Home Work 1

1) We have the equation:

$$x^2 y'' + pxy' + qy = 0 \quad (1)$$

with $x = e^z \Rightarrow z = \ln x$

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x} \Rightarrow \frac{dy}{dx} \cdot \frac{dz}{dy} = \frac{dz}{dx} = \frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$

also, $$y''_{(x)} = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d}{dx} \cdot \left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right)$$

$$= -\frac{1}{x^2} \cdot \frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} = \frac{1}{x^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$\Rightarrow$ We have:

$$x y' = x \frac{dy}{dx} = \frac{dy}{dz}$$

$$x^2 y'' = x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow (1) \Leftrightarrow \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) + p\frac{dy}{dz} + qy = 0$$

$$\Leftrightarrow \frac{d^2 y}{dz^2} + (p-1)\frac{dy}{dz} + qy = 0$$

$\Rightarrow$ We have already changed the original equation into an equation with constant coefficient are 1, $p-1$, & q.

Then applying for $2x^2 y'' + 10xy' + 8y = 0$

$$\Leftrightarrow x^2 y'' + 5xy' + 4y = 0$$

$$\Leftrightarrow \frac{d^2 y}{dz^2} + (5-1)\frac{dy}{dz} + 4y = 0$$

$$\Leftrightarrow \frac{d^2y}{dz^2} + 4\frac{dy}{dz} + 4y = 0$$

$\Rightarrow$ the auxiliary equation:

$$m^2 + 4m + 4 = 0 \Leftrightarrow (m+2)^2 = 0$$

$\Leftrightarrow$ repeated root: $m = -2$

$\Rightarrow$ the general solution:

$$y(z) = C_1 e^{-2z} + C_2 z e^{-2z}$$

Since $z = \ln x \Rightarrow e^{-2z} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$

$$\Rightarrow y(x) = C_1 \frac{1}{x^2} + C_2 \cdot \ln x \, \frac{1}{x^2}$$

$$\Rightarrow \boxed{y(x) = \frac{1}{x^2}\left(C_1 + C_2 \ln x\right) \quad (\text{with } x > 0)}$$

2) a) $x'' + x' - 2x = 0$ (1), $x(0) = x_0$, $x'(0) = v_0$

let $\begin{cases} y_1 = x \\ y_2 = x' \end{cases} \Rightarrow \begin{cases} y_1' = x' \\ y_2' = x'' \end{cases}$

$\Rightarrow$ (1) $\Leftrightarrow$ $x'' = 2x - x'$

$\Leftrightarrow y_2' = 2y_1 - y_2$

So, we have:

$$\begin{cases} y_1' = y_2 \\ y_2' = 2y_1 - y_2 \end{cases} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We look for solution of the form:

$$x(t) = e^{\lambda t} v$$

$$\Rightarrow (A - \lambda I) v = 0 \Rightarrow \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$\Rightarrow$ the auxiliary equation is:

$\lambda^2 - \tau \lambda + \Delta = 0$ with $\tau = TR(A) = -1$

$\Delta = \det A = -2$

$\Rightarrow \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = 1, -2$

* With $\lambda = 1$, we have the eigenvector:

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -v_1 + v_2 = 0 \\ 2v_1 - 2v_2 = 0 \end{cases} \Leftrightarrow v_1 - v_2 = 0$$

$$\Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

* With $\lambda = -2$, we have the eigenvector:

$$\begin{pmatrix} +2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow 2v_1 + v_2 = 0 \Leftrightarrow 2v_1 = -v_2$$

$$\Rightarrow \vec{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\Rightarrow$ the general solution:

$$\vec{x}(t) = C_1 e^{t} \vec{v_1} + C_2 e^{-2t} \vec{v_2}$$

$$= C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

* Used $x(0) = x_0$ & $x'(0) = v_0$

We have:

$$\vec{x}(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ x' \end{bmatrix}$$

$$\Rightarrow \vec{x}(0) = \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x(0) = C_1 + C_2 = x_0 \\ x'(0) = C_1 - 2C_2 = v_0 \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{2x_0 + v_0}{3} \\ C_2 = \dfrac{x_0 - v_0}{3} \end{cases}$$

$$\vec{x}(t) = \frac{2x_0 + v_0}{3} e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{x_0 - v_0}{3} e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ x' \end{bmatrix}$$

$\Rightarrow$ final Solution:

$$\boxed{x(t) = \frac{2x_0 + v_0}{3} e^{t} + \frac{x_0 - v_0}{3} e^{-2t}}$$

b) $x'' + x = 0$ (1), $x(0) = x_0$, $x'(0) = v_0$

Let $\begin{cases} y_1 = x \Rightarrow y_1' = x' \\ y_2 = x' \Rightarrow y_2' = x'' \end{cases}$

$\Rightarrow$ (1) $\Rightarrow x'' = -x \Rightarrow y_2' = -x = -y_1$

Then, we have:

$$\begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases}$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We look for solution of the form:

$$x(t) = e^{\lambda t} . v$$

$$\Rightarrow (A - \lambda I) v = 0 \Rightarrow \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

