Announcement: We will have only one midterm

Final 40%
Midterm 30%
HW 29.5%
Eval. 0.5%

Lecture 2

2nd order lin. DE with constant coefficients

$$y'' + p y' + q y = f \qquad y = y(t)$$

$$p, q \in \mathbb{R} \qquad f = f(t)$$

- If $f = 0$ the the DE is called homogeneous. Otherwise, it is called inhomogeneous

Example:

$$y'' + y' + y = 0 \quad \text{hom.}$$

$$y'' + 2y' + y = \sin(t) \quad \text{non-hom.}$$

How do we solve

$$y'' + py' + qy = f \quad ?$$

① Find the general sol. to the associated homo. DE

$$y_h'' + py_h' + qy_h = 0$$

② Find a particular solution y_p

* the method of undetermined coefficients
* the method of variation of parameters

③ The general sol. is

$$y(t) = y_h(t) + y_p(t)$$

This is guarantied by a Theorem covered in 33B

How do we solve

$\rightarrow \quad y_h'' + p\, y_h' + q\, y_h = 0$?

Educated guess! Set

$$y_h(t) = e^{\lambda t}$$

and insert

$$y_h'(t) = \lambda e^{\lambda t}$$

$$y_h''(t) = \lambda^2 e^{\lambda t}$$

$$\lambda^2 \cancel{e^{\lambda t}} + p\lambda \cancel{e^{\lambda t}} + q \cancel{e^{\lambda t}} = 0$$

Characteristic polynomial

$\rightarrow \quad \lambda^2 + p\lambda + q = 0$

General solution

$\rightarrow \quad \lambda_{+,-} = \frac{1}{2}\left[-p \pm \sqrt{\underbrace{p^2 - 4q}_{\Delta}}\right]$

We have three possibilities

$\Delta > 0$: 2 distinct real roots
$\lambda_1 \neq \lambda_2$

$$y_h(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$\Delta = 0$: 1 root of multiplicity 2

$$y_h(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

$\Delta < 0$: Complex conjugate
$\lambda = \alpha \pm i\beta$

$$y_h(t) = A_1 e^{(\alpha + i\beta)t} + A_2 e^{(\alpha - i\beta)t}$$

or

$$y_h(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

How do we solve

$$y'' - y' - 2y = 4x^2 \; ?$$

Method of UC

1. $y_h'' - y_h' - 2y_h = 0$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda_{+,-} = \frac{1}{2}\left[1 \pm \sqrt{1 - 4\cdot(-2)}\right]$$

$$= \frac{1}{2}[1 \pm 3] = \begin{cases} 2 \\ -1 \end{cases}$$

$$y_h(x) = c_1 e^{2x} + c_2 e^{-x}$$

2. Particular solution

$$y_p(x) = A + Bx + Cx^2$$

insert and get $C = -2,\ B = 2$

$A = -3$

3. $$y(x) = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2$$

$y(0) = 1$

$y'(0) = 2$

Planar systems

A planar system is an eqn. of the form

$$\dot{\underline{x}} = A\underline{x} \qquad \underline{x} \in \mathbb{R}^2$$

$$A = \begin{bmatrix} a & b \\ d & d \end{bmatrix} \qquad A : \mathbb{R}^2 \to \mathbb{R}^2$$

Task: Transform $\ddot{y} + p\dot{y} + qy = 0$ into a 1st order system (planar system)

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= \dot{y} \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} \dot{y}_1 &= \dot{y} \\ \dot{y}_2 &= \ddot{y} \\ &= -p\dot{y} - qy \\ &= -py_2 - qy_1 \end{aligned}$$

$$\dot{y}_1 = y_2$$

$$\Longrightarrow \quad \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

How do we solve planar systems?

We look for soln. of the form

$$\underline{x}(t) = e^{\lambda t}\,\underline{v}$$

$$\Rightarrow \quad (A - \lambda \mathbb{I})\,\underline{v} = \underline{0}$$

$$\Rightarrow \quad \lambda^2 - \tau\lambda + \Delta = 0 \qquad \tau = \operatorname{tr} A$$

$$\Delta = \det A$$

$$\Rightarrow \quad \lambda_{+,-} = \frac{1}{2}\left[\tau \pm \sqrt{\tau^2 - 4\Delta}\,\right]$$

Three cases:

Case I: $\tau^2 - 4\Delta > 0$ two distinct real numbers $\lambda_1 \neq \lambda_2$

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\,\underline{v}_1 + c_2 e^{\lambda_2 t}\,\underline{v}_2$$

Case II: $\tau^2 - 4\Delta < 0$ two complex conjugate numbers

$\lambda = a + ib \rightarrow \underline{v} = \underline{v_1} + i\,\underline{v_2}$

$\bar{\lambda} = a - ib$

$$\underline{x}(t) = c_1 e^{\lambda t}\underline{v} + c_2 e^{\bar{\lambda} t}\overline{\underline{v}}$$

or

$$\underline{x}(t) = c_1 e^{at}\left(\cos(bt)\,\underline{v_1} - \sin(bt)\,\underline{v_2}\right) + c_2 e^{at}\left(\sin(bt)\,\underline{v_1} + \cos(bt)\,\underline{v_2}\right)$$

Case III: $\tau^2 - 4\Delta = 0$ one root of multiplicity 2.

$\lambda \longrightarrow \underline{v}$

We need to find a generalized eigenvector $\underline{w}$
i.e. $(A - \lambda I)\underline{w} = \underline{v}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Soln

$$\underline{x}(t) = c_1 e^{\lambda t} \underline{v} + c_2 e^{\lambda t} [\underline{w} + t\underline{v}]$$