Home Work 6

$\boxed{\text{1) Section 37, problem 3}}$

* If one or both $\alpha$ & $\beta$ is null vector

$\Rightarrow |(\alpha, \beta)| = 0$ & $\|\alpha\|\|\beta\| = 0$

$\Rightarrow \boxed{(19) \text{ holds.}}$

* other wise, we check: with $\lambda = \dfrac{\langle\alpha, \beta\rangle}{\|\beta\|^2}$

$$F(\lambda) = \|\alpha - \lambda\beta\|^2 = (\alpha - \lambda\beta, \alpha - \lambda\beta)$$

$$= (\alpha, \alpha) - (\lambda\beta, \alpha) - (\alpha, \lambda\beta) + (\lambda\beta, \lambda\beta)$$

$$= \|\alpha\|^2 - \lambda(\beta, \alpha) - \bar{\lambda}(\alpha, \beta) + \lambda\bar{\lambda}(\beta, \beta)$$

$$= \|\alpha\|^2 - \lambda\overline{(\alpha, \beta)} - \bar{\lambda}(\alpha, \beta) + \lambda\bar{\lambda}\|\beta\|^2$$

$$= \|\alpha\|^2 - \frac{\langle\alpha, \beta\rangle}{\|\beta\|^2}\overline{(\alpha, \beta)} - \frac{\overline{(\alpha, \beta)}}{\|\beta\|^2}(\alpha, \beta) + \frac{\langle\alpha, \beta\rangle}{\|\beta\|^2}\frac{\overline{\langle\alpha, \beta\rangle}}{\|\beta\|^2}\|\beta\|^2$$

$$= \|\alpha\|^2 - \frac{\langle\alpha, \beta\rangle\overline{\langle\alpha, \beta\rangle}}{\|\beta\|^2} + \frac{\langle\alpha, \beta\rangle\overline{\langle\alpha, \beta\rangle}}{\|\beta\|^2} - \frac{\overline{(\alpha, \beta)}}{\|\beta\|^2}(\alpha, \beta)$$

$$= \|\alpha\|^2 - \frac{|(\alpha, \beta)|^2}{\|\beta\|^2}$$ Since $(\alpha, \beta)\overline{(\alpha, \beta)} = |(\alpha, \beta)|^2$

Also, $\|\alpha - \lambda\beta\|^2 \geq 0$

$$\Rightarrow \|\alpha\|^2 - \frac{|(\alpha, \beta)|^2}{\|\beta\|^2} \geq 0$$

$$\Rightarrow \|\alpha\|^2 \|\beta\|^2 \geq |(\alpha, \beta)|^2$$

$$\Rightarrow |(\alpha, \beta)|^2 \leq \|\alpha\|^2 \|\beta\|^2$$

$$\Rightarrow |\alpha, \beta| \leq \|\alpha\| \|\beta\|.$$

$\Rightarrow$ $\boxed{\text{19 holds}}$

$\boxed{2)\ \text{section } 37,\ \text{problem } 5}$

* Prove Pythagorean: with $f$ & $g$ are two orthogonal vectors

we have $(f,g) = f \cdot g = g \cdot f = (g,f) = 0$

then $\|f-g\|^2 = (f-g, f-g)$

$= (f,f) - (g,f) - (f,g) + (g,g)$

$= \|f\|^2 - 0 - 0 + \|g\|^2$

$\Rightarrow \boxed{\|f-g\|^2 = \|f\|^2 + \|g\|^2}$

* Prove its converse:

We have $\|f-g\|^2 = \|f\|^2 + \|g\|^2$

$\Leftrightarrow \|f\|^2 - (g,f) - (f,g) + \|g\|^2 = \|f\|^2 + \|g\|^2$

$\Leftrightarrow (g,f) + (f,g) = 0 \Leftrightarrow 2(g,f) = 0$

$\Leftrightarrow (g,f) = 0 \Rightarrow$ $\boxed{f \text{ \& } g \text{ are orthogonal}}$

$\boxed{\text{3) Section 3.8, Problem 1}}$

Given $f_n(x) = \begin{cases} 0 & 0 \le x \le 1/n \\ \sqrt{n} & 1/n < x < 2/n \\ 0 & 2/n \le x \le 1 \end{cases}$

a) With $x \in [0,1]$, choose $n_0 \in \mathbb{N}$ such that

$0 < \frac{2}{n_0} < x \le 1$

So, with $n \ge n_0 \Rightarrow \frac{2}{n} \le \frac{2}{n_0} \le x \le 1$

$\Rightarrow f_n(x) = 0 \Rightarrow |f_n(x) - 0| = 0$ as $n \to \infty$

$\Rightarrow$ by definition 6.5.1 (Point Wise Convergence), the sequence $\{f_n(x)\}$ converges point wise to the zero function on the Interval $[0,1]$

b) We have $[f(x) - f_n(x)]^2 = (\sqrt{n})^2$ with $\frac{1}{n} < x < \frac{2}{n}$

$\Rightarrow E_n = \int_{\frac{1}{n}}^{2/n} [f(x) - f_n(x)]^2 dx = \int_{1/n}^{2/n} n\, dx = n x \Big|_{1/n}^{2/n}$

