Home Work 4

1) Section 50, problem 5

Given ③, $L[y'] = pL[y] - y(0)$

Let $y(x) = \int_0^x f(x)\,dx$

$$\Rightarrow y'(x) = f(x)$$

$$\Rightarrow L[y'(x)] = L[f(x)]$$

$$\Rightarrow p \cdot Y(p) - y(0) = F(p)$$

$$\Rightarrow p \cdot Y(p) - \int_0^0 f(x)\,dx = F(p)$$

$$\Rightarrow pY(p) - 0 = F(p) \Rightarrow Y(p) = \frac{F(p)}{p} \quad ①$$

Also, $y(x) = \int_0^x f(x)\,dx \Rightarrow L[y(x)] = L\left[\int_0^x f(x)\,dx\right]$

$$\Rightarrow Y(p) = L\left[\int_0^x f(x)\,dx\right] \quad ②$$

From ① & ②, we have $\boxed{L\left[\int_0^x f(x)\,dx\right] = \frac{F(p)}{p}}$

Or $L^{-1}\left[\frac{F(p)}{p}\right] = \int_0^x f(x)\,dx$.

⊕ Need to verify $L^{-1}\left[\frac{1}{p(p+1)}\right]$ in two ways.

⊕ 1st Way: We have $\frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1}$

$$\Rightarrow L^{-1}\left[\frac{1}{p(p+1)}\right] = L^{-1}\left[\frac{1}{p}\right] - L^{-1}\left[\frac{1}{p+1}\right]$$

$$= \boxed{1 - e^{-x}} \quad \mathrm{Re}\{p\} > -1$$

⊕ 2nd Way:

We have $F(p) = \frac{1}{p+1} \Rightarrow \frac{1}{p(p+1)} = \frac{F(p)}{p}$

As the previous, we know $L^{-1}\left[\frac{F(p)}{p}\right] = \int_0^x f(x)\,dx$

with $F(p) = \frac{1}{p+1}$.

Since $L^{-1}[F(p)] = L^{-1}\left[\frac{1}{p+1}\right] = e^{-x}$

$$\Rightarrow f(x) = e^{-x} \Rightarrow L^{-1}\left[\frac{F(p)}{p}\right] = \int_0^x e^{-x}\,dx = \left(-e^{-x}\right)\Big|_0^x$$

$$= e^{-x}\Big|_x^0 = \boxed{1 - e^{-x}} \quad \mathrm{Re}\{p\} > -1$$

2) Section 50, problem 6

$$y' + 4y + 5\int_0^x y\,dx = e^{-x}, \quad y(0) = 0$$

$$\Rightarrow L\left[y' + 4y + 5\int_0^x y\,dx\right] = L[e^{-x}] = \frac{1}{s+1}$$

$$\Rightarrow L[y'] + 4L[y] + 5L\left[\int_0^x y\,dx\right] = \frac{1}{s+1}$$

$$\Rightarrow sY(s) - y(0) + 4Y(s) + 5 \cdot \frac{Y(s)}{s} = \frac{1}{s+1}$$

$$\Rightarrow sY(s) + 4Y(s) + \frac{5Y(s)}{s} = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 4s + 5)Y(s) = \frac{s}{s+1}$$

$$\Rightarrow Y(s) = \frac{s}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{Bs+C}{(s+2)^2+1}, \quad \mathrm{Re}\{s\} > -2$$

$$A = \left.\frac{s}{s^2+4s+5}\right|_{s=-1} = \frac{-1}{2}$$

$$\Rightarrow Y(s) = \frac{Bs+C}{s^2+4s+5} - \frac{1}{2(s+1)} = \frac{2(s+1)(Bs+C) - s^2 - 4s - 5}{2(s+1)(s^2+4s+5)}$$