The auxiliary equation:

$$\lambda^2 - \tau . \lambda + \Delta = 0 \quad \text{with } \tau = Tr(A) = 0$$

$$\Rightarrow \lambda^2 + 1 = 0 \qquad \Delta = \det A = 0 + 1 = 1$$

$$\Rightarrow \lambda = \pm j$$

With $\lambda = j$, we have the eigenvector:

$$\begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} -jv_1 + v_2 = 0 \\ -v_1 - jv_2 = 0 \end{cases}$$

$$\Leftrightarrow v_1 + jv_2 = 0 \Leftrightarrow v_1 = -jv_2$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -jv_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -j \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{v} = \begin{bmatrix} -j \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + j\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \& \quad v_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Also, $\lambda = \pm j \Rightarrow a = 0, \ b = 1$

$\Rightarrow$ the general solution:

$$\vec{x}(t) = C_1 e^{at}\left[\cos(bt)\, v_1 - \sin(bt)\, v_2\right] + C_2 e^{at}\left[\sin(bt)\, v_1 + \cos(bt) . v_2\right]$$

$$= C_1\left(\cos t . \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin t . \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) + C_2\left(\sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \cos t . \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right)$$

$$= C_1\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + C_2\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} C_1 \sin t - C_2 \cos t \\ C_1 \cos t + C_2 \sin t \end{pmatrix}$$

$$= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$$

$$\Rightarrow \vec{x}(0) = \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} -C_2 \\ C_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} C_2 = -x_0 \\ C_1 = v_0 \end{cases}$$

$\Rightarrow$ the final solution:

$$x(t) = C_1 \sin t - C_2 \cos t$$

$$\Leftrightarrow \boxed{x(t) = v_0 \sin t + x_0 \cos t}$$

(*) Using the method in Section 17.

a) $x'' + x' - 2x = 0, \quad x(0) = x_0, \; x'(0) = v_0$

We have the auxiliary equation:

$$m^2 + m - 2 = 0 \Leftrightarrow \quad m = 1, -2$$

$\Rightarrow$ the general solution:

$$x(t) = C_1 e^{t} + C_2 e^{-2t}$$

$$\Rightarrow x'(t) = C_1 e^{t} - 2C_2 e^{-2t}$$

$$\Rightarrow \begin{cases} x(0) = C_1 + C_2 = x_0 \\ x'(0) = C_1 - 2C_2 = v_0 \end{cases} \Rightarrow \begin{cases} C_1 = \dfrac{2x_0 + v_0}{3} \\ C_2 = \dfrac{x_0 - v_0}{3} \end{cases}$$

$$\Rightarrow \boxed{x(t) = \frac{2x_0 + v_0}{3} e^{t} + \frac{x_0 - v_0}{3} e^{-2t}}$$

As we can see, this result is same with the one using the system of first order linear differential equation.

b) $x'' + x = 0$, $x(0) = x_0$, $x'(0) = v_0$

the auxiliary equation:

$$m^2 + 1 = 0 \Rightarrow m = \pm j \Rightarrow \alpha = 0, \beta = 1$$

$\Rightarrow$ the general solution:

$$x(t) = C_1 e^{jt} + C_2 e^{-jt}$$

$$= e^{\alpha t}\left[A_1 \cos(\beta t) + A_2 \sin(\beta t)\right]$$

$$\Rightarrow x(t) = A_1 \cos t + A_2 \sin t$$

$$x'(t) = -A_1 \sin t + A_2 \cos t$$

$$\Rightarrow \begin{cases} x(0) = A_1 = x_0 \\ x'(0) = A_2 = v_0 \end{cases} \Rightarrow \boxed{x(t) = x_0 \cos t + v_0 \sin t}$$

This result is also matched with the one using the system of the first order linear differential equation.

3) $x' = |x|^{1/2}, \quad x(0) = 0$

$$\Leftrightarrow \frac{dx}{dt} = |x|^{1/2} = \begin{cases} x^{1/2} & x \geq 0 \\ (-x)^{1/2} & x < 0 \end{cases}$$

(*) With $x \geq 0$, we have: (*)

$$\frac{dx}{dt} = x^{1/2} \Rightarrow \frac{dx}{x^{1/2}} = dt$$

$$\Rightarrow \int \frac{dx}{\sqrt{x}} = \int dt \Rightarrow \int x^{-1/2} dx = \int dt$$

