Lecture 11

Derivates and integrals of L.T.

Theorem: If f is P.W. cont. on $[0,\infty)$ and of exp. order α and $\mathcal{L}\{f\}(s) = F(s)$ then

$$\frac{d^n}{ds^n}F(s) = \mathcal{L}\{(-1)^n t^n f(t)\}(s)$$

$s > \alpha$.

Proof: $\boxed{n=1}$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

this is valid thanks to Leibniz's rule

$$= \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right)dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}(-t f(t))\,dt$$

$$= \mathcal{L}\{-t f(t)\}(s)$$

So, by induction we get the desired result. $\blacksquare$

Example: Let $f(t) = \cos(\omega t)$

$$F(s) = \frac{s}{s^2+\omega^2}$$

Then

$$\mathcal{L}\{t \cos(\omega t)\}(s)$$

$$-\frac{d}{ds} F(s) = \mathcal{L}\{t f(t)\}(s)$$

$$-\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

$$\Rightarrow \mathcal{L}\{t \cos(\omega t)\}(s) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Example: $g(t) = 1$

$G(s) = \frac{1}{s}$

$\mathcal{L}\{t\}(s) = ?$

$$-\frac{d}{ds} G(s) = \mathcal{L}\{ t g(t)\}(s)$$

$$\rightarrow \quad \mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

Leibniz rule: Let $G(s,t) = e^{-st} f(t)$.

If $G(s,t)$ and $\partial_s G(s,t)$ are continuous (except possibly for a finite number of jump discontinuities in t) in t, s in some region of the (t,s) plane

And $\int_0^\infty G(s,t)\,dt < \infty$ and $\int_0^\infty \partial_s G(s,t)\,dt$ converges uniformely, then

$$\frac{d}{ds}\int_0^\infty G(s,t)\,dt = \int_0^\infty \partial_s G(s,t)\,dt$$

Check that this result can be applied in the proof of the theorem above.