$$= \frac{2s}{2(s+1)(s^2+4s+5)}$$

We have $2(s+1)(Bs+C)-s^2-4s-5$

$= 2Bs^2 + 2sC + 2Bs + 2C - s^2 - 4s - 5$

$= (2B-1)s^2 + (2C+2B-4)s + 2C-5 = 2s$

$$\Rightarrow \begin{cases} 2B-1=0 \\ 2C+2B-4=2 \\ 2C-5=0 \end{cases} \Rightarrow \begin{cases} B=\frac{1}{2} \\ C=\frac{5}{2} \end{cases}$$

$$\Rightarrow Y(s) = \frac{Bs+C}{(s+2)^2+1} + \frac{A}{s+1} = \frac{s}{2[(s+2)^2+1]} + \frac{5}{2[(s+2)^2+1]} - \frac{1}{2(s+1)}$$

$$= \frac{s+2}{2[(s+2)^2+1]} + \frac{3}{2[(s+2)^2+1]} - \frac{1}{2(s+1)}$$

We have $\cos x \xrightarrow{L} \frac{s}{s^2+1}$ Re $\{s\} > 0$

$$\Rightarrow e^{-2x}\cos x \xrightarrow{L} \frac{s+2}{(s+2)^2+1}$$

$$\Rightarrow \frac{e^{-2x}\cos x}{2} \xrightarrow{L} \frac{s+2}{2[(s+2)^2+1]}$$

$$\Rightarrow L^{-1}\left[\frac{s+2}{2[(s+2)^2+1]}\right] = \frac{e^{-2x}\cos x}{2}$$

Also, $\sin x \xrightarrow{L} \dfrac{1}{s^2+1}$ $\quad Re\{s\} > 0$

$$\Rightarrow e^{-2x}\sin x \xrightarrow{L} \frac{1}{(s+2)^2+1}$$

$$\Rightarrow \frac{3e^{-2x}\sin x}{2} \longrightarrow \frac{3}{2[(s+2)^2+1]}$$

$$\Rightarrow L^{-1}\left[\frac{3}{2[(s+2)^2+1]}\right] = \frac{3e^{-2x}\sin x}{2}$$

And $$L^{-1}\left[\frac{1}{2(s+1)}\right] = \frac{e^{-x}}{2}$$

$$\Rightarrow y(x) = L^{-1}[Y(s)] = L^{-1}\left[\frac{s+2}{2[(s+2)^2+1]} + \frac{3}{2[(s+2)^2+1]} - \frac{1}{2(s+1)}\right]$$

$$= \frac{e^{-2x}\cos x}{2} + \frac{3e^{-2x}\sin x}{2} - \frac{e^{-x}}{2}$$

$$\Rightarrow \boxed{y(x) = \frac{e^{-2x}\cos x + 3e^{-2x}\sin x - e^{-x}}{2}}$$

3) Section 51, problem 1

⊕ We have $L[\cos ax] = \dfrac{p}{p^2+a^2} = F(p)$, $Re\{p\}>0$

$$\Rightarrow L[x\cos ax] = (-1)F'(p) = -\frac{p^2+a^2-p(2p)}{(p^2+a^2)^2}$$

$$= \frac{p^2-a^2}{(p^2+a^2)^2} \Rightarrow \boxed{L[x\cos ax] = \frac{p^2-a^2}{(p^2+a^2)^2}}$$

⊕ Need to find $L^{-1}\left[\dfrac{1}{(p^2+a^2)^2}\right]$

We have $L[\sin ax] = \dfrac{a}{p^2+a^2} = F(p)$, $Re\{p\}>0$

$$\Rightarrow L[x\sin ax] = -F'(p) = -\frac{-a2p}{(p^2+a^2)^2} = \frac{2pa}{(p^2+a^2)^2}$$

$$\Rightarrow L[x\sin ax] = \frac{2pa}{(p^2+a^2)^2} = F(p)$$

& $f(x) = x\sin ax$

Since $L\left[\int_0^x f(x)\,dx\right] = \dfrac{F(p)}{p}$

$$\Rightarrow L\left[\int_0^x x\sin ax\, dx\right] = \frac{2a}{(p^2+a^2)^2}$$

$$\Rightarrow L\left[\frac{1}{2a}\int_0^x x\sin ax\, dx\right] = \frac{1}{(p^2+a^2)^2}$$

$$\Rightarrow L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] = \frac{1}{2a}\int_0^x x\sin ax\, dx$$

