Laplace Transforms

- For any function $f : [0, \infty) \to \mathbb{R}$ that is **locally integrable** and of **exponential order**, the corresponding **Laplace Transform** is defined by
$$L[f] = F(p) := \int_0^\infty e^{-px} f(x)dx.$$
- The Laplace Transform is (essentially) a bijection, meaning
$$L[f(x)] = L[g(x)] \iff f(x) = g(x).$$
- For our purposes, it is useful to have a collection of transform pairs already calculated for later use. Verifying each of these transform pairs is its own exercise.

TABLE 1

Simple Transform Pairs

$f(x)$	$F(p) = L[f(x)]$
1	$\frac{1}{p}$
x	$\frac{1}{p^2}$
x^n	$\frac{n!}{p^{n+1}}$
e^{ax}	$\frac{1}{p-a}$
$\sin ax$	$\frac{a}{p^2+a^2}$
$\cos ax$	$\frac{p}{p^2+a^2}$
$\sinh ax$	$\frac{a}{p^2-a^2}$
$\cosh ax$	$\frac{p}{p^2-a^2}$

TABLE 2

General Properties of $L[f(x)] = F(p)$

$L[\alpha f(x) + \beta g(x)] = \alpha F(p) + \beta G(p)$
$L[e^{ax} f(x)] = F(p-a)$
$L[f'(x)] = pF(p) - f(0)$; $L[f''(x)] = p^2F(p) - pf(0) - f'(0)$
$L\left[\int_0^x f(x)dx\right] = \frac{F(p)}{p}$
$L[-xf(x)] = F'(p)$; $L[(-1)^n x^n f(x)] = F^{(n)}(p)$
$L\left[\frac{f(x)}{x}\right] = \int_p^\infty F(p)dp$
$L\left[\int_0^x f(x-t)g(t)dt\right] = F(p)G(p)$

- Multiple entries in these table can be combined to make more complicated transform pairs. For example, let $f(x) = \sin(bx)$. Then we can combine the 2nd entry in Table 2 with the 5th entry in Table 1 to see
$$L[e^{ax}\sin(bx)] = \frac{b}{(p-a)^2 + b^2}.$$

Exercises

1. Solve $\begin{cases} y'' + 2y' + 5y = 3e^{-x}\sin(x) \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$ using Laplace transforms. (The Method of Undetermined Coefficients works too, yielding $y(x) = e^{-x}\sin(2x) + e^{-x}\sin(x)$.)

   Solution: First, Laplace transform both sides of the ODE:

   $$(p^2Y - 0s - 3) + 2(pY - 0) + 5Y = \frac{3}{(p+1)^2+1}.$$

   Then solve this algebraic equation for $Y(p)$:

   $$(p^2+2p+5)Y = \frac{3}{p^2+2p+2} + 3 \implies Y(p) = \frac{3}{(p^2+2p+2)(p^2+2p+5)} + \frac{3}{(p^2+2p+5)}.$$

   Now that we have the Laplace transform of the solution, we are done if we can reverse-transform it. Recall that the Laplace Transform is linear, so we may factor out constants and compute transforms term-by-term. Notice that the second term already appears on our table of known transforms with $a = 1$ and $b = 2$. Even though the first term is not in our table, some partial fraction decomposition can break it into terms that are. Since the denominator is already maximally factored (you may further use complex factors if you like, it will work either way), we are looking for constants $A, B, C, D \in \mathbb{R}$ such that

   $$\begin{aligned} \frac{3}{(p^2+2p+2)(p^2+2p+5)} &= \frac{Ap+B}{p^2+2p+2} + \frac{Cp+D}{p^2+2p+5} \\ 3 &= (Ap+B)\left(p^2+2p+5\right) + (Cp+D)\left(p^2+2p+2\right) \\ &= (A+C)\,p^3 + (2A+B+2C+D)\,p^2 + (5A+2B+2C+2D)\,p + (5B+2D)\,. \end{aligned}$$

   Match coefficients to find a linear system for the unkown coefficients:

   $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 5 & 2 & 2 & 2 \\ 0 & 5 & 0 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

   Some linear algebra reveals the only solution to this system is $A = 0$, $B = 1$, $C = 0$, and $D = -1$. Therefore we can more usefully write $Y(p)$ as

   $$Y(p) = \frac{1}{p^2+2p+2} + \frac{2}{p^2+2p+5}$$

   Notice that our transform pair from earlier, once with $(a,b) = (-1,1)$ and once with $(a,b) = (-1,2)$, gives

   $$Y(p) = L[e^{-x}\sin(x)] + L[e^{-x}\sin(2x)] = L[e^{-x}\sin(x) + e^{-x}\sin(2x)].$$

   Since the Laplace transform is (essentially) a bijection, we must have

   $$y(x) = e^{-x}\sin(x) + e^{-x}\sin(2x).$$

2. Prove the last transform pair in Table 2.

   Solution: By definition of the Laplace transform,

   $$L\left[\int_0^x f(x-t)g(t)dt\right] = \int_0^\infty e^{-px}\left(\int_0^x f(x-t)g(t)dt\right)dx = \int_0^\infty \left(\int_0^x e^{-px}f(x-t)g(t)dt\right)dx.$$

   Then change the order integration:

   $$= \int_0^\infty \int_t^\infty e^{-px}f(x-t)g(t)dxdt.$$

Within the inner integral (where t is fixed), make the substitution $u := x - t \implies du = dx$:

$$= \int_0^\infty \left(\int_0^\infty e^{-p(u+t)} f(u) g(t) du \right) dt.$$

Lastly, apply Fubini's Theorem and notice the remaining factors are Laplace transforms:

$$= \left(\int_0^\infty e^{-pu} f(u) du \right) \left(\int_0^\infty e^{-pt} g(t) dt \right) = F(p)G(p).$$

3. Assume $L[g(x)] = G(p)$ exists. Solve $\begin{cases} 4y'' + y = g(x) \\ y(0) = 3 \\ y'(0) = -7 \end{cases}$ using Laplace transforms.

Solution: Transform the ODE:

$$4(p^2 Y - 3p + 7) + Y = G.$$

Solve the algebraic equation:

$$(p^2 + 1)Y - 12p + 28 = G \implies Y(p) = \frac{12p - 28 + G}{4p^2 + 1}.$$

Rearrange the terms to use the table to reverse-transform:

$$\begin{aligned} Y(p) &= 3\frac{p}{p^2 + 1/4} - 14\frac{1/2}{p^2 + 1/4} + \frac{1}{2}G(p)\frac{1/2}{p^2 + 1/4} \\ &= 3L\left[\cos\left(\frac{x}{2}\right)\right] - 14L\left[\sin\left(\frac{x}{2}\right)\right] + \frac{1}{2}\underbrace{L[g(x)]L\left[\sin\left(\frac{x}{2}\right)\right]}_{=L[\int_0^x \sin(t/2)g(x-t)dt]} \\ &= L\left[3\cos\left(\frac{x}{2}\right) - 14\sin\left(\frac{x}{2}\right) + \frac{1}{2}\int_0^x \sin\left(\frac{t}{2}\right) g(x-t)dt\right] \end{aligned}$$

Since the Laplace Transform is (essentially) a bijection, we must have

$$y(x) = 3\cos(x/2) - 14\sin(x/2) + \frac{1}{2}\int_0^x \sin\left(\frac{t}{2}\right) g(x-t)dt.$$