

Improved bounds on the size of permutation codes under Kendall τ -metric

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Abstract

In order to overcome the challenges caused by flash memories and also to protect against errors related to reading information stored in DNA molecules in the shotgun sequencing method, the rank modulation is proposed. In the rank modulation framework, codewords are permutations. In this paper, we study the largest size $P(n, d)$ of permutation codes of length n , i.e., subsets of the set S_n of all permutations on $\{1, \dots, n\}$ with the minimum distance at least $d \in \{1, \dots, \binom{n}{2}\}$ under the Kendall τ -metric. By presenting an algorithm and some theorems, we managed to improve the known lower and upper bounds for $P(n, d)$. In particular, we show that $P(n, d) = 4$ for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$. Additionally, we prove that for any prime number n and integer $r \leq \frac{n}{6}$,

$$P(n, 3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2-8rn+20r^2}} \sqrt{\frac{(n-1)!}{n(n-r)!}}. \text{ This result greatly improves the upper bound of } P(n, 3) \text{ for all primes } n \geq 37.$$

Index Terms

Rank modulation, Kendall τ -Metric, Permutation codes.

I. INTRODUCTION

In order to overcome the challenges caused by flash memories and also to protect against errors related to reading information stored in DNA molecules in the shotgun sequencing method, the rank modulation is proposed (see [15] and [16], respectively). In the rank modulation framework, codewords are permutations. Within this framework, permutation codes were extensively examined using three metrics: the Kendall τ -metric [1], [15], [20], [21], [25], the Ulam metric [18] and the ℓ_∞ metric [17], [19]. This study specifically concentrates on permutation codes under the Kendall τ -metric.

A Permutation Code (PC) of length n represents a non-empty subset of S_n , which includes all permutations of the set $[n] := \{1, 2, \dots, n\}$. In the context of a permutation $\pi := [\pi(1), \pi(2), \dots, \pi(i), \pi(i+1), \dots, \pi(n)] \in S_n$, an adjacent transposition, denoted as $(i, i+1)$ for $1 \leq i \leq n-1$, transforms π into the permutation $[\pi(1), \pi(2), \dots, \pi(i+1), \pi(i), \dots, \pi(n)]$. The Kendall τ -distance between two permutations, ρ and π in S_n , is defined as the minimum number of adjacent transpositions required to express $\rho\pi^{-1}$ as their product. In the context of the Kendall τ -metric, a PC of length n with minimum distance d can correct up to $\frac{d-1}{2}$ errors induced by charge-constrained errors, as cited in [15].

A central question in the theory of PCs is determining the value of $P(n, d)$, that is the size of the largest code in S_n with minimum Kendall τ -distance d , for $d \leq \binom{n}{2}$. The exact value of $P(n, d)$ is determined for $d \in \{1, 2\}$ and $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$ [7] and also for $n = 5$ and for $n = 6$ when $d \neq 3$ [25]. Furthermore,

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n	[1, Theorem 1.1]	Theorem I.2
37	$36! - 15$	$36! - 62$
41	$40! - 16$	$40! - 330$
43	$42! - 17$	$42! - 456$
47	$46! - 18$	$46! - 2537$
53	$52! - 20$	$52! - 155518$
59	$58! - 22$	$58! - 195360$
61	$60! - 23$	$60! - 323371$

TABLE I: Comparing the upper bounds of $P(n, 3)$ obtained from Theorems [1, Theorem 1.1] and Theorem I.2.

several researchers have presented bounds on $P(n, d)$ (see [1], [3], [7], [15], [20], [21], [25]).

In this paper, we present a theorem supporting the value of $P(n, d)$ as follows:

Theorem I.1. $P(n, d) = 4$, for all $n \geq 6$ and $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$.

Moreover, we achieved significant improvements on the lower bound of $P(n, d)$ when $n \in \{7, 8\}$ by constructing new PCs from the subgroups of S_n (see Table II, below) and, in particular, we establish $P(7, 12) = 7$.

Utilizing sphere packing bound (see [15, Theorems 12 and 13]), $P(n, 3) \leq (n-1)!$. In [9, Corollary 2.5 and Theorem 2.6] and [7, Corollary 2], it is proved that if $n > 4$ is a prime number or $4 \leq n \leq 10$, then $P(n, 3) \leq (n-1)! - 1$. Enhancing this, in [1, Theorem 1.1], we improved the upper bound to $P(n, 3) \leq (n-1)! - \lceil \frac{n}{3} \rceil + 2 \leq (n-1)! - 2$ for all primes $n \geq 11$. Here we prove an additional upper bound on $P(n, 3)$ as follows:

Theorem I.2. For a prime number n and integer $r \leq \frac{n}{6}$,

$$P(n, 3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2-8rn+20r^2}} \sqrt{\frac{(n-1)!}{n(n-r)!}}. \quad (\text{I.1})$$

The upper bound for $P(n, d)$ derived from [1, Theorem 1.1] outperforms that from Theorem I.2 for all prime numbers $11 \leq n \leq 31$. However, considering that every prime number greater than 5 can be written in the form of $6n+1$ or $6n+5$, the following corollary shows that Theorem I.2 significantly enhances the upper bound of $P(n, 3)$ for all prime numbers $n \geq 37$.

Corollary I.3. Let $r \geq 6$. If $n = 6r+1$ is a prime number, then

$$P(n, 3) < (6r)! - (1.61)(5r+5)^{\frac{r-4}{2}},$$

and if $n = 6r+5$ is a prime number, then

$$P(n, 3) < (6r+4)! - 5(1.61)(5r+9)^{\frac{r-4}{2}}.$$

In Table I, a comparison is made between the upper bounds of $P(n, 3)$ obtained from [1, Theorem 1.1] and Theorem I.2 for prime numbers $37 \leq n \leq 61$.

The subsequent sections are organized as follows: In Section II, we provide the definitions and notations of PCs and summarize important results regarding bounds on $P(n, d)$. Section III presents a new table of values for lower bounds of $P(n, d)$ for $n \in \{5, 6, 7, 8\}$. In Section IV, we first prove Theorem I.1, and subsequently, using a specific method, we determine the exact value of $P(7, 12)$. Finally, in Section V, we proceed to prove Theorem I.2.

II. PRELIMINARIES

In this section, we first present some definitions and notations for PCs under Kendall τ -metric. Subsequently, we summarize key known results about the bounds used to determine the best known bounds on PCs under Kendall τ -metric in Table II.

Let n be a positive integer and let S_n denote the symmetric group on n letters, i.e., the set of all $n!$ permutations of $[n]$. Throughout this paper, for a permutation $\pi \in S_n$, we employ the vector notation of π as $[\pi(1), \pi(2), \dots, \pi(i), \pi(i+1), \dots, \pi(n)]$. The composition of two permutations π and σ in S_n , denoted by $\sigma\pi$, is defined as $\sigma\pi(i) = \pi(\sigma(i))$ for all $i \in [n]$. The identity element of S_n is denoted by $\xi := [1, 2, \dots, n]$. For distinct elements $i, j \in [n]$, (i, j) , which is called transposition, is the permutation obtained from exchanging i and j in ξ . For a permutation $\pi \in S_n$, let $I(\pi) := |\{(i, j) \in [n]^2 \mid i < j \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|$. In view of the parity of $I(\pi)$, π is called an even or odd permutation. For a set Q , $|Q|$ denotes the size of the set Q .

For two permutations π and ρ in S_n , $d_K(\rho, \pi)$ denotes the Kendall τ -distance between ρ and π . There exists a well-known equivalent expression for $d_K(\rho, \pi)$ [15] as follows: $d_K(\rho, \pi) = |\{(i, j) \in [n]^2 \mid \rho^{-1}(i) < \rho^{-1}(j) \wedge \pi^{-1}(i) > \pi^{-1}(j)\}|$. A PC \mathcal{C} of length n is called an (n, d) -PC, if $d_K(\pi, \sigma) \geq d$ for all distinct elements $\pi, \sigma \in \mathcal{C}$. The largest size of a (n, d) -PC is denoted by $P(n, d)$. It is known that $P(n, 1) = n!$, $P(n, 2) = \frac{n!}{2}$ and if $\frac{2}{3}\binom{n}{2} < d \leq \binom{n}{2}$, then $P(n, d) = 2$ (see [7, Theorem 10]). In the following, we review some results that determine the best known bounds on $P(n, d)$.

For a positive integer r and a permutation $\sigma \in S_n$, the ball of radius r which centered at σ in S_n under the Kendall τ -distance is denoted by $B_r(\sigma)$ defined by $B_r(\sigma) := \{\pi \in S_n \mid d_K(\sigma, \pi) \leq r\}$. Since the Kendall τ -metric is right invariant (i.e., for every three permutations $\sigma, \pi, \rho \in S_n$ we have $d_K(\sigma, \pi) = d_K(\sigma\rho, \pi\rho)$ [7]), the size of a ball of radius r is independent of its center and we denote it by $B_K(r)$. The Gilbert-Varshamov bound and sphere-packing bound for PCS under Kendall τ -metric are as follows:

Proposition II.1. [15, Theorems 12 and 13]

$$\frac{n!}{B_K(d-1)} \leq P(n, d) \leq \frac{n!}{B_K(\lfloor \frac{d-1}{2} \rfloor)}.$$

Let σ and τ be two permutations with $d_K(\sigma, \tau) = 1$. Then the double ball of radius r centered at σ and τ , denoted by $DB_r(\sigma, \tau)$, is defined by $DB_r(\sigma, \tau) := B_r(\sigma) \cup B_r(\tau)$. The size of $DB_r(\xi, [2, 1, 3, \dots, n])$ is denoted by $DB_{n,r}$. There are two useful results for bounds on $P(n, d)$, when d is even, as follows:

Proposition II.2. For all n and $t \geq 1$,

- (1) [7, Corollaries 5 & 6] $P(n, 2(t+1)) \leq \frac{n!}{DB_{n,t}}$. Especially $P(n, 4) \leq \frac{n!}{2(n-1)}$.
- (2) [15, Theorem 21] $P(n, 2t) \geq \frac{1}{2}P(n, 2t-1)$.

