Math 134 - Lecture 2
Fall 2021
Final Exam
12/9/2021

Name:

Student ID:

Please sign the following honor statement. **If you do not sign this, you will receive 0 points.**

> *I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.*

Signed:

This exam contains 12 pages (including this cover page) and 9 problems. There are a total of 100 points available.

- Attempt all questions.
- You may use additional blank pages as required.
- Please write legible.
- You can use 2 cheat sheets.
- If you do not properly justify your results you may loss points.
- You can use a simple calculator (i.e. with no plotting capabilites).
- For those of you taking your test through Zoom: **Posting problems to online forums or "tutoring" websites counts as interaction with another person so it is strictly forbidden.**

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1. (15 points) Consider the system

$$\begin{cases} \dot{x} = y + 2xy \\ \dot{y} = x + x^2 - y^2. \end{cases}$$

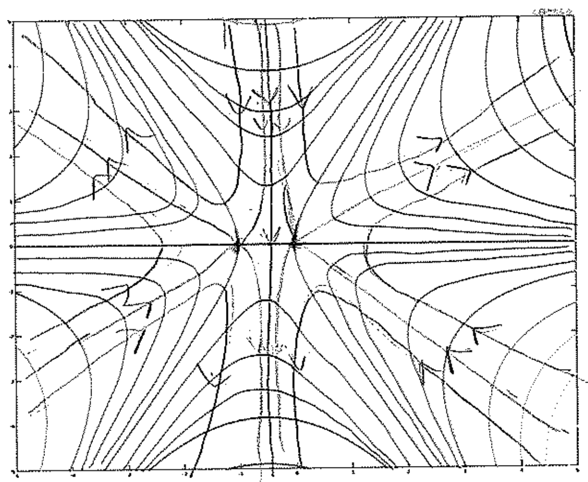
(a) Compute the critical points and classify them (find the eigenvalues and corresponding eigenvectors.)

(b) Show that the nonlinear system is a gradient flow.

(c) Can this system have a closed orbit?

(d) Compute V and classify its critical points.

(e) The following Figure illustrates the contour curves of V, sketch a plausible phase portrait. Which result covered in class are you using? (State it in a few words.)



a) $\dot{x} = y + 2xy = 0 \Rightarrow y(1+2x) = 0 \Rightarrow y = 0$ or $x = -1/2$

$\dot{y} = x + x^2 - y^2 = 0 \Rightarrow y = 0:\ x + x^2 = 0 \Rightarrow x(1+x) = 0 \Rightarrow x = 0,\ x = -1$

$x = -1/2:\ -\frac{1}{2} + \frac{1}{4} - y^2 = 0 \Rightarrow y^2 = -\frac{1}{4} \Rightarrow$ No real solutions

The critical points are $(x^*, y^*) = (0,0)$ and $(-1, 0)$

$$\nabla f = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 2y & 1+2x \\ 1+2x & -2y \end{bmatrix}$$

$(0,0)$:

$\nabla f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\tau = 0$, $\Delta = -1$ $\Rightarrow$ saddle

$\lambda = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) = \frac{1}{2}(\pm\sqrt{4}) = \pm 1$

$\lambda_1 = -1$:

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 1$:

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow \lambda_1 < 0$, $\lambda_2 > 0$ $\Rightarrow$ linearized system has a saddle (unstable)

$Re(\lambda_{1,2}) \neq 0 \Rightarrow$ Hyperbolic. Nonlinear system has saddle @ $(0,0)$

Hartmann grobmann

$(-1,0)$:

$\nabla f(-1,0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ $\tau = 0$, $\Delta = -1$ $\Rightarrow$ saddle

$\lambda_{1,2} = \pm 1$

$\lambda_1 = -1$:

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1$:

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

d) $f(x) = -\nabla V = -\begin{bmatrix} \partial_x V \\ \partial_y V \end{bmatrix}$

$-\partial_x V = y + 2xy$

$\Rightarrow V = -(xy + x^2 y + G(y))$

$-\partial_y V = x + x^2 - y^2 \Rightarrow V = -(xy + x^2 y - \frac{1}{3}y^3 + C)$

$\Rightarrow \boxed{V(x,y) = -xy - x^2 y + \frac{1}{3}y^3 + C}$

const

In a gradient system, solution curves are orthogonal to the level curves of V.

