# Math 135 - Final Exam - Practice problems

*In addition to these problems, your review should include all previous homework assignments, and the midterm.*

## Laplace transforms

1. Use the Laplace transform to solve the initial value problem

$$\begin{cases} y'' + 2y' + 2y = xe^{-x} \\ y(0) = 0 \\ y'(0) = -1. \end{cases}$$

   **Solution:** Let $Y = \mathcal{L}[y]$. Taking the Laplace transform we obtain

$$\begin{aligned} p^2 Y + 1 + 2pY + 2Y &= \frac{1}{(p+1)^2} \\ (p^2 + 2p + 2)Y &= -\frac{p^2 + 2p}{(p+1)^2} \\ Y &= -\frac{p^2 + 2p}{(p+1)^2\left[(p+1)^2 + 1\right]} \\ Y &= \frac{1}{(p+1)^2} - \frac{2}{(p+1)^2 + 1}, \end{aligned}$$

   where the final line follows from partial fractions. We may then invert the Laplace transform to obtain

$$y(x) = xe^{-x} - 2e^{-x}\sin(x).$$

2. Let $f\colon [0, \infty) \to \mathbb{R}$ be a continuous function of exponential order satisfying $f(0) = 0$ and

$$\mathcal{L}\left[f'\right] = p\,\tfrac{d}{dp}\left(\tfrac{4p}{p^2+4}\right).$$

   Find $f(x)$.

   **Solution:** We compute that

$$\begin{aligned} p\mathcal{L}[f] &= p\,\tfrac{d}{dp}\left(\tfrac{4p}{p^2+4}\right) \\ \mathcal{L}[f] &= \tfrac{d}{dp}\mathcal{L}[4\cos(2x)] \\ \mathcal{L}[f] &= \mathcal{L}[-4x\cos(2x)]. \end{aligned}$$

   Inverting the Laplace transform we obtain

$$f(x) = -4x\cos(2x).$$

3. Let $f(x)$ be a continuous function of exponential order. Use the Laplace transform to find the solution of

$$\begin{cases} y'(x) - 2y(x) + 17\displaystyle\int_0^x y(t)\,dt = f(x), \\ y(0) = 0. \end{cases}$$

**Solution:** Let $Y = \mathcal{L}[y]$. Taking the Laplace transform we have

$$\begin{aligned} pY - 2Y + \frac{17}{p}Y &= \mathcal{L}[f] \\ Y &= \left[\tfrac{p-1}{(p-1)^2+16} + \tfrac{1}{(p-1)^2+16}\right]\mathcal{L}[f] \\ Y &= \mathcal{L}\left[e^x\cos(4x) + \tfrac{1}{4}e^x\sin(4x)\right]\mathcal{L}[f]. \end{aligned}$$

Inverting the Laplace transform we have

$$y(x) = \int_0^x e^t\left[\cos(4t) + \tfrac{1}{4}\sin(4t)\right]f(x-t)\,dt.$$

4. Use the Laplace transform to solve the differential equation

$$\begin{cases} xy'' + xy' + (3-2x)y = 0, \\ y(0) = 0. \end{cases}$$

**Solution:** Let $Y = \mathcal{L}[y]$. Taking the Laplace transform we have

$$-(p+2)(p-1)Y' - 2(p-1)Y = 0.$$

For $p > 1$ we then have

$$\begin{aligned} Y' &= -\frac{2}{p+2}Y \\ Y &= \frac{C}{(p+2)^2}, \end{aligned}$$

for some $C \in \mathbb{R}$. Inverting the Laplace transform we have

$$y(x) = Cxe^{-2x},$$

for some $C \in \mathbb{R}$.

## Existence and uniqueness

5. (a) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose there exists $M > 0$ so that $|h'(y)| \le M$ for all $y \in \mathbb{R}$. Show that $h$ is Lipschitz on $\mathbb{R}$.

   *Hint: Use the Mean Value Theorem.*

(b) Let $g(x)$ be defined on $\mathbb{R}$, and let $h(y)$ be Lipschitz on $\mathbb{R}$. Show that the function

$$f(x, y) = g(x) + h(y)$$

is Lipschitz in $y$ on $\mathbb{R} \times \mathbb{R}$.

