Home Work 5

$\boxed{\text{1) Section 33 , problem 1 \& 7}}$

* Problem 1 :

$$f(x) = \begin{cases} \pi & -\pi \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le x \le \pi \end{cases}$$

We have : $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots +$

$+ b_1 \sin x + b_2 \sin 2x$

$$\oplus\ a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi\,dx + \underbrace{\frac{1}{\pi}\int_{\pi/2}^{\pi} 0\,dx}_{=0}$$

$$= \frac{1}{\pi}(\pi x)\Big|_{-\pi}^{\pi/2} = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

$$\oplus\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi/2} \pi \cos nx\,dx + 0$$

$$= \int_{-\pi}^{\pi/2} \cos nx\,dx = \frac{\sin nx}{n}\Big|_{-\pi}^{\pi/2} = \frac{1}{n}\Big[\sin\frac{n\pi}{2} + \underbrace{\sin n\pi}_{=0}\Big]$$

$$= \frac{1}{n}\sin\frac{n\pi}{2}$$

$$\oplus\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\Big[\int_{-\pi/2}^{\pi} \pi \sin nx\,dx + 0\Big]$$

$$= -\frac{\cos nx}{n}\Big|_{-\pi}^{\pi/2} = \frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} = \frac{1}{n}\left[\cos n\pi - \cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{n}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

$$\Rightarrow \begin{cases} a_n = \frac{1}{n}\sin\frac{n\pi}{2} \\ b_n = \frac{1}{n}\left[(-1)^n - \cos\frac{n\pi}{2}\right] \\ a_0 = \frac{3\pi}{2} \end{cases}$$

$\Rightarrow$ Fourier series:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$= \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n}\left[(-1)^n - \cos\frac{n\pi}{2}\right]\sin x\right]$$

$$= \boxed{\frac{3\pi}{4} + \frac{1}{n}\sum_{n=1}^{\infty}\left[\sin\frac{n\pi}{2}\cos nx + \left((-1)^n - \cos\frac{n\pi}{2}\right)\sin x\right]}$$

(*) Problem 7

From the problem, given:

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x \le \pi/2 \\ 0 & \pi/2 < x \le \pi \end{cases}$$

We have:

$\oplus$ $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\,dx + \int_{0}^{\pi/2} dx + \int_{\pi/2}^{\pi} 0\,dx\right]$$

$$= \frac{1}{\pi}\, x\Big|_0^{\pi/2} = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

$\oplus$ $$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\,dx + \int_{0}^{\pi/2} \cos nx\,dx + \int_{\pi/2}^{\pi} 0\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi/2} \cos nx\,dx + 0\right]$$

$$= \frac{1}{\pi}\cdot\frac{\sin nx}{n}\Big|_0^{\pi/2} = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin 0\right]$$

$$= \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$\oplus$ $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi/2} \sin nx\,dx = \frac{1}{\pi}\,\frac{\cos nx}{n}\Big|_{\pi/2}^{0}$$

$$= \frac{1}{n\pi}\left[\cos 0 - \cos\frac{n\pi}{2}\right] = \frac{1}{n\pi}\left[1 - \cos\frac{n\pi}{2}\right]$$

$$\Rightarrow \boxed{f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin nx\right]}$$

$\boxed{\text{2) Section 34, problem 5}}$ $f(x) = e^{x}, \ -\pi \le x < \pi$

a) $\oplus\, a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}dx = \frac{1}{\pi}\left[e^{x}\Big|_{-\pi}^{\pi} \right] = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$

$= \frac{2}{\pi} \cdot \frac{e^{\pi}-e^{-\pi}}{2} = \frac{2\sinh\pi}{\pi}$

$\oplus\, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\cos nx\, dx$

We also have:

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

applying for a_n with $a=1,\ b=n$

$$\Rightarrow a_n = \frac{1}{\pi}\cdot\frac{e^{x}}{n^2+1}\left[\cos nx + n\sin nx\right]\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}(\cos n\pi + \pi\underbrace{\sin n\pi}_{=0}) - \frac{e^{-\pi}}{n^2+1}(\cos n\pi - n\underbrace{\sin n\pi}_{=0})\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{n^2+1}(-1)^n - \frac{e^{-\pi}}{n^2+1}(-1)^n\right]$$

$$= \frac{(-1)^n}{\pi}\left[\frac{e^{\pi}-e^{-\pi}}{n^2+1}\right] = \frac{2(-1)^n}{\pi(n^2+1)}\sinh\pi$$