$\Rightarrow$ By hartman grobman $\Rightarrow$ Nonlinear system has a saddle (unstable) at $(-1,0)$

b) $\left.\begin{array}{l} \partial_y(y + 2xy) = 1 + 2x \\ \partial_x(x + x^2 - y^2) = 1 + 2x \end{array}\right\}$ equal $\Rightarrow$ gradient flow

c) Since the nonlinear system is a gradient flow $\Rightarrow$ NO closed orbits.

2. (10 points) Consider the differential equation

$$\ddot{x} = x^2 - 11x + 10$$

(a) Write the differential equation as a first order ODE system.

(b) Calculate all fixed points and classify them using linear stability analysis.

(c) Find a conserved quantity for the differential equation. Show that your quantity is indeed preserved.

(d) Classify the fixed points of the non-linear ODE.

a) Let $y = \dot{x}$

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^2 - 11x + 10 \end{cases}$$

b) $\dot{x} = y = 0 \Rightarrow y = 0$

$\dot{y} = x^2 - 11x + 10 = (x-1)(x-10) = 0 \Rightarrow x = 1, 10$

Fixed pts: $(1,0), (10,0)$

$$\nabla f = \begin{bmatrix} 0 & 1 \\ 2x - 11 & 0 \end{bmatrix} \qquad \frac{\pm\sqrt{4 \cdot 9}}{2} = \frac{\pm\sqrt{36}}{2} = \pm\frac{6}{2} = \pm 3$$

$(1,0)$: $\nabla f(1,0) = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}$, $\tau = 0$, $\Delta = 9 \Rightarrow$ linear center
$\lambda = \pm 3i$ (neutrally stable)
↳ not hyperbolic

$(10,0)$: $\nabla f(10,0) = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}$, $\tau = 0$, $\Delta = -9 \Rightarrow$ linear saddle
$\lambda = \pm 3$ $\Rightarrow$ Non-linear saddle
↳ Hyperbolic (Hartmann) (unstable)

c) let $x^2 - 11x + 10 = -\frac{dV}{dx} \Rightarrow V(x) = -\frac{1}{3}x^3 + \frac{11}{2}x^2 - 10x$

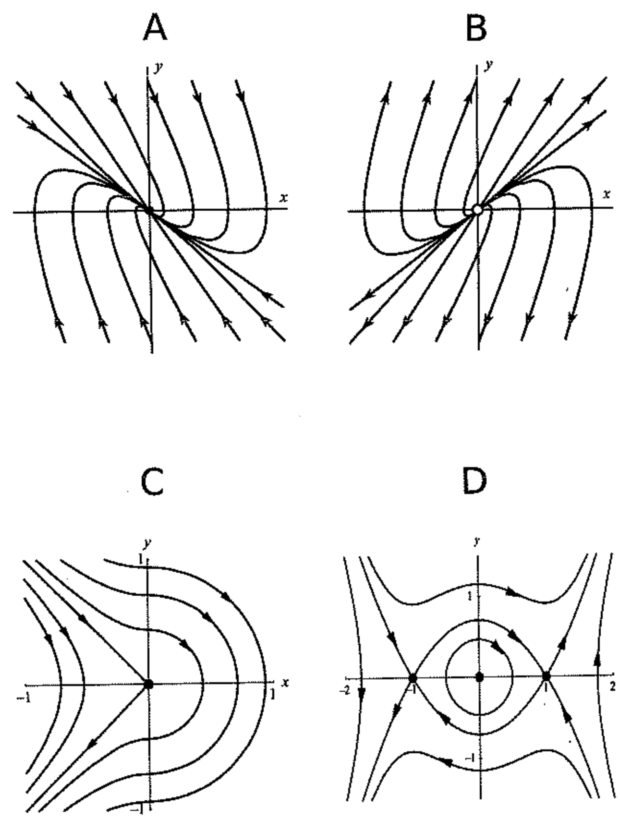
$\ddot{x} = -V'(x) \Rightarrow \ddot{x} + V'(x) = 0 \Rightarrow \ddot{x}\dot{x} + V'(x)\dot{x} = 0 \Rightarrow \frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{x}^2 + V(x)\right)}_{E(x)} = 0$

$E(x,y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + \frac{11}{2}x^2 - 10x$ is a conserved quantity

d) From Hartmann Grobman $(10,0)$ is a saddle point for the non-linear system since it is hyperbolic. (unstable)

$\Rightarrow$ B...

