Analysis Procedure

Solution steps

1. Calculate the global stiffness matrix for each element

$$K_G = \frac{EA}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$s = \sin\theta$$
$$c = \cos\theta$$

θ captures the orientation of each member

Solution steps

2. Build the primary stiffness matrix for the structure

   - Doesn't take account of supports
   - Just how elements relate to each other in the structure

$$K_p = \begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & \dots & \dots & n \end{array} & \\ \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ \dots \\ \dots \\ n \end{array} \end{array}$$

nxn Primary stiffness matrix for structure with n degrees of freedom

Solution steps

3. 'Impose' influence of support conditions to get structure stiffness matrix

$$K_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution steps

4. Solve for unknown nodal displacements

$$\{F\} = [K_S]\{U\}$$

$$[K_S]^{-1}\{F\} = \{U\}$$

Inverting the structure stiffness matrix, $[K_S]$ is 'expensive' → suits a computer

Solution steps

5. Multiply nodal displacements by primary stiffness matrix to find reaction forces

$$\{F\} = [K_P]\{U\}$$

Solution steps

6. Use nodal displacements to determine member forces

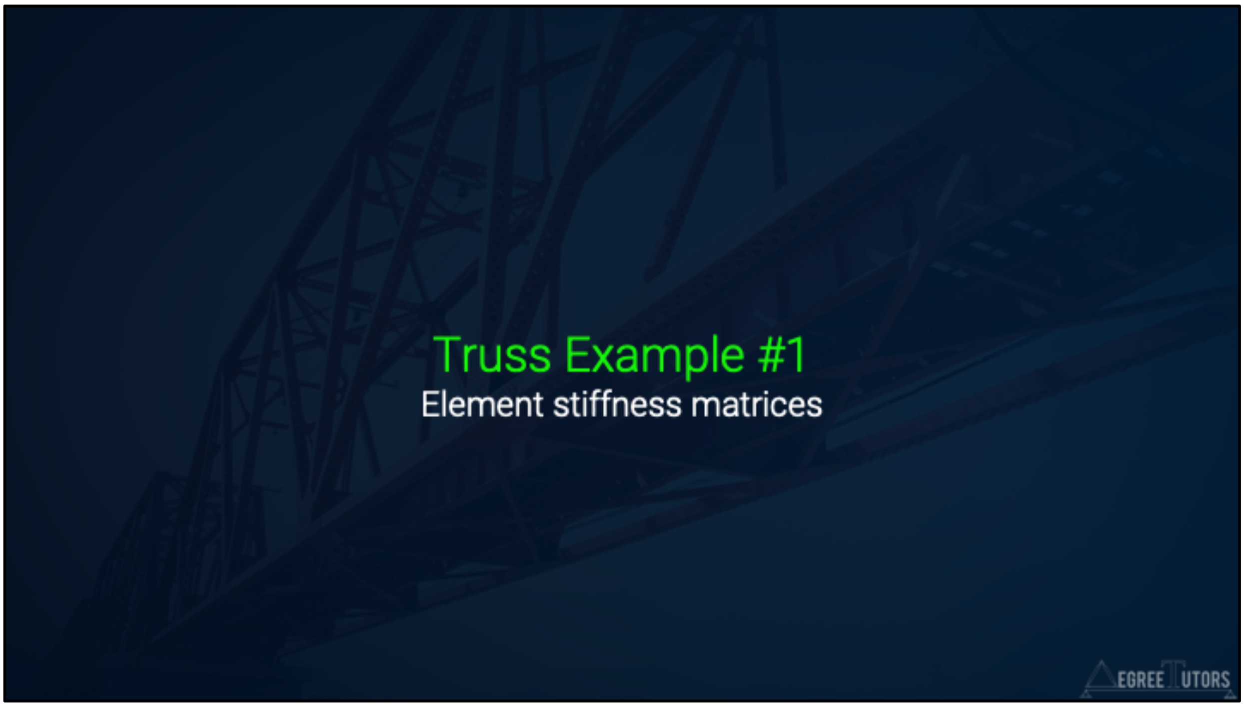
Force = stiffness × displacement

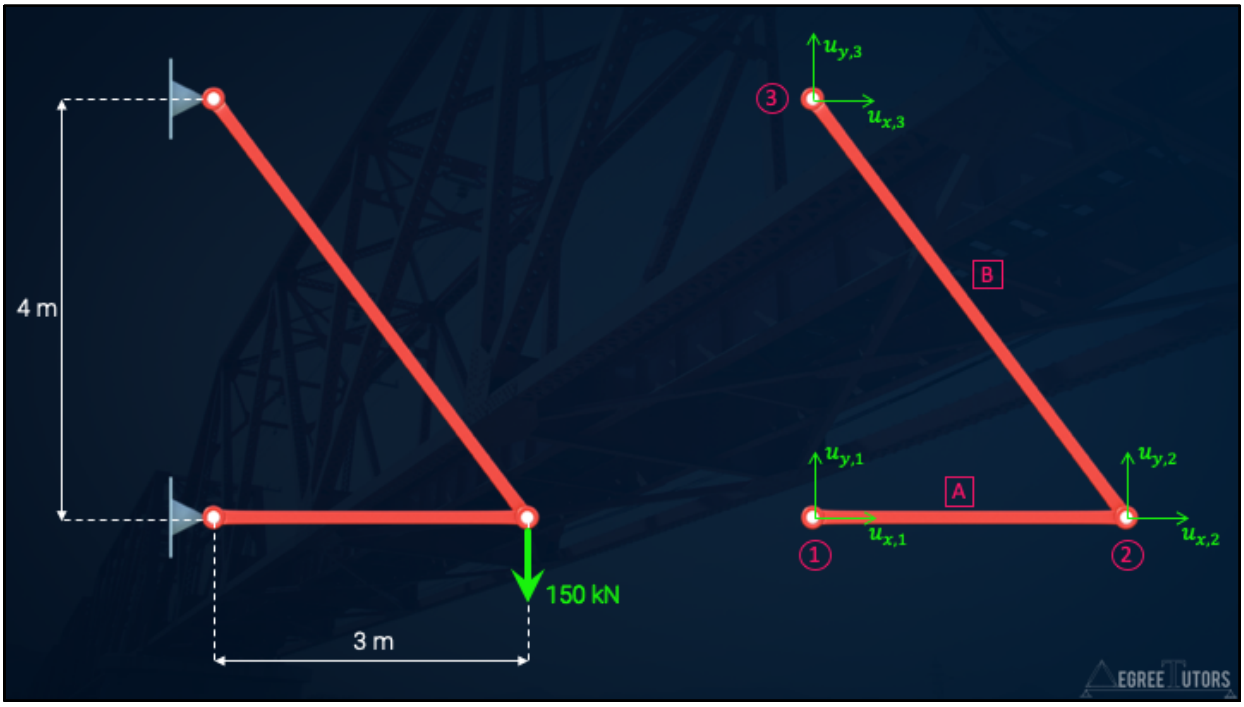
$$F_{i,j} = \frac{EA}{L}\underbrace{(u_j - u_i)}_{\text{Known local nodal displacements}}$$

