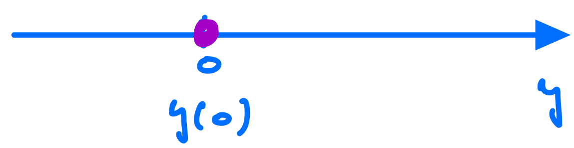
Lecture 13

Let's assume we have a mechanical/electrical system at rest subject to an external force $f(t)$

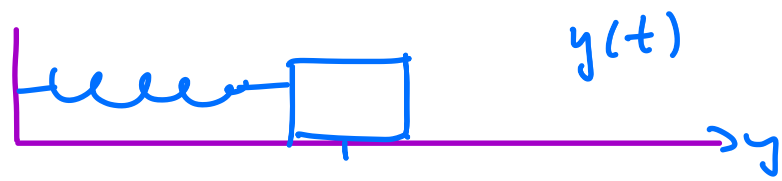
$$y'' + ay' + by = f(t), \qquad t \geq 0$$

$$\left.\begin{aligned} y(0) &= 0 \\ y'(0) &= 0 \end{aligned}\right\} \text{at rest ICs.}$$



The external force could be a sudden hammer blow in the mechanical system or a lightning stroke on a transmission line.

So, our external force can be a very irregular function.



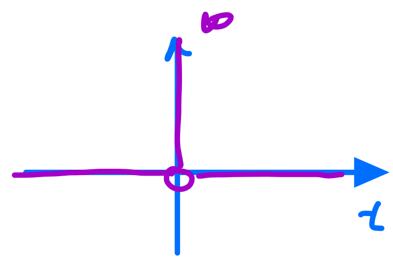
Dirac Delta "distribution" function

Def (Dirac Delta function)

- P. Dirac
- L. Schwartz

1950's

The Dirac Delta function (distribution) is characterized by the following two properties:



(1) $$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \infty, & \text{if } t = 0 \end{cases}$$

(2) $$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

for any function that is cont. on an open interval containing 0.

Remark:

a) By property (2), for $a \geq 0$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-a)\, dt = \int_{-\infty}^{\infty} f(u+a)\, \delta(u)\, du$$
$$= f(a)$$

$u = t - a \to t = u + a$

$du = dt$

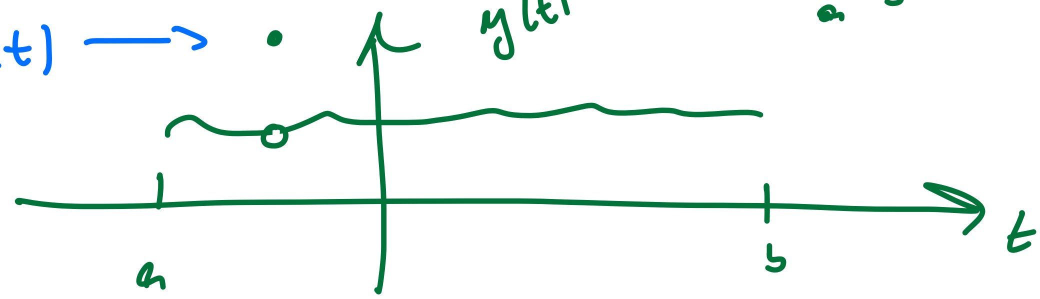
b) let's take $f(t) \equiv 1$

$$\int_{-\infty}^{\infty} 1\,\delta(t)\,dt = 1$$

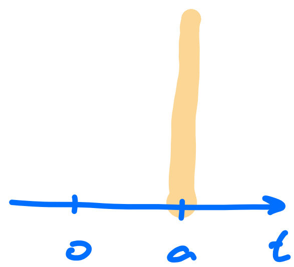
so $\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1}$

$\int_a^b y(t)\,dt$

$\tilde{y}(t) \longrightarrow$



c) L.T. of $\delta(t-a)$ for $a \geq 0$?



$$\mathcal{L}\{\delta(t-a)\}(s) = \int_0^{\infty} e^{-st}\,\delta(t-a)\,dt$$

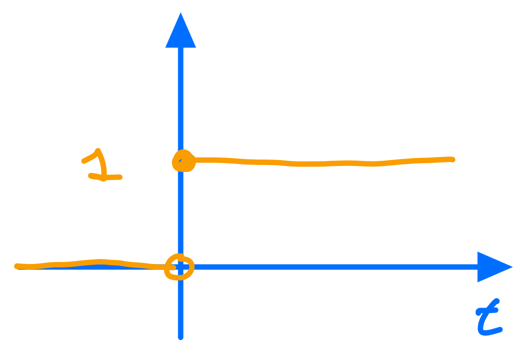
$$= \int_{-\infty}^{\infty} e^{-st}\,\delta(t-a)\,dt$$

$$= e^{-sa}$$

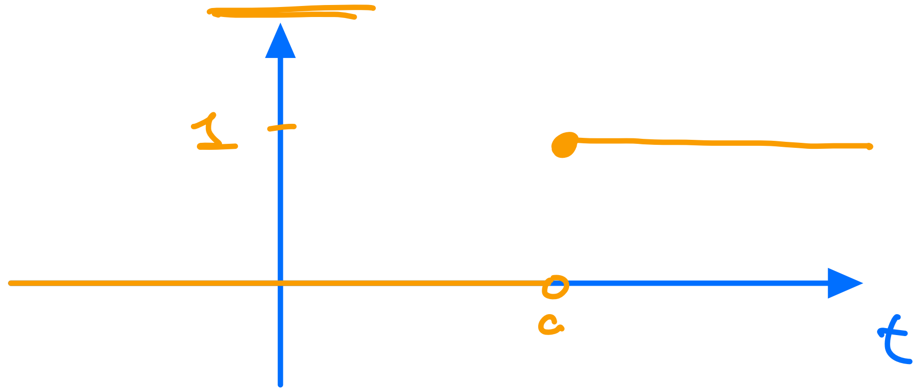
Unit step function

The unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



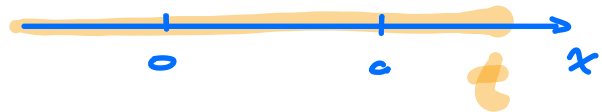
Plot $u(t-a)$ where $a > 0$.



