Revisiting the Phase Portrait

Recall that a 2D system is a DE of the form

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2)\\ f_2(x_1,x_2)\end{bmatrix}$$

$x_1 = x_1(t)$

$x_2 = x_2(t)$

$\in \mathbb{R}$

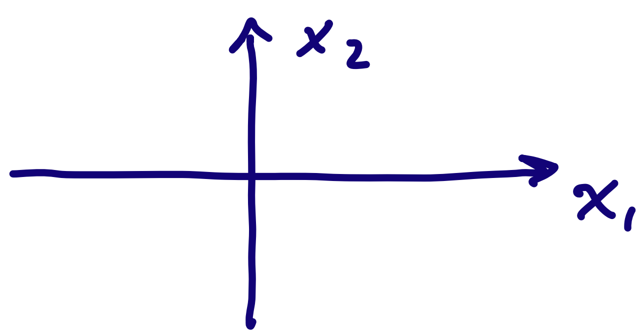
or in vector form

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

$$\left(\underline{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \quad \underline{f}(\underline{x}) = \begin{bmatrix} f_1(x)\\ f_2(x)\end{bmatrix} \right)$$

$\vec{x}$

The phase plane is the cartesian plane $x_1 - x_2$



Given a point $\underline{x_0}$ in the phase plane the $\underbrace{\text{solution of the IVP}}$

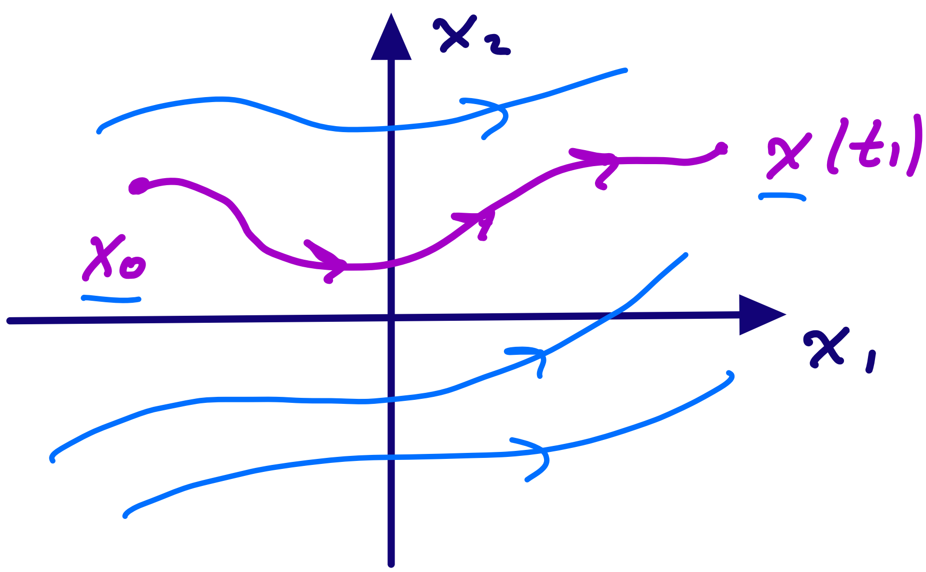
$$\begin{cases} \underline{\dot{x}} = f(\underline{x}) \\ \underline{x}(0) = \underline{x_0} \end{cases}$$

(This is warrantied by the Existence and Uniqueness Theorem that will be covered in the next lecture)

is denoted as $x(t; x_0)$ or for simplicity $x(t)$.

