Intro to Fourier Series

Fourier Analysis is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions.

Nowadays, Fourier Analysis can be considered a part of Harmonic Analysis (more about this later)

Fourier Analysis is a fundamental tool for applied mathematicians. It appears in:

- Signal processing
- Image analysis
- Image compression
- Sol. of PDEs
- . . .

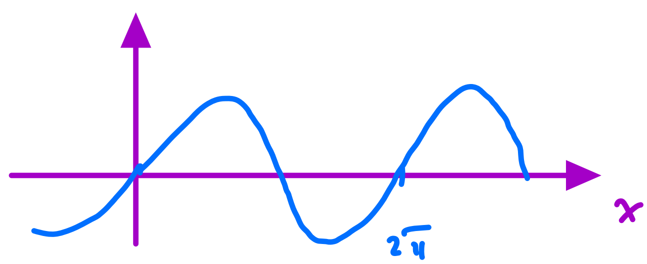
Recall: A function $f: \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number p such that

$$f(x+p) = f(x) \quad \forall x \in \mathbb{R}$$

Any such p is called the period of f.

Examples:

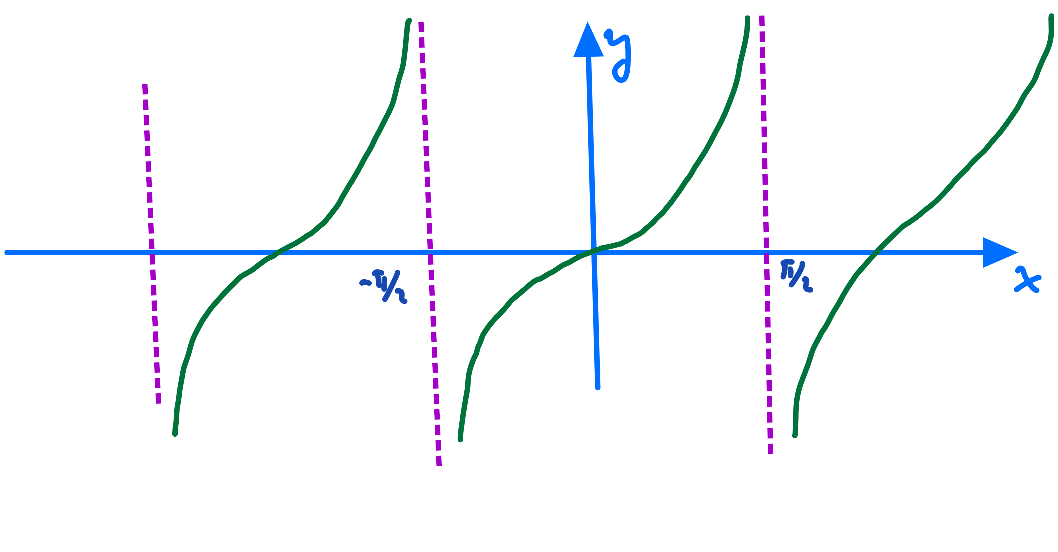
i) $f(x) = \sin(x)$



$p = 2\pi$

ii) $\cos(x)$

iii) $y(x) = \tan(x)$ period $= \pi$



Goal: Write a 2π-periodic function $f : [-\pi, \pi) \to \mathbb{R}$ as a sum of sines/cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

(Fourier series of f)

How do we compute the coefficients? (let's assume that we have unif. convergence of the series)

$$\int_{-\pi}^{\pi} f(x)\,dx =$$

$$\int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right] \right) dx$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx)\,dx$$

$$+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(nx)\,dx$$

and let's recall the following trig. identities

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx) \, dx = -\frac{\cos(nx)}{n} \Bigg|_{x=-\pi}^{x=\pi}$$

$$= -\frac{\cos(n\pi)}{n} + \frac{\cos(-n\pi)}{n}$$

$$= 0$$

So, let's get first a_0:

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx = \frac{a_0}{2}(2\pi)$$

$$\rightarrow \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$ (the average of f)

Trig. identities:

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m+n)x + \cos(m-n)x],$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

Now let's get a_1:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

Let's recall the following identities:

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$ for all $m, n \geq 1$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad m \neq n.$$ for all $m, n \geq 1$ and $m \neq n$

