Home Work 1

1. In 1918, Georg Duffing introduced a nonlinear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical problems, the equations reads as follows

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)\,, \quad (1)$$

where δ, α, β, and ω are constants. Show that this system can be expressed as a first order autonomous system of ODEs.

let $\begin{cases} x_1 = x \\ x_2 = x' \\ x_3 = t \end{cases} \Rightarrow \begin{cases} x_1' = x' = x_2 \\ x_2' = x'' = \gamma\cos(\omega t) - \delta x' - \alpha x - \beta x^3 \\ x_3 = t \end{cases}$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = \gamma\cos(\omega x_3) - \delta x_2 - \alpha x_1 - \beta x_1^3 \\ x_3' = 1 \end{cases}$$

Let $\begin{cases} f_1(x_1, x_2, x_3) = x_2 \\ f_2(x_1, x_2, x_3) = \gamma\cos(x_3 \omega) - \delta x_2 - \alpha x_1 - \beta x_1^3 \\ f_3(x_1, x_2, x_3) = 1 \end{cases}$

then the equation (1) can be expressed as a first order autonomous ODEs:

$$\boxed{\begin{cases} \dot{x}_1 = f_1(x_1, x_2, x_3) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = f_3(x_1, x_2, x_3) \end{cases}}$$

2. Suppose that a new short form video app, KotKit, has been launched. Write down a one-dimensional autonomous system for the growth of the app, taking $x(t)$ to be the fraction of the population that has downloaded the app at time t. Check that the fixed points and their stability match common sense.

Assume the numbers of user download the app will be increased unlimited, it means with the time is long, almost everyone will join the app. So, the growth of the app can be used by the logistic system that can be represented:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)$$ with r is the rate of the growth and k is the max number of users join the app.

$$\Leftrightarrow \dot{x} = rx\left(1 - \frac{x}{k}\right)$$ is a one-dimesional autonomous system for the growth of the app.

Then, the fixed point $\Leftrightarrow rx\left(1 - \frac{x}{k}\right) = 0 \Leftrightarrow \left[\begin{array}{l} x = 0 \\ x = k \end{array}\right.$

At $x = 0$, mean the first time the app is launched and at this time, people have not known the app so the app is not popular and need to take time to let people like and use it more than before.

However, at $x = K$, most people join the app & it becomes popular.

Based on these idea, we can have

$$\boxed{\begin{array}{l}\text{at } x=0 \text{ is an unstable fixed point} \\ \text{at } x=K \text{ is an stable fixed point}\end{array}}$$

3. Suppose that the one-dimensional autonomous ODE

$$\dot{x} = f(x),$$

has a fixed point x^* so that $a = f'(x^*) \neq 0$.

(a) Write down the linearization of the ODE about x^*.

(b) Show that the time required for the solution of the linearized equation found in part (a) to increase or decrease its value (depending on the sign of a) by a factor of $k > 0$ is a constant that depends only on a, k.

(c) The book defines the 'characteristic timescale' attendant to the fixed point x^* to be $1/|a|$. Using your answer to part (b), give an interpretation of this quantity.

a) Since x^* is a fixed point, $f(x^*) = 0$

Let $\eta = x - x^*$, then $\dot{\eta} = \dot{x} = f(x) = f(\eta + x^*)$

$$= \underbrace{f(x^*)}_{=0} + f'(x^*)\eta + \frac{1}{2} f''(x^*)\eta^2 + \dots \dots \text{(Taylor's theorem)}$$

$$= f'(x^*)\eta + O(\eta^2) \quad \text{as } \eta \to 0$$

$\Rightarrow$ the equation $\begin{cases} \dot{\eta} = f'(x^*)\eta \\ \eta(0) = \eta \end{cases}$ is the linearization of ODE about x^*

b) Solving the equation, we get the general solution:

$\eta(t) = \eta_0 e^{at}$ with T is the required time

& $\eta(0) = \eta_0$. Then, if $a > 0$, then η we will have

$k\eta_0 = e^{aT}\eta_0 \Rightarrow k = e^{aT} \Rightarrow \ln k = aT \Rightarrow T = \frac{1}{a}\ln k$

$\Rightarrow$ the solution $\boxed{\text{increase}}$ by a factor of k with $\boxed{T = \frac{1}{a}\ln k}$

If $a < 0$, then the solution $\boxed{\text{decrease}}$ by a factor of k and

$$\frac{1}{k} n_0 = e^{aT} \cdot n_0 \Rightarrow \ln \frac{1}{k} = aT \Rightarrow T = \frac{1}{a} \ln \frac{1}{k} = \boxed{\frac{-1}{a} \ln k}$$

c) From part b, we have $T = \frac{1}{|a|} \ln k$

$\Rightarrow$ the characteristic time scale attendant $\frac{1}{|a|}$ is a fraction to the time require for the solution of the linearized equation found in part a that can increase or decrease by a factor of $k > 0$.

