1) ⊕ $f(t) = t^3 . \sin(t)$, let $a > 0$ is a constant

We have $\lim_{t \to \infty} e^{-at} t^3 \sin(t) = \lim_{t \to \infty} \frac{t^3 \sin(t)}{e^{at}}$

Because $|\sin(t)| \leq 1$ when $t \to \infty$

So, we can check $\lim_{t \to \infty} \frac{t^3 \to \infty}{e^{at} \to \infty}$ => applying L'Hospital Rule

$= \lim_{t \to \infty} \frac{3t^2}{ae^{at}} = \lim_{t \to \infty} \frac{6t}{a^2 e^{at}} = \lim_{t \to \infty} \frac{6}{a^3 e^{at}} = 0$

$\Rightarrow \lim_{t \to \infty} e^{-at} t^3 \sin(t) = 0$ with $a > 0$ is a constant

$\Rightarrow$ $\boxed{\text{Hence, the function } f(t) = t^3 \sin(t) \text{ is of exponential order}}$

⊕ $f(t) = t^2 e^t$

Check $\lim_{t \to \infty} e^{-at} t^2 e^t = \lim_{t \to \infty} t^2 e^{-(a-1)t}$

$= \lim_{t \to \infty} \frac{t^2}{e^{(a-1)t}}$ let $a > 1$ is a constant, then

$t^2 \to \infty$ & $e^{(a-1)t} \to \infty$ when $t \to \infty \Rightarrow$ applying the L'Hospital rule, we have:

$$\lim_{t\to\infty} e^{-at} f(t) = \lim_{t\to\infty} \frac{2t}{(a-1)\,e^{(a-1)t}} = \lim_{t\to\infty} \frac{2}{(a-1)^2\,e^{(a-1)t}} = 0$$

$\Rightarrow$ Hence, the function $f(t) = t^2 e^t$ is of exponential order.

2) Given $f(t)$ be a piecewise continuous on $[0, \infty)$
of exponential order

We have: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

$$= \int_0^{t_0} e^{-st} f(t)\,dt + \int_{t_0}^\infty e^{-st} f(t)\,dt \text{, with } t_0 \in [0, \infty)$$

$$\Rightarrow |F(s)| = \left| \int_0^{t_0} e^{-st} \cdot f(t)\,dt + \int_{t_0}^\infty e^{-st} \cdot f(t)\,dt \right|$$

Also $|a+b| \leq |a| + |b|$

$$\Rightarrow |F(s)| \leq \underbrace{\left| \int_0^{t_0} e^{-st} f(t)\,dt \right|}_{A} + \underbrace{\left| \int_{t_0}^\infty e^{-st} \cdot f(t)\,dt \right|}_{B}$$

$$\circledast \; A = \left| \int_0^{t_0} e^{-st} \cdot f(t)\,dt \right| \leq \int_0^{t_0} \left| e^{-st} f(t)\,dt \right|$$

$$= \int_0^{t_0} e^{-st} |f(t)|\,dt$$

f is piece-wise continuous, so it bounded on $[0, t_0)$

Let $|f(t)| < M_1$ for $[0, t_0]$

$$\Rightarrow A \le M_1 \int_0^{t_0} e^{-st}\, dt = M_1 . (-\tfrac{1}{s}) e^{-st} \Big|_0^{t_0}$$

$$= M \frac{1}{s} . e^{-st} \Big|_{t_0}^{0} = M_1 \frac{1}{s} \left[1 - e^{-st_0} \right] = M_1 \frac{1}{s} \left[1 - \frac{1}{e^{st_0}} \right]$$

Also $\lim\limits_{s \to \infty} M_1 \frac{1}{s} \left[1 - \frac{1}{e^{st_0}} \right] = 0 \Rightarrow \lim\limits_{s \to \infty} A = 0$

* $B = \left| \int_{t_0}^{\infty} e^{-st} . f(t)\, dt \right| \le \int_{t_0}^{\infty} e^{-st} |f(t)|\, dt$