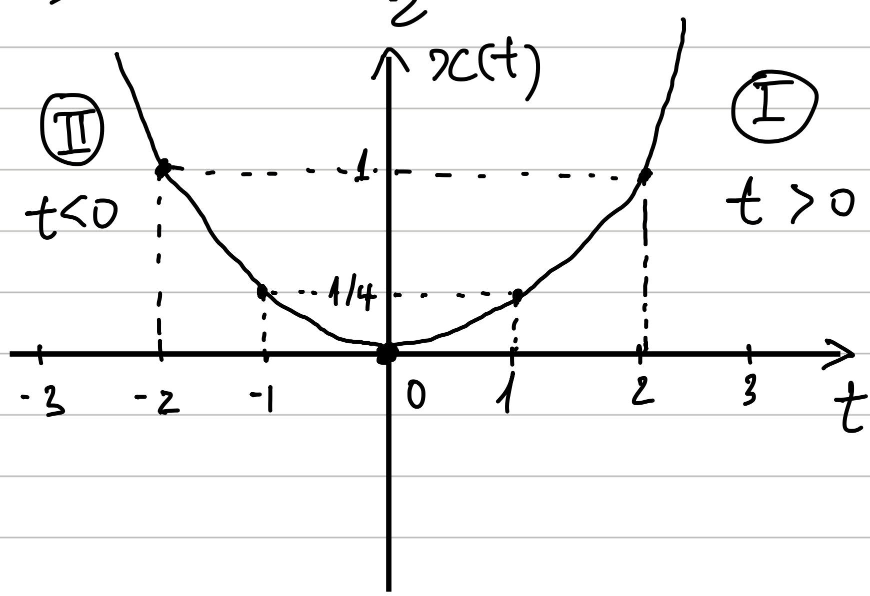
$$\Rightarrow 2x^{1/2} = t + C$$

$$\Rightarrow 2\sqrt{x} = t + C$$

$$x(0) = 0 \Rightarrow 2\sqrt{x(0)} = 0 + C \Rightarrow C = 0$$

$$\Rightarrow 2\sqrt{x} = t \Rightarrow \sqrt{x} = \frac{1}{2} t$$

$$\Rightarrow x(t) = \frac{1}{4} t^2$$



(*) With $x<0$, we have: (*)(*)

$$\frac{dx}{dt} = (-x)^{1/2} \Rightarrow \frac{dx}{\sqrt{(-x)}} = dt$$

$$\Rightarrow \int (-x)^{-1/2}\, dx = \int dt$$

$$\Rightarrow -2(-x)^{1/2} = t + C$$

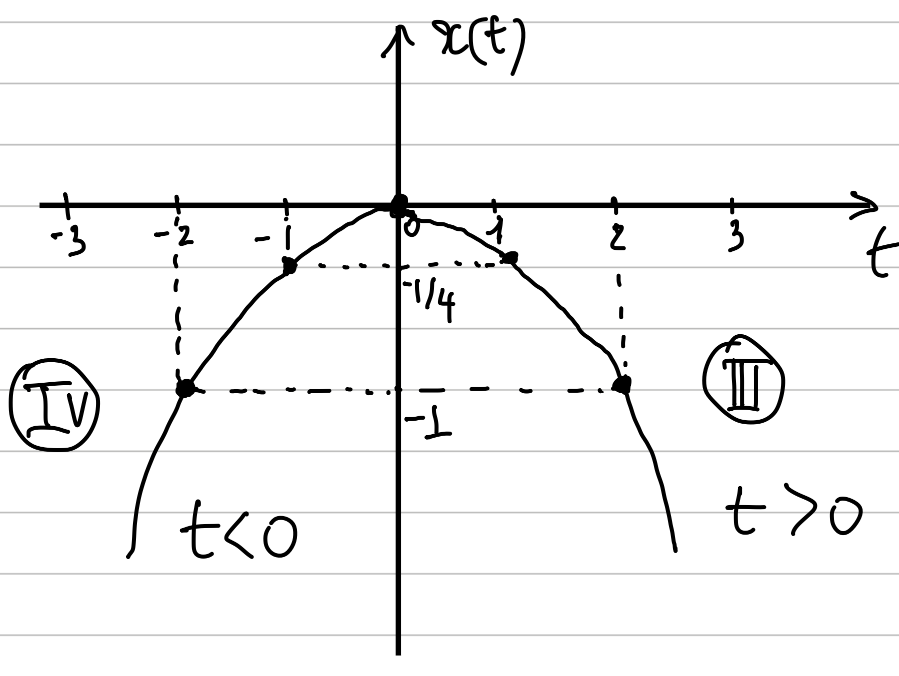
Since $x(0) = 0 \Rightarrow -2[-x(0)]^{1/2} = 0 + C$

$$\Rightarrow C = 0$$

$$\Rightarrow -2(-x)^{1/2} = t$$

$$\Rightarrow (-x)^{1/2} = -\frac{1}{2}t \Rightarrow -x = \frac{1}{4}t^2$$

$$\Rightarrow x = -\frac{1}{4}t^2 \quad (x<0)$$



Combine (*) & (*)(*) we can see the original equation has 4 solutions through the point (0, 0)

4) Given $y' = x + y$, $y(0) = 1$

a) $y_0(x) = e^x \Rightarrow y_0(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$

We have $\dfrac{dy}{dx} = x + y \Rightarrow dy = (x+y)\,dx$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} [t + y(t)]$$

$$\Rightarrow y(x) - y_0 = \int_{x_0}^{x} [t + y(t)]\,dt$$

$$\Rightarrow y(x) = y_0 + \int_{x_0}^{x} [t + y(t)]\,dt$$

$$\Rightarrow y(x) = 1 + \int_{0}^{x} [t + y(t)]\,dt$$

then apply Picard's method, we have:

$$y_{n+1}(x) = 1 + \int_0^x [t + y_n(t)]\,dt$$

$$y_1(x) = 1 + \int_0^x (t + e^t)\,dt = 1 + \left(\frac{t^2}{2} + e^t\right)\Big|_0^x$$

$$= 1 + \left(\frac{x^2}{2} + e^x\right) - 1 = \frac{x^2}{2} + e^x$$

$$\simeq \frac{x^2}{2} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$\simeq 1 + x + x^2 + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \simeq 1 + x + 2\times\frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$y_2(x) = 1 + \int_0^x \left[t + \frac{t^2}{2} + e^t \right] dt$$

