Math 135, Spring 2022

Lecture #16: Fourier series

Wednesday May ~~3rd~~ 4th

Last time

- Given an integrable function $f(x)$ on $[-\pi, \pi]$, we introduced the **Fourier coefficients**

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx.$$

- We then defined the **Fourier series**

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \Big[a_n \cos(nx) + b_n \sin(nx) \Big].$$

Learning objectives

Today we will discuss:

- Cosine and Sine series.

- Extending Fourier series to intervals of arbitrary length.

Fourier series

Review: An example

Suppose that $f(x)$ is an **odd** function on $[-\pi, \pi]$. What can you say about the Fourier coefficients a_n?

A) $a_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x) \cos(nx)\, dx$

B) $a_n = 0$

C) Some of the a_n do not converge

D) None of the above are true

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \overbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos(nx)}_{\text{even}}}^{\text{odd}} \, dx$$

An example

even

Suppose that $f(x)$ is an ~~odd~~ function on $[-\pi, \pi]$. What can you say about the Fourier coefficients b_n?

A) $b_n = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x)\sin(nx)\,dx$

B) $b_n = 0$

C) Some of the b_n do not converge

D) None of the above are true

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{even}}\ \underbrace{\sin(nx)}_{\text{odd}}\,dx$$

odd

Theorem: Let $f(x)$ be an integrable function on $[-\pi, \pi]$.

- If $f(x)$ is **even** we have

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \quad \text{and} \quad b_n = 0.$$

- If $f(x)$ is **odd** we have

$$a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

Remark: For even functions, the Fourier series contains only cosines. For odd functions, it contains only sines.

An example

Find the Fourier series for $f(x) = |x|$.

As $f(x) = |x|$ is even, $b_n = 0$ for all $n \geq 1$.

If $n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx$$

$$= \left[\frac{2}{n\pi}x\sin(nx)\right]\Big|_0^\pi - \int_0^\pi \frac{2}{n\pi}\sin(nx)\,dx$$

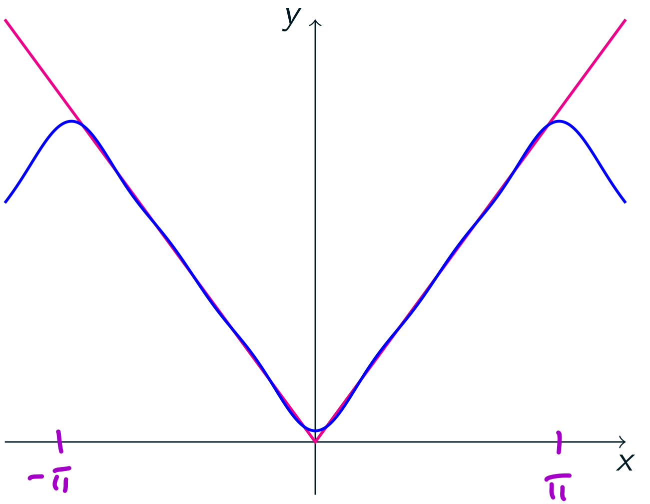
$$= \left[\frac{2}{n^2\pi}\cos(nx)\right]\Big|_0^\pi = \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \qquad \cos(n\pi) = (-1)^n$$

Also, $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$

So, the Fourier series for $f(x) = |x|$ is

$$\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos(nx)$$

$$= \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi}\cos((2n-1)x)$$



$f(x) = |x|$

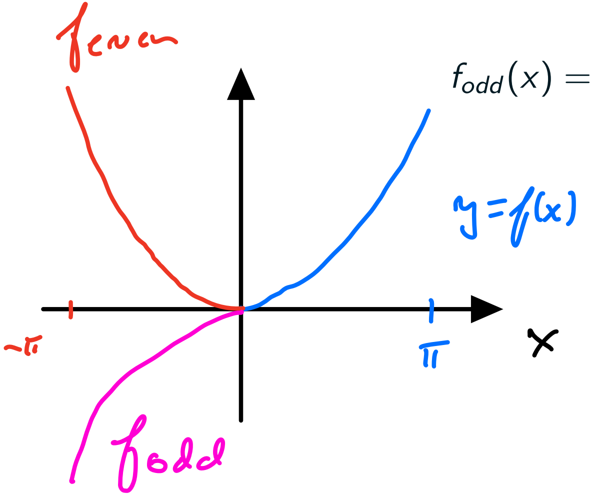
$$S_5(x) = \frac{\pi}{2} - \sum_{n=1}^{3} \frac{4}{\pi(2n-1)^2} \cos((2n-1)x)$$