The best known relation for the lower bound on $P(n, 3)$ is as follows:

Proposition II.3. $P(n, 3) \geq \frac{n!}{2n-1}$ [15, p. 2116] and in particular if $n-2$ is a prime power, then $P(n, 3) \geq \frac{n!}{2n-2}$ [3, Theorem 4.5].

Remark II.4. By the part (ii) of Proposition II.2 and Proposition II.3, $P(n, 4) \geq \frac{n!}{2(2n-2)}$ if $n-2$ is a prime power and $P(n, 4) \geq \frac{n!}{2(2n-1)}$ otherwise.

There is an important improvement of the lower bound on $P(n, d)$, when $n-2$ is a prime power and $d > 4$ as follows:

Proposition II.5. [20, Theorem 18] Let $m = ((n-2)^{t+1} - 1)/(n-3)$, where $n-2$ is a prime power. Then $P(n, 2t+1) \geq \frac{n!}{(2t+1)m}$ and so $P(n, 2t+2) \geq \frac{n!}{2(2t+1)m}$.

If $\frac{1}{2}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$, then the following bound may turn out to be better than the sphere packing upper bound or part (1) of Proposition II.2.

Proposition II.6. [20, Theorem 23] If $P(n, 2t) \geq M$, then $2\binom{M}{2}t \leq \binom{n}{2}\lfloor \frac{M}{2} \rfloor \lceil \frac{M}{2} \rceil$ and if $P(n, 2t+1) \geq M$, then $(2t+2)(\binom{\lfloor \frac{M}{2} \rfloor}{2} + \binom{\lceil \frac{M}{2} \rceil}{2}) + (2t+1)\lfloor \frac{M}{2} \rfloor \lceil \frac{M}{2} \rceil \leq \binom{n}{2}\lfloor \frac{M}{2} \rfloor \lceil \frac{M}{2} \rceil$.

III. CONSTRUCTING PERMUTATION CODES FROM COSETS OF SUBGROUPS

In this section, we initially devise an algorithm that determines the largest (n, d) -PC under Kendall τ -metric constructed by a subgroup and some of its left cosets (see Remark III.1, below) among all subgroups of S_n for integers n and d . Employing GAP [10] through this algorithm allows us to discover new (n, d) -PCs under Kendall τ -metric, as detailed in Appendix VI, which improve the lower bounds of $P(n, d)$ when $n \in \{7, 8\}$. Subsequently, Table II is presented, illustrating the best-known bounds on $P(n, d)$ for $n \in \{5, 6, 7, 8\}$. Recently, several improved lower bounds for $P(n, d)$ have been obtained in [4], using recursive techniques, automorphisms, and programs that combine randomness and greedy strategies. Notably, the bold and italic entries in the table represent results from the current paper and [4], respectively. Also the blue entries shows the best known of lower bounds for $P(n, d)$, $n \in \{7, 8\}$.

Remark III.1. If H is a subgroup of a finite group G and $g \in G$, then $Hg := \{hg \mid h \in H\}$ and $gH := \{gh \mid h \in H\}$ are called a right coset of H and a left coset of H , respectively, with the representative g . It is known that if X be the set of right (left) cosets of H in G , i.e., $\mathbf{X} := \{Hg \mid g \in G\}$ ($\mathbf{X} := \{gH \mid g \in G\}$), then \mathbf{X} partitions G , i.e., $G = \cup_{X \in \mathbf{X}} X$ and $X \cap X' = \emptyset$ for all distinct elements X and X' of \mathbf{X} , and $|\mathbf{X}| = |G|/|H|$.

Description of Algorithm 1: Algorithm 1 takes two input integers, n and d . It initializes G and T as the symmetric group on the set $[n]$ and all subgroups of G , respectively (using GAP [10], access to all subgroups of G is possible with “ConjugacyClassesSubgroups(G)”). The algorithm comprises three functions: Δ , Λ , and Θ . The first two return the minimum Kendall τ -distance between elements of a subgroup and a subset, respectively. The third function returns the minimum Kendall τ -distance between g and all elements of a set M . Notice that if H is a subgroup, then since the Kendall τ -metric is right invariant and $hh_0^{-1} \in H$ for all elements $h, h_0 \in H$, $\min\{d_K(h, h_0) \mid h \neq h_0, h, h_0 \in H\} = \min\{d_K(h, \xi) \mid h \in H\}$. Hence, in order to reduce computer calculations, the algorithm define a separate function for calculating the minimum kendall τ -distance between elements of a subgroup. It initializes two lists D and L to be empty lists. All subgroups of G that are (n, d) -PCs are added to the list D . For each $H \in D$, the algorithm initializes a list L_H as the set of left transversal set H in G (i.e., $\{xH \mid x \in L(H)\}$ is the set of all left cosets of H in G). The goal is to find the largest subset S_H of L_H such that $\xi \in S_H$ and $\cup_{x \in S_H} xH$ is an (n, d) -PC. For the latter, it first initializes two lists M_H and S_H to be list of elements of H and empty list, respectively. Next, for all $j \in L_H$, if jH is an (n, d) -PC and if the minimum Kendall τ -distance between j and all elements of M is at least d , then it add j to the set S_H and $M := M \cup jH$. Note that if there exist $x, y \in T$ such that xH and yH are two (n, d) -PC and $d_K(xh, y) \geq d$ for all $h \in H$, then the right invariant property of the Kendall τ -metric implies that $xH \cup yH$ is an (n, d) -PC. With this procedure, the algorithm creates the subset S_H of L_H to achieve its goal. Finally, for each subgroup H of G , it adds the list $[H, S_H]$ to L . Now by considering the elements of the list L we can find the largest (n, d) -PC is created by the algorithm 1.

It is worth noting that the construction of PCs using certain subgroups of symmetric groups and their right cosets under the Hamming metric has already been explored (see, for instance, [5]). Since the Hamming metric on S_n is left and right invariant, for adding each right (left) coset to the previously created PC of a certain subgroup and its right (left) cosets, it is enough to check the minimum distance of the representative of that coset with the previous PC. Also, in Algorithm 1, because the Kendall τ -metric is only right invariant, adding the left cosets to the previously constructed PC is used to reduce the calculations.

IV. THE VALUE OF $P(n, d)$ FOR CERTAIN VALUES OF d

Within this section, we initially establish the proof for Theorem I.1 and subsequently ascertain the exact value of $P(7, 12)$. The proof of Theorem I.1 relies on the following straightforward lemma.

Algorithm 1: Construction (n, d) -PCs from the subgroups and some their cosets.

Input: Integer numbers n and d .

Output: A list of elements as $[H, S_H]$ such that $\cup_{x \in \{\xi\} \cup S_H} xH$ is an (n, d) -PC.

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1:  $G \leftarrow$  symmetric group on  $n$  letters
2:  $T \leftarrow$  all subgroups of  $G$ 
3:  $\Delta \leftarrow$  a function whose input is a
   subgroup  $H$ 
4:  $S \leftarrow []$ 
5: for all  $i$  in  $H$  do
6:   add Kendall  $\tau$ -distance between  $\xi$  and  $i$  to  $S$ 
7: end for
8: return(minimum of the list  $S$ )
9: end function
10:  $\Lambda \leftarrow$  a function whose input is the
    subset  $N$ 
11:  $S \leftarrow []$ 
12: for all  $i$  and  $j$  in  $N$  do
13:   add Kendall  $\tau$ -distance between  $i$  and  $j$  to  $S$ 
14: end for
15: return(minimum of the list  $S$ )
16: end function
17:  $\Theta \leftarrow$  a function whose inputs are a subset  $M$ 
    and an element  $g$  of  $G$ 
18:  $S \leftarrow []$ 
19: for all  $i$  in  $M$  do
20:   add Kendall  $\tau$ -distance between  $g$  and  $i$  to  $S$ 
21: end for
22: return(minimum of the list  $S$ )
23: end function
24:  $D \leftarrow []$ 
25: for all  $i$  in  $T$  do
26:   if  $\Delta(i) \geq d$  then add  $i$  to the list  $D$ 
27:   end if
28: end for
29:  $L \leftarrow []$ 
30: for all  $H$  in  $D$  do
31:    $L_H \leftarrow$  left transversal set  $H$  in  $G$ 
32:    $M_H \leftarrow$  elements of  $H$ 
33:    $S_H \leftarrow []$ 
34:   for all  $j$  in  $L_H$  do
35:     if  $\Lambda(\text{elements of } (jH)) \geq d$  and
        $\Theta(M, j) \geq d$  then
38:       add  $j$  to the set  $S_H$ 
39:        $M \leftarrow$  union of  $M_H$  and the left coset  $jH$ 
       of  $H$  in  $G$ 
40:     end if
41:   end for
42:   add  $[H, S_H]$  to the set  $L$ 
43: end for

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Lemma IV.1. Let $n \geq 5$ be an integer. If $n \equiv 0, 2 \pmod{3}$ ($n \equiv 1 \pmod{3}$), then there exist 3 non-empty subsets with the same sumset which partitions $[n]$ ($[n] \setminus \{1\}$), respectively.

