Home Work 7

1) a) $\begin{cases} \dot{x} = x - y \\ \dot{y} = x^2 - 4 \end{cases} \Rightarrow \nabla f = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}$

$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Leftrightarrow \begin{cases} x = y = 2 \\ x = y = -2 \end{cases}$

$\Rightarrow \nabla f(2,2) = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$ & $\nabla f(-2,-2) = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$

* With $(x,y) = (2,2)$

$A = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} \Rightarrow \begin{matrix} trA = 1 = \tau \\ \det A = 4 = \Delta \end{matrix}$

Since $\tau^2 - 4\Delta = 1 - 4.4 = -15 < 0$

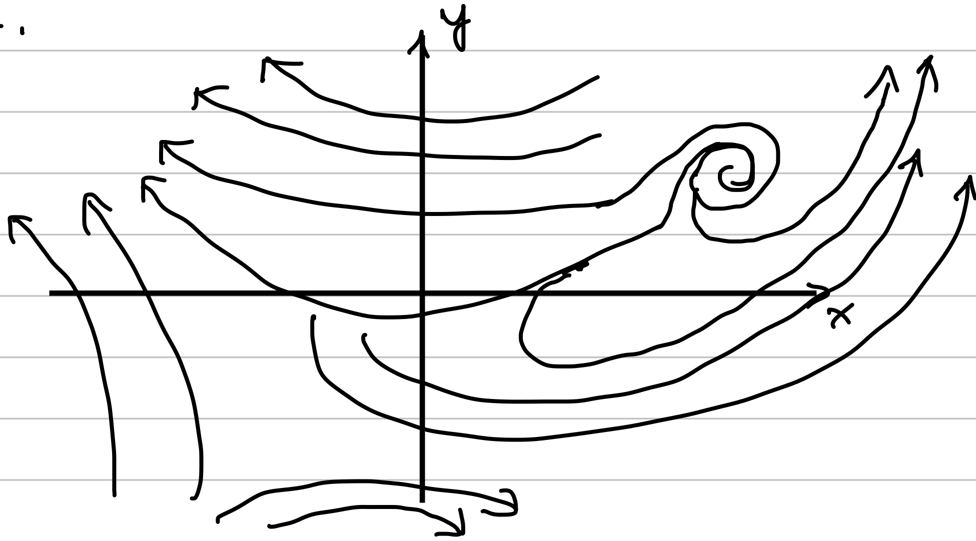
$\Rightarrow$ $\boxed{\text{fixed point } (2,2) \text{ is unstable spiral}}$

* With $(x,y) = (-2,-2)$ & $A = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix} \Rightarrow \begin{matrix} \tau = trA = 1 \\ \Delta = \det A = -4 \end{matrix}$

$\tau^2 - 4\Delta = 1 + 16 = 17 > 0$

$\Rightarrow$ $\boxed{\text{fixed point } (-2,-2) \text{ is a saddle point}}$

Phased potrait.



b) $\begin{cases} \dot{x} = y + x - x^3 = 0 \\ \dot{y} = -y = 0 \end{cases} \iff \begin{cases} x(x^2-1) = 0 \\ y = 0 \end{cases}$

$\Rightarrow (x, y) = (0,0), (1,0), (-1,0)$

$$\nabla f = \begin{bmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow \nabla f(0,0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = A \Rightarrow \tau = \text{TR}A = 0$

$\Delta = \det A = -1 < 0$

$\Rightarrow (0,0)$ is a Saddle point

$\Rightarrow \nabla f(1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} = A \Rightarrow \tau = \text{TR}A = -3$