**Solution:**

(a) Note that as $h$ is differentiable it is continuous. Given $y < z$ we may then apply the Mean Value Theorem to find some $w$ so that

$$h(y) - h(z) = h'(w)(y - z),$$

so taking absolute values and using that $|f'(w)| \leq M$ we have

$$|h(y) - h(z)| \leq |h'(w)||y - z| \leq M|y - z|.$$

An identical argument applies if $y > z$ and the case $y = z$ is vacuous. Consequently,

$$|h(y) - h(z)| \leq M|y - z|$$

for all $y, z \in \mathbb{R}$, so $f$ is Lipschitz on $\mathbb{R}$.

(b) As $h$ is Lipschitz on $\mathbb{R}$ there exists some constant $M > 0$ so that

$$|h(y) - h(z)| \leq M|y - z| \quad \text{for all} \quad y, z \in \mathbb{R}.$$

Given $x \in [a, b]$ and $y, z \in \mathbb{R}$ we then have

$$|f(x, y) - f(x, z)| = |h(y) - h(z)| \leq M|y - z|,$$

so $f$ is Lipschitz in $y$ on $[a, b] \times \mathbb{R}$.

6. Consider the differential equation

$$(*) \qquad \begin{cases} y''' + x^2y^2 + |y''| + e^x = 0, \\ y(0) = y'(0) = y''(0) = 1. \end{cases}$$

(a) Write the differential equation $(*)$ as a first order ODE system.

(b) Show that the differential equation $(*)$ has a unique local solution.

**Solution:**

(a) We introduce

$$\mathbf{z} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix},$$

and denoting the components of $\mathbf{z}$ by $z_j$ we take

$$\mathbf{f}(x, \mathbf{z}) = \begin{bmatrix} z_2 \\ z_3 \\ -x^2z_1^2 - |z_3| - e^x \end{bmatrix},$$

to obtain the first order equation

$$\begin{cases} \mathbf{z}' = \mathbf{f}(x, \mathbf{z}) \\ \mathbf{z}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{cases}$$

(b) The function $f(x, \mathbf{z})$ is continuously differentiable on the box

$$\mathcal{D} = [-1, 1] \times [\tfrac{1}{2}, \tfrac{3}{2}]^3$$

From our theorem in class, as $\mathcal{D}$ is closed and bounded, this suffices to show that $f(x, \mathbf{z})$ is continuous and Lipschitz in $\mathbf{z}$ on $\mathcal{D}$. We may then apply Picard's Theorem to obtain a unique local solution of $(*)$.

## Fourier series

7. Let $f(x) = x\mathbb{1}_{[0,\frac{\pi}{2}]}(x)$ for $x \in [-\pi, \pi]$ and let

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big\{a_n\cos(nx) + b_n\sin(nx)\Big\}$$

be the Fourier series for $f(x)$.

(a) Compute

$$a_2, a_6, \ldots, a_{4k+2}, \ldots, \qquad b_1, b_5, \ldots, b_{4k+1}, \ldots.$$

(b) Are there any $-\pi < x < \pi$ for which

$$f(x) \neq \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big\{a_n\cos(nx) + b_n\sin(nx)\Big\} \ ?$$

For each such $x$, to what value does the Fourier series converge?

<u>Remark:</u> *You may wish to use that the conclusion of Dirichlet's Theorem for the convergence of Fourier series applies to piecewise smooth functions.*

**Solution:**

(a) For $n \geq 1$ we have

$$\begin{aligned}
a_n &= \tfrac{1}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx \\
&= \tfrac{1}{n\pi}\big[x\sin(nx)\big]_0^{\frac{\pi}{2}} - \tfrac{1}{n\pi}\int_0^{\frac{\pi}{2}}\sin(nx)\,dx \\
&= \tfrac{1}{2n}\sin(n\tfrac{\pi}{2}) + \tfrac{1}{n^2\pi}\big[\cos(nx)\big]_0^{\frac{\pi}{2}} \\
&= \tfrac{1}{2n}\sin(n\tfrac{\pi}{2}) + \tfrac{1}{n^2\pi}\cos(n\tfrac{\pi}{2}) - \tfrac{1}{n^2\pi}.
\end{aligned}$$

