Math 135, Spring 2022

Lecture #20: Fourier series

Friday May 13th

Last time

- We discussed what it means to say a real vector space is an **inner product space**.
- We discussed the **norm** associated with an inner product.
- We introduced the inner product space $\mathcal{R}^2[a, b]$.
- We discussed what it means for a set to be orthogonal and orthonormal.

Learning objectives

Today we will discuss:

- What it means to say that a series converges in an inner product space.
- Generalized Fourier series.
- Convergence of Fourier series in mean.

Fourier series

Inner product spaces

- Recall that $\mathcal{R}^2[a,b]$ is the real vector space of square Riemann integrable functions on $[a,b]$ together with the "inner product"

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx,$$

and corresponding "norm"

$$\|f\| = \sqrt{\int_a^b |f(x)|^2\,dx}.$$

- Given a sequence $\{w_1, w_2, \dots\}$ of non-zero orthogonal functions in $\mathcal{R}^2[a,b]$, we define the projection of $v \in \mathcal{R}^2[a,b]$ onto $\operatorname{span}\big(\{w_1, w_2, \dots\}\big)$ to be

$$\sum_{n=1}^{\infty} \frac{\langle w_n, v \rangle}{\|w_n\|^2} w_n,$$

provided this series converges.

- Last time we showed that Fourier series can be interpreted as the case where $[a,b] = [-\pi, \pi]$ and our sequence is

$$\left\{ \tfrac{1}{2}\,,\ \cos(x)\,,\ \sin(x)\,,\ \cos(2x)\,,\ \sin(2x)\,,\ \dots \right\}$$

- This construction can be generalized to **any** sequence $\{\theta_1, \theta_2, \dots\}$ of non-zero orthogonal functions in $\mathcal{R}^2[a, b]$.

- Given a function $f \in \mathcal{R}^2[a, b]$, the corresponding sequence

$$\sum_{n=1}^{\infty} \frac{\langle \theta_n, f \rangle}{\|\theta_n\|^2} \theta_n$$

  is referred to as the **(generalized) Fourier series** of f with respect to the orthogonal sequence $\{\theta_1, \theta_2, \dots\}$.

- Depending on the application, different choices of sequence are often more useful than others.

$\int_{-1}^{1} (x^3)^2 dx < \infty$

An example

The **Legendre polynomials** are a sequence of orthogonal polynomials $P_0, P_1, P_2, \ldots$ in $\mathcal{R}^2[-1,1]$, where P_n is a polynomial of degree n. A. M. Legendre (around 1782) introduced these polynomials when studying the gravitational potential associated to a point mass. Later, these polynomials also proved to be quite useful for solving Schrödinger's equation.

The first three elements in this sequence are

$$W = \left\{ P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \right\}$$

For $f(x) = x^3$, what is the value of the partial sum $\displaystyle\sum_{n=0}^{2} \frac{\langle P_n, f\rangle}{\|P_n\|^2} P_n$? $\mathbb{P}_W[f]$

A) $\frac{1}{2}(5x^3 - 3x)$

B) $\frac{3}{5}$

C) $\frac{1}{3} + \frac{1}{4}x + \frac{1}{5}x^2$

D) $\frac{3}{5}x$

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

$f(x) = x^3$

$\langle P_0, f\rangle = \int_{-1}^{1} 1 \cdot x^3 \, dx = 0$

$\langle P_1, f\rangle = \int_{-1}^{1} x \cdot x^3 \, dx = 2\int_{0}^{1} x^4 \, dx = \frac{2}{5}$

$\langle P_2, f\rangle = \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) x^3 \, dx = 0$

$\mathbb{P}_W[f] = \sum_{n=0}^{2} \frac{\langle P_n, f\rangle}{\|P_n\|^2} P_n = \frac{2/5}{2/3} x = \frac{3}{5} x$

$\|P_1\| = \langle P_1, P_1\rangle = \int_{-1}^{1} x \cdot x \, dx = 2\int_{0}^{1} x^2 \, dx = \frac{2}{3}$

Convergence of Fourier series

Convergence of Fourier series

- We are now going to discuss the convergence of Fourier series.
- This is a very delicate topic and much of the analysis is well beyond this class.
- Today we will discuss **convergence in mean** which is different from pointwise convergence.

- Let $\{w_1, w_2, \dots\}$ be an orthogonal sequence of non-zero elements of a real inner product space V.

- Define an **orthonormal** sequence by taking

$$\phi_n = \frac{w_n}{\|w_n\|}.$$

$\|\phi_n\|^2 =$

$\langle \phi_n, \phi_n \rangle = \langle \frac{w_n}{\|w_n\|}, \frac{w_n}{\|w_n\|} \rangle$

$= \frac{1}{\|w_n\|^2} \langle w_n, w_n \rangle$

$=$

- We would like to show that the series

$$\sum_{n=1}^{\infty} \frac{\langle w_n, v \rangle}{\|w_n\|^2} w_n = \sum_{n=1}^{\infty} \langle \phi_n, v \rangle \phi_n$$

converges to v.

- This is **not true** in general: The sequence and the inner product space need to satisfy certain conditions.