Now

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(x)\,dx$$

$$+ \int_{-\pi}^{\pi} a_1 \cos(x)\cos(x)\,dx + \int_{-\pi}^{\pi} a_2 \cos(2x)\cos(x)\,dx + \cdots$$

$$+ \int_{-\pi}^{\pi} b_1 \sin(x)\cos(x)\,dx + \int_{-\pi}^{\pi} b_2 \sin(2x)\cos(x)\,dx + \cdots$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = \int_{-\pi}^{\pi} a_1 \cos(x)\cos(x)\,dx$$

$$= a_1 \int_{-\pi}^{\pi} \cos^2(x)\,dx$$

$$= a_1 \pi$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)\,dx$$

So, given a Riemann integrable function $f: [-\pi, \pi) \to \mathbb{R}$ we can always construct the Fourier series associated to f as:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

where

$$\begin{cases} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad , \; n \geq 1 \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \quad , \; n \geq 1 \\ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \end{cases}$$

Fourier coefficients

Remark a): If

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

then we could use

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \sin(nx) + b_n \cos(nx) \right]$$

as an approximation of $f(x)$. This is the starting point for numerical methods.

Remark b): Notice that

$$\frac{d}{dx} \sin(nx) = n \cos(nx)$$

$$\frac{d}{dx} \cos(nx) = -n \sin(nx)$$

so

$$f'(x) = \sum_{n=1}^{\infty} \left[a_n n \cos(nx) - b_n n \sin(nx) \right]$$

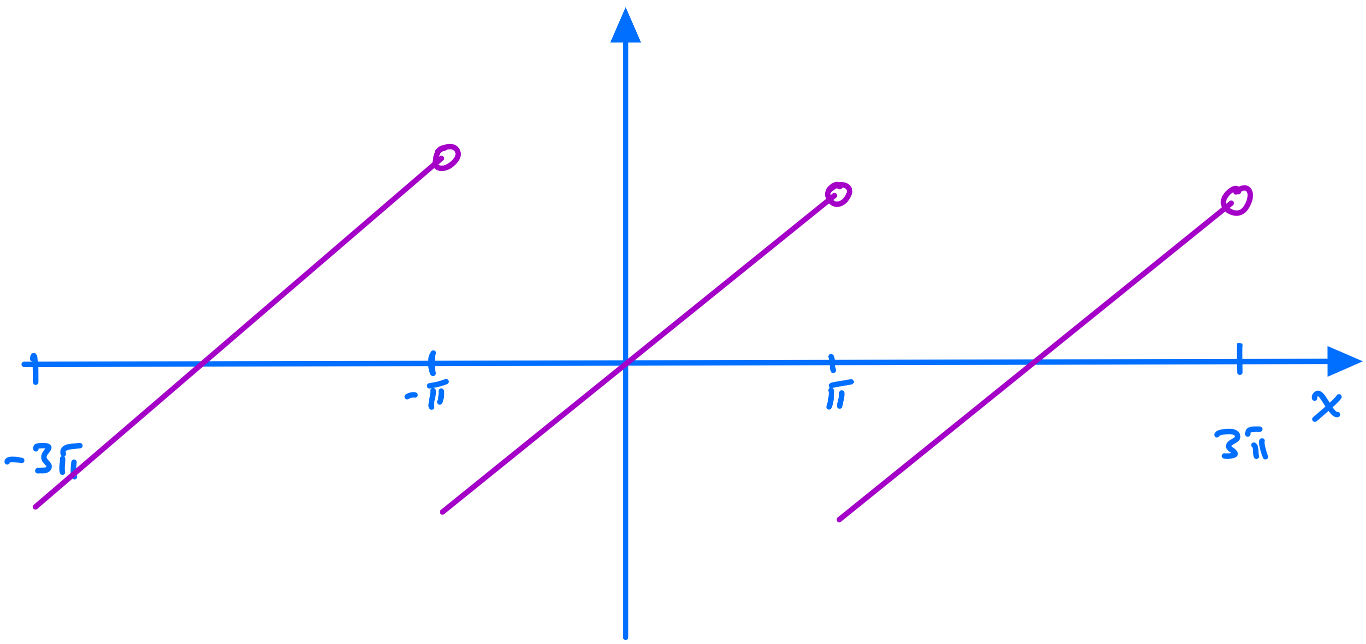
Remark c): We will see later how the decay of the Fourier coefficients is linked to how smooth our function f is.

