Homework 11

12.3.11. Claim: Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}.$$

It's characteristic polynomial is $(x-1)^2(x+1)^2$, and its rational canonical form and its Jordan canonical form are given below.

Proof: First, we calculate

$$xI - A = \begin{pmatrix} x-1 & 0 & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 2 & 2 & x & -1 \\ 2 & 0 & 1 & x+2 \end{pmatrix}.$$

The characteristic polynomial is the determinant of this matrix, which is $(x-1)^2(x+1)^2$. Following the row and column operations permitted, we diagonalize $xI - A$ to get

$$(xI - A)' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & (x-1)(x+1)^2 \end{pmatrix}.$$

This shows that the invariant factors of A are $a_1(x) = (x-1)$ and $a_2(x) = (x-1)^2(x+1)^2$. Rather than follow the algorithm further, we will now use the results of sections 12.2 and 12.3 to write down the companion matrices to $a_1(x)$ and $a_2(x)$, giving that the rational canononical form of A is

$$A' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Similarly, we can write the Jordan blocks for the linear factors of $a_1(x)$ and $a_2(x)$ and get the Jordan form of A is

$$A'' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

12.3.18. Claim: All possible Jordan canonical forms for a linear transformation with characteristic polynomial $(x-2)^3(x-3)^2$ are listed below.

Proof: The minimal polynomial for a matrix with characteristic polynomial must have both $(x-2)$ and $(x-3)$ as factors, and it must divide $(x-2)^3(x-3)^2$. The possible Jordan blocks associated to the factor $x-2$ are

$$J_{x-2} = \begin{pmatrix} 2 \end{pmatrix},$$

or

$$J_{(x-2)^2} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},$$

or

$$J_{(x-2)^3} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},$$

Similarly, the possible Jordan blocks associated to the factor $x-3$ are

$$J_{x-3} = \begin{pmatrix} 3 \end{pmatrix},$$

or

$$J_{(x-3)^2} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

The Jordan canonical form of the matrix will be built out of these blocks.

We can further consider the size of the matrix. Since it's characteristic polynomial has degree 5, it must be a 5×5 matrix. Furthermore, the matrix must have the same eigenvalues as the original matrix, up to multiplicity, so it must have three 2s and two 3s along the diagonal. This leads to the following possibilities, up to permutation of the elementary divisors.

1.
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

2.
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

3.
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

4.
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

5.
$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

6.
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

AM 15. Let A be a commutative ring with 1, and $X = \operatorname{Spec}(R)$. For any subset $E \subseteq A$, let $V(E)$ be the set of prime ideals of A which contain E.

(i) Claim: If $\mathfrak{a}$ is the ideal generated by E, and $r(\mathfrak{a})$ is the radical of $\mathfrak{a}$, then
$$V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a})).$$

Proof: First, since $E \subseteq \mathfrak{a}$, it is clear that $V(\mathfrak{a}) \subseteq V(E)$. Further, since $\mathfrak{a}$ is by definition the smallest ideal containing E, and prime ideal containing E must also contain $\mathfrak{a}$. Therefore $V(E) \subseteq V(\mathfrak{a})$.

Since $\mathfrak{a} \subseteq r(\mathfrak{a})$, we can see that $V(r(\mathfrak{a})) \subseteq V(\mathfrak{a})$. Suppose P is a prime ideal containing $\mathfrak{a}$. For any $r \in r(\mathfrak{a})$ there exists an integer $m > 0$ so that $r^m \in \mathfrak{a}$. Since P is prime, it must contain r as well. Therefore P contains $r(\mathfrak{a})$. Hence $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$.

(ii) Claim: $V(0) = X$, and $V(A) = \emptyset$.

Proof: First, every prime ideal of A contains 0, so $V(0) = X$. Second, no prime ideal contains the whole ring A, so $V(A) = \emptyset$.

(iii) Claim: If $(E_i)_{i \in I}$ is any family of subsets of A, then

$$V(\cup_{i \in I} E_i) = \cap_{i \in I} V(E_i).$$

Proof: If P is a prime ideal containing $\cup_{i \in I} E_i$, then P contains each E_i. Hence P will lie in $\cap_{i \in I} V(E_i)$. Therefore $V(\cup_{i \in I} E_i) \subseteq \cap_{i \in I} V(E_i)$.

If P is a prime ideal lying in $\cap_{i \in I} V(E_i)$, then P contains each set E_i. Therefore P contains the union $V(\cup_{i \in I} E_i)$. Hence $\cap_{i \in I} V(E_i) \subset V(\cup_{i \in I})$.

(iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.

Proof: First, suppose P is an ideal of A containing $\mathfrak{a}$ or $\mathfrak{b}$. Then P contains $\mathfrak{a} \cap \mathfrak{b}$, so $P \in V(\mathfrak{a} \cap \mathfrak{b})$. Therefore $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$. Now suppose P is a prime ideal containing $\mathfrak{a} \cap \mathfrak{b}$. If there exists an $a \in \mathfrak{a}$ and a $b \in \mathfrak{b}$ which are both not in P, then $ab \in P$ contradicts that P is prime. Hence either all of $\mathfrak{a}$ or all of $\mathfrak{b}$ is contained in P. This shows $V(\mathfrak{a} \cap \mathfrak{b}) \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$.

AM 17. For each $f \in A$, le X_f be the complement of $V(f)$ in X.

(i) Claim: $X_f \cap X_g = X_{fg}$.

Proof: $X_{fg} = V(fg)^c$, and $X_f \cap X_g = (V(f) \cup V(g))^c$. By 15i, $V(fg) = V((fg))$, $V(f) = V((f))$, and $V(g) = V((g))$. Since $(fg) = (f) \cap (g)$, 15iv shows that $V(fg) = V(f) \cup V(g)$. Hence the complements are equal.

(ii) Claim: $X_f = \emptyset$ if and only if f is nilpotent.

Proof: $X_f = \emptyset$ if and only if $V(f) = X$. That is, f lies in every prime ideal of A. Suppose f is nilpotent. Then $f^m = 0$ for some $m > 0$. Since 0 is in each ideal, f must lie in each prime ideal of A. Now suppose g lies in each prime ideal of A. Call this intersection of all prime ideals $\mathfrak{n}$. (It is called the *nilradical* of A.) Then as showed in class, g is nilpotent.

(iii) Claim: $X_f = X$ if and only if f is a unit.

Proof: $X_f = X$ if and only if no prime ideals contain f. Every nonunit in A lies in a maximal (hence prime) ideal. Therefore $X_f = X$ if and only if f is a unit.

(iv) Claim: $X_f = X_g$ if and only if $r((f)) = r((g))$.

Proof: $X_f = X_g$ if and only if $V(f) = V(g)$. By 15i, this holds if and only if $V(r((f))) = V(r((g)))$. This clearly holds if $r((f)) = r((g))$. Now suppose $V(r((f))) = V(r((g)))$. In other words, the set of prime ideals containing $r((f))$ is the same as the set of prime ideals containing $r((g))$. As showed in class, $r((f))$ is the intersection of this set of ideals, as is $r((g))$. Hence $r((f)) = r((g))$.