Definition: Let $\{\phi_1, \phi_2, \dots\}$ be an orthonormal sequence in a real inner product space V. We say that the series

$$\sum_{n=1}^{\infty} \langle \phi_n, v \rangle \phi_n$$

converges to $v \in V$ if the **partial sums**

$$S_N = \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n$$

satisfy

$$\lim_{N \to \infty} \| S_N - v \| = 0.$$

Hilbert spaces

- We will need to work with a special type of inner product space. **Hilbert space**: A vector space H with an inner product $< \cdot, \cdot >$ which is complete with respect to the norm $\|x\| = \sqrt{< x, x >}$, is called a Hilbert space.

- Hilbert spaces have the property that if the partial sums of our series satisfy

$$\lim_{N,M\to\infty} \|S_N - S_M\| = 0,$$

then there exists some $S \in V$ (not necessarily equal to v) so that

$$\lim_{N\to\infty} \|S_N - S\| = 0.$$

- The space $\mathbb{R}^n$ with inner product given by the dot product is a Hilbert space.

Complete sequences

- We say that an orthonormal sequence $\{\phi_1, \phi_2, \dots\}$ in a Hilbert space V is **complete** if, whenever

$$\langle \phi_n, v \rangle = 0 \quad \text{for all n}$$

we have $v = 0$.

Theorem: Let V be a (real) **Hilbert space** and $\{\phi_1, \phi_2 \ldots,\}$ be a **complete orthonormal sequence** in V.

Then the series

$$\sum_{n=1}^{\infty} \langle \phi_n, v \rangle \phi_n$$

converges to v and satisfies

$$\boxed{\|v\|^2 = \sum_{n=1}^{\infty} |\langle \phi_n, v \rangle|^2. \qquad \text{Parseval's identity}}$$

Proof (sketch): $S_N = \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n$

- $\|S_N\|^2 = \langle S_N, S_N \rangle = \left\langle \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n, \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n \right\rangle$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \phi_n, v \rangle \langle \phi_m, v \rangle \langle \phi_n, \phi_m \rangle$$

$$= \sum_{n=1}^{N} |\langle \phi_n, v \rangle|^2$$

- Let $M < N$ then

$$\|S_N - S_M\|^2 = \Big\| \sum_{n=M+1}^{N} \langle \phi_n, v \rangle \phi_n \Big\|^2 = \sum_{n=M+1}^{N} |\langle \phi_n, v \rangle|^2$$

- $\langle v - S_N, S_N \rangle = \langle v, S_N \rangle - \langle S_N, S_N \rangle$

$$= \Big\langle v, \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n \Big\rangle - \Big\langle \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n, \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n \Big\rangle$$

$$= \sum_{n=1}^{N} \langle \phi_n, v \rangle \langle v, \phi_n \rangle - \sum_{n=1}^{N} |\langle \phi_n, v \rangle|^2 = 0$$

- $\|v\|^2 = \|v - S_N + S_N\|^2 = \|v - S_N\|^2 + 2\underbrace{\langle v - S_N, S_N \rangle}_{=0} + \|S_N\|^2$

$$= \|v - S_N\|^2 + \sum_{n=1}^{N} |\langle v, \phi_n \rangle|^2 \geq \sum_{n=1}^{N} |\langle v, \phi_n \rangle|^2$$

- So $\sum_{n=1}^{\infty} |\langle \phi_n, v\rangle|^2$ converges (in $\mathbb{R}$) and

$$\sum_{n=1}^{\infty} |\langle \phi_n, v\rangle|^2 \le \|v\|^2$$ (Bessel's inequality)

- As a consequence $\lim_{\substack{M<N \\ M,N\to\infty}} \left[\sum_{n=M+1}^{N} |\langle \phi_n, v\rangle|^2\right] = 0$

- So $\lim_{M,N\to\infty} \|S_N - S_M\| = 0$ (i.e. $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence)

So, as V is a Hilbert space, there exists $S \in V$ so that

$$\lim_{N\to\infty} \|S_N - S\| = 0$$

- For fixed n, and $N \geq n$,

$$\langle S_N, \phi_n \rangle = \left\langle \sum_{m=1}^{N} \langle \phi_m, v \rangle \phi_m, \phi_n \right\rangle$$

$$= \sum_{m=1}^{N} \langle \phi_m, v \rangle \langle \phi_m, \phi_n \rangle = \langle \phi_n, v \rangle$$

Now send $N \to \infty$:

$$\langle S, \phi_n \rangle = \langle \phi_n, v \rangle$$

But then

$$\langle v - S, \phi_n \rangle = \langle v, \phi_n \rangle - \langle S, \phi_n \rangle = \langle v, \phi_n \rangle - \langle \phi_n, v \rangle$$
$$= 0$$

As $\{\phi_1, \phi_2, \dots\}$ is complete then $v = S$.

So $\lim_{N \to \infty} \| S_N - v \| = 0$ $\blacksquare$

Back to Fourier series

- We want to apply this result to Fourier series.

- Our first serious problem is that the space $\mathcal{R}^2[-\pi, \pi]$ is **not a Hilbert space**.

