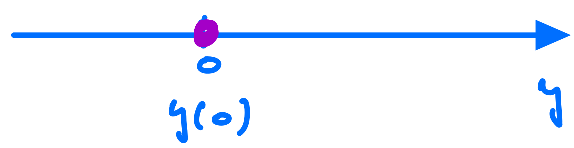
Lecture 13

Let's assume we have a mechanical/electrical system at rest subject to an external force $f(t)$

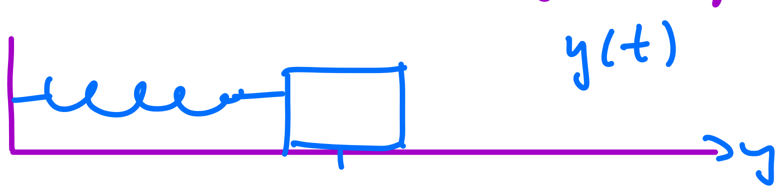
$$y'' + ay' + by = f(t), \qquad t \geq 0$$

$$\left.\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}\right\} \text{ at rest ICs.}$$



The external force could be a sudden hammer blow in the mechanical system or a lightning stroke on a transmission line.

So, our external force can be a very irregular function.

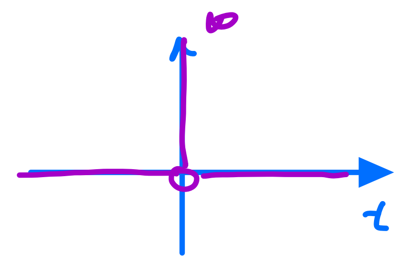


Dirac Delta "distribution"
function

Def (Dirac Delta function)

– P. Dirac
– L. Schwartz 1950's

The Dirac Delta function (distribution) is charactetized by the following two properties:



(1) $$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \infty, & \text{if } t = 0 \end{cases}$$

(2) $$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

for any function that is cont. on an open interval containing 0.

Remark:

a) By property (2), for $a \geq 0$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-a)\,dt = \int_{-\infty}^{\infty} f(u+a)\,\delta(u)\,du$$
$$= f(a)$$

$u = t-a \to t = u+a$

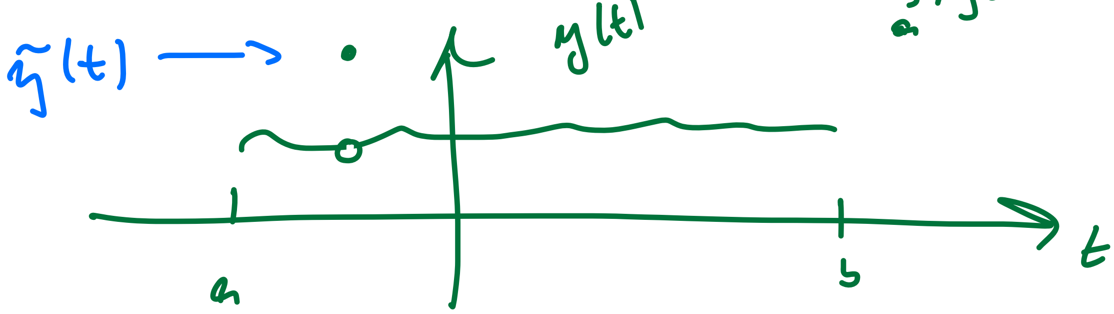
$du = dt$

b) let's take $f(t) \equiv 1$

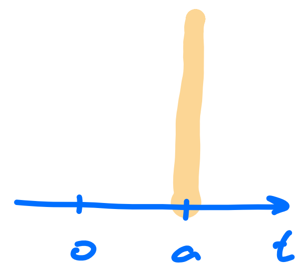
$$\int_{-\infty}^{\infty} 1\, \delta(t)\,dt = 1$$

so $$\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1}$$

$$\int_a^b y(t)\,dt$$



c) L.T. of $\delta(t-a)$ for $a \geq 0$?