Solution steps

1. Calculate the global stiffness matrix for each element
2. Build the primary stiffness matrix for the structure
3. 'Impose' influence of support conditions to get structure stiffness matrix
4. Solve for unknown nodal displacements
5. Substitute back in to calculate reaction forces
6. Use nodal displacements to determine member forces

Crying out for a programmatic solution!


Truss Example #1
Element stiffness matrices


4 m
150 kN
3 m
$u_{y,3}$
$u_{x,3}$
3
B
$u_{y,1}$
$u_{y,2}$
A
$u_{x,1}$
$u_{x,2}$
1
2

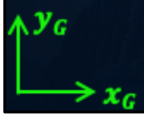
Member A

Measure all angles anticlockwise from the global positive x-axis with node i at the origin where $i < j$

$\theta = 0\ rads$

$$K_{A,G} = \frac{EA}{L_A}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$K_{A,G} = EA\begin{bmatrix} 0.333 & 0 & -0.333 & 0 \\ 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



Member B

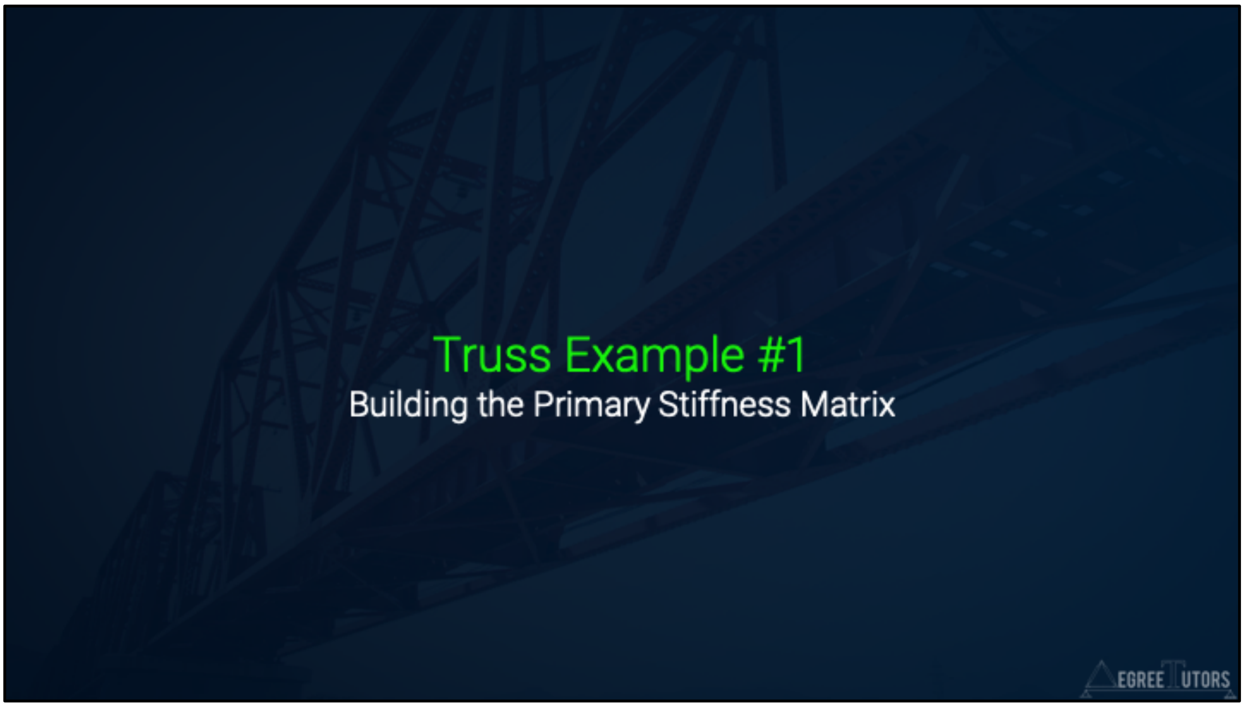
Measure all angles **anticlockwise** from the **global positive x-axis** with **node i at the origin** where $i < j$

$$\theta = 2.2143\ rads\ (126.9°)$$

$$K_{B,G} = \frac{EA}{L_B}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$$K_{B,G} = EA\begin{bmatrix} 0.072 & -0.096 & -0.072 & 0.096 \\ -0.096 & 0.128 & 0.096 & -0.128 \\ -0.072 & 0.096 & 0.072 & -0.096 \\ 0.096 & -0.128 & -0.096 & 0.128 \end{bmatrix}$$

y_G x_G


Truss Example #1
Building the Primary Stiffness Matrix
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Combine element stiffness matrices into a primary stiffness matrix

