Math 135, Spring 2022

Lecture #23: PDEs and boundary value problems

Friday May 19^{th}

Last time

- We considered the IBVP

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \\ y(t,0) = 0 = y(t,\pi) \\ y(0,x) = f(x) \\ \frac{\partial y}{\partial t}(0,x) = g(x) \end{cases}$$

- In the case $g(x) = 0$ we showed that (formally) the solution is given by

$$y(t,x) = \sum_{n=1}^{\infty} b_n \cos(nct)\sin(nx) \quad \text{where} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(x).$$

- The case that $f(x) = 0$ and $g(x) \neq 0$ is considered on your homework!

Learning objectives

Today we will discuss:

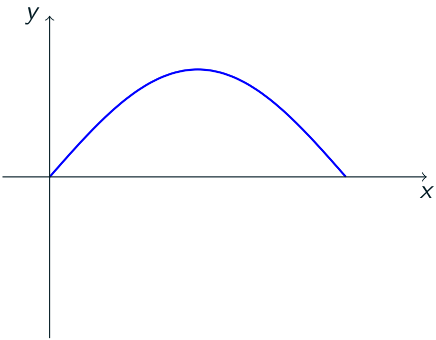
- An example of the initial boundary value problem for the $1d$ wave equation.
- Derivation of the $1d$ heat equation from the temperature of a thin insulated rod.
- The initial boundary value problem for the $1d$ heat equation.

PDEs and boundary value problems

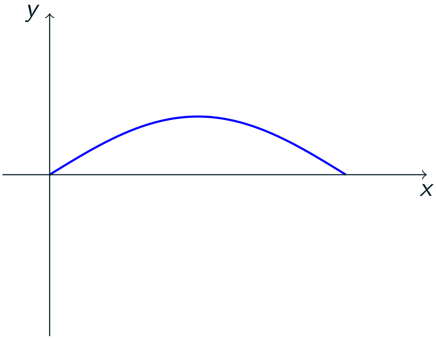
An example

Find the solution of the equation

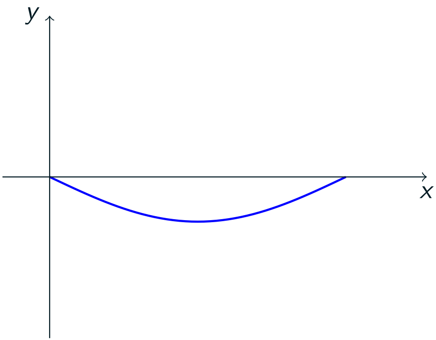
$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \\ y(t, 0) = 0 = y(t, \pi) \\ y(0, x) = \sin(x) \\ \frac{\partial y}{\partial t}(0, x) = 0 \end{cases}$$



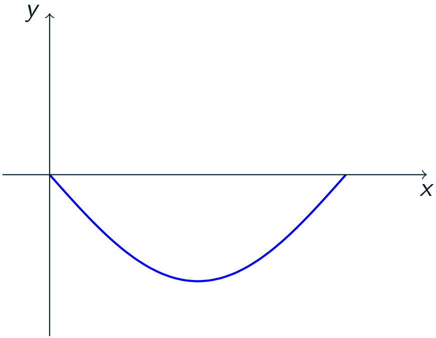
$$y(0, x) = \sin(x)$$



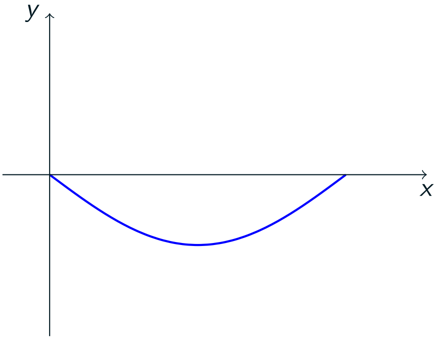
$$y(1,x) = \frac{1}{2}\left[\sin(x-1) + \sin(x-1)\right]$$



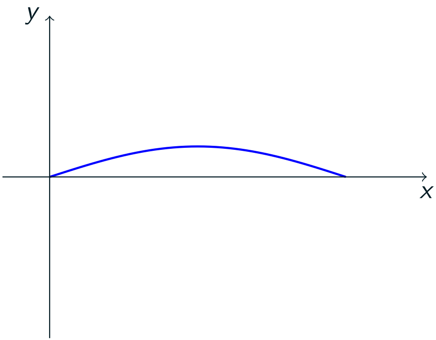
$$y(2,x) = \frac{1}{2}\left[\sin(x-2) + \sin(x-2)\right]$$



$$y(3,x) = \frac{1}{2}\left[\sin(x-3) + \sin(x-3)\right]$$



$$y(4,x)= \frac{1}{2}\left[\sin(x-4)+\sin(x-4)\right]$$



$$y(5,x) = \frac{1}{2}\left[\sin(x-5)+\sin(x-5)\right]$$

The $1d$ heat equation

The **$1d$ heat equation**:

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}$$

with the **boundary conditions**

$$w(t,0) = 0 = w(t,\pi)$$

and the **initial condition**

$$w(0,x) = f(x).$$

Note 1: Other boundary conditions will be considered in the homework.

