Lecture 5

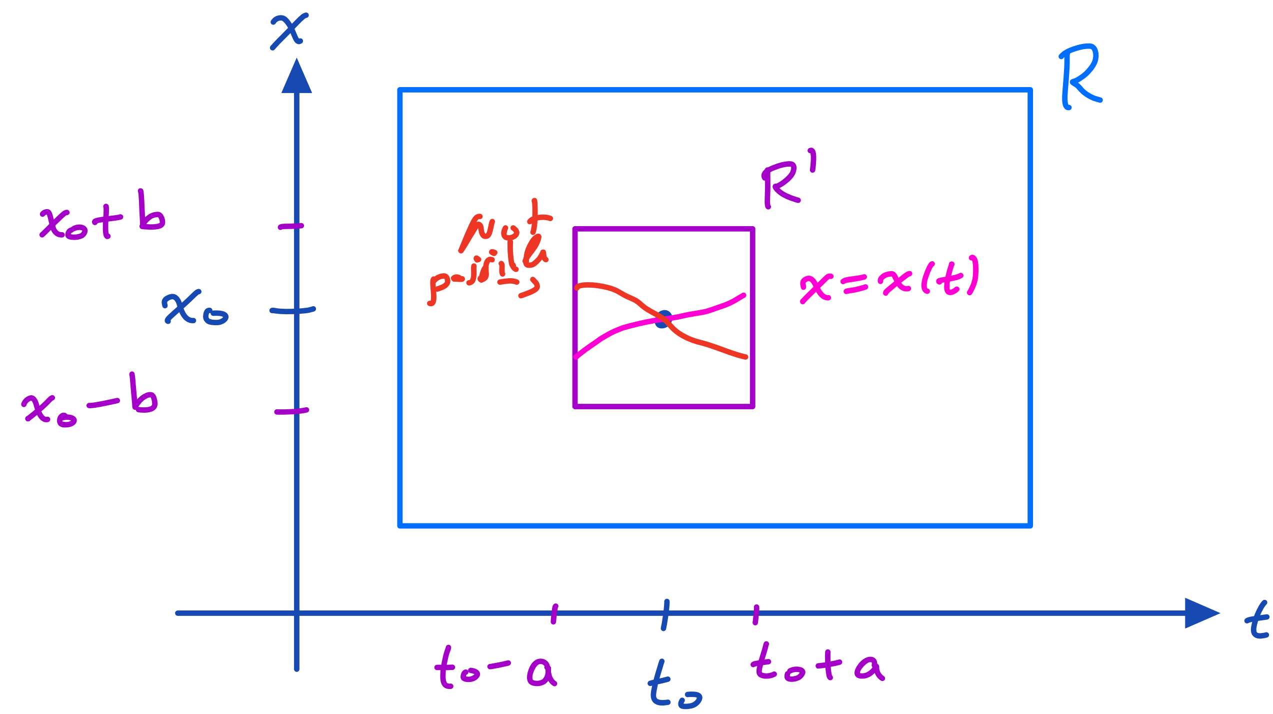
Today we will prove Picard's Theorem (also known as Picard–Lindelöf's Theorem)

$\boxed{\text{Picard's Theorem}}$

Let $f = f(x,t)$ and $\partial_x f(x,t)$ be cont. functions of x and t on a closed rectangle R with sides parallel to the axes. If (x_0, t_0) is in the interior point of R, then there exists a number $a > 0$ with the property that the IVP

$$\begin{cases} x' = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

has one and only one solution $x = x(t)$ on the interval $|t - t_0| \leq a$.



Note: Since R is closed and bounded then f and $\partial_x f$ are bounded.