4. Draw a phase portrait (cf. Figure 1 on p. 37 of Strogatz) for each of the following systems, including the values and stabilities of fixed points. Overlay a sketch of a potential function on each one.

(a) $\dot{x} = x(x-1)^2$

(b) $\dot{x} = 1 - |x|$

(c) $\dot{x} = \sin(3x)$

(d) $\dot{x} = \begin{cases} x \ln|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

a) $\dot{x} = x(x-1)^2 = 0 \Leftrightarrow x = 0, 1$ are the fixed points

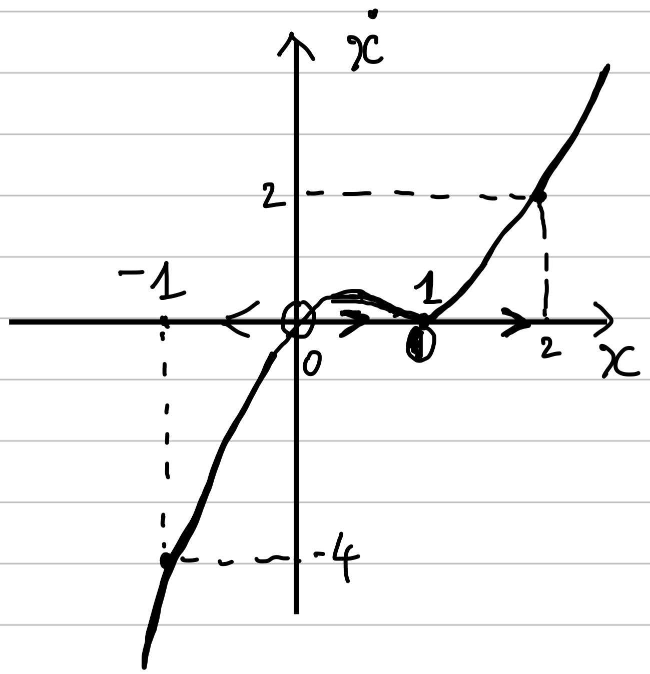
Also, $\dot{x} = x(x-1)^2 = -V'(x)$ with $V(x)$ is the potential function.

