Proving Existence and Uniqueness

Consider the initial value problem

$$\begin{cases} y'(t) = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

- A function $f(x)$ is **Lipschitz** on a domain D if for all $x_1, x_2 \in D$ there exists a *Lipschitz constant* $K > 0$ such that $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$. Equivalently, f has bounded secant slopes on D: $\left|\frac{f(x)-f(y)}{x-y}\right| \leq K$.

  Example: $f(x) = x^3$ is Lipschitz on $[0,1]$. To prove this, let $x, y \in [0,1]$. Then

  $$\left|\frac{f(x)-f(y)}{x-y}\right| = \left|\frac{x^3-y^3}{x-y}\right| = |x^2+xy+y^2| \leq |x^2| + |xy| + |y^2| \leq |x|^2 + |x|\ |y| + |y|^2 \leq 3.$$

  More generally, $f(x) = x^3$ is Lipschitz on any bounded subset of $\mathbb{R}$. (exercise!) However, $f(x) = x^3$ is not Lipschitz on $\mathbb{R}$. To see this, let x_n be any sequence that goes to infinity. Then the secant slopes between 0 and x_n grow without bound:

  $$\left|\frac{f(0)-f(x_n)}{0-x_n}\right| = |0^2 + 0x_n + x_n^2| = x_n^2 \to \infty.$$

- Let f be a differentiable function. Then

  $$|f'(x)| \leq K \text{ for all } x \in D \iff f \text{ is Lipschitz on } D \text{ with Lipschitz constant } K.$$

  Example: $f(x) = \sqrt{x}$ is not Lipschitz on any domain containing 0 because $f'(0)$ is infinite.

Picard's Theorem

- **Picard's Theorem** If there is a closed rectangle containing (t_0, y_0) on which f is continuous w.r.t t and Lipschitz w.r.t y, then there exists a unique solution in some neighborhood of t_0.

- Under the stated conditions, it can be proven that the sequence of *Picard iterates* converges to the unique solution:

  $$y_0(t) = y_0 \quad \text{and for all } n \geq 0, \quad y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau))d\tau$$

- **Corollary:** If f is

  (1) continuous w.r.t. t and

  (2) Lipschitz w.r.t y, where the Lipschitz constant is allowed to depend on t

  on an infinite strip $[a,b] \times \mathbb{R}^n$ where $a < t_0 < b$, then there exists a unique solution for all $t \in [a,b]$.

Exercises

1. Does the following IVP have a unique solution in any neighborhood of 0: $\begin{cases} y' = 2|y|^{1/2} \\ y(0) = y_0 \end{cases}$

   (a) if $y_0 = 0$? *Solution:* The assumptions of Picard's Theorem are not satisfied because f is not Lipschitz in any domain containing $y_0 = 0$, so we don't have any guarantees. It turns out that for any $a \leq 0$ and $b \geq 0$,

$$y_{a,b}(t) := \begin{cases} -(t-a)^2 & if\ t \leq a \\ 0 & if a \leq t \leq b \\ (t-b)^2 & if\ b \leq t \end{cases}$$

   is a solution. In particular, $y(t) = 0$ and $y_{0,0}(t)$ are two solutions. Since they differ everywhere except exactly at $t = 0$, there is no neighborhood of 0 on which there is a unique solution.

   (b) if $y_0 = 1$? *Solution:* Now the assumptions of Picard's Theorem are satisfied on any rectangle containing $(0, 1)$ that avoids $y = 0$, so yes a unique local solution exists. For example, the rectangle $[-1, 1] \times [0.5, 1.5]$ works. However, a global solution is still not unique because $y_{a,-1}(t)$ is a solution for any $a \leq -1$. This does not violate Picard's Theorem because all these solutions match for $t \in (-0.9, 0.9)$. (Any neighborhood around $t = 0$ that avoids $t = -1$ works.)

2. Show that $\begin{cases} y' = t + y \\ y(0) = 1 \end{cases}$ has a unique global solution.

   Solution: First, $f(t, y) = t + y$ is a polynomial w.r.t. t, so it is continuous w.r.t. t everywhere. Now we have to verify that it is Lipschitz w.r.t. y on $\mathbb{R}$. Let $t, y_1, y_2 \in \mathbb{R}$. Then

$$|f(t, y_1) - f(t, y_2)| = |t + y_1 - (t + y_2)| = |y_1 - y_2|,$$

   so f is Lipschitz w.r.t. y on $\mathbb{R}$ with Lipschitz constant $K = 1$. Then by the above corollary of Picard's Theorem, a unique global solution exists. (You can use the method of undetermined coefficients to find $y(t) = 2e^t - t - 1$)

3. Use Picard Iteration to solve the previous IVP.

   Solution:

   By definition,

$$\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= 1 + \int_0^t (\tau + 1) d\tau = \frac{t^2}{2} + t + 1 \\ y_2(t) &= 1 + \int_0^t \left(\tau + \frac{\tau^2}{2} + \tau + 1 \right) d\tau = \frac{t^3}{6} + t^2 + t + 1 \\ y_3(t) &= 1 + \int_0^t \left(\tau + \frac{\tau^3}{6} + \tau^2 + \tau + 1 \right) d\tau = \frac{t^4}{24} + \frac{t^3}{3} + t^2 + t + 1 \\ &\vdots \end{aligned}$$

   It appears that we have the pattern

$$y_n(t) = 2 \sum_{k=0}^{n+1} \frac{t^k}{k!} - \frac{t^{n+1}}{(n+1)!} - t - 1$$

   for all $n \in \mathbb{N}$. (This guess can be proven correct using induction.) Then as $n \to \infty$,

$$y(t) = \lim_{n \to \infty} y_n(t) = \lim_{n \to \infty} \left(2 \sum_{k=0}^{n+1} \frac{t^k}{k!} - \frac{t^{n+1}}{(n+1)!} - t - 1 \right) = 2e^t - t - 1.$$

Rewriting Higher Order Equations as a First Order System

We can rewrite any higher order ODE as a system of first order ODEs by introducing the dependent variables

$$y_j(t) := y^{(j)}(t) \quad \text{for } j = 0, \ldots, n-1.$$

($j = 0$ is really just renaming $y(t)$ to $y_0(t)$.) Then

$$y^{(n)}(t) = g(t, y, y', \ldots, y^{(n-1)}) \iff \underbrace{\begin{bmatrix} y_0' \\ y_1' \\ \vdots \\ y_{n-2}' \\ y_{n-1}' \end{bmatrix}}_{\vec{y}\,'} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ g(t, y_0, y_1, \ldots, y_{n-1}) \end{bmatrix}}_{\vec{f}(t,\vec{y})},$$

where $\vec{y}_i := y_i$ and the derivative is done component-wise. Furthermore, we can introduce one more component $y_n := t$ to make the system autonomous. In this case,

$$y^{(n)}(t) = g(t, y, y', \ldots, y^{(n-1)}) \iff \underbrace{\begin{bmatrix} y_0' \\ y_1' \\ \vdots \\ y_{n-2}' \\ y_{n-1}' \\ y_n' \end{bmatrix}}_{\vec{y}\,'} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ g(y_n, y_0, y_1, \ldots, y_{n-1}) \\ 1 \end{bmatrix}}_{\vec{f}(\vec{y})}.$$

- In both cases, the first component is the solution we are trying to find.
- If $g(t, y, y', \ldots, y^{(n-1)}) = a_0 y + a_1 y' + \cdots + a_{n-1} y^{(n-1)}$ for some $a_0, \ldots, a_{n-1} \in \mathbb{R}$, this system is linear and can be written as $\vec{y}\,' = A\vec{y}$.
- To prove the existence and uniqueness of a solution to a higher order IVP, write it as a system of first-order ODEs and apply Picard's Theorem. Use $||\cdot||_2$ instead of absolute value for the definition of Lipschitz.

Exercise

5. Without solving, prove there is a unique solution to $\begin{cases} y'' + t^2 y = 0 \\ y(\pi) = 1 \\ y'(\pi) = 0 \end{cases}$ for all $t \in \mathbb{R}$.

   Solution:

   First, rewrite the ODE as a first order system:

$$\vec{y}\,' = \underbrace{\begin{bmatrix} y_1 \\ -t^2 y_0 \end{bmatrix}}_{=\vec{f}(\vec{y})} \quad \text{with } \vec{y}(\pi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   (1) $\vec{f}$ is continuous w.r.t. t everywhere since each component is a polynomial in t.

   (2) To prove f is Lipschitz, let $(t, y_0, y_1), (t, x_0, x_1) \in \mathbb{R}^3$. Then

$$\begin{aligned} ||\vec{f}(\vec{x}) - \vec{f}(\vec{y})||_2 &= \left|\left| \begin{bmatrix} x_1 - y_1 \\ -t^2(x_0 - y_0) \end{bmatrix} \right|\right|_2 \\ &= \sqrt{(x_1 - y_1)^2 + t^4(x_0 - x_1)^2} \\ &\le \sqrt{\max(1, t^4)(x_1 - y_1)^2 + \max(1, t^4)(x_0 - x_1)^2} \\ &= \max(1, t^2)||\vec{x} - \vec{y}||_2, \end{aligned}$$

   By the corollary to Picard's Theorem with $a = -\infty$ and $b = \infty$, there is a unique solution on $[a, b] = \mathbb{R}$.