$= n\left[\frac{2}{n} - \frac{1}{n}\right] = 1 \neq 0$ when $n \to \infty$ $\Rightarrow$ the sequence $f_n(x)$ does not converge in the mean to the zero function on the interval $[0,1]$

$\boxed{\text{4) Section 38, problem 4}}$

Given the function $f(x) = 1$ is to be approximated on $[0, \pi]$

$$p(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + b_5 \sin 5x$$

$\int_0^\pi [1 - p(x)]^2 dx$ is minimized

$$p_n(x) = \sum_{n=1}^{5} b_n \sin nx \Rightarrow b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^\pi \sin nx \, dx = \frac{2}{\pi} \cdot \left(-\frac{1}{n} \cos nx\right)\Big|_0^\pi = \frac{2}{\pi n} \cos nx \Big|_\pi^0$$

$$= \frac{2}{\pi . n} \left[1 - \cos n\pi\right]$$

$$\Rightarrow b_1 = \frac{2}{\pi}\left[1 - \cos \pi\right] = \frac{2}{\pi}\left[1 + 1\right] = \frac{4}{\pi}$$

$$\Rightarrow b_2 = \frac{2}{2\pi}\left[1 - \cos(2\pi)\right] = 0$$

$$\Rightarrow b_3 = \frac{2}{3.\pi}\left[1 - \cos 3\pi\right] = \frac{4}{3\pi}$$

$$\Rightarrow b_4 = \frac{2}{4\pi}\left[1 - \cos 4\pi\right] = 0$$

$$\Rightarrow b_5 = \frac{2}{5\pi}\left[1 - \cos 5\pi\right] = \frac{4}{5\pi}$$

5) Section 38, problem 7

We have $x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$

$$= 2\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\sin nx}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\sin nx}{n}$$

$$\Rightarrow \quad a_0 = a_n = 0, \quad b_n = \frac{(-1)^{n+1} \cdot 2}{n} \Rightarrow b_n^2 = \frac{4}{n^2}$$

Applying Parserval's equation, we have:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2} a_0^2 + \sum_{1}^{\infty} [a_n^2 + b_n^2]$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{1}^{\infty} b_n^2 = \sum_{1}^{\infty} \frac{4}{n^2}$$

$$= \frac{1}{\pi} \frac{1}{3} x^3 \Big|_{-\pi}^{\pi} = \frac{1}{3\pi}[\pi^3 + \pi^3] = \sum_{1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \quad \frac{2}{3}\pi^2 = \sum_{1}^{\infty} \frac{4}{n^2} \Rightarrow \frac{\pi^2}{6} = \sum_{1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \quad \boxed{\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$* \quad x^2 = \frac{\pi^2}{3} + 4\sum_{1}^{\infty} (-1)^n \frac{\cos nx}{n^2} = \frac{2\pi^2}{2.3} + \sum_{1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx$$

$$\Rightarrow a_0 = \frac{2\pi^2}{3}, \quad a_n = \frac{(-1)^n \cdot 4}{n^2}, \quad b_n = 0 \Rightarrow a_n^2 = \frac{16}{n^4}$$

Then applying Parserval's equation:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{1}{2}a_0^2 + \sum_1^{\infty} [a_n^2 + b_n^2]$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{2} \cdot \frac{4\pi^4}{9} + \sum_1^{\infty} \frac{16}{n^4}$$

$$\Rightarrow \frac{1}{\pi} \cdot \frac{x^5}{5}\Bigg|_{-\pi}^{\pi} = \frac{1}{5\pi}[\pi^5 + \pi^5] = \frac{2\pi^4}{9} + \sum_1^{\infty} \frac{16}{n^4}$$

$$\Rightarrow \frac{2}{5}\pi^4 - \frac{2\pi^4}{9} = \sum_1^{\infty} \frac{16}{n^4}$$

$$\Rightarrow \frac{8}{45}\pi^4 = \sum_1^{\infty} \frac{16}{n^4} \Rightarrow \frac{1}{90}\pi^4 = \sum_1^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \boxed{\sum_1^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

6) Given $V$ is a real inner product space & $v, w \in V$ be non-zero

a) $J(t) = \|v - tw\|^2$ for $t \in \mathbb{R}$

$= (v - tw, v - tw) = (v, v) - (tw, v) - (v, tw) + (tw, tw)$

$= \|v\|^2 - 2(tw, v) + t^2\|w\|^2$

$= \|v\|^2 - 2t(w, v) + t^2\|w\|^2$

$\Rightarrow \frac{dJ}{dt} = -2(v, w) + 2t\|w\|^2$

$\Rightarrow \frac{d^2J}{dt^2} = 2\|w\|^2 > 0 \Rightarrow J$ min at $\frac{dJ}{dt} = 0$

$\Leftrightarrow \quad t\|w\|^2 = (v, w) \Rightarrow \boxed{t = \frac{(v, w)}{\|w\|^2}}$

b) We have:

$J_{min} = \|v - \frac{(v, w)}{\|w\|^2} w\|^2$, with the projection as given

$P_w(v) = \frac{\langle w, v\rangle}{\|w\|^2} w$

$\Rightarrow \boxed{J_{min} = \|v - P_w(v)\|^2}$