A REVIEW OF SOME LINEAR ALGEBRA

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We consider a 2×2 matrix
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
We will assume throughout that $A \neq 0$. Associated to the matrix A are two important quantities, the *trace*
$$\operatorname{tr} A = a + d$$
and the *determinant*
$$\det A = ad - bc.$$

We say that the columns $\mathbf{v}$ and $\mathbf{w}$ of A are *linearly dependent* if there exist constants q, p, not both zero, so that
$$q\mathbf{v} + p\mathbf{w} = 0.$$
We say the columns are *linearly indepednent* if they are not linearly dependent. We then have the following:

Lemma 1. *The determinant* $\det A \neq 0$ *if and only if the columns of* A *are linearly independent.*

Proof. The columns of A are linearly dependent if and only if there exist constants q, p, not both zero, so that
$$q \begin{bmatrix} a \\ c \end{bmatrix} + p \begin{bmatrix} b \\ d \end{bmatrix} = 0.$$
Without loss of generality, we can assume that $q \neq 0$. Then the columns are linearly dependent if and only if
$$\det A = ad - bc = \left(-\tfrac{p}{q} b\right) d - b \left(-\tfrac{p}{q} d\right) = 0.$$
$\square$

If $\det A \neq 0$, we say that the matrix A is *invertible*. The unique inverse matrix is given by
$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$
and satisfies
$$AA^{-1} = A^{-1}A = I,$$
where the *identity matrix*
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

satisfies

$$AI = IA = A \quad \text{for any matrix} \quad A.$$

We say that a constant $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if

$$\det(A - \lambda I) = 0.$$

Expanding the determinant and applying the quadratic formula, we see that[1]

$$\lambda = \tfrac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) \quad \text{where} \quad \tau = \operatorname{tr} A \quad \text{and} \quad \Delta = \det A.$$

In particular, taking λ to correspond to the choice of "+" and μ to correspond to the choice of "−" (note that $\lambda = \mu$ if $\tau^2 = 4\Delta$) we have

$$\operatorname{tr} A = \lambda + \mu \quad \text{and} \quad \det A = \lambda\mu.$$

We say that a non-zero vector $\mathbf{v}$ is an *eigenvector* of A associated to the eigenvalue λ if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

We then recall the following:

Lemma 2. *If λ is an eigenvalue of A then there is at least one eigenvector $\mathbf{v}$ associated to λ.*

Proof. If $A = \lambda I$ then any non-zero choice of $\mathbf{v}$ will suffice. Otherwise, suppose that

$$A - \lambda I = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

As the determinant of this matrix vanishes, we may apply Lemma 1 to find constants q, p, not both zero, so that

$$q \begin{bmatrix} x \\ z \end{bmatrix} + p \begin{bmatrix} y \\ w \end{bmatrix} = 0.$$

We may then take $\mathbf{v} = \begin{bmatrix} p \\ q \end{bmatrix}$. □

Our main application of linear algebra in this class will be to reduce any real 2×2 matrix to one of three *real canonical forms*. Precisely, we have the following:

Theorem 3. *Given any real 2×2 matrix A, there exists a real invertible matrix P so that*

$$A = PMP^{-1},$$

and M is one of the real canonical forms

$$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \qquad \begin{bmatrix} \sigma & 1 \\ 0 & \sigma \end{bmatrix}, \qquad \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$

where $\lambda \geq \mu$ and $\beta > 0$.

Proof. We first note that if $A = 0$ then we may take $P = I$ and to reduce to the first real canonical form with $\lambda = \mu = 0$. For $A \neq 0$, we divide into several cases depending on the trace $\tau = \operatorname{tr} A$ and determinant $\Delta = \det A$.

<u>Case 1: $\tau^2 > 4\Delta$.</u> Here we have two distinct eigenvalues

$$\lambda = \tfrac{1}{2}\left(\tau + \sqrt{\tau^2 - 4\Delta}\right) \quad \text{and} \quad \mu = \tfrac{1}{2}\left(\tau - \sqrt{\tau^2 - 4\Delta}\right),$$

[1] We have to be a little careful about how we define the square root if $(\tau^2 - 4\Delta) \notin [0, \infty)$. We will be deliberately ambiguous here and defer this discussion to classes on complex analysis.

which satisfy $\lambda > \mu$. Applying Lemma 2, we may find an eigenvector $\mathbf{v}$ associated to λ and an eigenvector $\mathbf{w}$ associated to μ. If we can show that $\mathbf{v}$ and $\mathbf{w}$ are linearly independent then the matrix $P = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ reduces A to the first real canonical form.

Now, suppose for a contradiction that $\mathbf{v}$ and $\mathbf{w}$ were linearly dependent. Then we can find constants q, p, not both zero, so that

$$q\mathbf{v} + p\mathbf{w} = 0.$$

Without loss of generality, assume that $q \neq 0$. Applying A to this expression and recalling that $\mathbf{v}$, $\mathbf{w}$ are eigenvalues yields

$$\lambda q\mathbf{v} + \mu p\mathbf{w} = 0.$$

Using that $p\mathbf{w} = -q\mathbf{v}$, we then obtain

$$(\lambda - \mu)q\mathbf{w} = 0,$$

which is a contradiction as $\lambda \neq \mu$, $q \neq 0$, and $\mathbf{w} \neq 0$.