$$= 1 + \left(\frac{t^2}{2} + \frac{t^3}{3.2} + e^t \right) \Big|_0^x = 1 + \frac{x^2}{2} + \frac{x^3}{2\times 3} + e^x - 1$$

$$= \frac{x^2}{2} + \frac{x^3}{2\times 3} + e^x \simeq 1 + x + 2\times\frac{x^2}{2!} + 2\times\frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x_n}{n!}$$

$$y_3(x) = 1 + \int_0^x \left[t + \frac{t^2}{2} + \frac{t^3}{6} + e^t \right] dt$$

$$= 1 + \left(\frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + e^t \right) \Big|_0^x$$

$$= 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + e^x - 1 = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + e^x$$

$$= \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!}$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\times 4} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!}$$

$$* \; y_4(x) = 1 + \int_0^x \left(t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + e^t\right) dt$$

$$= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + e^x$$

$$\simeq 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\times 4} + \frac{x^5}{3\times 4\times 5} + \frac{x^6}{6!} + \dots + \frac{x^n}{n!}$$

$$\Rightarrow y_n(x) = 1 + x + 2\times\frac{x^2}{2} + 2\times\frac{x^3}{6} + 2\times\frac{x^4}{2\times 3\times 4} + \dots$$

$$+ 2\times\frac{x^n}{2\times 3\times ..\times n} + \frac{x^{n+1}}{(n+1)!}$$

$$= 1 + x + 2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}\right] + \frac{x^{n+1}}{(n+1)!}$$

$$= 1 + x + 2(e^x - x - 1) + \frac{x^{n+1}}{(n+1)!}$$

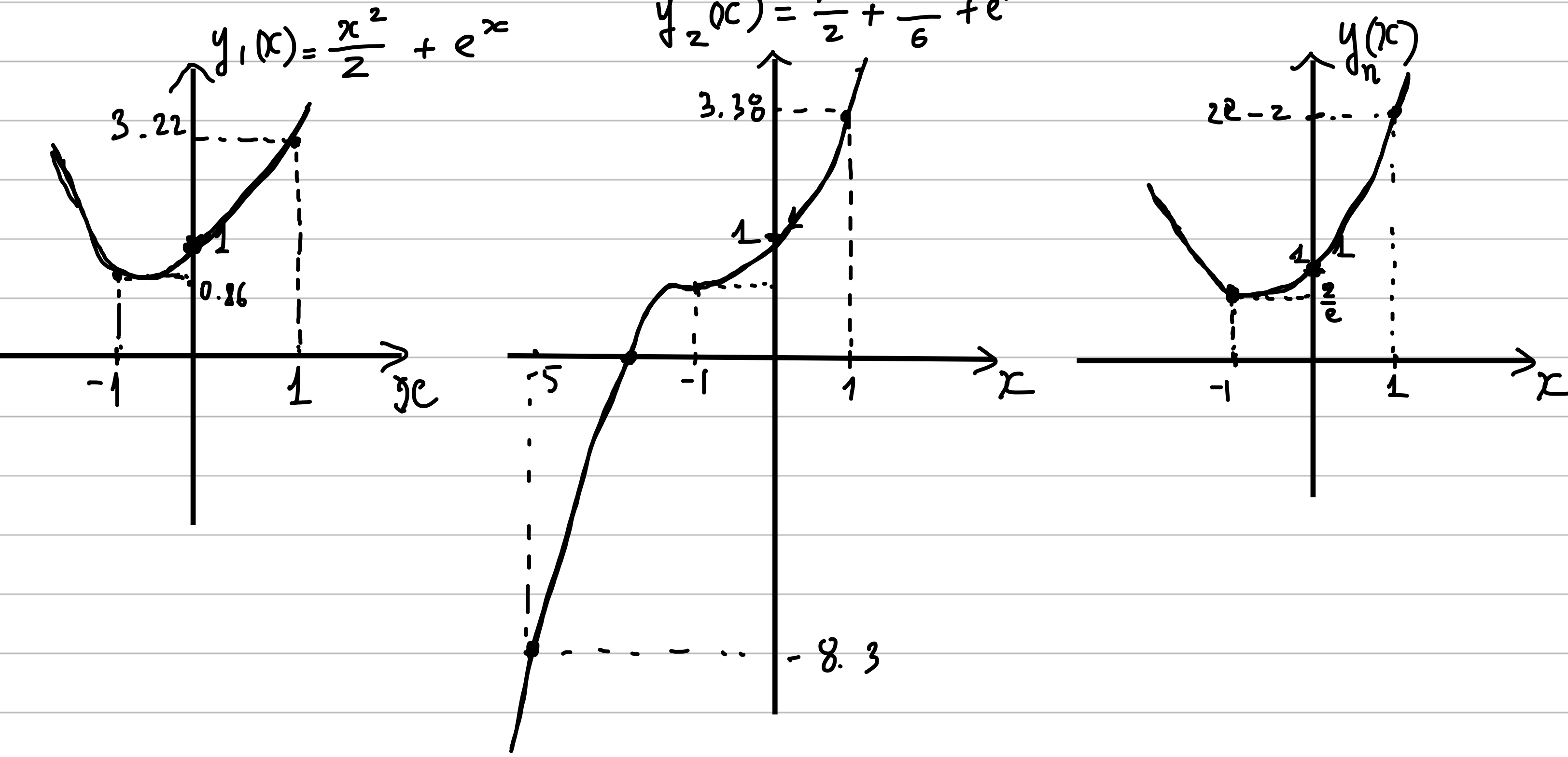
$$= \boxed{2e^x - x - 1 + \frac{x^{n+1}}{(n+1)!}}$$

