Math 135, Spring 2022

Lecture #24: PDEs and boundary value problems

Monday May 23rd

Last time

- We considered the IBVP

$$\begin{cases} \dfrac{\partial w}{\partial t} = a^2 \dfrac{\partial^2 w}{\partial x^2} \\ w(t,0) = 0 = w(t,\pi) \\ w(0,x) = f(x) \end{cases}$$

- We showed that (formally) the solution is given by

$$w(t,x) = \sum_{n=1}^{\infty} b_n e^{-a^2 n^2 t} \sin(nx) \quad \text{where} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Learning objectives

Today we will discuss:

- The Dirichlet problem for the unit disc.

The Laplace Equation

$2d$ models

$w = w(x,y)$

$\Delta w := \partial^2_{xx} w + \partial^2_{yy} w$

Laplacian of w

- We now consider our models in $2d$.
- The **wave equation** takes the form

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. = \Delta w$$

- The **heat equation** takes the form

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. = \Delta w$$

- In both cases, **steady states** are given by solutions of the **Laplace equation**

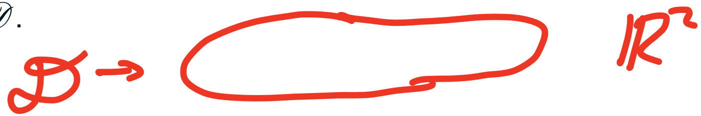
$\lim_{t \to \infty} w(x,t)$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0.$$

[w is harmonic]

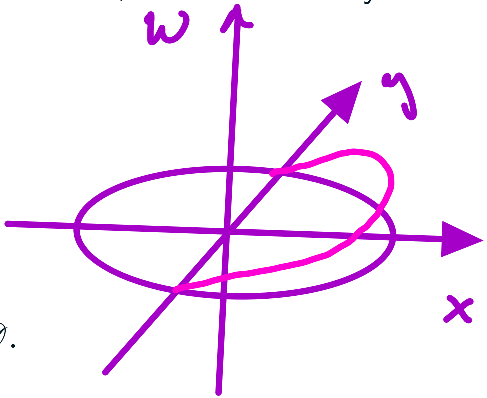
The Dirichlet problem

- Let $\mathscr{D} \subseteq \mathbb{R}^2$ be an open, simply connected region with simple, smooth, closed boundary $\partial\mathscr{D}$.



- The **Dirichlet problem** concerns finding solutions of

$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} = 0 & \text{for} \quad (x, y) \in \mathscr{D} \\ w(x, y) = f(x, y) & \text{for} \quad (x, y) \in \partial\mathscr{D}. \end{cases}$$



- We refer to $f(x, y)$ as the **boundary values** of $w(x, y)$.

A special case

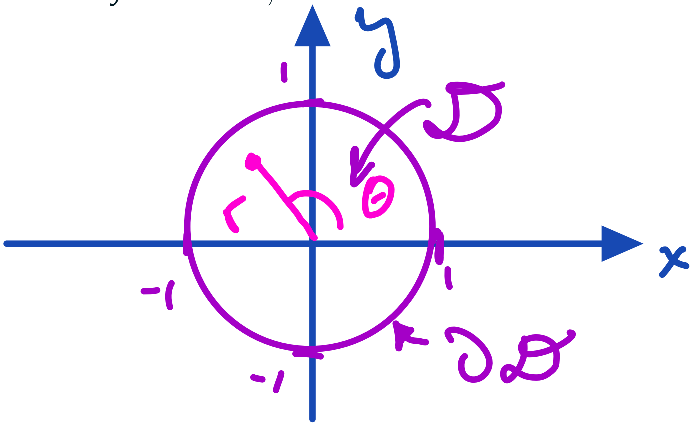
- We will restrict our attention to the case of the unit disc

$$\mathscr{D} = \{x^2 + y^2 < 1\} \quad \text{and} \quad \partial\mathscr{D} = \{x^2 + y^2 = 1\}.$$

- It then makes sense to switch to **polar coordinates**

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta,$$

where $r \geq 0$ and $-\pi < \theta \leq \pi$.



- Suppose that $w = w(r, \theta)$.

- An exercise in the multivariable chain rule (see homework) shows that the Laplace equation can be written as

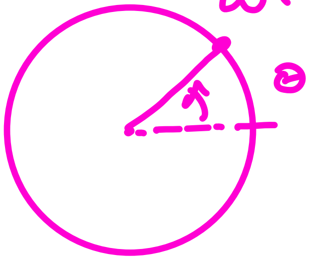
$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0,$$

for $0 < r < 1$ and $-\pi < \theta \leq \pi$.

