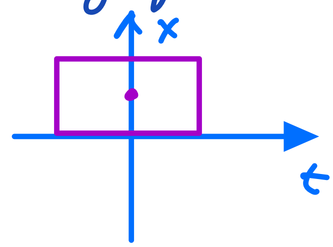
Math 135: Review

- Existence and uniqueness theory for ODEs



$$\begin{cases} \frac{d}{dt}\underline{x}(t) = f(\underline{x}(t), t) \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

here $x(t) \in \mathbb{R}^n$ for $t \geq 0$.

$$x(t) = x_0 + \int_0^t f(x(s), s)\, ds$$

- Laplace transform

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{kernel}} f(t)\, dt$$

$$\text{ODE} \xrightarrow{\mathcal{L}} \underset{\mathcal{L}\{y\}(s)}{\text{Algebra}} \xrightarrow{\mathcal{L}^{-1}} \text{Solution}$$

- Fourier Series

For a square Riemann integrable function f defined in $[-\pi, \pi)$ we can assign a Fourier Series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos(nx) + b_n \sin(nx) \}$$

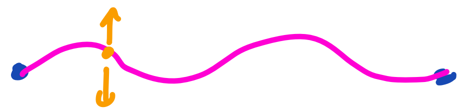
where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

- Partial Differential Equations

Wave Eqn. 

$$\begin{cases} \partial_{tt}^2 u(x,t) = c^2 \partial_{xx}^2 u(x,t) & \text{for } x \in [0,\pi] \text{ and } t \geq 0 \\ u(t,0) = u(t,\pi) = 0 \text{ for } t \geq 0 & \text{B.C.} \\ \dfrac{\partial u}{\partial t}(0,x) = g(x) \quad \text{for } x \in [0,\pi] & \text{I.C.} \\ u(0,x) = f(x) \quad \text{for } x \in [0,\pi] & \text{I.C.} \end{cases}$$

Heat Eqn.

$$\begin{cases} \partial_t u = a^2 \partial_{xx}^2 u & \\ u(t,0) = u(t,\pi) = 0 \qquad t \geq 0 & \text{B.C.} \\ u(0,x) = f(x) \qquad x \in [0,\pi] & \text{I.C.} \end{cases}$$

Laplace Eqn (Dirichlet problem)

$$\begin{cases} \Delta u(x,y) = 0 & \text{for } (x,y) \in \mathcal{D} \subset \mathbb{R}^2 \\ u(x,y) = f(x,y) & \text{for } (x,y) \in \partial\mathcal{D} \end{cases}$$

b.c.

We use the method of sep. of variables to solve these eqns. i.e

$$u(x,t) = v(t)\,\omega(x)$$

Plug in to the DE an get a Sturm-Liouville problem

Finally, express the sol. as a series

6. (20 points) Consider the equation IVP

$$\begin{cases} \frac{dx}{dt} = \sin(x) \\ x(0) = 1. \end{cases}$$

t_0

(a) Show that there exists a unique solution to this problem. Justify your answer.

(b) Write down the integral equation equivalent to this problem.

(c) Compute the first two Picard iterations.

a) The function $f(x) = \sin(x)$ is C^1 so it is Lipschitz at $x = 1$. Thus by the E. and U. theorem there exists a unique sol.

b) $$x(t) = x_0 + \int_{t_0}^{t} \sin(x(s))\,ds$$

$$= 1 + \int_{0}^{t} \sin(x(s))\,ds$$

c) $x_0 = 1$

$$x_1(t) = x_0 + \int_0^t \sin(x_0)\,ds$$

$$= 1 + \sin(1)\,t$$

$$x_2(t) = x_0 + \int_0^t \sin(x_1(s))\,ds$$

$$= 1 + \int_0^t \sin(1 + \sin(1)s)\,ds$$

$$= 1 - \left. \frac{\cos(1+\sin(1)s)}{\sin(1)} \right|_0^t$$

$$= 1 - \frac{\cos(1+\sin(1)t) - \cos(1)}{\sin(1)}$$

Theorem: Let V be a (real) **Hilbert space** and $\{\phi_1, \phi_2 \ldots,\}$ be a **complete orthonormal sequence** in V.