Systematic process…

Step 1: Break element stiffness matrices into quadrants

$$K_{A,G} = EA\begin{bmatrix} \boxed{\begin{matrix} 0.333 & 0 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} -0.333 & 0 \\ 0 & 0 \end{matrix}} \\ \boxed{\begin{matrix} -0.333 & 0 \\ 0 & 0 \end{matrix}} & \boxed{\begin{matrix} 0.333 & 0 \\ 0 & 0 \end{matrix}} \end{bmatrix} \qquad K_{B,G} = EA\begin{bmatrix} \boxed{\begin{matrix} 0.072 & -0.096 \\ -0.096 & 0.128 \end{matrix}} & \boxed{\begin{matrix} -0.072 & 0.096 \\ 0.096 & -0.128 \end{matrix}} \\ \boxed{\begin{matrix} -0.072 & 0.096 \\ 0.096 & -0.128 \end{matrix}} & \boxed{\begin{matrix} 0.072 & -0.096 \\ -0.096 & 0.128 \end{matrix}} \end{bmatrix}$$

$$K_{A,G} = EA\begin{bmatrix} \boxed{K_A 11} & \boxed{K_A 12} \\ \boxed{K_A 21} & \boxed{K_A 22} \end{bmatrix} \qquad K_{B,G} = EA\begin{bmatrix} \boxed{K_B 11} & \boxed{K_B 12} \\ \boxed{K_B 21} & \boxed{K_B 22} \end{bmatrix}$$

Step 2: Set up the primary stiffness matrix template

Number nodes in the structure

$$K_P = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k11 & k12 & k13 \\ k21 & k22 & k23 \\ k31 & k32 & k33 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Number nodes in the structure

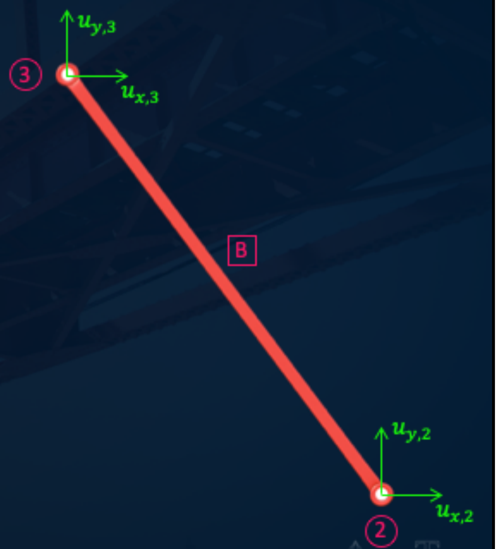
Each matrix element k represents a 2x2 matrix – so the template expands to become 6x6 when filled

Step 3: Fill in the template with the quadrants of the element stiffness matrices

Consider element B with nodes 2 and 3; element B's contribution to the primary stiffness matrix is…

$$K_{B,G} = EA\begin{bmatrix} \boxed{K_B11} & \boxed{K_B12} \\ \boxed{K_B21} & \boxed{K_B22} \end{bmatrix}$$

$$K_P = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} & & 2 & 3 \\ & \left[\begin{array}{c} k11 \\ k21 \\ k31 \end{array}\right. & \begin{array}{cc} k12 & k13 \\ \boxed{k22} & \boxed{k23} \\ \boxed{k32} & \boxed{k33} \end{array} & \left.\begin{array}{c} \\ \\ \\ \end{array}\right] \begin{array}{c} \\ 2 \\ 3 \end{array} \end{array}$$



Step 3: Fill in the template with the quadrants of the element stiffness matrices

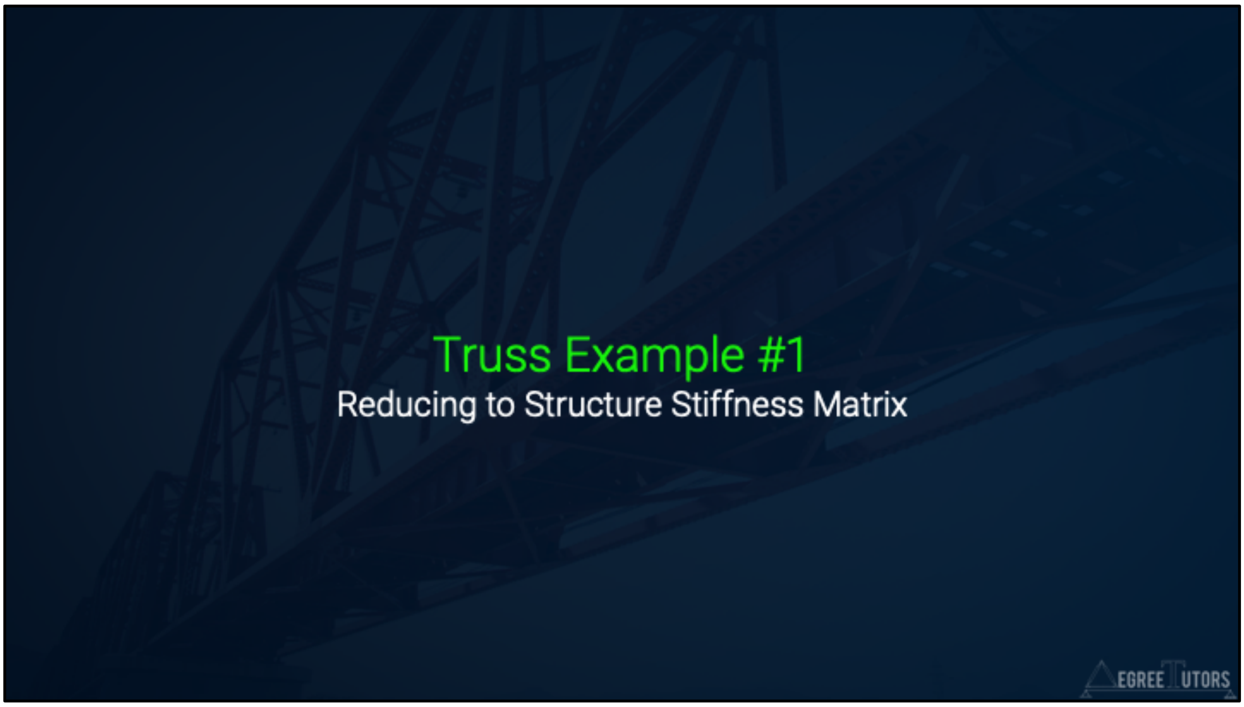
For both members... $K_P = K_{A,G} + K_{B,G}$

