Home Work 7

$\boxed{\text{1) Section 40, Exercise 1 b, e}}$

$y'' + \lambda y = 0 \Rightarrow$ auxilliary equation: $m^2 + \lambda = 0$

b) $y(0) = 0$, $y(2\pi) = 0$

⊕ When $\lambda > 0 \Rightarrow m^2 + \lambda = 0 \Rightarrow m = \pm i\sqrt{\lambda}$

$\Rightarrow$ general solution:

$$y(t) = A_1 \cos\sqrt{\lambda}t + A_2 \sin\sqrt{\lambda}t$$

$$y(0) = A_1 = 0$$

$$y(2\pi) = A_1 \cos 2\pi\sqrt{\lambda} + A_2 \sin 2\pi\sqrt{\lambda} = 0$$

$$= A_2 \sin 2\pi\sqrt{\lambda} = 0$$

If $\sqrt{\lambda} \notin \mathbb{Z}_+ = \{1, 2, 3, ..\} \Rightarrow A_2 = 0, A_1 = 0$

$\Rightarrow$ no non-trivial solution

If $\sqrt{n} \in \mathbb{Z}_+ = \{1, 2, 3, ..\} \Rightarrow \sin 2\pi\sqrt{\lambda} = 0$

$$\Rightarrow 2\pi\sqrt{\lambda} = n\pi \Rightarrow \sqrt{\lambda} = \frac{n}{2}, n \in \mathbb{N}$$

$$\Rightarrow \lambda = \frac{n^2}{4}, n \geq 1, n \in \mathbb{N}$$

$\Rightarrow$ the solution $y_n(t) = A_2 \sin \frac{n}{2}t$ is the

$$\boxed{\text{eigenfunction with the eigenvalue } \lambda_n = \frac{n^2}{4} \text{ with } n = 1, 2, 3, \ldots}$$

⊕ when $\lambda = 0$, the repeated root $m = 0$

$\Rightarrow$ the general solution:

$$y(t) = C_1 + C_2 t$$

$$y(0) = 0 \Rightarrow C_1 = 0$$

$$y(2\pi) = 0 \Rightarrow C_2 \cdot 2\pi = 0 \Rightarrow C_2 = 0$$

$\Rightarrow$ no non-trivial solution

⊕ when $\lambda < 0 \Rightarrow m^2 + \lambda = 0 \Rightarrow m^2 = -\lambda \Rightarrow m = \pm\sqrt{-\lambda}$

$\Rightarrow$ the general solution:

$$y(t) = C_1 e^{\sqrt{-\lambda}\, t} + C_2 e^{-\sqrt{-\lambda}\, t}$$

$$y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$

$$y(2\pi) = C_1 e^{\sqrt{-\lambda}\, 2\pi} + C_2 e^{-\sqrt{-\lambda}\, 2\pi} = 0$$

$$\Rightarrow C_1 e^{\sqrt{-\lambda} \cdot 2\pi} - C_1 e^{-\sqrt{-\lambda} \cdot 2\pi} = 0$$

$$\Rightarrow C_1 \Big[\underbrace{e^{\sqrt{-\lambda} \cdot 2\pi} - e^{-\sqrt{-\lambda}\, 2\pi}}_{\neq 0}\Big] = 0$$

$\Rightarrow C_1 = 0$ $\Rightarrow$ $\boxed{\text{No non-trivial solution}}$

e) $y(-L) = 0$, $y(L) = 0$ when $L > 0$

Similarly part b, we have:

⊕ When $\lambda > 0$, the general solution:

$y(t) = A_1 \cos\sqrt{\lambda}t + A_2 \sin\sqrt{\lambda}t$

* $y(-L) = A_1 \cos(-\sqrt{\lambda}L) + A_2 \sin(-\sqrt{\lambda}L) = 0$

$\Rightarrow A_1 \cos\sqrt{\lambda}L - A_2 \sin\sqrt{\lambda}L = 0$

* $y(L) = A_1 \cos\sqrt{\lambda}L + A_2 \sin\sqrt{\lambda}L = 0$

$\Rightarrow 2A_1 \cos\sqrt{\lambda}L = 0$

$\Rightarrow \begin{cases} A_1 \cos\sqrt{\lambda}L = 0 \\ A_2 \sin\sqrt{\lambda}L = 0 \end{cases}$

⊕ If both A_1 & A_2 are 0 $\Rightarrow$ no non-trivial solution.