$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^{\infty} e^{-st}\,\delta(t-a)\,dt$$

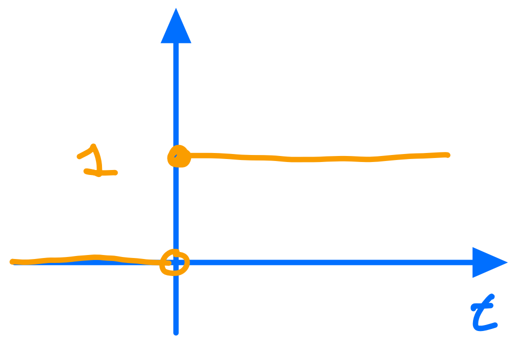
$$= \int_{-\infty}^{\infty} e^{-st}\,\delta(t-a)\,dt$$

$$= e^{-sa}$$

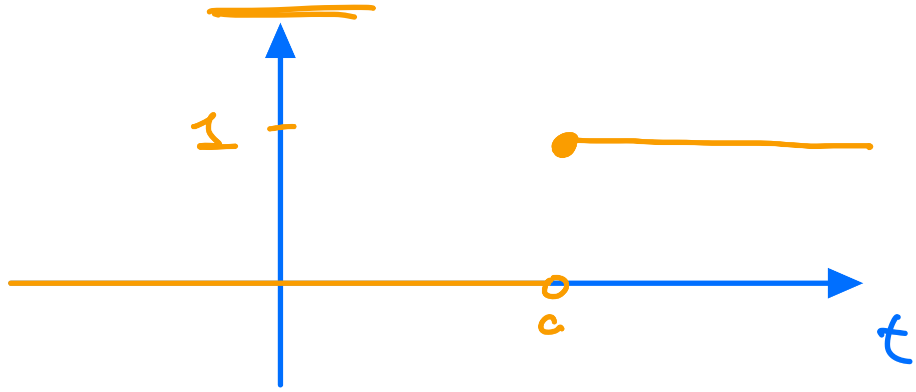
Unit step function

The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



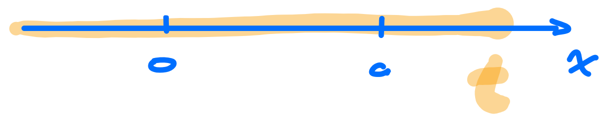
Plot $u(t-a)$ where $a > 0$.



Note that

$$\mathcal{L}\{u(t)\}(s) = \int_0^\infty e^{-st} u(t)\, dt = \frac{1}{s}, \quad s > 0$$

Remark:

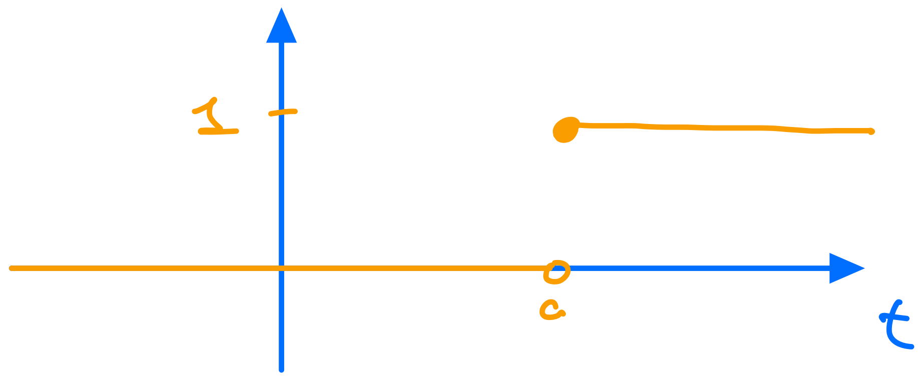


$$\int_{-\infty}^{t} \delta(x-a)\,dx = \begin{cases} 0, \text{ if } t<a \\ 1, \text{ if } t \geq a \end{cases}$$