$$K_{A,G} = EA\begin{bmatrix} K_A 11 & K_A 12 \\ K_A 21 & K_A 22 \end{bmatrix}$$

$$K_P = EA\begin{bmatrix} 0.333 & 0 & -0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.405 & -0.096 & -0.072 & 0.096 \\ 0 & 0 & -0.096 & 0.128 & 0.096 & -0.128 \\ 0 & 0 & -0.072 & 0.096 & 0.072 & -0.096 \\ 0 & 0 & 0.096 & -0.128 & -0.096 & 0.128 \end{bmatrix}$$

$$\begin{bmatrix} 0.405 & -0.096 \\ -0.096 & 0.128 \end{bmatrix}$$

$$K_{B,G} = EA\begin{bmatrix} K_B 11 & K_B 12 \\ K_B 21 & K_B 22 \end{bmatrix}$$


Truss Example #1
Reducing to Structure Stiffness Matrix

Full force-displacement relationship…

6 DoF problem

$$
\begin{array}{c}
1\\2\\3\\4\\5\\6
\end{array}
\begin{bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{bmatrix}
= EA
\begin{array}{c}
\begin{array}{cccccc} \;\;1\;\; & \;\;2\;\; & \;\;3\;\; & \;\;4\;\; & \;\;5\;\; & \;\;6\;\; \end{array}\\
\begin{bmatrix}
0.333 & 0 & -0.333 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-0.333 & 0 & 0.405 & -0.096 & -0.072 & 0.096 \\
0 & 0 & -0.096 & 0.128 & 0.096 & -0.128 \\
0 & 0 & -0.072 & 0.096 & 0.072 & -0.096 \\
0 & 0 & 0.096 & -0.128 & -0.096 & 0.128
\end{bmatrix}
\end{array}
\begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,2} \\ u_{y,2} \\ u_{x,3} \\ u_{y,3} \end{bmatrix}
$$

- 10 variables (6 displacements & 4 reactions)
- Not all unknown, we know horizontal & vertical displacement at node 1 and 3 = 0
- We now impose these known (zero) displacements on the system of equations
- DoF 1, 2, 5, 6 are zero

Impose boundary conditions on the system of equations

Place 1 in the diagonal corresponding to known zero displacement and 0 in in all other elements of corresponding rows and columns. Also place 0 in corresponding elements of force vector

$$
\begin{array}{c}
\\ ① \\ ② \\ ③ \\ ④ \\ ⑤ \\ ⑥
\end{array}
\begin{array}{c}
\\ \begin{bmatrix} 0 \\ 0 \\ F_{x,2} \\ F_{y,2} \\ 0 \\ 0 \end{bmatrix}
\end{array}
= EA
\begin{array}{c}
\begin{array}{cccccc} ① & ② & ③ & ④ & ⑤ & ⑥ \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.405 & -0.096 & 0 & 0 \\
0 & 0 & -0.096 & 0.128 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\begin{array}{c}
\\ \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,2} \\ u_{y,2} \\ u_{x,3} \\ u_{y,3} \end{bmatrix}
\end{array}
$$

Impose boundary conditions on the system of equations

$$\begin{bmatrix} 0 \\ 0 \\ F_{x,2} \\ F_{y,2} \\ 0 \\ 0 \end{bmatrix} = EA \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.405 & -0.096 & 0 & 0 \\ 0 & 0 & -0.096 & 0.128 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Structure stiffness matrix}} \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,2} \\ u_{y,2} \\ u_{x,3} \\ u_{y,3} \end{bmatrix}$$

Placing zeros in the primary stiffness matrix **rows** imposes boundary conditions