Check: $\int_0^x x\sin ax\, dx$ $\quad u = x \Rightarrow du = dx$

$dv = \sin ax\, dx$

$\Rightarrow v = -\frac{1}{a}\cos(ax)$

$$= \int_0^x u\,dv = uv\Big|_0^x - \int_0^x v\,du$$

$$= x\left(-\frac{1}{a}\right)\cos(ax) + \frac{1}{a}\int_0^x \cos ax\, dx$$

$$= \frac{x\cos(ax)}{a}\Big|_x^0 + \frac{1}{a}\cdot\frac{1}{a}\sin ax\Big|_0^x$$

$$= \frac{1}{a}\left[0 - x\cos ax\right] + \frac{1}{a^2}\left[\sin ax - 0\right]$$

$$= \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} \Rightarrow \int_0^x x\sin ax\, dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$$

$$\Rightarrow L^{-1}\left[\frac{1}{(p^2+a^2)^2}\right] = \frac{1}{2a}\left[\frac{\sin ax - ax\cos ax}{a^2}\right]$$

$$= \boxed{\frac{\sin ax - ax\cos ax}{2a^3}} \quad (\mathrm{Re}\{p\} > 0)$$

4) Section 52, problem 2a

$$y(x) = 1 - \int_0^x (x-t) \cdot y(t)\, dt \text{ , let } f(x) = x$$

$$\Rightarrow L[y(x)] = L[1] - L[f(x)]\, L[y(x)]$$

$$\Rightarrow Y(s) = \frac{1}{s} - L[x] \cdot Y(s) = \frac{1}{s} - \frac{1}{s^2} \cdot Y(s) \quad Re\{s\} > 0$$

$$\Rightarrow Y(s)\left[1 + \frac{1}{s^2}\right] = \frac{1}{s} \Rightarrow Y(s) \cdot \frac{s^2+1}{s^2} = \frac{1}{s}$$

$$\Rightarrow Y(s) = \frac{s^2}{s(s^2+1)} = \frac{s}{s^2+1}$$

Since $\cos x \xrightarrow{L} \frac{s}{s^2+1}$

$$\Rightarrow L^{-1}[Y(s)] = L^{-1}\left[\frac{s}{s^2+1}\right] = \cos x$$

$$\Rightarrow \boxed{y(x) = \cos x}$$

5) Section 52, problem 5:

Given $y'' + a^2 y = f(x)$ $\quad y(0) = y'(0) = 0$

$$\Rightarrow L[y''] + a^2 L[y] = L[f(x)]$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + a^2 Y(s) = F(s)$$

$$\Rightarrow (s^2 + a^2)\,Y(s) = F(s)$$

$$\Rightarrow Y(s) = \frac{F(s)}{(s^2+a^2)}$$

We have $L[f(x)] = F(s)$

$L[\sin ax] = \dfrac{a}{s^2+a^2} \Rightarrow L\left[\dfrac{1}{a}\sin ax\right] = \dfrac{1}{s^2+a^2} = G(s)$

$\operatorname{Re}\{s\} > 0$

with $f(x)$ & $g(x) = \dfrac{1}{a}\sin ax$

and $Y(s) = F(s) \cdot G(s)$

$$\Rightarrow L^{-1}[Y(s)] = L^{-1}[F(s) \cdot G(s)] = \int_0^x f(t)\,g(x-t)\,dt$$

$$\Rightarrow y(x) = \int_0^x f(t)\,\frac{1}{a}\sin a(x-t)\,dt$$

$$\Rightarrow \boxed{y(x) = \frac{1}{a}\int_0^x f(t)\sin a(x-t)\,dt \text{ is a solution}}$$

6) a) $(f * g)(t) = (g * f)(t)$

We have: $(f*g)(t) = \int_{-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$

Let $\alpha = t - \tau \Rightarrow \tau = t - \alpha \Rightarrow d\tau = -d\alpha$