⊕ When $A_1 = 0$ & $A_2 \neq 0$

$\Rightarrow \sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi$, $n = 1, 2, 3, \ldots$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2}{L^2}$

then $y(t) = A_2 \sin\left(\frac{n\pi}{L} \cdot t\right)$

⊛ When $A_1 \neq 0$, $A_2 = 0$

$\Rightarrow \cos\sqrt{\lambda} L = 0$

$\Rightarrow \sqrt{\lambda} L = \frac{\pi}{2} + n\pi$, $n = 1, 2, 3, \ldots$

$\Rightarrow \lambda = \left[\frac{1}{L}\left(\frac{\pi}{2} + n\pi\right)\right]^2 = \frac{\pi^2}{L^2}\left(n + \frac{1}{2}\right)^2$

$\Rightarrow y(t) = A_1 \cos\sqrt{\lambda}\, t$

$\Rightarrow y(t) = A_1 \cos\left[\frac{1}{L}\left(\frac{\pi}{2} + n\pi\right)t\right]$

⊛ When A_1 & $A_2 \neq 0$

$\Rightarrow \begin{cases} \cos\sqrt{\lambda} L = 0 \\ \sin\sqrt{\lambda} L = 0 \end{cases} \Rightarrow$ reject since $\cos^2\sqrt{\lambda}L + \sin^2\sqrt{\lambda}L = 1$

Therefore,

⊛ If the eigenvalue $\lambda_n = \frac{n^2\pi^2}{L^2}$, then the eigenfunctions

$y_n(t) = A_2 \sin\left(\frac{n\pi}{L} t\right)$

⊛ If the eigenvalue $\lambda_n = \frac{\pi^2}{L^2}\left(n + \frac{1}{2}\right)^2$, then the eigenfunction $y_n(t) = A_1 \cos\left[\frac{1}{L}\left(\frac{\pi}{2} + n\pi\right)t\right]$, $n = 1, 2, 3, \ldots$

⊛ When $\lambda = 0$, the general solution:

$$y(t) = C_1 + C_2 t \quad , \quad L > 0$$

$$\Rightarrow y(L) = C_1 + C_2 L = 0$$

$$\Rightarrow C_1 = 0, C_2 = 0$$

$$\Rightarrow y(-L) = C_1 - C_2 L = 0$$

$\Rightarrow$ $\boxed{\text{no non-trivial solution}}$

⊛ When $\lambda < 0$, the general solution:

$$y(t) = C_1 e^{\sqrt{-\lambda} t} + C_2 e^{-\sqrt{-\lambda} t}$$

$$y(L) = C_1 e^{\sqrt{-\lambda} L} + C_2 e^{-\sqrt{-\lambda} L} = 0$$

$$y(-L) = C_1 e^{-\sqrt{-\lambda} L} + C_2 e^{\sqrt{-\lambda} L} = 0$$

$$\Rightarrow C_1 \left(e^{\sqrt{-\lambda} L} - e^{-\sqrt{-\lambda} L}\right) + C_2 \left(e^{-\sqrt{-\lambda} L} - e^{\sqrt{-\lambda} L}\right) = 0$$

$$\Rightarrow \underbrace{\left(e^{\sqrt{-\lambda} L} - e^{-\sqrt{-\lambda} L}\right)}_{\neq 0} (C_1 - C_2) = 0$$

$$\Rightarrow C_1 = C_2 \Rightarrow C_1 = C_2 = 0$$

$\Rightarrow$ $\boxed{\text{no non-trivial solution}}$

$\boxed{\text{2) Section 40, Exercise 2}}$

$\begin{cases} y = F(x) \text{ is an arbitiary function} \\ y = F(x+at) \end{cases}$

$\begin{cases} y = G(x) \text{ is another arbitiary function} \\ y = G(x-at) \end{cases}$

$$y(x,t) = F(x+at) + G(x-at)$$

a) Equation 8: $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

$$\frac{\partial y}{\partial x} = F'(x+at) + G'(x-at)$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = F''(x+at) + G''(x-at)$$

