all questions.
- Solutions must be uploaded to Gradescope before 10am Pacific Time on October 28^{th}.
- You may complete the problems on a printout of this exam, blank paper, or a tablet/iPad.
- You may use additional blank pages as required.
- If you handwrite your solutions, please make sure your scan is clearly legible.
- The work submitted must be entirely your own: you may not collaborate or work with anyone else to complete the exam.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- Please note: **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

Please sign the following honor statement. **If you do not sign this, you will receive 0 points**.

> *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

Signed:

Print name:

1. (10 points)

   (a) Find a function $f(x)$ so that the system $\dot{x} = f(x)$ has **exactly** the following fixed points:

   - A stable fixed point at -1,
   - An unstable fixed point at 0,
   - A stable fixed point at 1,
   - A half-stable fixed point at 3.

   (b) Draw a phase portrait for your system.

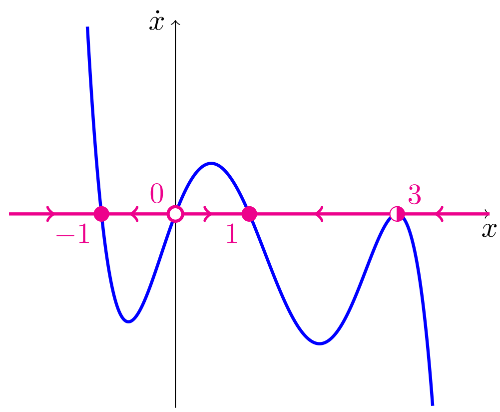
   (c) Find all values of $x(0)$ so that $x(t) \to 1$ as $t \to \infty$.

Solution:

(a) One possible solution is

$$f(x) = -(x+1)x(x-1)(x-3)^2.$$

(b) We sketch



(c) By inspection of the phase portrait, $x(t) \to 1$ as $t \to \infty$ if and only if $x(0) \in (0, 3)$.

2. (6 points) Consider the equation

$$\begin{cases} \dot{x} = \tan(x) \\ x(4) = \frac{\pi}{4}. \end{cases}$$

(a) Write down the integral equation equivalent to this problem.

(b) Use Euler's method with $\Delta t = \frac{1}{10}$ to approximate $x(4.1)$. *You should give the exact answer: No credit will be given for a decimal approximation.*

Solution:

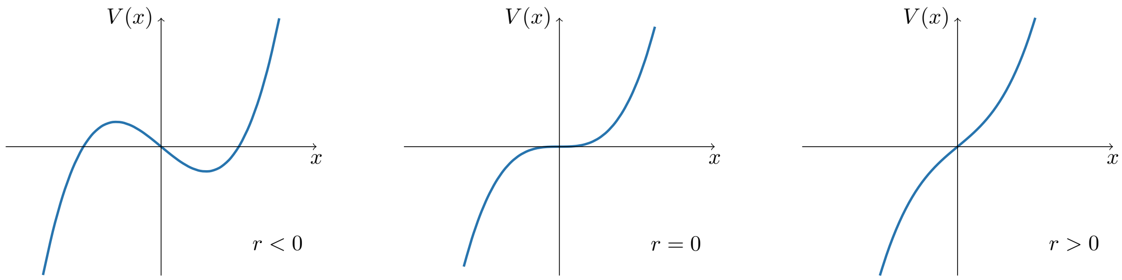
(a) The corresponding integral equation is

$$x(t) = \frac{\pi}{4} + \int_4^t \tan\big(x(s)\big)\, ds.$$

(b) Applying a single step of Euler's method starting at $t = 4$ and with $\Delta t = \frac{1}{10}$ we have

$$x(4.1) \approx x(4) + \tan\big(x(4)\big)\frac{1}{10} = \frac{\pi}{4} + \frac{1}{10}.$$

3. (6 points) A system $\dot{x} = f(x, r)$ has a bifurcation at $x = 0$, $r = 0$. A **potential function** for f has the following graphs:

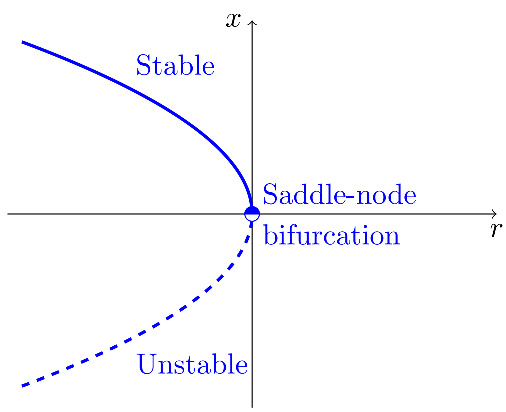


Sketch the bifurcation diagram and identify the type of bifurcation.