Case 2: $\tau^2 = 4\Delta$. Here we have a repeated eigenvalue $\sigma = \frac{1}{2}\tau$.

If there exists two linearly independent vectors $\mathbf{v}, \mathbf{w}$ associated to σ then, as any vector can be written as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ we have $A\mathbf{x} = \sigma\mathbf{x}$ for all vectors $\mathbf{x}$. Consequently, $A = \sigma I$ and we may take $P = I$ to reduce to the first canonical form with $\lambda = \mu = \sigma$.

If $A \neq \sigma I$ then, as in the proof of Lemma 2 we may write

$$A - \sigma I = \begin{bmatrix} x & y \\ z & -x \end{bmatrix},$$

where we have used that $\sigma = \frac{1}{2}\tau$ to simplify the diagonal entries. Moreover, as the determinant

$$0 = \det(A - \sigma I) = -x^2 - yz,$$

and $A - \sigma I \neq 0$, none of x, y, z can be zero. As the columns are linearly dependent, there exist constants q, p, not both zero, so that

$$q \begin{bmatrix} x \\ z \end{bmatrix} + p \begin{bmatrix} y \\ -x \end{bmatrix} = 0.$$

As none of x, y, z vanish, we must have that $q, p \neq 0$. As before, we may take our eigenvector to be $\mathbf{v} = \begin{bmatrix} q \\ p \end{bmatrix}$. We may then take our generalized eigenvector, which solves $(A - \sigma I)\mathbf{w} = \mathbf{v}$ to be

$$\mathbf{w} = \begin{bmatrix} y \\ -x + \frac{q}{y} \end{bmatrix}.$$

Taking $P = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$, we may readily verify that

$$\det P = \frac{qp}{x} \neq 0,$$

so P is indeed invertible. It is then straightforward to verify that conjugation by P reduces A to the second real canonical form.

Case 3: $\tau^2 < 4\Delta$. Taking $\alpha = \frac{1}{2}\tau$ and $\beta = \frac{1}{2}\sqrt{4\Delta - \tau^2}$, we see that the two eigenvalues are $\alpha \pm i\beta$. Applying Lemma 2, we may find an eigenvector $\mathbf{v} + i\mathbf{w}$ associated to the eigenvalue $\alpha + i\beta$, where both $\mathbf{v}, \mathbf{w}$ are real vectors. Note that, as A is real-valued, the eigenvector $\mathbf{v} - i\mathbf{w}$ is associated to the eigenvector $\alpha - i\beta$.

Arguing precisely as in Case 1, we may show that the vectors $\mathbf{v} \pm i\mathbf{w}$ are linearly independent. Consequently,

$$\mathbf{v} = \tfrac{1}{2}(\mathbf{v} + i\mathbf{w}) + \tfrac{1}{2}(\mathbf{v} - i\mathbf{w}) \quad \text{and} \quad \mathbf{w} = \tfrac{1}{2i}(\mathbf{v} + i\mathbf{w}) - \tfrac{1}{2i}(\mathbf{v} - i\mathbf{w}),$$

are linearly independent. We may then take $P = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix}$ to reduce to the third real canonical form.

$\square$

We note that the proof of our theorem gives us a construction of the matrix P. We demonstrate this fact with several examples of this decomposition:

Example 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

We compute that

$$\det(A - \lambda I) = (\lambda - 5)(\lambda + 1).$$

For the eigenvalue $\lambda = 5$ we have

$$A - 5I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix},$$

so we may take the corresponding eigenvector to be $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. For the eigenvalue $\lambda = -1$ we have

$$A + I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix},$$

so we may take the corresponding eigenvector to be $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We then take the matrix

$$P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix},$$

to obtain

$$A = P \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}.$$

Example 5. Consider the matrix

$$A = \begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix}$$

We compute that

$$\det(A - \lambda I) = (\lambda + 1)^2.$$

Computing that

$$A + I = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix},$$

we see that an eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. A corresponding generalized eigenvector is then $\mathbf{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We then take

$$P = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix},$$

to obtain

$$A = P \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} P^{-1}.$$

Example 6. Consider the matrix

$$A = \begin{bmatrix} 4 & 2 \\ -5 & -2 \end{bmatrix}.$$

We compute that

$$\det(A - \lambda I) = \lambda^2 - 2\lambda + 2,$$

and hence

$$\lambda = 1 \pm i.$$

To find the eigenvector(s), we consider

$$A - (1+i)I = \begin{bmatrix} 3-i & 2 \\ -5 & -3-i \end{bmatrix},$$

so we may take $\mathbf{v} + i\mathbf{w} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Taking our matrix

$$P = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix},$$

we then have

$$A = P \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} P^{-1}.$$