8 Autonomous first order ODE

There are different ways to approach differential equations. Prior to this lecture we mostly dealt with *analytical* methods, i.e., with methods that require a formula as a final answer. Another possible approach is the *numerical* methods, in this case we approximate our unknown solution with a solution of a related problem, which can be easily (in principle) solved, in most cases on a computer. However, the most fruitful way to tackle ODE problem is, arguable, a *qualitative* one, when we infer properties of solutions without actually solving the equation. This method was in the mainstream of modern mathematics for the last hundred years or so and produced a wealth of remarkable results. We are going only to touch on this method (you can learn much more in Math 484: Mathematical Methods in Biology).

8.1 Definitions and properties

In this lecture I will use t as an independent variable and $x = x(t)$ as the dependent variable. This convention comes from mechanics, where t usually refers to time and x to the displacement of a particle. The derivative, again following the classical mechanics, will be denoted as $\dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}$.

Definition 1. *First order ODE $\dot{x} = f(t, x)$ is called autonomous if the right hand side does not depend explicitly on t:*

$$\dot{x} = f(x). \tag{1}$$

We start with an example, and then generalize the properties deduced in this example to all autonomous equations.

Example 2. Consider the simplest autonomous equation

$$\dot{x} = x,$$

which is a separable equation, and whose solution is

$$x(t) = Ce^t,$$

where C is an arbitrary constant. If we had the initial condition $x(t_0) = x_0$, then our solution would be

$$x(t; x_0) = x_0 e^{t-t_0}.$$

Having the explicit formula for the solution it is very simple to sketch several integral curves (recall that an integral curve is the graph of a solution) and the direction field of this equation (see the figure below). From the figure it becomes obvious that the direction field of the equation in the example, as well as direction field of any autonomous equation, has the property that it is the same on any line parallel to t axis. Hence we can project the whole direction field onto the x-axis, without losing much information (I put an arrow that points in the positive direction if the slope is positive and an arrow that points in the negative direction if the slope is negative; what actually lost is the absolute values of the slopes). The picture on the x-axis, which consists of arrows and points, is called the *phase portrait* and the x axis itself is called the *phase line* or *phase space* (again, the terminology originated in mechanics). Note that for $x = 0$ the slope neither positive nor negative and hence I mark this point on the phase portrait with a bold dot.

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Fall 2013

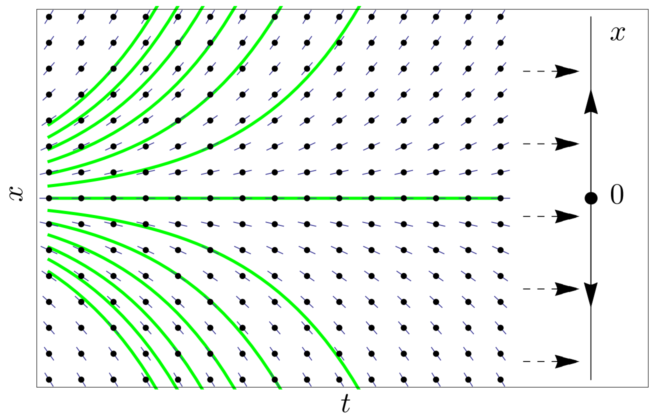

Figure 1: The direction field of $\dot{x} = x$ together with its phase portrait. The dashed arrows show the projection of the direction field onto the x-axis. The phase line here consists of three orbits: the one where the arrow points in the negative direction ($x \in (-\infty, 0)$), the origin, where the slope of the direction field is zero ($x = 0$), and the one where the arrow points in the positive direction ($x \in (0, \infty)$)

Can we come up with the phase portrait without looking at the direction field, which was known to us to the simplicity of the original equation? The answer is "yes." Consider the following figure. Here we look at the function $f(x)$, which is simply x in our case. Note that if $x > 0$ then $f(x) > 0$ hence

