Intro to Fourier Series

Fourier Analysis is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions.

Nowadays, Fourier Analysis can be considered a part of Harmonic Analysis (more about this later)

Fourier Analysis is a fundamental tool for applied mathematicians. It appears in:

- Signal processing
- Image analysis
- Image compression
- Sol. of PDEs
- ...

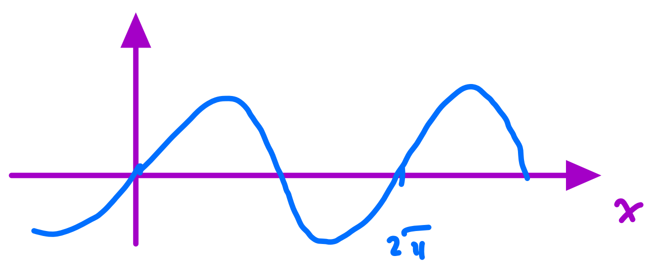
Recall: A function $f: \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number p such that

$$f(x+p) = f(x) \quad \forall x \in \mathbb{R}$$

Any such p is called the period of f.

Examples:

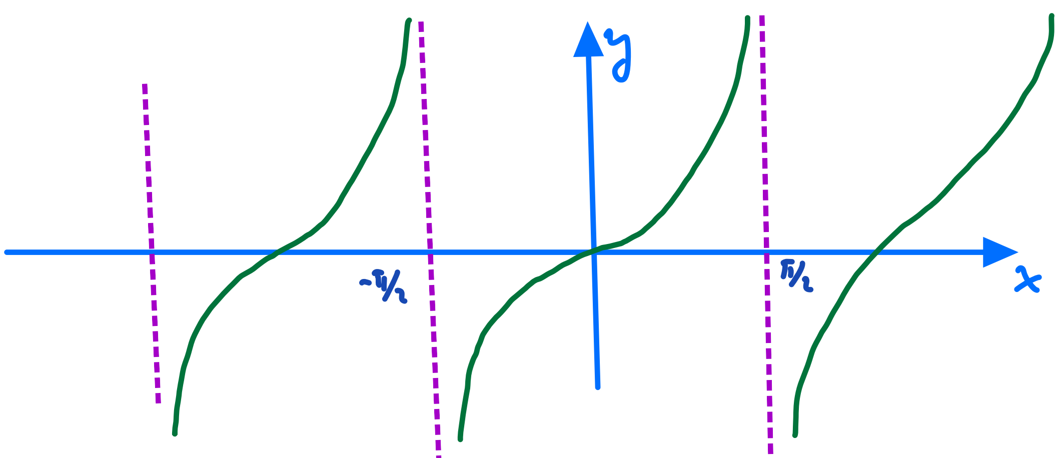
i) $f(x) = \sin(x)$



$p = 2\pi$

ii) $\cos(x)$

iii) $y(x) = \tan(x)$ period $= \pi$



Goal: Write a 2π-periodic function $f:[-\pi,\pi)\to\mathbb{R}$ as a sum of sines/cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

(Fourier series of f)

How do we compute the coefficients?
(let's assume that we have unif. convergence of the series)

$$\int_{-\pi}^{\pi} f(x)\,dx =$$

$$\int_{-\pi}^{\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]\right)dx$$

$$= \int_{-\pi}^{\pi}\frac{a_0}{2}\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos(nx)\,dx$$

$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n\sin(nx)\,dx$$

and let's recall the following trig. identities

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin nx\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\, dx = -\frac{\cos(nx)}{n}\Bigg|_{x=-\pi}^{x=\pi}$$

$$= -\frac{\cos(n\pi)}{n} + \frac{\cos(-n\pi)}{n}$$

$$= 0$$

So, let's get first a_0:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \frac{a_0}{2}(2\pi)$$

$$\rightarrow \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$ (the average of f)

Trig. identities:

$$\sin mx \cos nx = \frac{1}{2}[\sin (m+n)x + \sin (m-n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos (m+n)x + \cos (m-n)x],$$

$$\sin mx \sin nx = \frac{1}{2}[\cos (m-n)x - \cos (m+n)x],$$

Now let's get a_1:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

Let's recall the following identities:

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$ for all $m,n \geq 1$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad m \neq n.$$ for all $m,n \geq 1$ and $m \neq n$

