**MATH134 - Homework 3 Suggested Solutions.**

Suggested Solutions to Homework # 3.

**Homework Problem 1.** Consider the ODE:

$$\dot{x} = (\cosh(1+x) - 1)(x^2 + 2x - r).$$

(a) Find a value $r_c$ and corresponding $x_c$ at which a bifurcation occurs.

(b) Sketch the bifurcation diagram.

(c) Taking $y = x - x_c$ and $s = r - r_c$, find a function $f(y, s)$ so that

$$\dot{y} = f(y, s).$$

(d) Compute the Taylor series of $f$ at $(y, s) = (0, 0)$ to the fourth order.

(e) Use both your bifurcation diagram and Taylor series to explain why this is a new type of bifurcation.

**Solution.**
(a) As per usual, we attempt to solve the following system:

$$\begin{cases} f = 0 & \to (\cosh(1+x) - 1)(x^2 + 2x - r) = 0 \\ \frac{\partial f}{\partial x} = 0 & \to (\sinh(1+x))(x^2 + 2x - r) + 2(\cosh(1+x) - 1)(x+1) = 0 \end{cases}$$

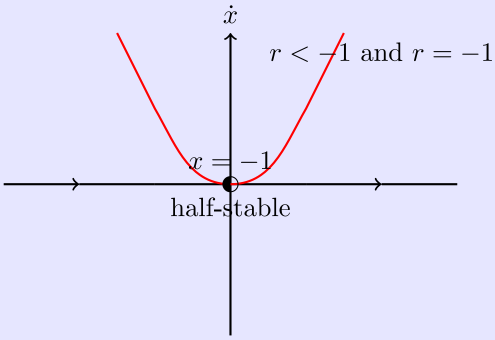
From the first equation, we deduce that either $\cosh(1+x) - 1 = 0$ or $x^2 + 2x - r = 0$. If $\cosh(1+x) - 1 = 0$, then $x = -1$ necessarily (recall that $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and thus attains its minimum at $x = 0$ with value 1). With such a value of $x$, it automatically satisfy the second equation, and we deduce that any possible value of $r \in \mathbb{R}$ is a possible bifurcation. Alternatively, if $x^2 + 2x - r = 0$, then $x = -1 \pm \sqrt{1+r}$ for $r \geq -1$ and has no solutions if $r < -1$. We substitute this values of $x$ into the second equation (and noting that $x^2 + 2x - r = 0$ is what we have set while solving for $x$) and solve for $r$ in the following way. Using such a condition, it remains to substitute $x$ in the following equation to solve for $r$:

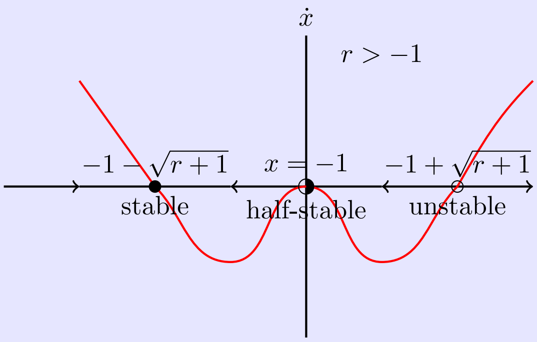
$$2(\cosh(1+x) - 1)(x - 1) = 0.$$

Note that both $\cosh(1+x) - 1$ and $x + 1$ are increasing functions, takes the value of 0 at $x = -1$, and both non-zero for $x \neq -1$. This implies that equality of the product of these two at zero can only be obtained at $x = -1$. In other words, we just have to solve $x = -1 = -1 \pm \sqrt{1+r}$, which gives $r = -1$ necessarily. As a summary:

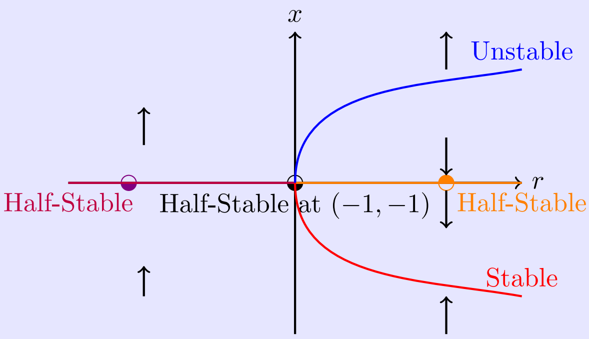
- Setting the first factor to zero gives $x = -1$ and $r \in \mathbb{R}$.
- Setting the second factor to zero gives $x = -1$ and $r = -1$.

One way to think about this is to see that for $x = -1$ and $r \neq -1$, such a possible bifurcation point is only caused by the first factor, while $x = -1$ and $r = -1$ is possibly caused by both factors, and it is more likely that $x = 1$ and $r = -1$ is a bifurcation point than say any other values of $r$. Indeed, one can check that $x = -1$ and $r \neq -1$, though being a **possible bifurcation point, it actually not**. However, $x = -1$ and $r = -1$ is an actual bifurcation point. Thus we have $x_c = -1$ and $r_c = -1$.
(b) The phase portraits for $r < -1$, $r = -1$ and $r > -1$ is given by:




with the following bifurcation diagram:

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(c) Let $y = x + 1$ and $s = r + 1$. Then, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(y-1) = (\cosh(1+x) - 1)((x+1)^2 - (r+1))$$
$$\dot{y} = (\cosh(y) - 1)(y^2 - s).$$

(d) Since $\cosh(y) = 1 + \frac{1}{2}y^2 + \frac{1}{24}y^4 + O(y^6)$, we have

$$\begin{aligned} f(y,s) &= (\cosh(y) - 1)(y^2 - s) \\ &= (\frac{1}{2}y^2 + \frac{1}{24}y^4 + O(y^6))(y^2 - s) \\ &= \frac{1}{2}y^4 - \frac{sy^2}{2} + h.o.t. \end{aligned}$$

(e) The bifurcation diagram shows a pitchfork-like bifurcation. However, the side tines are of different stability and the stability of the middle branch is always at half-stable (with a tilted orientation upon crossing $x = -1$). This is different from that of a pitchfork bifurcation, and from any of the bifurcation diagrams that we have observed in the lecture notes. Alternatively, from the Taylor's series, we see that the form does not match any of the normal forms of the bifurcation diagrams (saddle-node with $as + by^2$, transcritical with $asy + by^2$, and super/subcritical pitchfork with $asy + by^3$) that we have covered in class.

**Homework Problem 2.** (Strogatz Exercise 3.5.7). Consider the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right),$$

with initial condition $N_0$.

(a) This system has three dimensional parameters $r, K, N_0$. Find the dimensions of each of these parameters.

(b) Show that the system can be re-written in the dimensionless form:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x) \\ x(0) = x_0 \end{cases}$$

for appropriate choices of the dimensionless variable $x, x_0$ and $\tau$.

(c) Find a different nondimensionalization in terms of variables $u$ and $\tau$, where $u$ is chosen such that the initial condition is always $u(0) = 1$.

**Solution.** (a) We suppose that the population has a unit of 'ppl'. Then, we have $r \sim \frac{1}{\text{'time'}}$, $K \sim$ 'ppl' and $N_0 \sim$ 'ppl'.
(b) Set $x = \frac{N}{M}$, and $\tau = \frac{t}{T}$ for some $M$ and $T$ to be determined and that each of the $x$ and $\tau$ are dimensionless. Recall that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{T}\frac{\mathrm{d}}{\mathrm{d}\tau}.$$

Substitute these into the equation, we have:

$$\begin{cases} \frac{M}{T}\frac{\mathrm{d}x}{\mathrm{d}\tau} = rMx(1 - \frac{M}{K}x) \\ x(0) = \frac{N_0}{M}. \end{cases}$$

which simplifies to:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = rTx(1 - \frac{M}{K}x) \\ x(0) = \frac{N_0}{M}. \end{cases} .$$

