Math 135, Spring 2022

Lecture #25: PDEs and boundary value problems

Friday May 27th

Last time

- We considered the **Dirichlet problem** for the disc $\mathscr{D} = \{x^2 + y^2 < 1\}$

Laplace eqn.

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0 \quad \text{for} \quad (x, y) \in \mathscr{D} \\ w(x, y) = f(x, y) \quad \text{for} \quad (x, y) \in \partial\mathscr{D} \end{cases}$$

- We showed that in polar coordinates, the solution becomes

$$w(r, \theta) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \Big\{ a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \Big\}$$

where

$$f(\theta) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \Big\{ a_n \cos(n\theta) + b_n \sin(n\theta) \Big\}$$

Learning objectives

Today we will discuss:

- Complex numbers.
- The Poisson integral.
- Maximum principle for elliptic equations
- Uniqueness of solutions

The Poisson integral

Complex numbers

- A **complex number** is an object of the form

$$z = x + iy,$$

$i = \sqrt{-1}$

  where $x, y \in \mathbb{R}$ and i is the **imaginary unit**.

- We refer to x as the **real part** of z and y as the **imaginary part** of z and write

$$x = \operatorname{Re} z \quad \text{and} \quad y = \operatorname{Im} z.$$

- We add complex numbers using the rule

$$(x + iy) + (u + iv) = (x + u) + i(y + v).$$

- We multiply complex numbers using the rule

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu).$$

- Taking $x = u = 0$ and $y = v = 1$ we see that $i^2 = -1$.

An example

What is

$$(1+2i)(3-4i)?$$

A) $11-2i$

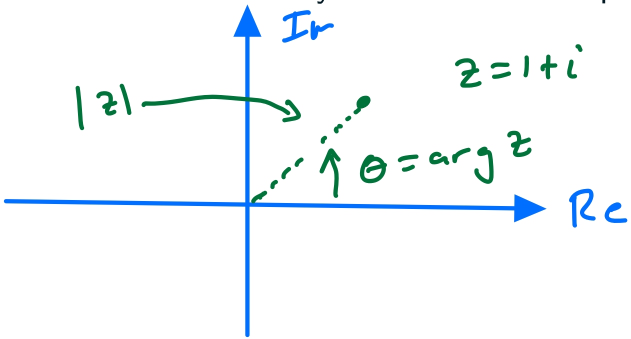
B) $-5+2i$

C) $11+2i$

D) $-5-2i$

$3-4i+6i+8$

- It is natural to identify the set $\mathbb{C}$ of complex numbers with the set $\mathbb{R}^2$:



- We can then introduce the **modulus**, $|z|$, and **argument**, $\arg z$, so that

$$\operatorname{Re} z = |z| \cos(\arg z) \quad \text{and} \quad \operatorname{Im} z = |z| \sin(\arg z)$$

- We define the **complex conjugate** of $z = x + iy$ to be

$$\bar{z} = x - iy$$

and have

$$|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}.$$

Theorem: If we define

$z \in \mathbb{C}$

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

then for $\theta \in \mathbb{R}$ we have Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i \sin\theta}$$

Proof: Skip

An application

Let $n \geq 1$ and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)[\cos(n\theta) - \sin(n\theta)i]\,d\theta$$

$$= \frac{1}{2}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta\right]$$

$$= \frac{1}{2}[a_n - ib_n]$$

If we take a_n, b_n to be the usual Fourier coefficients of f, which of the following expressions is true?

A) $c_n = a_n + ib_n$

B) $b_n = 2\,\mathrm{Im}\,c_n$

C) $2c_n = a_n - ib_n$

D) $c_n = 2a_n - 2ib_n$

$$c_n = \frac{1}{2}[a_n - b_n i]$$

Fourier series

Theorem: The Fourier series of a (real-valued) function $f(\theta)$ can be written as

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

where

So, for $n \geq 1$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\, d\theta.$$