Since with $n \in \mathbb{N}$, we have:

$$y(x) = \lim_{n\to\infty} y_n(x) = \lim_{n\to\infty}\left(2e^x - x - 1 + \frac{x^{n+1}}{(n+1)!}\right)$$

$$= 2e^x - x - 1 \Rightarrow \boxed{\text{the Picard iterates is converge}}$$

Plot some iterations:



b) $y_0(x) = 1 + x$, given $y' = x + y$, $y(0) = 1$

$x_0 = 0 \Rightarrow y_0 = 1$

Do the same process as part a,

then apply Picard's method, we have:

$$y_{n+1}(x) = 1 + \int_0^x [t + y_n(t)]\, dt$$

$$\Rightarrow y_1(x) = 1 + \int_0^x (t + 1 + t)\, dt = 1 + \int_0^x (2t + 1)\, dt$$

$$= 1 + (t^2 + t)\Big|_0^x = 1 + x^2 + x = 1 + x + x^2$$

$$\circledast\ y_2(x) = 1 + \int_0^x (t + 1 + t + t^2)\, dt$$

$$= 1+\int_0^x (t^2+2t+1)\,dt = 1+\left(\frac{t^3}{3}+t^2+t\right)\Big|_0^x$$

$$= 1+x+x^2+\frac{x^3}{3}$$

* $y_3(x) = 1+\int_0^x \left(t+1+t+t^2+\frac{t^3}{3}\right)dt$

$$= 1+\int_0^x \left(1+2t+t^2+\frac{t^3}{3}\right)dt$$

$$= 1+\left(t+t^2+\frac{t^3}{3}+\frac{t^4}{3\times 4}\right)\Big|_0^x$$

$$= 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}$$

* $y_4(x) = 1+\int_0^x \left(t+1+t+t^2+\frac{t^3}{3}+\frac{t^4}{12}\right)dt$

$$= 1+\int_0^x \left(1+2t+t^2+\frac{t^3}{3}+\frac{t^4}{12}\right)dt$$

$$= 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{12}+\frac{x^5}{60}$$

$$= 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{3\times 4}+\frac{x^5}{3\times 4\times 5}$$

* Similarity, $y_5 = 1+x+x^2+\frac{x^3}{3}+\frac{x^4}{3\times 4}+\frac{x^5}{3\times 4\times 5}+\frac{x^6}{3\times 4\times 5\times 6}$

$$\Rightarrow y_n(x) = 1 + x + x^2 + 2\times\frac{x^3}{2\times 3} + 2\times\frac{x^4}{2\times3\times4} + 2\times\frac{x^5}{2\times3\times4\times5}$$

$$+ 2\times\frac{x^6}{2\times3\times4\times5\times6} + \cdots + 2\times\frac{x^{n+1}}{2\times3\times4\times\cdots\times(n+1)}$$

$$= 1 + x + 2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}\right] + \frac{2.x^{n+1}}{(n+1)!}$$

Also, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$$\Rightarrow 2(e^x - 1 - x) = 2\times\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right]$$

$$\Rightarrow y_n(x) = 1 + x + 2(e^x - 1 - x) + \frac{2x^{n+1}}{(n+1)!}$$

$$\Rightarrow \boxed{y_n(x) = 2e^x - x - 1 + \frac{2x^{n+1}}{(n+1)!}}$$

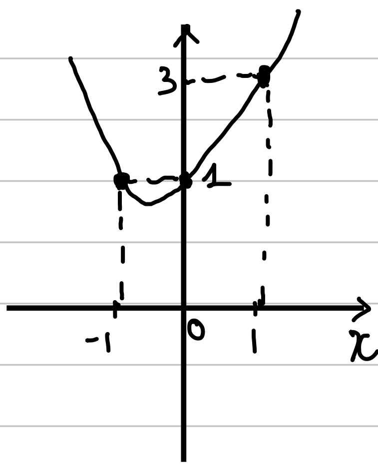
Since $\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty}\left[2e^x - x - 1 + \frac{2x^{n+1}}{(n+1)!}\right]$