$\oplus\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$

Similiar a_n, we also have:

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right], \quad a=1,\ b=n$$

$$\Rightarrow b_n = \frac{1}{\pi}\cdot\frac{e^x}{n^2+1}\left[\sin nx - n\cos nx\right]\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\Big[e^{\pi}(\underbrace{\sin n\pi}_{=0} - n\cos n\pi) - e^{-\pi}(\underbrace{\sin n\pi}_{=0} - n\cos n\pi)\Big]$$

$$= \frac{1}{\pi(n^2+1)}\left[-e^{\pi}n\cos n\pi + e^{-\pi}n\cos n\pi\right]$$

$$= \frac{n\cos n\pi}{\pi(n^2+1)}\left[e^{-\pi} - e^{\pi}\right] = \frac{-2n(-1)^n}{\pi(n^2+1)}\cdot\frac{e^{\pi}-e^{-\pi}}{2}$$

$$= \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh\pi.$$

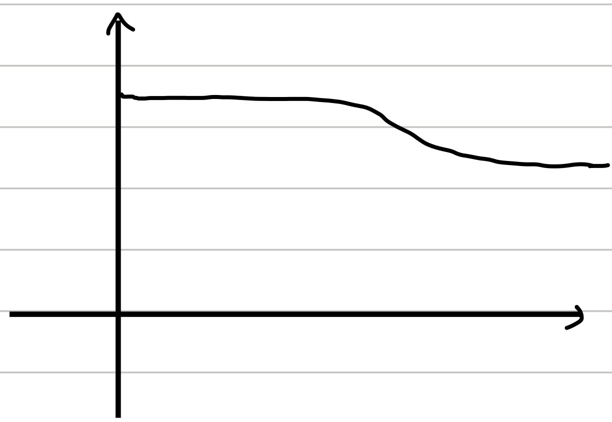
$$\Rightarrow f(x) = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n\sinh\pi}{\pi(n^2+1)}\cos nx + \frac{2n(-1)^{n+1}}{\pi(n^2+1)}\sinh\pi\sin nx\right]$$

$$= \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2+1}\cos nx + \frac{n(-1)^{n+1}}{n^2+1}\sin xn\right]$$

$$= \boxed{\frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty} \cdot \frac{(-1)^n}{n^2+1}\left[\cos nx - n\sin nx\right]}$$

b) with $-5\pi \le x \le 5\pi$

we have $n = [-5, 5]$



c) Based on part a, we have:

$$f(0) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}[1-0] = e^0 = 1$$

$$\Rightarrow \frac{\sinh\pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\right] = 1$$

$$\Rightarrow 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{\pi}{\sinh x} \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1} = \frac{1}{2}\left[\frac{\pi}{\sinh x} - 1\right]}$$

(+) for $x = \pi$,

$$f(\pi) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left[\cos n\pi - n\sin n\pi\right]$$

$$= \frac{\sinh\pi}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\cdot(-1)^n\right]$$

$$= \frac{\sinh \pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right] = e^{\pi} \quad 1$$

⊛ for $x = -\pi$

$$\Rightarrow f(-\pi) = \frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}\left(\underbrace{\cos n\pi}_{(-1)^n} + n\underbrace{\sin n\pi}_{0}\right)\right]$$

$$= \frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2+1}\right]$$

$$= \frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right] = e^{-\pi} \quad 2$$

Based on 1 & 2 $\Rightarrow \frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right]$

$$\Rightarrow \cosh\pi = \frac{\sinh\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{\pi\cosh\pi}{\sinh\pi} - 1\right] = \sum_{n=1}^{\infty}\frac{1}{n^2+1}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{1}{2}\left[\frac{\pi}{\tanh\pi} - 1\right]}$$

d) We have $f(x) = \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^n}{n^2+1}\left[\cos nx - n\sin nx\right]$

Since $\begin{cases} -1 \le \cos nx \le 1 \\ -n \le n\sin nx \le n \Rightarrow n \ge -n\sin nx \ge -n \end{cases}$