In particular, if we take $n = 4k + 2$ we obtain

$$a_{4k+2} = -\tfrac{1}{2\pi(2k+1)^2}.$$

Similarly, for $n \geq 1$ we have

$$\begin{aligned}
b_n &= \tfrac{1}{\pi}\int_0^{\frac{\pi}{2}} x\sin(nx)\,dx \\
&= -\tfrac{1}{n\pi}\big[x\cos(nx)\big]_0^{\frac{\pi}{2}} + \tfrac{1}{n\pi}\int_0^{\frac{\pi}{2}}\cos(nx)\,dx \\
&= -\tfrac{1}{2n}\cos(n\tfrac{\pi}{2}) + \tfrac{1}{n^2\pi}\big[\sin(nx)\big]_0^{\frac{\pi}{2}} \\
&= -\tfrac{1}{2n}\cos(n\tfrac{\pi}{2}) + \tfrac{1}{n^2\pi}\sin(n\tfrac{\pi}{2}),
\end{aligned}$$

so taking $n = 4k+1$ we obtain

$$b_{4k+1} = \tfrac{1}{(4k+1)^2\pi}.$$

(b) As $f(x)$ is piecewise smooth, the remark ensures that the Fourier series converges to

$$\tfrac{1}{2}\big[f(x-) + f(x+)\big]$$

for all $-\pi < x < \pi$. As $f(x)$ is continuous on $(-\pi, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$, the Fourier series converges to $f(x)$ at every $x \in (-\pi, \pi)$ *except* $x = \frac{\pi}{2}$. Here it converges to

$$\tfrac{1}{2}\big[f(\tfrac{\pi}{2}-) + f(\tfrac{\pi}{2}+)\big] = \tfrac{\pi}{4}.$$

8. Consider the function $f\colon [0,\pi] \to \mathbb{R}$,

$$f(x) = \begin{cases} 3 & \text{for } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{for } x = \frac{\pi}{2} \\ -1 & \text{for } \frac{\pi}{2} < x \leq \pi \end{cases}$$

(a) Show that the Fourier cosine series of $f(x)$ can be written as

$$\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos\big((2n-1)x\big)$$

and determine $\lambda_n \in \mathbb{R}$ for $n = 0, 1, 2, \ldots$.

(b) Draw the graph of the Fourier cosine series for $x \in [-\pi, \pi]$.

(c) Draw the graph of the Fourier sine series for $x \in [-\pi, \pi]$.

**Solution:**

(a) We start by writing the Fourier cosine seres as

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \tfrac{2}{\pi}\int_0^{\frac{\pi}{2}} 3\,dx - \tfrac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} dx = 2$$

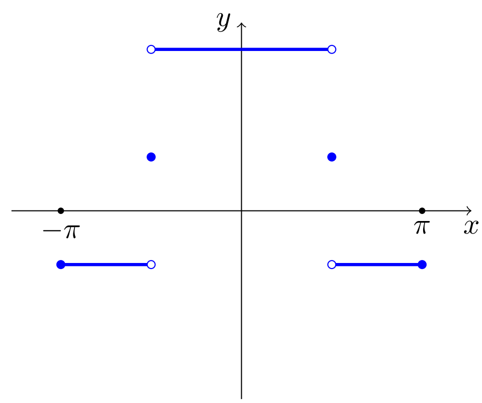
and for $n \geq 1$ we have

$$\begin{aligned} a_n &= \tfrac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx \\ &= \tfrac{2}{\pi} \int_0^{\frac{\pi}{2}} 3\cos(nx)\, dx - \tfrac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \cos(nx)\, dx \\ &= \tfrac{4}{n\pi} \sin(n\tfrac{\pi}{2}), \end{aligned}$$

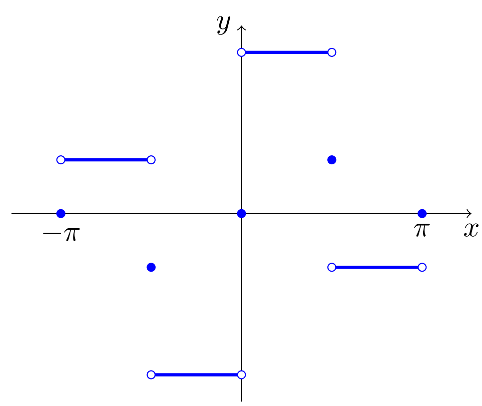
and hence we may take

$$\lambda_0 = 1 \quad \text{and} \quad \lambda_n = \tfrac{4}{(2n-1)\pi}(-1)^{n+1}$$

(b) We sketch




(c) We sketch




## PDEs & boundary value problems

9. Find a solution $y(t,x)$, for $t \geq 0$ and $0 \leq x \leq \pi$, to the boundary value problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} - y, \\ y(t,0) = 0 = y(t,\pi), \\ \dfrac{\partial y}{\partial t}(0,x) = 0, \;\; y(0,x) = f(x) \quad \text{for} \quad 0 < x < \pi. \end{cases}$$

using the method of separation of variables.