3. (6 points) Consider the following phase portraits:



In which of these phase portraits is the fixed point at (0; 0):

(a) Lyapunov stable? A, D

(b) Attracting? A

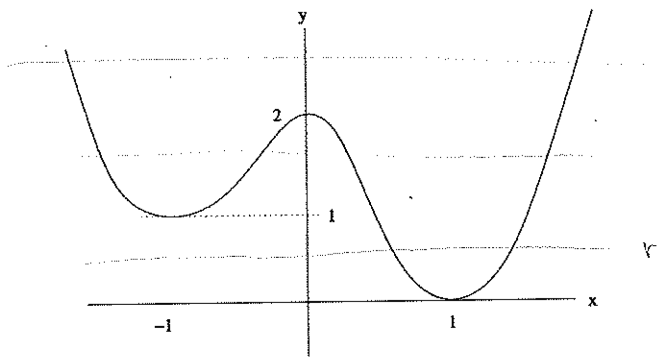
(c) Neutrally stable? D

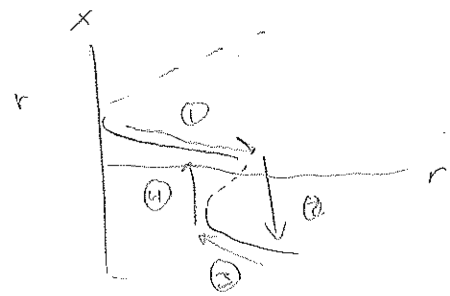
(d) Asymptotically stable? A

(e) Unstable? B, C

There may be multiple correct answers for each part. You must identify all correct answers to receive credit.

4. (15 points) The graph of $y = f(x)$ is sketched below





Consider the family of differential equations parameterized by r,

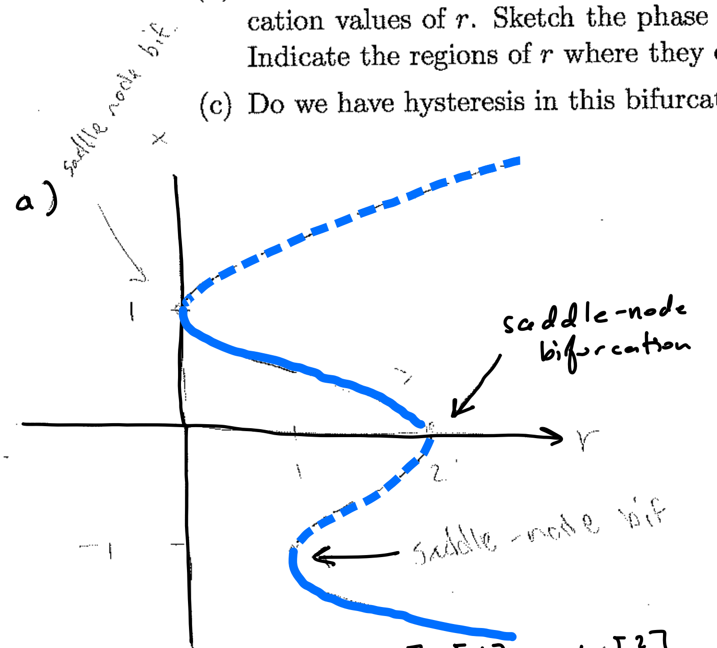
$$\dot{x} = f(x) - r, \qquad \text{for } -\infty < r < \infty$$

(a) (Bifurcation diagram.) In the (r, x)-plane with the r-axis horizontal, sketch the set of equilibria as functions of r.

(b) As r varies, there are qualitatively different phase portraits that occur separated by bifurcation values of r. Sketch the phase portraits that occur between the bifurcation values. Indicate the regions of r where they occur.