$\Rightarrow -n-1 \le \cos nx - n\sin nx \le n+1$

Also $\lim_{n\to\infty} f(x) = \lim_{n\to\infty}\left[\frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin nx)\right]$

$$= \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{(-1)^n(\cos nx + n\sin nx)}{n^2+1}$$

$$= \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\lim_{n\to\infty}\frac{(-1)^n(\cos nx + n\sin nx)}{n^2+1}\right]$$

Because $\cos nx - n\sin x \in [-n-1, n+1]$

$$\Rightarrow \lim_{n\to\infty}\frac{(-1)^n(\cos nx + n\sin nx)}{n^2+1} \rightarrow 0$$

$\Rightarrow$ $\boxed{f(x) \text{ will converge to } \frac{\sinh\pi}{\pi}}$

$\boxed{\text{3) Section 35, problem 8}}$ $f(x) = \pi - x$

a) We have:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left[\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(-\pi^2 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \pi^2 + \frac{\pi^2}{2}\right] = \frac{2\pi^2}{\pi} = 2\pi$$

$$\oplus\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\cos nx\,dx$$

Also, $\int (\pi - x)\cos nx\,dx = \int \pi \cos nx\,dx - \int x\cos nx\,dx$

$$= (\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}$$

then, $\left(\dfrac{(\pi - x)\sin nx}{n} - \dfrac{\cos nx}{n^2}\right)\Big|_{-\pi}^{\pi}$

$$= \left[\frac{(\pi - \pi)\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right] - \left[\frac{-(\pi + \pi)\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right]$$

$$= -\frac{\cos n\pi}{n^2} + \frac{\cos n\pi}{n^2} = 0$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\cos n\pi\,dx = 0$$

$\oplus\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)\sin nx\,dx.$

Same as a_n, we also have:

$$\int(\pi - x)\sin nx\,dx = \int \pi \sin x\,dx - \int x\sin nx\,dx$$

$$= \frac{(x-\pi)\cos nx}{n} - \frac{\sin nx}{n^2}$$

then, $\left[\frac{(x-\pi)\cos nx}{n} - \frac{\sin nx}{n^2}\right]\Bigg|_{-\pi}^{\pi}$

$$= \left[\frac{(\pi-\pi)\cos n\pi}{n} - \frac{\sin n\pi}{n^2}\right] - \left[\frac{(-\pi-\pi)\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right]$$

$$= -\left[-\frac{2\pi\cos n\pi}{n}\right] = \frac{2\pi}{n}(-1)^n$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{\pi}\cdot\frac{2\pi}{n}(-1)^n = \frac{2}{n}(-1)^n$$

$$\boxed{\Rightarrow f(x) = \pi + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^n\sin(nx)}$$

b) its cosine on $[0,\pi]$, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx)$

$\oplus\ a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)\Big|_0^{\pi}$

$$= \frac{2}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{2}{\pi} \cdot \frac{\pi^2}{2} = \pi$$

$$\theta \quad a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \cos nx \, dx$$

Also, $\int_0^{\pi} (\pi - x) \cos nx \, dx = \int_0^{\pi} \pi \cos nx \, dx - \int_0^{\pi} x \cos nx \, dx$

$$\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]\Bigg|_0^{\pi}$$

$$= \left[\frac{(\pi - \pi)\sin n\pi}{n} - \frac{\cos n\pi}{n^2}\right] - \left[\frac{\pi \sin 0}{n} - \frac{1}{n^2}\right]$$

$$= -\frac{\cos n\pi}{n^2} + \frac{1}{n^2} = \frac{1 - \cos n\pi}{n^2}$$

$$\Rightarrow a_n = \frac{2}{\pi} \cdot \frac{1 - \cos(n\pi)}{n^2} = \frac{2[1 - (-1)^n]}{\pi n^2}$$

$$\Rightarrow \boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{\pi n^2} \cos nx = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\cos(2n-1)x}{(2n-1)^2}}$$

c) its sine series on $0 < x \leq \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx$$

Also having:

$$\int_0^{\pi} (\pi - x)\sin nx\,dx = \int_0^{\pi} \pi \sin x\,dx - \int_0^{\pi} x\sin nx\,dx$$

