Linear, Constant-Coefficient, Homogeneous ODEs

- These are equations of the form $F[y] := a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$.
- The **general solution** is a linear combination of n linearly independent solutions. To see this, let y_1 and y_2 be two (linearly independent) solutions. Then $F[Ay_1 + By_2] = A\underbrace{F[y_1]}_{=0} + B\underbrace{F[y_2]}_{=0} = 0$.
- To find these n linearly independent solutions, make a guess that $y(t) = e^{rt}$. Plug this into the ODE and divide by e^{rt} to get the **auxiliary equation:**

$$p(r) := a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

- If there are n distinct solutions for r in this equation, say $r_1, \ldots, r_n$, then we have found n solutions, and the general solution is

$$y(t) = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}$$

  for any constants $c_1, \ldots, c_n \in \mathbb{R}$.
- If there are repeated roots, then it seems there are fewer than n linearly independent solutions, which is a problem. Here is how to discover the remaining solutions:

"Multiplication Rule"

Suppose $p(x)$ has a repeated root r of multiplicity m, meaning $p(r) = p'(r) = \cdots = p^{(m-1)}(r) = 0$. Just like in the non-repeated root case, since $p(r) = 0$, e^{rt} solves the ODE:

$$F[e^{rt}] = a_n r^n e^{rt} + a_{n-1}r^{n-1}e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} = p(r)e^{rt} = 0.$$

Now differentiate *with respect to r:*

$$F[te^{rt}] = \cancel{p(r)}te^{rt} + \cancel{p'(r)}e^{rt} = 0,$$

so te^{rt} is a solution. Differentiate another time with respect to r to find t^2e^{rt} is a solution:

$$F[t^2e^{rt}] = \cancel{p(r)}t^2e^{rt} + 2\cancel{p'(r)}te^{rt} + \cancel{p''(r)}e^{rt} = 0.$$

This can be inductively repeated $m-1$ times to find that $t^j e^{rt}$ is a solution for $j = 0, 1, \ldots, m-1$.

Exercises

Find the general solutions of the following equations:

1. $y^{(6)} - 10y^{(5)} + 40y^{(4)} - 82y''' + 91y'' - 52y' + 12y = 0$

   Hint: $r^6 - 10r^5 + 40r^4 - 82r^3 + 91r^2 - 52r + 12 = (r-1)^3(r-2)^2(r-3)$

   Solution: Make the exponential guess to find that r must be a root of the polynomial given in the hint. Since the hint gives the roots of the auxiliary polynomial with their corresponding multiplicities, the general solution is

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + c_4e^{2t} + c_5te^{2t} + c_6e^{3t}$$

   for any $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$.

2. $y^{(4)} + 2y'' + y = 0$

   Solution: Make the exponential guess to find the auxiliary polynomial is

$$p(r) = r^4 + 2r^2 + 1 = (r^2+1)^2 = (r+i)^2(r-i)^2.$$

   Therefore the general solution is $y(t) = c_1e^{it} + c_2te^{it} + c_3e^{-it} + c_4te^{-it}$ for any $c_1, c_2, c_3, c_4 \in \mathbb{R}$. It is usually more useful to rewrite complex exponentials using Euler's Formula, in which case

$$y(t) = C_1\sin(t) + C_2\cos(t) + C_3t\sin(x) + C_4t\cos(t)$$

   for any $C_1, C_2, C_3, C_4 \in \mathbb{R}$.

Inhomogeneous ODEs

We say that a linear ODE is **inhomogeneous** if there is a term not involving y, called the **forcing term**. (Note: It does not make sense to call a nonlinear ODE inhomogeneous.) For example, the ODE

$$y'' - 2y' + y = e^t$$

is an inhomogeneous ODE with forcing term e^t. To find the general solution of an inhomogeneous ODE:

1. Find the general solution, say $y_h(t)$, of the corresponding homogeneous ODE obtained by replacing the forcing term with 0.
2. Find any solution, say $y_p(t)$, to the inhomogeneous ODE. This is called a **particular solution**.

Then the general solution of our inhomogeneous ODE is $y_h(t)+y_p(t)$ because $F[y_h+y_p] = \cancel{F[y_h]}+F[y_p] = g(t)$. Usually, the most difficult step is finding a particular solution. If the ODE has constant coefficients, the **Method of Undetermined Coefficients** may be useful, in which you guess the solution and find the undetermined coefficients in your guess. Choose your guess to match the "form" of the forcing term. For more sophisticated equations, either a more sophisticated guess is needed, or this method may just not work.

