Lecture 6

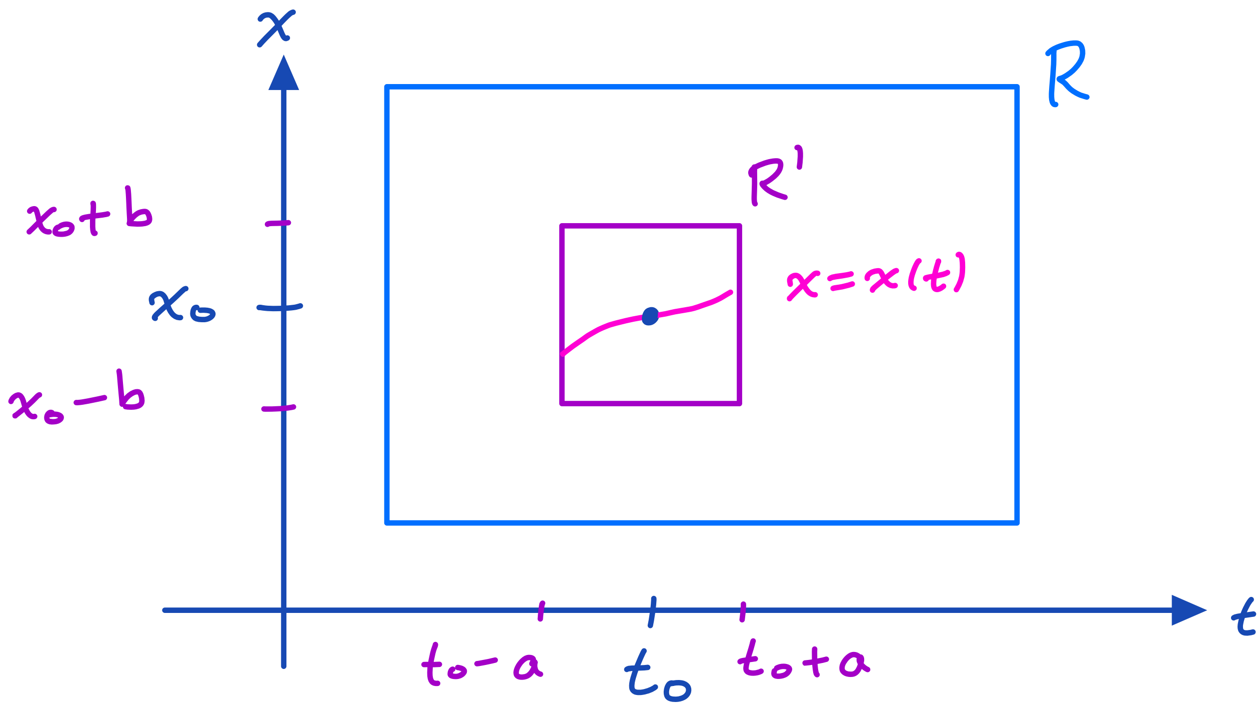
Remarks about Picard's Theorem

$\boxed{\text{Picard's Theorem}}$ (Picard-Lindelöf's theorem)

Let $f = f(x,t)$ and $\partial_x f(x,t)$ be cont. functions of x and t on a closed rectangle R with sides parallel to the axes. If (x_0, t_0) is in the interior of R, then there exists a number $a > 0$ with the property that the IVP

$$\begin{cases} x' = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

has one and only one solution $x = x(t)$ on the interval $|t - t_0| \leq a$.



$$aL<1 \qquad b=Ma$$

where

$$M = \max_{(x,t)\in R} |f(x,t)|$$

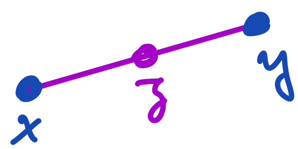
$$L = \max_{(x,t)\in R} |\partial_x f(x,t)|$$

Useful fact from Analysis: Let $\mathcal{C}$ be a closed bounded subset of $\mathbb{R}^n$, and $f: \mathcal{C} \to \mathbb{R}$ continuous, then the there exists $x_0 \in \mathcal{C}$ such that

$$|f(x)| \leq |f(x_0)| \quad \forall x \in \mathcal{C},$$

Thus, $f(x)$ is bounded in $\mathcal{C}$.