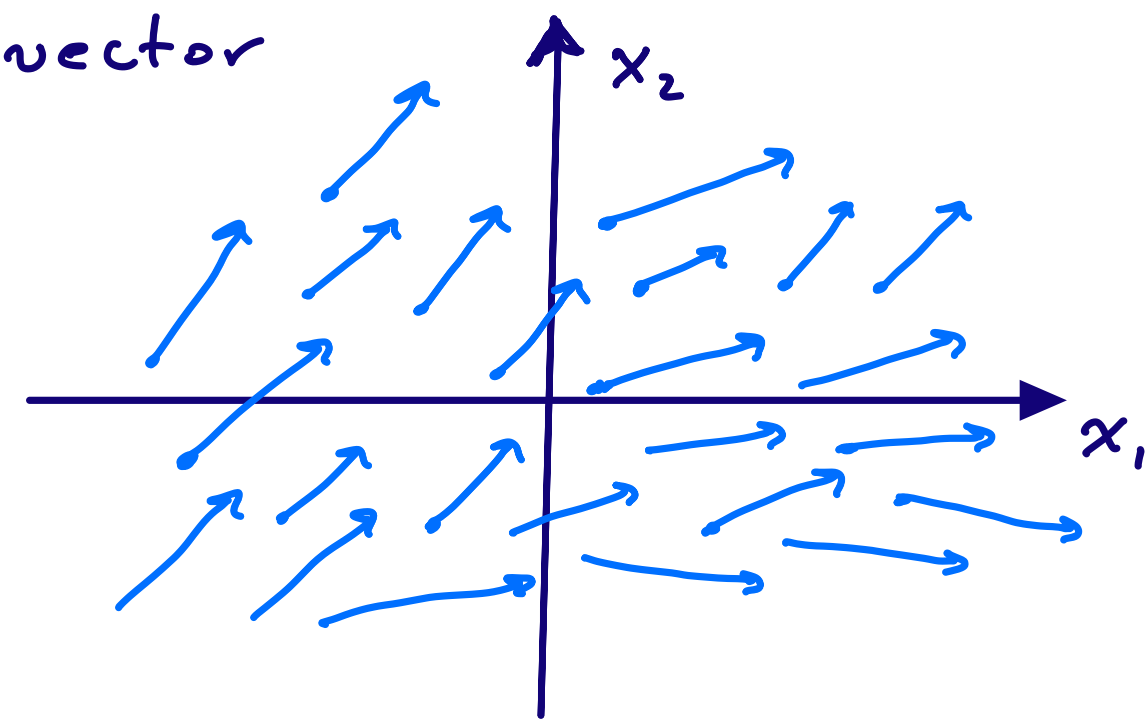
A trajectory in the phase plane is a curve defined as