Note that

$$\mathcal{L}\{u(t)\}(s) = \int_0^\infty e^{-st} u(t)\,dt = \frac{1}{s}, \quad s > 0$$

Remark:

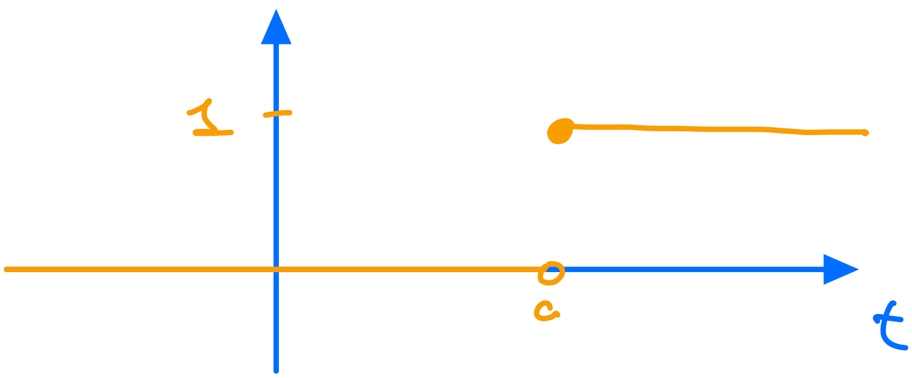


$$\int_{-\infty}^{t} \delta(x-a)\,dx = \begin{cases} 0, & \text{if } t<a \\ 1, & \text{if } t\geq a \end{cases}$$

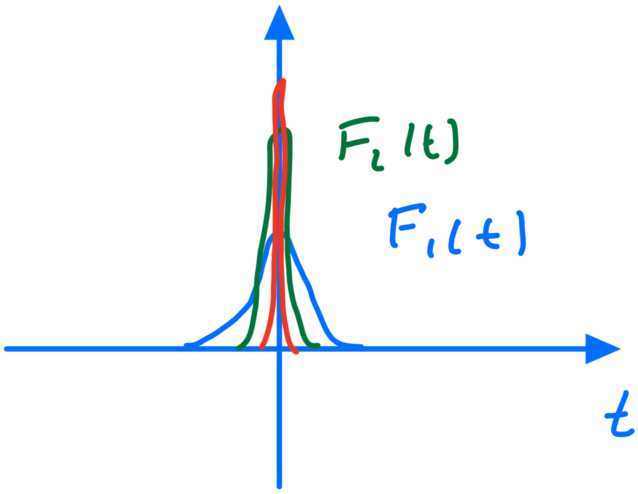
$$= u(t-a)$$

der. w.r.t. time

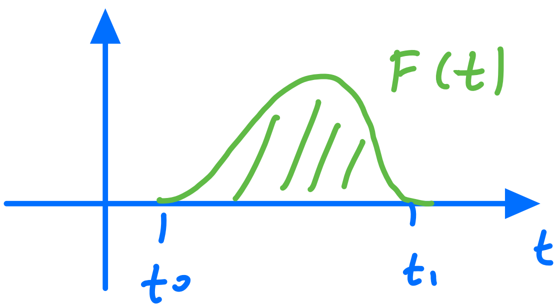
$$\delta(t-a) = u'(t-a)$$



So, the derivative of the step. funct. is the Dirac Delta distribution!!



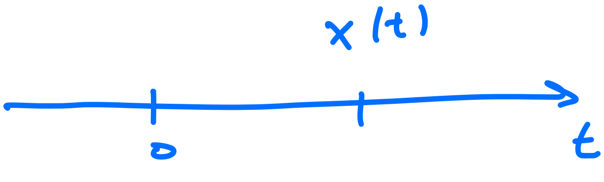
$$\delta(t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(t)\,dt$$

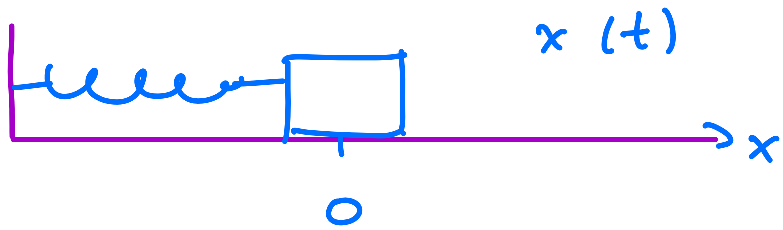


$$\text{Impulse} = \int_{t_0}^{t_1} F(t)\,dt$$

$$= \int_{t_0}^{t_1} m\dot{v}(t)\,dt$$

$$= mv(t_1) - mv(t_0$$





$$\begin{cases} x''(t) + x(t) = \delta(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases} \qquad x'' = -x$$

<u>Sol.</u> Taking the L.T.

$$(s^2+1)X(s) = 1$$

$$X(s) = \frac{1}{s^2+1}$$

$$\rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin(t$$