Lecture 4

We saw an example of an equation with f continuous having many solutions.

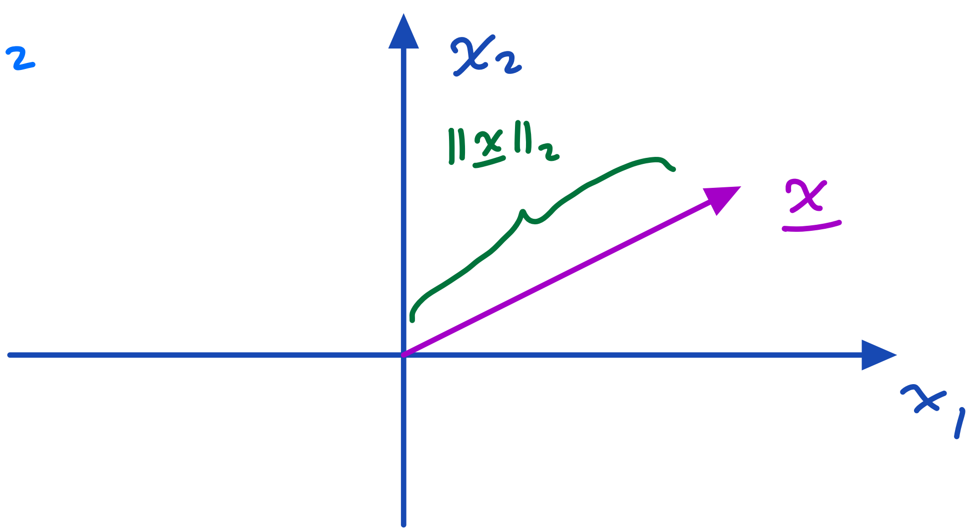
The concept that we need is Lipschitz continuity.

Recall the concept of norm from Calculus

for $\underline{x} \in \mathbb{R}^n$, define

$$\|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

In $\mathbb{R}^2$



Def. $f: \mathbb{R}^n \to \mathbb{R}^n$ is glob. Lipschitz continuous iff for all $x \neq y \in \mathbb{R}^n$

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} \leq L$$

Here L is a constant and it is called the Lipschitz const.

1D Lipschitz continuity: "gradient is bounded"

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L < \infty$$

Example 1: $f(x) = x$ on $\mathbb{R}$

$$\frac{|f(x)-f(y)|}{|x-y|} = \frac{|x-y|}{|x-y|} = \underbrace{1}_{L} < \infty$$

Example 2: $g(x) = x^2$ on $\mathbb{R}$

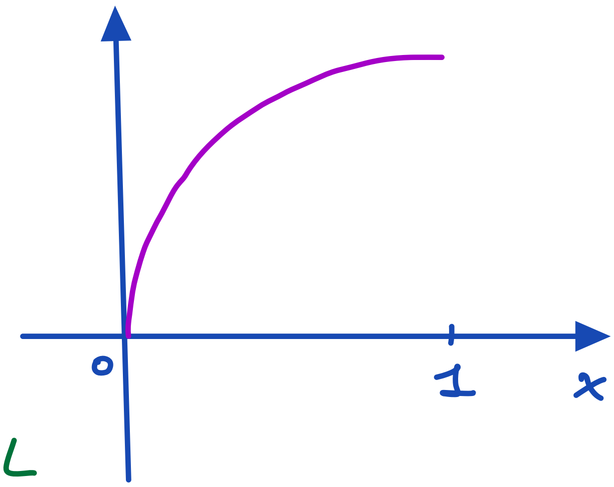
$$\frac{|g(x)-g(y)|}{|x-y|} = \frac{|x^2-y^2|}{|x-y|} = \frac{\cancel{(x-y)}(x+y)}{\cancel{x-y}}$$

$(x>y)$

$$= x+y < L$$

as $x,y \to \infty \to \underline{\underline{x+y \to \infty}}$ so this function is not glob. Lipschitz cont.

