Discussion 2

Wednesday, April 6, 2022 10:41 PM

Topics
1. Big O / Little O
2. Fixed point and Potential Functions
3. Homework

Big O / Little O

Big O – $f(x) = O(g(x))$ as $x \to x_0$ if $\exists C$ such that $|f(x)| \le |C \cdot g(x)|$ when x is near x_0

Little O – $f(x) = o(g(x))$ as $x \to x_0$ if $\frac{f(x)}{g(x)} \to 0$ as $x \to x_0$

Examples:

- $10x^2 = O(x^2)$ as $x \to 0$ (and $x \to \infty$) because $|10x^2| \le C \cdot |x^2|$ ($C = 10, 20, \dots$)
- $10x^2 = O(x)$ as $x \to 0$ because $|10x^2| \le 10 \cdot |x|$ if $x \in [-1,1]$
- $10x^2 = O(x^3)$ as $x \to \infty$ (not $x \to 0$) because $|10x^2| < |10x^3|$ when $x > 1$
- $10x^2 \ne o(x^2)$ as $x \to 0$ (or $x \to \infty$) because $\lim_{x\to 0} \frac{10x^2}{x^2} = 10 \ne 0$
- $10x^2 = o(x)$ as $x \to 0$ because $\lim_{x\to 0} \frac{10x^2}{x} = 10x = 0$
- $10x^2 = o(x^3)$ as $x \to \infty$ because $\lim_{x\to\infty} \frac{10x^2}{x^3} = \frac{10}{x} = 0$
- $\ln(x) = o(x^p)$ as $x \to \infty$ ($p > 0$) because $\lim_{x\to\infty} \frac{\ln(x)}{x^p} \overset{L'Hopital's\ Rule}{=} \lim_{x\to\infty} \frac{x^{-1}}{p \cdot x^{p-1}} = \lim_{x\to\infty} \frac{1}{p} \cdot \frac{1}{x^p} = 0$

- Applied to Taylor Series

Ex: "Show $e^x = 1 + x + \frac{x^2}{2} + O(x^3)$ as $x \to 0$"

Apply Taylor's Theorem to e^x about 0 on $[-1,1]$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{f'''(z)}{3!}x^3 \quad \text{where } z \in [0,x] \text{ or } [-x, 0]$$

If $z \in [-1,1]$, then $f'''(z) \in [e^{-1}, e^1]$. So, $\left| \frac{f'''(z)}{3!} \cdot x^3 \right| \le \frac{e}{3!} \cdot x^3$, and thus

$$e^x = 1 + x + \frac{x^2}{2!} + O(x^3)$$

Homework Notes

3. Increase by factor of k: Suppose we have (t^*, x^*) s.t. $x^* = e^{a \cdot t^*}$.
   How long does it take to get to (kx^*)? $kx^* = e^{a \cdot t}$ t?

6. $m\dot{v} = mg - kv^2$
   - Separation of Variables
   - Partial Fractions

$$\frac{m\,dv}{mg - kv^2} = dt$$

Factor $mg - kv^2 = (\sqrt{mg} - \sqrt{k}v)(\sqrt{mg} + \sqrt{k} \cdot v)$

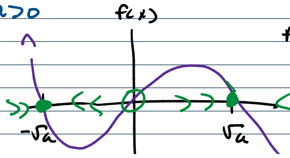
Fixed Points and Potentials

- Suppose we have $\dot{x} = f(x)$.

Fixed point – x s.t. $f(x) = 0$
Stable fixed point – Nearby trajectories converge (Minima of Potential functions)
Unstable fixed point – Nearby trajectories diverge (Maxima of Potential functions)
Half Stable – Stable on one side, unstable on other
Potential – If $\dot{x} = f(x)$, V s.t. $V' = -f(x)$.

Ex: Find and classify the fixed points of $f(x) = ax - x^3$ where $a > 0$, $a < 0$, or $a = 0$ and sketch a potential function for each case.

$a > 0$

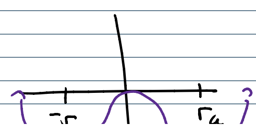


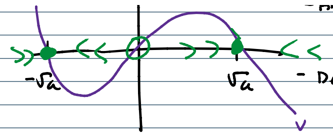
$f(x) = x(a - x^2)$
- Find Roots
- Determine Sign on each interval

$\longrightarrow$

$V(x) = -\left(\frac{a}{2}x^2 - \frac{x^4}{4}\right)$

$V(\pm\sqrt{a}) = -\frac{a^2}{4}$





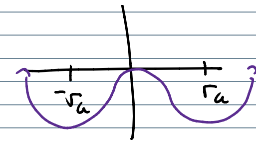
- the roots
- Determine sign on each interval

$\longrightarrow$

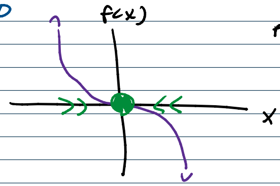
$V(x) = -(\frac{a}{2}x^2 - \frac{x^4}{4})$

$V(\pm\sqrt{a}) = -\frac{a^2}{4}$

$V(0) = 0$



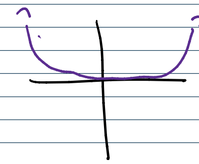
$a = 0$



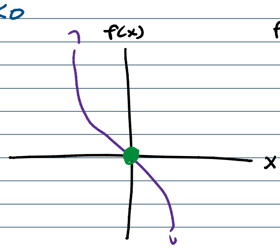
$f(x) = -x^3$

$\longrightarrow$

$V(x) = -\frac{1}{4}x^4$



$a < 0$

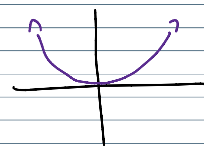


$f(x) = x(a - x^2)$

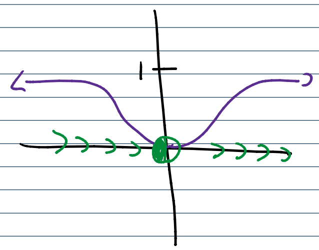
$- \; a - x^2 < 0$

$\longrightarrow$

$V(x) = -\left(\frac{a}{2}x^2 - \frac{x^4}{4}\right)$



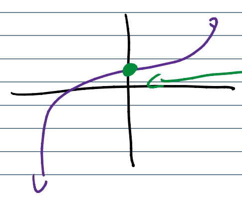
Ex: $\dot{x} = 1 - e^{-x^2}$



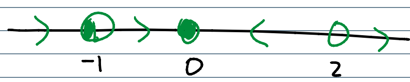
$e^{-x^2} = 1$ iff $x = 0$

$0 \leq e^{-x^2} \leq 1$ since $-x^2 \leq 0$

Note: $\int e^{-x^2}$ does not have closed form solution



Ex: "Find an equation ($\dot{x} = f(x)$) for the following phase portrait:"



$f(x) = (x+1)^2 \cdot x \cdot (x-2)$

Roots: $-1, 0, 2$

Look for something like $(x+1) \cdot x \cdot (x-2)$

Half-stable FP at $x = -1 \rightarrow x = -1$ is a double root

Check signs to make sure $f > 0$ when x is large