Proof. If n is 5, 6, 7, 8, 9 and 10, respectively, then $\{\{5\}, \{1, 4\}, \{3, 2\}\}$, $\{\{6, 1\}, \{5, 2\}, \{3, 4\}\}$, $\{\{2, 7\}, \{3, 6\}, \{4, 5\}\}$, $\{\{8, 4\}, \{7, 3, 2\}, \{1, 5, 6\}\}$, $\{\{6, 5, 4\}, \{9, 1, 2, 3\}, \{8, 7\}\}$ and $\{\{10, 8\}, \{9, 2, 7\}, \{3, 4, 6, 5\}\}$ are the partitions of $[n]$ or $[n] \setminus \{1\}$ satisfying the lemma. Now suppose that $n > 10$. Hence there exist $t > 0$ and $r \in \{5, 6, 7, 8, 9, 10\}$ such that $n = 6t + r$. Note that if $n \equiv 1 \pmod{3}$, then $r \in \{7, 10\}$.

TABLE II: Best known lower bound (LB) and upper bound (UB) on $P(n, d)$.

n/d		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17-18
5	LB=UB	20^i	12^i	6^i	5^i	2^c	2^c	2^c	2^c	—	—	—	—	—	—	—
6	UB	116^b	64^i	26^i	20^i	11^i	7^i	4^i	4^i	2^c	2^c	2^c	—	—	—	—
	LB	102^i	64^i	26^i	20^i	11^i	7^i	4^i	4^i	2^c	2^c	2^c	—	—	—	—
7	UB	716^b	420^c	186^a	120^c	66^a	45^c	28^a	21^c	10	7	4^g	4^g	2^c	2^c	2^c
	LB	—	336	126	84	42	—	—	13	8	7	4	—	—	—	—
	LB	—	315	126	84	42	28	15	12	8	7	4	4	—	—	—
	OLB	588^c	294^f	110^d	55^c	34^d	17^f	14^d	7^a	2^a	2^a	2^a	2^a	2^c	2^c	2^c
8	UB	5039^c	2880^c	1152^a	720^c	363^a	242^c	141^a	99^c	64^a	47^c	32^a	25^c	10^g	8^g	4^g
	LB	3752	2240	672	448	168	115	57	43	26	21	15	12	8	—	—
	LB	3696	2184	672	392	168	112	48	48	24	24	14	14	8	8	4
	OLB	2688^h	1344^a	142^a	76^a	33^a	20^a	12^a	7^a	6^a	4^a	3^a	3^a	1^a	1^a	1^a

Key to the superscripts used in Table

superscript a	Sphere packing bound
superscript b	Sphere packing bound+[1, Theorem 3.5]
superscript c	[7, Corollary 5 or Theorems 10,12 or 13]
superscript d	Lower bounds from [15]
superscript f	[15, Theorem 21]
superscript g	[20, Theorem 23]
superscript h	$P(n, 3) \geq \frac{n!}{2^{n-1}}$ [15]
superscript i	[25, Table II]
an entry in bold	Tables III and IV and Theorem IV.6
an entry in italic	Lower bounds from [4]
blue entries	Best known lower bounds for $P(n, d)$, $n \in \{7, 8\}$

Consider $t + 1$ subsets $\Theta_1, \dots, \Theta_{t+1}$ of $[n]$ as follows:

$$\underbrace{1, \dots, r}_{\Theta_1}, \underbrace{r+1, \dots, r+6}_{\Theta_2}, \dots, \underbrace{n-11, \dots, n-6}_{\Theta_t}, \underbrace{n-5, \dots, n}_{\Theta_{t+1}}$$

Clearly, for all $2 \leq i \leq t + 1$, $\Theta_i = \{k_i + 1, k_i + 2, \dots, k_i + 6\}$, where $k_i = r + (i - 2)6 + 1$. Hence, 3 sets $\Theta_{i1} := \{k_i + 1, k_i + 6\}$, $\Theta_{i2} := \{k_i + 2, k_i + 5\}$ and $\Theta_{i3} := \{k_i + 3, k_i + 4\}$ with the same sumset partition the set Θ_i . Therefore, since $r \in \{5, 6, 7, 8, 9, 10\}$, corresponding to each set Θ_i , $1 \leq i \leq t + 1$, there exists a partition of 3 sets Θ_{i1} , Θ_{i2} and Θ_{i3} with the same sumset for Θ_i . Let $\Delta_j := \cup_{i=1}^{t+1} \Theta_{ij}$ for all $j \in \{1, 2, 3\}$. So Δ_1 , Δ_2 and Δ_3 with the same sumset partition $[n]$ or $[n] \setminus \{1\}$ if $n \equiv 0, 2 \pmod{3}$ or $n \equiv 1 \pmod{3}$, respectively. This completes the proof. \square

Proof of Theorem I.1. It follows from [20, Theorem 23] that if $P(n, d) \geq 5$, then we must have $\binom{5}{2}d \leq 6 \times \binom{n}{2}$ and therefore $d \leq \frac{3}{5}\binom{n}{2}$. So for all $\frac{3}{5}\binom{n}{2} < d \leq \frac{2}{3}\binom{n}{2}$, $P(n, d) \leq 4$. Since $P(n, d + 1) \leq P(n, d)$, it is enough to show that there exists an $(n, \lfloor 2/3\binom{n}{2} \rfloor)$ -PC of size 4.

Let $N := \sum_{i=1}^{n-1} i = \binom{n}{2}$. It follows from Lemma IV.1 that there exist pairwise distinct subsets Δ_1, Δ_2 and Δ_3 of $[n - 1]$ or $[n - 1] \setminus \{1\}$ such that if $n - 1 \equiv 0, 2 \pmod{3}$ or $n - 1 \equiv 1 \pmod{3}$, respectively, then $\sum_{j \in \Delta_i} j = \frac{N}{3}$ or $\sum_{j \in \Delta_i} j = \frac{N-1}{3}$, for all $i \in \{1, 2, 3\}$. Now, suppose that for $n \geq 6$, the subsets Δ_1, Δ_2 and Δ_3 of $[n - 1]$ are determined. Corresponding to each Δ_i , we introduce a permutation α_i as follows: let $r_i := |\Delta_i|$, $\Delta'_i := \{n - j \mid j \in \Delta_i\}$ and $\Theta_i := [n] \setminus \Delta'_i$. Suppose that $j_1 < j_2 < \dots < j_{r_i}$ and $l_0 < l_1 < \dots < l_{n-r_i-1}$ are all elements of Δ'_i and Θ_i , respectively. Let $\alpha_i \in S_n$ such that $\alpha_i(t) = j_t$ and $\alpha_i(n - s) = l_s$ for all $t \in \{1, \dots, r_i\}$ and $s \in \{0, \dots, n - r_i - 1\}$. Let α_x and α_y be two distinct permutations corresponding to distinct subsets Δ_x and Δ_y , $x, y \in \{1, 2, 3\}$. In view of the definition of α_x , if $i < j$ are two elements of $[n]$, then $\alpha_x^{-1}(i) < \alpha_x^{-1}(j)$ if and only if $i \in \Delta'_x$. So, since $\Delta'_x \cap \Delta'_y = \emptyset$,

we have $(i, j) \in [n]^2$ satisfies $\alpha_x^{-1}(i) < \alpha_x^{-1}(j)$ and $\alpha_y^{-1}(i) > \alpha_y^{-1}(j)$, if and only if $(i, j) \in A \cup B$, where $A := \{(i, j) \mid i < j, i \in \Delta'_x\}$ and $B := \{(i, j) \mid i > j, j \in \Delta'_y\}$. Hence

$$\begin{aligned} d_K(\alpha_x, \alpha_y) &= |\{(i, j) \mid \alpha_x^{-1}(i) < \alpha_x^{-1}(j) \wedge \alpha_y^{-1}(j) > \alpha_y^{-1}(i)\}| \\ &= |A \cup B| = |A| + |B|. \end{aligned}$$

Therefore, $d_K(\alpha_x, \alpha_y) = \sum_{i \in \Delta_x} i + \sum_{i \in \Delta_y} i$ and so $d_K(\alpha_x, \alpha_y)$ is equal to $\frac{2N}{3}$ if $n - 1 \equiv 0, 2 \pmod{3}$ and otherwise is equal to $\frac{2(N-1)}{3} = \lfloor \frac{2N}{3} \rfloor$. Also it is easy to see that

$$\begin{aligned} d_K(\xi, \alpha_x) &= |\{(i, j) \mid i < j \wedge \alpha_x^{-1}(i) > \alpha_x^{-1}(j)\}| \\ &= |\{(i, j) \mid i < j, i \in \Theta_x\}|, \end{aligned}$$

and therefore $d_K(\xi, \alpha_x)$ is equal to $N - \frac{N}{3} = \frac{2}{3}N$ if $n - 1 \equiv 0, 2 \pmod{3}$ and is equal to $N - \frac{N-1}{3} = \frac{2N+1}{3} > \lfloor \frac{2N}{3} \rfloor$ if $n - 1 \equiv 1 \pmod{3}$. Hence, $\{\xi, \alpha_1, \alpha_2, \alpha_3\}$ is an $(n, \lfloor \frac{2N}{3} \rfloor)$ -PC of size 4. This completes the proof. \square

Example IV.2. Let $n = 14$. Suppose $\Delta_1 := \{2, 7, 8, 13\}$, $\Delta_2 := \{3, 6, 9, 12\}$ and $\Delta_3 := \{4, 5, 10, 11\}$. Then $\Delta_1, \Delta_2, \Delta_3$ have the same sumset 30 and partitions $\{2, 3, \dots, 13\}$. Hence by the proof of Theorem I.1, $\{\xi, \alpha_1, \alpha_2, \alpha_3\}$ is a $(14, 60)$ -PC, where

$$\begin{aligned} \alpha_1 &= [1, 6, 7, 12, 14, 13, 11, 10, 9, 8, 5, 4, 3, 2], \\ \alpha_2 &= [2, 5, 8, 11, 14, 13, 12, 10, 9, 7, 6, 4, 3, 1], \\ \alpha_3 &= [3, 4, 9, 10, 14, 13, 12, 11, 8, 7, 6, 5, 2, 1]. \end{aligned}$$

Definition IV.3. A permutation code \mathcal{C} is called equidistance (called EPC for short) under Kendall τ -distance whenever all two distinct permutations in \mathcal{C} have the same Kendall τ -distance. The maximum size of the largest EPC of length n and Kendall τ -distance d denoted by $EP(n, d)$. Also we denote by $P(n, d, m, d')$, the size of the largest PC with minimum Kendall τ -distance d in S_n such that contains an EPC of size m and Kendall τ -distance d' .