<u>Remark:</u> *You may use without justification that the Sturm–Liouville problem*

$$\begin{cases} u'' + \lambda u = 0, \\ u(0) = 0 = u(\pi) \end{cases}$$

*has eigenvalues* $\lambda_n = n^2$ *and eigenfunctions* $u_n(x) = \sin(nx)$ *for integers* $n \geq 1$.

**Solution:** We start by seeking a solution in the form

$$y(t, x) = v(t)u(x).$$

This yields the equation

$$\frac{v''}{v} = \frac{u''}{u} - 1.$$

As the left hand side depends only on $t$ and the right hand side depends only on $x$ they are both equal to a constant, say $-\lambda$. This yields the ODEs

$$u'' + (\lambda - 1)u = 0, \tag{1}$$

$$v'' + \lambda v = 0. \tag{2}$$

The boundary conditions yield the conditions

$$u(0) = 0 = u(\pi),$$

so from class we know that the corresponding eigenvalues for the Sturm–Liouville problem (1) are

$$\lambda_n = 1 + n^2 \quad \text{for integers} \quad n \geq 1,$$

with corresponding eigenfunctions

$$u_n(x) = \sin(nx).$$

Turning to the equation for $v$, the initial condition

$$\frac{\partial y}{\partial t}(0, x) = 0$$

yields the condition

$$v'(0) = 0,$$

and a non-trivial solution of (2) with $\lambda = \lambda_n$ satisfying this condition is

$$v_n(t) = \cos\big(\sqrt{1 + n^2}\, t\big).$$

Putting these together, for each integer $n \geq 1$ we have a solution of

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2} - y \\ y(t, 0) = 0 = y(t, \pi) \\ \dfrac{\partial y}{\partial t}(0, x) = 0 \end{cases}$$

given by

$$y_n(t,x) = \cos\big(\sqrt{1+n^2}\,t\big)\sin\big(nx\big).$$

We may then take linear combinations to obtain a solution (ignoring convergence)

$$y(t,x) = \sum_{n=1}^{\infty} b_n \cos\big(\sqrt{1+n^2}\,t\big)\sin\big(nx\big),$$

for any constants $b_1, b_2, \ldots$.

Utilizing the remaining initial condition we see that

$$f(x) = y(0,x) = \sum_{n=1}^{\infty} b_n \sin\big(nx\big),$$

so if we take

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx,$$

then the series

$$y(t,x) = \sum_{n=1}^{\infty} b_n \cos\big(\sqrt{1+n^2}\,t\big)\sin\big(nx\big),$$

yields a (formal) solution of our problem.

10. Determine the eigenfunctions $y_n(x)$ and eigenvalues $\lambda_n$ of the boundary value problem

$$\begin{cases} -y'' - 4y = \lambda y, \\ y'(0) = 0 = y'(1). \end{cases}$$

**Solution:** We first write our problem in the form.

$$y'' + (4+\lambda)y = 0.$$

If $\lambda < -4$, the general solution is given by

$$y(x) = C_1 e^{\sqrt{-(4+\lambda)}x} + C_2 e^{-\sqrt{-(4+\lambda)}x},$$

and the only solution satisfying the boundary conditions is $y(x) = 0$.

If $\lambda = -4$, the general solution is given by

$$y(x) = C_1 + C_2 x,$$

and hence, taking $C_2 = 0$ we obtain a non-trivial solution satisfying the boundary conditions. Consequently, our ground state energy is $\lambda_1 = -4$ and the corresponding ground state is

$$y_1(x) = 1.$$

If $\lambda > -4$, the general solutions is given by

$$y(x) = C_1 \cos\big(\sqrt{4+\lambda}\,x\big) + C_2 \sin\big(\sqrt{4+\lambda}\,x\big).$$

Using the boundary conditions, we have a non-trivial solution if and only if $\sqrt{4+\lambda} = n\pi$ for an integer $n \geq 1$. This leads us to take

$$\lambda_n = (n-1)^2\pi^2 - 4 \quad \text{for} \quad n \geq 2,$$

with corresponding eigenfunctions

$$y_n(x) = \cos\big((n-1)\pi x\big).$$