$\Rightarrow \quad t: -\infty \rightarrow +\infty$

$\Rightarrow \quad \alpha: +\infty \rightarrow -\infty$

$$\Rightarrow (f*g)(t) = \int_{+\infty}^{-\infty} f(t-\alpha)\, g(\alpha)\, (-d\alpha)$$

$$= \int_{-\infty}^{+\infty} g(\alpha)\, f(t-\alpha)\, d\alpha$$, just replace α by τ

$$= \int_{-\infty}^{+\infty} g(\tau)\, f(t-\tau)\, d\tau = (g * f)(t)$$

$$\Rightarrow \boxed{(f * g)(t) = (g * f)(t)}$$

b) Given f & g are piecewise continuous & exponential order on $[0, \infty)$, need to prove $(f*g)(t)$ is of exponential order on $[0, \infty)$

Since both f & g are piecewise continuous & exponential order

we have $|f(t)| < M_1 e^{\alpha_1 t}$

$|g(t)| < M_2 e^{\alpha_2 t}$

with $\exists\, M_1 > 0$ & $M_2 > 0$ & $\alpha_1 < \alpha_2$, for all $t \geq k > 0$

with $[0, \infty)$, we also have:

$$(f * g)(t) = \int_0^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$$\Rightarrow |(f*g)(t)| = \left| \int_0^{+\infty} f(\tau)\, g(t-\tau)\, d\tau \right|$$

$$\leq \int_0^{+\infty} |f(\tau)\, g(t-\tau)\, d\tau| \leq \int_0^{+\infty} |f(\tau)| . |g(t-\tau)|\, d\tau$$

$$< \int_0^{+\infty} M_1 e^{\alpha_1 \tau} . M_2 e^{\alpha_2 (t-\tau)}\, d\tau$$

$$= M_1 M_2 \int_0^{+\infty} e^{\alpha_1 \tau + \alpha_2 (t-\tau)}\, d\tau$$

$$= M_1 M_2 e^{\alpha_2 t} . \int_0^{+\infty} e^{(\alpha_1 - \alpha_2)\tau}\, d\tau = M_1 M_2 e^{\alpha_2 t} \frac{1}{\alpha_1 - \alpha_2} e^{(\alpha_1 - \alpha_2)\tau} \Big|_0^{+\infty}$$

$$= M_1 M_2 e^{\alpha_2 t} \frac{1}{\alpha_1 - \alpha_2} [0 - 1]$$

$$\Rightarrow |(f*g)(t)| < \frac{M_1 M_2}{\alpha_2 - \alpha_1} e^{\alpha_2 t}$$

Since $M_1 > 0$, $M_2 > 0$, $\alpha_2 > \alpha_1 \Rightarrow \alpha_2 - \alpha_1 > 0$

$\Rightarrow \exists\, M = \dfrac{M_1 M_2}{\alpha_2 - \alpha_1} > 0$ & α_2 for all $t > t_0 \in [0, \infty)$

so that $|(f*g)(t)| < M e^{\alpha_2 t}$

$\Rightarrow$ $\boxed{(f*g)(t) \text{ is also of exponential order on } [0,\infty)}$

7) Given $L\{f(t)\}(s) = F(s)$

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$

Need to prove: $L\{u_a(t) \cdot f(t-a)\}(s) = e^{-as} F(s). \ (a \geq 0)$

We have: $L\{\underbrace{u_a(t) \cdot f(t-a)}_{y(t)}\}(s) = \int_0^\infty e^{-st} y(t)\, dt$

$$= \int_0^a e^{-st} u_a(t) f(t-a)\, dt + \int_a^{+\infty} e^{-st} u_a(t) f(t-a)\, dt$$

$$= 0 + \int_a^{+\infty} e^{-st} f(t-a)\, dt \ (1)$$

let $\tau = t - a \Leftrightarrow t = \tau + a$

$\Rightarrow d\tau = dt$

$t: a \to +\infty$

$\Rightarrow \tau: 0 \to +\infty$

$$\Rightarrow (1) = \int_0^{+\infty} e^{-s(\tau+a)} f(\tau)\, d\tau = e^{-sa} \underbrace{\int_0^{+\infty} e^{-s\tau} f(\tau) d\tau}_{F(s)}$$

$$\Rightarrow (1) = e^{-sa} \cdot F(s)$$

$$\Rightarrow \boxed{L\{u_a(t) \cdot f(t-a)\}(s) = e^{-as} F(s) \quad (a \geq 0)}$$

