Lecture 12

Today: i) Integration of the L.T.
ii) More about I.V.P.
iii) Convolutions
iv) Integral Eqns.

Integration of the L.T.

Theorem: If f is P.W. cont. on $[0,\infty)$ and of exp. order α with $F(s) = \mathcal{L}\{f(t)\}(s)$ and $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists, then

$$\int_s^\infty F(x)\,dx = \mathcal{L}\left\{\frac{f(t)}{t}\right\}(s), \quad s>\alpha.$$

Proof: $F(x) = \int_0^\infty e^{-xt} f(t)\,dt$

Integrate both sides

$$\int_s^\infty F(x)\,dx = \int_s^\infty \int_0^\infty e^{-xt} f(t)\,dt\,dx$$

$$= \lim_{\omega \to \infty} \int_s^{\omega} \int_0^{\infty} e^{-xt} f(t)\, dt\, dx$$

$$= \lim_{\omega \to \infty} \int_0^{\infty} \left(\int_s^{\omega} e^{-xt} f(t)\, dx \right) dt$$

$$= \lim_{\omega \to \infty} \int_0^{\infty} \left(f(t) \frac{e^{-xt}}{-t} \Bigg|_{x=s}^{x=\omega} \right) dt$$

$$= \int_0^{\infty} \left(e^{-st} \frac{f(t)}{t} \right) dt$$

$$\underbrace{- \lim_{\omega \to \infty} \int_0^{\infty} e^{-\omega t} \frac{f(t)}{t}\, dt}_{=0}$$

$$= \mathcal{L}\left\{ \frac{f(t)}{t} \right\}(s) \qquad \lim_{\omega \to \infty} \mathcal{L}\left\{ \frac{f(t)}{t} \right\}(\omega) = 0$$

So,

$$\int_s^{\infty} F(x)\, dx = \mathcal{L}\left\{ \frac{f(t)}{t} \right\}(s), \quad s > \alpha.$$

If $G(x,t) = e^{-xt} f(t)$ is cont. and if $\int_0^{\infty} G(x,t)\, dt$ converges unif in x, then

$$\int \int (\cdot)\, dx\, dt = \int \int (\cdots)\, dt\, dx$$

Note: If $G(x,t)=e^{-xt}f(t)$ is cont. and if

$\int_0^\infty G(x,t)\,dt$ converges unif $\forall x>0$

then

$$\int_s^\infty \int_0^\infty G(x,t)\,dt\,dx = \int_0^\infty \int_s^\infty G(x,t)\,dx\,ds$$

Summary: For $s>\alpha$

i) $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$

ii) $\mathcal{L}\{\int_0^t f(\tau)\,d\tau\}(s) = \frac{\mathcal{L}\{f\}(s)}{s}$ $\leftarrow$ HW

iii) $\int_s^\infty F(x)\,dx = \mathcal{L}\{\frac{f(t)}{t}\}(s)$

iv) $\frac{d^n F(s)}{ds^n} = \mathcal{L}\{(-1)^n t^n f(t)\}(s)$

Example:

Solve the following IVP

$$\begin{cases} w''(t) - 2w'(t) + 5w(t) = -8e^{\pi - t} \\ w(\pi) = 2 \\ w'(\pi) = 12 \end{cases}$$

Sol. Set $\boxed{y(t) = w(t+\pi)}$

$y(0) = 2$

$y'(0) = 12$

Replacing t by $t+\pi$ in the DE yields

$$w''(t+\pi) - 2w'(t+\pi) + 5w(t+\pi)$$
$$= -8e^{\pi-(t+\pi)}$$
$$= -8e^{-t}$$

$$\Rightarrow \quad y''(t) - 2y'(t) + 5y(t) = -8e^{-t}$$

$y(0) = 2$

$y'(0) = 12$

Using L.T we get

$$y(t) = 3e^{-t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t}$$

Since $w(t+\pi) = y(t)$ then

$$w(t) = y(t-\pi)$$

Thus

$$w(t) = 3e^{-(t-\pi)}\cos(2(t-\pi)) + 4e^{t-\pi}\sin(2(t-\pi)) - e^{-(t-\pi)}$$

$\mathcal{L}$. T. of a convolution

<u>Def.</u> Given f and g, denote the convolution of f and g is defined as

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

The integration domain could be $[0,\infty)$, $(-\infty,\infty)$:

$$\int_0^\infty f(\tau)g(t-\tau)\,d\tau$$

$$\int_{-\infty}^\infty f(\tau)g(t-\tau)\,d\tau$$

The integration domain will be clear from the context.

Properties:

i) Commutative $\quad f*g = g*f$

ii) Distribution $\quad (f*g)*h = f*(g*h)$

<u>homework</u>

Theorem: If f and g are P.W. cont. on $[0,\infty)$ and of exp. order α, then

i) $\mathcal{L}\{f*g\}(s) = (\mathcal{L}\{f\}(s))(\mathcal{L}\{g\}(s))$

$s>\alpha$

ii) $(f*g)(t) = \mathcal{L}^{-1}\{F(s)G(s)\}(t)$

where $F(s)=\mathcal{L}\{f(t)\}(s)$ and

$G(s)=\mathcal{L}\{g(t)\}(s)$.

<u>Proof</u>:

$$\mathcal{L}\{f(t)\}(s)\,\mathcal{L}\{g(t)\}(s)$$

$$= \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-su} g(u)\,du\right)$$

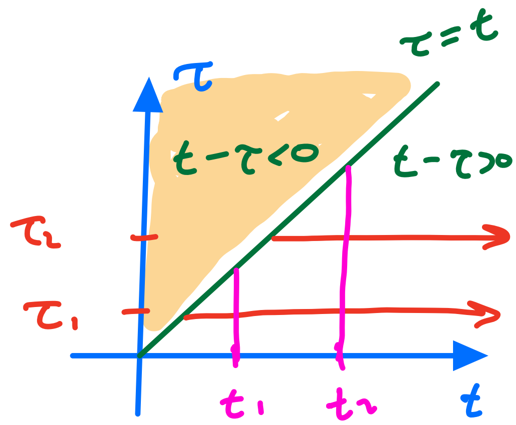
$$= \int_0^\infty \underbrace{\int_0^\infty e^{-s(\tau+u)} f(\tau)g(u)\,du}\,d\tau$$

Set $t=\tau+u \to u=t-\tau \to du=dt$

$u=0 \to t=\tau$

$$= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau) g(t-\tau)\, dt \, d\tau$$

$$= \int_0^\infty \int_0^t e^{-st} f(\tau) g(t-\tau)\, d\tau \, dt$$



$$= \int_0^\infty e^{-st} \left(\int_0^t f(\tau) g(t-\tau)\, d\tau \right) dt$$

$$= \int_0^\infty e^{-st} (f*g)(t)\, dt$$

$$= \mathcal{L}\{f*g\}(s)$$

So,

$$\mathcal{L}\{f(t)\}(s)\, \mathcal{L}\{g(t)\}(s) = \mathcal{L}\{f*g\}(s)$$

Example: Find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}(t)$$

Sol.

Notice that

$$\mathcal{L}\{t\}(s)=\frac{1}{s^2}=:F(s)$$

$$\mathcal{L}\{e^t\cdot 1\}(s)=\frac{1}{s-1}=:G(s)$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}(t)=(f*g)(t)$$

where, $f(t)=t$ and $g(t)=e^t$

$$(f*g)(t)=\int_0^t \tau e^{t-\tau}\,d\tau$$

$$=1-e^{-t}-te^{-t}$$