$$\{ \underline{x}(t) \in \mathbb{R}^2 \mid 0 \leq t \leq t_1 \}$$

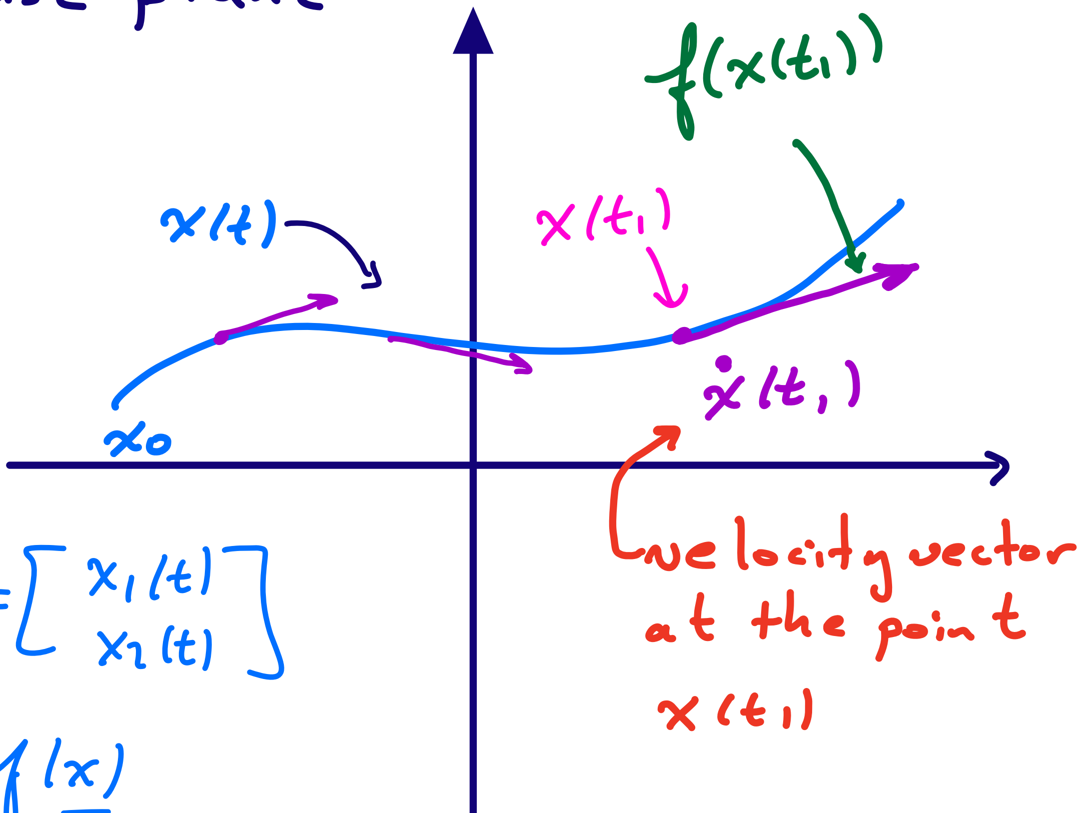


A sketch of different trajectories in the phase space is called a <u>phase portrait.</u>

Notice that the RHS of $\dot{\underline{x}} = \underline{f}(\underline{x})$ is a <u>vector field</u>, i.e. for each $x \in \mathbb{R}^2$, $f(x)$ is a vector



Let's consider a trajectory in the phase plane



$$\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

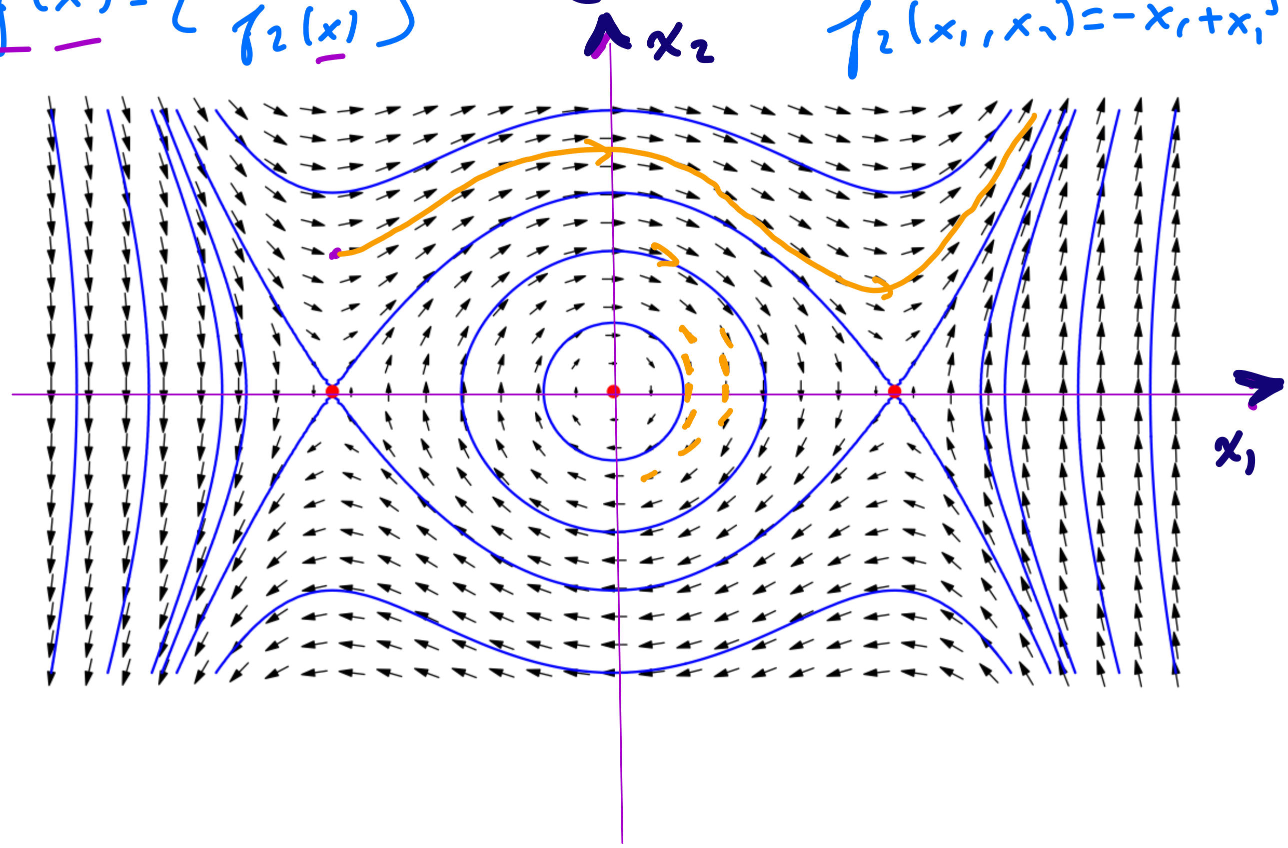
$$\begin{cases} \underline{\dot{x}} = \underline{f}(\underline{x}) \\ \underline{x}(0) = x_0 \end{cases}$$