Also, $\frac{\partial y}{\partial t} = aF'(x+at) - aG'(x-at)$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = a^2 F''(x+at) + a^2 G''(x-at)$$

$$= a^2 [F''(x+at) + G''(x-at)]$$

$$= a^2 \frac{\partial^2 y}{\partial x^2}$$

$\Rightarrow \boxed{a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}}$ that satifies equation 8.

b) $\alpha = x + at \quad , \quad \beta = x - at$

$$\left.\begin{array}{l} \dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial \alpha} \cdot \dfrac{\partial \alpha}{\partial x} = \dfrac{\partial y}{\partial \alpha} \cdot 1 \\[2ex] \dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial \beta} \cdot \dfrac{\partial \beta}{\partial x} = \dfrac{\partial y}{\partial \beta} \cdot 1 \end{array}\right\} \Rightarrow \dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial \alpha \partial \beta}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial t} = \frac{\partial y}{\partial \alpha} \cdot a$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial \beta} \cdot \frac{\partial \beta}{\partial t} = \frac{\partial y}{\partial \beta} (-a) = -\frac{\partial y}{\partial \beta} \cdot a$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = -a^2 \frac{\partial^2 y}{\partial \alpha \partial \beta}$$

$$\Rightarrow a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \Leftrightarrow a^2 \frac{\partial^2 y}{\partial \alpha \partial \beta} = -a^2 \frac{\partial^2 y}{\partial \alpha \partial \beta}$$

$$\Leftrightarrow 2a^2 \frac{\partial^2 y}{\partial \alpha \partial \beta} = 0$$ With $a \neq 0$ is a constant

$$\Leftrightarrow \frac{\partial^2 y}{\partial \alpha \partial \beta} = 0 \quad \text{OR} \quad \boxed{(8) \Leftrightarrow \frac{\partial^2 y}{\partial \alpha \partial \beta} = 0}$$

$\boxed{\text{3) Section 40, Exercise 3, a, b, c}}$

Initial condition: $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ & $y(x,0) = f(x)$

$$y(x,t) = F(x+at) + G(x-at)$$

$$\Rightarrow y(x,0) = F(x) + G(x) = f(x)$$

$$\frac{\partial y}{\partial t} = aF'(x+at) - aG'(x-at)$$

$$\Rightarrow \left.\frac{\partial y}{\partial t}\right|_{t=0} = aF'(x) - aG'(x) = 0$$

$$\Rightarrow F'(x) = G'(x) \Rightarrow \int F'(x)\,dx = \int G'(x)\,dx$$

$$\Rightarrow F(x) - F(x_0) = G(x) - G(x_0) + C$$

$$\Rightarrow F(x) - F(x_0) - G(x) + G(x_0) = C$$

Also, $F(x) + G(x) = f(x)$

$$\Rightarrow 2F(x) + G(x_0) - F(x_0) = C + f(x)$$

$$\Rightarrow F(x) = \frac{C + f(x) + F(x_0) - G(x_0)}{2}$$

$$\Rightarrow G(x) = f(x) - F(x) = \frac{f(x) - C - F(x_0) + G(x_0)}{2}$$

$$\Rightarrow F(x+at) = \frac{C + f(x+at) + F(x_0) - G(x_0)}{2}$$

$$G(x-at) = \frac{f(x-at) - C - F(x_0) + G(x_0)}{2}$$

$$\Rightarrow y(x,t) = F(x+at) + G(x-at)$$

$$= \frac{\cancel{C} + f(x+at) + \cancel{F(x_0)} - \cancel{G(x_0)} + f(x-at) - \cancel{C} - \cancel{F(x_0)} + \cancel{G(x_0)}}{2}$$

$$= \boxed{\frac{1}{2}\left[f(x+at) + f(x-at)\right]}$$

b) $y(0,t) = y(\pi,t) = 0$

We have $y(0,t) = \frac{1}{2}\left[f(at) + f(-at)\right] = 0$

$\Rightarrow f(at) = -f(-at)$ let $x = at$

$\Rightarrow f(x) = -f(-x) \Rightarrow$ $\boxed{f(x) \text{ is odd function}}$

Also, $y(\pi,t) = \frac{1}{2}\left[f(at+\pi) + f(\pi-at)\right] = 0$

$\Rightarrow f(\pi+at) + f(\pi-at) = 0$ ①

Sin $f(x)$ is an odd function

$\Rightarrow f(\pi - at) = -f(at - \pi)$, plug it into ①, then

$\Rightarrow f(\pi + at) - f(at - \pi) = 0$

$\Rightarrow f(at + \pi) = f(at - \pi)$ let $x = at - \pi$

$\Rightarrow x + 2\pi = at + \pi$

$\Rightarrow f(x + 2\pi) = f(x)$

$\Rightarrow$ $\boxed{\text{the function } f(x) \text{ is periodic with period } 2\pi}$