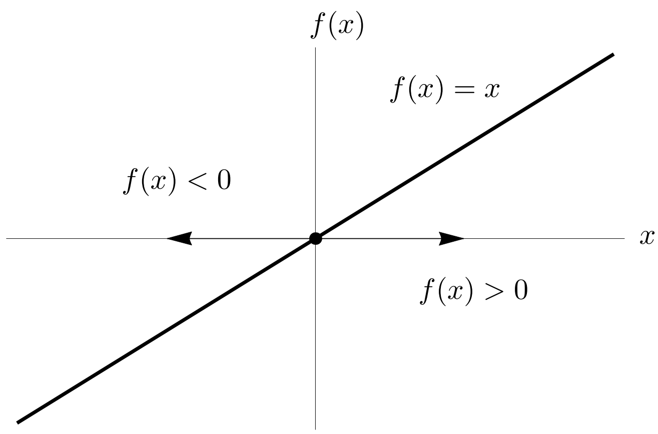

Figure 2: The phase portrait of $\dot{x} = x$

$\dot{x} > 0$ hence the solutions are increasing. Similarly, if $x < 0$, then $\dot{x} < 0$ and solutions are decreasing. I indicate these facts with arrows in the graph and obtain again the same phase portrait that we already saw in the previous figure. Hence the conclusion: We do not need actual solutions to the autonomous differential equation to figure out the phase portrait of this equation. And knowledge of the phase portrait allows to infer the asymptotic behavior of the solutions ("asymptotic" in this context means for $t \to \infty$). In our example we see that if the initial condition $x_0 > 0$ then $x(t; x_0) \to \infty$ for $t \to \infty$, and if $x_0 < 0$ then $x(t; x_0) \to -\infty$. Here and below the notation $x(t; x_0)$ means the solution to ODE with the initial condition x_0.

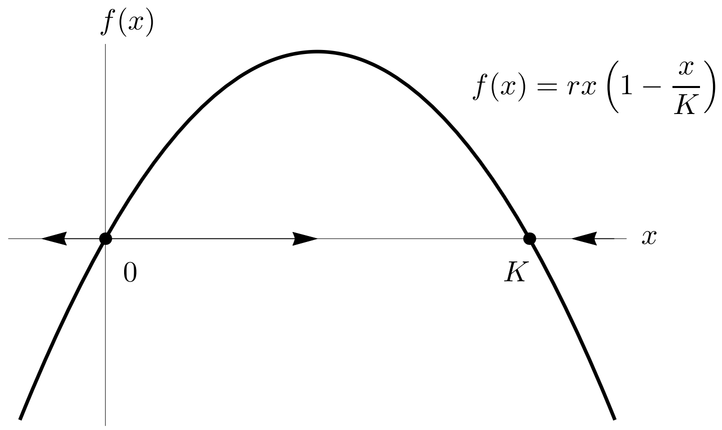

Figure 3: The phase portrait of $\dot{x} = rx(1 - x/K)$

Example 3. Here is another example: the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right), \quad r, K > 0.$$

Here r, K are positive parameters. We actually solved this equation before and studies its solutions. Here we will sketch several integral curves by studying its phase portrait. The graph of $f(x) = rx(1 - x/K)$ is a parabola with branches pointing down, which crosses x-axis at the points $\hat{x}_1 = 0$ and $\hat{x}_2 = K$ (see the figure).

We can see that $f(x)$ is negative when $x < 0$ and $x > K$ and positive for $x \in (0, K)$, hence the directions of the arrows. This means that the integral curves are increasing for $x \in (0, K)$ and degreasing for $x < 0$ and $x > K$. If $x = 0$ or $x = K$ we have that $f(x) = 0$, and hence the slope of the integral curves here is zero, these points in the phase correspond to the integral curves parallel t-axis. Having just this information we can sketch several integral curves (see the figure below). You should compare it with the previous figure.

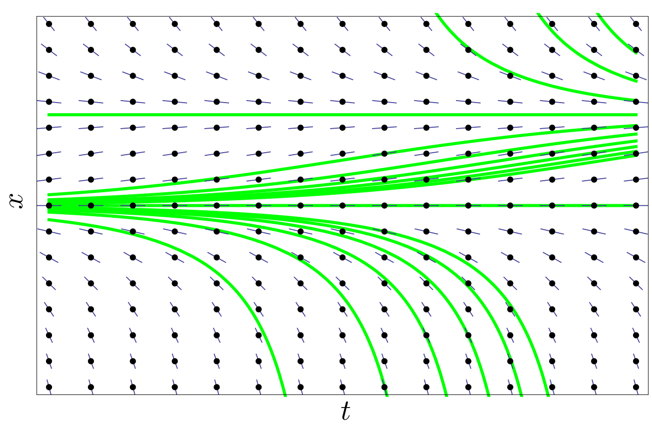

Figure 4: The direction field of the logistic equation $\dot{x} = rx(1 - x/K)$ and several integral curves (green). The two horizontal integral curves are $x = 0$ and $x = K$

Now we are in the position to formulate general properties of the autonomous ODE.

- The fact that the direction field of the autonomous ODE has the same slopes on any line parallel to t-axis can be formulated analytically as follows: If $x(t)$ solves the problem (1) then $x(t+c)$, where c is any constant, also a solution. This means that if we know the solution with the initial condition $x(0) = x_0$, then any solution with the initial condition $x(t_0) = x_0$ can be obtained by translation. This is why we can write $x(t; x_0)$ without specifying the time moment at which the initial condition is prescribed.

