UCLA MATH 135, WINTER 2022, MIDTERM EXAM 1 SOLUTIONS

Students MUST COPY AND SIGN the following honor pledge AT THE TOP of the paper submitted for their exam solutions or else receive a failing grade by department policy:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this exam.

Print name: Sign name:

This is an open-book and open-note examination. Please show all your work. Partial credit will be given to partial answers. There are 1+4 = 5 problems for a total of 40 points. Test is designed to be completed in 1 hour. Posted: January 28, 8:00am. To be completed and uploaded by January 29, 7:59am (Pacific standard times). Between these times, as much time as desired is allowed to work on the exam.

Please note that there is a table of common Laplace transforms that is posted on Canvas along with this exam. These may be of use for the problems below.

[0] ∞ *points*:

Did you sign the honor pledge? If not, turn back right now to the cover page and sign it.

[**1**] *Short answer questions, 2 points each for 10 points total*:

(a) *True or false?* If $L[f(x)] = F(p)$, then

$$L[f(\alpha x)] = \alpha F\left(\frac{p}{\alpha}\right)$$

for $\alpha > 0$. You do not need to justify your answer.

False. The correct scaling is $L[f(\alpha x)] = F(p/\alpha)/\alpha$.

(b) *True or false?* For all $x \geq 1$, the function $f(x) = \exp(\sqrt{x}/2)$ is of exponential order. You do not need to justify your answer.

True; $\sqrt{x}$ grows more slowly than x, and e^u is monotonic.

(c) Compute the Laplace transform $L[f]$ for $f(x) = x\ \sin(3x)$.

We know $L[xf(x)] = -F'(p)$, so that

$$\begin{aligned} L[x \sin(3x)] &= -\frac{d}{dp}L[\sin(3x)] \\ &= -\frac{d}{dp}\left(\frac{3}{p^2+9}\right) \\ &= \frac{6p}{(p^2+9)^2} \end{aligned}$$

(d) Suppose $h \in C^3$. What is the Laplace transform of

$$\frac{d^3h}{dx^3}?$$

$p^3H(p) - p^2h(0) - ph'(0) - h''(0)$.

(e) Suppose that $r = \pm 2i$ are the roots of the characteristic equation. What is the differential equation this corresponds to, and what is the general solution?

The ODE is $y'' + 4y = 0$, and the general solution is a superposition of $\sin(2x)$ and $\cos(2x)$.

Note: a superposition of $e^{\pm 2ix}$ is also acceptable.

[2] *10 points*: Suppose that $\exists M, c > 0$ such that

$$|f(x)| \leq Me^{cx}, \qquad \forall x$$

i.e. that f is of exponential order on the real line. Show that

$$\lim_{p \to \infty} \left| p\, F(p) \right| < \infty$$

must be finite, where $F(p)$ denotes the Laplace transform of f.

By definition,

$$\begin{aligned}
\lim_{p \to \infty} \left| p\, F(p) \right| &= \lim_{p \to \infty} \left| p \int_0^\infty e^{-px} f(x)\, dx \right| \\
&\leq \lim_{p \to \infty} p \int_0^\infty e^{-px} \left| f(x) \right| \, dx \\
&\leq \lim_{p \to \infty} p \int_0^\infty M e^{-px} e^{cx}\, dx \\
&= M \lim_{p \to \infty} \left[\frac{p}{c-p} \left(\underbrace{\lim_{x \to \infty} e^{(c-p)x}}_{=0} - 1 \right) \right] \\
&= M \lim_{p \to \infty} \left[\frac{p}{p-c} \right] \\
&= M < \infty
\end{aligned}$$

[3] *10 points*: Compute the Laplace transform of the square-wave function $s(x)$ given by

$$s(x) = \begin{cases} 1 & x \in [0,\pi) \cup [2\pi, 3\pi) \cup [4\pi, 5\pi) \cup \dots \\ 0 & x \in [\pi, 2\pi) \cup [3\pi, 4\pi) \cup [5\pi, 6\pi) \cup \dots \end{cases}$$

On HW2, Q7 we showed

$$L[s] = \frac{1}{1 - e^{-2\pi p}} \int_0^{2\pi} e^{-px} s(x)\, dx.$$

Since

$$\int_0^{2\pi} e^{-px} s(x)\, dx = \int_0^{\pi} e^{-px}\, dx = \frac{1}{p}(1 - e^{-\pi p})$$

we have in total

$$L[s] = \frac{1}{p}\left(\frac{1 - e^{-\pi p}}{1 - e^{-2\pi p}}\right)$$

[4] *10 points*: In class we loosely defined the Dirac delta distribution $\delta(x)$ to be the 'function' that was infinite at $x = 0$ and zero everywhere else, and we took as a more formal definition $\delta(x)$ to be the function such that

$$L[\delta(x)] = 1.$$

Compute the solution to the IVP given by

$$\begin{cases} y''(x) + 25y(x) = f(x) \\ y(0) = 1, y'(0) = 0 \end{cases}$$

where $f(x) = \delta(x - 3)$.

Note that $L[\delta(x-3)] = e^{-3p}$ by the shifting property of L. Taking L on both sides of the ODE gives

$$(p^2 + 25)Y(p) - p = e^{-3p} \implies Y(p) = \frac{p}{p^2+25} + \left(\frac{1}{p^2+25}\right) e^{-3p}.$$

Since the inverse transform L^{-1} is linear and we know

$$L[\cos(\alpha x)] = \frac{p}{p^2+\alpha^2},$$

we can say

$$y(x) = \cos(5x) + L^{-1}\left[\left(\frac{1}{p^2+25}\right) e^{-3p}\right].$$

To compute the second inverse transform, we first consider the transform of the function $H_\gamma(x)$ which equals 1 for $x \in [\gamma, \infty)$ and equals 0 for $x \in [0, \gamma)$ (this is just a simple step function which 'steps up' at $x = \gamma$). It has Laplace transform

$$L[H_\gamma] = \int_0^\infty e^{-px} H_\gamma(x)\, dx = \int_\gamma^\infty e^{-px}\, dx = \frac{1}{p} e^{-\gamma p}.$$

We also know that $L[\sin(5x)] = 5/(p^2 + 25) \implies$

$$L\left[\frac{1}{5}\sin(5x)\right] = \frac{1}{(p^2+25)}. \tag{1}$$

Since we know $L[1] = 1/p$ and we just showed that $L[1\, H_\gamma(x)] = 1/p\, e^{-\gamma p}$, this suggests that we could try to compute $L[1/5 \sin\big(5(x-\gamma)\big) H_\gamma(x)]$ for $\gamma = 3$. Trying it out gives:

$$L\left[\frac{1}{5}\sin\big(5(x-3)\big)H_3(x)\right] = \int_3^\infty \frac{1}{5}\sin\big(5(x-3)\big)e^{-px}\, dx = \int_0^\infty \frac{1}{5}\sin(5u)e^{-3p}e^{-pu}\, du = e^{-3p} L\left[\frac{1}{5}\sin(5u)\right]$$

where we used a simple u-substitution $u = x - 3$. Combining this with (1) then gives that

$$L^{-1}\left[\left(\frac{1}{p^2+25}\right) e^{-3p}\right] = \frac{1}{5}\sin\big(5(x-3)\big)H_3(x).$$

Putting it all together gives:

$$y(x) = \begin{cases} \cos(5x), & x \in [0, 3) \\ \cos(5x) + \frac{1}{5}\sin\big(5(x-3)\big), & x \geq 3 \end{cases}$$