Also, since $f(t)$ is of exponential order, then $\exists$ $M > 0$ & constant α such that $|f(t)| < M_2 e^{\alpha t}$, with $t > t_0$

$$\Rightarrow B \le \int_{t_0}^{\infty} e^{-st} M_2 e^{\alpha t}\, dt = \int_{t_0}^{\infty} M_2 e^{-t(s-\alpha)}\, dt$$

$$\Rightarrow B \le M_2 \int_{t_0}^{\infty} e^{-t(s-\alpha)}\, dt = M_2 \frac{-1}{s-\alpha} e^{-t(s-\alpha)} \Big|_{t_0}^{\infty}$$

$$= M_2 . \frac{1}{s-\alpha} e^{-t(s-\alpha)} \Big|_{\infty}^{t_0} = M_2 \frac{1}{s-\alpha} \left[e^{-t_0(s-\alpha)} - 0 \right]$$

$$= M_2 . \frac{e^{-t_0(s-\alpha)}}{s-\alpha}$$ with α is constant

Since $\lim_{s \to \infty} e^{-t_0(s-\alpha)} = \lim_{s \to \infty} \frac{1}{e^{t_0(s-\alpha)}} = 0$

$$\Rightarrow \lim_{s \to \infty} \frac{M}{s-\alpha} \cdot e^{-t_0(s-\alpha)} = 0$$

$$\Rightarrow \lim_{s \to \infty} B = 0$$

$$\Rightarrow \lim_{s \to \infty} |F(s)| \leq \lim_{s \to \infty} (A+B) = 0$$

$$\Rightarrow \boxed{\lim_{s \to \infty} F(s) = \lim_{s \to \infty} L\{f\}(s) = \lim_{s \to \infty} L(s) = 0}$$

3) Section 48, Problem 2

a) Given $p > |a|$,

we have $\sinh(ax) = \frac{1}{2}[e^{ax} - e^{-ax}]$

$\Rightarrow L[\sinh(ax)] = L[\frac{1}{2}(e^{ax} - e^{-ax})]$

$= \frac{1}{2}[L(e^{ax} - e^{-ax})] = \frac{1}{2}[L(e^{ax}) - L(e^{-ax})]$

Since $L(e^{ax}) = \frac{1}{p-a}$, $Re\{p\} > a$

$L(e^{-ax}) = \frac{1}{p+a}$, $Re\{p\} > -a$

$\Rightarrow L[\sinh(ax)] = \frac{1}{2}\left[\frac{1}{p-a} - \frac{1}{p+a}\right]$, with $\begin{cases} Re\{p\} > a \\ Re\{p\} > -a \end{cases}$

$\Leftrightarrow p > |a|$

$\Rightarrow L[\sinh(ax)] = \frac{1}{2}\left[\frac{p+a-p+a}{(p-a)(p+a)}\right]$ with $p > |a|$

$= \frac{1}{2} \cdot \frac{2a}{p^2-a^2} = \frac{a}{p^2-a^2}$

Hence, $\boxed{L[\sinh(ax)] = \frac{a}{p^2-a^2} \text{ with } p > |a|}$

b) $\cosh(ax) = \frac{1}{2}\left[e^{ax} + e^{-ax}\right]$

$\Rightarrow L[\cosh(ax)] = L\left[\frac{1}{2}(e^{ax} + e^{-ax})\right]$

$= \frac{1}{2}L(e^{ax}) + \frac{1}{2}L(e^{-ax})$ (1)

Similar part a, we also have:

(1) $\Leftrightarrow \frac{1}{2}\left[\frac{1}{p-a} + \frac{1}{p+a}\right]$ with $p > |a|$

$\Leftrightarrow \frac{1}{2} \cdot \frac{p+a+p-a}{(p-a)(p+a)} = \frac{p}{p^2-a^2}$ with $p > |a|$