$\Delta = \det A = 2$

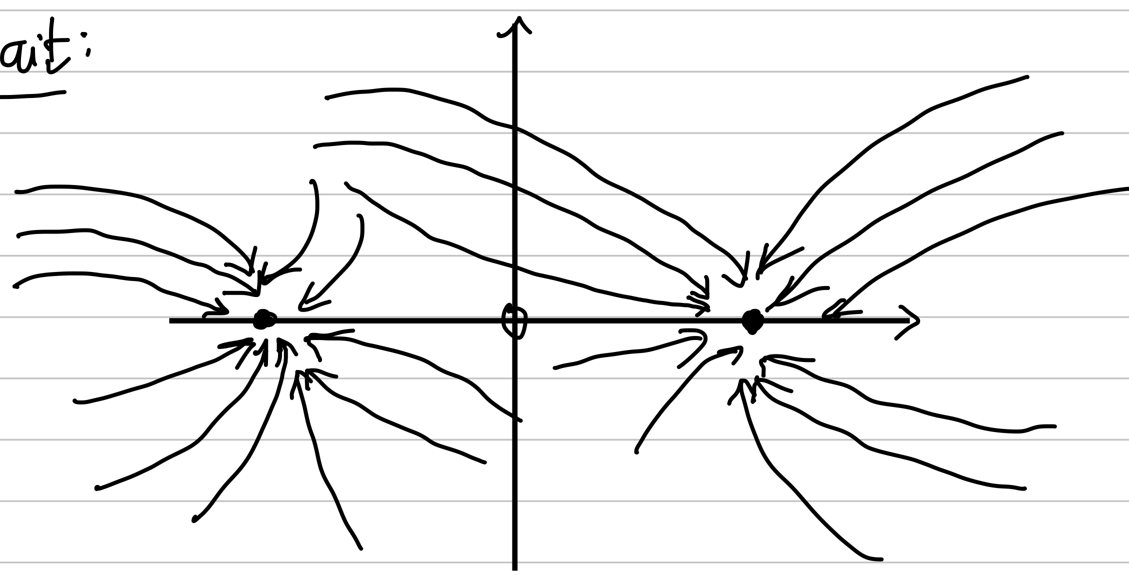
$\tau^2 - 4\Delta = 9 - 4 \times 2 = 1 > 0$

$\Rightarrow (1,0)$ is a stable point

$\Rightarrow \nabla f(-1,0) = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} = A \Rightarrow \tau^2 - 4\Delta = 1 > 0$

$\Rightarrow (-1,0)$ is a stable point

Phase potrait:



c) $\begin{cases} \dot{x} = xy - 1 = 0 \\ \dot{y} = x - y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} xy = 1 \\ x = y^3 \end{cases} \Rightarrow \begin{cases} x(y-1) = 1 - y^3 \quad (1) \\ xy = 1 \quad (2) \end{cases}$

$(1) \Leftrightarrow x(y-1) + y^3 - 1 = x(y-1) + (y-1)(y^2+y+1)$

$= (y-1)(x + y^2 + y + 1) = 0$

$\Rightarrow \left[\begin{array}{l} y = 1 \\ x + y^2 + y + 1 = 0 \end{array}\right.$

(*) When $y = 1$, (2) $xy = 1 \Rightarrow x = 1$.

$\Rightarrow (x, y) = (1, 1)$

(*) When $x + y^2 + y + 1 = 0$, (2) $xy = 1$

If $x = 0 \Rightarrow y^2 + y + 1 = 0 \Leftrightarrow \left(y + \frac{1}{2}\right)^2 + \frac{3}{4} = 0$ (rejected)

If $y = 0 \Rightarrow x = 0$ reject since $xy = 1 \neq 0$

With $x \neq 0$ & $y \neq 0$, $x = \frac{1}{y}$

$\Rightarrow \frac{1}{y} + y^2 + y + 1 = 0 \Rightarrow y^3 + y^2 + y + 1 = 0$

$\Rightarrow y^2(y+1) + y + 1 = 0 \Leftrightarrow (y+1)(y^2+1) = 0$

$\Rightarrow y = -1 \Rightarrow x = -1$

$\Rightarrow (x, y) = (-1, -1)$

Also, $\nabla f = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}$

⊛ $(x,y) = (1,1) \Rightarrow \nabla f = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$ $\begin{cases} \tau = 1-3 = -2 \\ \Delta = -4 < 0 \end{cases}$