The problem of determining bounds on EPCs under the Hamming metric back to the 1970s, beginning with a question of Bolton in [6]. Various studies, including [12], [13], [22], [24], have explored this topic due to its applications in powerline communications and balanced scheduling. For a brief overview of EPCs under the Hamming metric, you can refer to [8, Section VI.44.5]. However there is no special study on EPCs under the Kendall τ -metric, and only in [25, p. 3160], the number of permutations that have the same distance with the identity element has been studied. In the subsequent discussion, we leverage the notion of EPCs under the Kendall τ -metric to demonstrate that $P(7, 12) = 7$.

Proposition IV.4. 1) For each $1 \leq d \leq \binom{n}{2}$ and $\sigma \in S_n$, there exists an element $\pi \in S_n$ such that $d_K(\sigma, \pi) = d$.
 2) If d is an odd number, then $EP(n, d) = 2$.
 3) $EP(n, d) = 2$, for all $2/3 \binom{n}{2} < d \leq \binom{n}{2}$.
 4) $EP(n, 2/3 \binom{n}{2}) = 4$.

Proof. First, we show that for each $1 \leq t \leq \binom{n}{2}$, there exists a subset $A \subseteq [n-1]$ such that $\sum_{a \in A} a = t$. If $t < n$, then there is nothing to prove. So we assume $t \geq n$. Since $t \leq \binom{n}{2}$, there exists $i \in [n]$ such that $\sum_{j=1}^i (n-j) \leq t \leq \sum_{j=1}^{i+1} (n-j)$. Suppose that $s = t - \sum_{j=1}^i (n-j)$. So either $s = 0$ or $s < n-i$. Hence, if $s = 0$, then $A = \{n, n-1, \dots, n-i\}$ and if $s < n-i$, then $A = \{n, n-1, \dots, n-i, s\}$. Let $d \leq \binom{n}{2}$ and $A \subseteq [n-1]$ such that $\sum_{a \in A} a = \binom{n}{2} - d$. Also let $|A| = r$, $\Delta := \{n-j \mid j \in A\}$ and $\Theta := [n] \setminus \Delta$. Suppose that $j_1 < j_2 < \dots < j_r$ and $l_0 < l_1 < \dots < l_{n-r-1}$ are all elements of Δ and Θ , respectively. Let $\alpha \in S_n$ such that $\alpha(t) = j_t$ and $\alpha(n-s) = l_s$ for all $t \in \{1, \dots, r\}$ and $s \in \{0, \dots, n-r-1\}$. It is easy to see that $|\{(i, j) \in [n]^2 \mid i < j \wedge \alpha^{-1}(i) > \alpha^{-1}(j)\}| = \binom{n}{2} - \sum_{a \in A} a = d$.

Therefore $d = d_K(\xi, \alpha) = d_K(\sigma, \alpha\sigma)$ and this completes the proof of part (1). Since the composition of two odd (even) permutations is an even permutation, the right invariant property of the Kendall τ -metric implies that the Kendall τ -distance between two permutations of the same parity is even. Hence the part (2) follows from part (1). Also the part (3) follows from the part (1) and [7, Theorem 10] and the part (4) follows from the proof of Theorem I.1. This completes the proof. \square

Lemma IV.5. $EP(7, 12) = 7$ and $P(7, 11, 6, 12) = 7$.

Proof. Let \mathcal{C} be an $(7, 12)$ -EPC under the Kendall τ -metric of maximum size. Without loss of generality we may assume that $\xi \in \mathcal{C}$ as Kendall τ -metric is right invariant. Let $\mathcal{A} := \{\sigma \in S_n \mid d_K(\xi, \sigma) = 12\}$. Using GAP [10], $|\mathcal{A}| = 531$. It is sufficient that we find the maximum EPC with Kendall τ -distance 12 in \mathcal{A} . Let \mathcal{A}_i , $2 \leq i \leq 7$, be the set of all subsets of size i in \mathcal{A} such that the Kendall τ -distance between any pair of distinct elements in any of them is equal to 12. Using GAP, $|\mathcal{A}_2| = 27697$, $|\mathcal{A}_3| = 172629$, $|\mathcal{A}_4| = 131777$, $|\mathcal{A}_5| = 10862$, $|\mathcal{A}_6| = 9$ and $|\mathcal{A}_7| = 0$. So, the size of largest EPC in \mathcal{A} is 6 and therefore $|\mathcal{C}| = 7$.

Suppose that \mathcal{C} is a $(7, 11)$ -PC such that contains an EPC $\bar{\mathcal{C}}$ of size 6 and the Kendall τ -distance 12. Without loss of generality we may assume that $\xi \in \bar{\mathcal{C}}$. According to the proof of the first part, $\bar{\mathcal{C}} \in \mathcal{M} := \{\{\xi\} \cup A \mid A \in \mathcal{A}_5\}$. So there are 10862 distinct cases for $\bar{\mathcal{C}}$. Let $\mathcal{B}_M := \{\sigma \in S_n \mid d_K(m, \sigma) \geq 11, \forall m \in M\}$, for all $M \in \mathcal{M}$. Using GAP, for all $M \in \mathcal{M}$, $0 \leq |\mathcal{B}_M| \leq 14$ and if $|\mathcal{B}_M| \neq 0$ and $b_1, b_2 \in \mathcal{B}_M$ then $d_K(b_1, b_2) < 11$. This completes the proof. \square

Theorem IV.6. $P(7, 12) = 7$ and $8 \leq P(7, 11) \leq 10$.

Proof. Let \mathcal{C} be a $(7, d)$ -PC under the Kendall τ -metric and $\Sigma := \sum_{c_1, c_2 \in \mathcal{C}} d_K(c_1, c_2)$. By a same argument as in the proof of [20, Theorem 23], it can be seen that $\Sigma \leq \binom{n}{2} \lceil \frac{|\mathcal{C}|}{2} \rceil \lfloor \frac{|\mathcal{C}|}{2} \rfloor$. By Theorem II.6, $P(7, 12) \leq 8$. As shown in Table II, $P(7, 12) \geq 7$. Then it is sufficient that we show $P(7, 12) \neq 8$. Suppose for a contradiction that \mathcal{C} be an $(7, 12)$ -PC of size 8. So we must have $\Sigma \leq 336$. On the other hand, since $EP(7, 12) = 7$ and $|\mathcal{C}| = 8$, there exist c_1, c_2 in \mathcal{C} such that $d_K(c_1, c_2) > 12$. Hence, $\Sigma \geq \binom{8}{2} \times 12 + 1 = 337$ that is a contradiction. So $P(7, 12) = 7$.

As shown in Table II, $P(7, 11) \geq 8$. Theorem II.6 implies that $P(7, 11) \leq 12$. Suppose for a contradiction that \mathcal{C} be an $(7, 11)$ -PC of size 12 or 11. Let $\mathcal{C}_1 := \mathcal{C} \cap A_n$ and $\mathcal{C}_2 := \mathcal{C} \setminus \mathcal{C}_1$, where A_n denotes the set of all even permutation in S_n . Without loss of generality we may assume that $|\mathcal{C}_1| \geq |\mathcal{C}_2|$. Since the Kendall τ -distance between two permutations of the same parity is even, if c_1 and c_2 are two distinct elements in \mathcal{C}_1 or \mathcal{C}_2 , then $d_K(c_1, c_2) \geq 12$. So

$$\binom{|\mathcal{C}_1|}{2} \times 12 + \binom{|\mathcal{C}_2|}{2} \times 12 + 11 \times |\mathcal{C}_1| \times |\mathcal{C}_2| \leq \Sigma.$$

If $|\mathcal{C}| = 12$ and 11, then $\Sigma \leq 756$ and 630, respectively. Hence it can be seen that if $|\mathcal{C}| = 12$, then \mathcal{C}_1 and \mathcal{C}_2 must be two $(7, 12)$ -EPC of sizes 6 and also if $|\mathcal{C}| = 11$, then \mathcal{C}_1 and \mathcal{C}_2 must be two $(7, 12)$ -EPC of sizes 6 and 5, respectively. Therefore if $|\mathcal{C}| \in \{11, 12\}$, then \mathcal{C} is an $(7, 11)$ -PC such that contains a $(7, 12)$ -EPC of size 6 that is contradict with Lemma IV.5, This completes the proof. \square

V. NEW UPPER BOUND OF $P(n, 3)$

In this section, we will follow the definitions and notations outlined in [1]. Specifically, we adopt the following definitions: Let H be a subgroup of a finite group G and $X = \{Ha_1, \dots, Ha_m\}$ be the set of right cosets of H in G . Fix the ordering on X as $Ha_i < Ha_j$ whenever $i < j$. Then ρ_X^G is the map from G to $GL_m(\mathbb{Z})$ (the group of all $m \times m$ invertible matrices with integer entries) defined by $g \rightarrow P_g$, where P_g is the $m \times m$ matrix whose (i, j) entry is 1 if $Ha_i g = Ha_j$ and 0 otherwise. Furthermore, if $Y \subseteq G$, then $\widehat{Y \rho_X^G}$ represents the element $\sum_{y \in Y} y \rho_X^G = \sum_{y \in Y} P_y$ in $Mat_m(\mathbb{Z})$, the set of all $m \times m$ matrices over \mathbb{Z} .

