Lecture 3

Existence and Uniqueness of Solutions to ODEs (Chapter 13)

IVP $\begin{cases} y'(t) = f(y,t) & t \in \mathbb{R} \\ y(t_0) = y_0 & y \in \mathbb{R} \end{cases}$

Soln. A sol. to (IVP) is a function $\phi \in C^1(I)$ where $I \subset \mathbb{R}$ is an interval, such that

$$\phi'(t) = f(\phi(t), t) \text{ for all } t \in I.$$

Notation:

$C^1(I) = C^1(I, \mathbb{R})$ is the set of all cont. functions with domain I and range $\mathbb{R}$

$$g : I \to \mathbb{R}$$

that are once differentiable
and the derivative is cont.

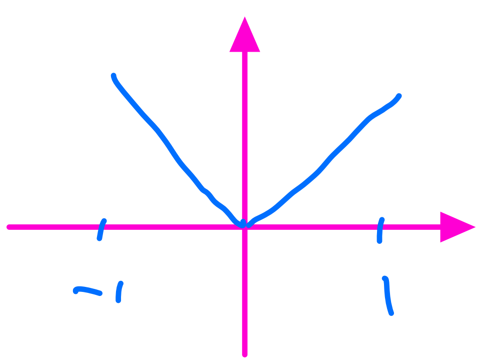
$I = (a,b), [a,b], [a,b)$
$(a,\infty), (-\infty, b)$

$C^1((-1,1);\mathbb{R})$

$g_1(x) = x$ $\quad$ $g_1 \in C^1((-1,1);\mathbb{R})$ ✓

$g_1'(x) = 1$

$g_2(x) = |x|$ $\quad$ $g_2 \notin C^1((-1,1);\mathbb{R})$

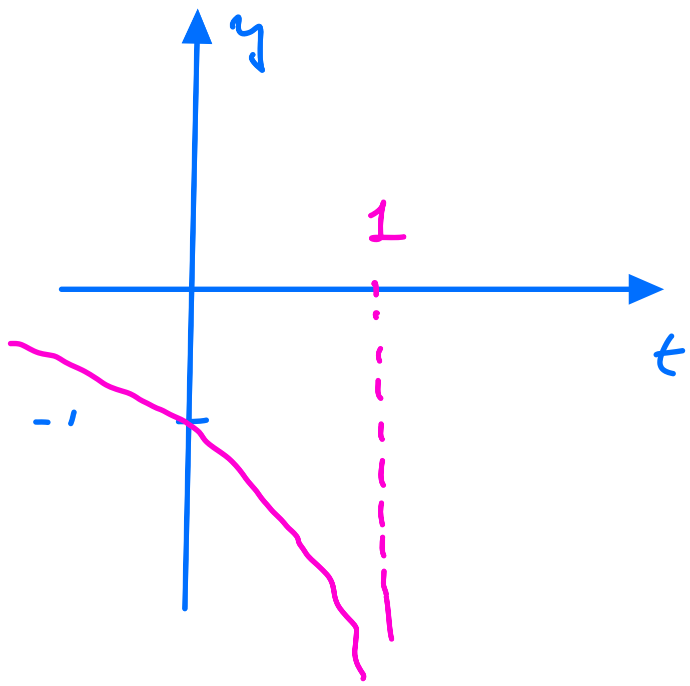


$C^2((-1,1);\mathbb{R})$

Example 1: $\begin{cases} \dot{y}(t) = -y^2 \\ y(0) = -1 \end{cases}$

cont.