Notice that MVT



$$\begin{aligned} |f(x,t) - f(y,t)| &= |\partial_x f(\zeta,t)(x-y)| \\ &= |\partial_x f(\zeta,t)|\,|x-y| \\ &\leq \underbrace{\left(\max_{(x,t)\in R} |\partial_x f(x,t)|\right)}_{L} |x-y| \end{aligned}$$

for all $(x,t), (y,t) \in R$

What does this imply? f is Lipschitz in x

Remark: Picard's Theorem is only local!

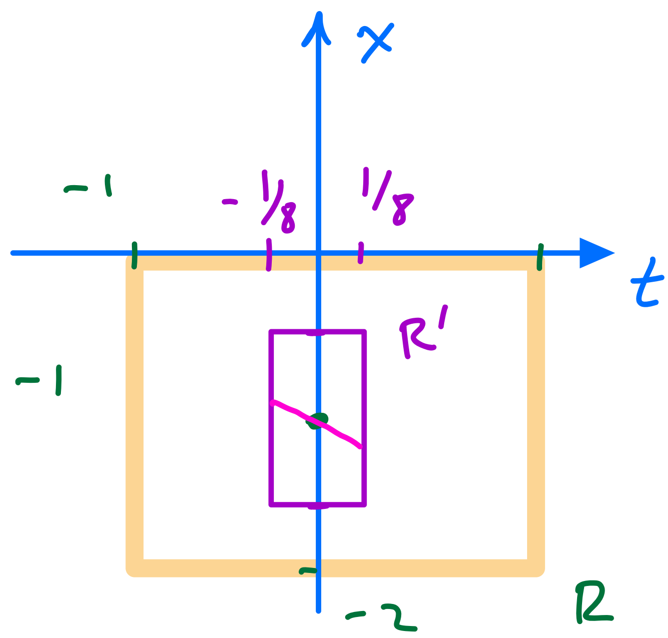
Consider the IVP

$$\begin{cases} \dot{x} = -x^2 \\ x(0) = -1 = x_0 \end{cases}$$

Can we apply Peano's Theorem?