We have: $V'(x) = -x(x-1)^2 = -x(x^2 - 2x + 1)$

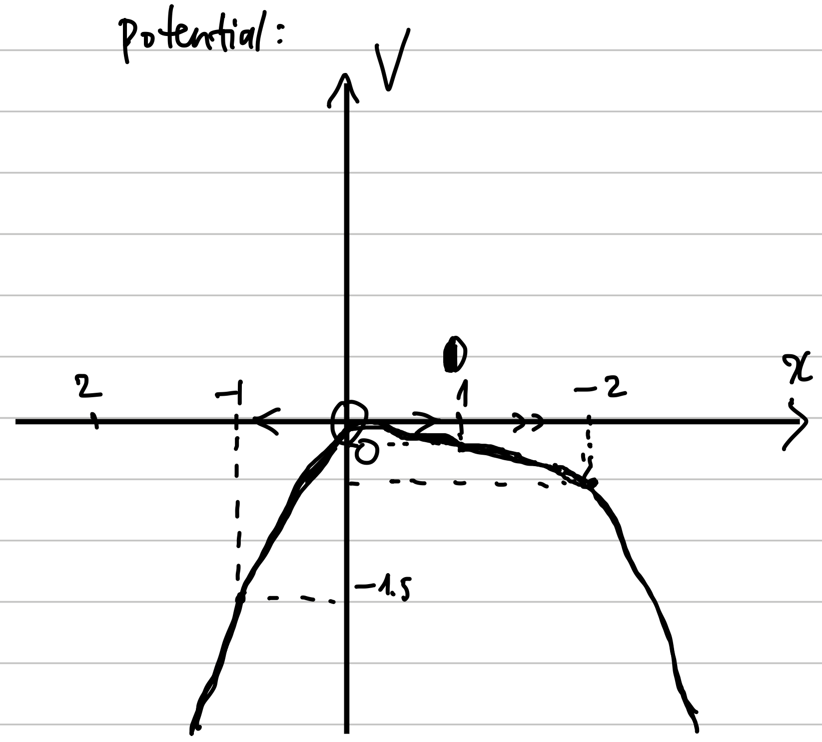
$\Rightarrow V'(x) = -x^3 + 2x^2 - x$

$\Rightarrow V(x) = -\dfrac{x^4}{4} + \dfrac{2x^3}{3} - \dfrac{x^2}{2}$, and $\dot{x} = f(x) = x^3 - 2x^2 + x$

Phase portrait



Potential:



$$\boxed{\begin{array}{l}\text{As we can see at } x=0 \text{ is unstable fixed point} \\ x=1 \text{ is a half-stable fixed point}\end{array}}$$

b) $\dot{x} = 1 - |x| = f(x) = \begin{cases} 1-x, & x \geq 0 \\ 1+x, & x < 0 \end{cases}$

$\dot{x} = f(x) = 0 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$ are fixed points

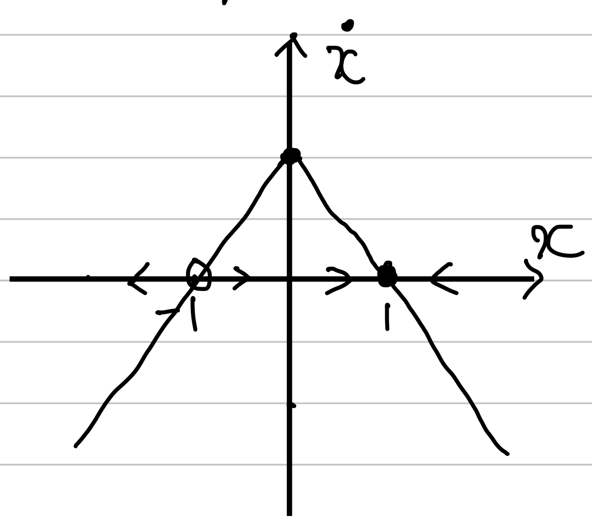
The potential function $V(x)$ with $\dot{x} = f(x) = -V'(x)$

$\Rightarrow V'(x) = -f(x) = |x| - 1$

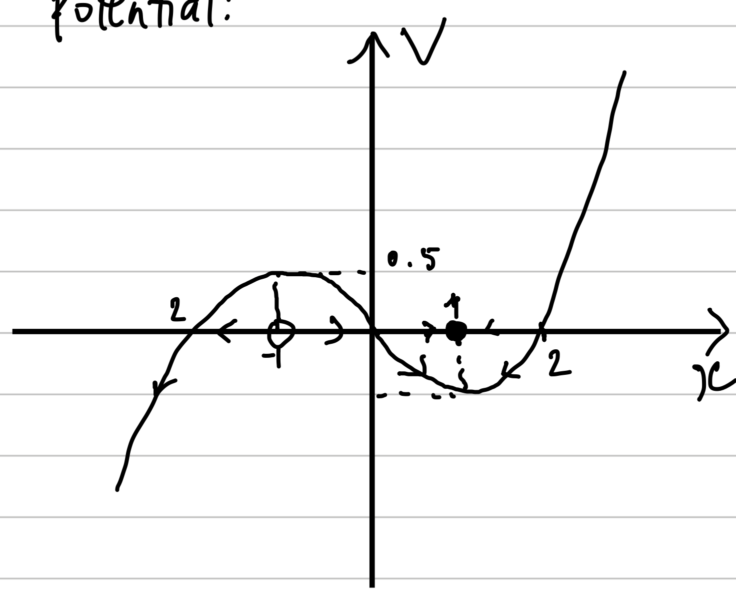
(*) With $x \geq 0$, $V'(x) = x - 1 \Rightarrow V(x) = \frac{x^2}{2} - x$

(*) With $x < 0$, $V'(x) = -x - 1 \Rightarrow V(x) = -\frac{x^2}{2} - x$

Phase portrait:



Potential:



$$\boxed{\begin{array}{l}\text{As we can see at } x=1 \text{ is a stable fixed point} \\ \text{and } x=-1 \text{ is an unstable fixed point}\end{array}}$$

c) $\dot{x} = \sin 3x = f(x)$

$\dot{x} = 0 \Rightarrow \sin 3x = 0 \Rightarrow 3x = k\pi \Rightarrow x = \frac{k\pi}{3}$

with $k \in \mathbb{Z}$ are the fixed point.

