Lecture 10

Today:

- Laplace Transform of derivatives
- Inverse of LT
- Solving ODEs with the L.T.
- Partial fraction decomposition

- **Laplace Transform of derivatives**

Theorem: Suppose f is cont on $[0,\infty)$ and of exp. order α and f' is PW cont. Then,

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)$$

Proof:

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)\,dt$$

$d f(t)$

$$= \lim_{\tau \to \infty} \int_0^\tau e^{-st} f'(t)\,dt$$

Int. by parts

$$= \lim_{\tau \to \infty} \left[e^{-st} f(t) \Big|_{t=0}^{t=\tau} - \int_0^\tau (-s) e^{-st} f(t)\,dt \right]$$

$$= \left[\lim_{\tau \to \infty} e^{-s\tau} f(\tau) \right] - f(0) + s\mathcal{L}\{f(t)\}(s)$$

Let's notice the following:

$$|e^{-s\tau} f(\tau)| \leq |e^{-s\tau} M e^{\alpha\tau}| = M e^{-(s-\alpha)\tau} \to 0$$

as $\tau \to \infty$

for $\tau > t_0$ for $s > \alpha$

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0)$$

If f, f' are cont. and of exp. order α

f'' is PW cont.

$$\mathcal{L}\{f''(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f'(0)$$

$$(f'(t))' = s(s\mathcal{L}\{f(t)\}(s) - f(0)) - f'(0)$$
$$= s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\}(s) = s^3\mathcal{L}\{f(t)\}(s) - s^2 f(0) - sf'(0) - f''(0)$$

In general, if $f(t), f'(t), \dots, f^{(n-1)}(t)$ are cont. on $[0,\infty)$ and of exp. order α while $f^{(n)}(t)$ is PW cont. on $[0,\infty)$. Then,

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

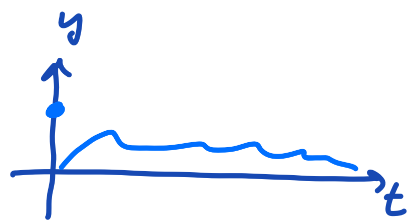
Inverse L.T.

If $\mathcal{L}\{f(t)\}(s) = F(s)$ then

$$\mathcal{L}^{-1}\{F(s)\}(t) = f(t), \quad t > 0.$$

What can go wrong?

$$g(t) = \begin{cases} \sin(\omega t), & t > 0 \\ 1, & t = 0 \end{cases}$$



$$\mathcal{L}\{\sin(\omega t)\}(s) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{g(t)\}(s) = \frac{s}{s^2+\omega^2} \qquad \int_0^\infty e^{-st}\sin(\omega t)\,dt$$

If we restrict to functions that are cont. on $[0,\infty)$, then the inv L.T.

$\mathcal{L}^{-1}\{F(s)\} = f(t)$ is uniquely defined

and we can talk about the inverse.

Properties about $\mathcal{L}^{-1}$

$\mathcal{L}^{-1}$ is linear

$$\mathcal{L}^{-1}\{\alpha F + \beta G\} = \alpha \mathcal{L}^{-1}(F) + \beta \mathcal{L}^{-1}(G)$$

for all $\alpha, \beta \in \mathbb{R}$.

Example: $\mathcal{L}^{-1}\left\{\frac{1}{2(s-1)} + \frac{1}{2(s+1)}\right\}(t)$

Recall that

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$$
$$\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{2(s-1)} + \frac{1}{2(s+1)}\right\}(t)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}(t) + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)$$
$$= \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} = \cosh(t), \quad t > 0.$$

Sol. of IVP.

Example: Solve the IVP

$$\begin{cases} y'' + 4y = 4x \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$$

Sol.

$$\mathcal{L}\{y'' + 4y\}(s) = \mathcal{L}\{4x\}(s)$$

$$\to \mathcal{L}\{y''\}(s) + 4\mathcal{L}\{y\}(s) = \mathcal{L}\{4x\}(s)$$

$$\to (s^2 Y(s) - s y(0) - y'(0)) + 4Y(s) = 4\frac{1}{s^2}$$

$$\to (s^2 Y(s) - s \quad - 5 \quad) + 4Y(s) = 4\frac{1}{s^2}$$

$$(s^2 + 4) Y(s) = s + 5 + 4\frac{1}{s^2}$$

$$Y(s) = \frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{(s^2+4)(s^2)}$$

by partial function dec.

$$\frac{4}{(s^2+4)s^2} = \frac{1}{s^2} - \frac{1}{s^2+4}$$

$$\mathcal{L}^{-1}\{Y(s)\}(x) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{5}{s^2+4} + \frac{4}{(s^2+4)(s^2)}\right\}(s)$$

$$= \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)(x) + \mathcal{L}^{-1}\left(\frac{5}{s^2+4}\right)(x)$$

$$+ \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+4}\right\}(x)$$

$$\mathcal{L}^{-1}\{Y(s)\}(x) = \cos(2x) + 2\sin(2x) + x$$

$$\to \quad y(t) = \cos(2x) + 2\sin(2x) + x$$

Summary: ① Take L.T.

② Solve for $\mathcal{L}\{y\} = F(s)$

③ Take the in. L.T.

Partial Frac. decomposition

$$F(s) = \frac{P(s)}{Q(s)}$$

(1) $\deg Q > \deg P$
(2) No common factor

i) $Q(s) = as+b \Rightarrow F(s) = \dfrac{A}{as+b}$, A const.

ii) Repeated lin. factor $Q(s) = (as+b)^n$

then, $F(s) = \dfrac{A_1}{as+b} + \dfrac{A_2}{(as+b)^2} + \cdots + \dfrac{A_n}{(as+b)^n}$

iii) Quadratic $Q(s) = as^2+bs+c$

$$F(s) = \frac{As+B}{as^2+bs+c}$$

Example 1: $\frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$

$\Rightarrow \quad 1 = A(s-3) + B(s-2)$

$= As - 3A + Bs - 2B$

$= (A+B)s + (-3A-2B)$

$\Rightarrow \quad \begin{cases} A+B=0 \\ -3A-2B=1 \end{cases}$

$\Rightarrow \quad A=-1 \;,\; B=1$

Example 2)

$\frac{s+1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$

$\rightarrow \quad s+1 = As(s-1) + B(s-1) + Cs^2$

taking $s=1 \rightarrow C=2$

taking $s=0 \rightarrow B=-1$

$\rightarrow \quad A+C=0 \rightarrow A=-2$

(equate coef. of s^2)

Example 3)

$$\frac{2s^2}{(s^2+1)(s-1)^2} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$$

$$\Rightarrow \quad 2s^2 = (As+B)(s-1)^2 + C(s^2+1)(s-1) + D(s^2+1)$$

Set $s=1 \rightarrow D=1$

Equate coeff. of s^3 and s we get

$C=1 \quad B=0 \quad A=-1$