Exercises

Find the general solution of the following ODEs:

1. $y'' - y' - 6y = 3\sin(3t)$.

   Solution: The corresponding homogeneous ODE is $y''-y'-6y = 0$. To find the general solution, make the exponential guess to find the auxiliary polynomial $r^2 - r - 6 = (r-3)(r+2)$, which has 2 simple roots: $r = 3$ and $r = -2$. Then the general solution to the homogeneous ODE is $c_1e^{3t} + c_2e^{-2t}$ for any $c_1, c_2 \in \mathbb{R}$.

   All that remains is to find a particular solution. Since the forcing term is a combinations of sines and cosines, guess $y_p = A\cos(3t) + B\sin(3t)$. (Sines and cosines go together!) Plugging this guess into the ODE gives

   $$(-9A\cos(3t) - 9B\sin(3t)) - (-3A\sin(3t) + 3B\cos(3t)) - 6(A\cos(3t) + B\sin(3t)) = 3\sin(3t).$$

   This is satisfied if the number of cosines and sines match on each side:

   $$\begin{cases} -15A - 3B = 0 \\ 3A - 15B = 3 \end{cases} \implies \begin{cases} A = \frac{1}{26} \\ B = -\frac{5}{26}. \end{cases}$$

   Therefore

   $$y(t) = c_1e^{3t} + c_2e^{-2t} + \frac{1}{26}\cos(3t) - \frac{5}{26}\sin(3t).$$

2. $y'' - y' - 6y = 20e^{-2t}$

   Solution: The corresponding homogeneous ODE $y'' - y' - 6y = 0$ has auxiliary polynomial $r^2 - r - 6 = (r-3)(r+2)$, so its general solution is $c_1e^{3t} + c_2e^{-2t}$. It would be problematic to guess $y_p = Ae^{-2t}$ (try it!) because this is part of the homogeneous equation's general solution. When this happens, multiply your guess by t. (This is motivated by the "multiplication rule".) That is, guess $y_p(t) = Ate^{-2t}$. Plugging this into the ODE gives

   $$20e^{-2t} = A(\cancel{4te^{-2t}} - 4e^{-2t}) - A\left(\cancel{-2te^{-2t}} + e^{-2t}\right) - \cancel{6(Ate^{-2t})} = -5Ae^{-2t} \implies A = -4.$$

   Therefore the general solution is

   $$y(t) = c_1e^{3t} + c_2e^{-2t} - 4te^{-2t}.$$

3. $y'' - 2y' = 12t - 10$

   Solution: The corresponding homogeneous ODE, $y'' - 2y' = 0$, has auxiliary polynomial $r^2 - 2r = r(r-2)$, so its general solution is $c_1 + c_2e^{2t}$. Since the forcing term is a polynomial, guess $y_p(t) = At^3 + Bt^2 + Ct + D$. Plugging in this guess gives

   $$12t - 10 = (6At + 2B) - 2(3At^2 + 2Bt + C) = -6At^2 + (6A - 4B)t + (2B - 2C).$$

   Matching coefficients gives us equations for A, B, and C: $-6A = 0$, $6A - 4B = 12$, and $2B - 2C = -10$. Solving these equations gives $A = 0$, $B = -3$, and $C = 2$. Notice that any value works for D, so lets choose $D = 0$. Then the general solution is

   $$y(t) = c_1 + c_2e^{2t} - 3t^2 + 2t.$$

Integrating Factor Method

This method works for any linear, first order ODE, i.e. equations of this form:

$$y' + p(t)y = q(t)$$

for some functions $p(t)$ and $q(t)$. To solve, multiply both sides by the **integrating factor** $\mu(t) := e^{\int p(t)dt}$ to get

$$\mu y' + \mu p y = \mu q.$$

By the product rule, we can rewrite the left side:

$$(\mu y)' = \mu q.$$

Lastly, integrate to get

$$\mu y = \int \mu(t)q(t)dt + C$$

and divide by μ :

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)q(t)dt + \frac{C}{\mu(t)}.$$

