Home Work 2

1) * Given $|x|<1$ & $|y|<1$ & $f(x,y)=x^2|y|$

We have:

$$\frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|} = \frac{|x^2|y_1|-x^2|y_2||}{|y_1-y_2|}$$

$$= \frac{x^2\,||y_1|-|y_2||}{|y_1-y_2|}$$. Since $||y_1|-|y_2|| \leq |y_1-y_2|$

$$\Rightarrow \frac{||y_1|-|y_2||}{|y_1-y_2|} \leq 1$$. Also $|x|<1$

$\Rightarrow x^2 < 1$

$$\Rightarrow \frac{x^2\,||y_1|-|y_2||}{|y_1-y_2|} \leq 1 < \infty \text{ (bounded)}$$

$\Rightarrow$ $\boxed{f(x,y) = x^2|y| \text{ satisfied a Lipschitz condition.}}$

* $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2|y|) = x^2 \cdot \frac{y}{|y|}$$

Because $\lim_{y \to 0^+} \left(x^2 \cdot \frac{y}{|y|}\right) = x^2$

$\lim_{y \to 0^-} \left(x^2 \frac{y}{|y|}\right) = -x^2$

$\Rightarrow$ they are different

$\Rightarrow \frac{\partial f}{\partial y}$ does not exist at $(x, 0)$ $(x \neq 0)$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y} \text{ fail to exist.}}$$

2) Given $f(x,y) = xy$

a) Rectangle : $a \le x \le b$ & $c \le y \le d$

We have: $\dfrac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \dfrac{|x(y_1 - y_2)|}{|y_1 - y_2|}$

$= \dfrac{|x|\,|y_1 - y_2|}{|y_1 - y_2|} = |x| \le \underbrace{\max\{|a|, |b|\}}_{\text{const}} < \infty$ (bounded)

$\Rightarrow$ $\boxed{f(x,y) \text{ satisfies Lipschitz condition}}$

b) Given $a \le x \le b$ & $-\infty < y < \infty$

Check: $\dfrac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \dfrac{|x(y_1 - y_2)|}{|y_1 - y_2|}$

$= |x| \le \underbrace{\max\{|a|, |b|\}}_{\text{constant}} < \infty$ (bounded)

$\Rightarrow$ $\boxed{\text{So it sastifies Lipschitz condition}}$

c) Given the entire plane, $x, y \in \mathbb{R}$

We have $\dfrac{\partial f}{\partial y} = x$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = |x| \to \infty \quad \text{if } x \to \infty$$

(not bounded)

$\Rightarrow$ $\boxed{\text{it does not satisfy a Lipschitz condition}}$

3) Given $f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$

$\Rightarrow f\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -y_1 + y_1 y_2 \\ y_2 - y_1 y_2 \end{bmatrix}$

Check $\|f(x) - f(y)\|_2 = \left\| \begin{matrix} -x_1 + x_1 x_2 + y_1 - y_1 y_2 \\ x_2 - x_1 x_2 - y_2 + y_1 y_2 \end{matrix} \right\|_2$

$= \sqrt{(-x_1 + x_1 x_2 + y_1 - y_1 y_2)^2 + (x_2 - x_1 x_2 - y_2 + y_1 y_2)^2}$

$= \sqrt{(x_1 - x_1 x_2 + y_1 y_2 - y_1)^2 + (x_2 - y_2 + y_1 y_2 - x_1 x_2)^2}$

$= \sqrt{[(x_1 - y_1) + (y_1 y_2 - x_1 x_2)]^2 + [(x_2 - y_2) + (y_1 y_2 - x_1 x_2)]^2}$ (1)

* Let $x_1 \geq y_1 \geq 0$, $x_2 \geq y_2 \geq 0$

$\Rightarrow x_1 x_2 \geq y_1 y_2 \Rightarrow y_1 y_2 - x_1 x_2 \leq 0$

$\Rightarrow \begin{cases} (x_1 - y_1) + (y_1 y_2 - x_1 x_2) \leq x_1 - y_1 \\ (x_2 - y_2) + (y_1 y_2 - x_1 x_2) \leq x_2 - y_2 \end{cases}$

$$\Rightarrow ① \le \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Also, $\|x - y\|_2 = \left\| \begin{matrix} x_1 - y_1 \\ x_2 - y_2 \end{matrix} \right\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

$$\Rightarrow \frac{\|f(x) - f(y)\|_2}{\|x - y\|_2} \le \frac{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}} = \underset{\substack{\uparrow \\ \text{constant}}}{1} < \infty$$

Therefore, with domain: $x_1 \ge y_1 \ge 0$, $x_2 \ge y_2 \ge 0$, the given function $f(x)$ is Lipschitz Continuous.

