Lecture 7

Laplace Transforms.

$$\begin{tikzcd}
\boxed{\text{Algebraic Equation (A E)}} \arrow[r] & \boxed{\text{Solution to A.E}} \arrow[d, "\text{Inverse LT}"] \\
\boxed{\text{Differential Eq}^n} \arrow[u, "\text{Laplace transform (LT)}"] \arrow[r, "\text{Difficult}"] & \boxed{\text{Solution to DE}}
\end{tikzcd}$$

Given $f: \mathbb{R}_+ \to \mathbb{R}$, let $s \in \mathbb{C}/\mathbb{R}$ define Laplace integral of f as

$$(\mathcal{L}f)(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{kernel of the Laplace Transform}} f(t)\, dt = F(s)$$

$\mathcal{L}\{f\}(s)$

(1) The integral above is called Laplace transform of f if the integral converges (exists)

$\mathcal{L}$ is an operator, similar to T defined for Picard iterates.

$\mathcal{L}$: Functions of time $\longrightarrow$ Functions of $\mathbb{R}(\mathbb{C})$
(Domain - t) (domain - s)

NOTE:

	Universal Terminology	Textbook Notation
Laplace variable :	s $\longrightarrow$	p
Time :	t $\longrightarrow$	x

Note: We will focus later on a criterion that will help us identify function for which the Lap transf. exists.

Notation: Small letters are reserved for functions of time E.g. $f(t), g(t)$.
Capital letters are reserved for functions of $s \in \mathbb{R}$. E.g. $F(s)$, $G(s)$.

Note that $\mathcal{L}$ is a linear operator.

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L} f + \beta \mathcal{L} g, \quad \alpha, \beta \in \mathbb{R}$$

Eg 1. $f(t) = 1$, $t \geq 0$, $s \in \mathbb{R}$

$$\mathcal{L} f = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \lim_{T \to \infty} \int_0^T e^{-st} f(t)\, dt = \lim_{T \to \infty} \left. \frac{e^{-st}}{-s} \right|_0^T$$

$$= \lim_{T \to \infty} \frac{e^{-sT}}{-s} + \frac{1}{s}$$

The limit $\lim_{\tau\to\infty} \frac{e^{-s\tau}}{-s}$ converges only if $s>0$, otherwise $e^{-s\tau} \to \infty$ as $\tau\to\infty$.

Thus, $\mathcal{L}(1) = \frac{1}{s}$ $(s>0)$. $\square$

Eg2 Show that $\mathcal{L}(1) = \frac{1}{s}$ for $s \in \mathbb{C}$ s.t. $Re(s) > 0$.

Note that domain of F $(= \mathcal{L}f)$ is $\mathbb{R}_+$ and right half of $\mathbb{C}$.

Eg3. $\mathcal{L}(e^{i\omega t}) = \int_0^\infty e^{-st} e^{i\omega t}\, dt.$

$$= \lim_{\tau\to\infty} \int_0^\tau e^{(-s+i\omega)t}\, dt$$

$$= \lim_{\tau\to\infty} \left. \frac{e^{(-s+i\omega)t}}{-s+i\omega} \right|_0^\tau$$

$s = Re(s) + i\, Im(s)$

$$= \lim_{\tau\to\infty} \left. \frac{e^{\overbrace{(-Im(s)+\omega)}^{=A} i\tau} e^{-Re(s)\tau}}{-s+i\omega} \right|_0^\tau$$

$$= \lim_{\tau\to\infty} \frac{e^{Ai\tau} e^{-Re(s)t}}{-s+i\omega} - \frac{1}{-s+i\omega}$$

Recall: $|e^{i\theta}| = 1$, so $|e^{iAt}| = 1$

The limit, again, will tend to 0 if $Re(s) > 0$.

Thus, $\mathcal{L}(e^{i\omega t}) = \dfrac{1}{s - i\omega}$ if $Re(s) > 0$ $\square$

Eg 4. Similarly, $F(s) = \mathcal{L}(e^{-i\omega t}) = \dfrac{1}{s + i\omega}$, $Re(s) > 0$.

Eg 5. $\mathcal{L}(\cos \omega t) = \mathcal{L}\left(\dfrac{e^{i\omega t} + e^{-i\omega t}}{2}\right)$ ⊛

$\mathcal{L}$ is linear

$$= \frac{1}{2}\left[\mathcal{L}(e^{i\omega t}) + \mathcal{L}(e^{-i\omega t})\right]$$

$$= \frac{1}{2}\left(\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right)$$

$$= \frac{s}{s^2 + \omega^2}, \quad Re(s) > 0$$

Eg 6. $\mathcal{L}(\sin \omega t) = \mathcal{L}\left(\dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)$ ⊛

$$= \frac{\omega}{s^2 + \omega^2}, \quad Re(s) > 0.$$

⊛ Recall formulae for $\sin t$, $\cos t$, $\sinh t$, $\cosh t$.

Standard Results:

$\mathcal{L}(1) = 1/s \qquad s > 0$

$\mathcal{L}(t) = 1/s^2 \qquad s > 0$

$\mathcal{L}(t^n) = \dfrac{n!}{s^{n+1}} \qquad s > 0$

$\mathcal{L}(e^{at}) = \dfrac{1}{s-a} \qquad s > a$

$\omega \in \mathbb{R}$, $\mathcal{L}(\cos \omega t) = \dfrac{s}{s^2+\omega^2} \qquad s > 0$

$\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2} \qquad s > 0$

Always note the domain of s.

Translation property of Laplace transforms → s-domain (Below)
→ t-domain (HW)

If $F(s) = \mathcal{L}(f)(t)$ for $s > 0$ then

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a), \quad a \in \mathbb{R}$$

Multiplication by e^{at} results in shift by a.

pf:
$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt$$
$$= \int_0^\infty e^{-st} f(t)\, e^{at}\, dt$$
$$= \mathcal{L}\{e^{at} f(t)\}(s)$$

Eg. $\mathcal{L}\{t\}(s) = \dfrac{1}{s^2}, \quad s > 0$

$$\mathcal{L}(te^{at}) = \frac{1}{(s-a)^2}, \quad s > a$$

Eg. $\mathcal{L}(\sin \omega t) = \dfrac{\omega}{s^2+\omega^2}, \quad s > 0, \ \omega \in \mathbb{R}$

$$\mathcal{L}(e^{2t} \sin 3t) = \frac{3}{(s-2)^2+9} \quad s > 2.$$