Lecture 9

RECALL:

Define $\mathcal{L}$: functions with (time) domain ($\mathbb{R}$) $\longrightarrow$ functions with domain ($\mathbb{R}$)

For $s \in \mathbb{R}$ define

$$(\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Function of time $t \in \mathbb{R}$

Function of $s \in \mathbb{R}$.

For those $s \in \mathbb{R}$ for which the integral converges we call F to be the Laplace transform of f.

Today: Which functions have a Laplace transform?

Preliminaries:

- Integral is said to <u>converge</u> <u>absolutely</u> if $\int_0^\infty |f(x)|\,dx < \infty$

- Comparison test : $|f(x)| < g(x)$

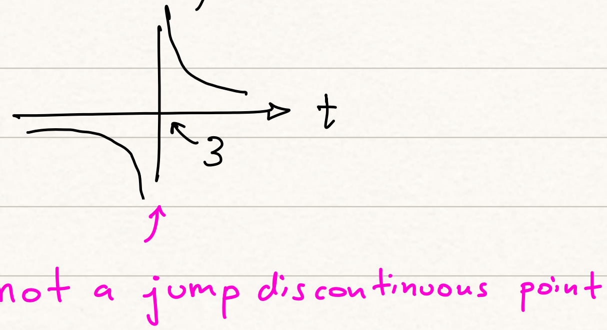
  and $\int_0^\infty g(x) < \infty$. Then, $\int_0^\infty f(x) < \infty$

- Piecewise continuous functions (P.W) on $[0,\infty)$.

  i) If $\lim_{t\to 0^+} f(t) = f(0^+)$ exists

  ii) f is continuous on every finite interval except at finite number of jump discontinuity points

$f(t) = \dfrac{1}{t-3}$ is not p-w. continuous!



- A function f has **exponential growth** of order α if $\exists\, M > 0$ & α st $|f(t)| < Me^{\alpha t}$ for all $t \geq t_0$ (some t_0).
  (↳ there exists)

Examples:

↳ $\sin t$, $\cos t$ - order 0

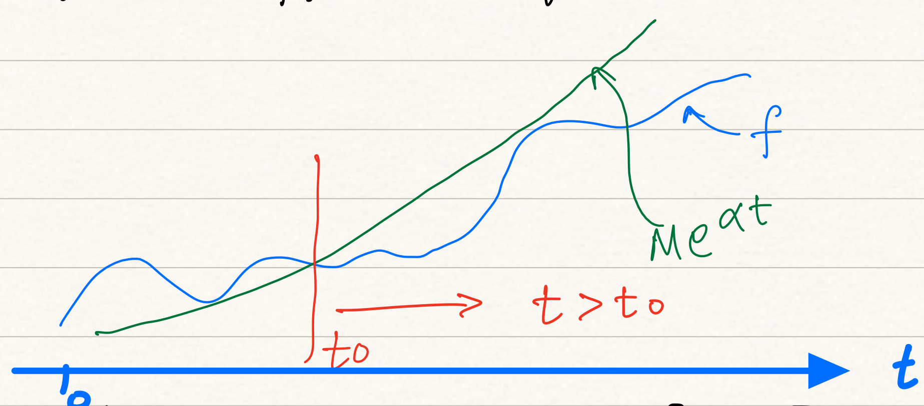
↳ e^{-t} - order -1

↳ $e^{t^2} \not< e^{\alpha t}$ No order as t^2 is Non linear & can't be bounded by linear f.

↳ t^n - order 1 (this holds true because $\lim_{t\to\infty} t^n/e^t = 0$)

Thm: If f is piecewise continuous on $[0,\infty)$ and of exponential order α, then the Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.

Pf:



$|f(t)| < M_1$ for $t \in [0, t_0]$

$$|f(t)| < M_2 e^{\alpha t}, \quad \alpha \in \mathbb{R}, \quad t > t_0$$

Choose M large enough so that

$$|f(t)| < M e^{\alpha t}$$

$$\int_0^\infty |e^{-st} f(t)|\, dt \leq M \int_0^\infty |e^{-st} e^{\alpha t}|\, dt$$

$s \in \mathbb{R}$ so,

$$\leq M \int_0^\infty |e^{-(s-\alpha)t}|\, dt$$

$$= \lim_{\tau \to \infty} M \left. \frac{e^{-(s-\alpha)}}{-(s-\alpha)} \right|_0^\tau$$

Why the limit to ∞ ?

Because ∞ is NOT a number! $G(\infty)$ is an abuse of notation. should be interpreted as $\lim_{\tau \to \infty} G(\tau)$.

$$= \frac{M}{s-\alpha} - \underbrace{\lim_{\tau \to \infty} \frac{e^{-(s-\alpha)}}{s-\alpha}}_{=0 \text{ if } s > \alpha}$$

Therefore, by the comparison test, the Laplace integral converges absolutely. $\square$

Aside: In fact, it converges uniformly!
This means that given $\varepsilon > 0$, there exists $T > 0$ st $\int_\tau^\infty |e^{-st} f(t)| dt < \varepsilon$ for all $\tau > T$.

and all $s > \alpha$.

Note that piecewise continuity + exponential order are only sufficient conditions NOT necessary.

Eg. $f(t) = \dfrac{1}{\sqrt{t}}$ discontinuous at 0.
(Not p.w continuous)

However the Laplace transform exists (proof in textbook).

Theorem: If f is piecewise continuous on $[0, \infty)$ & has exponential order α then $F(s) \to 0$ as $s \to \infty$.

proof: Homework.