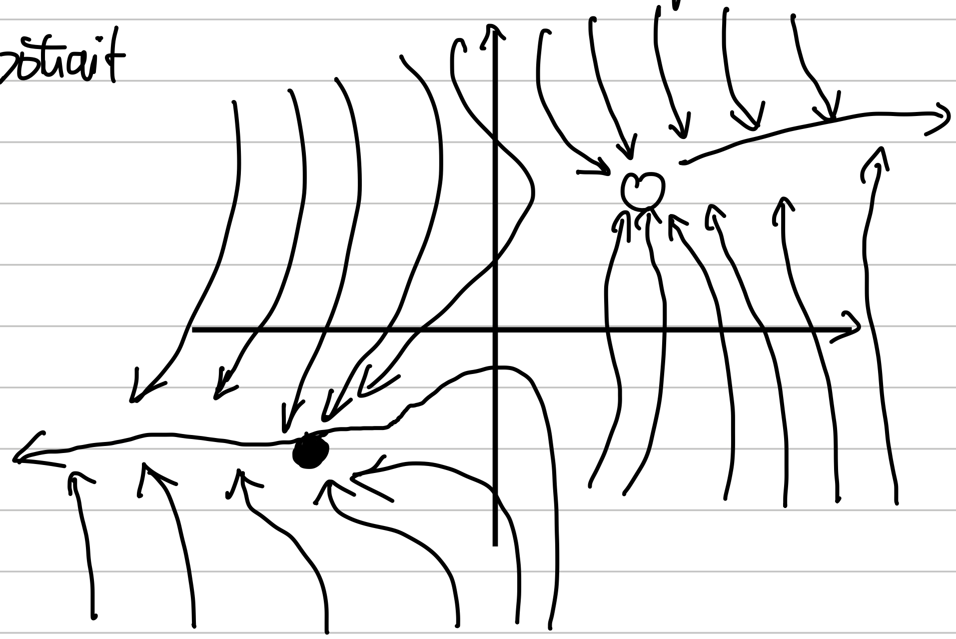
$\Rightarrow$ $(1,1)$ is a saddle point

⊛ $(x,y) = (-1,-1) \Rightarrow \nabla f = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$ $\tau = -4$
$\Delta = 3+1 = 4$

$\tau^2 - 4\Delta = 16 - 4 \cdot 4 = 0$

$\Rightarrow$ $(-1,-1)$ is a stable degenerate point

⊛ Phase portrait



2) $\begin{cases} \dot{x} = y^3 - 4x = 0 \\ \dot{y} = y^3 - y - 3x = 0 \end{cases} \Leftrightarrow \begin{cases} y^3 = 4x \\ y^3 - y - 3x = 0 \end{cases}$

a)

$\Leftrightarrow \begin{cases} \dfrac{y^3}{4} = x & ① \\ \dfrac{y^3 - y}{3} = x & ② \end{cases}$ Based on ① & ②, $\dfrac{y^3}{4} = \dfrac{y^3 - y}{3}$

$\Rightarrow 3y^3 = 4y^3 - 4y$

$\Rightarrow y^3 - 4y = 0 \Leftrightarrow y(y^2 - 4) = 0 \Leftrightarrow y(y-2)(y+2) = 0$

$\Rightarrow \begin{cases} y = 0 \Rightarrow x = 0 \\ y = 2 \Rightarrow x = 2 \\ y = -2 \Rightarrow x = -2 \end{cases}$ $\Rightarrow$ fixed point $(x,y) = (0,0), (2,2), (-2,-2)$

$\nabla f = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2 - 1 \end{bmatrix}$

With $(x,y) = (0,0) \Rightarrow \nabla f(0,0) = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix}$

$\tau = \text{TR}A = -5$
$\Delta = \det A = 4$ $\Rightarrow \tau^2 - 4\Delta = 25 - 4.4 = 9 > 0$