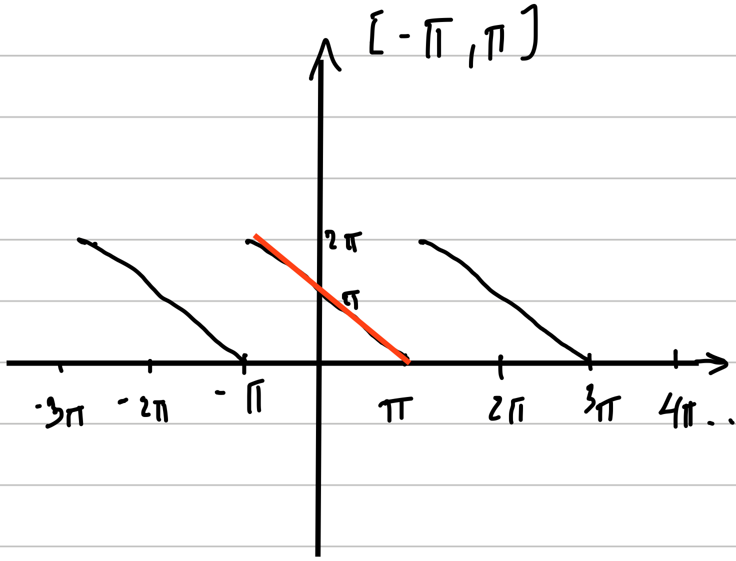
$$= \left[\frac{(x-\pi)\cos nx}{n} - \frac{\sin nx}{n^2}\right]\Bigg|_0^{\pi}$$

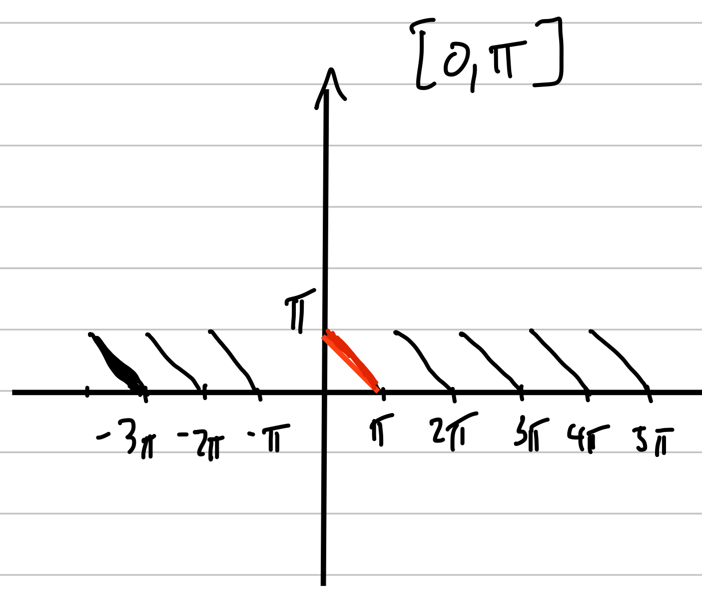
$$= \left[\frac{(\pi-\pi)\cos n\pi}{n} - \frac{\sin n\pi}{n^2}\right] - \left[\frac{-\pi\cos 0}{n} - \frac{\sin 0}{n^2}\right]$$

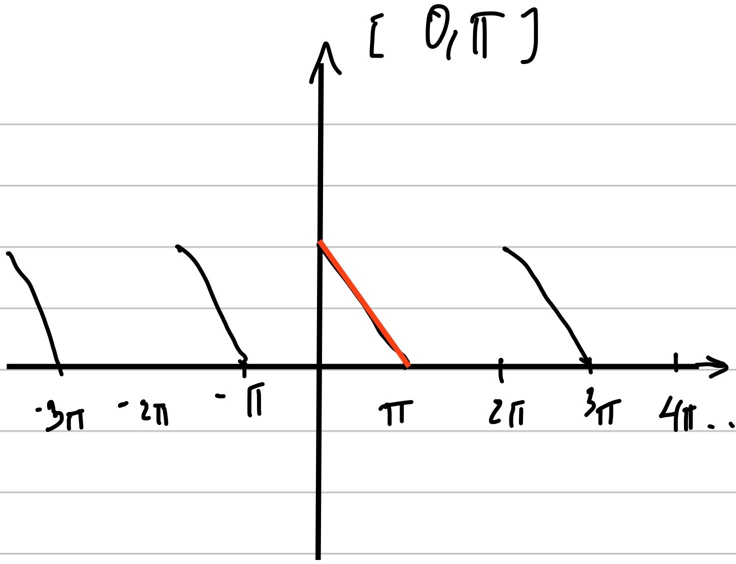
$$= \frac{\pi}{n} \Rightarrow b_n = \frac{2}{\pi}\cdot\frac{\pi}{n} = \frac{2}{n}$$