$$\Longrightarrow \quad \max_{(x,t)\in R} |f(x,t)| = M$$

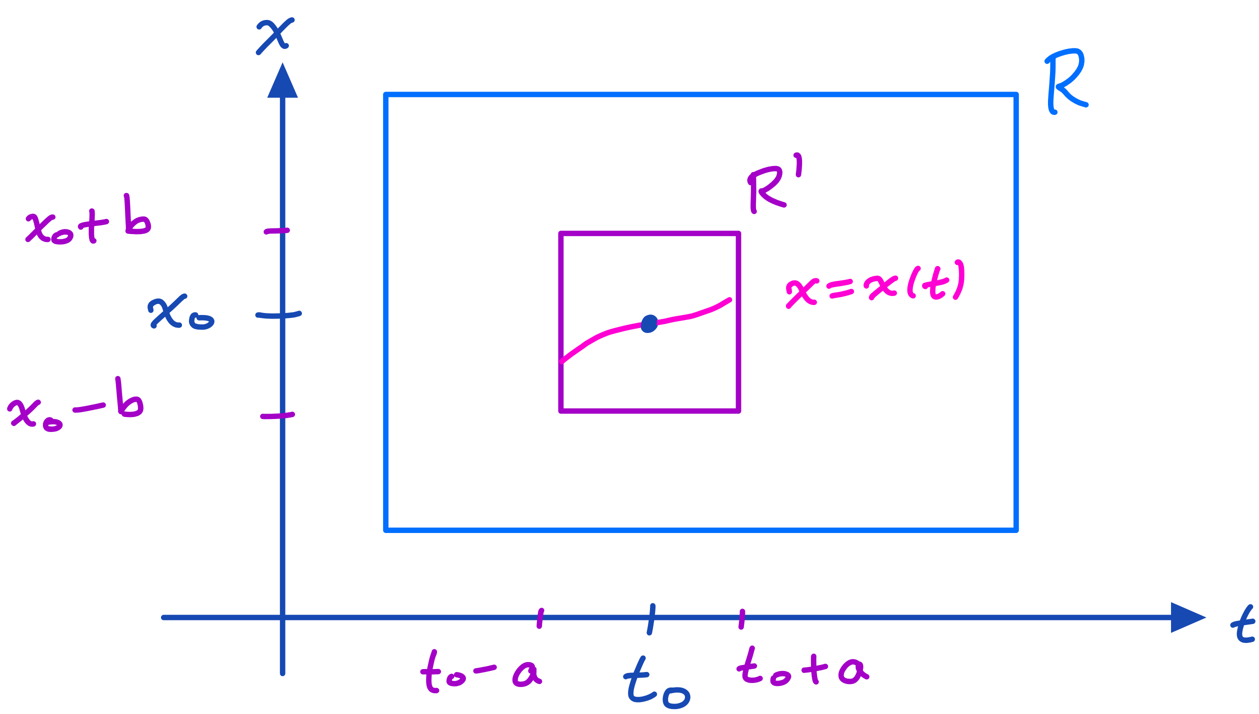
$$\max_{(x,t)\in R} |\partial_x f(x,t)| = L$$

so, the function f is global Lipschitz cont. in x!

How big is R'?

We pick a s.t. $aL < 1$ and

$b = Ma$



Proof outline:

Step 0:

Solving IVP $\iff$ Find a cont. sol. to the IE

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases} \qquad x(t) = x_0 + \int_{t_0}^{t} f(x(\tau),\tau)\,d\tau$$

Step 1: Approximate sol. using Picard iteration

$$x_0(t) = x_0 , \qquad x_n(t) = x_0 + \int_{t_0}^{t} f(x_{n-1}(\tau),\tau)\,d\tau$$

Step 2: Prove that Picard iterates converge $\lim_{n\to\infty} x_n(t) = \bar{x}(t)$

and $\bar{x}(t)$ is cont.

Step 3: Prove that $\bar{x}(t)$ solves the integral equation.

Step 4: Prove that $\bar{x}(t)$ is unique.

Step 2: Prove that Picard iterates converge $\lim_{n\to\infty} x_n(t) = \bar{x}(t)$

2a) Let's note that

$$x_n(t) = x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

$x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})$

telescoping sum

we cast a sequence as a series!

2b) We will show that the sequence

$$x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

(p. 205 in Ross)

converges by Weierstrass M-test

- If $|f_n(t)| \le M_n$, $\forall\, n \ge 0$ and $\forall\, t \in D$

and $\sum_{n=0}^{\infty} M_n$ converges

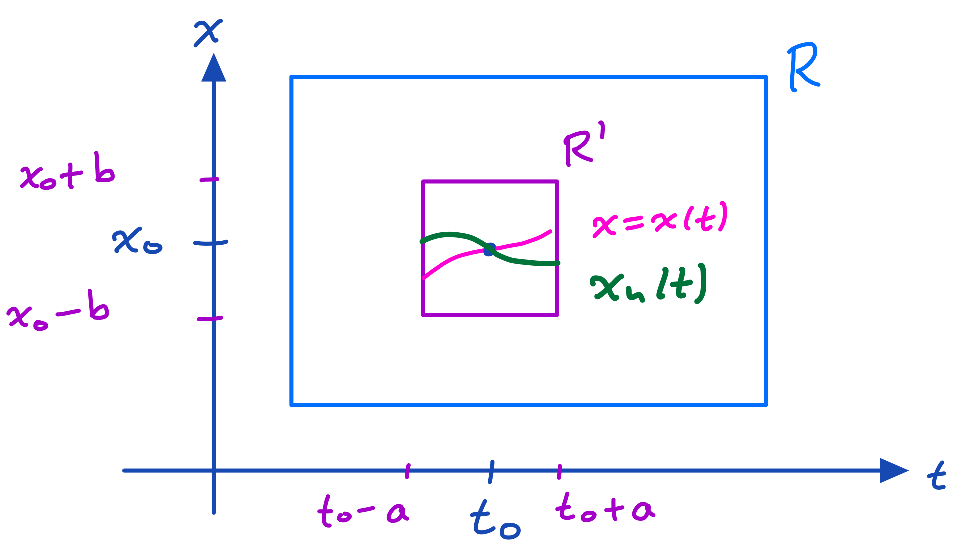
then $\sum_{n=0}^{\infty} f_n(t)$ converges absolutely and uniformely (see p. 193 in Ross)

• Exercise: Check why this theorem cannot be applied to $f_n(t) = t^n$ on $[0,1]$.