c) Since $f(x)$ is odd & periodic with period 2π

$\Rightarrow f(x) = f(x+2\pi) = -f(-x)$

$f(x) = -f(-x) \Rightarrow f(0) = -f(0) \Rightarrow 2f(0) = 0$

$\Rightarrow \boxed{f(0) = 0}$

Also, $f(\pi) = -f(-\pi) = -f(-\pi + 2\pi) = -f(\pi)$

$\Rightarrow 2f(\pi) = 0$

$\Rightarrow \boxed{f(\pi) = 0}$

$\Rightarrow$ $\boxed{f(x) \text{ is necessarily vanishes at } 0 \text{ \& } \pi}$

$\boxed{\text{4) Section 40, Ex 5 a, b}}$

Given $y(x,0) = f(x)$

$$(*) \quad \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{Let } y(x,t) = u(x)\,v(t)$$

then we get:

$$u'' + \lambda u = 0 \quad \& \quad v'' + \lambda a^2 v = 0 \quad , \quad \lambda = n^2$$
$$n = 1, 2, 3, \ldots$$

$$\Leftrightarrow u'' + n^2 u = 0 \quad \& \quad v'' + n^2 a^2 v = 0$$

$\rightarrow$ general solutions:

$$\begin{cases} u(x) = C_1 \cos nx + C_2 \sin nx \\ v(t) = C_3 \cos nat + C_4 \sin nat \end{cases}$$

For vibrating problem, we have $\begin{cases} y(t,0) = y(\pi,t) = 0 \\ \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0 \end{cases}$

$$\Rightarrow \begin{cases} U(0) = 0 \\ U(\pi) = 0 \\ v'(0) = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_4 = 0 \end{cases}$$

$$\Rightarrow y(x,t) = C_2 C_3 \sin nx \cos nat$$

$\Rightarrow$ general solution: $y_n = \sum_{n=1}^{\infty} b_n \sin nx \cos nat$

Also, $y(x,0) = f(x)$

$$\Rightarrow \quad y_n(x,0) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} \frac{2cx}{\pi}\sin nx\,dx\right.$$

$$\left. + \int_{\pi/2}^{\pi} \frac{2c(\pi-x)}{\pi}\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2c}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2c}{\pi}\int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\right]$$

Check $\int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx$ $\quad t = \pi - x \Rightarrow dt = -dx$

$\circledast$ $\sin(nx) = \sin n(\pi-t) = \sin(n\pi - nt)$

$$= -\sin(nt - n\pi)$$

$$= (-1)^{n+1}\sin(nt)$$

$$= -\int_{\pi/2}^{0} t\sin(nt)(-1)^{n+1}\,dt$$

$\circledast$ $x = \pi/2 \to \pi$

$\Rightarrow t = \frac{\pi}{2} \to 0$

$$= \int_0^{\pi/2} (-1)^{n+1} t\sin nt\,dt$$

$$= \int_0^{\pi/2} (-1)^{n+1} x\sin(nx)\,dx$$

$$\Rightarrow b_n = \frac{4c}{\pi^2}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_0^{\pi/2} (-1)^{n+1} x \sin(nx)\, dx\right]$$

* If b_n is even $\Rightarrow (-1)^{n+1} = -1$

$\Rightarrow b_n = 0$

* If n is odd $\Rightarrow (-1)^{n+1} = 1$

$$\Rightarrow b_n = \frac{4c}{\pi^2} \cdot 2 \int_0^{\pi/2} x \sin(nx)\, dx = \frac{8c}{\pi^2} \int_0^{\pi/2} x \sin(nx)\, dx$$

$$= \frac{8c}{\pi^2}\left[\frac{\sin nx - nx \cos nx}{n^2}\right]\Bigg|_0^{\pi/2}$$

$$= \frac{8c}{n^2\pi^2}\left[\left(\sin \frac{n\pi}{2} - n\frac{\pi}{2}\cos\frac{n\pi}{2}\right)\right]$$ let $n = 2k+1$

