Problems for review:

Problem 1

Given the system

$$\begin{cases} \dot{x} = -x(x^2+y^2+1) \\ \dot{y} = y(x^2-1) \end{cases}$$

can you conclude through linearization that the origin is stable/unstable?

Taking $V(x,y) = x^2+y^2$ as a Lyapunov function, can you conclude that the origin is stable/unstable?

Sol. $\boxed{\text{Using linearization}}$

Let

$$f(x,y) = \begin{bmatrix} -x(x^2+y^2+1) \\ y(x^2-1) \end{bmatrix}$$

$$\nabla f(0,0) = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A}$$

why?

so $\operatorname{tr} A = -2$ and $\det A = 1$

$\Rightarrow$ the origin is asymp. stable

$\boxed{\text{With Lyapunov function}}$

- A strong Lyapunov function is a continuously differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$ so that for all $\boldsymbol{x} \in B(\boldsymbol{x}^*, \delta)$ we have:

  i) $F(\boldsymbol{x}) \geq 0$ with equality if and only if $\boldsymbol{x} = \boldsymbol{x}^*$.

  ii) $\nabla F(\boldsymbol{x}) \cdot \boldsymbol{f}(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \neq \boldsymbol{x}^*$.

$\frac{d}{dt} F(x(t)) < 0$ for all $x \neq x^*$

i) $V(x,y) = x^2 + y^2 \geq 0$ with equality only if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ii) $\frac{d}{dt} V(x(t), y(t)) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$

$$= -2x^2(x^2+y^2+1) + 2y^2(x^2-1)$$

$$= -2x^4 - 2x^2y^2 - 2x^2 + 2x^2y^2 - 2y^2$$

$$= -2(x^4+x^2+y^2) < 0 \text{ for all } \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Theorem: If $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has an isolated fixed point $\boldsymbol{x}^*$ and a corresponding strong Lyapunov function F, then it is asymptotically stable.

The only thing left to prove is that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an isolated fixed point:

$$\begin{cases} -x(x^2+y^2+1) = 0 \\ y(x^2-1) = 0 \end{cases}$$

$$y(x^2-1) = 0 \iff y = 0 \text{ or } x^2 - 1 = 0$$

$$x = \pm 1$$

If $y=0$ then $-x(x^2+1)=0 \iff x=0$

$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If $x=1$ then $-(1^2+y^2+1)=0$, no sol.

If $x=-1$ then $(1+y^2+1)=0$, no sol.

so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the unique critical point, thus it is isolated.

So we can conclude thanks to the Theorem above that the origin is asymptotically stable.

Problem (St 7.2.10)

Show that the following nonlinear system doesn't have closed orbits

$$\begin{cases} \dot{x} = y - x^3 \\ \dot{y} = -x - y^3 \end{cases}$$

Show that $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only critical point.

$$V(x,y) = ax^2 + by^2$$

$$\frac{d}{dt} V(x(t), y(t)) = \partial_x V \cdot \dot{x} + \partial_y V \cdot \dot{y}$$

$$= 2ax(y - x^3) + 2by(-x - y^3) < 0 ?$$

6.5.12 (Why we need to assume *isolated* minima in Theorem 6.5.1) Consider the system $\dot{x} = xy$, $\dot{y} = -x^2$.

a) Show that $E = x^2 + y^2$ is conserved.

b) Show that the origin is a fixed point, but not an isolated fixed point.

c) Since E has a local minimum at the origin, one might have thought that the origin has to be a center. But that would be a misuse of Theorem 6.5.1; the theorem does not apply here because the origin is *not* an isolated fixed point. Show that in fact the origin is not surrounded by closed orbits, and sketch the actual phase portrait.

Sol.

