# Lecture 8

## Laplace Transform

$$t\text{-space} \xrightarrow{\mathcal{L}} s\text{-space}$$

Example 1: $f(t) = 1$

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)\,dt$$

$$= \frac{1}{s}, \quad s > 0$$

Example 2: Let $s \in \mathbb{C}$ and $f(t) = 1$

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \text{ for } \operatorname{Re}(s) > 0$$

Exercise!

Example 3: $g(t) = e^{i\omega t}$ where $\omega \in \mathbb{R}$.

$$\mathcal{L}\{e^{i\omega t}\}(s) := \int_0^\infty e^{-st} e^{i\omega t}\,dt \quad \text{where } s \in \mathbb{C}.$$

$$= \lim_{\tau\to\infty} \int_0^\tau e^{-st} e^{i\omega t}\,dt$$

$$= \lim_{\tau \to \infty} \int_0^{\tau} e^{(-s+i\omega)t}\, dt$$

$[s = Re(s) + Im(s)\, i]$

$$= \lim_{\tau \to \infty} \frac{e^{(-s+i\omega)t}}{-s+i\omega} \Bigg|_{t=0}^{t=\tau}$$

$$= \lim_{\tau \to \infty} \frac{e^{(-Im(s)+\omega)t\, i}\; e^{-Re(s)\, t}}{-s+i\omega} \Bigg|_{t=0}^{t=\tau}$$

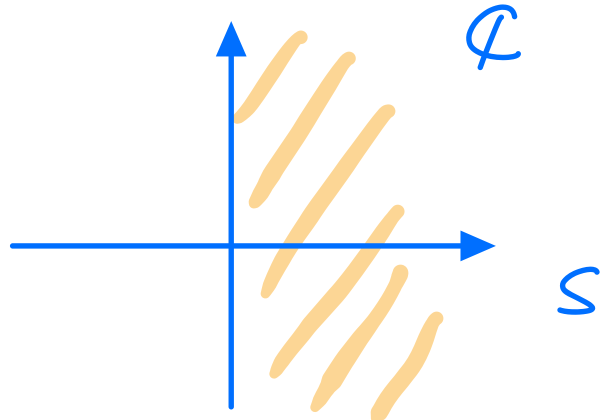
$$= -\frac{1}{-s+i\omega}$$ for all $s \in \mathbb{C}$ for which $Re(s) > 0$.

$$= \frac{1}{-s+i\omega}$$

Recall: $|e^{i\theta}| = 1$ where $\theta \in \mathbb{R}$

$e^{i\theta} = \cos\theta + i \sin(\theta)$

$$\underset{t}{[0,\infty)} \xrightarrow{\mathcal{L}}$$



Example 4:

$$\mathcal{L}\{e^{-i\omega t}\}(s) = \frac{1}{s+i\omega}, \quad Re(s)>0$$

Proven as in Example 3.

Example 5:

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\mathcal{L}\{\cos(\omega t)\}(s) = \mathcal{L}\left\{\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right\}(s)$$

$$= \frac{1}{2}\mathcal{L}\{e^{i\omega t}\}(s) + \frac{1}{2}\mathcal{L}\{e^{-i\omega t}\}(s)$$

$$\underset{\text{by thm.}}{=} \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right)$$

$$= \frac{s}{s^2+\omega^2}, \quad Re(s)>0$$

Example 6:

$$\mathcal{L}\{\sin(\omega t)\}(s) = \mathcal{L}\left\{\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right\}(s)$$

$$\cancel{\cos(\omega t)} + i\sin(\omega t)$$

$$-\left[\cancel{\cos(\omega t)} - \sin(\omega t)\, i\right]$$

$$= \frac{\omega}{s^2+\omega^2}, \quad Re(s) > 0.$$

Table of Laplace Transforms:

$$\mathcal{L}\{0\}(s) = 0$$

$$\mathcal{L}\{1\}(s) = 1/s \qquad s>0$$

$$\mathcal{L}\{t\}(s) = 1/s^2 \qquad s>0$$

$$\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a} \qquad s>a$$

$\omega \in \mathbb{R}$

$$\mathcal{L}\{\cos(\omega t)\}(s) = \frac{s}{s^2+\omega^2}, \quad s>0$$

$$\mathcal{L}\{\sin(\omega t)\}(s) = \frac{\omega}{s^2+\omega^2}, \quad s>0$$

Translation property of Laplace Transforms

If $F(s) = \mathcal{L}\{f(t)\}(s)$ for $s>0$ then

$\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)$, for $a \in \mathbb{R}$.

multiplication by $e^{at}$ results in a shift by $a$.

Proof:

$$F(s-a) := \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \underbrace{f(t)\, e^{at}}\, dt$$

$$= \mathcal{L}\{e^{at} f(t)\}(s)$$

$\boxed{}$

Example: $\mathcal{L}\{t\}(s) = \frac{1}{s^2}$, $s>0$

$$\mathcal{L}\{te^{at}\}(s) = \frac{1}{(s-a)^2}, \quad s>a$$

Example: $\mathcal{L}\{\sin(\omega t)\}(s) = \frac{\omega}{s^2+\omega^2}$, $s>0$, $\omega \in \mathbb{R}$

$$\mathcal{L}\{e^{3t}\sin(2t)\}(s) = \frac{2}{(s-3)^2+4}, \quad s>3$$

# Technical tools

→ Preliminaries:

- An integral is said to be absolutely convergent if

$$\int_0^\infty |f(x)|\,dx < \infty$$

- Comparison test: Let's assume that

$$|f(x)| \leq |g(x)| \quad \text{for all } x \geq 0$$

and $$\int_0^\infty g(x)\,dx < \infty$$

$$\Rightarrow \int_0^\infty f(x)\,dx < \infty$$