Math 134, Spring 2022

Lecture #6: Existence and uniqueness & an Intro to bifurcations.

Friday April 7th

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Learning objectives

Today we will discuss:

- What it means to say a function is (globally) Lipschitz continuous.
- Properties of Lipschitz functions.
- Global solutions.
- Comparing solutions of ODEs.
- Intro to bifurcation theory

Existence and uniqueness

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Global and local Lipschitz functions

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Finite time blowup: An example

Does the solution of

$$\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$$

blow up in finite time?

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous.

Then there exists a unique **global** solution $x : \mathbb{R} \to \mathbb{R}$ of

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0. \end{cases}$$

Proof: We will omit it.

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An example

Is $f(x) = x^2$ Lipschitz continuous on $\mathbb{R}$?

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Comparing solutions

Let $f \leq g$ be smooth and let $x_0 \leq y_0$. Suppose that x and y are solutions of the ODEs

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{y} = g(y) \\ y(0) = y_0 \end{cases}$$

on an interval $[0, T]$. Then $x(t) \leq y(t)$ for all $t \in [0, T]$.

Proof(Sketch)

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Comparing solutions: Example

An Intro to bifucations

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External parameters

- Consider the ODE
$$\dot{x} = f(x, r)$$
where r is a parameter of the model.

- **Question:** How do the dynamics vary as we vary r?

An example

$$\dot{x} = r + x^2$$

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Definition!

Consider the following autonomous system

$$\dot{x} = f(x, \lambda)$$

where $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. A **bifurcation** occurs at parameter $\lambda = \lambda_0$ if there are parameter values λ_1 arbitrarily close to λ_0 with dynamics topologically inequivalent from those at λ_0.

See you next time!

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