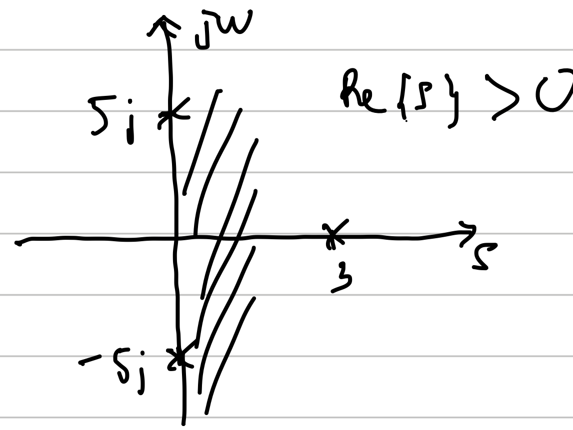
$\Rightarrow \boxed{L[\cosh ax] = \frac{p}{p^2-a^2}, \ p > |a|}$

4) Section 48, problem 4c

(f) Given $f(x) = 2e^{3x} - \sin 5x$

$$\Rightarrow L\{f(x)\} = L[2e^{3x} - \sin 5x] = 2L(e^{3x}) - L(\sin 5x)$$

$$= \underbrace{\frac{2}{s-3}}_{Re\{s\} > 3} - \underbrace{\frac{5}{s^2+5^2}}_{Re\{s\} > 0}$$



$$\Rightarrow \boxed{L\{f(x)\} = \frac{2}{s-3} - \frac{5}{s^2+25} \quad \text{with } Re\{s\} > 0}$$

5) Section 48, problem 5c.

c) $F(p) = \frac{4}{p^3} + \frac{6}{p^2+4}$

We have: $L(x^n) = \frac{n!}{p^{n+1}}$, $Re\{p\} > 0$

$\Rightarrow L(x^2) = \frac{2!}{p^3} \Rightarrow \frac{1}{2} L(x^2) = \frac{1}{p^3}$

$\Rightarrow 2L(x^2) = \frac{4}{p^3} \Rightarrow L(2x^2) = \frac{4}{p^3}$

$\Rightarrow L^{-1}\left(\frac{4}{p^3}\right) = 2x^2$ (1)

also, $L(\sin \alpha x) = \frac{\alpha}{p^2+\alpha^2}$ $Re\{p\} > 0$

$\Rightarrow L(\sin 2x) = \frac{2}{p^2+4}$

$\Rightarrow L(3\sin 2x) = \frac{6}{p^2+4}$

$\Rightarrow L^{-1}\left\{\frac{6}{p^2+4}\right\} = 3\sin 2x$ (2)

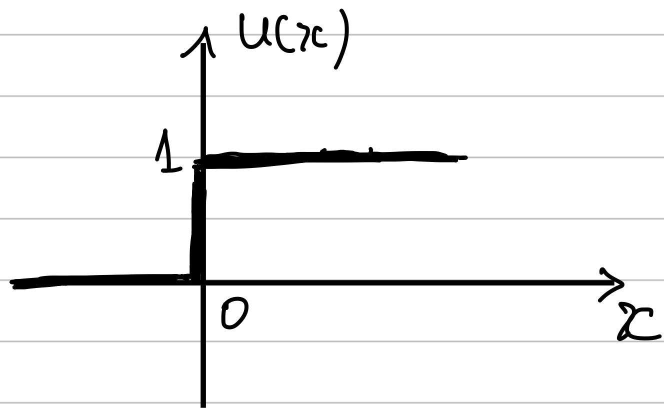
From (1) & (2) $\Rightarrow L^{-1}\left\{\frac{4}{p^3}+\frac{6}{p^2+4}\right\} = 2x^2 + 3\sin 2x$