Example 3: $g(x) = \sqrt{x}$ on $[0,1]$



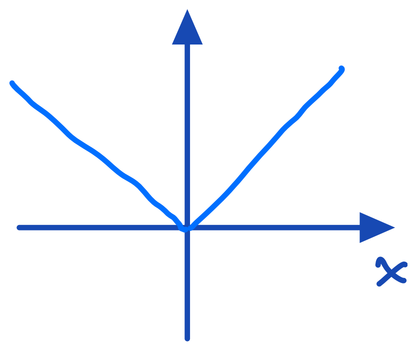
$$g'(x) = \frac{1}{2\sqrt{x}}$$

$$g'(x) \xrightarrow[x \to 0]{} \infty$$

$$\frac{|g(x)-g(y)|}{|x-y|} < L$$

Can this function be Lipschitz?

Example 4: $h(x) = |x|$ on $\mathbb{R}$.



$$\frac{\big|\,|x|-|y|\,\big|}{|x-y|} \leq \frac{|x-y|}{|x-y|} = 1 =: L$$

so $h(x) = |x|$ is glob. Lip. cont.

In Picard's iteration

$u_0, u_1, u_2, \ldots, u_n, \ldots$

if u_i is continous for $i = 0, 1, 2, \ldots$

and assuming that

$$u_n \to y$$

how do we know that y is cont?

Let's examine the following example:

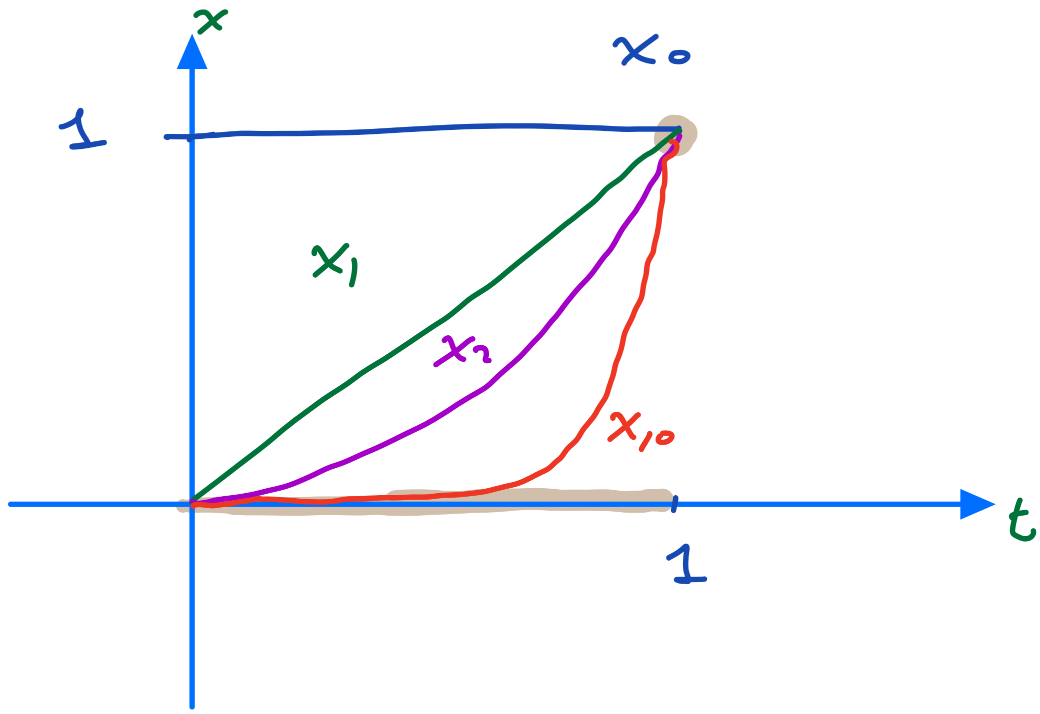
$x_n(t) = t^n$ on $[0,1]$

Is x_n cont. for all $n \geq 0$?