- Given a function $f(x)$ defined on $[0,\pi]$ we can define its **even extension** to $[-\pi,\pi]$ by

$$f_{even}(x) = \begin{cases} f(x) & \text{if} \quad 0 \le x \le \pi \\ f(-x) & \text{if} \quad -\pi \le x < 0 \end{cases}$$

- Given a function $f(x)$ defined on $[0,\pi]$ we can define its **odd extension** to $[-\pi,\pi]$ by

$$f_{odd}(x) = \begin{cases} f(x) & \text{if} \quad 0 < x \le \pi \\ 0 & \text{if} \quad x = 0 \\ -f(-x) & \text{if} \quad -\pi \le x < 0 \end{cases}$$



An example

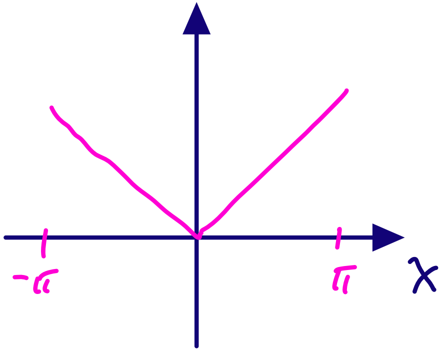
If $f(x) = x$ defined on $[0, \pi]$. What is its **even extension** to $[-\pi, \pi]$?

A) x

B) $|x|$

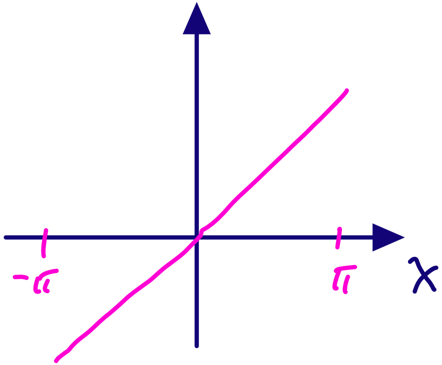
C) $\frac{1}{2}(x + |x|)$

D) None of the above



An example

If $f(x) = x$ defined on $[0, \pi]$. What is its **odd extension** to $[-\pi, \pi]$?



A) x

B) $|x|$

C) $\frac{1}{2}(x + |x|)$

D) None of the above

Definition: Given an integrable function $f(x)$ defined on $[0, \pi]$ we define:

- The **cosine series** of $f(x)$ to be the **Fourier series** of $f_{even}(x)$:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{where} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

- The **sine series** of $f(x)$ to be the **Fourier series** of $f_{odd}(x)$:

$$\sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

Ex: $f(x) = x$ on $[0, \pi]$

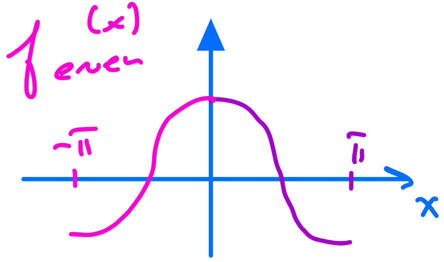
Last time we computed the sine series

Just now we computed the cosine series.

An example

Let $f(x) = \cos(x)$ be defined on $[0, \pi]$.

What is its cosine series? of the even extension



A) $\displaystyle\sum_{n=1}^{\infty} b_n \sin(nx)$ where $\displaystyle b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(nx)\,dx$

B) 0

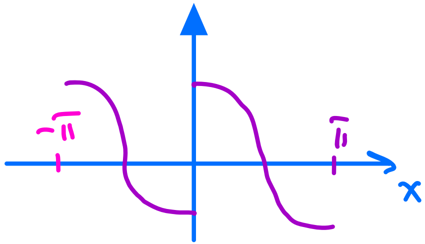
C) $\displaystyle\frac{1}{2} + \sum_{n=1}^{\infty} \cos(nx)$

D) $\cos(x)$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x)\cos(nx)\,dx = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

Let $f(x) = \cos(x)$ be defined on $[0, \pi]$.



What is its sine series?