Remark d): What conditions can we impose on f such that

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \sin(nx) + b_n \cos(nx) \right] \xrightarrow[N\to\infty]{} f(x) \ ?$$

In which sense?

Example: Let's take $f(x) = x$ for $x \in [-\pi, \pi)$ and extend it by periodicity, i.e.



$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0$$

$$u = x \qquad dv = \cos(nx)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]\Bigg|_{x=-\pi}^{x=\pi}$$

$$= \frac{1}{\pi}\left[\pi\,\cancel{\frac{\sin(n\pi)}{n}} + \cancel{\frac{\cos(n\pi)}{n^2}}\right]$$

$$- \frac{1}{\pi}\left[-\pi\,\cancel{\frac{\sin(-n\pi)}{n}} + \cancel{\frac{\cos(-n\pi)}{n^2}}\right]$$

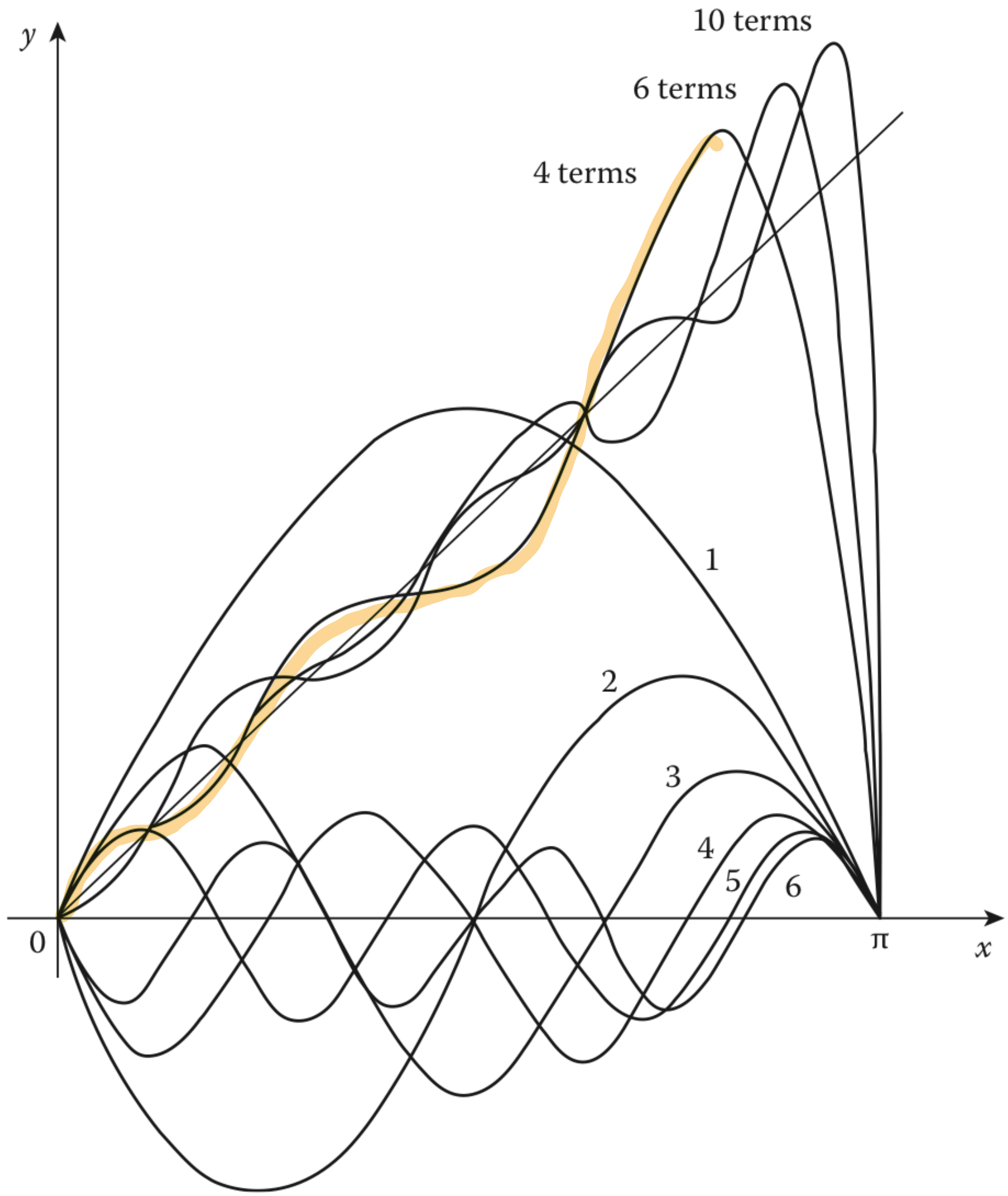
$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{x=-\pi}^{x=\pi}$$

