Math 135, Spring 2022

Lecture #22: PDEs and boundary value problems

Wednesday May 18^{th}

Recap

- We finished the topic “Fourier Series”.

So far we have covered

- Existence and uniqueness theory for ODEs
- Laplace transform
- Fourier series

And we are now going to deal with

- Partial Differential Equations (PDEs)

Learning objectives

Today we will discuss:

- Fourier series in an arbitrary domain.
- Derivation of the $1d$ wave equation as a model for a vibrating string.
- Solving the initial boundary value problem for the $1d$ wave equation by separation of variables.

Last topic on Fourier series

Scaling

Suppose that $f(x)$ is defined on $[-L, L]$, where $L > 0$.
We want to construct the Fourier series for $f(x)$.

Let $g(t) = f(Lt/\pi)$ be defined on $[-\pi, \pi]$

The Fourier series for $g(t)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{ a_n \cos(nt) + b_n \sin(nt) \}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt)\,dt \quad \text{and} \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(nt)\,dt$$

As $x = Lt/\pi$, take our Fourier series for $f(x)$ to be

$$t = \frac{\pi}{L}x$$

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n\cos\left(n\frac{\pi x}{L}\right) + b_n\sin\left(n\frac{\pi x}{L}\right)\right\}$$

where

$$t = \frac{\pi}{L}x \longrightarrow dt = \frac{\pi}{L}dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(Lt/\pi)\cos(nt)\,dt = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi x}{L}\right)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(Lt/\pi)\sin(nt)\,dt = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Definition: Let $f(x)$ be an integrable function on the interval $[-L, L]$, where $L > 0$. We define the **Fourier series** for $f(x)$ on $[-L, L]$ to be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right],$$

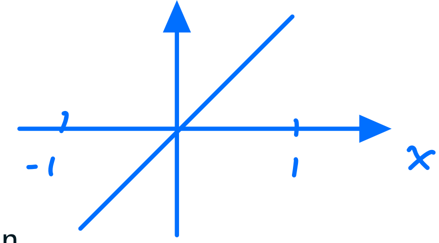
where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right) dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right) dx.$$

An example

odd

Find the Fourier series for $f(x) = x$ on $[-1, 1]$.



Step 1: Use any available symmetries to simplify the computation.

$a_n = 0$ for all n as $f(x)$ is odd

Step 2: Compute

$$b_n = 2\int_0^1 x\sin(n\pi x)\,dx$$

$$b_n = \int_{-1}^{1} \underbrace{x\sin(n\pi x)}_{\text{even}}\,dx = 2\int_0^1 x\sin(n\pi x)\,dx$$

As $n \geq 1$,

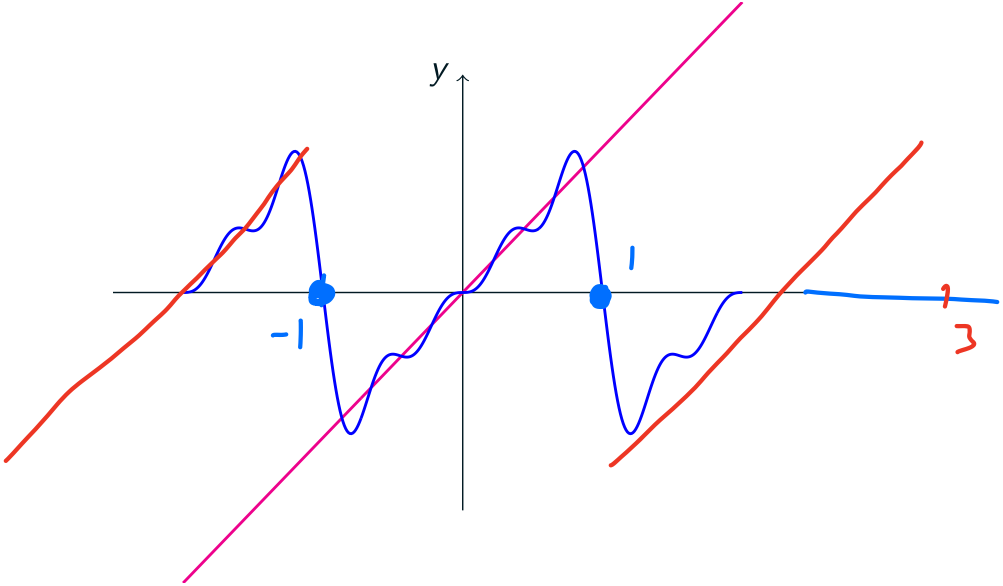
$$b_n = \left[-\frac{2}{n\pi}x\cos(n\pi x)\right]\Big|_{x=0}^{x=1} + \int_0^1 \frac{2}{n\pi}\cos(n\pi x)\,dx$$

$$= -\frac{2}{n\pi}(-1)^n$$

$$= -\frac{2}{n\pi}(-1)^{n+1}$$

So, the Fourier series for $f(x) = x$ on $[-1, 1]$ is

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x)$$



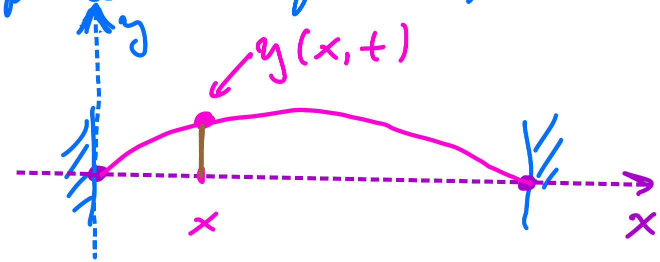
$$f(x) = x$$

$$S_4(x) = \sum_{n=1}^{4} (-1)^{n+1} \frac{2}{n\pi} \sin(nx)$$

PDEs and boundary value problems

The vibrating string

Let $y = y(x,t)$ be the vertical displacement at time t of a tight string constraint at its ends