Proof:

$$c_n e^{in\theta} + c_{-n} e^{-in\theta} = \left(\tfrac{1}{2}a_n - \tfrac{i}{2}b_n\right)\left(\cos(n\theta) + i\sin(n\theta)\right)$$
$$+ \left(\tfrac{1}{2}a_n + \tfrac{i}{2}b_n\right)\left(\cos(n\theta) - i\sin(n\theta)\right)$$
$$= a_n\left[\tfrac{1}{2}\cos(n\theta) + \cancel{\tfrac{i}{2}\sin(n\theta)} + \tfrac{1}{2}\cos(n\theta) - \cancel{\tfrac{i}{2}\sin(n\theta)}\right]$$
$$+ b_n\left[\cancel{-\tfrac{i}{2}\cos(n\theta)} + \tfrac{1}{2}\sin(n\theta) + \cancel{\tfrac{i}{2}\cos(n\theta)} + \tfrac{1}{2}\sin(n\theta)\right]$$
$$= a_n\cos(n\theta) + b_n\sin(n\theta)$$

So $$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\theta) + b_n\sin(n\theta)\right]$$

The Poisson integral

Theorem: Let

$$w(r,\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right\}$$

be a solution of the Dirichlet problem, where the a_n, b_n are the Fourier coefficients of a continuous, (real-valued,) 2π-periodic function $f(\theta)$. Then, for $0 \leq r < 1$ we have

$$w(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}}_{K(\theta,\phi,r)} f(\phi)\,d\phi.$$

Boundary Element Methods

Proof: Skip

An example

Let $f(\theta) = \sin^2\theta$ and let w solve the corresponding Dirichlet problem on the unit disc. What can you say about w at $(x, y) = (0, 0)$?

A) It is less than 0

B) It is equal to $\frac{1}{2}$

C) It is undefined

D) None of the above are true

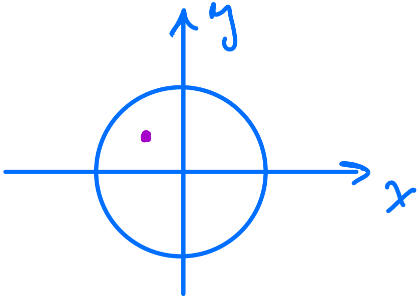
Hint: Use the Poisson integral $w(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\, d\phi$.

$$\lim_{r\to 0} w(r,\theta) = \lim_{r\to 0} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \sin^2\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2(\theta)\, d\theta = \frac{1}{2}$$

An example

Let $f(\theta)$ be a non-negative function and let w solve the corresponding Dirichlet problem on the unit disc. What can you say about w for all points on the unit disc?



A) It is non-negative

B) It is non-positive

C) It is unbounded

D) None of the above are true

Hint: Use the Poisson integral $w(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\, d\phi.$

$\omega(r,\theta) \geq 0$

Some properies of harmonic functions

$\Delta u = \partial^2_{xx} u + \partial^2_{yy} u$

Definition

We say a function u is harmonic in an open set $\mathscr{U} \subset \mathbb{R}^{\cancel{x}\,2}$ if $u \in C^2(U)$ and $\Delta u(x) = 0$ for each $x \in \mathscr{U}$.

Laplacian

Example: $u(x, y) = x^2 - y^2$ is harmonic in any open set of $\mathbb{R}^2$. Why?

$$\Delta u = \partial^2_{xx} u + \partial^2_{yy} u = 2 - 2 = 0$$

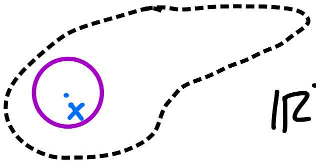
Another example: $u(x,y) = x + y$

$$\Delta u = \partial^2_{xx} u + \partial^2_{yy} u = 0$$

(Mean-Value Property) Suppose $u \in C^2(\mathscr{U})$ where $\mathscr{U}$ is an open set of $\mathbb{R}^2$. Then u is harmonic in U if and only if it has the mean-value property:

$$u(x) = \frac{1}{2\pi r} \int_{\gamma \in \partial B(x,r)} u(\gamma) d\gamma$$

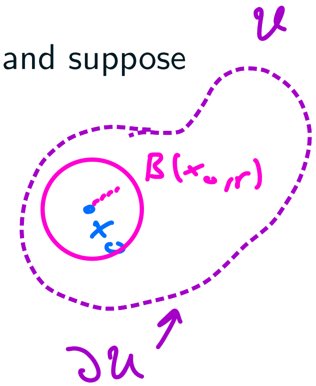
for every ball $B(x, r) \in \mathscr{U}$.