$$= \frac{2}{n}(-1)^{n+1}$$

since $\cos(n\pi) = (-1)^n$

$$x = 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots\right)$$



$$y = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x.$$

Even and odd functions

- A function $f: [-a,a] \to \mathbb{R}$, where $a>0$, is called even if

$$f(-x) = f(x) \quad \forall x \in [-a,a]$$

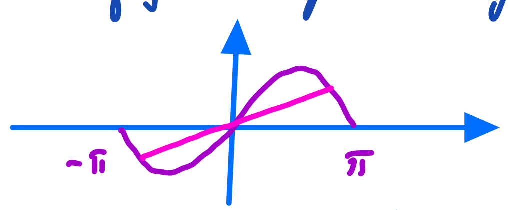
- A function $h: [-a,a] \to \mathbb{R}$, where $a>0$, is called odd if

$$h(-x) = -h(x) \quad \forall x \in [-a,a]$$

Examples: Classify the following functions

i) $\sin(x)$

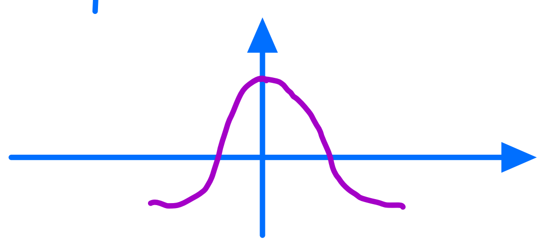
$\sin(-x) = -\sin(x)$? odd



ii) $\cos(x)$

$\cos(-x) = \cos(x)$

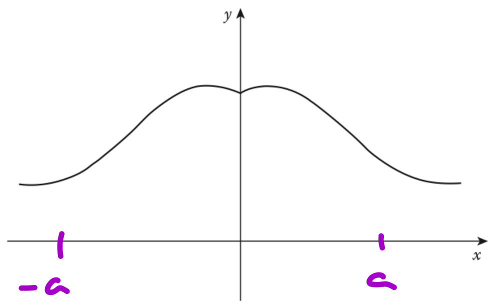
even!



iii) x^2 even

iv) x^3 odd

In general, for any even function $f : [-a, a] \to \mathbb{R}$



$$y = -x \qquad dy = -dx$$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-y)\,(-dy)$$

$$= -\int_{a}^{0} f(y)\,dy$$

$$= \int_{0}^{a} f(y)\,dy$$

we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx$$

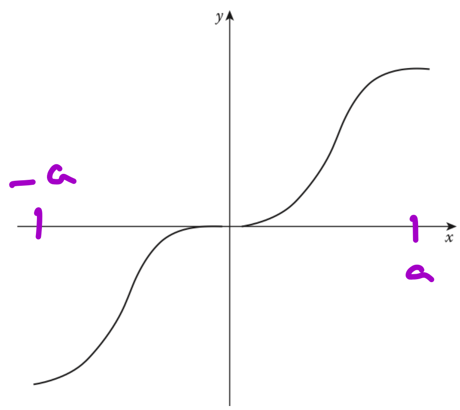
Also, the product of even functions is an even function:

$$f, g : [-a, a] \to \mathbb{R}$$

We need to show that $f(x)\,g(x)$ is even

$$f(-x)\,g(-x) = f(x)\,g(x)$$

In general, for any odd function
$f : [-a, a] \to \mathbb{R}$



we have

$$\int_{-a}^{a} f(x) = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 0$$

Also, the product of odd functions is an even function:

odd → $f, g : [-a, a] \to \mathbb{R}$ ↖ odd

We need to show that $f(x)\,g(x)$ is even

$$f(-x)\,g(-x) = [-f(x)][-g(x)] = f(x)\,g(x)$$

Similarly,

(even)(odd) = odd   (odd)(odd) = even
(+1) (−1) (−1)   (−1) (−1) (+1)