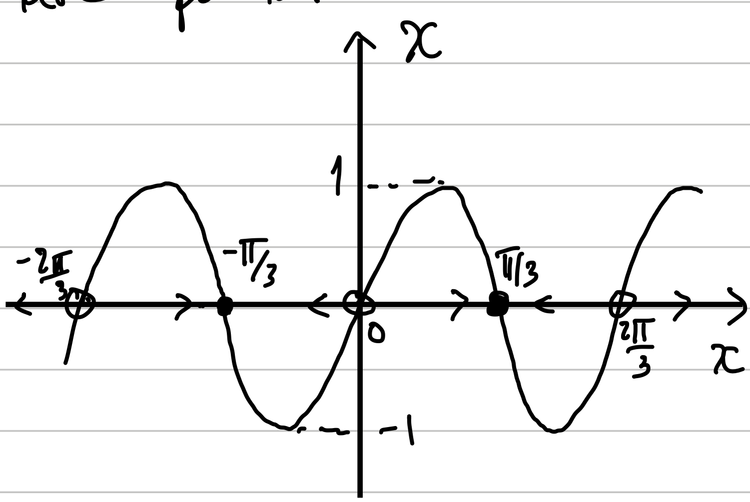
The potential function $V(x)$ such that $f(x) = -V'(x)$

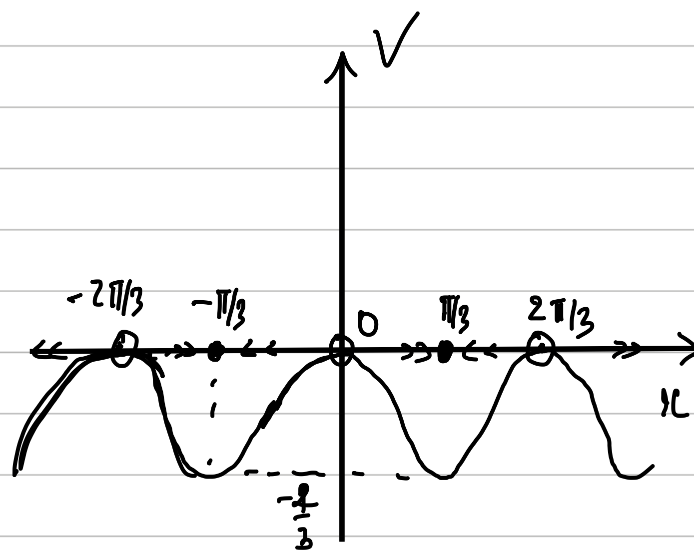
$\Rightarrow V'(x) = -f(x) = -\sin(3x)$

$\Rightarrow V(x) = -\int_0^x \sin(3y)\,dy = \frac{1}{3}\cos(3y)\Big|_0^x$

$\Rightarrow V(x) = \frac{1}{3}(\cos 3x - 1)$

Phase portrait





$$\boxed{\begin{array}{l}\text{So, at } x = \frac{k\pi}{3} \text{ with } k \text{ is odd } \& \ k \in \mathbb{Z}, \text{ they are stable fixed point} \\ \text{At } x = \frac{k\pi}{3} \text{ with } k \text{ is even } \& \ k \in \mathbb{Z}, \text{ they are unstable fixed point}\end{array}}$$

d) $\dot{x} = \begin{cases} x \ln|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

+ With $x \neq 0$, $\dot{x} = 0 \Leftrightarrow x \ln|x| = 0 \Leftrightarrow \ln|x| = 0$