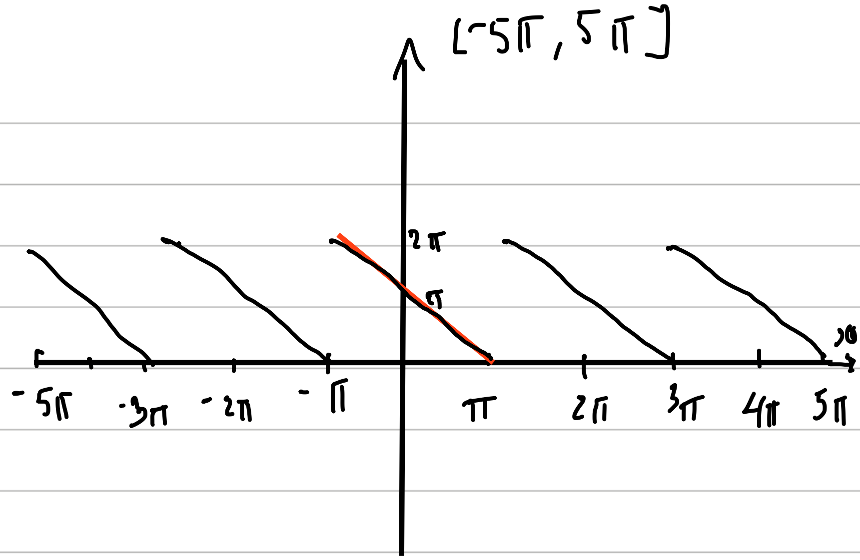
$$\Rightarrow \boxed{f(x) = \sum_{n=1}^{\infty} \frac{2\sin nx}{n}}$$

Sketch: $f(x) = \pi - x$






[0, π]
-3π
-2π
-π
π
2π
3π
4π . .


[-5π, 5π]
2π
π
-5π
-3π
-2π
-π
π
2π
3π
4π
5π

4) Section 35, problem 3

(*) Prove: $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even

Since $f(x)$ is even $\Rightarrow f(x) = f(-x)$

then $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

Check $\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} f(-x)dx = -\int_{-a}^{0} f(-x)\,d(-x)$

let $t = -x \Rightarrow$ $x: -a \to 0$
$t: a \to 0$

$\Rightarrow \int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(t)\,dt = \int_{0}^{a} f(t)dt = \int_{0}^{a} f(x)dx$

$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)dx$

$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

(*) Prove: $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is odd.

Similarly, $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$

$f(x)$ is odd $\Rightarrow f(x) = -f(-x)$

Check $\int_{-a}^{0} f(x)\,dx = -\int_{-a}^{0} f(-x)\,dx = \int_{-a}^{0} f(-x)\,d(-x)$

Put $t = -x \Rightarrow dt = d(-x)$ & $\begin{cases} x : -a \to 0 \\ t : a \to 0 \end{cases}$

$$\Rightarrow \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(t)\,dt = -\int_{0}^{a} f(t)\,dt$$

$$= -\int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

5) Given: f is piece-wise continuous on $[0, \pi]$ &

$f(\theta) = f(\pi - \theta)$

We have $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta$

$$= \frac{1}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta = \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\cos(n\theta)\, d\theta + \int_{\pi/2}^{\pi} f(\theta)\cos n\theta \, d\theta\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\cos n\theta \, d\theta + \int_{\pi/2}^{\pi} f(\pi-\theta)\cos(n\theta)\, d\theta\right]$$

