Unit Step Function

$u_a(t) := \begin{cases} 0 \text{ if } t < a \\ 1 \text{ if } t > a \end{cases}$ is the unit step function at a. We can compute

$$L[u_0(t)] = \frac{1}{s}.$$

Notice this is the same as $L[1]$ which makes sense because the Laplace transform only knows what $f(t)$ does for $t > 0$.
Translation property (homework exercise)

$$L[u_a(t)f(t-a)] = e^{-as}F(s)$$

Exercise

1. Solve $\begin{cases} y''(x) + 2y'(x) + y(x) = f(x) \\ y(0) = 0 \\ y'(0) = 1, \end{cases}$ where $f(x) = \begin{cases} 1 \text{ if } 1 \leq x \leq 2 \\ 0 \text{ otherwise.} \end{cases}$

Dirac Delta "Function"

$\delta(t)$ is the derivative of $u_0(t)$, meaning $\delta(t) = \begin{cases} 0 \text{ if } t \neq 0 \\ \infty \text{ if } x = 0, \end{cases}$ where the ∞ is "the right size" so that when it is integrated over, the result is 1. More generally, $\delta(t)$ satisfies the **Sifting Property**

$$\int_{-}^{+} \delta(t)f(t)dt = f(0),$$

where the integral is done over any interval containing $t = 0$. Let $f(t) = e^{-st}$ to see $L[\delta(t)] = e^{(-0 \cdot s)} = 1$.

Fundamental Solution (a.k.a. Impulse Response) to a Linear ODE

Suppose you want to solve the linear ODE $y'' + y = f(t)$ for many different choices of $f(t)$. Instead of re-solving for each f, find the **fundamental solution**: which is the solution when $f(t) = \delta(t)$. To be consistent with the textbook, let's call the fundamental solution $h(t)$, which for this ODE means

$$h'' + h = \delta(t).$$

If you can solve for h (usually using Laplace transforms), the original ODE is solved by

$$y(t) = h(t) * f(t).$$

Therefore you do not need to re-solve the ODE for each f; simply compute a convolution for each f. (This formula is not specific to this ODE, I just didn't want to write a general linear ODE as $L[y] = f(t)$ because that is confusing with out notation for Laplace transforms.)

Exercise

2. Solve $My'' + ky = f(t)$ for any $f(t)$ using the fundamental solution.
3. Solve $\begin{cases} y'' + 2y' + y = f(t) \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ for any f using the impulse response. Express your final answer without using integrals, but you may use F, the Laplace Transform of $f(t)$.