$\Rightarrow |x| = 1 \Leftrightarrow x = \pm 1$

+ With $x = 0 \Rightarrow \dot{x} = 0 \Rightarrow x = 0$ is a fixed point

$\Rightarrow$ the fixed points: $x = 1, -1, 0$

The potential function $V(x)$ such that $\dot{x} = f(x) = -V'(x)$

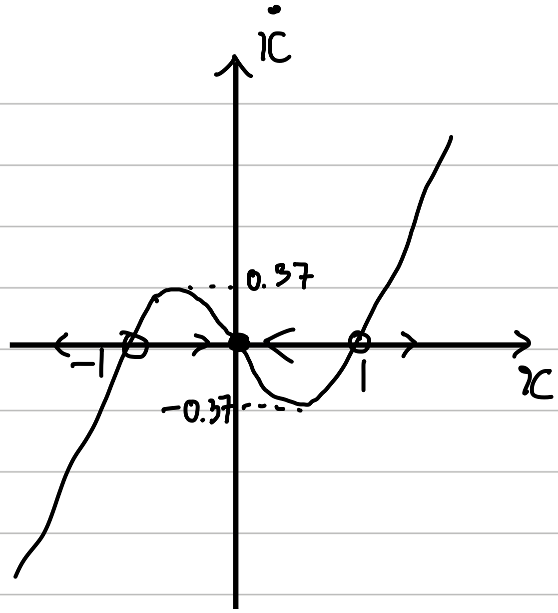
$\Rightarrow V'(x) = -f(x)$

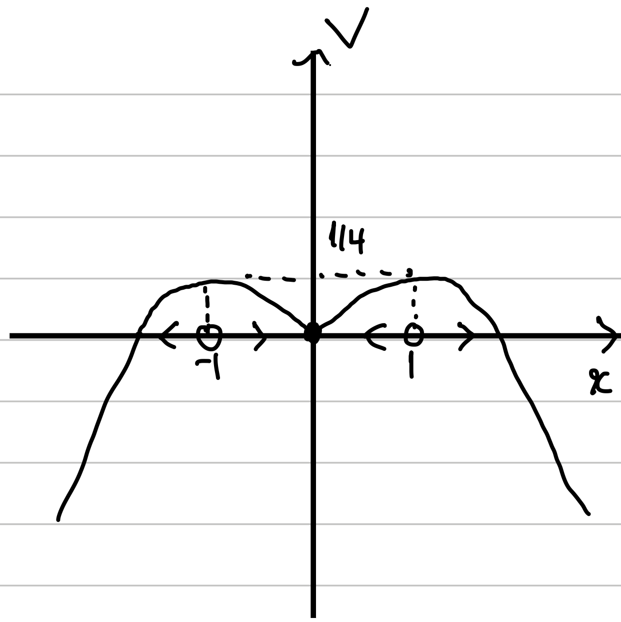
$$V(x) = -\int_0^x f(y)\,dy = -\int_0^x y \ln|y|\,dy$$

$$= -\ln|y| \cdot \frac{y^2}{2} \Big|_0^x + \int_0^x \frac{1}{y} \cdot \frac{y^2}{2}\,dy$$

$$= -\ln|y| \cdot \frac{y^2}{2} \Big|_0^x + \int_0^x \frac{y}{2}\,dy = -\ln|x| \frac{x^2}{2} + \frac{x^2}{4}$$

Phase potrait: & potential V:





At $x = 0$ is a stable fixed point

At $x = \pm 1$ are unstable fixed point

5. For each integer $k = 1, 2, 3, \ldots$ and each choice of $+$ or $-$, determine the stability of $x = 0$ as a fixed point of the equation

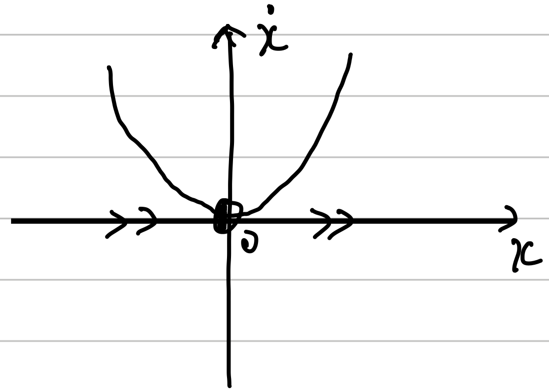
$$\dot{x} = \pm x^k$$

Restricting to the cases where $x = 0$ is stable, does making k larger result in faster or slower convergence to the fixed point? Give both a heuristic explanation and one in terms of the exact solutions. (Note that this equation is separable.)