In (b), we set $rT = 1$ and $\frac{M}{K} = 1$, ie

$$T = \frac{1}{r} \qquad \text{and} \qquad K = M$$

and thus

$$\tau = rt \qquad \text{and} \qquad x = \frac{N}{K}.$$

The system thus reduces to

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-x) \\ x(0) = x_0, \end{cases}$$

with

$$x_0 = \frac{N_0}{M} = \frac{N_0}{K}.$$

In (c), we set $\frac{N_0}{M} = 1$ instead to obtain $M = N_0$. We can keep $T = \frac{1}{r}$. This implies that we have

$$\tau = rt \qquad \text{and} \qquad x = \frac{N}{N_0}$$

and the system reduces to

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = x(1-\kappa x) \\ x(0) = 1. \end{cases}$$

with

$$\kappa = \frac{N_0}{K}.$$

To reply to the question, we relabel $x \to u$.

**Homework Problem 3.** (Based on Strogatz Exercise 3.5.6). Consider a bead of mass $m > 0$ constrained to slide on a striaght, vertical wire. A spring of relaxed length $L_0 > 0$ and spring constant $k > 0$ is attached form the top of the mass to a point $P$ on the wire (see the diagram). Let $x$ be the displacement of the mass from the point $P$ and $g$ be the gravitational constant. Finally, suppose that the motion of the bead is opposed by a viscous damping force $b\dot{x}$, where $b > 0$. (See diagram in the original homework set.)

(a) Let $x$ be the displacement of the mass from the top of the wire. Use Newton's law to write down an ODE for $x$.

(b) Show that it is possible to find dimensionaless variables $\tau, u,$ and $\varepsilon$ so that the ODE from part (a) takes the form:

$$\varepsilon \frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + \frac{\mathrm{d}u}{\mathrm{d}\tau} + u = 0$$

(c) Use your knowledge in 33B to solve the equation above with initial conditions $u(0) = 1$ and $\frac{\mathrm{d}u}{\mathrm{d}\tau}(0) = 0$.

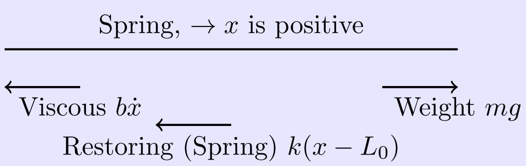
(d) Now suppose $\varepsilon \ll 1$. Show that there are two widely separated time scales in the problem, and estimate them in terms of $\varepsilon$.

(e) Graph the solution of $u(\tau)$ for $\varepsilon \ll 1$, and indicate the two time scales on the graph.

(f) What can you conclude about the validity of replacing the equation with its singular limit? The singular limit is given by:

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} + u = 0.$$

**Solution.**
(a) Using the given diagram, we set downwards to be positive. Rotating the diagram 90 degrees counter-clockwise, we have




By Newton's Second Law (by matching the direction of the arrows), we have

$$m\ddot{x} = -b\dot{x} - k(x - L_0) + mg.$$

(b) Note that $kL_0$ and $mg$ are constants. Thus, by rearranging the equation, we obtain:

$$m\ddot{x} = -b\dot{x} - k\left(x - L_0 - \frac{mg}{k}\right).$$

Thus, using the technique of nondimensionalizing, we assume

$$u = \frac{x - L_0 - \frac{mg}{k}}{L} \quad \text{and} \quad \tau = \frac{t}{T},$$

for $L$ and $T$ to be chosen. Reusing the relation for the differential in $\tau$ as in Question 2, we also have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{1}{T^2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}.$$

