# Math 135, Spring 2022

Lecture #22: PDEs and boundary value problems

Wednesday May 18th

# Recap

- We finished the topic "Fourier Series".

So far we have covered

- Existence and uniqueness theory for ODEs
- Laplace transform
- Fourier series

And we are now going to deal with

- Partial Differential Equations (PDEs)

## Learning objectives

Today we will discuss:

- Fourier series in an arbitrary domain.

- Derivation of the $1d$ wave equation as a model for a vibrating string.

- Solving the initial boundary value problem for the $1d$ wave equation by separation of variables.

# Last topic on Fourier series

# Scaling

Suppose that $f(x)$ is defined on $[-L, L]$, where $L > 0$.
We want to construct the Fourier series for $f(x)$.
Let $g(t) = f(Lt/\pi)$ be defined on $[-\pi, \pi]$
The Fourier series for $g(t)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos(nt) + b_n \sin(nt)\}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt$$

As $x = Lt/\pi$, take our Fourier series for $f(x)$ to be

$$t = \frac{\pi}{L} x$$

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(n \frac{\pi x}{L}\right) + b_n \sin\left(n \frac{\pi x}{L}\right) \right\}$$

where $$t = \frac{\pi}{L} x \longrightarrow dt = \frac{\pi}{L} dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(Lt/\pi) \cos(nt)\, dt = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(n \frac{\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(Lt/\pi) \sin(nt)\, dt = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

**Definition:** Let $f(x)$ be an integrable function on the interval $[-L, L]$, where $L > 0$. We define the **Fourier series** for $f(x)$ on $[-L, L]$ to be

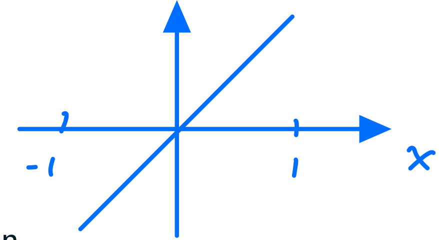
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\tfrac{n\pi}{L}x\right) + b_n \sin\left(\tfrac{n\pi}{L}x\right)\right],$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{n\pi}{L}x\right)dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{n\pi}{L}x\right)dx.$$

# An example

Find the Fourier series for $\underbrace{f(x)}_{\text{odd}} = x$ on $[-1, 1]$.




**Step 1:** Use any available symmetries to simplify the computation.

$a_n = 0$ for all $n$ as $f(x)$ is odd

**Step 2:** Compute

$$b_n = 2\int_0^1 x\sin(n\pi x)\,dx$$

$$b_n = \int_{-1}^{1} \underbrace{x\sin(n\pi x)}_{\text{even}}\,dx = 2\int_0^1 x\sin(n\pi x)\,dx$$

As $n \geq 1$,

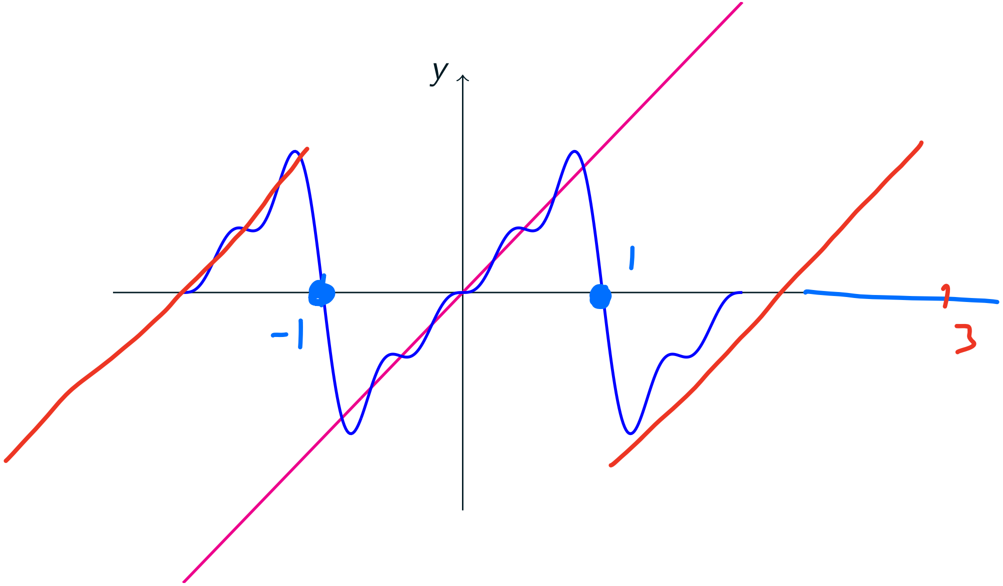
$$b_n = \left[-\frac{2}{n\pi}x\cos(n\pi x)\right]\Bigg|_{x=0}^{x=1} + \int_0^1 \frac{2}{n\pi}\cos(n\pi x)\,dx$$