By a number partition λ of n (with the length m) we mean an m -tuple $(\lambda_1, \dots, \lambda_m)$ of positive integers such that $\lambda_1 \geq \dots \geq \lambda_m$ and $n = \sum_{i=1}^m \lambda_i$. According to [1, Definition 3.1 and Remark 3.2], the Young subgroup corresponding to a partition $(\lambda_1, \dots, \lambda_m)$ of a positive integer n refers to the subgroup H of S_n defined as $H := S_{\Delta_1} \times \dots \times S_{\Delta_m} = \{\sigma_1 \dots \sigma_m \mid \sigma_i \in \Delta_i, 1 \leq i \leq m\}$, where $(\Delta_1, \dots, \Delta_m) = (\{1, \dots, \lambda_1\}, \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \dots, \{n - \lambda_m + 1, \dots, n\})$, and S_{Δ_i} denotes the symmetric group on the set Δ_i for all $i = 1, \dots, m$.

Lemma V.1. *Let H be the Young subgroup of S_n corresponding to the partition $\lambda := (n - r, \underbrace{1, \dots, 1}_r)$ and X be the set of right cosets of H in S_n . If $S = \{(i, i + 1) \mid 1 \leq i \leq n - 1\}$ and $T := S \cup \{\xi\}$, then $T^{\rho_X^{S_n}}$ is a symmetric matrix $A = (a_{ij})_{\ell \times \ell}$, where $\ell = \frac{n!}{(n-r)!}$, with the following properties:*

- 1) $a_{ii} \geq n - 2r$ for all $i \in [\ell]$.
- 2) $a_{ij} \in \{0, 1\}$ for all $i \neq j \in [\ell]$.
- 3) $\sum_{j=1}^{\ell} a_{ij} = n$ for all $i \in [\ell]$.

Proof. In view of [1, Remark 3.2], without loss of generality we may assume that λ is the partition $\{[n - r], [n - r + 1], [n - r + 2], \dots, [n]\}$ of n and therefore $H \cong S_{n-r}$. Let $\mathcal{F} := \{(f_1, f_2, \dots, f_r) \in [n]^r \mid \forall i \neq j, f_i \neq f_j\}$. Corresponding to each ordered r -tuple $F = (f_1, \dots, f_r) \in \mathcal{F}$, let $S_n^F := \{\sigma \in S_n \mid \sigma(n - r + 1) = f_1, \sigma(n - r + 2) = f_2, \dots, \sigma(n) = f_r\}$. Clearly, $S_n^F = Hg$, where $g = (n - r + 1, f_1)(n - r + 2, f_2) \dots (n, f_r)$, is a right coset of H and if F and \bar{F} are two distinct elements of \mathcal{F} , then $S_n^F \cap S_n^{\bar{F}} = \emptyset$. Hence, since $|\mathcal{F}| = \ell$, in view of Remark III.1, $X = \{S_n^F \mid F \in \mathcal{F}\}$ is the set of all right cosets of H in S_n . Suppose that $F_1, F_2, \dots, F_{\ell}$ are all ordered r -tuples in \mathcal{F} . Fix the ordering of X such that $S_n^{F_i} < S_n^{F_j}$ if $i < j$, for all $i, j \in [\ell]$. In view of [1, Definition 2.10], the (i, j) entry of $T^{\rho_X^{S_n}}$ is equal to $|\mathcal{O}_{ij}|$, where $\mathcal{O}_{ij} := \{t \in T \mid S_n^{F_i} t = S_n^{F_j}\}$. Since $\mathcal{O}_{ij} = \mathcal{O}_{ji}$ for all $i, j \in [\ell]$, A is a symmetric matrix. Let $(i, i + 1) \in T$ and let $F = (f_1, \dots, f_r)$ and $\bar{F} = (\bar{f}_1, \dots, \bar{f}_r)$ be two distinct elements of \mathcal{F} . The sufficient condition for $S_n^F(i, i + 1) = S_n^{\bar{F}}$ is $\{i, i + 1\} \cap \{f_1, \dots, f_r\} = \emptyset$. So $a_{ss} \geq n - 2r$ for all $s \in [\ell]$. Now suppose for a contradiction that there exists $(j, j + 1) \in T \setminus \{(i, i + 1)\}$ such that $S_n^F(i, i + 1) = S_n^{\bar{F}} = S_n^F(j, j + 1)$. Since $F \neq \bar{F}$ we have $P_1 := \{f_1, \dots, f_r\} \cap \{i, i + 1\} \neq \emptyset$ and $\{f_1, \dots, f_r\} \cap \{j, j + 1\} \neq \emptyset$. Suppose that $i \in P_1$ and $f_m = i$ for some $m \in [r]$. Then for all $\sigma \in S_n^F$, $(\sigma(i, i + 1))(n - r + m) = i + 1$ and $(\sigma(j, j + 1))(n - r + m)$ is equal to j if $i = j + 1$ and is equal to i if $\{i, i + 1\} \cap \{j, j + 1\} = \emptyset$. So $S_n^F(i, i + 1) \neq S_n^F(j, j + 1)$ that is a contradiction. Now suppose that $i + 1 \in P_1$ and $f_d = i + 1$ for some $d \in [r]$. then by the same argument it can be seen that $(\sigma(i, i + 1))(n - r + d) \neq (\sigma(j, j + 1))(n - r + d)$, for all $\sigma \in S_n^F$. Hence, $S_n^F(i, i + 1) \neq S_n^F(j, j + 1)$ that is a contradiction. Hence, $a_{ij} \in \{0, 1\}$ for all $i \neq j \in [\ell]$. Note that for each $x \in [\ell]$, since $\cup_{y=1}^{\ell} \mathcal{O}_{xy} = T$ and $\mathcal{O}_{xy} \cap \mathcal{O}_{xy'} = \emptyset$ for all $y \neq y' \in [\ell]$, we have $\sum_{j=1}^{\ell} a_{ij} = n$ for all $i \in [\ell]$. This completes the proof. \square

Here, we provide some notations used in the proof of Theorem I.2. The transposition of a matrix or vector is denoted by \cdot^t . The inner product of two vectors $\mathbf{x} = (x_1, \dots, x_n)^t$ and $\mathbf{y} = (y_1, \dots, y_n)^t$ in \mathbb{R}^n is defined as $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^t \mathbf{y} = \sum_{i=1}^n x_i y_i$, the notation $\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ denotes the 2-norm of vector \mathbf{x} and the notation $\|\mathbf{x}\|_1 := \sum_{i=1}^n |x_i|$ denotes the 1-norm of vector \mathbf{x} , where $|a|$ denotes the absolute value of real number a .

Definition V.2. [11] A polyhedral cone is a subset $\mathcal{C} \subset \mathbb{R}^n$ of the form $\mathcal{C} := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{0}\}$, for a matrix $A \in \mathbb{R}^{m \times n}$ and column vector $\mathbf{0}$ of order $n \times 1$ whose entries are equal to 0.

Remark V.3. Let $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{0}\}$ be a polyhedral cone for a non-singular matrix $A \in \mathbb{R}^{n \times n}$. In view of [11, p. 104-105], the vector $\mathbf{d} \in \mathbb{R}^n$ is called an extreme ray of \mathcal{C} , if there exists $1 \leq i \leq n$ such that $A_i \mathbf{d} = \mathbf{0}$ and $a_i \mathbf{d} \leq 0$, where a_i denotes the i -th row of the matrix A and A_i is the submatrix of A obtained by removing a_i . We say that two extreme rays \mathbf{d} and \mathbf{d}' of \mathcal{C} are equivalent, and denote it by $\mathbf{d} \sim \mathbf{d}'$, if one is a positive multiple of the other. In view of [11, p. 101-105], the number of equivalence

classes of extreme rays in \mathcal{C} is finite. Also according to [11, p. 105], if $\{\mathbf{w}_1, \dots, \mathbf{w}_s\}$ is a complete set of representatives of all equivalence classes of extreme rays in \mathcal{C} , then $\mathcal{C} = \{\sum_{i=1}^s \lambda_i \mathbf{w}_i \mid \lambda_i \geq 0\}$.