- The boundary values can be written as

$$w(1, \theta) = f(\theta).$$

$w(1,\theta) = f(\theta)$



Separation of variables

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = 0$$

Look for solutions $w(r,\theta) = u(r)v(\theta)$

Then

$$u''(r)v(\theta) + \frac{1}{r}u'(r)v(\theta) + \frac{1}{r^2}u(r)v''(\theta) = 0$$

$$\rightarrow \quad \frac{r^2u'' + ru'}{u} = -\frac{v''}{v} = \lambda \quad \text{constant}$$

$$\rightarrow \quad \begin{cases} v'' + \lambda v = 0 \\ r^2u'' + ru' - \lambda u = 0 \end{cases}$$

$\leftarrow$ Euler's equidimensional equation

θ-values

$v(\theta) = v(\theta + 2\pi)$

We want to find non-trivial solutions of

$$v'' + \lambda v = 0$$

that are 2π-periodic.

What should we take λ to be?

A) n for $n \in \mathbb{N}$

B) n^2 for $n \in \mathbb{N}$

C) $-n^2$ for $n \in \mathbb{N}$

D) $-n$ for $n \in \mathbb{N}$

If $\lambda > 0$:

$\mu^2 + \lambda = 0$

$\mu^2 = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$

$v(\theta) = c_1 \sin(\sqrt{\lambda}\,\theta) + c_2 \cos(\sqrt{\lambda}\,\theta)$

$\sqrt{\lambda} = n, \; n \geq 1$

If $\lambda < 0$:

$\mu^2 + \lambda = 0$

$\mu = \pm\sqrt{-\lambda}$

$v(\theta) = c_1 e^{\sqrt{-\lambda}\,\theta} + c_2 e^{-\sqrt{-\lambda}\,\theta}$

NOT periodic

What is the general solution to

$$v'' + n^2 v = 0$$

when $n \in \mathbb{N}$?

$n \geq 1$

A) $C \sin(n\theta)$

B) $C \cos(n\theta)$

$\longrightarrow$ C) $C_1 \cos(n\theta) + C_2 \sin(n\theta)$

D) $C_1 e^{n\theta} + C_2 e^{-n\theta}$

Because v is 2π periodic

$\lambda = 0 \rightarrow v'' = 0 \rightarrow v(\theta) = C_1 + C_2\theta \Rightarrow \underline{C_2 = 0}$

Summary: $v'' + \lambda v = 0$ has a non-trivial, 2π-periodic solution of the form

$$v_n = \begin{cases} \frac{1}{2} a_0 & \text{if } \lambda = 0 \\ a_n \cos(n\theta) + b_n \sin(n\theta) & \text{if } \lambda = n^2 \end{cases}$$

$n \in \mathbb{Z}_+$
$= \{1, 2, 3, \dots\}$

r-values

If we write

$$u(r) = \phi(\rho) \quad \text{for} \quad \rho = \ln r,$$

what is

$$r^2 u'' + ru' \qquad ?$$

A) $e^{-2\rho}\phi'' - e^{-\rho}\phi' + \phi$

B) ϕ''

C) $e^{2\rho}\phi'' + e^{\rho}\phi'$

D) None of the above

Hint: We can write $r^2 u'' + ru' = r(ru')'$.

$u(r) = \phi(\rho) \qquad \rho = \ln r$

$r^2 u'' + r u' = r(ru')'$

$$ru' = r\frac{du}{dr} = r\frac{d\phi}{d\rho}\frac{d\rho}{dr} = \cancel{r}\,\phi'\frac{1}{\cancel{r}} = \phi'(\rho)$$

$$r(ru')' = r\frac{d}{dr}(ru') = r\frac{d}{dr}\left(\phi'(\rho)\right) = r\phi''(\rho)\frac{d\rho}{dr}$$
$$= \cancel{r}\,\phi''(\rho)\frac{1}{\cancel{r}}$$
$$= \phi''(\rho)$$

So

$$r^2 u'' + r u' = \phi''(\rho)$$

What is the general solution of

$$r^2u'' + ru' = 0 \qquad ?$$

$r^2u'' + ru' - \cancel{\lambda u} = 0$

$\phi''(\xi) = 0$

$\rightarrow$

$\phi(\xi) = c_1 + c_2\xi$

$= c_1 + c_2 \ln r$

A) C_1

B) $C_1 r + C_2 r^{-1}$

C) $C_1 \cos(r) + C_2 \sin(r)$

$\rightarrow$ D) $C_1 + C_2 \ln r$

What is the general solution of

$$\underbrace{r^2 u'' + r u'}_{\phi''(\varrho)} - n^2 u = 0,$$

where $n \geq 1$?