$$\Rightarrow \boxed{f(x) = 2x^2 + 3\sin 2x \ , \ Re\{p\} > 0}$$

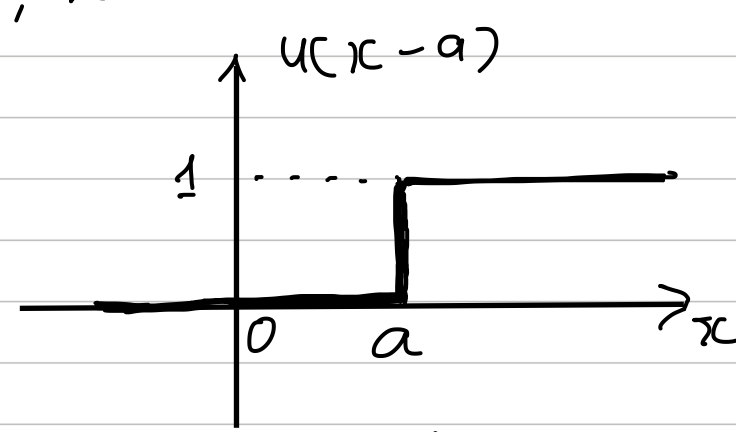
6) Section 49, problem 2

a) $f(x) = u(x-a)$, $a > 0$

We have $u(x) = \begin{cases} 0 , & x < 0 \\ 1 , & x \geq 0 \end{cases}$



$\Rightarrow$



We have: $L\{u(x)\} = \int_0^\infty e^{-sx} f(x)\,dx = \int_0^\infty e^{-sx} u(x)\,dx$

$= \int_0^\infty e^{-sx}\,dx = \frac{1}{s} e^{-sx} \Big|_\infty^0 = \frac{1}{s}$

$\Rightarrow L\{u(x)\} = \frac{1}{s} = F(s) \quad \text{Re}\{s\} > 0$

Also, $L\{u(x-a)\} = \int_0^\infty e^{-sx} u(x-a)\,dx$

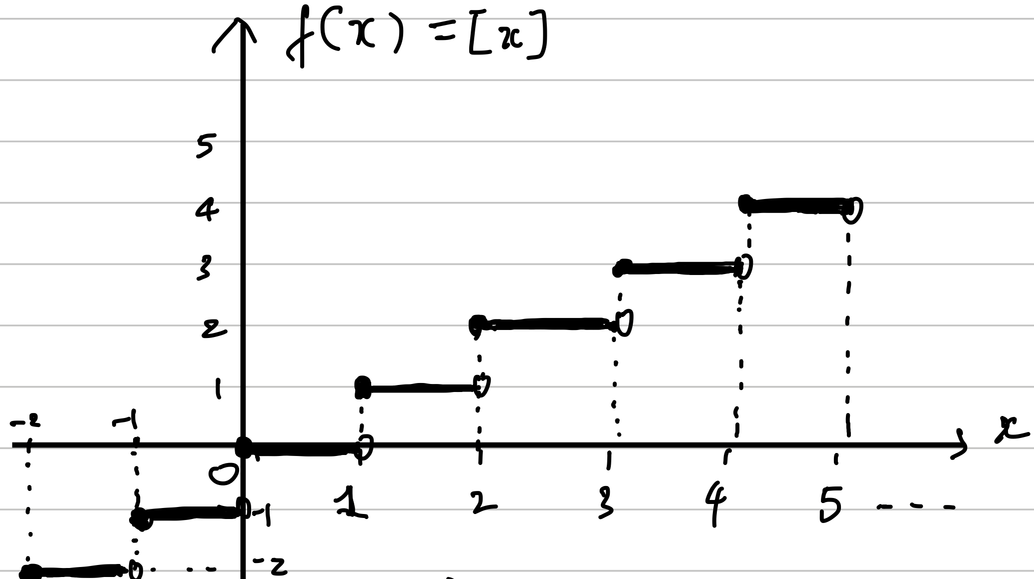
$= \int_0^a e^{-sx} u(x-a)\,dx + \int_a^\infty e^{-sx} u(x-a)\,dx$