Exercise

1. Find the general solution of $ty' + y = t$ for $t > 0$.

   Solution: First, divide by t so that the coefficient of y' is 1:

   $$y' + \frac{y}{t} = 1.$$

   After multiplying by the integrating factor

   $$\mu(t) = e^{\int \frac{1}{t}dt} = e^{\ln(t)} = t,$$

   the product rule allows us to rewrite the ODE as $(ty)' = t$ (this is basically how the equation was originally given). Integrating gives

   $$ty = \frac{t^2}{2} + C \implies y(t) = \frac{t}{2} + \frac{C}{t}.$$

Separation of Variables

A first order ODE is **separable** if it can be written in the form

$$f(y)y' = g(t).$$

To solve, integrate both sides with respect to t and make the chain rule on the left side:

$$\int f(y)dy = \int g(t)dt.$$

If possible, rearrange the resulting equation to solve for y explicitly.

Exercise

1. Solve $\begin{cases} y' = ty(y-2) \\ y(1) = 1. \end{cases}$

   Solution: Separate and integrate:

   $$\int \frac{dy}{y(y-2)} = \int t dt \implies \frac{\ln(|y-2|) - \ln(|y|)}{2} = \frac{t^2}{2} + C \implies y(t) = \frac{2}{1 + e^{(t^2-1)}}$$

Note:

- The y integral is done with **partial fraction decomposition**, where we guessed $\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2}$. Multiplying by $y(y-2)$ gives $1 = A(y-2) + By = (A+B)y - 2A$. This can only happen if $\begin{cases} A + B = 0 \\ -2A = 1 \end{cases} \implies \begin{cases} A = -\frac{1}{2} \\ B = \frac{1}{2}. \end{cases}$

Rewriting Higher Order Equations as a First Order System

We can rewrite any higher order ODE as a system of first order ODEs by introducing the vector $\vec{y} := \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$, where $y_j := y^{(j)}$ for $j = 1, \ldots, n-1$. Then

$$F(y, y', y'', \ldots, y^{(n-1)}, y^{(n)}, t) = 0 \iff \begin{cases} y_0' = y_1 \\ y_1' = y_2 \\ \vdots \\ y_{n-2}' = y_{n-1} \\ F(y_0, y_1, y_2, \ldots, y_{n-1}, y_{n-1}', t) = 0. \end{cases}$$

If the differential equation can be rearrange to solve for $y^{(n)}$, say $y^{(n)} = G(y, y', \ldots, y^{(n-1)}, t)$, then the system can be written as

$$\vec{y}\,' = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \\ G(y_0, \ldots, y_{n-1}, t) \end{bmatrix}$$

Furthermore, we can introduce one more component $y_n := t$ to make the system autonomous. In this case

$$y^{(n)} = F(y, y', y'', \ldots, y^{(n-1)}, t) \iff \vec{y}\,' = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-2} \\ G(y_0, \ldots, y_{n-1}, y_n) \\ 1 \end{bmatrix}.$$

- In both cases, the first component is the solution we are actually interested in.
- If $G(y, \ldots, y^{(n-1)}, t) = a_0 y + a_1 y' + \cdots + a_{n-1} y^{(n-1)}$ for some $a_0, \ldots, a_{n-1} \in \mathbb{R}$, this system is linear and can be written as $\vec{y}\,' = A\vec{y}$. The general solution to this system is $\vec{y}(t) = e^{At}\vec{y}(0) := \sum_{n=0}^{\infty} \frac{(At)^n}{n!}\vec{y}(0)$

Example

1. Rewrite $\begin{cases} y'''y''y'y = t \\ y(\pi) = 0 \\ y'(\pi) = 1 \\ y''(\pi) = 2 \end{cases}$ as a first order system.

   Solution: As recommended above, define $y_j := y^{(j)}$ for $j = 0, 1, 2$. Then the equivalent first order system is

   $$\vec{y}\,' = \begin{bmatrix} y_1 \\ y_2 \\ t(y_0 y_1 y_2)^{-1} \end{bmatrix} \quad \text{with} \quad \vec{y}(\pi) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

   To furthermore make this autonomous, we can introduce $y_3 := t$. In this case

   $$\vec{y}\,' = \begin{bmatrix} y_1 \\ y_2 \\ y_3(y_0 y_1 y_2)^{-1} \\ 1 \end{bmatrix} \quad \text{with} \quad \vec{y}(\pi) = \begin{bmatrix} 0 \\ 1 \\ 2 \\ \pi \end{bmatrix}.$$