$$= \boxed{2e^x - x - 1} \quad (n \in N)$$

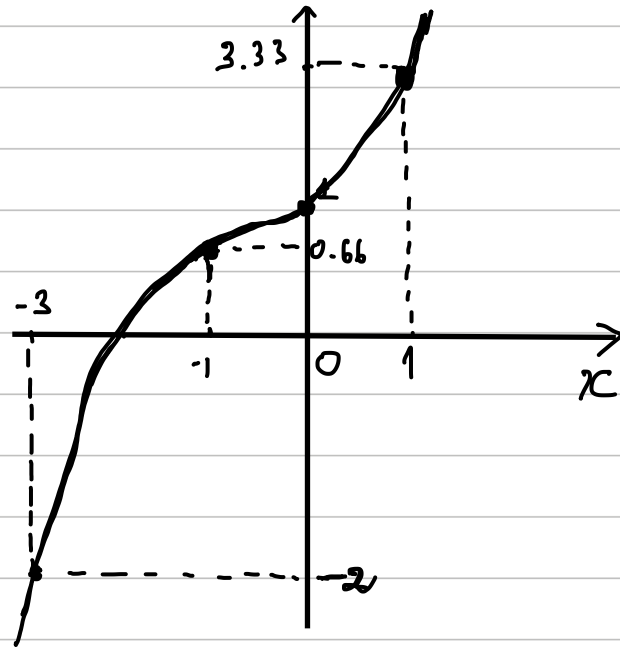
$\Rightarrow$ $\boxed{\text{the Picard iterates is converge}}$

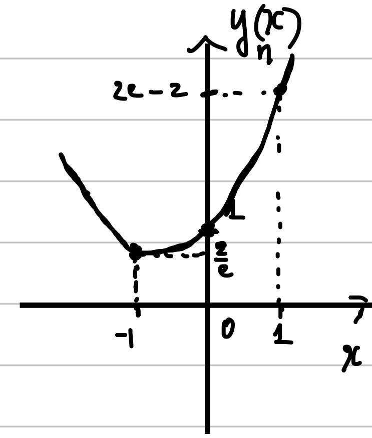
Plot some iterations:

$y_1(x) = 1 + x + x^2$



$y_2(x) = 1 + x + x^2 + \frac{x^3}{3}$





c) $y_0(x) = \cos x$, given $y' = x + y$, $y(0) = 1$

Do the same process as part a,

then apply Picard's method, we have:

$$y_{n+1}(x) = 1 + \int_0^x [t + y_n(t)]\, dt$$

$$\Rightarrow y_1(x) = 1 + \int_0^x (t + \cos t)\, dt = 1 + \left(\frac{t^2}{2} + \sin t\right)\Big|_0^x$$

$$= 1 + \frac{x^2}{2} + \sin x = (\sin x - x) + 1 + x + \frac{x^2}{2}$$

$$\Rightarrow y_2(x) = 1 + \int_0^x \left(t + 1 + \frac{t^2}{2} + \sin t\right) dt$$

$$= 1 + \left(t + \frac{t^2}{2} + \frac{t^3}{6} - \cos t\right)\Big|_0^x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \cos x + 1$$

$$= -\left(\cos x - 1 + \frac{x^2}{2}\right) + 1 + x + 2\times\frac{x^2}{2!} + \frac{x^3}{6}$$

$$\Rightarrow y_3(x) = 1 + \int_0^x \left(t + 1 + t + \frac{t^2}{2} + \frac{t^3}{6} - \cos t + 1\right) dt$$

$$= 1 + \int_0^x \left(2 + 2t + \frac{t^2}{2} + \frac{t^3}{6} - \cos t\right) dt$$

$$= 1 + \left(2t + t^2 + \frac{t^3}{6} + \frac{t^4}{24} - \sin t\right)\Big|_0^x$$

$$= 1 + 2x + x^2 + \frac{x^3}{3!} + \frac{x^4}{4!} - \sin(x)$$

$$= -(\sin x - x) + 1 + x + x^2 + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$= -\left(\sin x - x + \frac{x^3}{3!}\right) + 1 + x + x^2 + 2\times\frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\Rightarrow y_4(x) = 1 + \int_0^x \left(t + 1 + 2t + t^2 + \frac{t^3}{3!} + \frac{t^4}{4!} - \sin t\right) dt$$

$$= 1 + \left(t + \frac{3}{2}t^2 + \frac{t^3}{3} + \frac{t^4}{24} + \frac{t^5}{5!} + \cos t\right)\Big|_0^x$$

$$= x + \frac{3}{2}x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \frac{x^5}{5!} + \cos x$$

$$= \left(\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{4!}\right) + 1 + x + x^2 + \frac{x^3}{3} + 2\times\frac{x^4}{4!} + \frac{x^5}{5!}$$

$$= \left(\cos x - 1 + \frac{x^2}{2} - \frac{x^4}{4!}\right) + 1 + x + x^2 + 2\times\frac{x^3}{3!} + 2\times\frac{x^4}{4!} + \frac{x^5}{5!}$$

Doing similarity, we have:

$$y_5(x) = \left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}\right) + 1 + x + x^2 + 2\times\frac{x^3}{3!} + 2\times\frac{x^4}{4!}$$

$$+ 2\times\frac{x^5}{5!} + \frac{x^6}{6!}$$

$$y_6(x) = -\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}\right) + 1 + x + 2\times\frac{x^2}{2} +$$

$$2\times\frac{x^3}{3!} + 2\times\frac{x^4}{4!} + 2\times\frac{x^5}{5!} + 2\times\frac{x^6}{6!} + \frac{x^7}{7!}$$