Note 2: The 2nd law of thermodynamics is used to derive the 1d heat equation among other assumptions. See your textbook for a detailed discussion.

Note 3: The constant $a^2 := \frac{k}{c\rho} > 0$, where k is the thermal conductivity, c is the specific heat and ρ is the density.

Linearity

Theorem: If $w_1(t,x)$ and $w_2(t,x)$ are solutions of the linear heat equation

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}$$

with the **boundary conditions**

$$w(t,0) = 0 = w(t,\pi)$$

and C_1, C_2 are constants, then so is

$$w(t,x) = C_1 w_1(t,x) + C_2 w_2(t,x).$$

Proof: Easy computation.

Separation of variables

What is the general solution of

$$v'(t) + a^2 n^2 v(t) = 0 \quad ?$$

A) $C_1 \cos(nat) + C_2 \sin(nat)$

B) $C_1 e^{nat} + C_2 e^{-nat}$

C) $Ce^{-n^2 a^2 t}$

D) None of the above

Solving our PDE

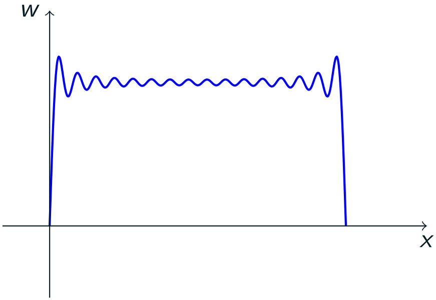
An example

Find the solution of the equation

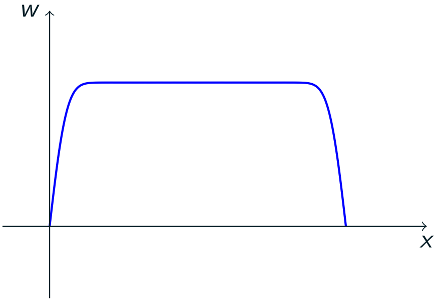
$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2} \\ w(t, 0) = 0 = w(t, \pi) \\ w(0, x) = 1. \end{cases}$$

Recall:

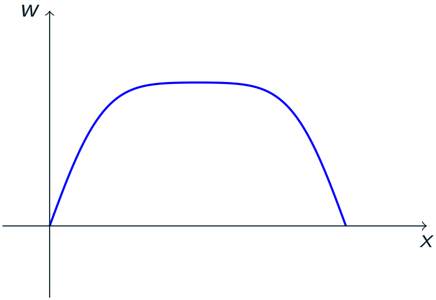
$$y(t, x) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx).$$



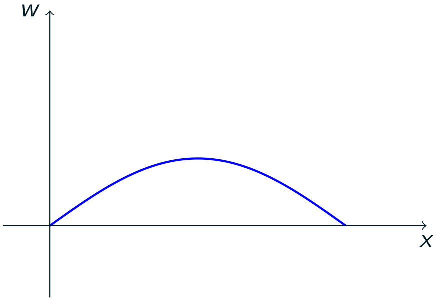
$$w(0,x) \approx \frac{4}{\pi}\sum_{n=1}^{16}\frac{1}{2n-1}\sin\big((2n-1)x\big)$$



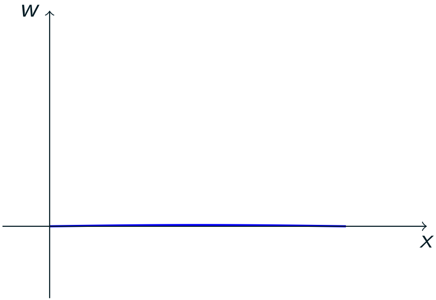
$$w\left(\frac{1}{100}, x\right) \approx \frac{4}{\pi} \sum_{n=1}^{16} \frac{1}{2n-1} e^{-\frac{1}{100}(2n-1)^2} \sin\big((2n-1)x\big)$$



$$w\left(\tfrac{1}{10},x\right)\approx \frac{4}{\pi}\sum_{n=1}^{16}\frac{1}{2n-1}e^{-\frac{1}{10}(2n-1)^2}\sin\big((2n-1)x\big)$$



$$w(1,x) \approx \frac{4}{\pi} \sum_{n=1}^{16} \frac{1}{2n-1} e^{-(2n-1)^2} \sin\big((2n-1)x\big)$$



$$w(5,x) \approx \frac{4}{\pi}\sum_{n=1}^{16}\frac{1}{2n-1}e^{-5(2n-1)^2}\sin\big((2n-1)x\big)$$

See you next time!