Math 134 - Homework 3

1. (Strogatz Exercises from 3.1, 3.2, 3.4) For each of the following, sketch all the qualitatively different vector fields that occur as r is varied. Find the values of r at which bifurcation occurs and classify as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.
   (a) $\dot{x} = 1 + rx + x^2$
   (b) $\dot{x} = rx - \frac{x}{1+x^2}$
   (c) $\dot{x} = x - rx(1-x)$
   (d) $\dot{x} = r - \cosh x$
   (e) $\dot{x} = rx - \sinh x$
   (f) $\dot{x} = x + \tanh(rx)$
   (g) $\dot{x} = rx - 4x^3$
   (h) $\dot{x} = x(r - e^x)$
   (i) $\dot{x} = rx + \frac{x^3}{1+x^2}$

2. (Strogatz Exercise 3.4.16 + extra part) In parts (a)–(d), let $V(x)$ be the potential, in the sense that $\dot{x} = -V'(x)$. Sketch the potential as a function of r. Be sure to show all the qualitatively different cases, including bifurcation values of r.
   (a) $\dot{x} = r - x^2$
   (b) $\dot{x} = rx - x^2$
   (c) $\dot{x} = rx - x^3$
   (d) $\dot{x} = rx + x^3 - x^5$

3. (Strogatz 3.4.11) Consider the system $\dot{x} = rx - \sin x$.
   (a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.
   (b) Show that when $r = 1$, there is only one fixed point. What kind of fixed point is it?
   (c) As r decreases from ∞ to 0, classify all the bifurcations that occur.
   (d) For $0 < r \ll 1$, find an approximate formula for values of r at which bifurcations occur
   (e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.
   (f) Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

4. The following examples have 'possible bifurcation points' that turn out not to be actual bifurcations. Find them and verify that there is indeed no change in fixed point structure at these points.
   (a) $\dot{x} = r^3 - x^3$.
   (b) $\dot{x} = r^2x + x^3$.