Exercises

1. Solve $e^{-x} = y(x) + 2\int_0^x \cos(x-t)y(t)dt$.

   Solution: Laplace transform:

$$\frac{1}{p+1} = Y + 2\left(\frac{p}{p^2+1}\right)Y = \frac{p^2+2p+1}{p^2+1}Y \implies Y(p) = \frac{p^2+1}{(p+1)^3}.$$

   Motivated by the denominator, notice $Y(p) = F(p+1)$, where

$$F(p) := \frac{(p-1)^2+1}{p^3} = \frac{p^2-2p+2}{p^3} = \frac{1}{p} - \frac{2}{p^2} + \frac{2}{p^3} = L[1-2x+x^2] = L[(x-1)^2].$$

   Then by the shifting property,

$$Y(p) = L[e^{-x}(x-1)^2] \implies y(x) = e^{-x}(x-1)^2.$$

2. Solve $\begin{cases} xy'' + (3x-1)y' - (4x+9)y = 0 \\ y(0) = 0. \end{cases}$

   Solution: Start by Laplace transforming the ODE:

$$\begin{aligned} -\frac{d}{dp}(p^2Y - y'(0)) - 3\frac{d}{dp}(pY) - (pY) + 4\frac{d}{dp}(Y) - 9Y &= 0 \\ -p^2Y' - 2pY - 3pY' - 3Y - pY + 4Y' - 9Y &= 0 \\ (-p^2 - 3p + 4)Y' + (-3p - 12)Y &= 0 \\ (p-1)Y' + 3Y &= 0. \end{aligned}$$

   It is unfortunate that we still need to solve an ODE, but the advantage is that this new ODE is first order (and still linear). To solve, use the Integrating Factor Method. Start by dividing by $(p-1)$ to write it in the desired form:

$$Y' + \frac{3}{p-1}Y = 0.$$

   Then the integrating factor is $e^{\int \frac{3}{p-1}dp} = e^{3\ln(p-1)} = (p-1)^3$. Multiply by this factor and use the product rule to rewrite the left side:

$$\frac{d}{dp}((p-1)^3Y) = 0 \implies (p-1)^3Y(p) = C \implies Y(p) = \frac{C}{(p-1)^3}.$$

   Lastly, reverse Laplace transform to get $y(x)$. By our table, $L[x^2] = \frac{2}{p^3}$, so the shifting formula gives $L[e^x x^2] = \frac{2}{(p-1)^3}$. Multiply this by $\frac{C}{2}$ using linearity of the Laplace transform to conclude

$$L\left[\frac{C}{2}e^x x^2\right] = \frac{C}{(p-1)^3} = Y(p) \implies y(x) = \frac{C}{2}e^x x^2.$$

   At this point, initial conditions may help determine the arbitrary constants, but our initial condition is satisfied for any choice of C, so it can remain arbitrary. (Since $C \in \mathbb{R}$ is an arbitrary constant, so is $c := \frac{C}{2}$. If preferred, we can write $y(x) = ce^x x^2$.)

3. Let $a, b > 0$. Compute $\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx$.

   Solution: Notice we are trying to compute

$$\int_0^\infty e^{-px}\frac{e^{-ax}-e^{-bx}}{x}dx\Big|_{p=0} = L\left[\frac{e^{-ax}-e^{-bx}}{x}\right]_{p=0}.$$

   By our tables,

$$L\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \int_p^\infty \left(\frac{1}{p+a} - \frac{1}{p+b}\right)dp = [\ln(p+a) - \ln(p+b)]_p^\infty = \left[\ln\left(\frac{p+a}{p+b}\right)\right]_p^\infty = -\ln\left(\frac{p+a}{p+b}\right).$$

   Lastly, plug in $p = 0$ to get $-\ln\left(\frac{a}{b}\right) = \ln\left(\frac{b}{a}\right)$.

4. Let $a, b > 0$. Compute $\int_0^\infty \frac{e^{-ax}\sin(bx)}{x}dx$.

Solution: Notice this is definition of $L\left[\frac{\sin(bx)}{x}\right]$ where we are using the letter a instead of p. By our tables,

$$L\left[\frac{\sin(bx)}{x}\right] = \int_a^\infty \frac{b}{a^2+b^2}da = \left[\arctan\left(\frac{a}{b}\right)\right]_a^\infty = \frac{\pi}{2} - \arctan\left(\frac{a}{b}\right).$$

5. Let $x \in \mathbb{R}$. Prove $\int_0^\infty \frac{\cos(xt)}{1+t^2}dt = \frac{\pi}{2}e^{-x}$

Solution: Let $f(x) = \int_0^\infty \frac{\cos(xt)}{1+t^2}dt$. Then

$$L[f(x)] = \int_0^\infty e^{-px}\left(\int_0^\infty \frac{\cos(xt)}{1+t^2}dt\right)dx = \int_0^\infty \frac{1}{1+t^2}\underbrace{\left(\int_0^\infty e^{-px}\cos(xt)dx\right)}_{=L[\cos(xt)]}dt = \int_0^\infty \frac{1}{1+t^2}\cdot\frac{p}{p^2+t^2}dt.$$

Now use partial fraction decomposition on the integrand, viewing it as a function of t. Make the guess

$$\frac{1}{1+t^2}\cdot\frac{p}{p^2+t^2} = \frac{At+B}{1+t^2} + \frac{Ct+D}{p^2+t^2}.$$

Multiply by the denominator to find

$$p = (At+B)\left(p^2+t^2\right) + (Ct+D)\left(1+t^2\right) = (A+C)t^3 + (B+D)t^2 + \left(Ap^2+C\right)t + \left(Bp^2+D\right).$$

The requirement that the cubic and linear coefficients must be 0 gives a system of 2 equations that can be solved to find $A = C = 0$. Since the linear coefficient must be 0, we see $D = -B$. Lastly, matching the constant terms gives

$$p = B(p^2-1) \implies B = \frac{p}{p^2-1}.$$

Using our partial fraction decomposition, we can simplify further:

$$L[f(x)] = \frac{p}{p^2-1}\int_0^\infty\left(\frac{1}{1+t^2} - \frac{1}{t^2+p^2}\right)dt = \frac{p}{p^2-1}\left[\arctan(t) - \frac{1}{p}\arctan\left(\frac{t}{p}\right)\right]_{t=0}^{t=\infty} = \frac{p}{p^2-1}\underbrace{\left(1-\frac{1}{p}\right)}_{=(p-1)/p}\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{1}{p+1}\right).$$

Lastly, reverse Laplace transform to find $f(x) = \frac{\pi}{2}e^{-x}$.