$= 0 + \int_a^\infty e^{-sx} \cdot 1\,dx = \int_a^\infty e^{-sx}\,dx = \frac{1}{s} e^{-sx} \Big|_\infty^a$

$$= \frac{1}{s}\left[e^{-sa} - 0\right] = \frac{e^{-sa}}{s}$$

$$\Rightarrow \boxed{L\{u(x-a)\} = \frac{e^{-sa}}{s}}$$

b) $f(x) = [x]$ where $[x]$ denote the greatest integer $\le x$



$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx$$

$$= \int_0^1 e^{-sx} . 0\, dx + \int_1^2 e^{-sx}\, dx + \int_2^3 2e^{-sx}\, dx + \int_3^4 3e^{-sx}\, dx$$

$$+ \int_4^5 4e^{-sx}\, dx + \cdots$$

$$= -\frac{1}{s} e^{-sx}\Big|_1^2 + 2\left(-\frac{1}{s}\right) e^{-sx}\Big|_2^3 + 3\left(-\frac{1}{s}\right) e^{-sx}\Big|_3^4$$

$$+4\cdot\left(-\frac{1}{s}\right)e^{-sx}\Big|_4^5+\cdots$$

$$=\frac{1}{s}e^{-sx}\Big|_2^1+\frac{2}{s}e^{-sx}\Big|_3^2+\frac{3}{s}e^{-sx}\Big|_4^3$$

$$+4\frac{1}{s}e^{-sx}\Big|_5^4+\cdots$$

$$=\frac{1}{s}\left[e^{-s}-e^{-2s}\right]+\frac{2}{s}\left[e^{-2s}-e^{-3s}\right]+\frac{3}{s}\left[e^{-3s}-e^{-4s}\right]$$

$$+\frac{4}{s}\left[e^{-4s}-e^{-5s}\right]+\cdots$$

$$=\frac{1}{s}e^{-s}+\frac{1}{s}e^{-2s}+\frac{1}{s}e^{-3s}+\frac{1}{4}e^{-4s}+\cdots$$

$$=\frac{1}{s}\left[e^{-s}+e^{-2s}+e^{-3s}+e^{-4s}+\cdots\right]$$

$$=\frac{e^{-s}}{s}\left(1+e^{-s}+e^{-2s}+e^{-3s}+\cdots\right)$$

Also, we have: $\frac{1}{1-x}\simeq 1+x+x^2+\cdots$

plug $x = e^{-s} \Rightarrow \frac{1}{1-e^{-s}} = 1 + e^{-s} + e^{-2s} + e^{-3s} + \dots$

$$\Rightarrow L\{f(x)\} = \frac{e^{-s}}{s} \cdot \frac{1}{1-e^{-s}} = \frac{1}{s.e^{s}(1-e^{-s})} = \frac{1}{s(e^{s}-1)}$$