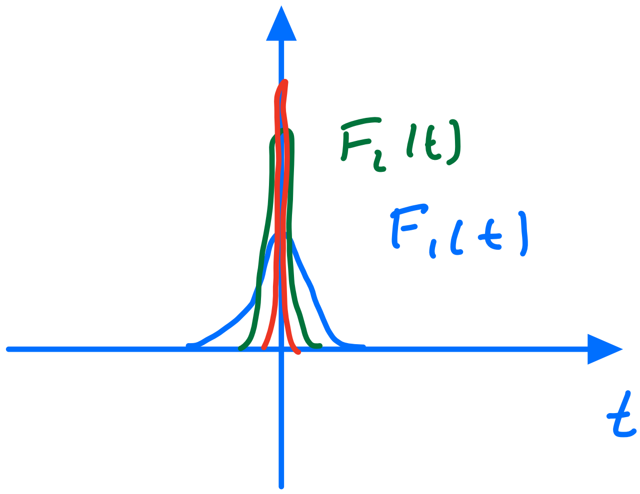
$$= u(t-a)$$

der. w.r.t. time

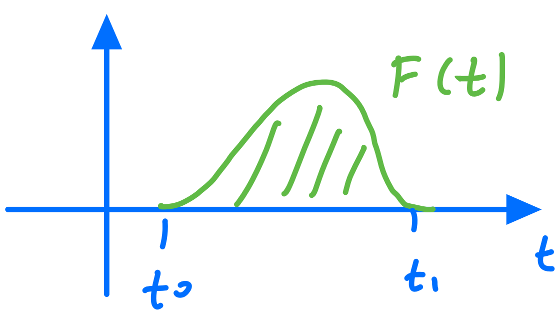
$$\delta(t-a) = u'(t-a)$$



So, the derivative of the step. funct. is the Dirac Delta distribution!!



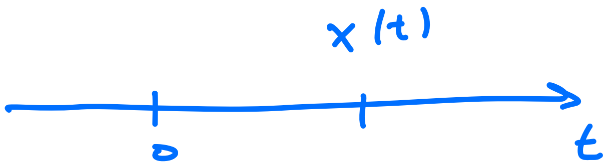
$$\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(t)\,dt}$$

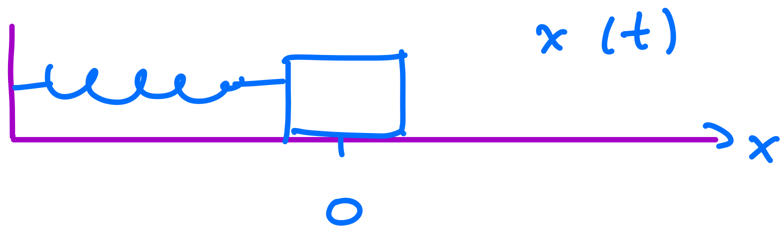


$$\text{Impulse} = \int_{t_0}^{t_1} F(t)\,dt$$

$$= \int_{t_0}^{t_1} m\dot{v}(t)\,dt$$

$$= mv(t_1) - mv(t_0)$$





$$\begin{cases} x''(t) + x(t) = \delta(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases} \qquad x'' = -x$$

Sol. Taking the L.T.