Solution: By inspection of the graph of the potential function, we see that:

- For $r < 0$ we have two fixed points, an unstable fixed point at a value $x < 0$ and a stable fixed point at a value $x > 0$;
- For $r = 0$ we have a single half stable fixed point at $x = 0$;
- For $r > 0$ we have no fixed points.

Consequently, our system has a saddle-node bifurcation at $(x, r) = (0, 0)$ and the following bifurcation diagram:

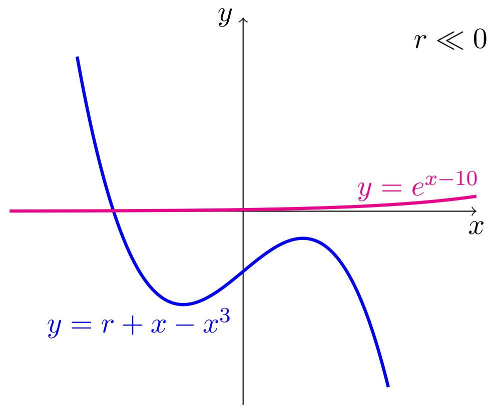


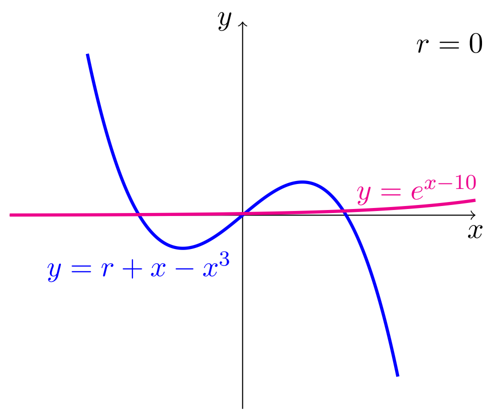
4. (10 points) Sketch a global bifurcation diagram for the system

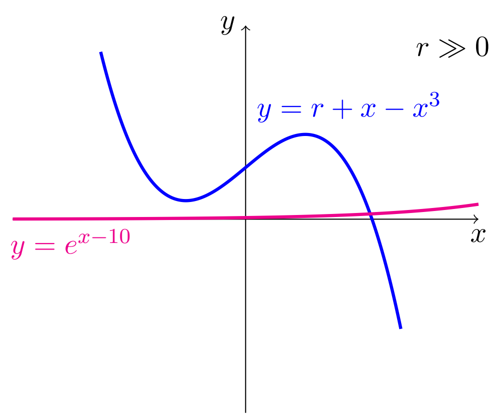
$$\dot{x} = r + x - x^3 - e^{x-10}.$$

Be sure to indicate the stability of each branch and identify each type of bifurcation. *You are not expected to be able to find the exact location of the bifurcation points.*

Solution: Let us consider the graphs of $r + x - x^3$ and e^{x-10} for different values of r:

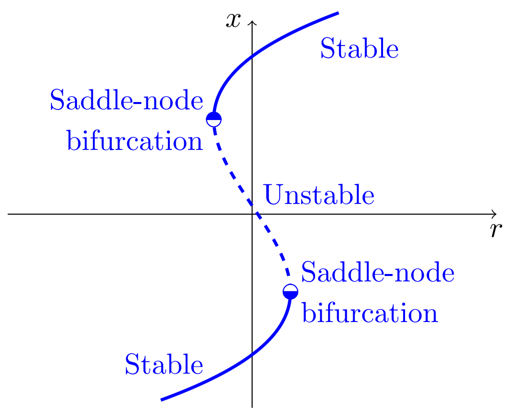






By inspection, for $r \ll 0$ we have a single single stable fixed point for some $x < 0$. As we increase r, we translate the graph of $y = r + x - x^3$ upwards. At a value $r = r_1 < 0$ a saddle-node bifurcation will occur. As we increase r further, we will have two stable fixed points, one at $x < 0$ and one at $x > 0$, and an unstable fixed point between them, at a value of x that decreases with r. At a value $r = r_2 > 0$, the unstable fixed point will reach the stable fixed point where $x < 0$, and another saddle-node bifurcation will occur. For $r > r_2$ we will simply have a single stable fixed point at a value $x > 0$.