*So, if n is odd, $n = 2k+1$, $k \in \mathbb{N}$, we have:

$$y_n(x) = (-1)^k\cdot\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots \pm \frac{x^n}{n!}\right)$$

$$+ 1 + x + 2\times\frac{x^2}{2!} + 2\times\frac{x^3}{3!} + \dots + 2\times\frac{x^5}{5!} + \dots + 2\times\frac{x^n}{n!}$$

$$+ \frac{x^{n+1}}{(n+1)!}$$

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\Rightarrow \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \pm \frac{x^n}{n!} = 0$$

$$\Rightarrow y_n(x) = 0 + 1 + x + 2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right] + \frac{x^{n+1}}{(n+1)!}$$

$$= 1 + x + 2\left(e^x - 1 - x\right) + \frac{x^{n+1}}{(n+1)!}$$

$$= 2e^x - x - 1 + \frac{x^{n+1}}{(n+1)!} \quad \mathrm{I}$$

* if n is even, $n = 2k$, $k \in \mathbb{N}$, we have:

$$y_n(x) = (-1)^k \left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots \pm \frac{x^n}{n!}\right)$$

$$+ 1 + x + 2 \cdot \frac{x^2}{2!} + 2 \times \frac{x^3}{3!} + \dots + 2 \times \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}$$

Since $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$

$$\Rightarrow \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{6} - \dots = 0$$

$$\Rightarrow y_n(x) = 0 + 1 + x + 2\left[\frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right] + \frac{x^{n+1}}{(n+1)!}$$

$$\Rightarrow y_n(x) = 1 + x + 2(e^x - x - 1) + \frac{x^{n+1}}{(n+1)!}$$

$$\Rightarrow y_n(x) = 2e^x - x - 1 + \frac{x^{n+1}}{(n+1)!} \quad (\text{II})$$

From (I) & (II), with any value of $n \in \mathbb{N}$, we have:

$$\boxed{y_n(x) = 2e^x - x - 1 + \frac{x^{n+1}}{(n+1)!}}$$

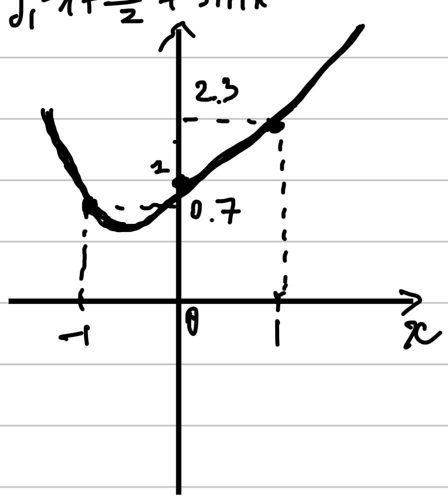
Since $\lim\limits_{n\to\infty} y_n(x) = \lim\limits_{n\to\infty}\left(2e^x - x - 1 + \frac{x^{n+1}}{(n+1)!}\right)$

$$= 2e^x - x - 1$$

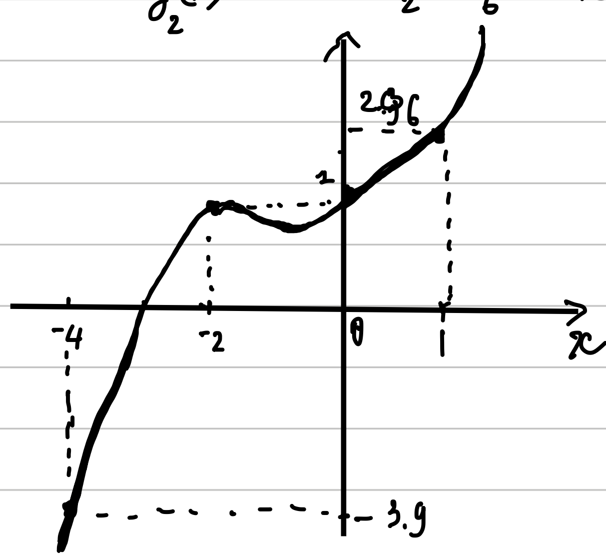
$\Rightarrow$ $\boxed{\text{the Picard iterates is converge}}$

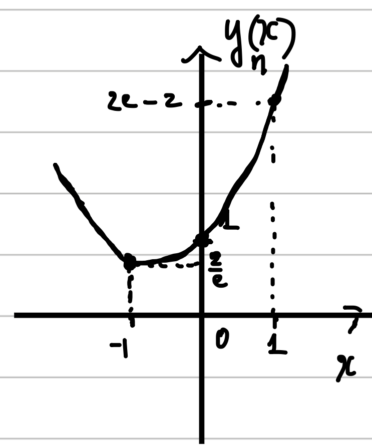
Plot some iterations:

$y_1 = 1 + \frac{x^2}{2} + \sin x$



$y_2(x) = 2 + x + \frac{x^2}{2} + \frac{x^3}{6} - \cos x$





5) Given:

$$\begin{cases} \dfrac{dy}{dx} = z & y(0) = 1 & \text{let } f_1(x,y,z) = z \,,\ x_0 = 0 \,,\ y_0 = 1 \\ \dfrac{dz}{dx} = -y & z(0) = 0 & \text{let } f_2(x,y,z) = -y \,,\ x_0 = 0 \,,\ z_0 = 0 \end{cases}$$

Using Picard's method, we have.