$0 = 1 \times u_{x,1}$

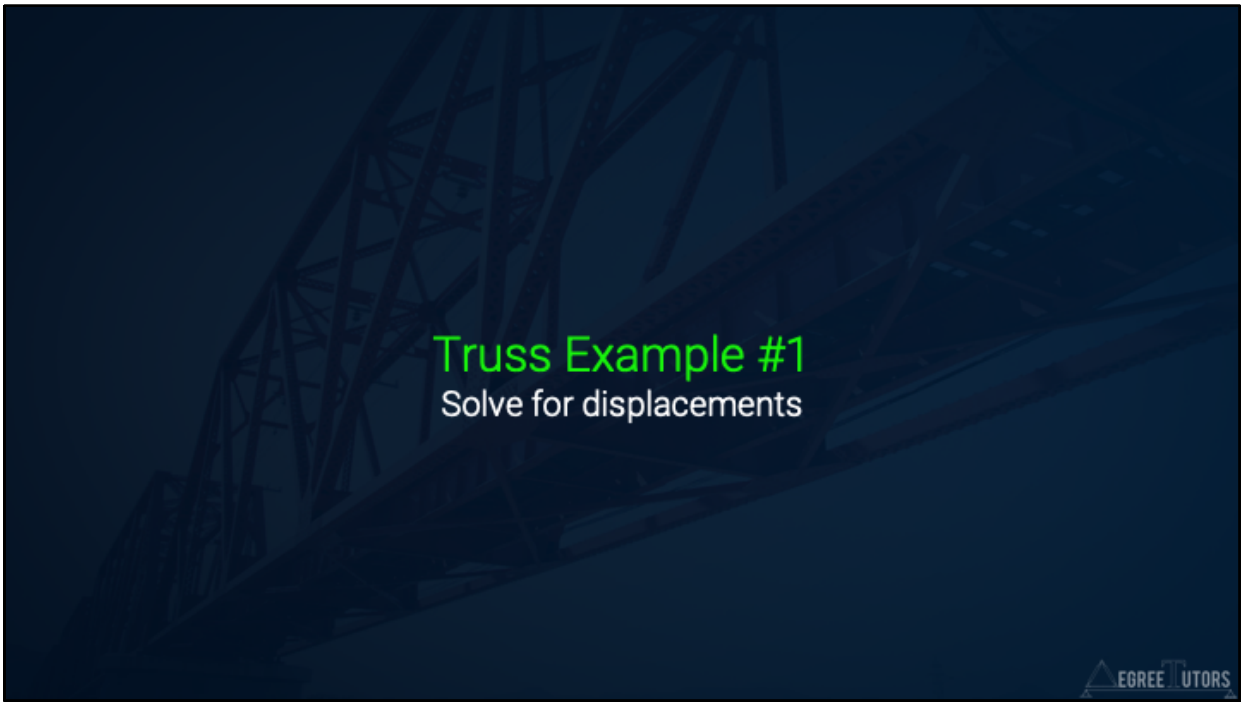
$0 = 1 \times u_{y,1}$

$0 = 1 \times u_{x,3}$

$0 = 1 \times u_{y,3}$

Placing zeros in the primary stiffness matrix columns ensures known displacements remain zero in subsequent calculations, e.g. consider row 3...

$$F_{x,2} = EA\left[(0 \times u_{x,1}) + (0 \times u_{y,1}) + (0.405 \times u_{x,2}) + (-0.096 \times u_{y,2}) + (0 \times u_{x,3}) + (0 \times u_{y,3})\right]$$


Truss Example #1
Solve for displacements

2 unknown displacements…the two DoF at node 2, $[u_{x,2} \quad u_{y,2}]$

$$\{F\} = [K_S]\{U\}$$

$$\begin{bmatrix} 0 \\ 0 \\ F_{x,2} \\ F_{y,2} \\ 0 \\ 0 \end{bmatrix} = EA \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.405 & -0.096 & 0 & 0 \\ 0 & 0 & -0.096 & 0.128 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,2} \\ u_{y,2} \\ u_{x,3} \\ u_{y,3} \end{bmatrix}$$

$$\begin{bmatrix} F_{x,2} \\ F_{y,2} \end{bmatrix} = EA \begin{bmatrix} 0.405 & -0.096 \\ -0.096 & 0.128 \end{bmatrix} \begin{bmatrix} u_{x,2} \\ u_{y,2} \end{bmatrix}$$

2 unknown displacements…the two DoF at node 2, $[u_{x,2} \quad u_{y,2}]$

$$[K_S]^{-1}\{F\} = \{U\}$$

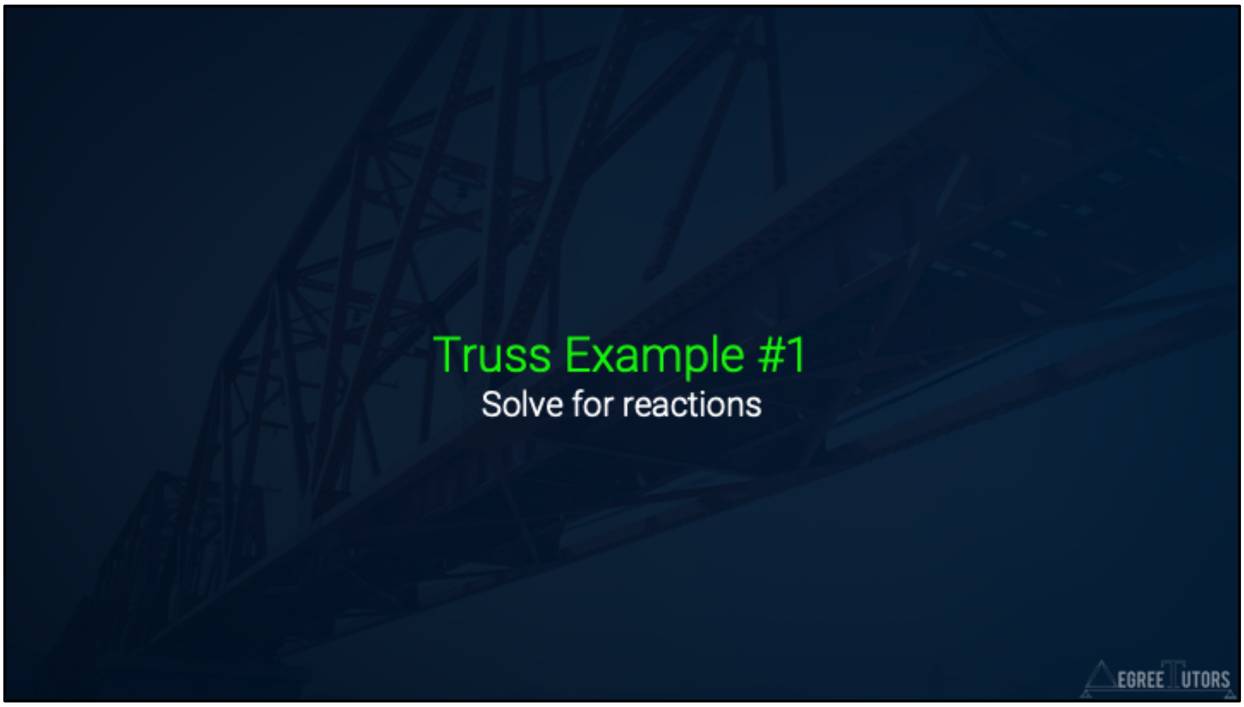
$$\frac{1}{EA}\begin{bmatrix} 0.405 & -0.096 \\ -0.096 & 0.128 \end{bmatrix}^{-1} \begin{bmatrix} F_{x,2} \\ F_{y,2} \end{bmatrix} = \begin{bmatrix} u_{x,2} \\ u_{y,2} \end{bmatrix}$$

