Math 134, Lec 1, Spring 2022

Lecture #2: Flows on the line

Wednesday March 30^{th}

Slides and lecture recording

- The lecture will be recorded and posted to the Canvas page after class. You are not allowed to store or record the lectures by any other means.
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Learning objectives

Today we will discuss:

- How to reduce ODEs to first order autonomous systems.
- How to draw and analyze the phase portrait of a continuous flow on the line.
- The definition of a fixed point of a continuous flow on the line.
- What it means to say a fixed point is stable, unstable, and half-stable.

Introduction to dynamical systems (cont.)

Last time

- A **first order autonomous** system is an ODE of the form

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n) \end{cases}$$

$\dot{x}_i = \frac{dx_i}{dt}$

Leibniz

Newton

$x = x(t)$

$\frac{d^2 x(t)}{dt^2} = \ddot{x}(t)$

An example: seasonal epidemics

SIR

$$\begin{cases} \dot{x} = -\kappa(t)xy \\ \dot{y} = \kappa(t)xy - \delta y \\ \dot{z} = \delta y \end{cases}$$

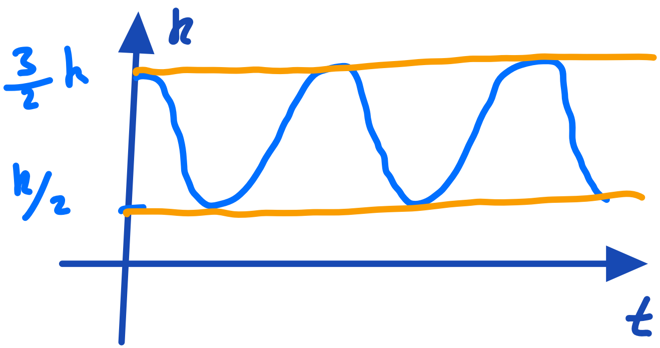
Is this system autonomous?
NO

where

$$\kappa(t) = k\left(1 + \tfrac{1}{2}\cos(t)\right)$$

Non-autonomous
$\Downarrow$
autonomous

for a constant $k > 0$.



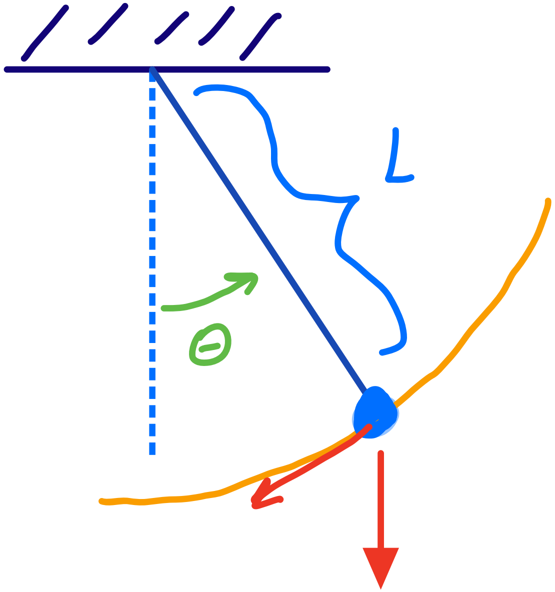
Introduce a new variable
$\tau = \tau(t) = t$

$$\begin{cases} \dot{x} = -k(\tau)xy \\ \dot{y} = k(\tau)xy - \delta y \\ \dot{z} = \delta y \\ \dot{\tau} = 1 \end{cases}$$

Another example: the pendulum

Model the angle θ of a pendulum of length $L > 0$ by

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$



Newton's 2nd Law:

$$mL\ddot{\theta} = -mg\sin(\theta) \quad (+F(t))$$

$$\theta = \theta(t)$$

Trick: $\begin{cases} x = \theta \\ y = \dot{\theta} \end{cases}$

$$\Rightarrow \begin{cases} \dot{x} = \dot{\theta} = y \\ \dot{y} = \ddot{\theta} = -\frac{g}{L}\sin(\theta) \\ \qquad = -\frac{g}{L}\sin(x) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L}\sin(x) \end{cases}$$

$$\ddot{\theta} + \cancel{\frac{g}{L}\sin}\,\theta = 0$$

What if we add an external force?

$$\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{1}{m}F(t)$$

$$\begin{cases} x = \theta \\ y = \dot{\theta} \\ z = t \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{\theta} = y \\ \dot{y} = \ddot{\theta} = -\frac{g}{L}\sin(\theta) + \frac{1}{m}F(t) \\ \qquad = -\frac{g}{L}\sin(x) + \frac{1}{m}F(z) \\ \dot{z} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{L}\sin(x) + \frac{1}{m}F(z) \\ \dot{z} = 1 \end{cases}$$

$$\frac{d^k x}{dt^k} = f\left(x, \frac{dx}{dt}, \dots, \frac{d^{(k-1)}x}{dt^{(k-1)}}\right)$$

$$z_1 = x,\ z_2 = \frac{dx}{dt},\ \dots,\ z_k = \frac{d^{k-1}x}{dt^{k-1}}$$

Flows on the line

$$\Rightarrow \begin{cases} \dfrac{d}{dt} z_1 = z_2 \\ \dfrac{d z_2}{dt} = z_3 \\ \quad\vdots \\ \dfrac{d z_k}{dt} = f(z_1, z_2, \dots, z_k) \end{cases}$$

We now consider systems of the form

$$\dot{x} = f(x)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.

