Fourier Series

- A function $f : [a, b] \to \mathbb{R}$ has **bounded variation** if $f = u - v$ for some bounded, non-decreasing functions u and v.
- If $f : [a, b] \to \mathbb{R}$ is bounded, piecewise continuous, and has finitely many local maxima and minima, then f has bounded variation. This accounts for most of the examples we will encounter.
- **Dirichlet's Conditions** (for a periodic function f):

  (i) f is (absolutely) integrable over a period.

  (ii) f has bounded variation over a period.
- **Dirichlet's Theorem:** If a periodic function f satisfies the Dirichlet Conditions, then there is a corresponding Fourier series that converges to $\frac{f(x^+)+f(x^-)}{2}$ for each x. Notice that if f is continuous at x, then $\frac{f(x^+)+f(x^-)}{2} = f(x)$.
- A **Fourier Series** is a sum of the form $\sum_{n=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ for some coefficients $a_n, b_n \in \mathbb{R}$.
- If a continuous periodic function f satisfies the Dirichlet conditions, then

$$f(x) = \sum_{n=0}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$$

for some coefficients. Suppose the period is 2π, although this process works for any period. If $n \neq 0$, Multiply the equation by $\cos(mx)$ for some $m \in \mathbb{N}$ and integrate over $[-\pi, \pi]$:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \sum_{n=0}^{\infty}\left(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx}_{=\pi\delta_{mn}} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx}_{=0}\right) = \pi a_m.$$

Dividing by π gives

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx.$$

If $n = 0$, this still works except that the surviving integral is $\int_{-\pi}^{\pi}\cos(0x)\cos(0x)dx = \int_{-\pi}^{\pi} 1dx = 2\pi$, so $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx$. Rather than remembering 2 separate formulas, we often use the for a_m formula for all $m \in \mathbb{N}$ and write the Fourier series as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

Similarly, we could instead multiply by $\sin(mx)$ and integrate to find

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx.$$

There is no need to worry about b_0 because $\sin(0x) = 0$.

Exercises

1. Prove that the periodic extension (over $[-\pi, \pi]$) of $f(x) = e^x$ has a Fourier series.

   Solution: Check the Dirichlet Conditions. First, f is absolutely integrable over a period because

   $$\int_{-\pi}^{\pi} |e^x| dx = \int_{-\pi}^{\pi} e^x dx = [e^x]_{x=-\pi}^{x=\pi} = e^{\pi} - e^{-\pi} < \infty.$$

   Second, e^x has bounded variation over a period because it is already and non-decreasing. Specifically, we can write it as $e^x = e^x - 0$. Then by Dirichlet's Theorem, there is a Fourier Series for the periodic extension of f.

2. Compute the Fourier series.

   Solution: We will need integration by parts:

   $$\int_{-\pi}^{\pi} e^x \cos(nx) dx = \underbrace{\left[e^x \frac{\cancel{\sin(nx)}}{n} + e^x \frac{\cos(nx)}{n^2} \right]_{x=-\pi}^{x=\pi}}_{= \frac{(-1)^n (e^{\pi} - e^{-\pi})}{n^2}} - \int_{-\pi}^{\pi} e^x \frac{\cos(nx)}{n^2} dx \implies \int_{-\pi}^{\pi} e^x \cos(nx) dx = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{n^2 + 1}.$$

   Dividing by π gives the cosine coefficients:

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(nx) dx = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(n^2 + 1)}.$$

   A similar calculation gives the sine coefficients:

   $$\int_{-\pi}^{\pi} e^x \sin(nx) dx = \underbrace{\left[e^x \frac{-\cos(nx)}{n} + e^x \frac{\cancel{\sin(nx)}}{n^2} \right]_{x=-\pi}^{x=\pi}}_{= \frac{-(-1)^n (e^{\pi} - e^{-\pi})}{n}} - \int_{-\pi}^{\pi} e^x \frac{\sin(nx)}{n^2} dx \implies \int_{-\pi}^{\pi} e^x \cos(nx) dx = \frac{-(-1)^n (e^{\pi} - e^{-\pi}) n}{n^2 + 1}.$$

   Lastly, divide by π :

   $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin(nx) dx = \frac{-(-1)^n (e^{\pi} - e^{-\pi}) n}{\pi(n^2 + 1)}.$$

   Then the Fourier series is

   $$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{e^{\pi} - e^{-\pi}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} (\cos(nx) - n \sin(nx)).$$

3. Use this Fourier series to compute

   $$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}.$$

   Hint: plug in $x = \pi$.

   Solution: As the hint suggests, plug in $x = \pi$ to the Fourier series. Since the periodic extension of f is discontinuous at $x = \pi$, Dirichlet's Theorem guarantees

   $$\begin{aligned}
   \frac{f(x^+) + f(x^-)}{2} &= \frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{e^{\pi} - e^{-\pi}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} (\underbrace{\cos(n\pi)}_{=(-1)^n} - n \cancel{\sin(n\pi)}) \\
   \frac{e^{\pi} + e^{-\pi}}{2} &= \frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{e^{\pi} - e^{-\pi}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \\
   \frac{e^{\pi} + e^{-\pi}}{2} - \frac{e^{\pi} - e^{-\pi}}{2\pi} &= \frac{e^{\pi} - e^{-\pi}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \\
   \frac{\pi}{2} \left(\frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} \right) - \frac{1}{2} &= \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}
   \end{aligned}$$

4. Repeat (1) and (2) with $f(x) = x^2$. Use the resulting Fourier series to show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.