$$(s^2+1)X(s) = 1$$

$$X(s) = \frac{1}{s^2+1}$$

$$\rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin(t)$$

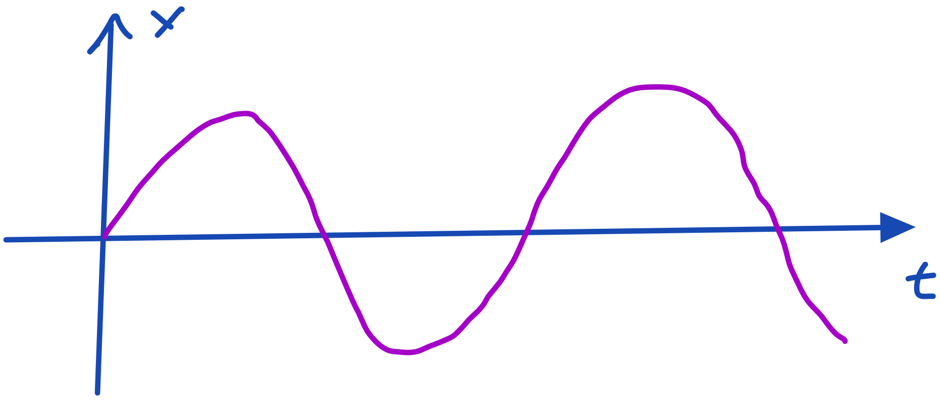
Remark: $x(0) = 0$

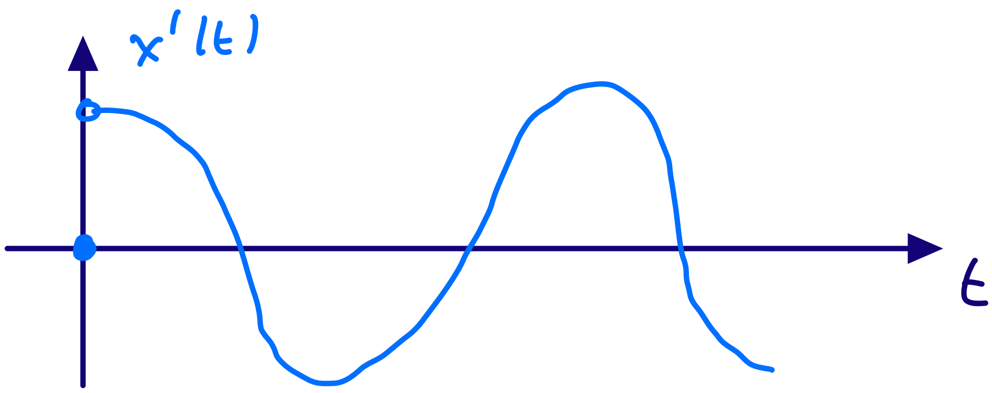
$$x'(0) = \cos(t)\big|_{t=0} = 1 \neq 0$$

momentum goes from $m x'(0) = 0$

$$\downarrow$$

$$\lim_{t \to 0^+} m\, x'(t) = 1$$

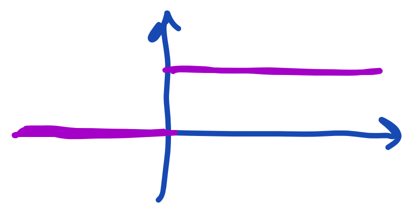




→ Now let's consider the 2nd order Lin D.E

$$\begin{cases} A''(t) + aA'(t) + bA(t) = u(t) \\ A(0) = 0 \\ A'(0) = 0 \end{cases}$$

↑ step function



$a, b \in \mathbb{R}$.

Applying the L.T. we get

$$s^2 \mathcal{L}\{A\}(s) + a s \mathcal{L}\{A\}(s) + b \mathcal{L}\{A\}(s)$$
$$= \mathcal{L}\{u(t)\}(s) = \frac{1}{s}$$

$$\to \mathcal{L}\{A\}(s) = \frac{1}{s} \frac{1}{s^2 + as + b}$$

The function A is called "indicial response" and it will help us to solve the general problem

$$\begin{cases} y''(t) + a y'(t) + b y(t) = f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$s^2 \mathcal{L}\{y\}(s) + a s \mathcal{L}\{y\}(s) + b \mathcal{L}\{y\}(s) = \mathcal{L}\{f\}(s)$$

$$\rightarrow \mathcal{L}\{y\}(s) = \frac{\mathcal{L}\{f\}(s)}{s^2 + as + b}$$

$$\rightarrow \frac{1}{s}\mathcal{L}\{y\}(s) = \frac{1}{s}\,\frac{1}{s^2+as+b}\,\mathcal{L}\{f\}(s)$$