Now

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = \int_{-\pi}^{\pi} \cancel{\frac{a_0}{2}\cos(x)\,dx}$$

$$+ \int_{-\pi}^{\pi} a_1 \cos(x)\cos(x)\,dx + \int_{-\pi}^{\pi} \cancel{a_2 \cos(2x)\cos(x)\,dx} + \cdots$$

$$+ \int_{-\pi}^{\pi} b_1 \cancel{\sin(x)\cos(x)\,dx} + \int_{-\pi}^{\pi} \cancel{b_2 \sin(2x)\cos(x)\,dx} + \cdots$$

$$\int_{-\pi}^{\pi} f(x)\cos(x)\,dx = \int_{-\pi}^{\pi} a_1 \cos(x)\cos(x)\,dx$$

$$= a_1 \int_{-\pi}^{\pi} \cos^2(x)\,dx$$

$$= a_1 \pi$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(x)\,dx$$

So, given a Riemann integrable function $f : [-\pi, \pi) \to \mathbb{R}$ we can always construct the Fourier series associated to f as:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

where

$$\begin{cases} a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, & n \geq 1 \\ b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin(nx)\, dx, & n \geq 1 \\ a_0 = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\, dx \end{cases}$$

Fourier coefficients

Remark a): If

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

then we could use

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \sin(nx) + b_n \cos(nx) \right]$$

as an approximation of $f(x)$. This is the starting point for numerical methods.

Remark b): Notice that

$$\frac{d}{dx} \sin(nx) = n \cos(nx)$$

$$\frac{d}{dx} \cos(nx) = -n \sin(nx)$$

so

$$f'(x) = \sum_{n=1}^{\infty} \left[a_n n \cos(nx) - b_n n \sin(nx) \right]$$

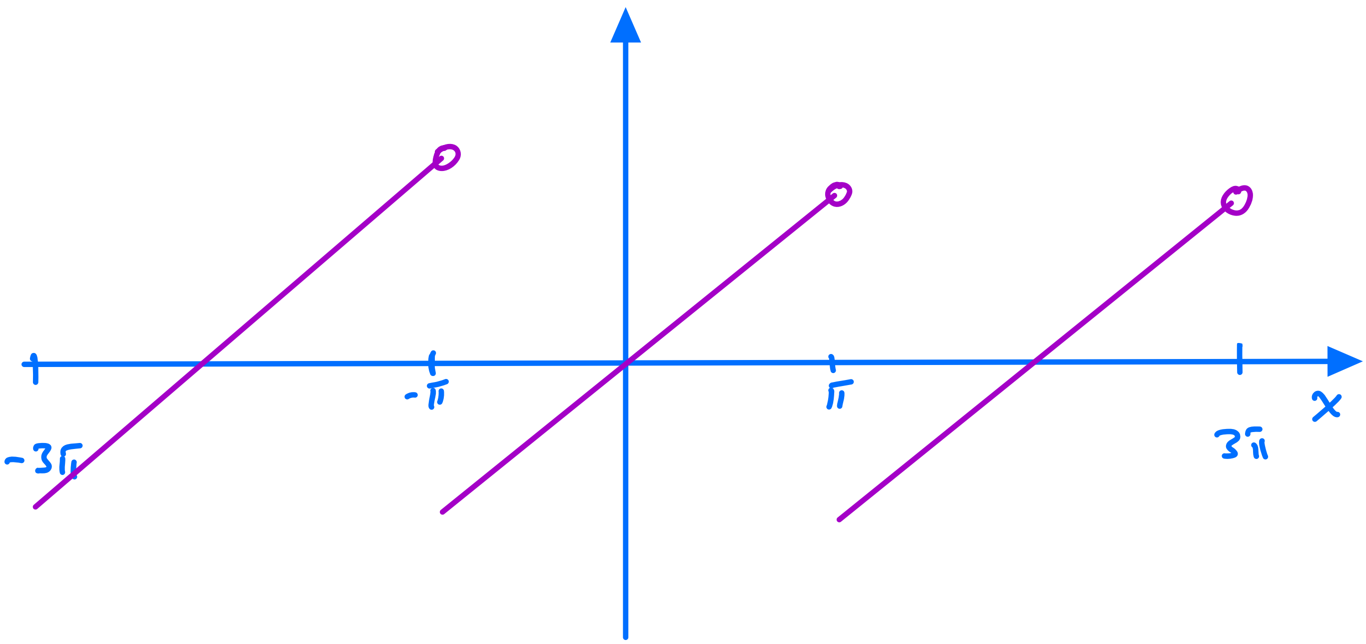
Remark c): We will see later how the decay of the Fourier coefficients is linked to how smooth our function f is.