$x_0(t) = t^0 = 1$ ✓

$x_1(t) = t$ ✓

$\vdots$



$$\bar{x}(t) = \lim_{n\to\infty} x_n(t) = \begin{cases} 0, & \text{if } 0 < t < 1 \\ 1, & \text{if } t = 1 \end{cases}$$

Is $\bar{x}$ cont. or discont.?

<u>Not cont.!</u>

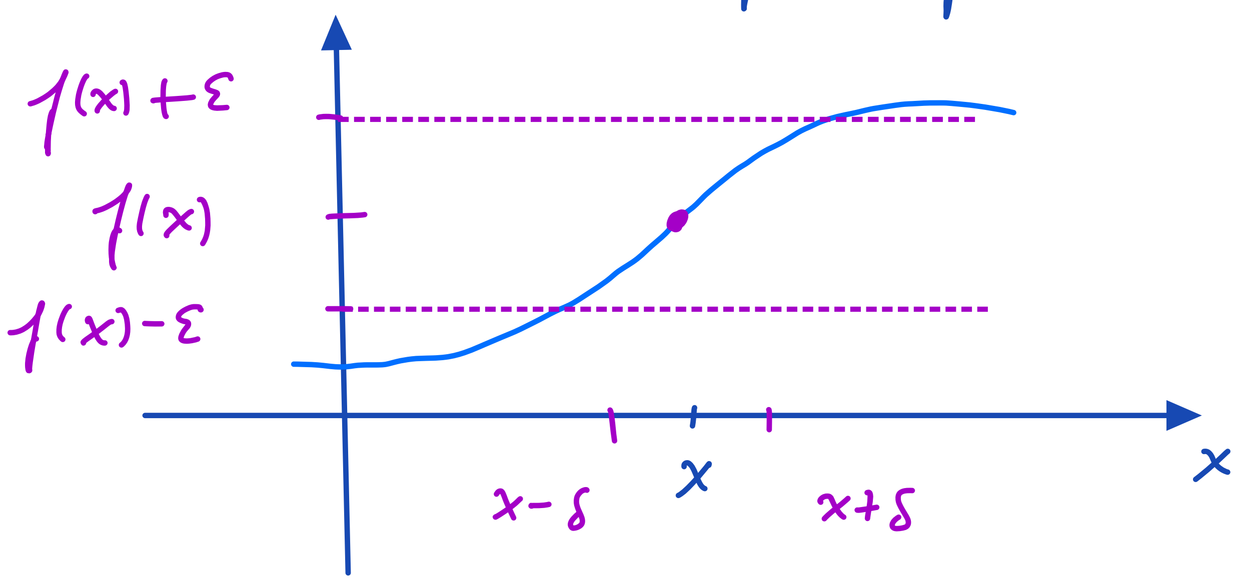
We need to study continuity in more detail!!

Different types of continuity

Definition of continuity of a function $f(x)$ at a point.

A function $f = f(x)$ is cont. at a point x iff for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x - \tilde{x}| < \delta \Rightarrow |f(x) - f(\tilde{x})| < \varepsilon$$



$$\delta = \delta(x, \varepsilon)$$

Example 1 : $f(x) = x$ is cont. on $\mathbb{R}$

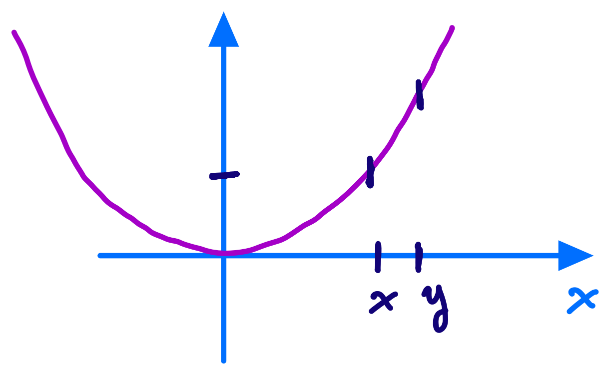
Given $x \in \mathbb{R}$ and $\varepsilon > 0$