- The solutions to the autonomous equation are monotone functions. In particular, the first order autonomous equations cannot have periodic solutions.

- There are special and very important solutions, which can be found as the roots of $f(x) = 0$. If $\hat{x}$ is such that $f(\hat{x}) = 0$ then $\hat{x}$ is called an *equilibrium point* (or stationary point, or critical point, or rest point, or simply equilibrium, or fixed point). If $\hat{x}$ is an equilibrium, then $x(t) = \hat{x}$ is a solution to (1), which corresponds to the integral curve parallel t-axis (look at the examples above).

- The asymptotic behavior of the solutions to the autonomous ODE (1) can be inferred from the phase portrait; there are only three options: Firstly, a solution can approach an equilibrium, secondly, a solution can be an equilibrium itself, and finally, a solution can tend to either plus or minus infinity.

- The last point can be rephrased in the following form: the phase portrait is a union of equilibrium points and *orbits* (intervals of $\mathbf{R}$ with specific directions).

If we look again at the example with the logistic equation, we can see that there are two equilibria: $\hat{x}_1 = 0$ and $\hat{x}_2 = K$, but the behavior of the orbits around these points is manifestly different: the point $\hat{x}_1$ repels orbits, whereas $\hat{x}_2$ attracts orbits (look at the directions of the arrows). The mathematical formalization that distinguishes these points is the notion of stability.

Definition 4. *Equilibrium $\hat{x}$ of the autonomous first order ODE* (1) *is Lyapunov stable if for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for any initial condition x_0 satisfying*

$$|x_0 - \hat{x}| < \delta,$$

it follows that

$$|x(t; x_0) - \hat{x}| < \varepsilon.$$

If, additionally,

$$|x(t; x_0) - \hat{x}| \to 0, \quad t \to \infty,$$

then $\hat{x}$ is called asymptotically stable.

Otherwise, $\hat{x}$ is called unstable.

This definition is quite general. For the first order equations we mostly will meet asymptotically stable and unstable equilibria (can you think of an example of a Lyapunov stable equilibrium?). This definition is difficult to apply for concrete examples, since it involves actual solutions to (1). However, even perfunctory inspection of the phase portrait of the logistic equation should bring you the idea of a simple analytical test for stability.

Proposition 5. *Let $\hat{x}$ be an equilibrium of* (1)*. If $f'(\hat{x}) > 0$ then $\hat{x}$ is unstable; if $f'(\hat{x}) < 0$ then $\hat{x}$ is asymptotically stable.*

In this proposition $f'(\hat{x})$ means the derivative of $f(x)$ evaluated at the point $x = \hat{x}$. I leave the proof of this proposition to a mathematically inclined reader (as a hint I can say that the Taylor series and the mean value theorem can be handy here).

For example, if we consider again the logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{K}\right),$$

then we have

$$f'(x) = r\left(1 - \frac{x}{K}\right) - \frac{rx}{K}.$$

Therefore

$$f'(\hat{x}_1) = f'(0) = r > 0,$$

hence the origin is unstable, and

$$f'(\hat{x}_2) = f'(K) = -r < 0,$$

therefore $\hat{x}_2 = K$ is asymptotically stable.

A very good question to think about is what happens is $f'(\hat{x}) = 0$.

8.2 Mathematical models of harvesting

Let us assume that dynamics of a fish population in the lake is governed by the logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right), \quad r, K > 0,$$

such that in the long term run we have that the population size stabilizes at $N(t) = K$, the carrying capacity of the lake. Now assume that we would like to start harvesting the fish in the lake. We need to optimize two conditions: First, we would like to guarantee that the fish does not go extinct in the lake, and second, we would like to maximize the yield. Consider two possible strategies:

- *Fixed yield.* This means that we fix the quote for the time period (say, 500 pounds per year).
- *Proportional yield.* This means that we fix the proportion of the fish population we would like to harvest during the time period (say, 25 percent of the current population per year).

Which strategy is better?

Let us start with the fixed yield strategy. The equation that governs the dynamics of population now reads