Remark d): What conditions can we impose on f such that

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \sin(nx) + b_n \cos(n x) \right] \xrightarrow[N \to \infty]{} f(x) \;?$$

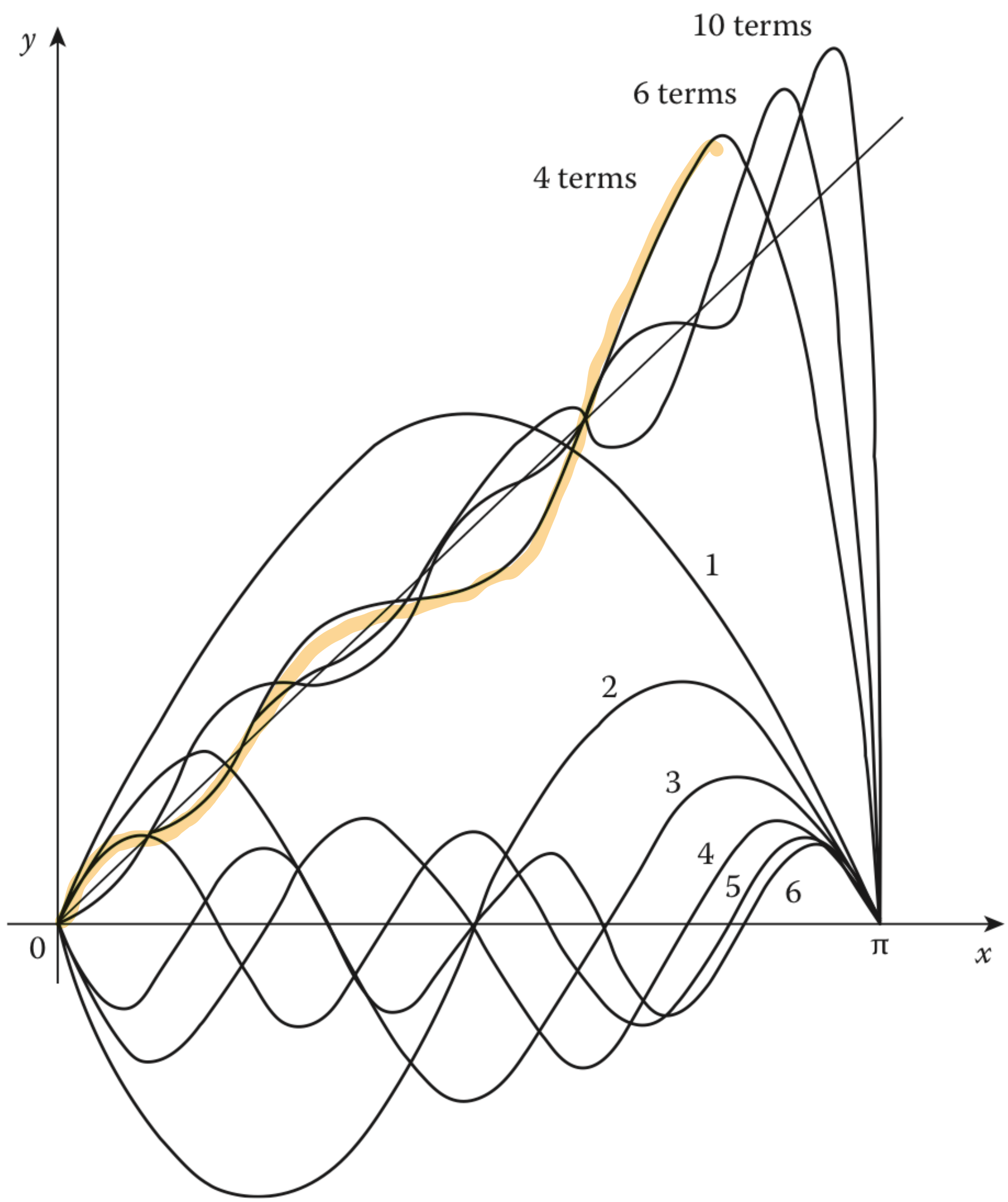
In which sense?

Example: Let's take $f(x) = x$ for $x \in [-\pi, \pi)$ and extend it by periodicity, i.e.



$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx$$



$$y = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x.$$

So, how do we compute the Fourier coefficients?

Analogy with finite dimensions

Let's consider a vector $\underline{u} \in \mathbb{R}^N$

$$\underline{u} = \sum_{i=1}^{N} u_i \underline{e_i} \qquad \text{where} \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

how do we compute u_i?

$$\underline{u} \cdot \underline{e_j} = \left(\sum_{i=1}^{N} u_i \, \underline{e_i} \right)$$

inner product

What's the analog of this process in ∞ dimensions?

$$N \to \langle \underline{u}, \underline{v} \rangle = \underline{u} \cdot \underline{v} = \sum_{i=1}^{N} u_i v_i$$

$$\infty \to \langle f, g \rangle = \int_a^b f(x) g(x)\, dx$$

inner product in ∞ dimensions!