$k = 0, 1, 2, \ldots$

$$= \frac{8c}{n^2\pi^2}\left[\sin\frac{(2k+1)\pi}{2} - \frac{n\pi}{2}\underbrace{\cos\frac{(2k+1)\pi}{2}}_{0}\right]$$

$$= \frac{8c}{(2k+1)^2\pi^2} \cdot \sin\left(\frac{\pi}{2} + k\pi\right) = \frac{8c\,(-1)^k}{(2k+1)^2\pi^2}$$

$$\Rightarrow b_n = \frac{8c}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \boxed{y_n = \sum_{n=1}^{\infty} \frac{8c}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin(nx) \cos(nat)}$$

b) $f(x) = \frac{1}{\pi} x(\pi - x)$

Similarly part a, we have

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin(nx)\, dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{\pi} x(\pi - x) \sin nx\, dx$$

$$= \frac{2}{\pi^2}\int_0^{\pi} \left[x\pi \sin nx - x^2 \sin nx\right] dx$$

$$= \frac{2}{\pi^2}\left[\pi\int_0^{\pi} x \sin nx\, dx - \int_0^{\pi} x^2 \sin nx\, dx\right]$$

* Check $\displaystyle\int_0^{\pi} x \sin(nx)\, dx = \left.\frac{\sin nx - nx\cos nx}{n^2}\right|_0^{\pi}$

$$= \frac{1}{n^2}\left[0 - n\pi\cos n\pi\right] = \frac{-\pi\cos n\pi}{n}$$

$$\Rightarrow \pi\int_0^{\pi} x \sin(nx)\, dx = -\frac{\pi^2 \cos(n\pi)}{n} = \frac{(-1)^{n+1}\pi^2}{n}$$

* Check $\displaystyle\int_0^{\pi} x^2 \sin nx\, dx = \left.\frac{2nx \sin nx + (2 - n^2x^2)\cos nx}{n^3}\right|_0^{\pi}$

$$= \frac{(2-n^2\pi^2)\cos(n\pi) - 2}{n^3} = \frac{(-1)^{n+1}(n^2\pi^2 - 2) - 2}{n^3}$$

$$\Rightarrow \pi \int_0^{\pi} x \sin nx\, dx - \int_0^{\pi} x^2 \sin nx\, dx =$$

$$\frac{(-1)^{n+1}\pi^2}{n} - \frac{(-1)^{n+1} n^2\pi^2 - 2.(-1)^{n+1} - 2}{n^3}$$

$$= \frac{(-1)^{n+1}.\pi^2.n^2 - (-1)^{n+1}\pi^2 n^2 + 2(-1)^{n+1} + 2}{n^3}$$

$$= \frac{2}{n^3}\left[(-1)^{n+1} + 1\right]$$

$$\Rightarrow b_n = \frac{2}{\pi^2} \cdot \frac{2}{n^3}\left[(-1)^{n+1} + 1\right]$$

$$\Rightarrow b_n = \frac{4}{\pi^2 n^3}\left[(-1)^{n+1} + 1\right]$$

$$\Rightarrow \boxed{y_n = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^3}\left[(-1)^{n+1} + 1\right] \sin(nx) \cos(nat)}$$

$\boxed{\text{5) Section 40, Exercise 7}}$

Given $\left.\frac{\partial y}{\partial t}\right|_{t=0} = g(x)$ & $y(x,0) = 0$

$y(0,t) = 0 = y(\pi, t)$

Equation 8) $a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$

Similarly, we also have general solution as question 4.

$$\begin{cases} u(x) = C_1 \cos nx + C_2 \sin nx \\ v(t) = c_3 \cos nat + C_4 \sin nat \end{cases}$$