Assumptions:

- Motion is strictly vertical
- No resistance to bending
- Small vibrations
- Constant density ρ
- Constant tension T

The $1d$ wave equation

We are lead to consider the $1d$ **wave equation**:

$y = y(x,t)$

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

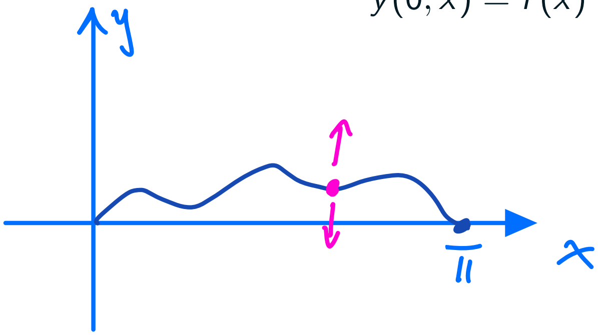
$c^2 = \frac{T}{\rho}$

with the **boundary conditions**

$$y(t,0) = 0 = y(t,\pi)$$

and the **initial conditions**

$$y(0,x) = f(x) \quad \text{and} \quad \frac{\partial y}{\partial t}(0,x) = g(x).$$



Linearity

Theorem: If $y_1(t,x)$ and $y_2(t,x)$ are solutions of the linear wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with the **boundary conditions**

$$y(t,0) = 0 = y(t,\pi)$$

and C_1, C_2 are constants, then so is

$$y(t,x) = C_1 y_1(t,x) + C_2 y_2(t,x).$$

Proof: Easy computation.

Separation of variables

$$\begin{cases} \partial^2_{tt} y = c^2 \partial^2_{xx} y \\ y(0,t) = y(\pi,t) = 0 \end{cases}$$

Try to find a solution of the form

$$y(x,t) = v(t)\,u(x) \quad \text{where } u(0) = u(\pi) = 0$$

(educated guess)

Plug this into: $v''(t)\,u(x) = c^2 v(t)\,u''(x)$

$$\frac{1}{c^2}\frac{v''(t)}{v(t)} = \frac{u''(x)}{u(x)} = -\lambda \quad \text{a constant!}$$

Spatial oscillations

$$\frac{u''(x)}{u(x)} = -\lambda$$

If $\lambda < 0$, what is the general solution of

$$u''(x) + \lambda u(x) = 0 \qquad ?$$

boundary value problem

$u(0) = u(\pi) = 0$

A) $C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

B) $C_1 \cos(\sqrt{-\lambda}x) + C_2 \sin(\sqrt{-\lambda}x)$

C) $C_1 x + C_2$

D) None of the above

$u(x) = e^{\mu x}$

Charact. poly.: $\mu^2 + \lambda = 0 \Rightarrow \mu^2 = -\lambda > 0$

so $\mu = \pm\sqrt{-\lambda}$

$u'' + \lambda u = 0$

If $\lambda < 0$: $u(x) = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$

Boundary conditions:

$0 = u(0) = c_1 + c_2 \longrightarrow c_2 = -c_1$

$0 = u(\pi) = c_1 e^{\sqrt{-\lambda}\,\pi} + c_2 e^{-\sqrt{-\lambda}\,\pi}$

$= c_1 \underbrace{\left(e^{\sqrt{-\lambda}\,\pi} - e^{-\sqrt{-\lambda}\,\pi}\right)}_{\neq 0} \longrightarrow c_1 = 0$

$\rightarrow c_2 = 0$

No non-trivial solutions.

If $\lambda > 0$, what is the general solution of

$$u''(x) + \lambda u(x) = 0 \qquad ?$$

A) $C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

B) $C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

C) $C_1 x + C_2$

D) None of the above

Eigenfunctions

- We have shown that we only have non-trivial solutions of the **boundary value problem**

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = 0 = u(\pi) \end{cases}$$

  when $\lambda = n^2$ for $n = 1, 2, \ldots$.

- We can think of $-\frac{d^2}{dx^2}$, together with the boundary conditions $u(0) = 0 = u(\pi)$ as a **linear operator**, i.e. $-\frac{d^2}{dx^2} : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}, \mathcal{Y}$ are certain space functions.

- We call the functions

$$u_n(x) = \sin(nx)$$

  eigenfunctions of this operator and we call the values $\lambda_n = n^2$ the corresponding **eigenvalues**.

Time oscillation

What is the general solution of

$$v''(t) + c^2 n^2 v(t) = 0 \qquad ?$$

A) $C_1 e^{nct} + C_2 e^{-nct}$

B) $C_1 \cos(nct) + C_2 \sin(nct)$

C) $C_1 nct + C_2$

D) None of the above

- Suppose we take $\frac{\partial y}{\partial t}(0,x) = g(x) = 0$.

- Consequently, for each $n \geq 1$ we have a corresponding solution of

$$\begin{cases} v'' + n^2 c^2 v = 0 \\ v'(0) = 0 \end{cases}$$

given by

$$v_n(t) = \cos(nct).$$

Solving our PDE

See you next time!