$$= -\frac{2}{n\pi}(-1)^n$$

$$= -\frac{2}{n\pi}(-1)^{n+1}$$

So, the Fourier series for $f(x) = x$ on $[-1, 1]$ is

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x)$$



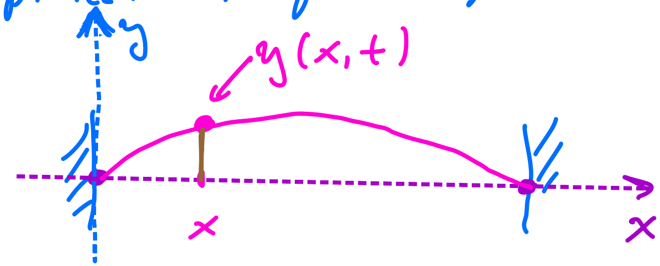

$f(x) = x$

$$S_4(x) = \sum_{n=1}^{4} (-1)^{n+1} \frac{2}{n\pi} \sin(nx)$$

# PDEs and boundary value problems

# The vibrating string

Let $y = y(x,t)$ be the vertical displacement at time $t$ of a tight string constraint at its ends




**Assumptions:**

- Motion is strictly vertical
- No resistance to bending
- Small vibrations
- Constant density $\rho$
- Constant tension $T$

# The $1d$ wave equation

We are lead to consider the $1d$ **wave equation**:

$y = y(x,t)$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$c^2 = \frac{T}{\rho}$

with the **boundary conditions**

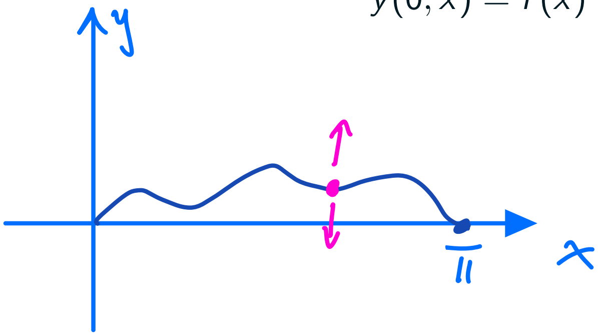
$$y(t,0) = 0 = y(t,\pi)$$

$T$ tension

$\rho$ density

and the **initial conditions**

$$y(0,x) = f(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(0,x) = g(x).$$



# Linearity

**Theorem:** If $y_1(t, x)$ and $y_2(t, x)$ are solutions of the linear wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with the **boundary conditions**

$$y(t, 0) = 0 = y(t, \pi)$$

and $C_1, C_2$ are constants, then so is

$$y(t, x) = C_1 y_1(t, x) + C_2 y_2(t, x).$$

**Proof:** Easy computation.

# Separation of variables

$$\begin{cases} \partial_{tt}^2 y = c^2 \partial_{xx}^2 y \\ y(0,t) = y(\pi,t) = 0 \end{cases}$$

Try to find a solution of the form

$$y(x,t) = v(t)\,u(x) \quad \text{where } u(0) = u(\pi) = 0$$

(educated guess)

Plug this into: $v''(t)\,u(x) = c^2 v(t)\,u''(x)$

$$\frac{1}{c^2}\,\frac{v''(t)}{v(t)} = \frac{u''(x)}{u(x)} = -\lambda \quad \text{a constant!}$$

# Spatial oscillations

$$\frac{u''(x)}{u(x)} = -\lambda$$

If $\lambda < 0$, what is the general solution of

$$u''(x) + \lambda u(x) = 0 \qquad ?$$

boundary value problem

$u(0) = u(\pi) = 0$

A) $C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

B) $C_1 \cos(\sqrt{-\lambda}x) + C_2 \sin(\sqrt{-\lambda}x)$

C) $C_1 x + C_2$

D) None of the above

$u(x) = e^{\mu x}$

Charact. poly.: $\mu^2 + \lambda = 0 \Rightarrow \mu^2 = -\lambda > 0$

so $\mu = \pm\sqrt{-\lambda}$

$u'' + \lambda u = 0$

If $\lambda < 0$: $\quad u(x) = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$

Boundary conditions:

$0 = u(0) = c_1 + c_2 \longrightarrow c_2 = -c_1$

$0 = u(\pi) = c_1 e^{\sqrt{-\lambda}\,\pi} + c_2 e^{-\sqrt{-\lambda}\,\pi}$

$= c_1 \big( \underbrace{e^{\sqrt{-\lambda}\,\pi} - e^{-\sqrt{-\lambda}\,\pi}}_{\neq 0} \big) \longrightarrow c_1 = 0$

$\rightarrow c_2 = 0$

No non-trivial solutions.