Horizontal force at node 2

$$\frac{1}{EA}\begin{bmatrix} 0.405 & -0.096 \\ -0.096 & 0.128 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -150 \end{bmatrix} = \begin{bmatrix} u_{x,2} \\ u_{y,2} \end{bmatrix}$$

Vertical force at node 2

$$\begin{bmatrix} u_{x,2} \\ u_{y,2} \end{bmatrix} = \frac{1}{EA}\begin{bmatrix} -337.5 \\ -1425 \end{bmatrix}$$


Truss Example #1
Solve for reactions
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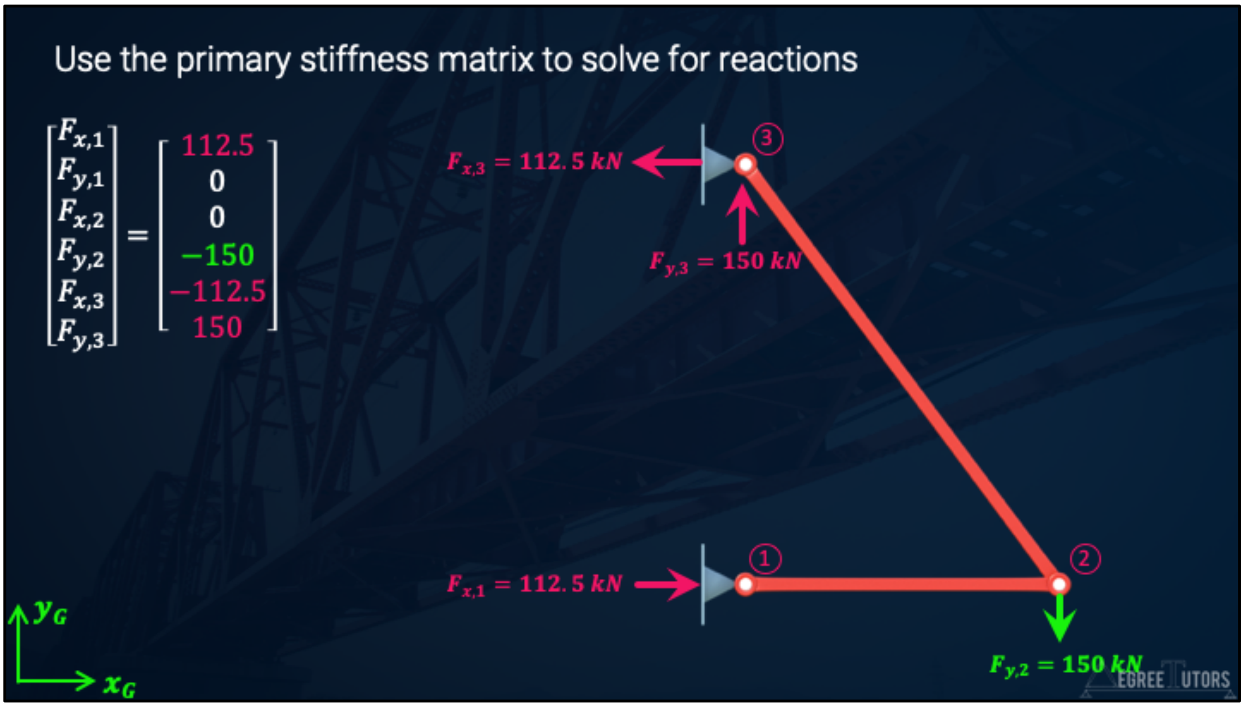
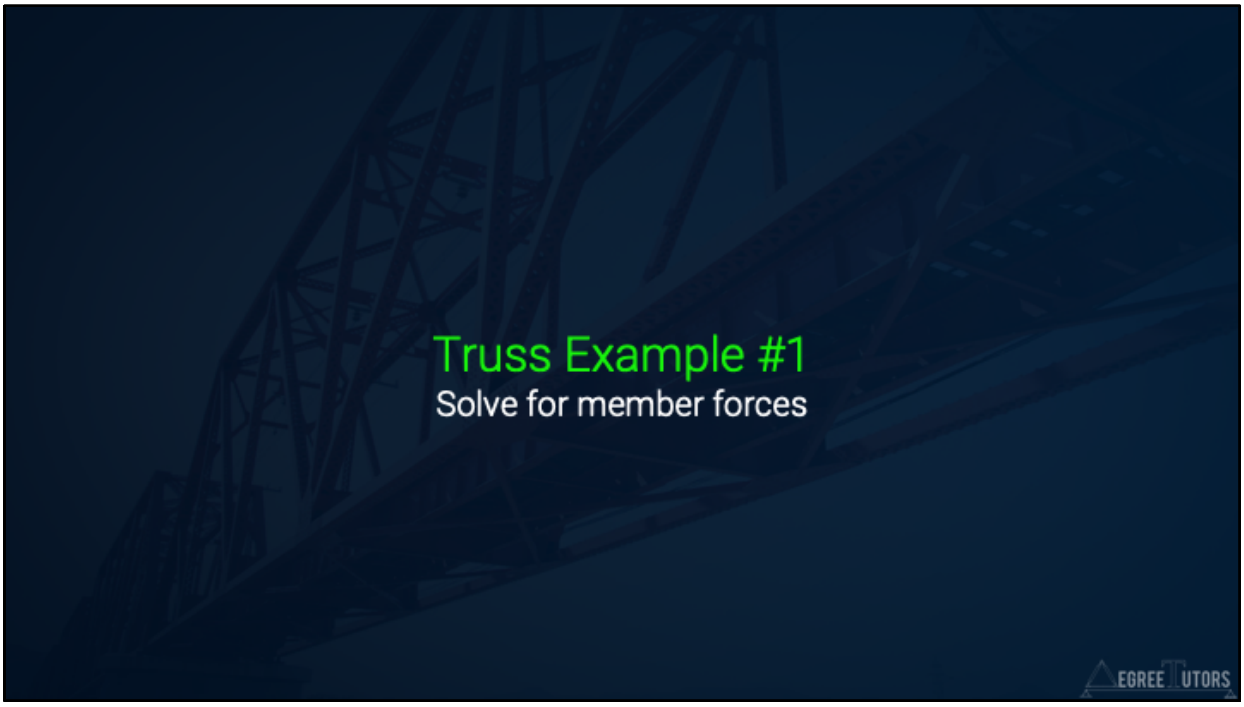
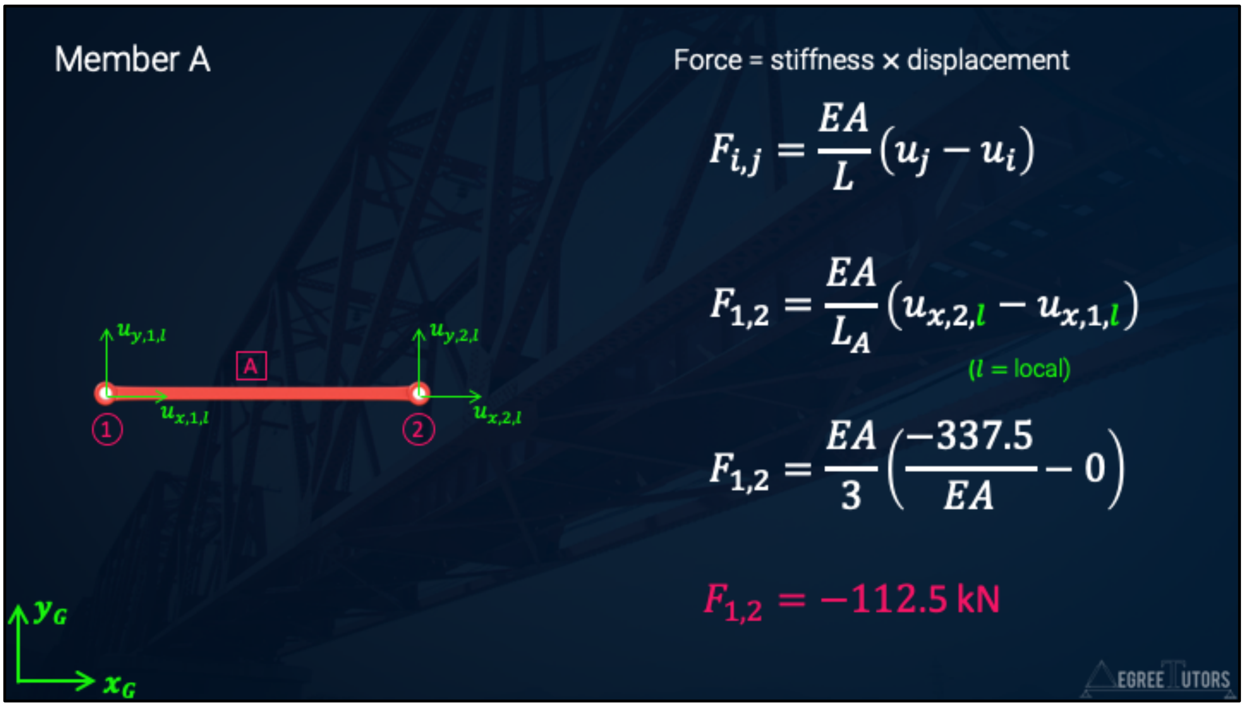
Use the primary stiffness matrix to solve for reactions

$$\begin{bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{bmatrix} = EA \underbrace{\begin{bmatrix} 0.333 & 0 & -0.333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 0.405 & -0.096 & -0.072 & 0.096 \\ 0 & 0 & -0.096 & 0.128 & 0.096 & -0.128 \\ 0 & 0 & -0.072 & 0.096 & 0.072 & -0.096 \\ 0 & 0 & 0.096 & -0.128 & -0.096 & 0.128 \end{bmatrix}}_{\text{Primary stiffness matrix}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ \left(\frac{-337.5}{EA}\right) \\ \left(\frac{-1425}{EA}\right) \\ 0 \\ 0 \end{bmatrix}}_{\text{Known displacements}}$$

$$\begin{bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{bmatrix} = \begin{bmatrix} 112.5 \\ 0 \\ 0 \\ -150 \\ -112.5 \\ 150 \end{bmatrix}$$


Use the primary stiffness matrix to solve for reactions
$\begin{bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{bmatrix} = \begin{bmatrix} 112.5 \\ 0 \\ 0 \\ -150 \\ -112.5 \\ 150 \end{bmatrix}$
$F_{x,3} = 112.5\ kN$
$F_{y,3} = 150\ kN$
$F_{x,1} = 112.5\ kN$
$F_{y,2} = 150\ kN$
1
2
3
y_G
x_G