Theorem V.4. Let r and n be integers such that $r < \frac{n}{4}$ and $n \nmid (n-r)!$. Then

$$P(n, 3) \leq (n-1)! - \frac{n-6r}{\sqrt{n^2 - 8rn + 20r^2}} \sqrt{\frac{(n-1)!}{n(n-r)!}}.$$

Proof. Let C be a PC in S_n with minimum Kendall τ -distance 3. Let H be the Young subgroup of S_n corresponding to the partition $\lambda := (n-r, \underbrace{1, \dots, 1}_r)$ and Y be the set of right cosets of H in S_n . If

$S = \{(i, i+1) \mid 1 \leq i \leq n-1\}$ and $T := S \cup \{\xi\}$, then by Lemma V.1, $\widehat{T^{\rho_Y^{S_n}}}$ is a matrix $A = (a_{ij})_{\ell \times \ell}$, $\ell = \frac{n!}{(n-r)!}$, with properties specified in Lemma V.1. Theorem [1, 2.14] implies that the optimal value of the objective function of the following integer programming problem gives an upper bound on $|C|$

$$\begin{aligned} & \text{Maximize} \quad \sum_{i=1}^{\ell} x_i, \\ & \text{subject to} \quad A(x_1, \dots, x_{\ell})^t \leq |H|\mathbf{1} = (n-r)!\mathbf{1}, \\ & \quad \quad \quad x_i \in \mathbb{Z}, \quad x_i \geq 0, \quad i \in \{1, \dots, \ell\}, \end{aligned}$$

where $\mathbf{1}$ is a column vector of order $\ell \times 1$ whose entries are equal to 1. Let α be a feasible solution for the above linear inequality system that achieves the optimum of the objective function and $\beta := \frac{(n-r)!}{n}\mathbf{1}$. It follows from the part (3) of Lemma V.1 that the sum of every row in A is equal to n and so $A\beta = (n-r)!\mathbf{1}$. Since $n \nmid (n-r)!$ we have $\alpha \neq \beta$. It is clear that $\sum_{i=1}^{\ell} \alpha_i \leq (n-1)!$, where α_i denotes the i -th entry of α , and suppose that $\sum_{i=1}^{\ell} \alpha_i = (n-1)! - k$ for a non-negative integer k . Consider two vectors $\vec{\beta\alpha} := \alpha - \beta$ and $-\mathbf{1}$. We let

$$\begin{aligned} \mu &:= \frac{\langle -\mathbf{1}, \vec{\beta\alpha} \rangle}{\|-\mathbf{1}\| \|\vec{\beta\alpha}\|} = \frac{\langle -\mathbf{1}, \alpha - \beta \rangle}{\|-\mathbf{1}\| \|\alpha - \beta\|} \\ &= \frac{\langle -\mathbf{1}, \alpha \rangle + \langle -\mathbf{1}, -\beta \rangle}{\|-\mathbf{1}\| \|\alpha - \beta\|} \\ &= \frac{\ell \frac{(n-r)!}{n} - \sum_{i=1}^{\ell} \alpha_i}{\sqrt{\ell} \sqrt{\sum_{i=1}^{\ell} (\alpha_i - \beta_i)^2}} \\ &= \frac{k}{\sqrt{\ell} \sqrt{\sum_{i=1}^{\ell} (\alpha_i - \beta_i)^2}}, \end{aligned}$$

where β_i denotes the i -th entry of β . Since for each $i \in [\ell]$, α_i is an integer number, we have $|\alpha_i - \beta_i| \geq \frac{1}{n}$. Hence,

$$k \geq \mu \sqrt{\ell} \sqrt{\frac{\ell}{n^2}} = \mu \frac{\ell}{n} = \frac{(n-1)!}{(n-r)!} \mu. \quad (\text{V.1})$$

Let $\mathcal{C} := \{\mathbf{x} \in \mathbb{R}^{\ell} \mid A\mathbf{x} \leq (n-r)!\mathbf{1}\} = \{\mathbf{x} \in \mathbb{R}^{\ell} \mid A(\mathbf{x} - \beta) \leq \mathbf{0}\}$. In view of Definition V.2, \mathcal{C} is a polyhedral cone. Note that since $r < \frac{n}{4}$, Lemma V.1 implies that $A = (a_{ij})_{\ell \times \ell}$ is a matrix such that $a_{ii} > \sum_{i \neq j=1}^{\ell} a_{ij}$ for all $1 \leq i \leq \ell$. Therefore Levy-Desplanques Theorem [14, p. 125] implies A is a non-singular matrix. Also, since $\lambda_0 \mathbf{u} + (1 - \lambda_0) \mathbf{v} \in \mathcal{C}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ and $\lambda_0 \in [0, 1]$, \mathcal{C} is a convex set. It is clear that $\beta, \alpha \in \mathcal{C}$ and so the vector $\vec{\beta\alpha}$ belongs to \mathcal{C} . Suppose that $\{\mathbf{w}_1, \dots, \mathbf{w}_s\}$ is a complete set of representatives of all equivalence classes of extreme rays in \mathcal{C} such that $\|\mathbf{w}_i\| = 1$ for all $1 \leq i \leq s$.

Since $\vec{\beta\alpha} \in \mathcal{C}$, it follows from Remark V.3 that there exist non-negative real numbers $\lambda_1, \dots, \lambda_s$ such that $\vec{\beta\alpha} = \sum_{i=1}^s \lambda_i \mathbf{w}_i$. Then

$$\mu = \frac{\langle -\mathbf{1}, \vec{\beta\alpha} \rangle}{\|\mathbf{1}\| \|\vec{\beta\alpha}\|} = \frac{\langle -\mathbf{1}, \sum_{i=1}^s \lambda_i \mathbf{w}_i \rangle}{\|\mathbf{1}\| \|\sum_{i=1}^s \lambda_i \mathbf{w}_i\|}.$$

Since $\|\sum_{i=1}^s \lambda_i \mathbf{w}_i\| \leq \sum_{i=1}^s \lambda_i \|\mathbf{w}_i\|$,

$$\mu \geq \frac{\sum_{i=1}^s \lambda_i \langle -\mathbf{1}, \mathbf{w}_i \rangle}{\|\mathbf{1}\| (\sum_{i=1}^s \lambda_i \|\mathbf{w}_i\|)},$$

and since $\|\mathbf{w}_i\| = 1$ for all $1 \leq i \leq s$,

$$\begin{aligned} \mu &\geq \sum_{i=1}^s \frac{\lambda_i \langle -\mathbf{1}, \mathbf{w}_i \rangle}{(\sum_{j=1}^s \lambda_j) \|\mathbf{1}\|} \\ &= \sum_{i=1}^s \frac{\lambda_i}{\sum_{j=1}^s \lambda_j} \frac{\langle -\mathbf{1}, \mathbf{w}_i \rangle}{\|\mathbf{1}\|} \geq \sum_{i=1}^s \frac{\lambda_i}{\sum_{j=1}^s \lambda_j} \mu_0 = \mu_0, \end{aligned} \tag{V.2}$$

where $\mu_0 := \min \left\{ \frac{\langle -\mathbf{1}, \mathbf{w}_i \rangle}{\|\mathbf{1}\|} \mid 1 \leq i \leq s \right\}$.

Suppose that $\mu_0 = \frac{\langle -\mathbf{1}, \mathbf{w}_r \rangle}{\|\mathbf{1}\|}$ for some $1 \leq r \leq s$. Hence it follows from Remark V.3 that there exists $i \in [n]$ such that $A_i \mathbf{w}_r = \mathbf{0}$ and $a_i \mathbf{w}_r \leq 0$, where a_i is the i -th row of the matrix A and A_i is the matrix obtained by removing a_i of the matrix A . According to the properties of the matrix A , without loss of generality, we may assume that $i = \ell$. Suppose that $\boldsymbol{\rho}$ is the ℓ -th column of A_ℓ and J is the $(\ell-1) \times (\ell-1)$ matrix obtained by removing the column $\boldsymbol{\rho}$ of the matrix A_ℓ . Levy-Desplanques Theorem implies J is a non-singular matrix. Hence, $A_\ell(x_1, \dots, x_\ell)^t = J(x_1, \dots, x_{\ell-1})^t + \boldsymbol{\rho} x_\ell = \mathbf{0}$ implies $(x_1, \dots, x_{\ell-1})^t = -J^{-1} \boldsymbol{\rho} x_\ell$.