$$\underline{\dot{x}}(t_1) = \underline{f}(\underline{x}(t_1))$$

Example: $\begin{cases} \dot{x}_1 = x_2 & \rightarrow f_1(x_1,x_2) = x_2 \\ \dot{x}_2 = -x_1 + x_1^3 & \rightarrow f_2(x_1,x_2) = -x_1 + x_1^3 \end{cases}$

$\dot{\underline{x}} = \underline{f}(\underline{x})$

$\underline{f}(\underline{x}) = \begin{bmatrix} f_1(\underline{x}) \\ f_2(\underline{x}) \end{bmatrix}$



When plotting the phase portrait it often helps to see a grid of representative vectors in the vector field.

Unfortunately, the arrowheads and different lengths can clutter such pictures. A plot of the direction field is clearer: short line segments are used to indicate the local direction of the flow.

So, why do we care about plotting the phase portrait? Because explicit solutions, even if available may NOT BE USEFUL!!

Example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = xy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1 - x^2 + y^2}{2}$$

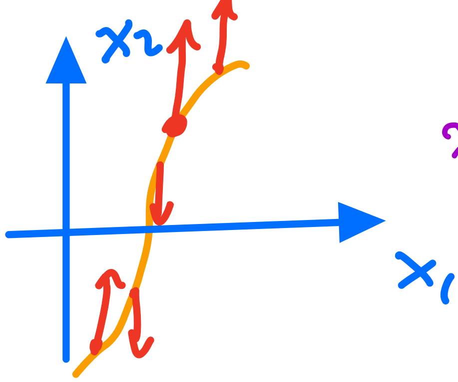
with initial condition $x(0) = x_0, y(0) = y_0$ is

$$x(t) = \frac{2x_0}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2)\cos t - 2y_0 \sin t},$$
$$y(t) = \frac{2y_0 \cos t + (1 - x_0^2 - y_0^2)\sin t}{1 + x_0^2 + y_0^2 + (1 - x_0^2 - y_0^2)\cos t - 2y_0 \sin t}.$$

In this case, a phase portrait is much more useful.

To sketch the phase portraits it is helpful to plot the nullclines, defined as the curves where either $\dot{x}_1 = 0$ or $\dot{x}_2 = 0$

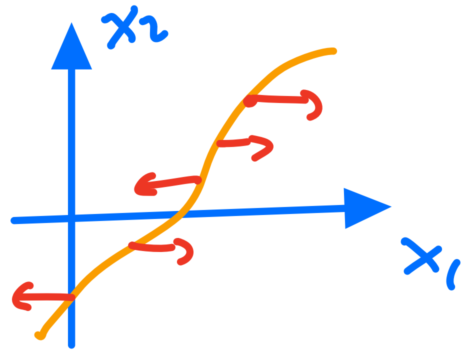
- If $\dot{x}_1 = f_1(x_1, x_2) = 0$ we might obtain some function g such that $x_2 = g(x_1)$ $\qquad \dot{x}_1 = 0$



x_1-nullcline (as the function does not "climb" in the x direction)

- Similarly, if $\dot{x}_2 = f_2(x_1, x_2) = 0$ along the curve in which $f_2(x_1, x_2) = 0$, the vector field will only point in the horizontal

(x- direction)



We will see soon some examples how to use the nullclines to help sketch the phase portrait.