Check $\int_{\pi/2}^{\pi} f(\pi-\theta)\cos(n\theta)\, d\theta$

let $t = \pi - \theta \Rightarrow dt = -d\theta \Rightarrow -dt = d\theta$

$\theta : \pi/2 \to \pi$

$t : \frac{\pi}{2} \to 0$

$$\Rightarrow \int_{\pi/2}^{\pi} f(\pi-\theta)\cos(n\theta)\, d\theta = -\int_{\pi/2}^{0} f(t)\cos n(\pi - t)\, dt$$

$$= \int_0^{\pi/2} f(t)\cos(nt - n\pi)\, dt = \int_0^{\pi/2} f(\theta)\cos(n\theta - n\pi)\, d\theta$$

If n is even $\Rightarrow \cos(n\theta - n\pi) = \cos(n\theta)$

If n is odd $\Rightarrow \cos(n\theta - n\pi) = -\cos(n\theta)$

$$\Rightarrow \int_{\pi/2}^{\pi} f(\pi-\theta)\cos n\theta \, d\theta = \int_0^{\pi/2} f(\theta)\cos(n\theta - n\pi)\, d\theta$$

$$= \begin{cases} \displaystyle\int_0^{\pi/2} f(\theta)\cos(n\theta)\, d\theta & \text{if } n \text{ is even} \\ \displaystyle -\int_0^{\pi/2} f(\theta)\cos(n\theta)\, d\theta & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\cos n\theta \, d\theta + \int_{\pi/2}^{\pi} f(\pi-\theta)\cos(n\theta)\, d\theta\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\cos n\theta \, d\theta + \int_0^{\pi/2} f(\theta)\cos(n\theta - n\pi)\, d\theta\right]$$

$$= \begin{cases} \displaystyle\frac{1}{\pi}\left[2\int_0^{\pi/2} f(\theta)\cos n\theta \, d\theta\right] & \text{if } n \text{ is even} \\ \displaystyle\frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\cos n\theta \, d\theta - \int_0^{\pi/2} f(\theta)\cos(n\theta)\, d\theta\right] = 0 & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow$ $\boxed{a_n = 0 \text{ if } n \text{ is odd}}$ Check on $[0,\pi]$

$*$ Also, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta = \frac{1}{\pi}\int_{0}^{\pi} f(\theta)\sin n\theta \, d\theta$

$= \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta + \int_{\pi/2}^{\pi} f(\theta)\sin n\theta \, d\theta\right]$

(*) Check $\int_{\pi/2}^{\pi} f(\theta)\sin n\theta \, d\theta = \int_{\pi/2}^{\pi} f(\pi-\theta)\sin n\theta \, d\theta$

Let $t = \pi - \theta \Rightarrow dt = -d\theta$ & $\theta = \pi - t$

$\Rightarrow$ $\theta: \pi/2 \rightarrow \pi$

$t: \pi/2 \rightarrow 0$

$\Rightarrow \int_{\pi/2}^{\pi} f(\pi-\theta)\sin n\theta \, d\theta = -\int_{\pi/2}^{0} f(t)\sin(n\pi - nt)\, dt$

$= \int_0^{\pi/2} f(t)\sin(n\pi - nt)\, dt = -\int_0^{\pi/2} f(t)\sin(nt - n\pi)\, dt$

$= -\int_0^{\pi/2} f(\theta)\sin(n\theta - n\pi)\, d\theta$

If n is even, $\sin(n\theta - n\pi) = \sin n\theta$

if n is odd, $\sin(n\theta - n\pi) = -\sin n\theta$

$$\Rightarrow \int_{\pi/2}^{\pi} f(\pi-\theta)\sin n\theta \, d\theta = \begin{cases} -\int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta, & n \text{ is even} \\ \int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta, & n \text{ is odd} \end{cases}$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta + \int_{\pi/2}^{\pi} f(\theta)\sin n\theta \, d\theta\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta + \int_{\pi/2}^{\pi} f(\pi-\theta)\sin n\theta \, d\theta\right]$$

$$= \begin{cases} \frac{1}{\pi}\left[\int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta - \int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta\right] = 0, & n \text{ is even} \\ \frac{1}{\pi}\left[2\int_0^{\pi/2} f(\theta)\sin n\theta \, d\theta\right], & n \text{ is odd} \end{cases}$$

$$\Rightarrow \boxed{b_n = 0, \text{ if } n \text{ is even}}$$

Therefore,

$$\boxed{\begin{cases} a_n = 0 & \text{if } n \text{ is odd} \\ b_n = 0 & \text{if } n \text{ is even} \end{cases}}$$