$$\Rightarrow \boxed{L\{f(x)\} = \frac{1}{s(e^{s}-1)}} \qquad Re\{s\} > 0$$

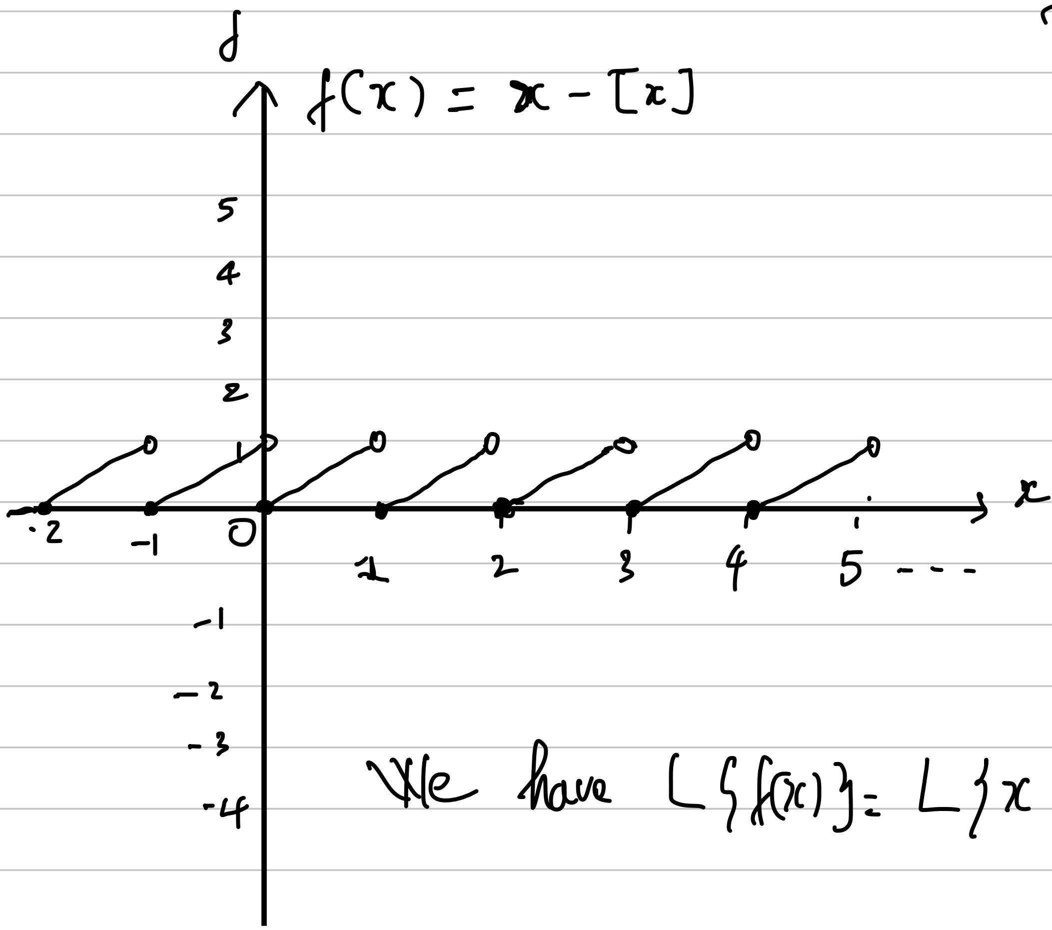
c) $f(x) = x - [x]$

let $[x] = m \Leftrightarrow m \leq x < m+1$

$\Rightarrow f(x) = x - [x] = x - m \Rightarrow$

$0 \leq f(x) = x - m \leq 1$

$f(x)$ $\quad f(x)$



We have $L\{f(x)\} = L\{x - [x]\} = L\{x\} - L\{[x]\}$

From part b, we have $L\{[x]\} = \dfrac{1}{s(e^s-1)}$ $\mathrm{Re}\{s\} > 0$

also, $L\{x\} = \int_0^\infty e^{-sx}\, x\, dx$ $\quad u = x,\ dv = e^{-sx}dx$

$\Rightarrow v = -\frac{1}{s}e^{-sx}$

$$= x\left(-\frac{1}{s}\right)e^{-sx}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-sx}\,dx$$

$$= \frac{1}{s}\,xe^{-sx}\Big|_\infty^0 + \frac{1}{s}\left(-\frac{1}{s}\right)e^{-sx}\Big|_0^\infty$$

$$= \frac{1}{s}\left[0-0\right] + \frac{1}{s^2}\,e^{-sx}\Big|_\infty^0 = \frac{1}{s^2}\left[e^0 - 0\right]$$