$\Rightarrow (0,0)$ is a stable point

With $(x,y) = (2,2) \Rightarrow \nabla f(2,2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}$

$\Rightarrow \begin{cases} \tau = \text{TR}A = 7 \\ \Delta = \det A = -44 + 36 = -8 < 0 \end{cases} \Rightarrow (2,2)$ is a saddle point

With $(x,y) = (-2,-2) \Rightarrow \nabla f(-2,-2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix}$

Similarly, $(x,y) = (-2,-2)$ is also a saddle point

b) $x = y \Rightarrow \dot{x} = y^3 - 4x = x^3 - 4x = x^3 - 4y$

$= x^3 - 3y - y = x^3 - y - 3y = x^3 - y - 3x = y^3 - y - 3x$

Also $\dot{y} = y^3 - y - 3x$

$\Rightarrow \dot{x} = \dot{y} \Rightarrow x(0) = y(0) \Rightarrow x(t) = y(t)$

or $x = y$ is invariant

c) with $u = x - y \Rightarrow \dot{u} = \dot{x} - \dot{y} = y^3 - 4x - (y^3 - y - 3x)$

$= -x + y = -u \Rightarrow \dot{u} + u = 0 \Rightarrow \frac{du}{dt} = -u$

$\Rightarrow \frac{du}{u} = -dt \Rightarrow \int \frac{du}{u} = \int - dt = -t + C$

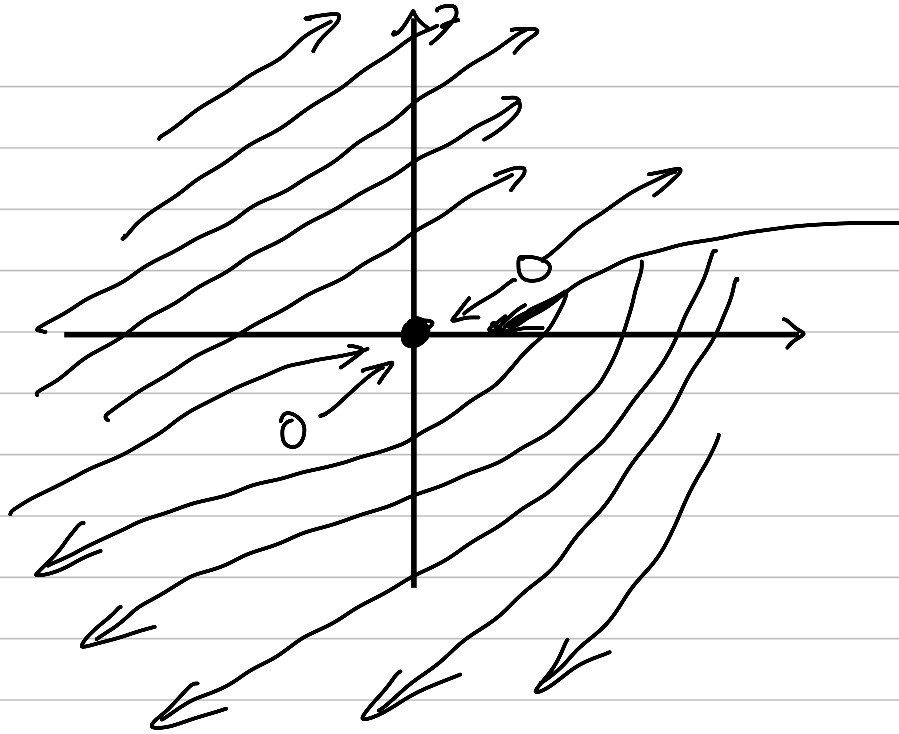
$\Rightarrow \ln u = -t + C \Rightarrow u = Ce^{-t}$

Also, $|x(t) - y(t)| = |u(t)| = |Ce^{-t}|$

$\Rightarrow \lim_{t \to \infty} |x(t) - y(t)| = \lim_{t \to \infty} |Ce^{-t}| = 0$