We are now ready to state the following result:

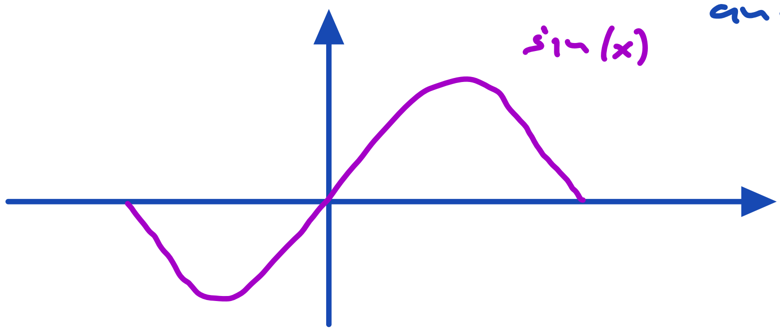
Theorem. *Let f(x) be an integrable function defined on the interval* $-\pi \le x \le \pi$. *If f(x) is even, then its Fourier series has only cosine terms and the coefficients are given by*

$$\rightarrow \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \qquad b_n = 0. \tag{5}$$

And if f(x) is odd, then its Fourier series has only sine terms and the coefficients are given by

$$a_n = 0, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx. \tag{6}$$

Proof: Notice that sin(nx) is odd for any $n \ge 1$



and $\cos(nx)$ is even for $n \geq 1$

Assume that $f(x)$ is even

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx)\, dx$$

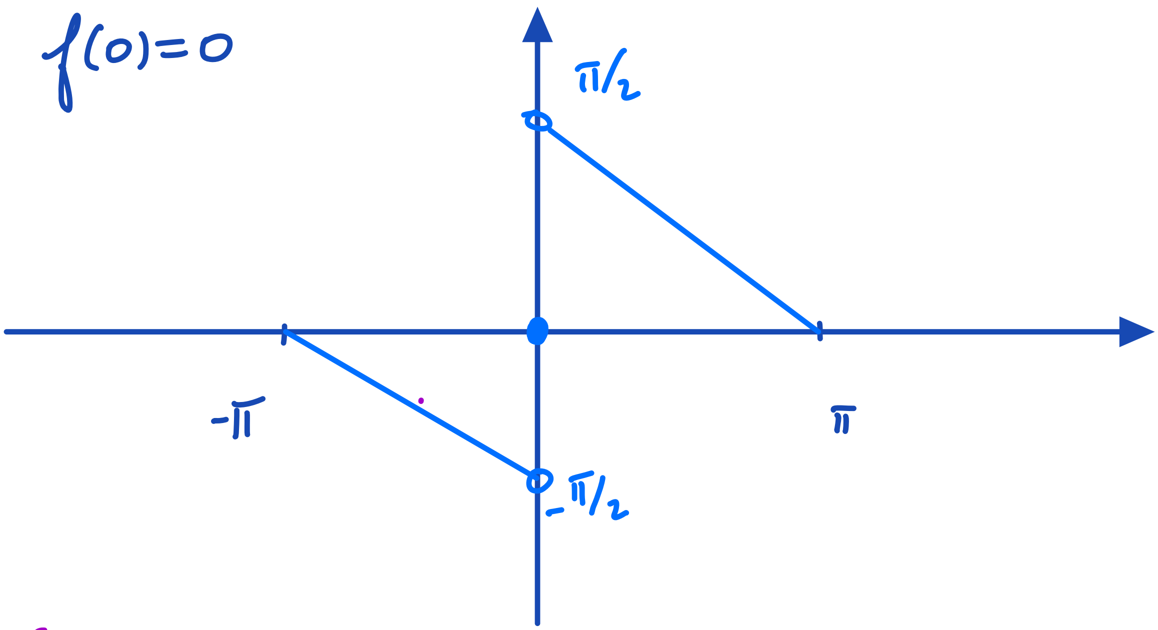
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = 0$$