$y(t) = \dfrac{1}{t-1}$

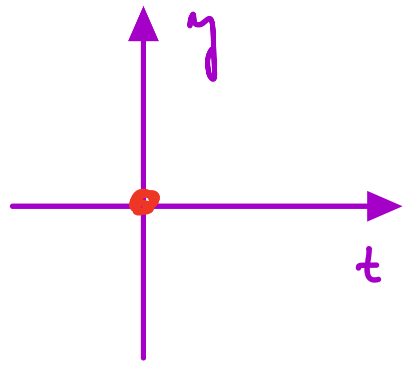


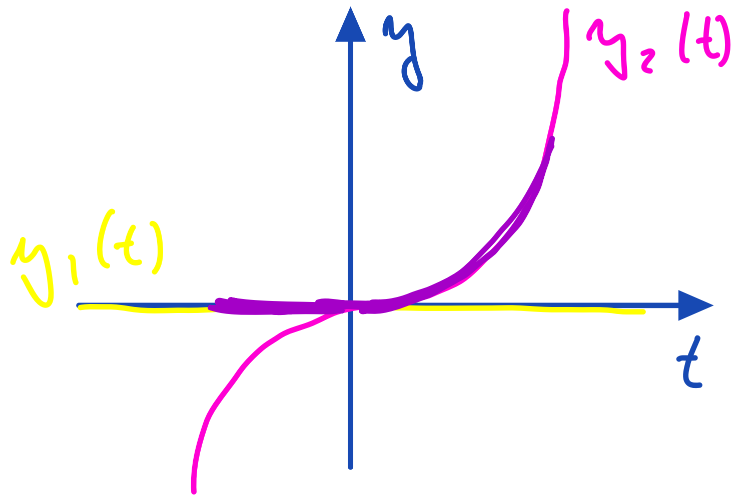
cont. of f $\not\Rightarrow$ solutions exist for all $\mathbb{R}$.

Example 2:

$$\begin{cases} y'(t) = y^{2/3} \\ y(0) = 0 \end{cases}$$

$f(y,t)$





$y_1(0) = 0$

$y_1(t) = \frac{t^3}{27}$

$y_2'(t) = \frac{3t^2}{27} = \frac{t^2}{9}$

$\left(\frac{t^3}{27}\right)^{2/3} = \left(\frac{t}{3}\right)^2 = \frac{t^2}{9}$

$$y_3(t) = \begin{cases} \frac{t^3}{27} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$y_4(t) = \begin{cases} \frac{t^3}{27} & , t \leq 0 \\ 0 & , t > \end{cases}$$

What about linear systems?

$$\underline{\dot{y}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \underline{y}$$

$$\underline{y}(t) = c_1 e^{t} \underline{v_1} + c_2 e^{2t} \underline{v_2}$$

Is the sol. well defined for all $t \in \mathbb{R}$? Yes

Lesson: Not all ODEs have unique sol. that exist for all time.

Goal: Derive conditions on f that guarantee the existence of a unique (local) solution.

$(t_0-\delta, t_0+\delta)$

How do we achieve this?

Step 1 Derive an algorithm to construct solutions - Picard iterates.

Step 2 Prove this algorithm converges

Step 1: Today

Let's first consider the 1D case, i.e.

$$(1) \quad \begin{cases} \dot{y}(t) = f(y,t) & dt \\ y(t_0) = y_0 \end{cases}$$

$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous

$y(t) \in \mathbb{R}$ for all $t \in I$

Integrating (1) we get

$\frac{dy}{dt}$ $\rightarrow$ $dy = f(y,t)\,dt$

$$\int_{y_0}^{y} d\tilde{y} = \int_{t_0}^{t} f(y(\tau),\tau)\,d\tau$$

$$\rightarrow \quad y - y_0 = \int_{t_0}^{t} f(y(\tau),\tau)\,d\tau$$

$$\rightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(y(\tau),\tau)\,d\tau$$

Note:

(2) $$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau)\, d\tau$$

is not a sol. because $y(t)$ is inside the integral.

Eqn. (2) is an example of an integral eqn.

Remark:

y is a cont. sol. of (2) (Integral eqn.) $\Longleftrightarrow$ y is a solution to (1) (Diff. eqn.)

($\Longleftarrow$ easy!)