The equation thus simplifies to:

$$\frac{m}{T^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + \frac{b}{T}\frac{\mathrm{d}u}{\mathrm{d}\tau} + ku = 0,$$

or by dividing by $k$ (valid since $k > 0$ as given in the question)

$$\frac{m}{kT^2}\frac{\mathrm{d}^2u}{\mathrm{d}\tau^2} + \frac{b}{kT}\frac{\mathrm{d}u}{\mathrm{d}\tau} + u = 0.$$

Note that we now face a case in which we have a higher degree of freedom than what we can simplify, as according to the question, we would just want to set the coefficient of the term in the second derivative to be $\varepsilon$ rather than 1. This implies that no matter what we do, at the end of the day, the required $\varepsilon$ is obtained by merely setting the coefficient of $\frac{\mathrm{d}^2u}{\mathrm{d}\tau^2}$. This implies that we only have to set the coefficient of $\frac{\mathrm{d}u}{\mathrm{d}\tau}$ to 1. This constitutes to having 1 equation and 2 free choices (of $L$ and $T$ from out side). Thus, we can just pick $L = L_0$ (since $L_0$ has the unit of length), and demand that

$$\frac{b}{kT} = 1 \to T = \frac{b}{k}.$$

With these choices of $T$ and $L_0$, we then set

$$\frac{m}{kT^2} = \varepsilon = \frac{m}{k\left(\frac{b}{k}\right)^2} = \frac{mk}{b^2}.$$

In summary, as required, our dimensionless variables are given by

$$\tau = \frac{kt}{b}, \qquad u = \frac{x - L_0 - \frac{mg}{k}}{L_0}, \qquad \text{and} \qquad \varepsilon = \frac{mk}{b^2}.$$

(c) This is a second-order ODE with constant coefficients. For an in-depth discussion, see notes from 33B and/or Discussion Supplement 1. The strategy is to set $u = e^{\lambda\tau}$ for $\lambda$ to be determined and obtain the characteristic equation:

$$\varepsilon\lambda^2 + \lambda + 1 = 0.$$

The roots (in $\mathbb{C}$) are given by

$$\lambda = -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon}.$$

Furthermore, note that $\varepsilon > 0$ since $\varepsilon = \frac{mk}{b^2}$, with each term $> 0$ as stated in the question. Thus, it is sufficient to look at cases for which $\varepsilon > 0$. In particular, we look at different regions of $\varepsilon$ in which the number of roots changes[a] as follows:

- For $0 < \varepsilon < \frac{1}{4}$, we have two distinct roots. This implies that the solution to the ODE is given by
$$u(\tau) = Ae^{-\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon}\right)\tau} + Be^{-\left(\frac{1}{2\varepsilon} + \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon}\right)\tau},$$
which upon using $u(0) = 1$ and $\frac{\mathrm{d}u}{\mathrm{d}\tau}(0) = 0$, we have
$$u(\tau) = \left(\frac{1+\sqrt{1-4\varepsilon}}{2\sqrt{1-4\varepsilon}}\right)e^{-\left(\frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon}\right)\tau} + \left(\frac{\sqrt{1-4\varepsilon}-1}{2\sqrt{1-4\varepsilon}}\right)e^{-\left(\frac{1}{2\varepsilon} + \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon}\right)\tau}.$$

- For $\varepsilon = \frac{1}{4}$, we have the situation of a repeated roots. Thus, we have the solution given by
$$u(\tau) = Ae^{-2\tau} + B\tau e^{-2\tau},$$
which upon solving for the constants with the initial conditions, we have
$$u(\tau) = e^{-2\tau} + 2\tau e^{-2\tau}.$$

- For $\varepsilon > \frac{1}{4}$, we have imaginary roots. From your ODE class, we note that since we can write $\lambda$ as
$$\lambda = -\frac{1}{2\varepsilon} \pm i\frac{1}{2\varepsilon}\sqrt{4\varepsilon - 1},$$

our solution takes the form:

$$u(\tau) = e^{-\frac{\tau}{2\varepsilon}} \left( A\cos(\frac{1}{2\varepsilon}\sqrt{4\varepsilon - 1}\tau) + B\sin(\frac{1}{2\varepsilon}\sqrt{4\varepsilon - 1}\tau) \right)$$

which upon solving for the constants with the initial conditions, we have

$$u(\tau) = e^{-\frac{\tau}{2\varepsilon}} \left( \cos\left( \frac{1}{2\varepsilon}\sqrt{4\varepsilon - 1}\tau \right) \right).$$