4) Given: $f, h : R \to R$ are locally Lipschitz over some bounded Domain D

We have:

$$\frac{|f(x) - f(y)|}{|x-y|} \leq K_1 \, , \; K_1 > 0 \text{ \& is a constant} \quad \text{I}$$

$$x, y \in R$$

$$\frac{|h(x) - h(y)|}{|x-y|} \leq K_2 \, , \; K_2 > 0 \text{ \& is a constant} \quad \text{II}$$

$$x, y \in \mathbb{R}.$$

* <u>Check $f + h$</u>:

$$\frac{|(f+h)(x) - (f+h)(y)|}{|x-y|} = \frac{|f(x) + h(x) - f(y) - h(y)|}{|x-y|}$$

$$= \frac{|f(x) - f(y) + h(x) - h(y)|}{|x-y|} \leq \frac{|f(x) - f(y)| + |h(x) - h(y)|}{|x-y|}$$

$$= \underbrace{\frac{|f(x) - f(y)|}{|x-y|}}_{\text{From I}} + \underbrace{\frac{|h(x) - h(y)|}{|x-y|}}_{\text{From II}} \leq K_1 + K_2$$

$$\Rightarrow \frac{|(f+h)(x) - (f+h)(y)|}{|x-y|} \leq \underbrace{K_1 + K_2}_{\text{constant}} < \infty$$

$$\Rightarrow \boxed{f + h \text{ is also locally Lipschitz over some bounded domain } D.}$$

* **Check** $f.h$.

We have:

$$\frac{|(fh)(x) - (fh)(y)|}{|x-y|} = \frac{|f(x)h(x) - f(y)h(y)|}{|x-y|}$$

$$= \frac{|f(x)h(x) - h(x)f(y) + h(x)f(y) - f(y)h(y)|}{|x-y|}$$

$$= \frac{|h(x)[f(x) - f(y)] + f(y)[h(x) - h(y)]|}{|x-y|}$$

$$\leq \frac{|h(x)[f(x) - f(y)]| + |f(y)[h(x) - h(y)]|}{|x-y|}$$

$$= \frac{|h(x)|\,|f(x) - f(y)|}{|x-y|} + \frac{|f(y)|\,|h(x) - h(y)|}{|x-y|}$$

$$\leq |h(x)| K_1 + |f(y)| K_2$$

$$= AK_1 + BK_2 < \infty$$

Since f & h are locally Lipschitz over some bounded domain D, so A & B are respectively bounded constant number of h & f. $\Rightarrow AK_1 + BK_2$ is also a constant

$$\Rightarrow \frac{|(fh)(x) - (fh)(y)|}{|x-y|} \leq \underbrace{AK_1 + BK_2}_{\text{constant}} < \infty$$

$\Rightarrow$ $\boxed{fh \text{ is locally Lipschitz over some bounded domain } D}$

* Check $f \circ h$:

We have:

$$\frac{|(f \circ h)(x) - (f \circ h)(y)|}{|x-y|} = \frac{|f(h(x)) - f(h(y))|}{|x-y|}$$

$$= \frac{|f(h(x)) - f(h(y))| \cdot |h(x) - h(y)|}{|h(x) - h(y)| \cdot |x-y|}$$

From (II), we already have $\frac{|h(x) - h(y)|}{|x-y|} \leq K_2$

f is also locally Lipschitz over some bounded domain D

$$\Rightarrow \frac{|f(h(x)) - f(h(y))|}{|h(x) - h(y)|} \le K \quad \text{with } K > 0 \text{ \& is constant}$$

$$\Rightarrow \frac{|(f \circ h)(x) - (f \circ h)(y)|}{|x - y|} \le \underbrace{K \cdot K_2}_{\text{const}} < \infty$$

$$\boxed{\Rightarrow f \circ h \text{ is also locally Lipschitz over some bounded domain } D}$$