$y(0,t) = 0$ & $y(\pi,t) = 0 \Rightarrow u(0) = u(\pi) = 0$

$\Rightarrow C_1 = 0 \Rightarrow U(x) = C_2 \sin nx$

$\Rightarrow y_n = c_2 \sin nx \left[C_3 \cos nat + C_4 \sin nat \right]$

$= \sin nx \left[a_1 \cos nat + a_2 \sin nat \right]$

$a_1 = C_2 C_3, \; a_2 = C_2 C_4$

$y(x,0) = 0 \Rightarrow \sin nx \left[a_1 + 0 \right] = 0$

$\Rightarrow a_1 = 0$

$\Rightarrow y = a_2 \sin nx \sin nat$

$\Rightarrow y_n = \sum_{n=1}^{\infty} a_n \sin nx \sin nat$

or $\boxed{y_n(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx) \sin(nat)}$

Also, $$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \cdot \sin(nx)(na)\cos(nat)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} naC_n \sin(nx)\cos(nat)$$

$$= \sum_{n=1}^{\infty} naC_n \sin(nx) \quad = g(x)$$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty} \underbrace{naC_n}_{b_n} \sin(nx)$$

And $b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx = naC_n$

$$\Rightarrow \frac{2}{\pi na}\int_0^{\pi} g(x)\sin(nx)\,dx = C_n$$

$$\Rightarrow \boxed{C_n = \frac{2}{\pi na}\int_0^{\pi} g(x)\sin(nx)\,dx}$$

Therefore we have:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \sin(nx)\sin(nat) \text{ with } C_n = \frac{2}{n\pi a}\int_0^{\pi} g(x)\sin nx\,dx}$$

6) Section 41, Exercise 2

Given : $w(x,0) = f(x)$, $w(0,t) = w_1$, $w(\pi,t) = w_2$

Let $w(t) = W(x,t) + g(x)$

Check $W(x,t)$ with $W(0,t) = 0 = W(\pi,t)$

$$a^2 \frac{\partial^2 W}{\partial x^2} = \frac{\partial W}{\partial t}, \text{ Let } W(x,t) = u(x)\,v(t)$$

We have two differential equation:

$$u''(x) + \lambda u(x) = 0 \quad \& \quad v'(t) + a^2 n^2 v(t) = 0 \qquad \lambda = n^2$$

$$\Rightarrow \begin{cases} u(x) = C_1 \cos nx + C_2 \sin nx \\ v(t) = C_3 e^{-n^2 a^2 t} \end{cases}$$

Since $W(0,t) = W(\pi,t) = 0 \Leftrightarrow u(0) = u(\pi) = 0$

$$\Rightarrow C_1 = 0 \rightarrow W(x,t) = C_2 C_3 e^{-n^2a^2t} \sin nx$$

$$\Rightarrow W_n(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2a^2t} \sin(nx)$$

Also, follow the steady state condition, we have:

$$g(x) = a_1 x + a_2$$

Since $g(0,t) = a_2 = w_1$, $g(\pi,t) = a_1\pi + a_2 = w_2$

$$\Rightarrow \quad a_1 \pi + w_1 = w_2 \Rightarrow a_1 = \frac{w_2 - w_1}{\pi}$$

$$\Rightarrow g(x) = \frac{w_2 - w_1}{\pi} x + w_1$$

$$\boxed{\begin{array}{l} \Rightarrow \text{the general solution :} \\ w(x,t) = w_1 + \dfrac{w_2 - w_1}{\pi} x + \displaystyle\sum_{n=1}^{\infty} B_n e^{-n^2 a^2 t} \sin nx \end{array}}$$

Also, $w(0,x) = f(x)$

$$\Rightarrow f(x) = w_1 + \frac{w_2 - w_1}{\pi} x + \sum_{n=1}^{\infty} B_n \sin nx$$

$$\Rightarrow \underbrace{f(x) - g(x)}_{h(x)} = \sum_{n=1}^{\infty} B_n \sin nx$$

$$\Rightarrow \boxed{B_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin(nx)\, dx}$$

$\boxed{\text{7\&8) Section 41, exercise 5 and exercise 6}}$

Given, the initial temperature of the rod before being put in the ice is $100°C$ $\Rightarrow \boxed{w(x,t) = 100}$ is a constant in this case. We also want to check from $0 \to \pi$. And in this situation, the $w(x,t)$ is a constant at lower & upper limit

$$\Rightarrow \boxed{\frac{\partial w}{\partial x}\Big|_{x=0} = \frac{\partial w}{\partial x}\Big|_{x=\pi} = 0}$$

Also, with the equation:

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} \quad \text{and} \quad u(x,0) = f(x)$$

we will have 2 solutions based on $\begin{cases} u'' + \lambda u = 0 \\ v' + \lambda a^2 v = 0 \end{cases}$ $(\lambda = n^2)$

$$\begin{cases} u_n(x) = C_1 \cos nx + C_2 \sin nx \\ v_n(t) = C_3 e^{-n^2 a^2 t} \end{cases} \quad \& \quad w(x,t) = u(x).v(t)$$

Using the condition from question 7, we have:

$$\oplus \ \frac{\partial w}{\partial x}\Big|_{x=0} = 0 \Rightarrow u'(0) = 0$$

$$u'(x) = -C_1 n \sin nx + nC_2 \cos nx$$

$$\Rightarrow u'(0) = nC_2 = 0 \Rightarrow C_2 = 0 \text{ or } n = 0$$

$\oplus$ $\left.\frac{\partial w}{\partial x}\right|_{x=\pi} = 0 \Rightarrow u'(\pi) = 0$

$\Rightarrow -C_1 n \sin n\pi + nC_2 \cos n\pi = 0$

$\Rightarrow C_2 = 0$ or $n = 0$

$\Rightarrow u(x) = C_1 \cos nx \,,\quad n = 0, 1, 2, \ldots$

$\Rightarrow w_n(x,t) = C_1 C_3 e^{-n^2 a^2 t} \cos(nx)$

$= C e^{-n^2 a^2 t} \cos(nx)$

$\Rightarrow w(x,t) = \sum_{n=0}^{\infty} C_n e^{-n^2 a^2 t} \cos(nx) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n e^{-n^2 a^2 t} \cos(nx)$

Also $w(x,0) = f(x)$

$\Rightarrow w_n(x,0) = \sum_{n=0}^{\infty} C_n \cos(nx) = f(x)$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} C_n \cos(nx)$ is cosine series

$\Rightarrow C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$

$$\boxed{\begin{array}{l} \Rightarrow w(x,t) = \frac{1}{2} C_0 + \sum_{n=1}^{\infty} C_n e^{-n^2 a^2 t} \cos(nx) \\ \text{with } C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \,,\; n = 0, 1, 2, \ldots \end{array}}$$

9) Section 41, Exercise 7

Given: $a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial w}{\partial t}$

$w(0,y) = 0$, $w(0,x) = f(x)$, $w(\pi, y) = 0$

$\lim_{y\to\infty} w(x,y) = 0$

At the steady-state, we have

$\partial^2_{xx} w + \partial^2_{yy} w = 0$ let $w(x,y) = u(x)\, v(y)$

$\Rightarrow$ We have two equation:

$u'' + \lambda u = 0$ & $v'' - \lambda v = 0$, $\lambda = n^2$

$\Rightarrow u = C_1 \cos nx + C_2 \sin nx$

$w(0,y) = 0 \Rightarrow u(0) = 0 \Rightarrow C_1 + 0 = 0 \Rightarrow C_1 = 0$

$\Rightarrow u_n = C_2 \sin nx$

And $v_n = C_3 e^{ny} + C_4 e^{-ny}$

Since $\lim_{y\to\infty} w(x,y) = 0 \Rightarrow \lim_{y\to\infty} v(y) = 0$

$\Rightarrow \lim_{y\to\infty}\left[C_3 e^{ny} + C_4 e^{-ny}\right] = 0$

$\rightarrow C_3 \lim_{y\to\infty} e^{ny} + C_4 \lim_{y\to\infty} e^{-ny} = 0$

Since $C_4 \lim_{y\to\infty} e^{-ny} = 0 \Rightarrow C_3 \lim_{y\to\infty} e^{ny} = 0 \Rightarrow C_3 = 0$

$$\Rightarrow v_n(y) = C_4 e^{-ny}$$

$$\Rightarrow w_n(x,y) = \underbrace{C_2 C_4}_{b_n} \sin nx\, e^{-ny}$$

$$\Rightarrow w_n(x,y) = b_n e^{-ny} \sin nx$$

$$\Rightarrow w(x,y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx$$

Because $w(x,0) = f(x)$

$$\Rightarrow f(x) = w(x,0) = \sum_{n=1}^{\infty} b_n \sin nx$$

with $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$\Rightarrow \boxed{\begin{array}{l} w(x,y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin(nx) \\ \text{with } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \end{array}}$$