Math 134, Spring 2022

Lecture #7: Bifurcations.

Monday April 11$^{\text{th}}$

Learning objectives

Today we will discuss:

- What it means to say a value of a parameter is a bifurcation point.
- Saddle-node bifurcations.
- How to draw a bifurcation diagram.
- Transcritical bifurcations.

Bifurcations

External parameters

- Consider the ODE
$$\dot{x} = f(x, r)$$
where r is a parameter of the model.

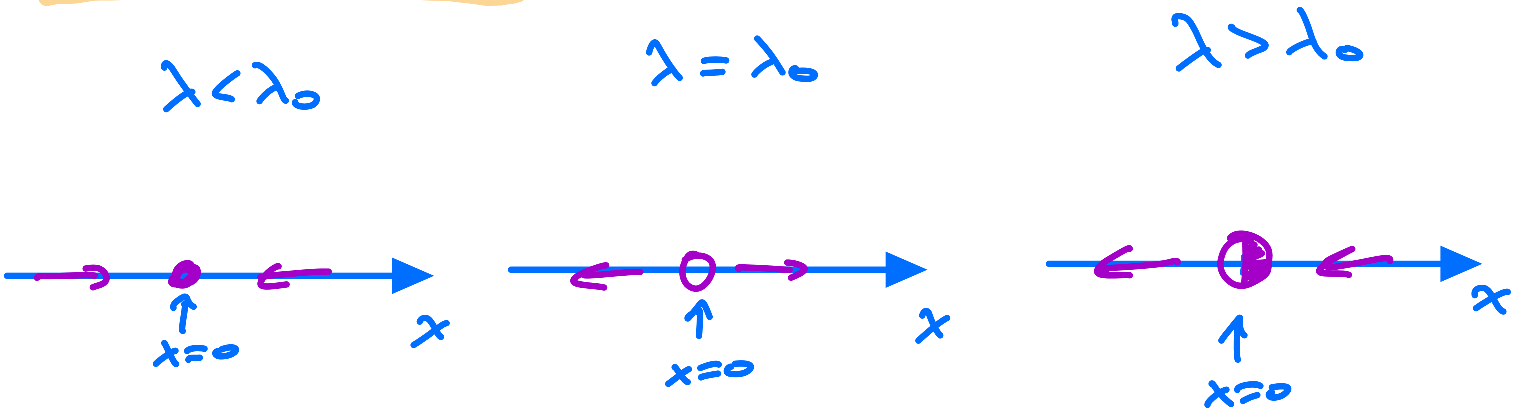
- **Question:** How do the dynamics vary as we vary r?

Definition!

Consider the following autonomous system

$$\dot{x} = f(x, \lambda)$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. A **bifurcation** occurs at parameter $\lambda = \lambda_0$ if there are parameter values λ_1 arbitrarily close to λ_0 with dynamics topologically inequivalent from those at λ_0.

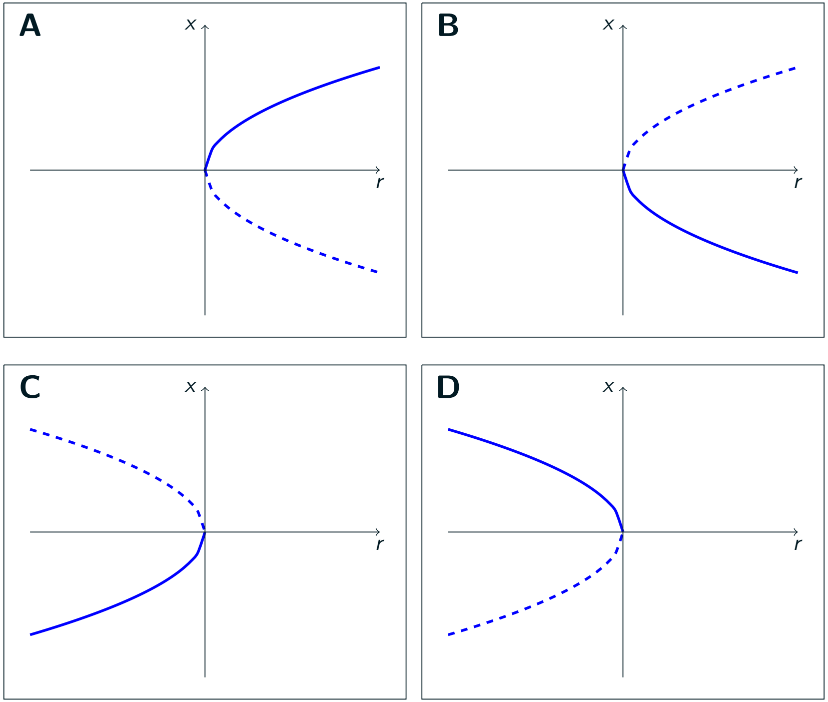


An example

Consider the equation

$$\dot{x} = r - x^2$$

Which of the following is the correct bifurcation diagram?