$\mathbb{R}^2$

Some properies of harmonic functions

(Tha maximum principle **)** Let $\mathscr{U} \subset \mathbb{R}^2$ be open, bounded, connected set, and suppose $u \in C^2(\mathscr{U}) \cap C(\bar{\mathscr{U}})$. Then either $u = \textit{constant}$ in $\bar{\mathscr{U}}$, or



$$u(x) < \max_{y \in \partial \mathscr{U}} u(y), \qquad \text{and ,}$$

$$\min_{y \in \partial \mathscr{U}} u(y) < u(x), \qquad \text{for all } x \in \mathscr{U}\ .$$

Proof: Let $x_0 \in U$ such that

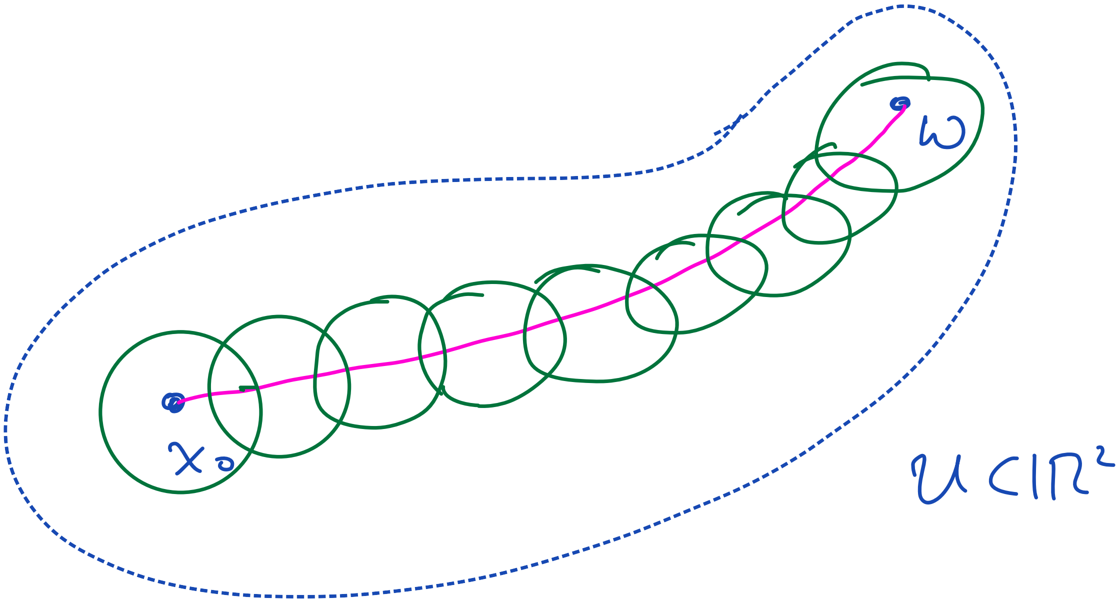
$$u(x_0) = \max_{x \in \bar{u}} u(x) = M$$

Choose r so that $B(x_0, r) \subset \mathcal{U}$. Then by the mean-value theorem

$$u(x_0) = M = \frac{1}{2\pi r} \int_{\gamma \in \partial B(x_0, r)} u(\gamma)\, d\gamma$$

But $u(y) \leq M$ everywhere, so we get that

$$u(z) = M \text{ for every } z \in \partial B(x_0, r)$$



$\Rightarrow \quad u(w) = M$ for every $w \in U$

Recall the Dirichlet problem: Find a solution u to the following boundary value problem on a bounded open set $\mathscr{U} \subset \mathbb{R}^2$

Laplace equation $\longrightarrow$

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad (x,y) \in \mathscr{U} \\ u(x,y) = g(x,y) \quad \text{for} \quad (x,y) \in \partial\mathscr{U} \end{cases}$$