Another example: Sawtooth wave

$f(x) = -\frac{\pi}{2} - \frac{x}{2}, \quad -\pi \le x < 0,$

$f(x) = \frac{\pi}{2} - \frac{x}{2}, \quad 0 < x \le \pi$

$f(0) = 0$



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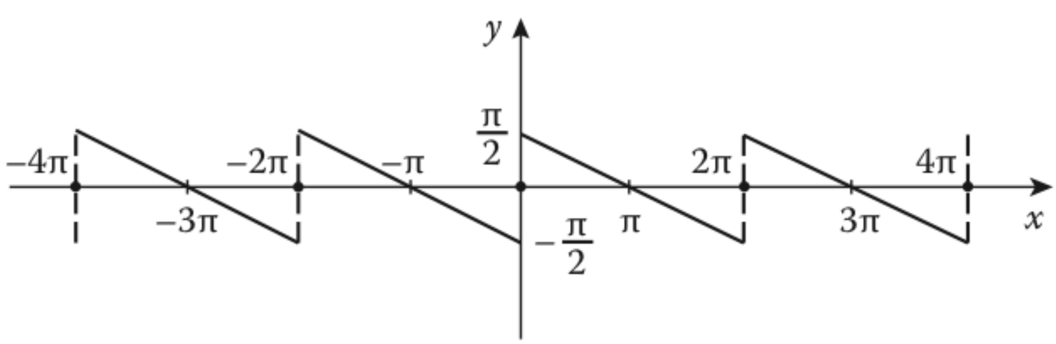


Computing the Fourier Series we get

$$f(x) = \sin(x) + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots$$

Let's try to address the question: Is this series convergent? In which sense?

There are several tests that exists to assess whether a given series of functions converges pointwise or uniformely.

For instance, Weierstrass M-test

$$\left(|f_n(x)| \leq M_n, \quad \sum_{n=1}^{\infty} M_n < \infty \Rightarrow \sum_{n=1}^{\infty} f_n(x) \right.$$

converges absolutely and uniformely)

Does it work in our case?

$$|\sin(x)| \leq 1$$

cont.

So, we cannot conclude with the Weierstrass M-test

Recall the following definitions:

Definition 6.5.1. *(*Pointwise Convergence*). An infinite series* $\sum_{n=1}^{\infty} f_n(x)$ *is called* pointwise convergent *in* $a < x < b$ *to* $f(x)$ *if it converges to* $f(x)$ *for* each x *in* $a < x < b$. *In other words, for each* x *in* $a < x < b$, *we have*

$$|f(x) - s_n(x)| \to 0 \qquad as \quad n \to \infty,$$

where $s_n(x)$ *is the* n*th partial sum defined by* $s_n(x) = \sum_{k=1}^{n} f_k(x)$.

Definition 6.5.2. *(*Uniform Convergence*). The series* $\sum_{n=1}^{\infty} f_n(x)$ *is said to* converge uniformly *to* $f(x)$ *in* $a \le x \le b$ *if*

$$\max_{a \le x \le b} |f(x) - s_n(x)| \to 0 \qquad as \quad n \to \infty.$$

Evidently, uniform convergence implies pointwise convergence, but the converse is not necessarily true.

In our case,

$$S_n(x) = \sum_{i=1}^{n} f_n(x)$$

From book

Linear Partial
Differential Equations
for Scientists and Engineers

by

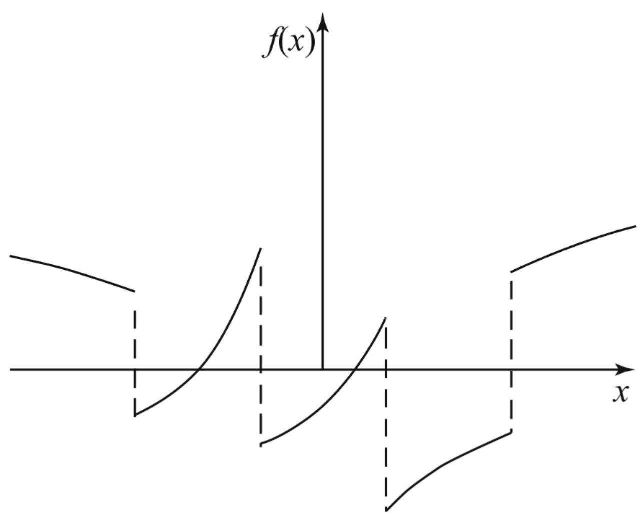
Tyn Myint-U
Lokenath Debnath

Definitions:

A function f is said to be *piecewise continuous* in an interval $[a, b]$ if there exist finitely many points $a = x_1 < x_2 < \ldots < x_n = b$, such that f is continuous in the intervals $x_j < x < x_{j+1}$ and the one-sided limits $f(x_j+)$ and $f(x_{j+1}-)$ exist for all $j = 1, 2, 3, \ldots, n-1$.

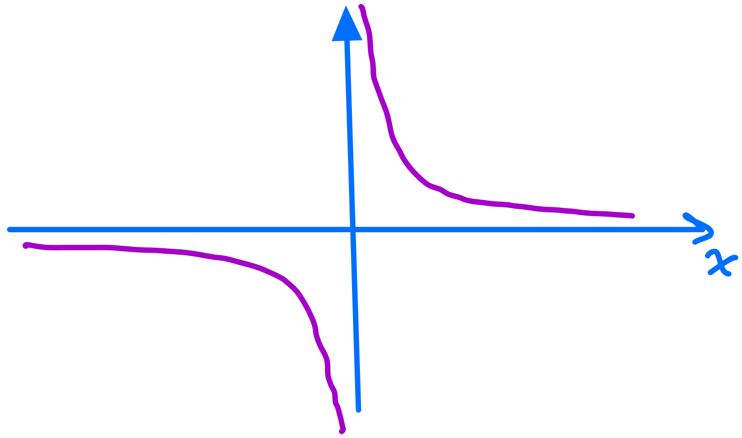
Example:

i)

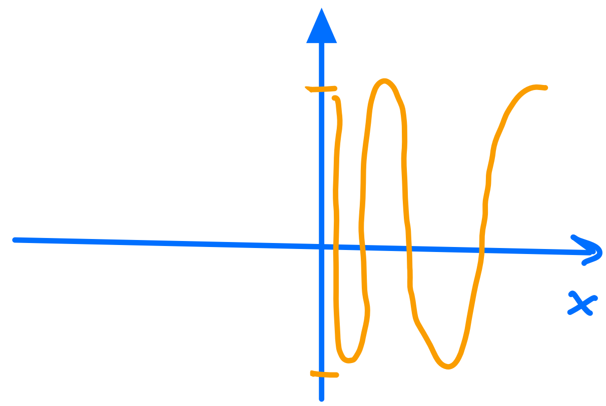


ii)

$f(x) = \frac{1}{x}$



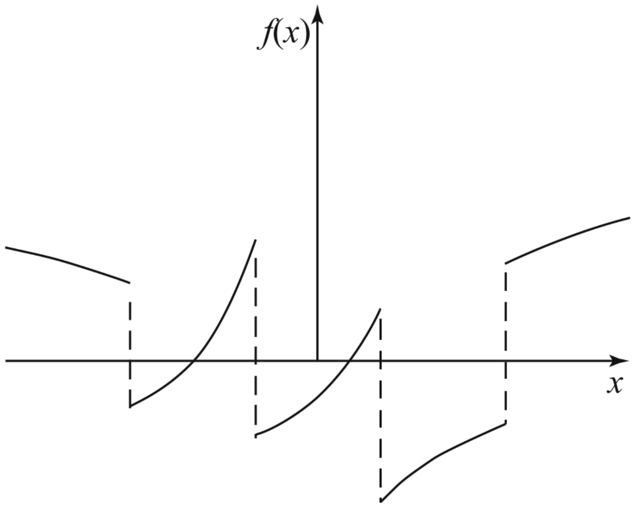
iii) $h(x) = \sin(1/x)$



If f is piecewise continuous in an interval $[a, b]$ and if, in addition, the first derivative f' is continuous in each of the intervals $x_j < x < x_{j+1}$, and the limits $f'(x_j+)$ and $f'(x_j-)$ exist, then f is said to be *piecewise smooth*.