* Chose + & k is even:

$\dot{x} = x^k$, $k = 2, 4, 6, \ldots$

With $k = 2$, $\dot{x} = x^2$, We have:

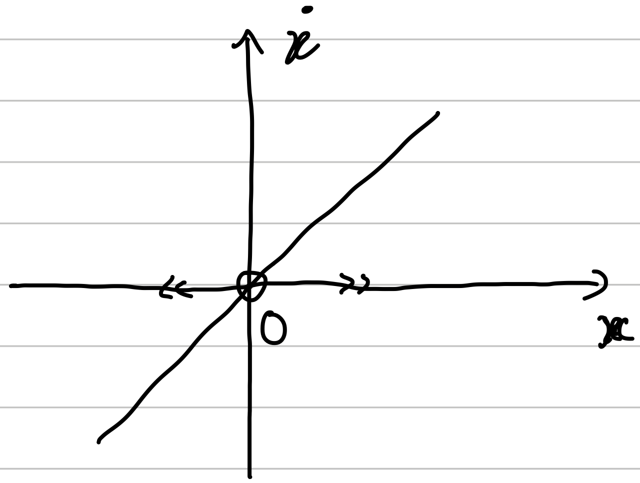


as we can see, $x = 0$ is a half-stable fixed point

* Chose + & k is odd, $k = 1, 3, 5, 7$

$\dot{x} = x^k$, chose $k = 1$, we have,

$\dot{x} = x$, then:

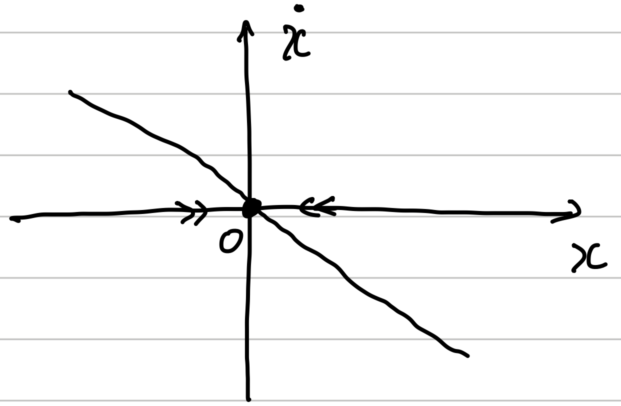


at $x = 0$ will be an unstable fixed point

* Chose $\ominus$ & k is odd

then $\dot{x} = -x^k$, $k = 1, 3, 5, 7, \ldots$

let $k = 1 \Rightarrow \dot{x} = -x$, then.

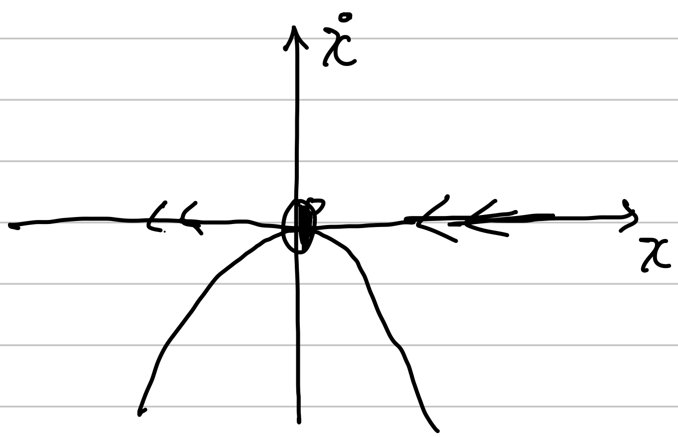


$\boxed{\text{we can see } x = 0 \text{ is a stable-fixed point.}}$

* Chose $\ominus$ & k is even

$\dot{x} = -x^k$, $k = 2, 4, 6, \ldots$

Chose $k = 2 \Rightarrow \dot{x} = -x^2$, then:



$\boxed{\text{at } x = 0, \text{ it is a half-stable fixed point}}$

Combine 4 cases, we can see that with choice $\ominus$ and k is odd, $x = 0$ is a stable fixed point. We

$$\dot{x} = -x^k \Rightarrow \frac{dx}{dt} = -x^k$$

Let a_1 & a_2 , $a_1, a_2 > 0$ & are odd numbers with $a_1 > a_2$ & $|x| < 1$, then

$\Rightarrow$ $|x|^{a_1} < |x|^{a_2}$

$\Rightarrow$ $|x|^{a_1}$ will be slower convergence than $|x|^{a_2}$

$\Rightarrow$ with k is larger will result in slower convergence to the fixed point