$\Rightarrow |x(t) - y(t)| \to 0$ as $t \to \infty$

d) Phase potrait:



3) Let $\frac{y}{x} = u$, $\theta = \tan^{-1}(u)$

$$\Rightarrow \dot{\theta} = \frac{\partial \theta}{\partial u} \cdot \frac{\partial u}{\partial t} = \frac{1}{1+u^2} \cdot \frac{\partial u}{\partial t} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\dot{y}x - \dot{x}y}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{x\dot{y} - y\dot{x}}{x^2} = \frac{x\dot{y} - y\dot{x}}{x^2+y^2} = \frac{x\dot{y} - y\dot{x}}{r^2}$$

(Since Let $x^2+y^2 = r^2$ & $x = r\sin\theta$, $y = r\cos\theta$)

4) $\begin{cases} \dot{x} = -y - x^3 = 0 \\ \dot{y} = x = 0 \end{cases} \Rightarrow \begin{cases} y = -x^3 \\ x = 0 \end{cases} \Leftrightarrow (x,y) = (0,0)$

$\nabla f = \begin{bmatrix} -3x^2 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \nabla f(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{cases} \tau = 0 \\ \Delta = 1 \end{cases}$

Since $\tau^2 - 4\Delta = 0 - 4 = -4 < 0$

$\Rightarrow$ fixed point $(x,y) = (0,0)$ is a stable center following to the linearization

Besides, $r^2 = x^2 + y^2 \Rightarrow \frac{dr^2}{dt} = \frac{d(x^2+y^2)}{dt}$

$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y} = 2x(-y-x^3) + 2yx = -2x^4$

$\Rightarrow r\dot{r} = -x^4 \Rightarrow \dot{r} = \frac{-x^4}{r} = \frac{-x^4}{\sqrt{x^2+y^2}} < 0$

$\Rightarrow$ r is a non-increasing function $\Rightarrow$ $(0,0)$ is a only fixed point

$\Rightarrow$ $(0,0)$ is a spiral $\Rightarrow$ the original $(0,0)$ is a spiral even the linearization predict a center

5) $\begin{cases} \dot{x} = -y + ax^3 \\ \dot{y} = x + ay^3 \end{cases} \Rightarrow \nabla f(x,y) = \begin{bmatrix} 3ax^2 & -1 \\ 1 & 3ay^2 \end{bmatrix}$

$\Rightarrow \begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{cases} ax^3 = y \\ ay^3 = -x \end{cases} \Rightarrow (x,y) = (0,0)$ is a fixed point

$\nabla f(0,0) = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A}$ $\left.\begin{array}{l} T = \mathrm{Tr} A = 0 \\ \Delta = \det A = 1 \end{array}\right\} \Rightarrow \lambda = \dfrac{\pm 2j}{2} = \pm j$

$\Rightarrow \alpha = 0,\ \beta = 1 \Rightarrow (0,0)$ is a origin stable center

We have $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\Rightarrow \begin{cases} \dot{x} = -r\sin(\theta) + a(r\cos\theta)^3 \\ \dot{y} = r\cos\theta + a(r\sin\theta)^3 \end{cases} \Rightarrow \begin{array}{l} \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} = -r\sin\theta + ar^3\cos^3\theta \quad (1) \\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta} = r\cos\theta + ar^3\sin^3\theta \quad (2) \end{array}$

$\Rightarrow \dot{r} = ar^3(\sin^4\theta + \cos^4\theta)$ (adding (1) & (2))

Also, $\dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{r^2} = \dfrac{x(x + ay^3) - y(ax^3 - y)}{r^2}$

$= \dfrac{x^2 + y^2 + axy^3 - ayx^3}{r^2} = \underbrace{\dfrac{x^2 + y^2}{r^2}}_{1} + \dfrac{axy^3 - ayx^3}{r^2}$