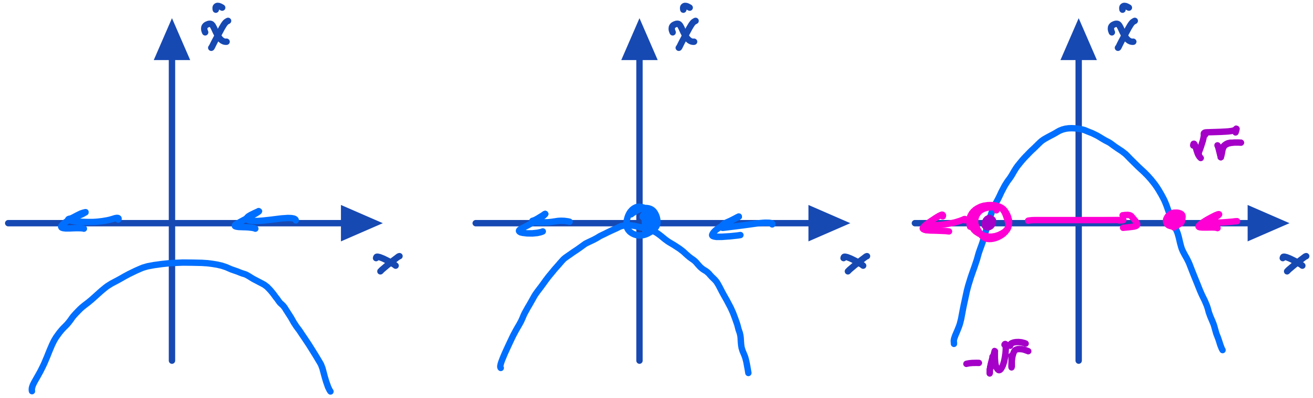


$\dot{x} = r - x^2$

$r = 0$

$r - x^2 = 0$

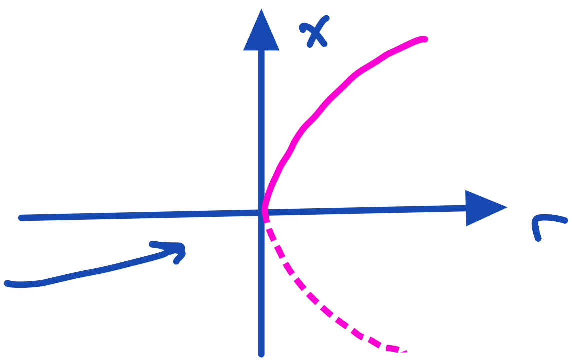
$x = \pm\sqrt{r}$



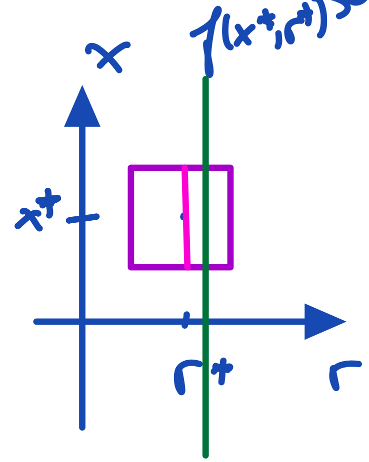
$\dot{x} = -x^2$

Bifurcation diagram

saddle-node bifurcation at $(x, r) = (0, 0)$



Identifying bifurcations



If the system

$$\dot{x} = f(x, r)$$

has a bifurcation at $(x, r) = (x^*, r^*)$ then

$$f(x^*, r^*) = 0 \quad \text{and} \quad \frac{\partial f}{\partial x}(x^*, r^*) = 0$$

Warning! The converse is not necessarily true. You will find a counter example in the next homework.

Proof: (sketch) $\eta = x - x^*$

$$\dot{\eta} = \frac{\partial f}{\partial x}(x^*, r^*)\,\eta \implies \begin{cases} \text{stable if } \dfrac{\partial f}{\partial x}(x^*, r^*) < 0 \\ \text{unstable if } \dfrac{\partial f}{\partial x}(x^*, r^*) > 0 \end{cases}$$

so there is a change in stability only if

$$\frac{\partial f}{\partial x}(x^*, r^*) = 0$$

An example

$$\dot{x} = r + x - \ln(1 + x)$$

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An example

Consider the equation

$$\dot{x} = r + \frac{1}{4}x - \frac{x}{1+x}$$

At what value of r do we have a saddle-node bifurcation?

A) $r = \frac{9}{4}$

B) $r = 1$

C) $r = \frac{1}{4}$

D) $r = 0$

Answer $r = \frac{9}{4}$ and $r = \frac{1}{4}$.

Homework: Find the bifurcation diagram.

See you next time!

Image credits:

xkcd: `https://xkcd.com/435/`