If $\lambda > 0$, what is the general solution of

$$u''(x) + \lambda u(x) = 0 \qquad ?$$

A) $C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

B) $C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

C) $C_1 x + C_2$

D) None of the above

Charact. poly. $\Rightarrow$ $\mu^2 = -\lambda < 0$ so $\mu = \pm i\sqrt{\lambda}$

$$\begin{cases} u'' + \lambda u = 0, & \lambda > 0 \\ u(0) = u(\pi) = 0 \end{cases}$$

$$u(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

B.C.

$$0 = u(0) = C_1$$

$$0 = u(\pi) = C_2 \sin(\sqrt{\lambda}\, \pi)$$

If $\sqrt{\lambda} \notin \mathbb{Z}_+ = \{1, 2, \dots\}$ then $c_1 = c_2 = 0$: no non-trivial solutions

If $\sqrt{\lambda} \in \mathbb{Z}_+ = \{1, 2, \dots\}$, i.e. $\lambda = n^2$ for $n \geq 1$

so we have non-trivial solutions

$$u(x) = C_2 \sin(nx)$$

# Eigenfunctions

- We have shown that we only have non-trivial solutions of the **boundary value problem**

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = 0 = u(\pi) \end{cases}$$

$-\frac{d^2}{dx^2} u = \lambda u$

when $\lambda = n^2$ for $n = 1, 2, \ldots$. [exercise: Check what happens when $\lambda = 0$]

- We can think of $-\frac{d^2}{dx^2}$, together with the boundary conditions $u(0) = 0 = u(\pi)$ as a **linear operator**, i.e. $\underbrace{-\frac{d^2}{dx^2}}_{L} : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}, \mathcal{Y}$ are certain space functions.

$Lu = \lambda u$ $\qquad$ $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{v} = \lambda \vec{v}$

- We call the functions

$$u_n(x) = \sin(nx)$$

**eigenfunctions** of this operator and we call the values $\lambda_n = n^2$ the corresponding **eigenvalues**.

$\boxed{L u_n = n^2 u_n}$

We are trying to solve

$$\frac{v''}{c^2 v} = \frac{u''}{u} = -\lambda$$

If $\lambda = n^2$, $\frac{u''}{u} = -n^2$ has a nontrivial sol.

Need to solve

$$\frac{v''}{c^2 v} = -n^2$$

$$\boxed{v'' + c^2 n^2 v = 0}$$

# Time oscillation

What is the general solution of

$$v''(t) + c^2 n^2 v(t) = 0 \qquad ?$$

A) $C_1 e^{nct} + C_2 e^{-nct}$

B) $C_1 \cos(nct) + C_2 \sin(nct)$

C) $C_1 nct + C_2$

D) None of the above

- Suppose we take $\frac{\partial y}{\partial t}(0, x) = g(x) = 0$.

(The case where $f(x)=0$
$g(x) \neq 0$ is on the HW)

- Consequently, for each $n \geq 1$ we have a corresponding solution of

$$\begin{cases} v'' + n^2 c^2 v = 0 \\ v'(0) = 0 \end{cases}$$

given by

$$v_n(t) = \cos(nct).$$

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$v(t) = c_1 \cos(nct) + c_2 \sin(nct)$

$y(x,t) = v(t)u(x)$, $\frac{\partial y}{\partial t}(x,0) = 0 = v'(t)u(x) \Rightarrow v'(0) = 0$

$v'(t) = -c_1 nc \sin(nct) + c_2 nc \cos(nct)$

So, $v'(0) = c_2 nc \Rightarrow c_2 = 0$

# Solving our PDE

Trying to solve
$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \\ y(0,t) = 0 = y(\pi,t) \\ \dfrac{\partial y}{\partial t}(x,t) = 0 \\ y(x,0) = f(x) \end{cases}$$

For each $n \geq 1$, $u_n(x) = \sin(nx)$ and $v_n(t) = \cos(nct)$

solve
$$\frac{v_n''}{c^2 v_n} = \frac{u_n''}{u_n} = -n^2$$

So $y_n(x,t) = v_n(t)\, u_n(t)$ is a sol
$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \\ y(0,t) = y(\pi,t) = 0 \\ \dfrac{\partial y}{\partial t}(x,0) = 0 \end{cases}$$

But then,

$$y(x,t) = \sum_{n=1}^{\infty} b_n y_n(x,t) = \sum_{n=1}^{\infty} b_n \cos(nct)\sin(nx)$$

and it solves

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \\ y(0,t) = y(\pi,t) = 0 \\ \dfrac{\partial y}{\partial t}(x,0) = 0 \end{cases}$$

Initial condition:

$$y(x,t) = \sum_{n=1}^{\infty} b_n \cos(nct)\sin(nx)$$

So $$f(x) = y(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\boxed{\text{The sine series for } f(x)}$$

# See you next time!