$$f(x,t) = -x^2$$

$$\partial_x f(x,t) = -2x$$



$$R = [-1,1] \times [-2,0]$$

$$M = \max_{(x,t)\in R} |\overbrace{f(x,t)}^{x^2}| = 4$$

$$L = \max_{(x,t)\in R} |\overbrace{\partial_x f(x,t)}^{|-2x|}| = 4$$

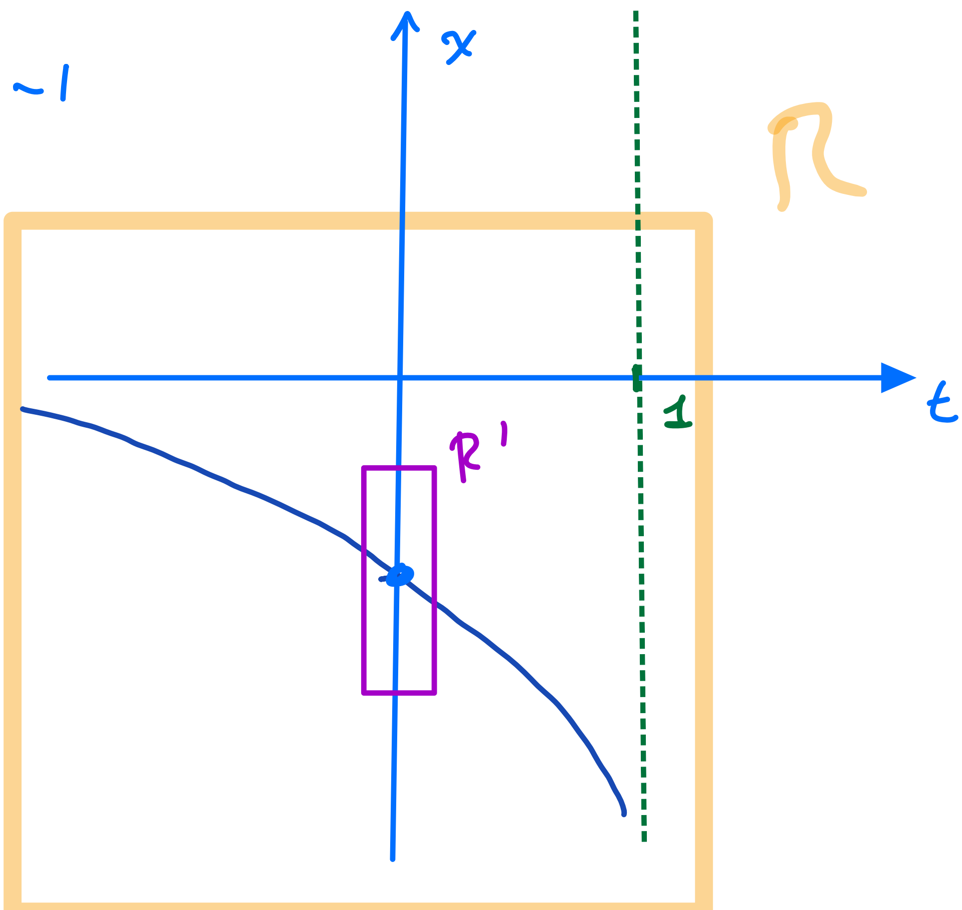
$$aL < 1 \qquad b = Ma$$

$$a = \frac{1}{8} \qquad b = 4 \times \frac{1}{8} = \frac{1}{2}$$

So, yes! We can apply Picard's Theorem.
Using separation of variables

$$\rightarrow \quad x(t) = \frac{1}{t-1}$$

$x(0) = -1$



Remark: Geometric implication of Picard's Theorem.

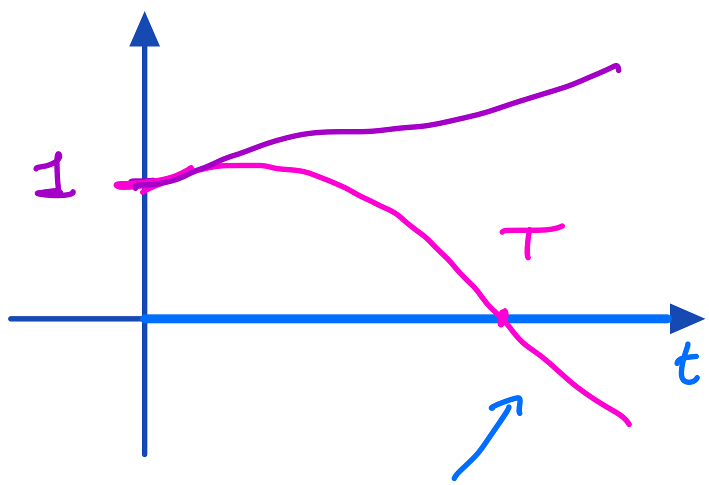
Suppose that x is a solution to the IVP

$$\begin{cases} x' = x\cos^2(t) \leftarrow f(x,t) \\ x(0) = 1 \end{cases} \qquad \partial_x f(x,t) = \cos^2 t$$

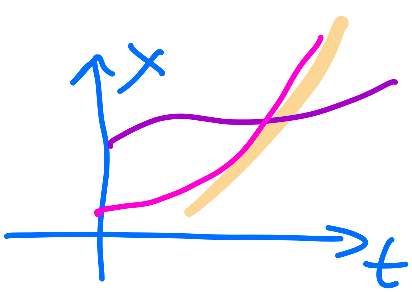
Show that $x(t) > 0$ for all t for which x is defined.

Solution

$x_1(t) = 0$ is a sol.



$$\begin{cases} y' = y\cos^2(t) \\ y(0) = 0 \end{cases}$$



$$\begin{cases} u' = u\cos^2(t) \\ u(T) = 0 \end{cases}$$

Remark

Picard's Theorem

Let $f = f(x,t)$ and $\partial_x f(x,t)$ be cont. functions of x and t on a closed rectangle R with sides parallel to the axes. If (x_0, t_0) is in the interior point of R, then there exists a number $a > 0$ with the property that the IVP

$$\begin{cases} x' = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

has one and only one solution $x = x(t)$ on the interval $|t - t_0| \leq a$.

Picard's Theorem holds true if instead of requiring $\partial_x f(x,t)$ to be cont. on R, we ask

$f(x,t)$ is globally Lipschitz continuous in x on R, i.e.

$$|f(x,t) - f(y,t)| \leq L\,|x-y|$$

for all $(x,t), (y,t) \in R$.

An example of a function for which $\partial_x f$ doesn't exist but it is globally Lipschitz continuous will come in the homework.