An example

$$\dot{x} = \underbrace{x(x+1)(x-1)^2}_{f(x)}$$

Solve it? Yes

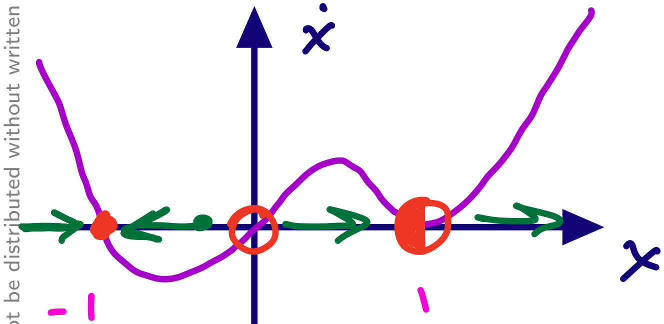
Note: Solution methods

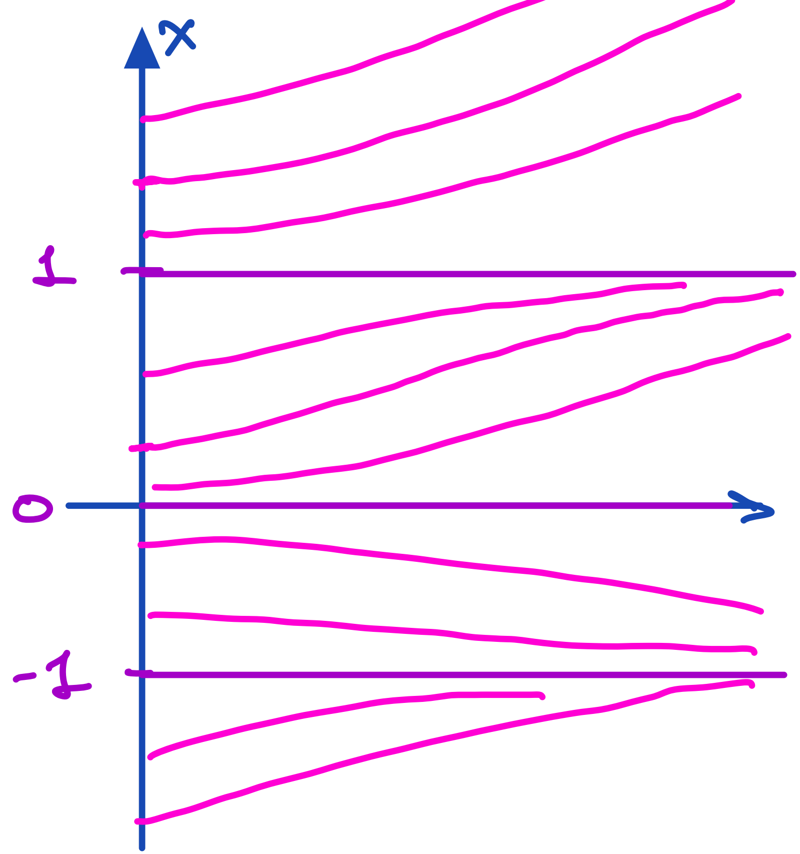
- Analitic methods (sep. of variables)
- Geometric methods (direction field)
- Numerical methods (Euler's method)

$$\dot{x} = x(x+1)(x-1)^2$$

$x_1(t) = 1$

Phase space





critical points

$-1, 0, 1$

Fixed points

- We say that x^* is a **fixed point** of the system

$$\dot{x} = f(x)$$

  if $f(x^*) = 0$.

  $x^* \in \mathbb{R}$.

- If x^* is a fixed point then the system has a constant solution $x(t) = x^*$.

- **Other names:** Equilibrium solutions, stationary points, rest points, critical points, steady states.

An example

Consider

$$\dot{x} = \cos x + 1$$

$\cos(x) + 1 = 0$

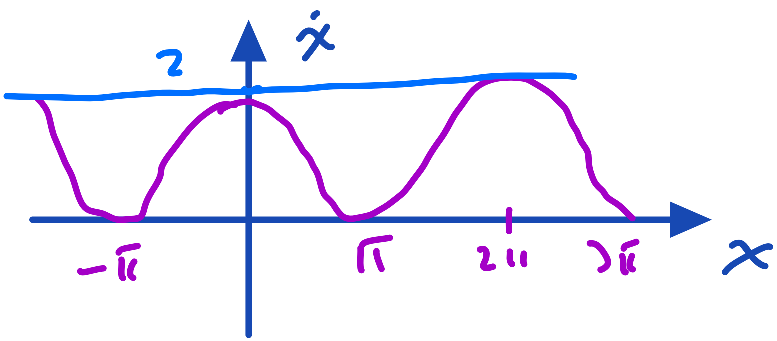
Which of the following describes all fixed points of the system?

A) $(2n-1)\pi$ for integers n

B) $2n\pi$ for integers n

C) 0

D) $n\pi$ for integers n



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Stability

Let x^* be a fixed point of the system

$$\dot{x} = f(x).$$

For now, we say that x^* is:

- **Stable** if solutions starting close to x^* approach x^* as $t \to \infty$.
- **Unstable** if solutions starting close to x^* diverge from x^* as $t \to \infty$.
- **Half-stable** if solutions starting close approach x^* from one side, but diverge from the other side.

An example

Consider the system

$$\dot{x} = \sin x$$

Which of the following statements is false?

A) There are fixed points at $n\pi$ for all integers n

B) There are stable fixed points at $(2n+1)\pi$ for all integers n

C) There are unstable fixed points at $2n\pi$ for all integers n

D) There are half-stable fixed points at $n\frac{\pi}{2}$ for all integers n

See you next time!