Then the series

$$\| \sum_{n=1}^{N} \langle \phi_n, v \rangle \phi_n - v \| \to 0 \quad N \to \infty$$

$$v = \sum_{n=1}^{\infty} \langle \phi_n, v \rangle \phi_n$$

converges to v and satisfies

$$\|v\|^2 = \sum_{n=1}^{\infty} |\langle \phi_n, v \rangle|^2.$$

Parseval's identity

- In this slightly larger Hilbert space, the sequence

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \ldots \right\},$$

  is a **complete orthonormal sequence**.

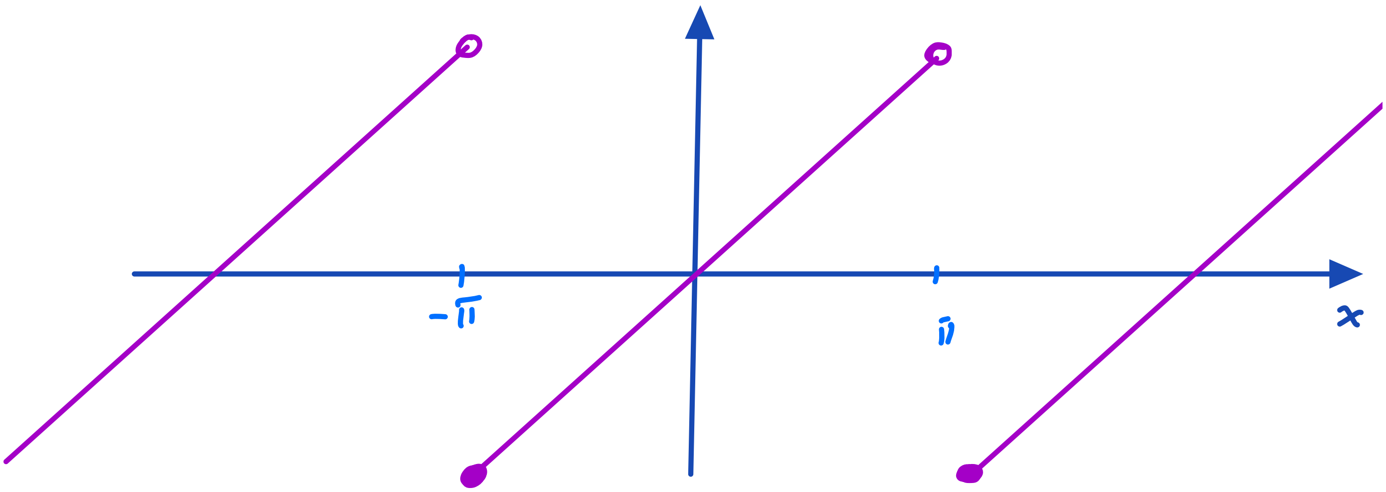
- Applying our result, given any $f(x) \in \mathcal{R}^2[-\pi, \pi]$ the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\},$$

  converges to $f(x)$.

- We say that the Fourier series **converges in mean** to $f(x)$.

Example: Let $f(x) = x$ for $x \in [-\pi, \pi)$
and extend it by periodicity



By Dirichlet's theorem

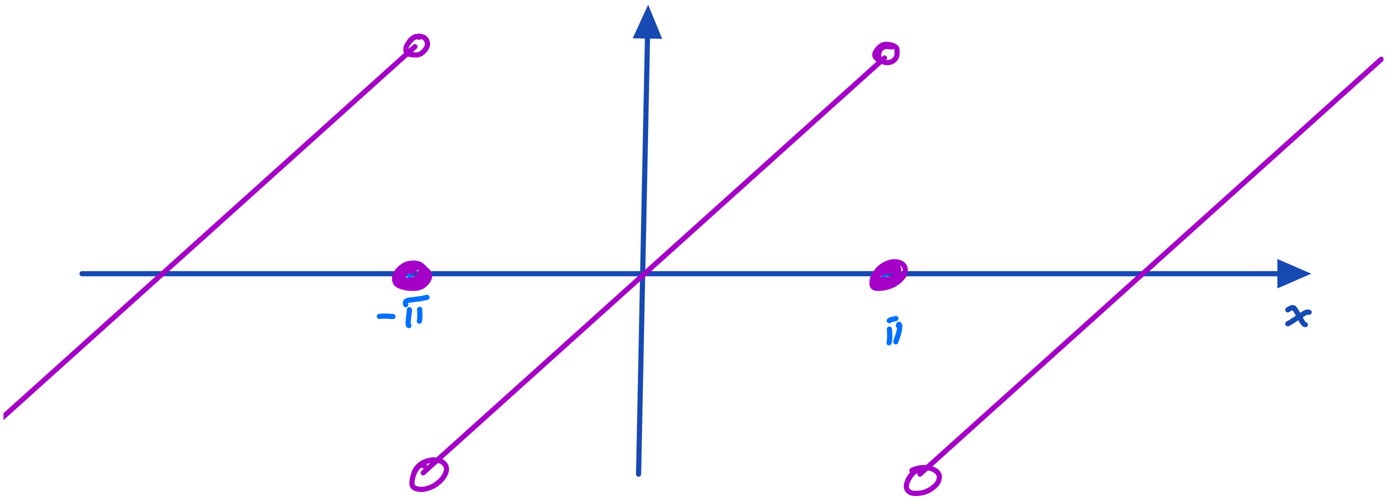
$$2\sin(x) - \sin(2x) + \cdots \longrightarrow x \quad \text{for } x \in (-\pi, \pi)$$