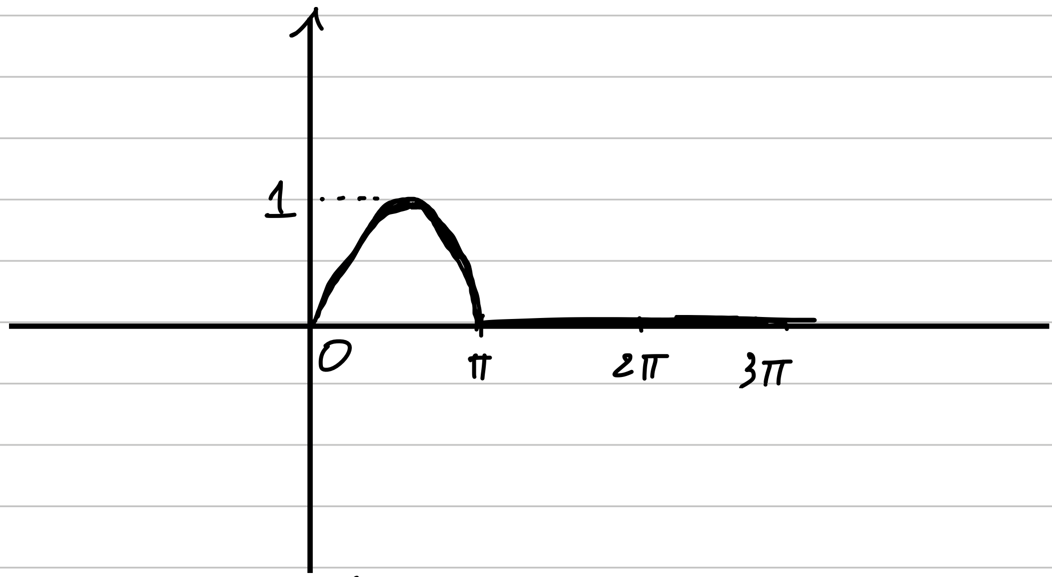
$$= \frac{1}{s^2} \Rightarrow L\{x\} = \frac{1}{s^2} \quad \mathrm{Re}\{s\} > 0$$

$$\Rightarrow L\{f(x)\} = L\{x - [x]\} = L\{x\} - L\{[x]\}$$

$$= \frac{1}{s^2} - \frac{1}{s(e^s-1)} = \frac{1}{s}\left[\frac{1}{s} - \frac{1}{e^s-1}\right] = \frac{e^s-1-s}{s^2(e^s-1)}$$

$$\Rightarrow \boxed{L\{f(x)\} = \frac{e^s-1-s}{s^2(e^s-1)}, \quad \mathrm{Re}\{s\} > 0}$$

d) $f(x) = \begin{cases} \sin x , & 0 \le x \le \pi \\ 0 , & x > \pi \end{cases}$



$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx = \int_0^\pi e^{-sx} \sin x \, dx = L$$

$u = e^{-sx}$, $dv = \sin x \, dx \Rightarrow v = -\cos x$

$du = -se^{-sx} dx$

$$\Rightarrow L\{f(x)\} = e^{-sx}(-\cos x)\Big|_0^\pi - \int_0^\pi \cos x \, se^{-sx} \, du$$

$$= e^{-sx} \cos x \Big|_\pi^0 - s\int_0^\pi \cos x \, e^{-sx} \, dx$$

$$= 1 + e^{-s\pi} - s\int_0^\pi \cos x \, e^{-sx} \, dx.$$

Let $A = \int_0^\pi \cos x \, e^{-sx} \, dx$ $\quad u = e^{-sx} \Rightarrow du = -se^{-sx} dx$

$$dv = \cos x\,dx \Rightarrow v = \sin x$$

$$\Rightarrow A = \sin x \cdot e^{-sx}\Big|_0^{\pi} + s\int_0^{\pi} \sin x\, e^{-sx}\,dx$$

$$= 0 + s\int_0^{\pi} \sin x \cdot e^{-sx}\,dx = 0 + s.L$$

$$\Rightarrow A = sL$$

$$\Rightarrow L\{f(x)\} = 1 + e^{-s\pi} - sA$$

$$= 1 + e^{-s\pi} - s.sL$$

$$\Rightarrow L = 1 + e^{-s\pi} - s^2L \Leftrightarrow L + s^2L = 1 + e^{-s\pi}$$

$$\Rightarrow L(1+s^2) = 1 + e^{-s\pi}$$

$$\Rightarrow \boxed{L = L\{f(x)\} = \frac{1+e^{-s\pi}}{1+s^2} \quad \mathrm{Re}\{s\} > 0}$$