(c) Do we have hysteresis in this bifurcation diagram? Explain your answer.

a)



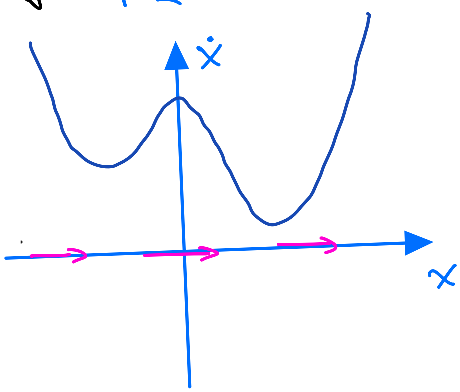
bifurcations at $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\0\end{bmatrix}$

$f(x) - r = 0 \Rightarrow f(x) = r$

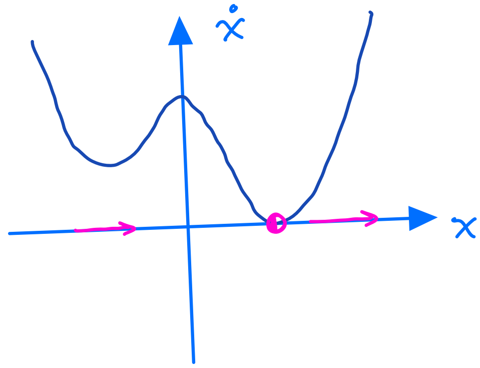
c) Yes we have hysterisis.
If we start on the stable branch at $(x,r) = (1,0)$ and increase r, we eventually will "fall" down to the bottom branch. If we reverse r at that point we can't go back to where we started without "jumping" at $r \neq 1$
Irreversibility with variation of r!

b)