Truss Example #1
Solve for member forces



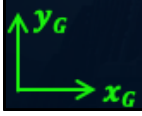
Member A
Force = stiffness × displacement
$F_{i,j} = \frac{EA}{L}(u_j - u_i)$
$F_{1,2} = \frac{EA}{L_A}(u_{x,2,l} - u_{x,1,l})$
(l = local)
$F_{1,2} = \frac{EA}{3}\left(\frac{-337.5}{EA} - 0\right)$
$F_{1,2} = -112.5$ kN
$u_{y,1,l}$
$u_{y,2,l}$
A
$u_{x,1,l}$
$u_{x,2,l}$
1
2
y_G
x_G

Member B $\theta = 2.2143\ rads\ (126.9°)$

$$\begin{bmatrix} u_{x,2,l} \\ u_{x,3,l} \end{bmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}}_{\text{Transformation matrix}} \begin{bmatrix} u_{x,2,G} \\ u_{y,2,G} \\ u_{x,3,G} \\ u_{y,3,G} \end{bmatrix}$$

$$\begin{bmatrix} u_{x,2,l} \\ u_{x,3,l} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \begin{bmatrix} -337.5 \\ -1425 \\ 0 \\ 0 \end{bmatrix} \frac{1}{EA}$$

$$\begin{bmatrix} u_{x,2,l} \\ u_{x,3,l} \end{bmatrix} = \frac{1}{EA} \begin{bmatrix} -937.5 \\ 0 \end{bmatrix}$$

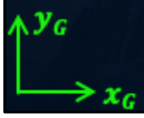


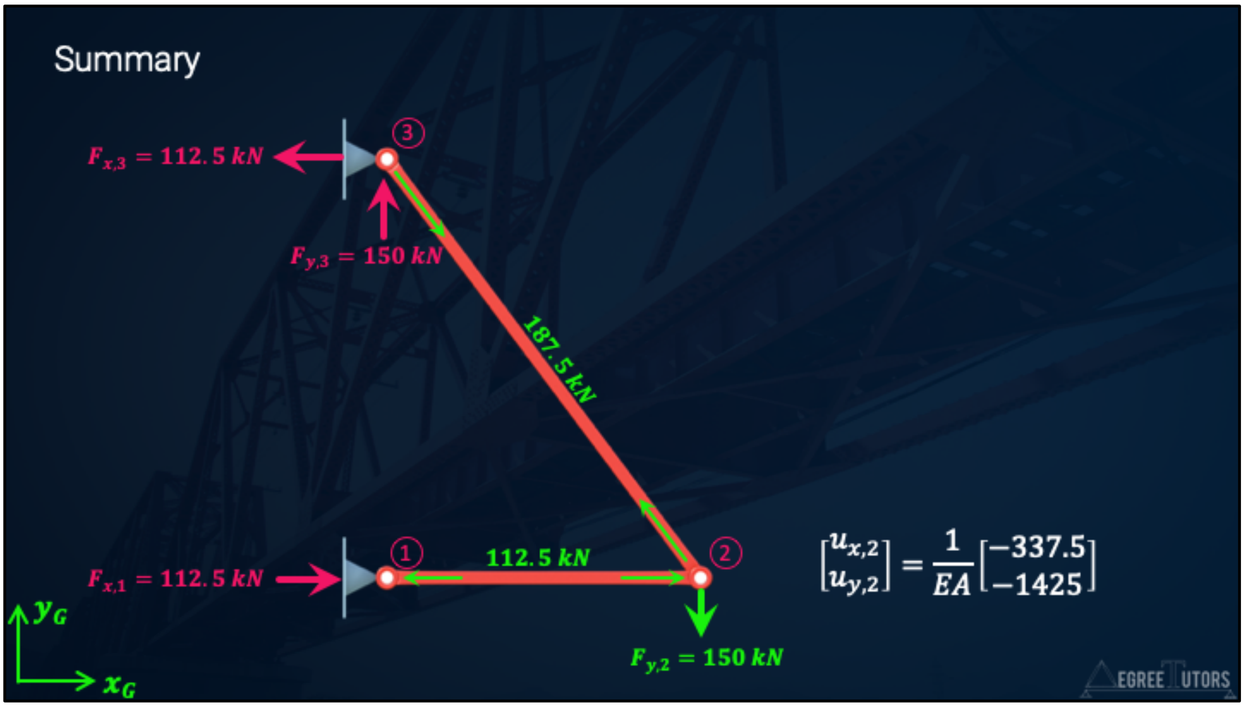
Member B $\theta = 2.2143\ rads\ (126.9°)$

$$F_{2,3} = \frac{EA}{L_B}(u_{x,3} - u_{x,2})$$

$$F_{2,3} = \frac{EA}{5}\left(0 - \frac{-937.5}{EA}\right)$$

$F_{2,3}$ =187.5 kN




Summary
$F_{x,3} = 112.5\ kN$
$F_{y,3} = 150\ kN$
187.5 kN
$F_{x,1} = 112.5\ kN$
112.5 kN
$F_{y,2} = 150\ kN$
$\begin{bmatrix} u_{x,2} \\ u_{y,2} \end{bmatrix} = \frac{1}{EA}\begin{bmatrix} -337.5 \\ -1425 \end{bmatrix}$
y_G
x_G
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Concept summary

- Simple idea: Force = stiffness × displacement
- One equation for each member - element stiffness matrix, K_G
- Combine all elements to build a model of the structure, K_P, (just a linear system of simultaneous equations)
- Impose known boundary conditions, K_S, (displacement at supports is zero)
- Solve for unknowns, $\{F\}$, $\{U\}$ and member forces

Now let's get the computer to do the hard work!