The M-Weierstrass Theorem together with the Uniform Limit Theorem guarantee that

$$\sum_{n=0}^{\infty} f_n(t) \quad \text{is continuous}$$

2b part 1) We will first show that all $x_n(t)$ are in R'



Recall that $x_0(t) = x_0$

$\Rightarrow \quad |x_0(t) - x_0| = 0$ so $x_0(t) \in R'$

Now for $x_1(t)$ we have

$$|x_1(t) - x_0| = \left| \int_{t_0}^{t} f(x_0(\tau), \tau)\, d\tau \right|$$

$$\left| \int_{t_0}^{t} g(\tau)\, d\tau \right| \leq \int_{t_0}^{t} |g(\tau)|\, d\tau$$

$$\leq \int_{t_0}^{t} |f(x_0(\tau), \tau)|\, d\tau$$

$$\leq \int_{t_0}^{t} M\, d\tau = M(t - t_0)$$

$$\leq M a = b$$

so, $x_1(t)$ is in R'

By induction, we can show that

$x_n(t)$ is in R'!

2b part 2) Apply the Weierstrass M-Theorem to the series of functions

$$x_0(t) + \sum_{k=1}^{n} x_k(t) - x_{k-1}(t)$$

Let's set $\Delta = \max\limits_{t\in(x_0-a,\, x_0+a)} |x_1(t) - x_0|$

Take $k=1$

$$|x_2(t) - x_1(t)| = \left| \int_{t_0}^{t} f(x_1(\tau),\tau) - f(x_0(\tau),\tau)\, d\tau \right|$$

$$\leq \int_{t_0}^{t} \left| f(x_1(\tau),\tau) - f(x_0(\tau),\tau) \right| d\tau$$

$$\leq \int_{t_0}^{t} L\, |x_1(\tau) - x_0(\tau)|\, d\tau$$

$$\leq \int_{t_0}^{t} L\Delta\, d\tau \leq \Delta L (t - t_0)$$

$$\leq \Delta L a$$

Similarly, for $k=2$

$$|x_3(t)-x_2(t)| = \left|\int_{t_0}^{t} f(x_2(\tau),\tau) - f(x_1(\tau),\tau)\, d\tau\right|$$

$$\ldots$$

$$\leq \Delta (aL)^2$$

By induction, we can show that $\forall n \geq 1$

$$|x_n(t)-x_{n-1}(t)| \leq \Delta (aL)^{n-1}$$

for all $t \in (t_0-a, t_0+a)$

$\Rightarrow$ Thus, since

$$\sum_{n=0}^{\infty} \Delta (aL)^n = \frac{\Delta}{1-aL} < \infty$$

if $aL < 1$

(geometric series)

So, thanks to the Weierstrass M-test

$$\lim_{n\to\infty} x_n(t) = \bar{x}(t)$$

also, $\bar{x}(t)$ is in R'

By the uniform convergence theorem we get that $\bar{x}(t)$ is cont.

Step 3 Show that $\bar{x}(t)$ solves the I.E.

Idea: We know that $|\bar{x}-x_n|$ gets small as $n\to\infty$, we will use this fact.

$$\left|\bar{x}(t)-x_0-\int_{t_0}^{t} f(\bar{x}(\tau),\tau)\,d\tau\right|$$

$$=\left|\bar{x}(t)-x_n(t)+x_n(t)-x_0-\int_{t_0}^{t} f(\bar{x}(\tau),\tau)\,d\tau\right|$$

$$\le \underbrace{|\bar{x}(t)-x_n(t)|+\left|x_n-x_0-\int_{t_0}^{t} f(\bar{x}(\tau),\tau)\,d\tau\right|}$$

$$=|\bar{x}(t)-x_n(t)|+\left|\int_{t_0}^{t} f(x_{n-1}(\tau),\tau)-f(\bar{x}(\tau),\tau)\,d\tau\right|$$

$$\le |\bar{x}(t)-x_n(t)|+\int_{t_0}^{t}\left|f(x_{n-1}(\tau),\tau)-f(\bar{x}(\tau),\tau)\right|d\tau$$

$$\le |\bar{x}(t)-x_n(t)|+\int_{t_0}^{t} L\,|x_{n-1}(\tau)-\bar{x}(t)|\,d\tau$$

$$\leq |\bar{x}(t) - x_n(t)| + (La)\left(\max |x_{n-1}(\tau) - \bar{x}(\tau)|\right)$$

Now, since $x_n \xrightarrow[n\to\infty]{} \bar{x}$ uniformely we get that

$$|\bar{x}(t) - x_n(t)| + (La)\left(\max |x_{n-1}(\tau) - \bar{x}(\tau)|\right) \xrightarrow[n\to\infty]{} 0$$

Thus

$$\bar{x}(t) - x_0 - \int_{t_0}^{t} f(\bar{x}(\tau), \tau)\, d\tau = 0$$

what we wanted to show!

Step 4: We will show that the sol. is unique

Skip! (This part can be obtained very easily using the Grönwall's ineq.)

Next time Laplace Transform!