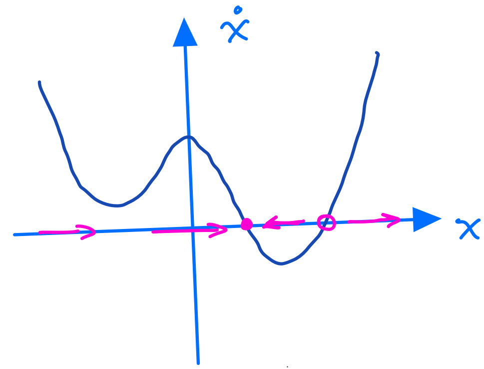
$r < 0$



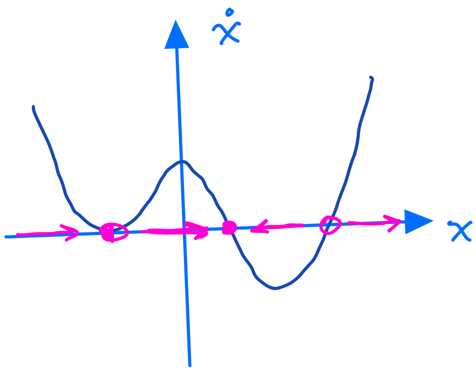
$r = 0$



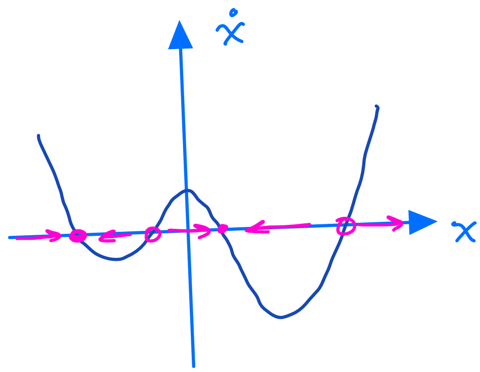
$0 < r < 1$



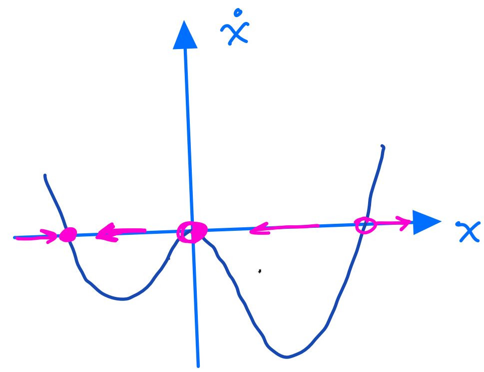
$r = 1$



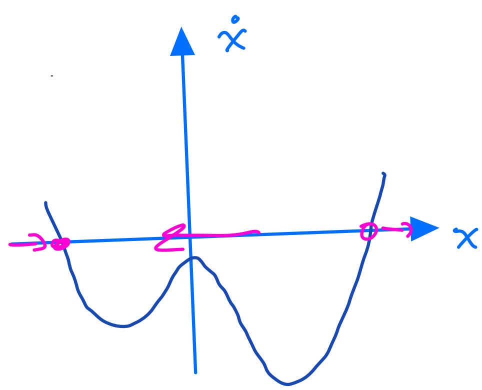
$1 < r < 2$



$r = 2$



$r > 2$



5. (15 points) Consider the system

$$\begin{cases}\dot{x} = 2y - x(x^2+y^2), & = 2y - x^3 - xy^2 \\ \dot{y} = -2x - y(x^2+y^2). & = -2x - yx^2 - y^3\end{cases}$$

(a) Linearize around $(0,0)$. Can you conclude what is the nature of the origin of the nonlinear system?

(b) Change to polar coordinates and show that the origin is a stable spiral.

a) $$\nabla f = \begin{bmatrix} -3x^2 - y^2 & 2 - 2xy \\ -2 - 2xy & -x^2 - 3y^2 \end{bmatrix}$$

$$\nabla f(0,0) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \quad \tau = 0,\ \Delta = 4 \Rightarrow \text{Linear center}$$

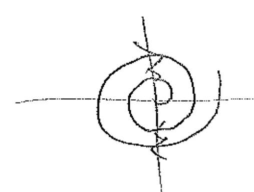
$$\lambda_{1,2} = \frac{\pm\sqrt{-4\cdot 4}}{2} = \pm 4i$$

No, we cannot conclude the nature of the origin of the non-linear system since it is not hyperbolic $(\mathrm{Re}(\lambda_{1,2}) = 0)$

b) $r^2 = x^2 + y^2 \Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \Rightarrow \dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \dot{x}\cos\theta + \dot{y}\sin\theta$

$x = r\cos\theta \qquad \dot{x} = 2r\sin\theta - r^3\cos\theta$

$y = r\sin\theta \qquad \dot{y} = -2r\cos\theta - r^3\sin\theta$



$$\dot{r} = 2r\sin\theta\cos\theta - r^3\cos^2\theta - 2r\sin\theta\cos\theta - r^3\sin^2\theta$$

$$= -r^3(\sin^2\theta + \cos^2\theta) = -r^3 \Rightarrow \dot{r} < 0 \text{ since } r > 0$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \dot{\theta} = \frac{\frac{x\dot{y} - y\dot{x}}{x^2}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x\dot{y} - y\dot{x}}{x^2 + y^2} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

$$\dot{\theta} = \frac{\dot{y}\cos\theta - \dot{x}\sin\theta}{r} = \frac{1}{r}\left(-2r\cos^2\theta - r^3\sin\theta\cos\theta - 2r\sin^2\theta + r^3\sin\theta\cos\theta\right)$$

$$= \frac{1}{r}\left(-2r(\sin^2\theta + \cos^2\theta)\right) = -2 \neq 0$$

$\Rightarrow \dot{\theta} = -2 \neq 0,\ \dot{r} = -r^3 < 0 \Rightarrow$ we spiral towards the origin as $t \to \infty$

Thus the origin is a stable spiral

6. (20 points) Consider the system

$$\begin{cases} \dot{x} = -5x^n - y \\ \dot{y} = 4x - y^3 \end{cases}$$

where $n \geq 1$ is an integer.

(a) For $n = 1$, use the Hartman-Grobman Theorem to sketch a local phase-portrait near $(0, 0)$. You should compute and depict the eigendirections near the fixed point!

(b) Explain why we cannot use the Hartman-Grobman Theorem to understand the local behavior near $(0, 0)$, when $n \geq 2$.