(d) Now, if $\varepsilon \ll 1$, we are looking at the case of $0 < \varepsilon < \frac{1}{4}$. In fact, for small $\varepsilon$, the exponent of the first solution is approximately:

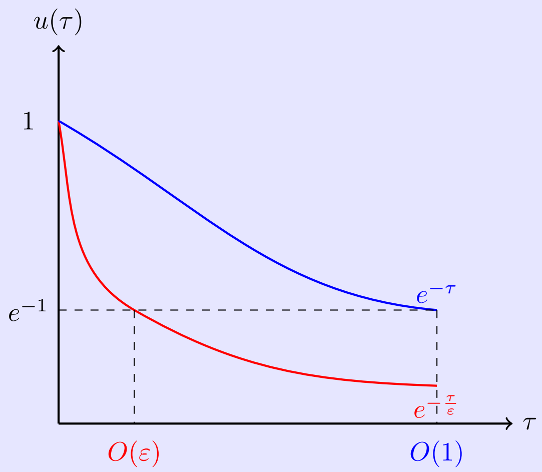
$$-\left( \frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon} \right)\tau \approx -\left( \frac{1}{2\varepsilon} - \frac{1}{2\varepsilon}(1 - 2\varepsilon + O(\varepsilon^2)) \right)\tau \approx -\tau$$

while the exponent of the second solution is approximately:

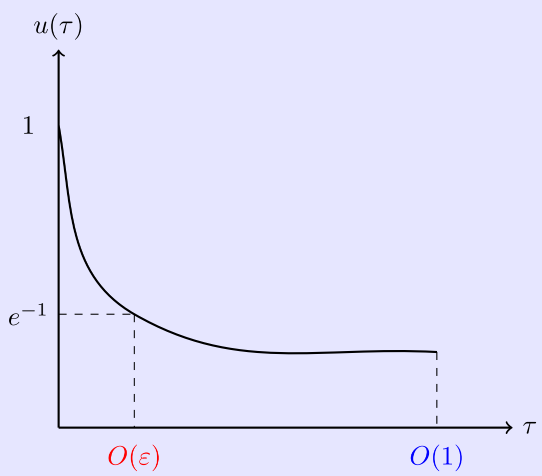
$$-\left( \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon}\sqrt{1-4\varepsilon} \right)\tau \approx -\left( \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon} \right)\tau \approx -\frac{\tau}{\varepsilon}.$$

The widely different time scales can thus be observed by looking at its characteristic time. This can be observed by looking at the time taken for $u$ to decrease by a factor of $e$. In the first case, $t_c = 1 = O(1)$. In the second case, $t_c = \varepsilon = O(\varepsilon)$. Thus, we see that the second solution exhibits a fast decay.

(e) Here, we note the shape of the two different linearly independent solutions here:




Actual Solution:



Here, we can observe that the solution rapidly decays on $O(\varepsilon)$ time scale, and then proceed to decay normally as in an $O(1)$ time scale.
(f) If we drop the term in $\varepsilon$, the resulting equation has solution

$$u(\tau) = e^{-\tau}$$

for $u(0) = 1$. Only on the time scales of $O(\varepsilon)$ do we miss out on the rapidly decaying solution and thus the two solutions would not coincide. Apart from that, on normal time scales such as $O(1)$, we see that the solutions actually agree. Thus, such an approximation is valid.
(Note: It depends on how you argue about "validity" of the approximation. For the purpose of this class, we refer to a "valid" approximation as one in which the solution mostly agrees.)

[a]In some sense, we can refer to this as a "bifurcation of roots"!