Hint: $\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin\big((m+n)x\big) + \sin\big((m-n)x\big)\Big]$

If $n \geq 2$, $b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)\cos(x)\,dx$

$$= \frac{1}{\pi}\int_0^\pi \sin((n+1)x) + \sin((n-1)x)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos((n+1)x) - \frac{1}{n-1}\cos((n-1)x)\right]\Big|_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1}[1-(-1)^{n+1}] + \frac{1}{n-1}[1-(-1)^{n-1}]\right]$$

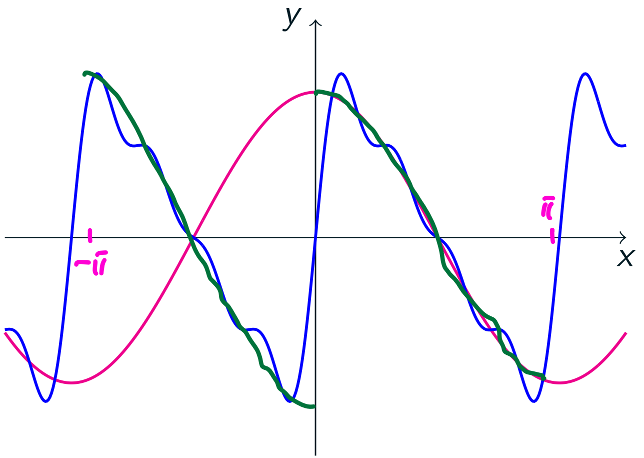
$$= \frac{2n}{\pi(n^2-1)}[1-(-1)^{n+1}]$$

Also, $b_1 = \frac{2}{\pi} \int_0^{\pi} \cos(x)\sin(x)\,dx = \frac{1}{\pi} \int_0^{\pi} \sin(2x)\,dx = 0$

The sine series is

$$\sum_{n=2}^{\infty} \frac{2n}{\pi(n^2-1)} \left[1 - (-1)^{n+1}\right] \sin(nx)$$

$$= \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} \sin(2nx)$$



$$f(x) = \cos(x)$$

$$S_8(x) = \sum_{n=1}^{4} \frac{8n}{\pi(4n^2-1)} \sin(2nx)$$

Scaling

~~Next time.~~

go to convergence of Fourier series

Definition: Let $f(x)$ be an integrable function on the interval $[-L, L]$, where $L > 0$. We define the **Fourier series** for $f(x)$ on $[-L, L]$ to be

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\tfrac{n\pi}{L}x\right) + b_n \sin\left(\tfrac{n\pi}{L}x\right)\right],$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\left(\tfrac{n\pi}{L}x\right)\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\tfrac{n\pi}{L}x\right)\,dx.$$

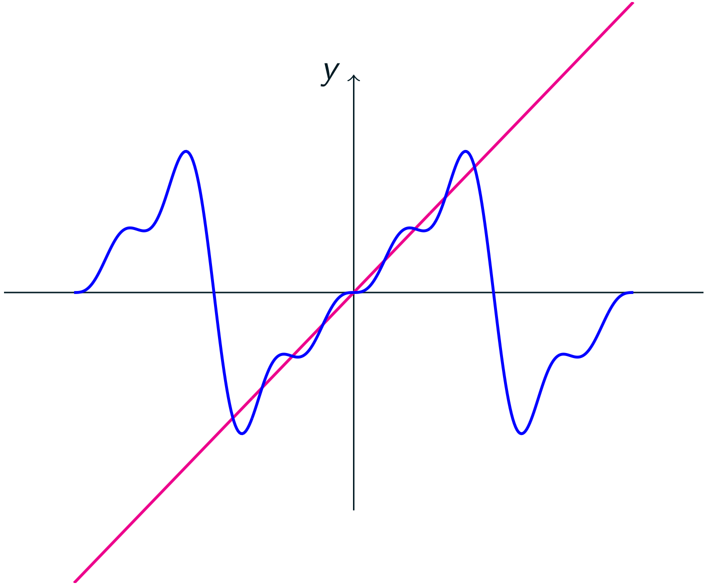
An example

Find the Fourier series for $f(x) = x$ on $[-1, 1]$.

Step 1: Use any available symmetries to simplify the computation.

Step 2: Compute

$$b_n = 2\int_0^1 x\,\sin(n\pi x)\,dx$$



$$f(x) = x$$

$$S_4(x) = \sum_{n=1}^{4} (-1)^{n+1} \frac{2}{n\pi} \sin(nx)$$

See you next time!