8) $y'' - y = t-2 \quad , y(2) = 3 \ \& \ y'(2) = 0$

Set $\quad g(t) = y(t+2)$

$\Rightarrow \begin{cases} g(0) = y(2) = 3 \\ g'(0) = y'(2) = 0 \end{cases}$

Then replacing t by $t+2$ in the DE yields:

$$y''(t+2) - y(t+2) = t+2-2 = t$$

$\Rightarrow \quad g'' - g = t \quad \Rightarrow \quad L[g''] - L[g] = L[t]$

$\Rightarrow \quad s^2 G(s) - s\,g(0) - g'(0) - G(s) = \frac{1}{s^2} \qquad Re\{s\} > 0$

$\Leftrightarrow \quad s^2 G(s) - 3s - G(s) = \frac{1}{s^2}$

$\Leftrightarrow \quad (s^2-1)\, G(s) = \frac{1}{s^2} + 3s = \frac{3s^2+1}{s^2}$

$\Leftrightarrow \quad G(s) = \frac{3s^2+1}{s^2(s^2-1)} = \frac{3}{s^2-1} + \frac{1}{s^2(s^2-1)} \qquad Re\{s\} > 1$

$$= \frac{3}{(s-1)(s+1)} + \frac{1}{s^2(s-1)(s+1)}$$

$\circledast$ Check $\quad \frac{1}{(s-1)(s+1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$

$$\Rightarrow \frac{3}{(s-1)(s+1)} = \frac{3}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \frac{3}{2(s-1)} - \frac{3}{2(s+1)}$$

(*) Check $\frac{1}{s^2(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2}$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s} + \frac{D}{s^2}$$

$$A = \frac{1}{s^2(s+1)}\bigg|_{s=1} = \frac{1}{2} \quad ; \quad D = \frac{1}{(s-1)(s+1)}\bigg|_{s=0} = -1$$

$$B = \frac{1}{s^2(s-1)}\bigg|_{s=-1} = -\frac{1}{2} \quad ; \quad C = \left[\frac{1}{s^2-1}\right]'\bigg|_{s=0} = -\frac{2s}{(s^2-1)^2}\bigg|_{s=0}$$

$$= 0$$

$$\Rightarrow \frac{1}{s^2(s-1)(s+1)} = \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2}$$

$$\Rightarrow G(s) = \frac{3}{2(s-1)} - \frac{3}{2(s+1)} + \frac{1}{2(s-1)} - \frac{1}{2(s+1)} - \frac{1}{s^2}$$

$$= \frac{2}{s-1} - \frac{2}{s+1} - \frac{1}{s^2}$$

Also, $L(e^t) = \frac{1}{s-1} \Rightarrow L(2e^t) = \frac{2}{s-1}$, $Re\{s\} > 1$

$L(e^{-t}) = \frac{1}{s+1} \Rightarrow L(2e^{-t}) = \frac{2}{s+1}$, $Re\{s\} > -1$

$$L(t) = \frac{1}{s^2} \quad Re\{s\} > 0$$

$$\Rightarrow g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{2}{s-1}\right] - L^{-1}\left[\frac{2}{s+1}\right] - L^{-1}\left[\frac{1}{s^2}\right]$$

$$= 2e^{t} - 2e^{-t} - t$$

Because $g(t) = y(t+2)$

$$\Rightarrow g(t-2) = y(t)$$

$$\Rightarrow y(t) = g(t-2) = 2e^{t-2} - 2e^{-t+2} - t + 2$$

So, $\boxed{y(t) = 2e^{t-2} - 2e^{-t+2} - t + 2}$