5) Initial Value Problem:

$$\begin{cases} y' = |y| = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

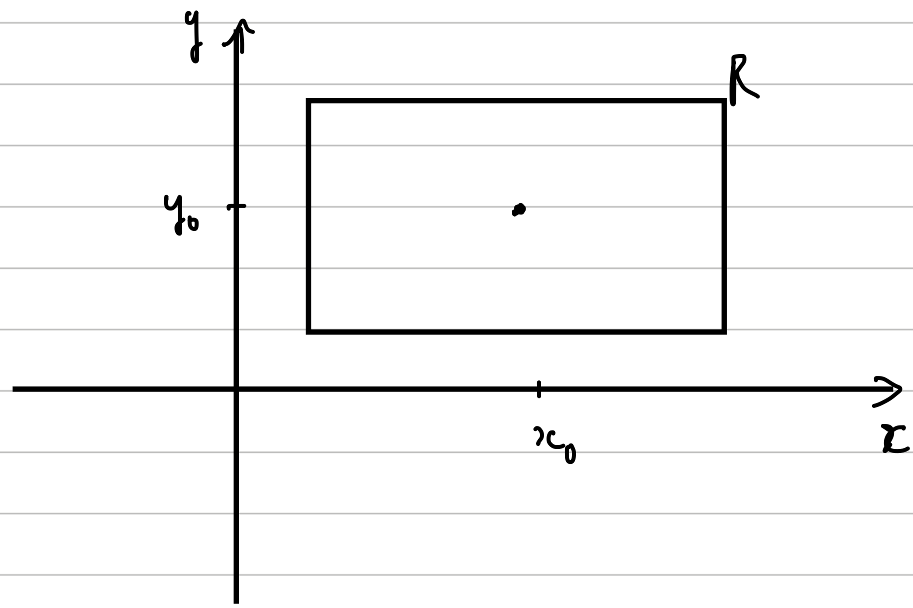
a) Firstly, $f(x,y) = |y| = \begin{cases} y & (y \geq 0) \\ -y & (y < 0) \end{cases}$ is continuous

for all value of $x \in \mathbb{R}$ & $y \in \mathbb{R}$

Next, $\frac{\partial f}{\partial y} = \frac{y}{|y|}$ is not continuous at $y = 0$

or $\frac{\partial f}{\partial y}$ is continuous with $x \in \mathbb{R}$ & $y \in \mathbb{R}, y \neq 0$

$\Rightarrow$ let D: $\beta < x < \alpha$ & $\beta_1 < y < \alpha_1$



$\Rightarrow f(x,y)$ & $\frac{\partial f}{\partial y}$ be continuous function of x & y

on a close rectangle R, $\Rightarrow$ this problem has a unique solution $y = y(x)$ on the interval $|x - x_0| \leq h$, $h > 0$

with $\boxed{(x_0, y_0) \text{ such that } x_0 \in R \ \& \ y_0 \in R \ \& \ y_0 \neq 0}$

b) From part a, we can see with $(x_0, y_0) \in D$ such that $x_0 \in R$, $y_0 \in R$ & $y_0 \neq 0$, the problem have a unique solution on some interval $|x - x_0| \leq h$, $h > 0$

We have:

$$\frac{dy}{dx} = |y| \Rightarrow \frac{dy}{|y|} = dx$$

* If $y > 0$, $\frac{dy}{y} = dx \rightarrow \int \frac{dy}{y} = \int dx$

$$\Rightarrow \ln y = x + C \Rightarrow C = \ln y - x$$

Since $y(x_0) = y_0 \Rightarrow C = \ln y_0 - x_0$

$$\Rightarrow \ln y = x + \ln y_0 - x_0$$

$$\Rightarrow y = e^{x + \ln y_0 - x_0} = y_0 e^{x - x_0}$$, Since $y > 0 \Rightarrow y_0 > 0$

* If $y < 0 \Rightarrow -\frac{dy}{y} = dx \Rightarrow \int \frac{dy}{-y} = \int dx$

$$\Rightarrow \quad -\ln(-y) = x + C \Rightarrow -\ln(-y_0) = x_0 + C$$

$$\Rightarrow \quad -\ln(-y_0) - x_0 = C$$

$$\Rightarrow \quad -\ln(-y) = x - x_0 - \ln(-y_0)$$

$$\Rightarrow \quad \ln(-y) = -x + x_0 + \ln(-y_0)$$

$$\Rightarrow \quad y = y_0 e^{-x+x_0}$$. Since $y < 0 \Rightarrow y_0 < 0$

$\boxed{\begin{array}{l}\Rightarrow \text{With } x_0 \in R \ \& \ y_0 \in R \ \& \ y_0 > 0 \text{, the IVP} \\ \text{has unique solution respectively for } y = f(x) > 0 \\ \text{and with } x_0 \in R \ \& \ y_0 \in R \ \& \ y_0 < 0 \text{, the IVP} \\ \text{has unique solution } y = f(x) < 0\end{array}}$

