Project 4
Math 228, Winter 2023, Prof. Knut Solna

Below B, W are standard independent Brownian motions.

Problem 1 (Black-Scholes Price)

Let S_t be the standard Black-Scholes model under pricing measure solving

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad S_0 = s_o, \tag{1}$$

with r the short rate and σ the constant volatility.

a) Derive the explicit expression for S_t.

b) In Eq. (1) replace the constant σ by a deterministic known function of time $\sigma(t)$. Find an expression for S_t in this case.

c) For the price given by the standard Black-Scales model in Eq. (1) derive the standard Black-Scholes formula for the price of a European call option with strike K and maturity T, that is with a payoff at maturity T: $h(S_T) = \max(S_T - K, 0)$.

Problem 2 (Ito's Rule)

a) For $f(t,x) = tx^2$ find $df(t, B_t)$.

b) For $f(x,y) = xy^2$ find $df(B_t, Z_t)$ when $Z_t = \rho B_t + \sqrt{1-\rho^2}W_t,\ \rho \in (0,1)$.

Problem 3 (Implied Volatility)

For the attached option data *SPX_option.txt* which are call option prices on S&P500 compute the implied volatility as function strike, use the short rate 4% per annum in your calculation. In the file the first column is the strike and the second column is the option price, the underlying value is $ 4137 and taken the current time is March 13 2023 and the time to maturity is 123 days. Download from Yahoo finance historical underlying prices for a period before the option data and estimate the historical volatility from these. How does it compare with the computed implied volatility? (Plot the implied volatility as function of strike and mark the current value for the underlying in the plot, plot also the historical pricing data you downloaded and attach the code, to compute the implied volatility you do not have to worry about efficiency, an exhaustive search for a best fit over a grid, set of volatility values, is ok).

Problem 4 (Extra Credit, Portfolio Optimization)

Consider the 5 attached price series corresponding to daily closing prices.

a) Compute the efficient frontier as shown in class using all the data to estimate the asset return statistics.

b) Assume that you additionally can invest in a risk free asset with annual return 4 %, compute the efficient frontier in this case.

c) Use historical pricing data (last 60 days) to estimate the rolling optimal portfolio, using the method above, and plot the corresponding total return process as function of time.

d) Can you do better than in c) (for instance via a neural net) ?