Math 134 - Homework 2

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function so that $\frac{d^n}{dx^n} f$ is bounded for $n = 0, 1, 2$ and consider the ODE

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

   Let x_1 be the approximation to $x(\Delta t)$ obtained from the improved Euler method. Using Taylor's Theorem, show that the local truncation error $e_1 = x(\Delta t) - x_1$ satisfies

$$|e_1| \leq C(\Delta t)^3$$

   for some constant $C > 0$.

2. Suppose that $f\colon (a, b) \to \mathbb{R}$ is Lipschitz. Show that f is continuous on (a, b).

   Solution

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz and let x^* be a fixed point of the ODE

$$\dot{x} = f(x).$$

   Show that there cannot exist a solution with $x(0) = x_0 \neq x^*$ that reaches the fixed point x^* in finite time.

   <u>Hint:</u> Suppose for a contradiction that such a solution exists. What can you say about uniqueness?

4. (Exercise 2.5.3 in Strogatz) Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $|x(t)| \to \infty$ in finite time, starting from any initial condition $x_0 \neq 0$.

5. Solve problem 2.5.2 in Strogatz.

6. Consider the equation

$$\dot{x} = r + \frac{1}{4}x - \frac{x}{1+x}.$$

   At what value of r do we have a saddle-node bifurcation?