Exactly solutions:

with $k=1$, we have the solution

$$x(t) = Ce^{-t}$$

Also, check $k = 2n+1$ is an odd number, we have: $(n \in \mathbb{N})$

$$\dot{x} = -x^{2n+1} \Rightarrow \frac{dx}{dt} = -x^{2n+1}$$

$$\Rightarrow -\frac{dx}{x^{2n+1}} = dt \Leftrightarrow -\int \frac{dx}{x^{2n+1}} = \int dt$$

$$\Rightarrow -\int x^{-(2n+1)} dx = \int dt = t + C$$

$$\Rightarrow \frac{1}{2n} x^{-2n} = t + C$$

$$\Rightarrow x^{-2n} = 2n(t+c) = 2nt + \underbrace{2nc}_{C_1}$$

$$\Rightarrow x^{-2n} = 2nt + C_1$$

$$\Rightarrow x^{2n} = \frac{1}{2nt + C_1}, \quad \text{let } x(0) = x_0$$

$$\Rightarrow x^{2n}(0) = \frac{1}{C_1} = x_0^{2n} \Rightarrow C_1 = x_0^{-2n}$$

$$\Rightarrow x^{2n}(t) = \frac{1}{2nt + x_0^{-2n}} = \frac{x_0^{2n}}{2n x_0^{2n} + 1}$$

$$\Rightarrow x = \left(\frac{x_0^{2n}}{2n x_0^{2n} + 1}\right)^{\frac{1}{2n}}$$ $\Rightarrow$ with n is bigger, x will approach 0 more slowly

Since $k = 2n + 1 \Rightarrow$ with n is larger, k is bigger

$\boxed{\Rightarrow \text{We conclude that with } k \text{ larger will result in slower convergence to the fixed point.}}$

6. (Derived from Strogatz Exercise 2.2.13) The velocity $v(t)$ of a skydiver falling to the ground is governed by

$$m\dot{v} = mg - kv^2, \quad ①$$

where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

(a) Find the exact solution for $v(t)$ when $v(0) = 0$.

(b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.

(c) Draw a phase portrait for this problem, and thereby re-derive a formula for the terminal velocity.

a) ① $\Leftrightarrow$ $m\dfrac{dv}{dt} = mg - kv^2$

$$\Rightarrow \frac{m}{mg - kv^2}\, dv = dt$$

$$\Leftrightarrow \frac{m}{(\sqrt{mg} - v\sqrt{k})(\sqrt{mg} + v\sqrt{k})}\, dv = dt$$

$$\Leftrightarrow \left(\frac{A}{\sqrt{mg} - v\sqrt{k}} + \frac{B}{\sqrt{mg} + v\sqrt{k}}\right) dv = dt$$

With :

$$\frac{m}{(\sqrt{mg} - v\sqrt{k})(\sqrt{mg} + v\sqrt{k})} = \left(\frac{A}{\sqrt{mg} - v\sqrt{k}} + \frac{B}{\sqrt{mg} + v\sqrt{k}}\right)$$

$$\Leftrightarrow \begin{cases} m = \sqrt{mg}\,(A+B) \\ (A - B)\, v\sqrt{k} = 0 \end{cases} \Leftrightarrow \begin{cases} A = B \\ A + B = \sqrt{\dfrac{m}{g}} \end{cases}$$

$$(\Rightarrow)\quad A = B = \frac{\sqrt{m}}{2\sqrt{g}}$$

then, we have:

$$\int \frac{\sqrt{m}}{2\sqrt{g}} \left(\frac{1}{\sqrt{mg} - v\sqrt{k}} + \frac{1}{\sqrt{mg} + v\sqrt{k}} \right) dv = \int dt = t + C$$

$$\Leftrightarrow \frac{\sqrt{m}}{2\sqrt{g}} \left[\frac{1}{\sqrt{k}} \ln\left(\sqrt{mg} + v\sqrt{k}\right) - \frac{1}{\sqrt{k}} \ln\left(\sqrt{mg} - v\sqrt{k}\right) \right] = t + C$$

$$\Leftrightarrow \frac{\sqrt{m}}{2\sqrt{gk}} \ln\left(\frac{\sqrt{mg} + v\sqrt{k}}{\sqrt{mg} - v\sqrt{k}} \right) = t + C, \text{ let } a = \frac{2\sqrt{gk}}{\sqrt{m}}$$

$$\Leftrightarrow \frac{\sqrt{mg} + v\sqrt{k}}{\sqrt{mg} - v\sqrt{k}} = \exp\left[(t+C) \frac{2\sqrt{gk}}{\sqrt{m}} \right]$$

$$\Leftrightarrow \frac{\sqrt{mg} + v\sqrt{k}}{\sqrt{mg} - v\sqrt{k}} = e^{a(t+C)}$$

$$\Rightarrow \sqrt{mg} + v\sqrt{k} = \sqrt{mg}\, e^{a(t+C)} - v\sqrt{k}\, e^{a(t+C)}$$

$$\Rightarrow v\left[\sqrt{k} + \sqrt{k}\, e^{a(t+C)} \right] = \sqrt{mg}\left(e^{a(t+C)} - 1 \right)$$

$$\Rightarrow v(t) = \frac{\sqrt{mg}}{\sqrt{k}} \cdot \frac{e^{a(t+C)} - 1}{e^{a(t+C)} + 1}$$