(c) Show that $(0, 0)$ is asymptotically stable for all odd integers $n \geq 1$, by finding a Lyapunov function. *Hint: Try a function of the form $L(x, y) = ax^2 + by^2$ for suitable a, b.*

a) $n=1$;

$$\begin{cases} \dot{x} = -5x - y \\ \dot{y} = 4x - y^3 \end{cases}$$

$$\nabla f = \begin{bmatrix} -5 & -1 \\ 4 & -3y^2 \end{bmatrix}, \quad \nabla f(0,0) = \begin{bmatrix} -5 & -1 \\ 4 & 0 \end{bmatrix}$$

$\tau = -5, \ \Delta = 4$

$(-5)^2 - 4 \cdot 4 = 9 > 0 \Rightarrow$ stable node (hyperbolic)

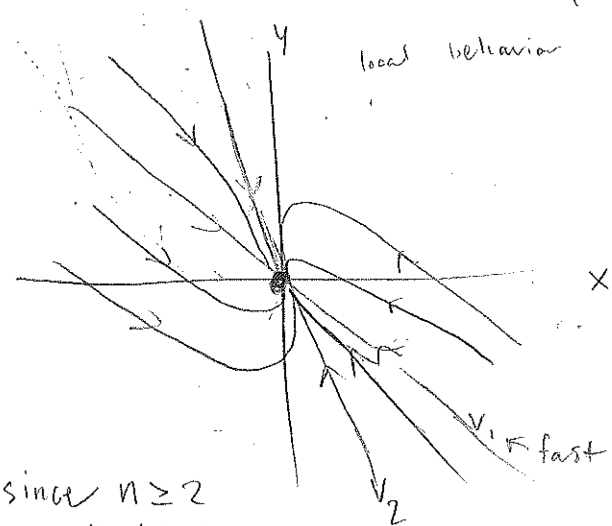
$$\lambda = \frac{1}{2}\left(-5 \pm \sqrt{(-5)^2 - 4 \cdot 4}\right) = \frac{1}{2}(-5 \pm 3) = -4, -1$$

$\lambda_1 = -4$:

$$\begin{bmatrix} -1 & -1 \\ 4 & 4 \end{bmatrix} V_1 = 0 \Rightarrow V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda_2 = -1$:

$$\begin{bmatrix} -4 & -1 \\ 4 & 1 \end{bmatrix} V_2 = 0 \Rightarrow V_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$



v_1 ← fast eigendir

v_2 ← slow eigendiv

b) $n \geq 2$;

$$\begin{cases} \dot{x} = -5x^2 - y \\ \dot{y} = 4x - y^3 \end{cases}$$

$$\nabla f = \begin{bmatrix} -5nx^{n-1} & -1 \\ 4 & -3y^2 \end{bmatrix}, \quad \nabla f(0,0) = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$$

since $n \geq 2$, $n - 1 \geq 1$

$\tau = 0, \ \Delta = 4 \Rightarrow$ center

$$\lambda_{1,2} = \frac{\pm\sqrt{-4 \cdot 4}}{2} = \pm 2i \Rightarrow Re(\lambda_{1,2}) = 0 \Rightarrow$$ So the point is not hyperbolic $\Rightarrow$ we can't use hartmann grobman for $n \geq 2$.

c) omitted

skip

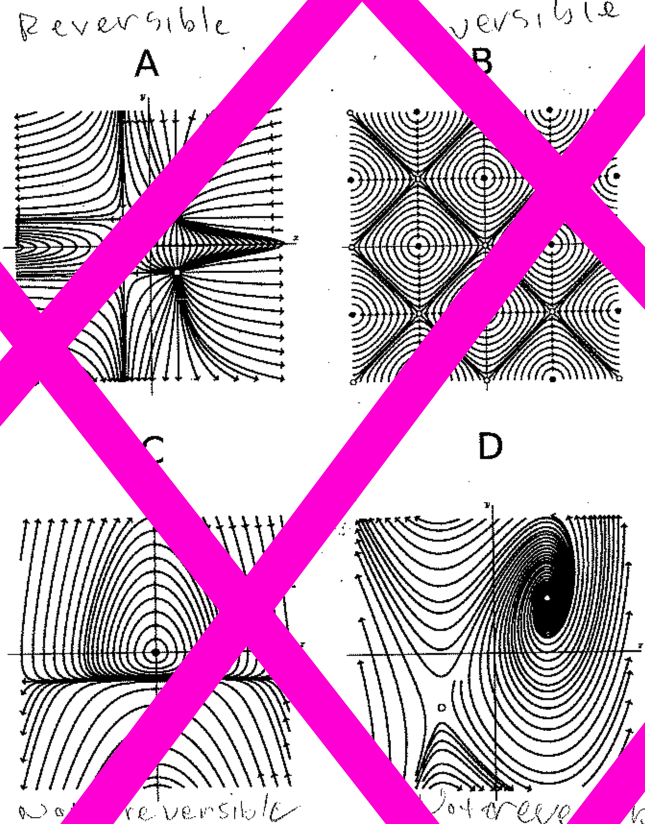
7. (10 points) (a) Show that

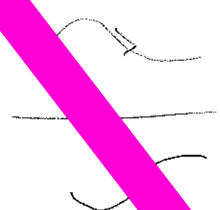
$$\begin{cases} \dot{x} = y(1-x^2) \\ \dot{y} = x(1-y^2) \end{cases}$$

is a reversible system.