a) $\frac{d}{dt} E(x(t), y(t)) =$

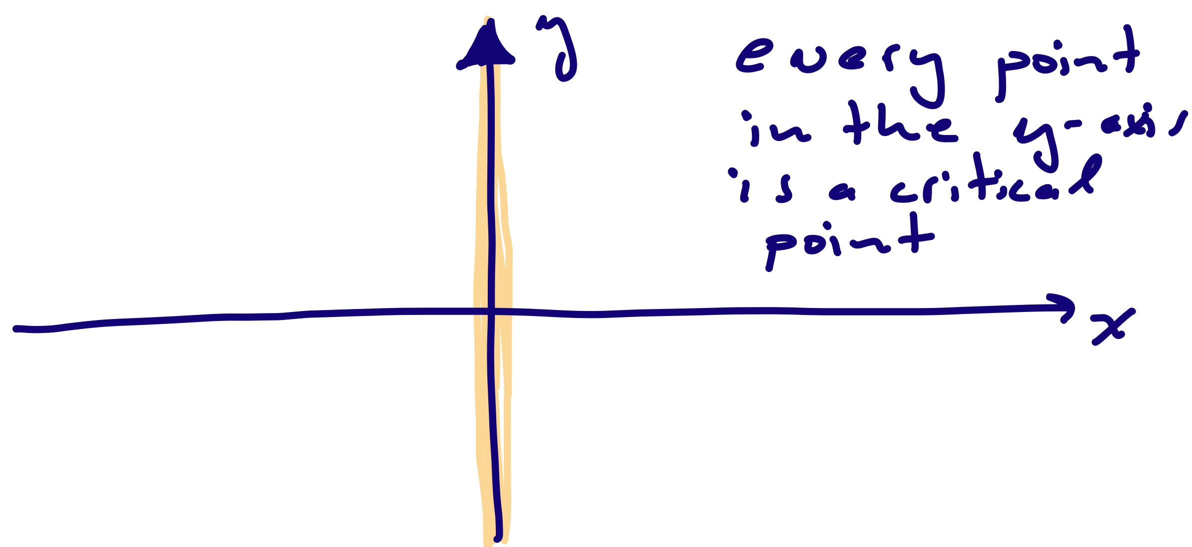
$$= \partial_x E \cdot \dot{x} + \partial_y E \cdot \dot{y}$$

$$= 2x(xy) + 2y(-x^2) = 0$$

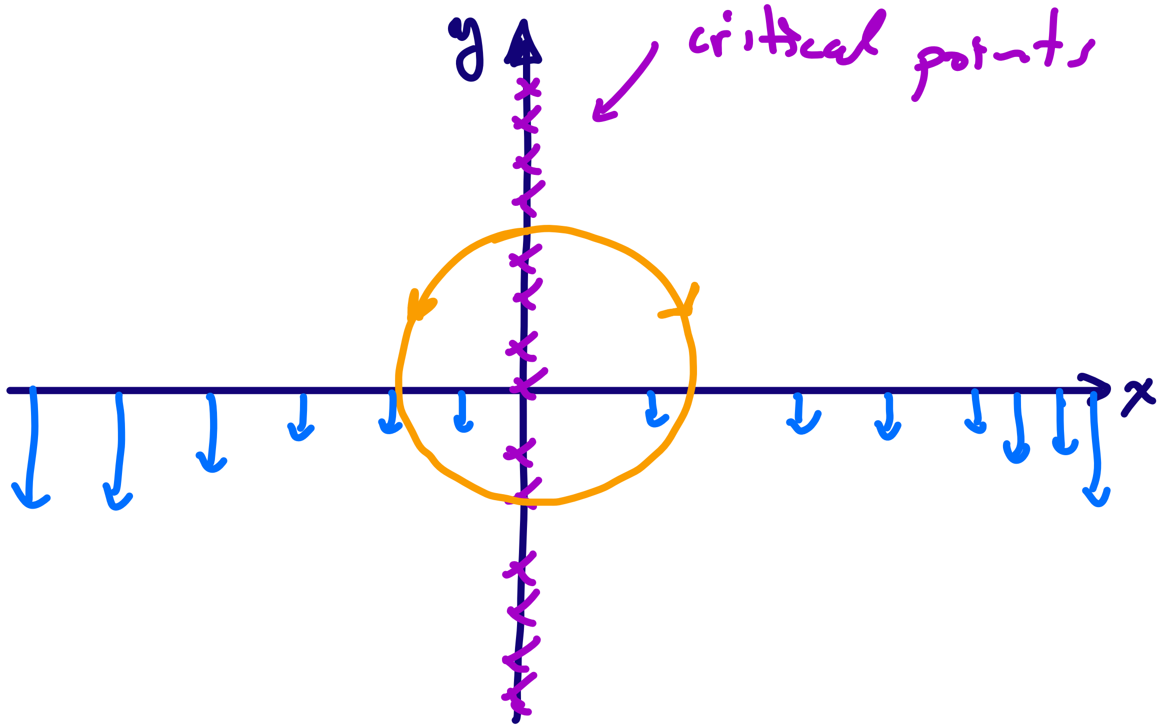
b) Fixed points:

$$xy = 0$$

$$-x^2 = 0 \Rightarrow x = 0 \quad y \in \mathbb{R}$$



c) $E = x^2 + y^2$ has a local minimum at $(0,0)$ but it is not isolated!!! Thus we cannot conclude that the origin is a center.



x - nullcline?

$\dot{x} = 0 \qquad xy = 0 \qquad x = 0$ or $y = 0$

for $y = 0 \qquad \dot{y} = -x^2$