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - Y_0, \quad r, K, Y_0 > 0,$$

where Y_0 is the yield that we plan to acquire during the time unit. We could find the equation for the equilibrium points and study their stability analytically, but the geometric picture in this case is much easier to deal with. The right hand side here is

$$f(N) = rN\left(1 - \frac{N}{K}\right) - Y_0,$$

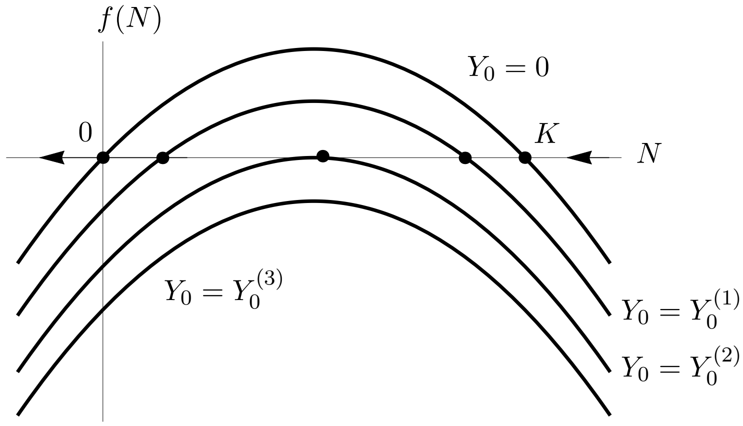

Figure 5: The phase portraits for the model with the fixed yield. $Y_0^{(1)} < Y_0^{(2)} < Y_0^{(3)}$. The maximal possible yield corresponds to $Y_0^{(2)}$, note that in this case we have only one equilibrium. For $Y_0^{(3)}$ there are no positive equilibria and the population goes extinct

whose graph is the parabola defined by $rN\left(1-\frac{N}{K}\right)$ and shifted down by Y_0. If Y_0 is small enough, then we still have two equilibria, let us call them $\hat{N}_1 > 0$ and $\hat{N}_2 < K$, such that the former is unstable and the latter one is asymptotically stable. Our task is to determine the maximum possible Y_0 in terms of the population parameters r, K. It is clear that for some Y_0 the parabola will touch N-axis and after that, for any Y_0 bigger than that, there will be no positive equilibria in the system, which corresponds to extinction (see the figure). Hence, the maximal possible yield corresponds to the moment when parabola touches N-axis, which happens when the discriminant of the quadratic polynomial

$$rN\left(1-\frac{N}{K}\right) - Y_0 = -\frac{rN^2}{K} + rN - Y_0$$

is equal to zero:

$$r^2 - 4\frac{rY_0}{K} = 0 \implies Y_0 = \frac{rK}{4}\,.$$

This is the maximal possible yield in this model.

Now let us go back to the proportional yield model. The equation reads

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - hN = N\left(r - h - \frac{rN}{K}\right).$$

We have here two equilibria

$$\hat{N}_1 = 0,$$

which is always unstable, and

$$\hat{N}_2 = \frac{K(r-h)}{r}\,,$$

which is always asymptotically stable (check!). To guarantee that $\hat{N}_2 > 0$, we must require that $h < r$. If the population at the stationary point $\hat{N}_2$, then our yield is

$$h\hat{N}_2 = \frac{Kh(r-h)}{r}\,,$$

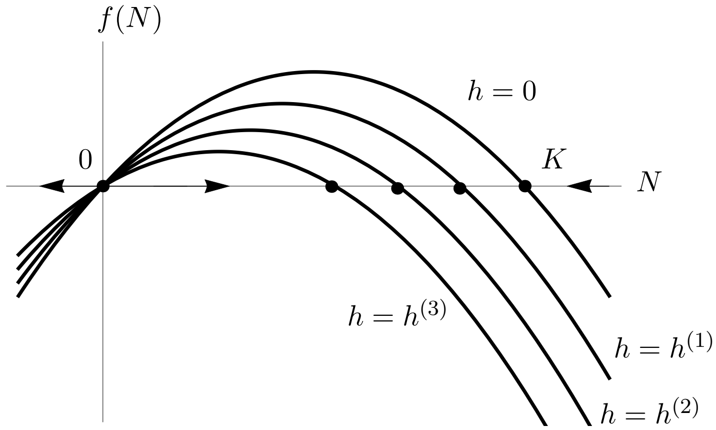

Figure 6: The phase portraits for the model with the proportional yield. $h^{(1)} < h^{(2)} < h^{(3)} < r$. Note that for any $h < r$ we still have two equilibria, the right of which is asymptotically stable

and we are free to pick any $h < r$. Let us maximize the last expression with respect to h: the maximum is attained when $h = r/2$ (prove this using Calculus), hence the maximal yield for this strategy is

$$\frac{rK}{4},$$

which is exactly the same as in the model with fixed yield. So is there any difference? The fixed yield at the maximal value leads to an inevitable catastrophe because sooner or later Y_0 will be such that there will no equilibria. The population will collapse. The mathematical term for this event here (two equilibria collide and disappear) is *bifurcation* (or even more precise the *saddle-node bifurcation*). For the model with the proportional yield there is no bifurcation since if we somehow exceed the best possible value $h = r/2$, then nothing dramatic happens, we will harvest slightly less fish (see the figure) and can restore the fish population already during the next time period.