(b) Which of the following phase portraits cannot be the one corresponding to a reversible system?

Reversible





Mirror over x axis
rev. dir

a) Let $X(t) = x(-t)$, $Y(t) = -y(-t)$

$\dot{X}(t) = -\dot{x}(-t) = -y(-t)(1-(x(-t))^2) = Y(t)(1-(X(t))^2) = Y(1-X^2)$

$\dot{Y}(t) = \dot{y}(-t) = x(-t)(1-(y(-t))^2) = X(t)(1-(-Y(t))^2) = X(1-Y^2)$

=> The system is invariant under the map $(x,y,t) \mapsto (x,-y,-t)$

=> The system is reversible

b) C and D do not correspond to a reversible system

8. (5 points) Consider the system

$$\begin{cases} \dot{x} = xf(x,y) \\ \dot{y} = g(x,y). \end{cases}$$

where $f(x,y)$ and $g(x,y)$ are continuously differentiable. Is there a trajectory that passes through both $(-1,0)$ and $(1,0)$? Explain your answer.

Since $f(x,y)$ and $g(x,y)$ are cont. differentiable for all $(x,y)\in\mathbb{R}^2$ exists a unique solution for any $(x_0,y_0)\in\mathbb{R}^2$

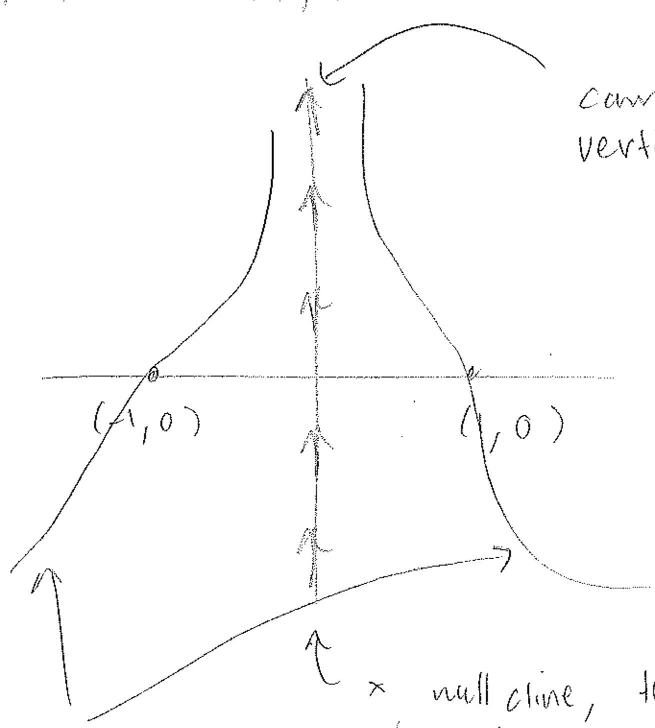
$x=0 \Rightarrow \begin{cases} \dot{x}=0 \\ \dot{y}=g(0,y) \end{cases}$

We have that a x nullcline is given by $x=0$,

so that means that any solution cannot cross the y-axis. since $(x_0,y_0)=(0,y_0)$ gives a unique solution, and crossing would violate uniqueness

Thus, a solution that passes through $(-1,0)$ is a unique solution, and a solution that passes through $(1,0)$ is a different unique solution.

$\Rightarrow$ $\boxed{\text{No.}}$ You cannot have trajectories that pass through both $(-1,0)$ and $(1,0)$



cannot cross over vertical axis

x null cline, trajectory must move vertically (up/down doesn't matter)

2 distinct solutions

9. (4 points) Which was your favorite result covered in this course? Explain with a few words why.