6) Given :

$$\begin{cases} y' = y|y| = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

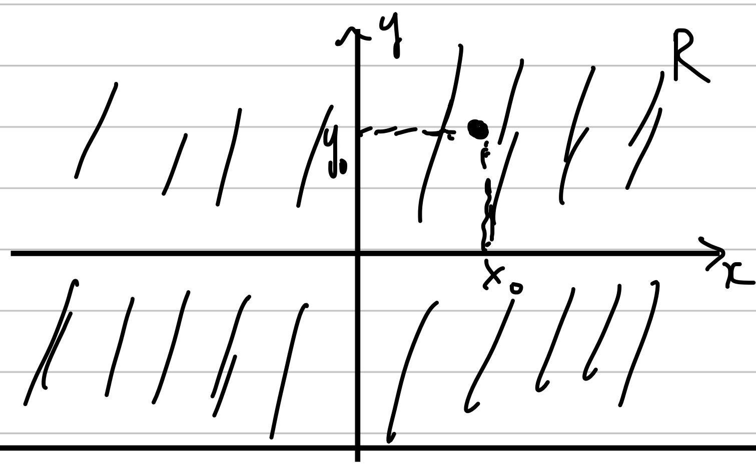
Firstly, $y' = f(x,y) = y|y|$ is continuous on $x \in R$ & $y \in R$

$$\frac{\partial f}{\partial y} = |y| + y \cdot \frac{y}{|y|} = |y| + \frac{y^2}{|y|} = |y| + \frac{(|y|)^2}{|y|}$$

$\Rightarrow \frac{\partial f}{\partial y} = 2|y|$ is also continuous on $x \in R$, $y \in R$

$\Rightarrow$ With $(x_0, y_0) \in \mathbb{R}^2$, then the intial value problem will has a unique solution on some interval $|x - x_0| \leq h$, $h > 0$

(by using result of theorem A)



$x_0 \in \mathbb{R}$
$y_0 \in R$

7) Given :

$$\begin{cases} \dfrac{dy}{dt} = ty^3 = f(t,y) \\ y(0) = y_0 \end{cases}$$

Firstly, $f(t,y) = ty^3$ is continuous on $t \in \mathbb{R}$, $y \in \mathbb{R}$

Also, $\dfrac{\partial f}{\partial y} = 3y^2 t$ is also continuous on $t \in \mathbb{R}$, $y \in \mathbb{R}$

$\Rightarrow$ Both $f(t,y)$ & $\dfrac{\partial f}{\partial y}$ are continuous of x & y on

whole $\mathbb{R}^2$: $t \in \mathbb{R}$ & $y \in \mathbb{R}$, and $(x_0, y_0) = (0, y_0) \in \mathbb{R}^2$

$\Rightarrow$ there exist a number $h > 0$ such that the IVP

has a unique solution $y = y(x)$ on the interval

$|t - t_0| \leq h$

$$\boxed{\text{$\rightarrow$ In this problem, we can apply the Picard's theorem and the IVP has a unique solution we mentioned above.}}$$

We also have:

$$\frac{dy}{dt} = ty^3 \Rightarrow \int \frac{dy}{y^3} = \int t\,dt$$

$$\Rightarrow \frac{-1}{2} \cdot y^{-2} = \frac{t^2}{2} + C$$

$$y(0) = y_0 \Rightarrow -\frac{1}{2} y_0^{-2} = C$$

$$\Rightarrow -\frac{1}{2} y^{-2} = \frac{t^2}{2} - \frac{1}{2} y_0^{-2}$$

$$\Rightarrow \frac{1}{y^2} = \frac{1}{y_0^2} - t^2 = \frac{1 - t^2 y_0^2}{y_0^2}$$

$$\Rightarrow y^2 = \frac{y_0^2}{1 - t^2 y_0^2} \Rightarrow 1 - t^2 y_0^2 > 0$$

$$\Rightarrow t^2 y_0^2 < 1 \Rightarrow t^2 < \frac{1}{y_0^2} \Rightarrow \left(t - \frac{1}{y_0}\right)\left(t + \frac{1}{y_0}\right) < 0$$

$$\Rightarrow \frac{1}{-|y_0|} < t < \frac{1}{|y_0|} \quad (y_0 \neq 0)$$

$\Rightarrow$ the maximum interval of existence of the solution is:

$\left(-\frac{1}{|y_0|}, \frac{1}{|y_0|}\right)$ $(y_0 \neq 0)$ We can see that when the absolute value of y_0 $(|y_0|)$ is larger, the interval will be smaller.

But the range is constant that is $\frac{2}{|y_0|}$