In the sequel, we show that $a_\ell(J^{-1} \boldsymbol{\rho}, -1)^t \leq 0$ and therefore by placing $x_\ell = -1$ we have $(-J^{-1} \boldsymbol{\rho} x_\ell, x_\ell)^t \sim \mathbf{w}_r$. It follows from [23, Theorem 1] and Lemma V.1 that if $\Delta := \min\{|J_{ii}| - \sum_{j=1, j \neq i}^{\ell-1} |J_{ij}| \mid 1 \leq i \leq \ell-1\}$, then $\|J^{-1}\|_\infty := \max\{\sum_{j=1}^{\ell-1} |(J^{-1})_{ij}| \mid 1 \leq i \leq \ell-1\} \leq \frac{1}{\Delta}$. So Lemma V.1 implies $\|J^{-1}\|_\infty \leq \frac{1}{n-4r}$. Also if $|A| := (|a_{ij}|)_{n \times n}$ for a matrix $A = (a_{ij})_{n \times n}$, then we have

$$\|J^{-1} \boldsymbol{\rho}\|_1 = \text{tr}(|J^{-1} \boldsymbol{\rho}| \mathbf{1}) \leq \text{tr}(|J^{-1}| \boldsymbol{\rho} \mathbf{1}).$$

Since the inverse of a symmetric matrix is a symmetric matrix, J^{-1} is a symmetric matrix. Suppose that ρ_i denotes the i -th entry of $\boldsymbol{\rho}$. It follows from Lemma V.1 that $\rho_i \in \{0, 1\}$ for all $1 \leq i \leq \ell-1$ and if $\tau := \{i \in [\ell-1] \mid \rho_i = 1\}$, then the size of τ is at most $2r$. Then we have

$$\begin{aligned} \text{tr}(|J^{-1}| \boldsymbol{\rho} \mathbf{1}) &= \sum_{i=1}^{\ell-1} \sum_{j \in \tau} |(J^{-1})_{ij}| \\ &= \sum_{j \in \tau} \sum_{i=1}^{\ell-1} |(J^{-1})_{ij}| \\ &= \sum_{j \in \tau} \sum_{i=1}^{\ell-1} |(J^{-1})_{ji}| \\ &\leq \sum_{j \in \tau} \|J^{-1}\|_\infty \leq 2r \|J^{-1}\|_\infty, \end{aligned}$$

and therefore,

$$\|J^{-1}\boldsymbol{\rho}\|_1 \leq \frac{2r}{n-4r}. \quad (\text{V.3})$$

So, parts (1) and (2) of Lemma V.1 and $r < \frac{n}{4}$ imply that

$$a_\ell(J^{-1}\boldsymbol{\rho}, -1)^t \leq \|J^{-1}\boldsymbol{\rho}\|_1 - (n-2r) \leq 0$$

and so $(J^{-1}\boldsymbol{\rho}, -1)^t \sim \mathbf{w}_r$. Hence,

$$\begin{aligned} \mu_0 &= \frac{\langle -\mathbf{1}, (J^{-1}\boldsymbol{\rho}, -1)^t \rangle}{\|\mathbf{1}\| \|(J^{-1}\boldsymbol{\rho}, -1)^t\|} = \frac{1 - \langle \mathbf{1}, J^{-1}\boldsymbol{\rho} \rangle}{\sqrt{\ell} \sqrt{1 + \|J^{-1}\boldsymbol{\rho}\|^2}} \\ &\geq \frac{1 - \|J^{-1}\boldsymbol{\rho}\|_1}{\sqrt{\ell} \sqrt{1 + \|J^{-1}\boldsymbol{\rho}\|_1^2}}. \end{aligned} \quad (\text{V.4})$$

Hence, relations (V.3) and (V.4) imply

$$\mu_0 \geq \frac{n-6r}{\sqrt{\ell} \sqrt{n^2 - 8rn + 20r^2}}, \quad (\text{V.5})$$

and therefore the result follows from relations (V.1), (V.2) and (V.5). This completes the proof. \square

Proposition V.5. *For integers $n \geq 10$ and $r \leq \frac{n}{2}$, if $n \nmid (n-r)!$, then n is a prime number.*

Proof. Suppose for a contradiction that n is not prime. Hence there exist $n_1, n_2 \in \mathbb{N} \setminus \{1\}$ such that $n = n_1 n_2$. Suppose first that $n_1 \neq n_2$ and $n_1 < n_2$. If $n_2 \leq n-r$, then $n \mid (n-r)!$ that is a contradiction. So $n_2 > n-r$. Since $r \leq \frac{n}{2}$,

$$\frac{n}{2} \leq n-r < n_2 = \frac{n}{n_1},$$

and therefore $n_1 < 2$ that is a contradiction. Now suppose that $n = n_1^2$. Since $n \nmid (n-r)!$, $n-r < 2n_1$ and so

$$\frac{n_1^2}{2} = \frac{n}{2} \leq n-r < 2n_1,$$

and therefore $n_1 < 4$ that is a contradiction. This completes the proof. \square

Remark V.6. *In view of Proposition V.5, the only numbers that satisfy the assumptions of Theorem V.4 are prime numbers. Thus, given that $P(n, 3) \leq (n-1)!$, Theorem V.4 is interchangeable with Theorem I.2.*

Proof of Corollary I.3: First suppose that $n = 6r+1 \geq 37$ is a prime number. It follows from inequality I.1 that

$$P(6r+1, 3) \leq (6r)! - \sqrt{\frac{(6r)(6r-1) \cdots (5r+2)}{48r^3 + 32r^2 + 10r + 1}}.$$

Clearly, $\frac{(5r+2)(5r+3)(5r+4)}{48r^3 + 32r^2 + 10r + 1} > 2.6$. Therefore,

$$\begin{aligned} P(6r+1, 3) &< (6r)! - \sqrt{2.6(5r+5)}^{\frac{r-4}{2}} \\ &< (6r)! - \left\lceil \frac{6r+1}{3} \right\rceil + 2. \end{aligned}$$

Now suppose that $n = 6r+5 \geq 41$ is a prime number. It follows from inequality I.1 that

$$P(6r+5, 3) \leq (6r+4)! - 5\sqrt{\frac{(6r)(6r-1) \cdots (5r+6)}{48r^3 + 160r^2 + 250r + 125}}.$$

Clearly, $\frac{(5r+6)(5r+7)(5r+8)}{48r^3+160r^2+250r+125} > 2.6$. Therefore,

$$\begin{aligned} P(6r+5, 3) &< (6r+4)! - 5\sqrt{2.6}(5r+9)^{\frac{r-4}{2}} \\ &< (6r+4)! - \left\lceil \frac{6r+5}{3} \right\rceil + 2. \end{aligned}$$

This completes the proof. \square

REFERENCES

- [1] A. Abdollahi, J. Bagherian, F. Jafari, M. Khatami, F. Parvaresh and R. Sobhani, “New upper bounds on the size of permutation codes under Kendall τ -metric,” *Cryptogr. Commun.*, vol. 15, pp. 891-903, Jun. 2023.
- [2] A. Abdollahi, J. Bagherian, F. Jafari, M. Khatami, F. Parvaresh and R. Sobhani, “New table of bounds on permutation codes under Kendall τ -metric,” 2022 10th Iran Workshop on Communication and Information Theory (IWCIT), Iran, Islamic Republic of, 2022, pp. 1-3.
- [3] A. Barg and A. Mazumdar, “Codes in permutations and error correction for rank modulation,” *IEEE Trans. Inform. Theory*, vol. 56, no. 7, pp. 3158-3165, Jul. 2010.
- [4] S. Bereg, W. Bumpass, M. Haghpanah, B. Malouf, and I. H. Sudborough, “Improved Permutation Arrays for Kendall Tau Metric,” *arXiv:2301.11423*, 2023.
- [5] S. Bereg, A. Levy and I. H. Sudborough, “Constructing permutation arrays from groups,” *Des. Codes Cryptogr.*, vol. 86, no. 7, pp. 1095-1111, Jun. 2018.
- [6] D.W. Bolton, “Problem, in *Combinatorics*,” D.J.A. Welsh and D. R. Woodall, eds., *Instit. Math. Applic.*, Southendon-Sea, Essex, England, 1972, pp. 351-352.
- [7] S. Buzaglo and T. Etzion, “Bounds on the size of permutation codes with the Kendall τ -metric,” *IEEE Trans. Inform. Theory*, vol. 61, no. 6, pp. 3241-3250, Jun. 2015.
- [8] C.J. Colbourn and J.H. Dinitz, “The CRC Handbook of Combinatorial Designs,” 2nd Edition, CRC Press, Boca Raton, 2006.
- [9] P. H. Edelman and D. White, Codes, “transforms and the spectrum of the symmetric group,” *Pacific J. Math.*, vol. 143, pp. 47-67, May. 1990.
- [10] The GAP Group, GAP-Groups, Algorithms, and Programming, version 4.11.1, 2021. (<http://www.gap-system.org>)
- [11] A. Schrijver, “Theory of linear and integer programming,” John Wiley and Sons, 1998.
- [12] F. Jafari, A. Abdollahi, J. Bagherian, M. Khatami and R. Sobhani, “Equidistant permutation group codes,” *Des. Codes Cryptogr.*, pp. 1-19, Jan. 2022.
- [13] K. Heinrich and G. H. J. van Rees, “Some constructions for equidistant permutation arrays of index one,” *Util. Math.*, vol. 13, pp. 193-200, 1978.
- [14] R. A. Horn and C. R. Johnson, “Matrix analysis,” Cambridge: Cambridge Univ. Press, 1991.
- [15] A. Jiang, R. Mateescu, M. Schwartz, and J. Bruck, “Correcting charge-constrained errors in the rank-modulation scheme,” *IEEE Trans. Inform. Theory*, vol. 56, no. 5, pp. 2112-2120, May. 2010.
- [16] H. M. Kiah, G. J. Puleo, and O. Milenkovic, “Codes for DNA sequence profiles,” *IEEE Trans. Inf. Theory*, vol. 62, no. 6, pp. 3125-3146, Jun. 2016.
- [17] T. Klove, T.T. Lin, S.C. Tsai and W.G. Tzeng, “Permutation arrays under the Chebyshev distance,” *IEEE Trans. Inform. Theory*, vol. 56, no. 6, pp. 2611-2617, Jun. 2010.
- [18] F. Farnoud, V. Skachek and O. Milenkovic, “Error-correction in flash memories via codes in the Ulam metric,” *IEEE Trans. Inform. Theory*, vol. 59, no. 5, pp. 3003-3020, May. 2013.
- [19] X. Wang, F. Fu, “Snake-in-the-box codes under the l_∞ -metric for rank modulation,” *Des. Codes Cryptogr.*, vol. 83, no. 8, pp. 455-465, Aug. 2019.
- [20] X. Wang, Y. Zhang, Y. Yang and G. Ge, “New bounds of permutation codes under Hamming metric and Kendall’s τ -metric,” *Des. Codes Cryptogr.*, vol. 85, no. 3, pp. 533-545, Nov. 2017.
- [21] X. Wang, Y. Wang, W. Yin and F-W. Fu, “Nonexistence of perfect permutation codes under the Kendall τ -metric,” *Des. Codes Cryptogr.*, vol. 89, no. 11, pp. 2511-2531, Sept. 2021.
- [22] G. H. J. Van Rees and S. A. Vanstone, “Equidistant permutation arrays: A bound,” *J. Austral. Math. Soc. Ser. A*, vol. 33, pp. 262-274, Oct. 1982.
- [23] J.M. Varah, “A lower bound for the smallest singular value of a matrix,” *Linear Algebra Appl.*, vol. 11, no. 1, pp. 3-5, 1975.
- [24] S. A. Vanstone, “The asymptotic behaviour of equidistant permutation arrays,” *Canad. J. Math.*, vol. 21, pp.45-48, Nov. 1979.
- [25] S. Vijayakumaran, “Largest permutation codes with the Kendall τ -metric in S_5 and S_6 ,” *IEEE Comm. Letters*, vol. 20, no. 10, pp. 1912-1915, Oct. 2016.