$$= \mathcal{L}\{A\}(s)\,\mathcal{L}\{f\}(s)$$

$$= \mathcal{L}\{(A * f)(t)\}(s)$$

$$= \mathcal{L}\{(f * A)(t)\}(s)$$

$$= \mathcal{L}\left\{\int_0^t f(\tau) A(t-\tau)\, d\tau\right\}(s)$$

$$\rightarrow \mathcal{L}\{y\}(s) = s\,\mathcal{L}\left\{\int_0^t f(\tau) A(t-\tau)\, d\tau\right\}(s)$$

$$\left[\mathcal{L}\{h'(t)\}(s) = s\mathcal{L}\{h(t)\}(s) - h(0)\right]$$

$$\rightarrow \mathcal{L}\{y\}(s) = \mathcal{L}\left\{\frac{d}{dt}\int_0^t f(\tau)A(t-\tau)\,d\tau\right\}(s)$$

$$\rightarrow y(t) = \frac{d}{dt}\int_0^t f(\tau)A(t-\tau)\,d\tau$$

Leibniz's rule

$$F(t) = \int_{u(t)}^{v(t)} G(t,x)\,dx$$

$$\frac{d}{dt}F(t) = G(t,v(t))\,v'(t) - G(t,u(t))\,u'(t) + \int_u^v \partial_t G(t,x)\,dx$$

$$y(t) = f(t)A(t-t) + \int_0^t \partial_t\big(f(\tau)A(t-\tau)\big)\,d\tau$$

$$= f(t)A(0) + \int_0^t f(\tau)A'(t-\tau)\,d\tau$$

or

~~typo~~

$$y(t) = \int_0^t A(t-\tau)f'(\tau)\,d\tau + f(0)A(t)$$

Example: Solve

$$\begin{cases} y'' - y' - 6y = \underline{\underline{2e^{3t}}} & f(t) \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Sol. First step: get the "indicial response" function $A(t)$

$$\mathcal{L}\{A\}(s) = \frac{1}{s(s^2+s-6)}$$

$1^2 - 4(-6) = 25 > 0$

Applying part. fraction decomp.

$$A(t) = -\frac{1}{6} + \frac{1}{15}e^{-3t} + \frac{1}{10}e^{2t}$$

$$\boxed{y(t) = \int_0^t A(t-\tau) f'(\tau)\, d\tau + f(0) A(t)}$$

$$y(t) = \int_0^t \left[-\frac{1}{6} + \frac{1}{15} e^{-3(t-\tau)} + \frac{1}{10} e^{2(t-\tau)} \right] \times \underline{6 e^{3\tau}}\, d\tau$$

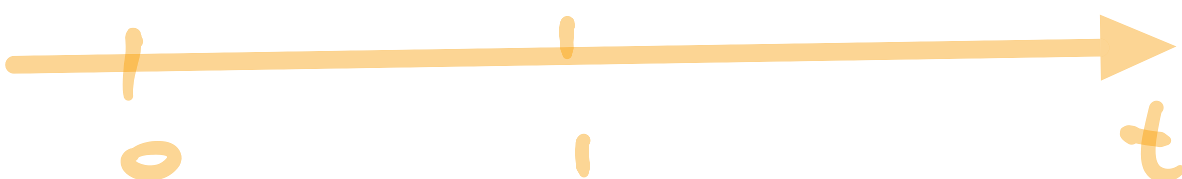
$$+\,2\left[-\frac{1}{6}+\frac{1}{15}e^{-3t}+\frac{1}{10}e^{2t}\right]$$

$$=\frac{1}{3}e^{3t}+\frac{1}{15}e^{-3t}-\frac{2}{5}e^{2t}$$

Finally:

$$f(t)=\begin{cases}0, & 0<t\le 1\\ e^{t}, & t>1\end{cases}$$

$$\mathcal{L}\{f(t)\}(s)=\int_0^\infty e^{-st}f(t)\,dt$$



Final Remark:

There are entire books just devoted to the Laplace Transform. Here we have just scratched the surface.

- If you would like to see more examples of partial fraction decomposition consult the book

Fundamentals of DE and boundary value problems by Nagle, Saff and Snider