Also, $v(0) = 0$

$$\Rightarrow 0 = \frac{\sqrt{mg}}{\sqrt{k}} \frac{e^{ac} - 1}{e^{ac} + 1} \Rightarrow e^{ac} - 1 = 0$$

$$\Rightarrow e^{ac} = 1 \Rightarrow ac = 0 \Rightarrow c = 0$$

Since $a = \frac{2\sqrt{gk}}{m}$ is constant

$$\Rightarrow \boxed{v(t) = \frac{\sqrt{mg}}{\sqrt{k}} \frac{e^{at} - 1}{e^{at} + 1} \quad \text{with } a = \frac{2\sqrt{gk}}{\sqrt{m}}}$$

b) $$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{\sqrt{mg}}{\sqrt{k}} \left(\frac{e^{at} - 1}{e^{at} + 1}\right)$$

$$= \frac{\sqrt{mg}}{\sqrt{k}} \lim_{t\to\infty} \left(\frac{\overbrace{e^{at} - 1}^{\to\infty}}{\underbrace{e^{at} + 1}_{\to\infty}}\right)$$ $\Rightarrow$ apply l'Hospital Rule:

$$= \frac{\sqrt{mg}}{\sqrt{k}} \lim_{t\to\infty} \frac{ae^{at}}{ae^{at}} = \frac{\sqrt{mg}}{\sqrt{k}}$$

$$\Rightarrow \boxed{\lim_{t\to\infty} v(t) = \frac{\sqrt{mg}}{\sqrt{k}}}$$ is terminal velocity

c) We have:

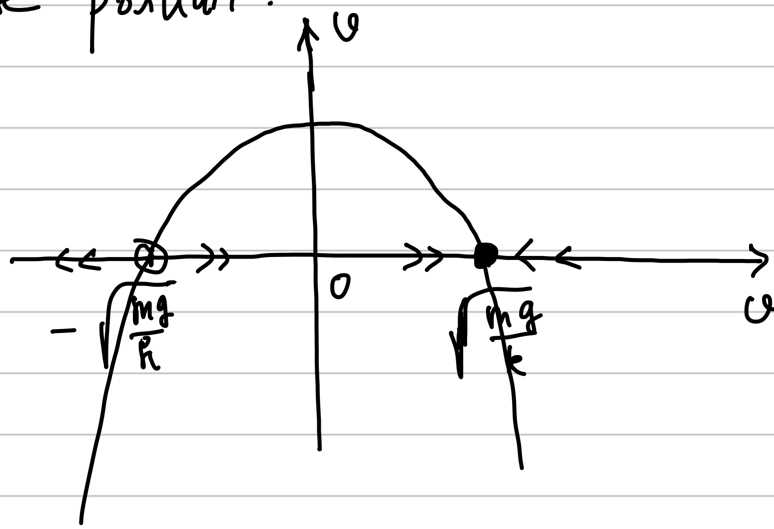
$$m\dot{v} = mg - kv^2 \Rightarrow \dot{v} = \frac{mg - kv^2}{m}$$

$$\Rightarrow \dot{v} = -\frac{k}{m} v^2 + g$$

So, $\dot{v} = 0 \Rightarrow mg - kv^2 = 0 \Rightarrow kv^2 = mg$

$\Rightarrow \quad v = \pm\sqrt{\frac{mg}{k}}$ are fixed points

phase portrait:



$\Rightarrow$

- $v = \sqrt{\frac{gm}{k}}$ is a stable fixed point
- $v = -\sqrt{\frac{gm}{k}}$ is an unstable point

Based on the stable fixed point, we can realize that the terminal velocity occurs at the stable fixed point

So, we have the value of terminal velocity is:

$$\boxed{v = \sqrt{\frac{gm}{k}}}$$