take $\delta = \delta(x, \varepsilon) = \varepsilon$

so, if $|x - \tilde{x}| < \delta(x, \varepsilon)$

$$\Rightarrow \underbrace{|f(x) - f(\tilde{x})|}_{=\, |x - \tilde{x}|} < \varepsilon$$

Example 2: $f(x) = x^2$ is cont. on $\mathbb{R}$



Let's assume that

$|x^2 - y^2| < \varepsilon$

and let's pick $\delta(x, \varepsilon) = \varepsilon$

Then $|x - y| < \delta = \varepsilon$

$$\Rightarrow |x^2 - y^2| = |(x-y)(x+y)| < \delta |x+y|$$

So, as x and y get larger we need to pick a smaller δ!

Now, let's set $\delta = \dfrac{\varepsilon}{2\max(|x|,|y|)}$

$$|x^2 - y^2| = |(x-y)(x+y)| < \underbrace{\frac{\varepsilon\,|x+y|}{2\max(|x|,|y|)}}_{\leq\, \varepsilon}$$

So, $\delta = \delta(x, \varepsilon)$

δ depends on x!

Uniform continuity

(a,b)

A function $f: I \to \mathbb{R}$ is said to be uniformely cont. if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$

Example a: $f(x) = x^2$ is not unif. cont. on $\mathbb{R}$ (as we saw above) BUT IT IS unif. cont. on $(a,b) \subset \mathbb{R}$

Why? Mean Value Theorem

Let $h \in C^1((a,b);\mathbb{R})$ then

$$h(x) - h(y) = h'(\xi)(x-y)$$

$$|x^2 - y^2| = |2\xi||x-y| \leq \left(\max_{\xi \in (a,b)} 2|\xi|\right)|x-y|$$

So, given $\varepsilon > 0$ take $\delta = \dfrac{\varepsilon}{\max\limits_{z \in (a,b)} 2|z|}$

$\Rightarrow$ δ doesn't dep. on x!

$$|x^2 - y^2| = |x-y|\,|x+y| < \frac{\varepsilon}{\max\limits_{z \in (a,b)} 2|z|}\,|x+y|$$

$$\leq \varepsilon$$

So, unif. cont. depends on the domain that we consider.

The global picture in any bounded interval $[a,b]$ is the following:

continuously differentiable $\subset$ global Lipschitz $\subset$ unif. cont. $=$ cont.

$f(x) = |x|$ on $[-1,1]$: global Lipschitz but not diff.

$h(x) = \sqrt{|x|}$ on $[-1,1]$: unif. cont. but not global Lipschitz

Local Lipschitz continuity

<u>Def.</u> Let $f : D \to \mathbb{R}^n$, $(D \subset \mathbb{R}^n)$ if each point D has a neighborhood $N(x)$ such that f satisfies

$$\|f(x) - f(y)\| \leq L \|x - y\|$$

for all $y \in N(x)$ and some constant L.

Note: For a locally Lipschitz function the constant L can vary with the point and indeed become very large.

Example:

$f(x) = x^2$ is not globally Lipschitz in $\mathbb{R}$
but it is locally Lipschitz.

How do we show this?

Let $x \in \mathbb{R}$, and get $N(x) = (x-h, x+h)$

$$\frac{|f(x) - f(y)|}{|x-y|} = \frac{|(x-y)(x+y)|}{|x-y|}$$

$$\leq \max(|x-h|, |x+h|)$$

For a more in-depth treatment of results covered here consult

Elementary Analysis: The theory of Calculus

by K. Ross, Springer (free!)