7) Section 49, Problem 4:

Need to show explicity $L[x^{-1}]$ does not exist

Assume $L[x^{-1}]$ exists, then

$$L[x^{-1}] = \int_0^\infty x^{-1} e^{-sx}\,dx = \int_0^\infty \frac{e^{-sx}}{x}\,dx$$

$$= \underbrace{\int_0^1 \frac{e^{-sx}}{x}\,dx}_{A} + \underbrace{\int_1^\infty \frac{e^{-sx}}{x}\,dx}_{B} \quad \text{is convergence}$$

$\Rightarrow$ A & B are both convergence

Check $x \in [0,1] \in [0,\infty)$ & $s \geq 0$, then we

have $A = \int_0^1 \frac{e^{-sx}}{x}\,dx$ is also converge ①

However, we have with $x \in [0,1]$ & $s \geq 0$

$$\Rightarrow \quad sx \leq s \quad \Rightarrow \quad -sx \geq -s$$

$$\Rightarrow e^{-sx} \geq e^{-s} \Rightarrow \frac{e^{-sx}}{x} \geq \frac{e^{-s}}{x} = e^{-s} \cdot \frac{1}{x}$$

$$\Rightarrow \int_0^1 \frac{e^{-sx}}{x}\,dx \geq \int_0^1 \frac{e^{-s}}{x}\,dx = e^{-s} \int_0^1 \frac{1}{x}\,dx$$

From ①, we have $\int_0^1 \frac{e^{-sx}}{x} dx$ is converge

$\Rightarrow \int_0^1 e^{-s} \frac{1}{x} dx$ is also converge (comparison test)

However $\int_0^1 e^{-s} \frac{1}{x} dx = e^{-s} \ln(x) \Big|_0^1$

$= e^{-s} \lim_{t \to 0} \left[\ln(x) \Big|_t^1 \right] = e^{-s} \lim_{t \to 0} [\ln 1 - \ln t]$

$= -e^{-s} \cdot \lim_{t \to 0} (\ln t) = e^{s} \cdot \lim_{t \to 0} \left[\ln \frac{1}{t} \right]$

Since $t \to 0$, $\frac{1}{t} \to \infty \Rightarrow \ln\left(\frac{1}{t}\right) \to \infty$

$\Rightarrow \int_0^1 e^{-s} \frac{1}{x} dx = e^{s} \lim_{t \to \infty} \left[\ln\left(\frac{1}{t}\right) \right] \longrightarrow \infty$

$\Rightarrow \int_0^1 e^{-s} \frac{1}{x} dx$ is divergence. This is contradict with

① means $L[x^{-1}]$ can not be convergence

$\Rightarrow L[x^{-1}] = \int_0^\infty e^{-st} f(x) dx$ is divergence

$\Rightarrow$ $\boxed{L[x^{-1}] \text{ does not exist}}$

8) Section 50, problem 3a

Given $y' + y = 3e^{2x}$, $y(0) = 0$

$\Rightarrow L\{y'+y\} = L\{3e^{2x}\} = 3L\{e^{2x}\}$

$\Rightarrow L\{y'\} + L\{y\} = 3L\{e^{2x}\}$

$\Rightarrow sY(s) - y(0) + Y(s) = 3 \cdot \frac{1}{s-2}$

$\Rightarrow Y(s)[s+1] = \frac{3}{s-2} \Rightarrow Y(s) = \frac{3}{(s+1)(s-2)}$

$\Rightarrow Y(s) = \frac{1}{s-2} - \frac{1}{s+1}$

$\Rightarrow L^{-1}[Y(s)] = L^{-1}\left(\frac{1}{s-2}\right) - L^{-1}\left(\frac{1}{s+1}\right)$

$\Rightarrow \boxed{y(t) = e^{2t} - e^{-t}}$

or $\boxed{y(x) = e^{2x} - e^{-x}}$