$$\begin{cases} y = y_0 + \int_0^x f_1(t, y(t), z(t))\, dt \\ z = z_0 + \int_0^x f_2(t, y(t), z(t))\, dt \end{cases}$$

$$\Rightarrow \begin{cases} y = 1 + \int_0^x z(t)\, dt \\ z = 0 + \int_0^x -y(t)\, dt \end{cases} \Rightarrow \begin{cases} y_{n+1} = 1 + \int_0^x z_n(t)\, dt \\ z_{n+1} = -\int_0^x y_n(t)\, dt \end{cases}$$

$$\Rightarrow \begin{cases} y_1 = 1 + \int_0^x z_0(t)\, dt = 1 \\ z_1 = -\int_0^x y_0\, dt = -\int_0^x 1.\, dt = -t\Big|_0^x = -x \end{cases}$$

$$\begin{cases} y_2 = 1 + \int_0^x z_1(t)\, dt = 1 + \int_0^x (-t)\, dt = 1 - \int_0^x t\, dt \\ z_2 = -\int_0^x y_1(t)\, dt = -\int_0^x dt = -x \end{cases}$$

$$\Rightarrow \begin{cases} y_2 = 1 - \left.\frac{t^2}{2}\right|_0^x \\ z_2 = -x \end{cases} \Rightarrow \begin{cases} y_2 = 1 - \frac{x^2}{2} \\ z_2 = -x \end{cases}$$

$$\text{*} \begin{cases} y_3 = 1 + \int_0^x z_2(t)\,dt = 1 + \int_0^x (-t)\,dt = 1 - \frac{x^2}{2} \\ z_3 = -\int_0^x y_2(t)\,dt = -\int_0^x \left(1 - \frac{t^2}{2}\right)dt = -x + \frac{x^3}{3!} \end{cases}$$

$$\text{*} \begin{cases} y_4 = 1 + \int_0^x z_3(t)\,dt = 1 + \int_0^x \left(\frac{t^3}{6} - t\right)dt \\ z_4 = -\int_0^x y_3(t)\,dt = -\int_0^x \left(1 - \frac{t^2}{2}\right)dt \end{cases}$$

$$\Rightarrow \begin{cases} y_4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ z_4 = -x + \frac{x^3}{3!} \end{cases}$$

Doing same processes; we have:

$$\begin{cases} y_5 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\ z_5 = -x + \frac{x^3}{3!} - \frac{x^5}{5!} \end{cases}$$

then :

$$\begin{cases} y_n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \\ z_n = -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = -\sum_{k=0}^{n} \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!} \end{cases}$$

$$\Rightarrow \boxed{\begin{cases} y_n(x) = \cos x \\ z_n(x) = -\sin x \end{cases} \quad \text{at } n \to \infty, \ n \in \mathbb{N}}$$

(*) Find exact solution:

$$\begin{cases} y' = z \\ z' = -y \end{cases} \Rightarrow \begin{cases} y' = z & , \ y(0) = 1 \\ z' = -y = y'' & , \ z(0) = 0 = y'(0) \end{cases}$$

$$\Rightarrow y'' + y = 0 \quad \text{with } y(0) = 1 \ , y'(0) = 0$$

Let $\begin{cases} y_1 = y \\ y_2 = y' \end{cases} \Rightarrow \begin{cases} y_1' = y' = y_2 \\ y_2' = y'' = -y = -y_1 \end{cases}$

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = -y_1 \end{cases} \Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow (A - \lambda I)\,v = 0 \Leftrightarrow \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

then the auxiliary equation:

$$\lambda^2 - \tau\lambda + \Delta = 0 \quad \text{with } \tau = 0, \ \det A = 1 = \Delta$$

$$\Leftrightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm j \Rightarrow \boxed{a = 0,\ b = 1}$$

with $\lambda = j$, the eigenvector:

$$\begin{bmatrix} -j & 1 \\ -1 & -j \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} -v_1 j + v_2 = 0 \\ -v_1 - j v_2 = 0 \end{cases}$$

$$\Leftrightarrow v_1 j = v_2 \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ j v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$\Leftrightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + j\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors

$$\Rightarrow \vec{y}(x) = C_1 e^{ax}\left[\cos(bx).\,v_1 - \sin(bx)\,v_2\right]$$

$$+ C_2 e^{at} \left[\sin(bt)\, v_1 + \cos(bt) .\, v_2 \right]$$

$$= C_1 \left(\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + C_2 \left(\sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\Rightarrow \vec{y}(x) = \begin{pmatrix} C_1 \cos x + C_2 \sin x \\ -C_1 \sin x + C_2 \cos x \end{pmatrix} = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix}$$

$$\Rightarrow \begin{cases} y(x) = C_1 \cos x + C_2 \sin x \\ y'(x) = -C_1 \sin x + C_2 \cos x \end{cases}$$ (general solution)

Also have : $y(0) = 1$, $y'(0) = 0$

$$\Rightarrow \begin{cases} y(0) = C_1 = 1 \\ y'(0) = C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$$

$$\Rightarrow y(x) = \cos x \quad \& \quad z(x) = \frac{dy}{dx} = -\sin x$$

$$\Rightarrow \boxed{\begin{cases} y(x) = \cos x \\ z(x) = -\sin x \end{cases}}$$

(exactly solution)

And we can see this solution matches with the Picard's method.