The corresponding bifurcation diagram is then:



5. (10 points) A marketing firm models the average rating x, between 0 and 100, of a new TV show using the equation

$$\dot{x} = x(R - x)(V - x)(R + 4V - 5x),$$

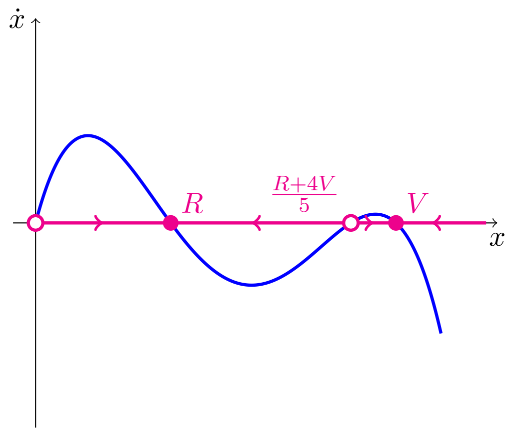
where $0 < R < V \leq 100$ are constants. The constant R represents the 'real' rating of the show, whereas V represents the rating if it becomes 'viral.'

(a) Sketch the phase portrait, indicating the location and stability of any fixed points.

(b) For $R = 80$ and $V = 90$, sketch trajectories with initial conditions $x(0) = 70$, $x(0) = 85$, and $x(0) = 89$.

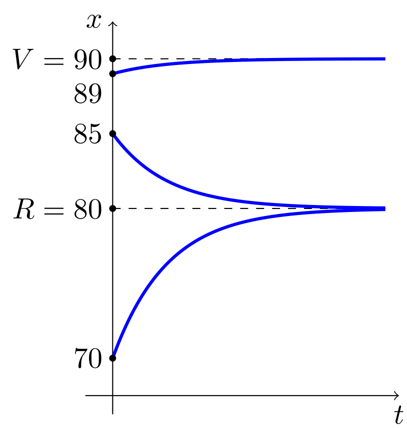
Solution:

(a) We sketch:



so we have unstable fixed points at $x = 0, \frac{R+4V}{5}$ and stable fixed points at $x = R, V$.

(b) We note that in this case the unstable fixed point $\frac{R+4V}{5} = 88$ and hence the tracjectories starting at $x(0) = 70, 85$ will converge to $R = 80$, whereas the trajectory starting at $x(0) = 89$ will converge to $V = 90$.



6. (8 points)

   (a) Let $a > 0$. Show that the solution of

$$\begin{cases} \dot{x} = ax^2 \\ x(0) = 1 \end{cases}$$

   blows up in finite time and find the time at which it blows up.

   (b) Let $0 < \varepsilon < 1$. Show that the solution of

$$\begin{cases} \dot{x} = x^2\big(1 + \varepsilon \sin(x)\big) \\ x(0) = 1 \end{cases}$$

   blows up at a time T satisfying $\frac{1}{1+\varepsilon} \leq T \leq \frac{1}{1-\varepsilon}$.

Solution:

(a) This equation is separable:

$$\int \frac{dx}{x^2} = \int a\,dt$$
$$-\frac{1}{x} = at + C,$$

where the constant C can be determined to be -1 from the initial condition $(t, x) = (0, 1)$. As a consequence, the solution is

$$x(t) = \frac{1}{1 - at} \quad \text{for} \quad t < \frac{1}{a},$$

which blows up at time $T = \frac{1}{a}$.

(b) Let us consider the ODEs

$$\begin{cases} \dot{y} = (1-\varepsilon)y^2 \\ y(0) = 1 \end{cases} \qquad \text{and} \qquad \begin{cases} \dot{z} = (1+\varepsilon)z^2 \\ z(0) = 1 \end{cases}$$

From part (a), we have

$$y(t) = \frac{1}{1 - (1-\varepsilon)t} \quad \text{for} \quad t < \frac{1}{1-\varepsilon},$$
$$z(t) = \frac{1}{1 - (1+\varepsilon)t} \quad \text{for} \quad t < \frac{1}{1+\varepsilon},$$

which blow up at times $\frac{1}{1-\varepsilon}$ and $\frac{1}{1+\varepsilon}$, respectively.

Using that $\sin(x) \geq -1$, we may apply our comparison theorem to show that

$$y(t) \leq x(t)$$

for all times $t \geq 0$ for which both $y(t), x(t)$ exist. As $y(t)$ is positive for $0 \leq t < \frac{1}{1-\varepsilon}$ and blows up at time $\frac{1}{1-\varepsilon}$, there must exist some time $T > 0$ so that $x(t)$ is positive for $0 \leq t < T$ and blows up at T.

However, using that $\sin(x) \leq 1$, we may again apply our comparison theorem to show that

$$x(t) \leq z(t)$$

for all times $t \geq 0$ for which both $x(t), z(t)$ exist. As $z(t)$ is finite for all times $0 \leq t < \frac{1}{1+\varepsilon}$, the blow-up time T for $x(t)$ must satisfy $T \geq \frac{1}{1+\varepsilon}$.

Combining these, $x(t)$ blows up at a time T satisfying $\frac{1}{1+\varepsilon} \leq T \leq \frac{1}{1-\varepsilon}$.