VI. APPENDIX

Tables III and IV contain information on the generators of subgroup H in S_n (i. e., a subset of the elements H such that every element of the H can be expressed as a combination of finitely many elements of the subset and their inverses) and S_H for $n \in \{7, 8\}$ as obtained from Algorithm 1, along with additional software checking details. In these tables, C_n^d indicates the size of the set of all subgroups of S_n that are (n, d) -codes under Kendall τ -metric and λ_H indicates the number of left cosets of H that are (n, d) -codes under Kendall τ -metric. In fact in these tables, for every pair (n, d) , $\cup_{x \in S_H \cup \xi} xH$ forms a new (n, d) -PC.

d	$ C_7^d $	Generators of H	$ H $	λ_H	Elements of S_H
4	5565	[4, 2, 1, 3, 7, 5, 6] [3, 5, 2, 6, 7, 1, 4]	21	240	[1, 2, 3, 7, 4, 6, 5] [1, 2, 3, 5, 7, 6, 4] [1, 2, 6, 3, 4, 7, 5] [1, 2, 4, 7, 3, 5, 6] [1, 2, 5, 4, 3, 7, 6] [1, 2, 5, 6, 3, 4, 7] [1, 2, 4, 6, 5, 3, 7] [1, 2, 6, 7, 5, 3, 4] [1, 2, 7, 4, 6, 5, 3] [1, 4, 2, 3, 6, 7, 5] [1, 6, 2, 7, 4, 3, 5] [1, 5, 2, 7, 6, 3, 4] [1, 6, 4, 2, 3, 5, 7] [1, 6, 5, 2, 3, 7, 4]
5	3651	[3, 4, 1, 2, 6, 5, 7] [5, 2, 1, 7, 3, 4, 6]	42	57	[1, 2, 5, 3, 7, 6, 4] [1, 2, 7, 6, 4, 3, 5]
6	2811	[7, 2, 1, 5, 6, 4, 3] [3, 4, 2, 5, 7, 1, 6]	21	166	[1, 2, 3, 7, 6, 5, 4] [1, 2, 7, 4, 6, 5, 3] [1, 6, 2, 5, 4, 7, 3]
7	1684	[6, 2, 4, 3, 7, 1, 5] [1, 3, 6, 5, 7, 2, 4]	42	3	—
8	1181	[2, 5, 7, 3, 4, 1, 6]	7	624	[1, 2, 7, 6, 3, 5, 4] [1, 5, 7, 2, 4, 3, 6] [1, 5, 6, 3, 2, 7, 4]
9	686	[2, 5, 7, 4, 1, 3, 6]	3	1418	[1, 7, 2, 4, 6, 5, 3] [1, 6, 3, 4, 7, 5, 2] [4, 6, 2, 1, 7, 3, 5] [3, 6, 5, 1, 2, 7, 4]
10	475	[2, 4, 7, 5, 3, 6, 1]	6	92	[3, 6, 4, 2, 5, 1, 7]
11	219	[6, 5, 3, 7, 2, 1, 4]	2	1400	[1, 7, 3, 5, 6, 4, 2] [5, 4, 3, 2, 1, 7, 6] [7, 2, 1, 4, 6, 5, 3]
12	163	[2, 4, 7, 6, 3, 5, 1]	7	40	—
13	83	[1, 7, 6, 4, 5, 3, 2]	2	198	[6, 2, 4, 5, 7, 3, 1]
14	66	[6, 5, 3, 4, 2, 1, 7]	2	266	[7, 5, 1, 4, 3, 6, 2]

TABLE III: New $(7, d)$ –codes and some details of software checking.

d	$ C_7^d $	Generators of H	$ H $	λ_H	Elements of S_H
3	105236	$[4, 2, 5, 1, 3, 8, 7, 6]$ $[8, 6, 3, 4, 1, 7, 2, 5]$	336	120	$[1, 2, 3, 4, 5, 8, 7, 6]$ $[1, 2, 3, 4, 6, 8, 5, 7]$ $[1, 2, 3, 7, 4, 5, 6, 8]$ $[1, 2, 3, 8, 4, 7, 5, 6]$ $[1, 2, 3, 7, 4, 8, 6, 5]$ $[1, 2, 3, 6, 5, 4, 8, 7]$ $[1, 2, 3, 8, 5, 4, 6, 7]$ $[1, 2, 3, 8, 6, 4, 5, 7]$ $[1, 2, 3, 7, 5, 8, 4, 6]$ $[1, 2, 3, 7, 8, 6, 5, 4]$
4	89682	$[7, 1, 8, 3, 4, 2, 6, 5]$ $[6, 5, 4, 2, 3, 8, 1, 7]$	168	240	$[1, 2, 3, 4, 8, 5, 7, 6]$ $[1, 2, 3, 4, 6, 8, 7, 5]$ $[1, 2, 3, 7, 4, 5, 8, 6]$ $[1, 2, 3, 5, 8, 4, 6, 7]$ $[1, 2, 3, 6, 5, 4, 8, 7]$ $[1, 2, 3, 6, 7, 4, 5, 8]$ $[1, 2, 3, 8, 7, 4, 6, 5]$ $[1, 2, 3, 5, 7, 6, 4, 8]$ $[1, 2, 3, 8, 5, 7, 6, 4]$ $[1, 2, 3, 6, 8, 7, 5, 4]$ $[1, 2, 5, 4, 3, 6, 8, 7]$ $[1, 2, 8, 5, 4, 3, 6, 7]$
5	66442	$[7, 2, 8, 6, 5, 4, 1, 3]$ $[6, 4, 3, 5, 2, 8, 7, 1]$	336	16	$[1, 2, 3, 8, 4, 7, 5, 6]$
6	54709	$[8, 3, 4, 6, 5, 7, 1, 2]$ $[5, 2, 4, 8, 3, 1, 6, 7]$	56	672	$[1, 2, 3, 8, 4, 6, 7, 5]$ $[1, 2, 3, 7, 6, 4, 8, 5]$ $[1, 2, 3, 5, 8, 6, 7, 4]$ $[1, 2, 6, 5, 3, 4, 8, 7]$ $[1, 2, 7, 5, 3, 6, 8, 4]$ $[1, 2, 7, 8, 6, 3, 5, 4]$
7	37499	$[8, 5, 4, 1, 6, 3, 7, 2]$ $[7, 2, 1, 3, 6, 8, 5, 4]$	56	390	$[1, 2, 7, 6, 3, 4, 5]$ $[1, 2, 4, 6, 7, 8, 5, 3]$
8	29249	$[5, 3, 6, 1, 2, 8, 7, 4]$ $[7, 2, 6, 8, 4, 5, 3, 1]$	56	390	$[1, 2, 4, 8, 5, 7, 3, 6]$
9	18352	$[4, 1, 7, 6, 8, 3, 5, 2]$ $[5, 1, 7, 3, 2, 6, 4, 8]$	48	12	—
10	13529	$[6, 1, 3, 5, 7, 2, 4, 8]$ $[8, 7, 1, 6, 3, 2, 4, 5]$	24	260	$[1, 2, 7, 6, 5, 3, 8, 4]$
11	8135	$[5, 6, 8, 7, 1, 2, 4, 3]$ $[1, 3, 7, 8, 5, 2, 4, 6]$	12	212	$[2, 8, 3, 1, 6, 4, 7, 5]$
12	6163	$[7, 8, 5, 6, 3, 4, 1, 2]$ $[4, 8, 5, 2, 7, 3, 6, 1]$	24	12	—
13	3169	$[4, 6, 1, 5, 8, 3, 7, 2]$	7	708	$[3, 7, 4, 6, 2, 5, 1, 8]$
14	2324	$[4, 6, 1, 5, 8, 3, 7, 2]$	7	708	$[3, 7, 4, 6, 2, 5, 1, 8]$
15	810	$[4, 6, 7, 8, 1, 3, 5, 2]$	8	168	—
16	607	$[4, 7, 6, 8, 1, 2, 3, 5]$ $[7, 4, 5, 3, 2, 1, 8, 6]$	8	96	—
17	252	$[7, 3, 8, 2, 6, 1, 5, 4]$	4	112	—
18	189	$[7, 3, 8, 2, 6, 1, 5, 4]$	4	112	—

TABLE IV: New $(8, d)$ —codes and some details of software checking.