(Implication of the maximum principle) Suppose g is continuous and $u \in C^2(\mathscr{U}) \cap C(\bar{\mathscr{U}})$ is a solution of the Dirichlet problem above. If $\mathscr{U}$ is connected and g satisfies $g(x) > 0$ for all $x \in \partial\mathscr{U}$, then

$$u(x) > 0, \qquad \text{for all } x \in \mathscr{U}.$$

Proof:

$0 < \min_{y \in \partial U} g(y) < \underline{\underline{u(x)}}$ for every $x \in U$

Uniqueness of Solutions of Boundary Value Problems

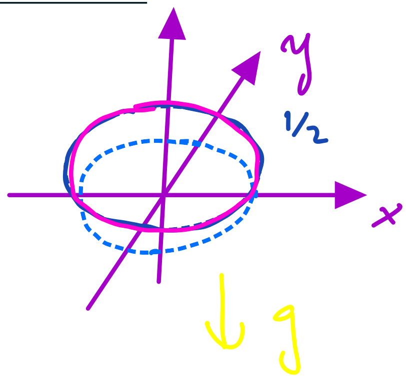
Another problem related to the Dirichlet one is Poisson's equation:

LHS

$-\Delta u$

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f(x,y) & \text{for} \quad (x,y) \in \mathscr{U} \\ u(x,y) = g(x,y) & \text{for} \quad (x,y) \in \partial\mathscr{U} \end{cases}$$

where $f \in C(\mathscr{U})$, and $g \in C(\partial\mathscr{U})$.



$f(x,y) = -g$

$\Delta u = 1$

Example: Find a solution to

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 1 & \text{for} \quad (x,y) \in \mathscr{B}(0,1) \\ u(x,y) = \cancel{1/2} & \text{for} \quad (x,y) \in \partial\mathscr{B}(0,1) \end{cases}$$

$1/4$

Let's use our physical intuition ...

$$\boxed{u(x,y) = (x^2+y^2)/4}$$

$$\Delta u = \partial^2_{xx} u + \partial^2_{yy} u$$

$$= (2+2)/4 = 1$$

$$u(1,0) = 1^2/4$$

Uniqueness of Solutions of Boundary Value Problems

Poisson's equation:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = f(x,y) \quad \text{for} \quad (x,y) \in \mathscr{U} \\ u(x,y) = g(x,y) \quad \text{for} \quad (x,y) \in \partial\mathscr{U} \end{cases}$$

where $f \in C(\mathscr{U})$, and $g \in C(\partial\mathscr{U})$.

Uniqueness of solutions: There is at most one solution $u \in C^2(\mathscr{U}) \cap C(\bar{\mathscr{U}})$ of the Poisson problem above.

Proof: Let's assume that we have two solutions:

$$\begin{cases} -\Delta u_1 = f \\ u_1(x) = g(x) \end{cases} \qquad \begin{cases} -\Delta u_2 = f \\ u_2(x) = f(x) \end{cases}$$

$$\begin{cases} -\Delta u_1 + \Delta u_2 = 0 & \text{in } U \\ u_1(x) - u_2(x) = 0 & \text{on } \partial U \end{cases}$$

$\Rightarrow$

$$\begin{cases} -\Delta(u_1 - u_2) = 0 & \text{in } U \\ \underbrace{(u_1 - u_2)}_{\omega} = 0 & \text{on } \partial U \end{cases}$$

$$\Rightarrow \begin{cases} -\Delta \omega = 0 & \text{in } U \\ \omega = 0 & \text{on } \partial U \end{cases}$$

$0 \leq \omega(x) \leq 0 \quad x \in U$

$\Rightarrow \omega(x) = 0$ for every $x \in U$

(maximum principle)

Another proof using "Energy methods": Set $w = u_1 - u_2$ and let's recall Green's identity

$$\Rightarrow \begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

Let's recall Green's identity

$$\int_U w \underbrace{\Delta w}_{=0} dx = \int_{\partial U} \underbrace{w}_{=0} \frac{\partial w}{\partial \nu} ds - \int_U |\nabla w|^2 dx$$

$$\Rightarrow \quad -\int_U |\nabla w|^2 dx = 0$$

$$\Rightarrow |\nabla w|^2 = 0 \Rightarrow w = \text{constant} = 0$$

See you next time!