($\Longrightarrow$ some work)

($\Longrightarrow$ some work)

y is a cont. sol. of (2) (Integral eqn.) $\Longrightarrow$ y is a solution to (1) (Diff. eqn.)

Proof (Sketch)

Take $t = t_0$ in

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau)\, d\tau$$

What do we get? $y(t_0) = y_0$

So, it satisfies the I.C.!

If $y = y(t)$ is a continuous sol. to

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau)\, d\tau$$

then $f(y(\tau), \tau)$ is cont.

Why? Composition of cont. functions

Result from calculus: If $g(\tau)$ is cont. then

$$G(t) = \int_{t_0}^{t} g(\tau)\, d\tau \qquad G'(t) = g(t)$$

is differentiable!

So $y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau)\, d\tau$

is differentiable

and

$$y'(t) = f(y(t), t) \quad !$$

Operators: Functions that act upon functions!

Example: $\underline{X} = C^1((-1,1); \mathbb{R})$

$T : f \to f'$

$T : \underline{X} \to \underline{X}$

$x^2 \overset{T}{\longmapsto} 2x$

$\sin(x) \longmapsto \cos(x)$

Example: $\underline{Y} = C^0(\mathbb{R}, \mathbb{R})$

$T : \underline{Y} \to \underline{Y}$

$f(x) \xrightarrow{T} f(x-1)$

$x \overset{T}{\longmapsto} x-1$

Example: $\underline{Y} = C^0(\mathbb{R}, \mathbb{R})$

$T : \underline{Y} \to \underline{Y}$

$f(x) \xrightarrow{T} \int_0^x f(\mu)\, d\mu$

$T(x) = \int_0^x \mu \, d\mu = \frac{x^2}{2}$

Let $u \in C^0(I, \mathbb{R})$

$$Tu(t) = y_0 + \int_{t_0}^{t} f(u(\tau), \tau)\, d\tau$$

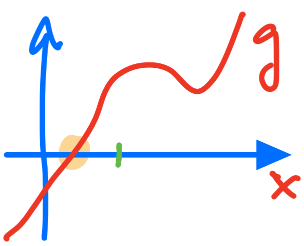
Remark if y^* is a sol. to

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau)\, d\tau$$

then $T(y^*) = y^*$

Picard Iteration:

Similar in spirit to Newton's method



$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$x_0, x_1, x_2, \ldots$

$x_n \xrightarrow[n \to \infty]{} x^*$ where $g(x^*) = 0$

Picard iterates

$$u_0 \to \underbrace{T(u_0)}_{u_1} \to \underbrace{T(T(u_0))}_{u_2} \to \cdots$$

Note that if $u_0 \in C(I, \mathbb{R})$

then (by a result above)

$$u_1 \in C^1(I;\mathbb{R}) \subset C^0(I;\mathbb{R})$$

Apply T again

$$T(u_1) \in C^1(I;\mathbb{R}) \subset C^0(I;\mathbb{R})$$

and so on!

Does $u_k \xrightarrow[k\to\infty]{} y^*$?

Example: $\begin{cases} y'(t) = y \\ y(0) = 1 \end{cases}$

$$T(u) = 1 + \int_0^t u(\tau)\, d\tau$$

Picard Iteration:

$u_0(t) = 1$ initial guess comes from the initial condition.

$$u_1(t) = T u_0(t) = 1 + \int_0^t 1\, d\tau = 1 + t$$

$$u_2(t) = T(u_1)(t) = 1 + \int_0^t (1+t)\, dt = 1 + t + \frac{t^2}{2}$$

$$u_3(t) = 1 + \int_0^t (1 + t + t^2/2)\, dt = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}$$

$$\Rightarrow \quad u_n(t) = 1 + t + \frac{t^2}{2} + \dots + \frac{t^n}{n!}$$

So we expect that

$$u_n \xrightarrow[n\to\infty]{} e^t$$

which is a sol. of our IVP

Next: We want to find conditions on f such that the Picard iterates converge! Also, in which sense does it converge?