$r^2 u'' + r u' - \underbrace{n^2}_{=\lambda} u = 0$

A) $C_1 r^n$

$\rightarrow$ B) $C_1 r^n + C_2 r^{-n}$

C) $C_1 \cos(nr) + C_2 \sin(nr)$

D) $C_1 + C_2 \ln(nr)$

$\varrho = \ln r$

$\rightarrow \quad \phi'' - n^2 \phi = 0$

$\rightarrow \quad \phi(\varrho) = c_1 e^{n\varrho} + c_2 e^{-n\varrho}$

$= c_1 r^n + c_2 r^{-n}$

Solving our PDE

Trying to solve $\partial^2_{r^2} w + \frac{1}{r}\partial_r w + \frac{1}{r^2}\partial^2_{\theta^2} w = 0$

If we set $w(r,\theta) = u(r)\,v(\theta)$ the we have non-trivial when

$$\frac{r^2 u'' + r u'}{u} = -\frac{v''}{v} = n^2$$

Then:

$$v_n(\theta) = \begin{cases} \frac{1}{2} a_0 & \text{if } n = 0 \\ a_n \cos(n\theta) + b_n \sin(n\theta) & \text{if } n \geq 1 \end{cases}$$

$$u_n(r) = r^n \qquad c_1 r^n + \cancel{c_2 r^{-n}}$$

imposing cond. at $r = 0$

$n=1$ $\quad (a_1 \cos(\theta) + b_1 \sin(\theta))(r)$

is a sol. $\quad \partial_{r^2}^2 w + \frac{1}{r}\partial_r w + \frac{1}{r^2}\partial_{\theta^2}^2 w = 0$

$n=2$ $\quad (a_2 \cos(2\theta) + b \sin(2\theta))(r^2)$

$\vdots$

more generally

$$w(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right)$$

$w(1,\theta) = f(\theta)$ so the series is the Fourier series of $f(\theta)$

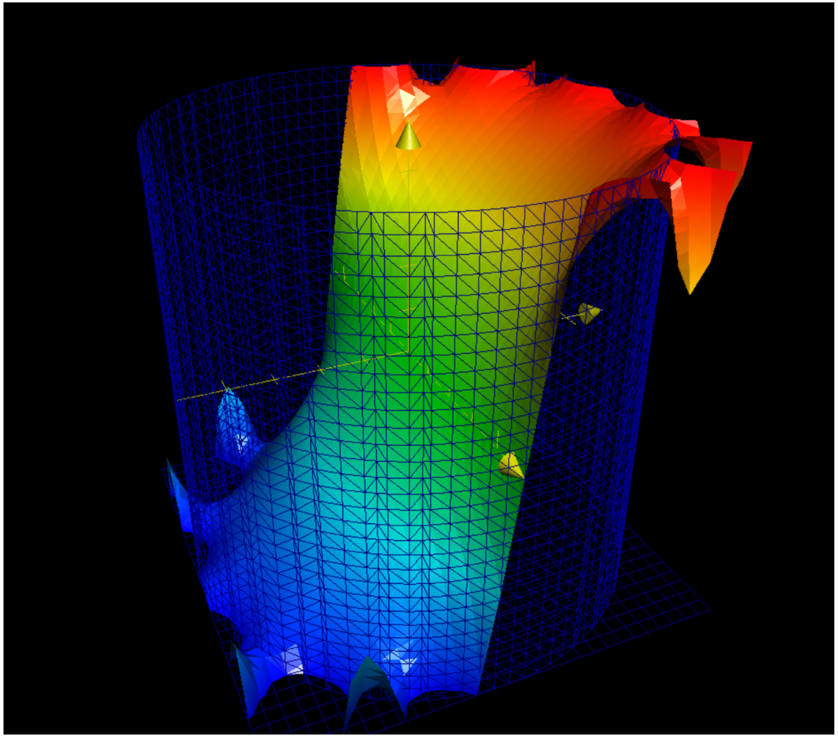
An example

Use the expression

$$w(r,\theta) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big\{a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\Big)$$

to solve the Dirichlet problem for the unit circle with boundary values

$$f(\theta) = \begin{cases} 1 & \text{if} \quad 0 \le \theta \le \pi \\ 0 & \text{otherwise.} \end{cases}$$



$$w(r,\theta) \approx \tfrac{1}{2} + \sum_{n=1}^{11} \frac{2}{2\pi(2n-1)} \sin\big((2n-1)\theta\big)$$

See you next time!