Moreover, if we only require f to be continuous we obtain:

Peano's Theorem

Let $f = f(x,t)$ be cont. functions of x and t on a closed rectangle R with sides parallel to the axes. If (x_0, t_0) is in the interior point of R, then there exist at least one solution to the IVP

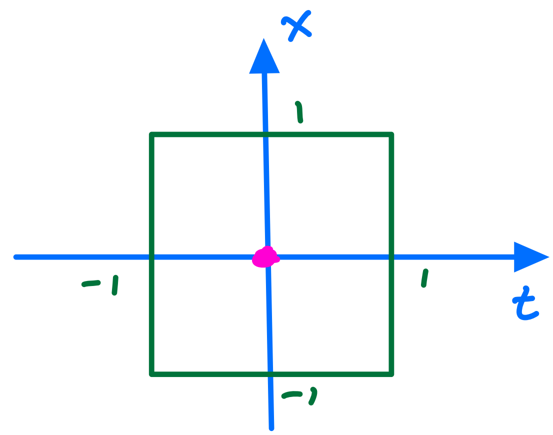
$$\begin{cases} x' = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

Proof: Skip

Example:

$$\begin{cases} x' = 3x^{2/3} \\ x(0) = 0 \end{cases}$$

$f(x,t)$



$R = [-1, 1] \times [-1, 1]$

$f(x,t) = 3x^{2/3}$

f is not Lipschitz continuous w.r.t. x

Why?

$x > 0$

$$\frac{|f(x,0) - f(0,0)|}{|x - 0|} = \frac{3x^{2/3}}{x} = \underbrace{\frac{3}{x^{1/3}}}_{\text{unbounded as } x \to 0}$$

$x_2'(t) = 3t^2$

$3(t^3)^{2/3} = 3t^2$

$x_2(0) = 0$

Are $x_1(t) = 0$ and $x_2(t) = t^3$ sol.?

What about systems?

Let's consider a cont. function $f = f(\underline{x}, t)$

$$f : \mathcal{U} \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

$$\underset{\underline{x}}{\Downarrow} \qquad \underset{t}{\Downarrow}$$

where $\mathcal{U}$ is a bounded closed set and assume that f is globally Lipschitz with respect to $\underline{x}$, i.e. there exists a $K > 0$

$$\| f(\underline{x}_1, t) - f(\underline{x}_2, t) \| \leq K \| \underline{x}_1 - \underline{x}_2 \|$$

$\forall (\underline{x}_1, t), (\underline{x}_2, t) \in \mathcal{U}$.

Then the IVP $\qquad \underline{x}(t) \in \mathbb{R}^n$

$$\begin{cases} \underline{x}' = f(\underline{x}, t) \\ \underline{x}(t_0) = \underline{x}_0 \end{cases}$$

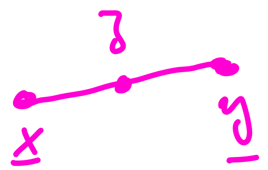
where $(\underline{x}_0, t_0) \in \mathcal{U}$ has a unique sol.

Proof: The same steps as for the 1D case.

Remark: Let $\mathcal{U}$ be a closed bounded subset of $\mathbb{R}^n \times \mathbb{R}$. If the function $f \in C^1(\mathcal{U}; \mathbb{R}^n)$ then $\| D_{\underline{x}} f(\underline{x}, t) \| \leq C$ for all

$(\underline{x}, t) \in \mathcal{U}$ and f is globally Lipschitz continuous w.r.t. x. The proof relies on the MVT

$$\| f(\underline{x}, t) - f(\underline{y}, t) \| \leq \underbrace{\left(\max_{(\underline{x}, t) \in D} | D_{\underline{x}} f(\underline{x}, t) | \right)}_{L} \| x - y \|$$



Example: Consider the IVP

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ y(x) \end{bmatrix} = \begin{bmatrix} x^2 y \, t \\ y^2 x \end{bmatrix} =: \underline{f}\left(\begin{bmatrix} x \\ y \end{bmatrix}, t \right)$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\mathcal{U} = [-1, 1] \times [-1, 1] \times [-1, 1] \subset \mathbb{R}^2 \times \mathbb{R}$

Is $\underline{f} \in C^1(\mathcal{U}; \mathbb{R}^2)$?

<u>exercise</u>

$D_{\bar{x}} f(\bar{x}, t) =$

The end of existence and uniqueness results!

Next time: Laplace Transform