  <u>Lesbegue</u> $L^2([-\pi, \pi])$

- This can be rectified by working in a slightly larger space and slightly redefining the integral. **We will not worry about this**.

- In this slightly larger Hilbert space, the sequence

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \dots \right\},$$

  is a **complete orthonormal sequence**.

- Applying our result, given any $f(x) \in \mathcal{R}^2[-\pi, \pi]$ the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\},$$

  converges to $f(x)$.

$$\lim_{N \to \infty} \left\| \frac{1}{2}a_0 + \sum_{n=1}^{N} \{a_n \cos(nx) + b_n \sin(nx)\} - f(x) \right\| = 0$$

- We say that the Fourier series **converges in mean** to $f(x)$.

A consequence

Theorem: If $f(x) \in \mathcal{R}^2[-\pi, \pi]$ then

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left\{a_n^2 + b_n^2\right\}.}$$

(Parseval's identity)

Proof:

We know that if $\{\phi_1, \phi_2, \dots\}$ is a complete orthonormal sequence in a Hilbert space, then

$$\|v\|^2 = \sum_{n=1}^{\infty} |\langle \phi_n, v\rangle|^2$$

For the Fourier basis, we take

$$\{\phi_1, \phi_2, \dots\} = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(x), \frac{1}{\sqrt{\pi}}\sin(x), \dots\right.$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \left|\left\langle \frac{1}{\sqrt{2\pi}}, f\right\rangle\right|^2 + \sum_{n=1}^{\infty} \left\{ \left\langle \frac{1}{\sqrt{\pi}} \cos(nx), f \right\rangle^2 + \left\langle \frac{1}{\sqrt{\pi}} \sin(nx), f \right\rangle^2 \right\}$$

As, $\left\langle \frac{1}{\sqrt{2\pi}}, f\right\rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} f(x) dx$

$$= \frac{1}{\sqrt{2\pi}} a_0$$

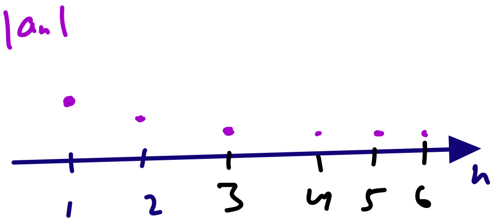
$$\left\langle \frac{1}{\sqrt{\pi}} \cos(nx), f\right\rangle = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \cos(nx) f(x) dx = \sqrt{\pi}\, a_n$$

$$\left\langle \frac{1}{\sqrt{\pi}} \sin(nx), f\right\rangle = \cdots = \sqrt{\pi}\, b_n$$

$$\Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \{ a_n^2 + b_n^2$$

A secondary consequence

Suppose that $f(x) \in \mathcal{R}^2[-\pi, \pi]$ and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$. What must be true about the sequence a_n?



A) They are decreasing

B) $\lim_{n\to\infty} a_n = 0$

C) $a_n = 0$ for all sufficiently large n

D) None of the above

A second issue

- Recall that a Fourier series **converges in mean** if

$$\lim_{N\to\infty}\left\|\frac{1}{2}a_0+\sum_{n=1}^{N}\left\{a_n\cos(nx)+b_n\sin(nx)\right\}-f(x)\right\|=0.$$

mean convergence $\not\Rightarrow$ pointwise convergence.

- This **does not** mean that

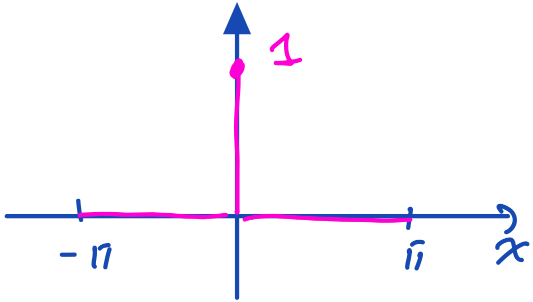
$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left\{a_n\cos(nx)+b_n\sin(nx)\right\}$$

for every $x\in[-\pi,\pi]$.

An example

$\mathbb{1}_{\{0\}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

Consider $\mathbb{1}_{\{0\}}(x) \in \mathcal{R}^2[\overset{-\pi}{\cancel{-1}}, \overset{\pi}{\cancel{1}}]$. What is $\|\mathbb{1}_{\{0\}}\|$?



A) 0

B) 1

C) 2

D) Undefined

$$\|\mathbb{1}_{\{0\}}(x)\|^2 = \int_{-\pi}^{\pi} (\mathbb{1}_{\{0\}}(x))^2 \, dx = \int_{-\pi}^{\pi} \mathbb{1}_{\{0\}}(x) \, dx = 0$$

- **Definition:** We say that a function $h \in \mathcal{R}^2[a,b]$ is a **null function** if

$$\|h\| = 0.$$

- The fact the Fourier series converge in mean then says that for $f \in \mathcal{R}^2[-\pi,\pi]$ we have

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big\{a_n\cos(nx) + b_n\sin(nx)\Big\} + \textbf{a null function}.$$

See you next time!