$= 1 + \dfrac{ar\cos\theta\, r^3\sin^3\theta - ar\sin\theta\, r^3\cos^3\theta}{r^2}$

$$= 1 + \frac{a r^4 \cos\theta \sin^3\theta - a r^4 \sin\theta \cos^3\theta}{r^2}$$

$$= 1 + \frac{a r^4 \cos\theta \sin\theta (\sin^2\theta - \cos^2\theta)}{r^2}$$

$$= 1 + a r^2 \cos\theta \sin\theta (\sin^2\theta - \cos^2\theta)$$

Since $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$= 1 - a r^2 \cos\theta \sin\theta \cos 2\theta$$

$$= 1 - \frac{a r^2 \cdot 4 \cos\theta \cdot \sin\theta \cdot \cos 2\theta}{4} \quad , \quad 2\sin\theta\cos\theta = \sin 2\theta, \quad 2\sin 2\theta \cdot \cos 2\theta = \sin 4\theta$$

$$= 1 - \frac{a r^2 \sin 4\theta}{4}$$

$\Rightarrow$ We have:

$$\begin{cases} \dot{r} = a r^3 (\sin^4\theta + \cos^4\theta) \\ \dot{\theta} = 1 - \dfrac{a r^2 \sin 4\theta}{4} \end{cases}$$

If $a>0$, then $r(t) \to 0$ as $t \to \infty$ $\Rightarrow (0,0)$ is a stable spiral

If $a=0$, then $r(t) = r_0$ for all t and $(0,0)$ is a center

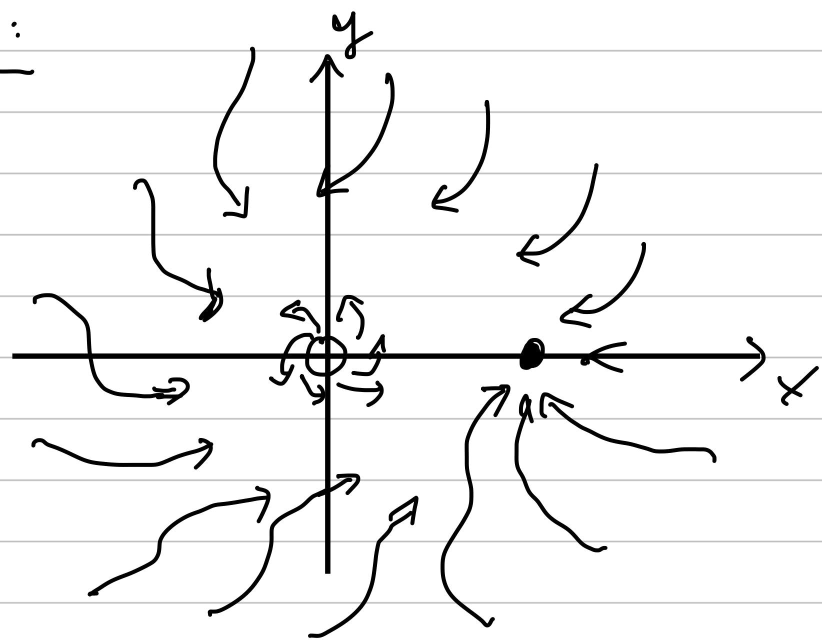
If $a>0$, then $r(t) \to \infty$ as $t \to \infty$ $\Rightarrow (0,0)$ is unstable spiral.

6) $\begin{cases} \dot{r} = r(1-r^2) = 0 \\ \dot{\theta} = 1 - \cos\theta = 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} r = 0 \\ r = 1 \\ r = -1 \text{ (rejected since } r \geq 0) \end{array}\right. \\ \cos\theta = 1 \end{cases}$

Because $\cos\theta = 1 \Rightarrow \theta = 2k\pi \quad k = 0, 1, 2, \dots$

$\Rightarrow$ fixed point $(r, \theta) = (0, 2k\pi), (1, 2k\pi), k = 0, 1, 2, \dots$

Phase portrait:



Based on phase potrait, we can see $(1, 0)$ is not Liapunov stable, but it is attracting