and

$$\lim_{x \to \pi^+} f(x) = -\pi$$

$$2\sin(x) - \sin(2x) + \cdots \longrightarrow \underbrace{\frac{1}{2}\Big[\overbrace{f(\pi^+)}^{-\pi} + \overbrace{f(\pi^-)}^{\pi}\Big]}_{=0}$$

So, the pointwise limit of the F. series is



And the "convergence in mean" is $f(x)$
i.e. (L^2 convergence)

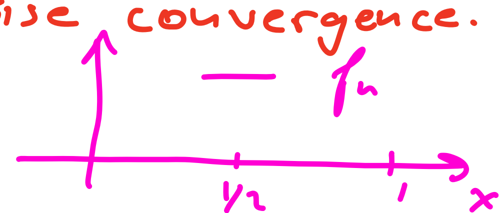
$$\| S_N - f \|_2 \xrightarrow[N \to \infty]{} 0 \quad , \quad \text{or}$$

$$\left(\int_{-\pi}^{\pi} [S_N(x) - f(x)]^2 \, dx \right)^{1/2} \to 0 , \; N \to \infty$$

NOTE: The pointwise limit is not the same as the L^2 limit !!!

Remark 1: Pointwise convergence doesn't imply convergence in mean (homework problem.)

Remark 2: Convergence in mean doesn't imply pointwise convergence.



Example:

$$f_n(x) = \begin{cases} 1 & \text{if } \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} + \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

on the interval $[0,1]$.

Show that f_n converges in mean to 0.

$$\|f_n - 0\|_{L^2} \to 0, \quad n \to \infty$$

$$\|f_n\|_{L^2}^2 = \int_0^1 [f_n(x)]^2 dx = \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}+\frac{1}{n}} dx = \frac{2}{n} \xrightarrow[n\to\infty]{} 0$$

Does $f_n \xrightarrow[n\to\infty]{} 0$ pointwise? NO

$f_n\left(\frac{1}{2}\right) = \frac{1}{2}$ for $n \geq 1$.

Elliptic Eqns:

Problem: Let's assume that in 2D $u = u(r,\theta)$ satisfies Laplace eqn $\Delta u = 0$ in the disk $B(0,1)$, and on the boundary it satisfies

$$u(1,\theta) = 1 + \sin(\theta)$$

→ What is the value of u at the origin?

→ Where do the maximum and minimum of u occur in the closed ball $\overline{B(0,1)}$?

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\, d\phi$$

$$\lim_{r\to 0} u(r,0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{r\to 0} \left(\quad \right) f(\phi)\, d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+\sin(\phi))\, d\phi$$

$$= \frac{2\pi}{2\pi} = 1$$

$$\overline{B(0,1)} = \{ x^2+y^2 \le 1 \}$$

$$\max_{x \in \overline{B(0,1)}} u(x) = \max_{x \in \partial \overline{B(0,1)}} u(x) = \max_{\theta \in [-\pi,\pi)} (1+\sin\theta)$$

$$= 2$$

$$\min_{x \in \overline{B(0,1)}} u(x) = \min_{x \in \partial \overline{B(0,1)}} u(x) = \min_{\theta \in [-\pi,\pi)} (1+\sin\theta)$$

$$= 0$$