Example

i)



ii) $h(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0 \\ x+1, & \text{if } x < 0 \end{cases}$

?

Dirichlet's Theorem

Theorem 6.10.1. ***(Pointwise Convergence Theorem).*** *If $f(x)$ is piecewise smooth and periodic function with period 2π in $[-\pi, \pi]$, then for any x*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \frac{1}{2}\left[f(x+) + f(x-)\right], \quad (6.10.8)$$

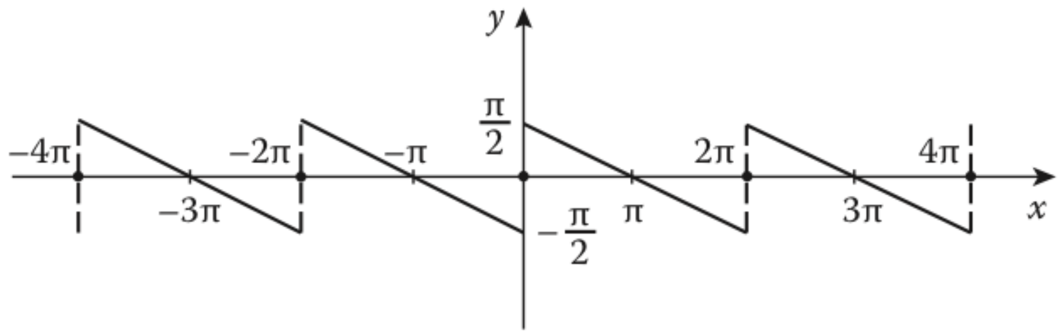
where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos kt \, dt, \qquad k = 0, 1, 2, \ldots, \quad (6.10.9)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin kt \, dt, \qquad k = 1, 2, 3, \ldots. \quad (6.10.10)$$

What does this imply at a continuous point?

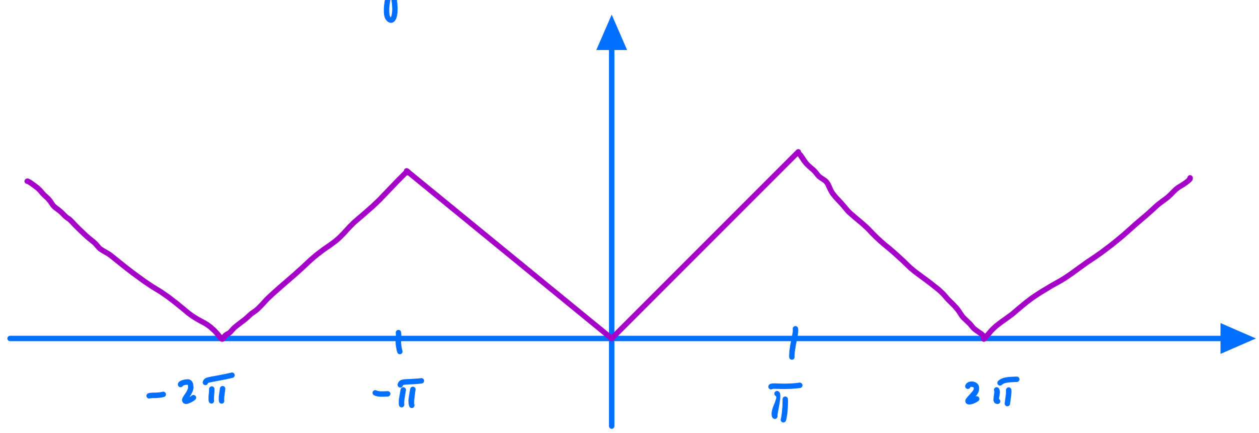
Example: Sawtooth function



Another example:

$$f(x) = |x|, \qquad -\pi \leq x \leq \pi$$

- $f(x)$ is even
- Extend f on $\mathbb{R}$



If we compute the Fourier Series we get, by Dirichlet's theorem,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

we have pointwise convergence for $x \in [-\pi, \pi]$