Assignment 1

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1 Introduction

In this assignment we look at a time series of foreign auto sales over the period January 1970 till December 2019 in the US. We will examine the characteristics of the time series and we will look at multiple models to see which describes and forecasts the model best.

2 Data

The Excel file "FAUTO.xlsx" contains monthly observations on the retail sales of foreign autos (in thousands of units) in the US over the period January 1970 – December 2019.[1] Column B contains the original data, while column C contains the seasonally adjusted time series. A dummy variable DREC is also included, which takes the value 1 during recession periods and 0 during expansions.

3 Results

In this section we discuss the results of our findings, which are the answers on the questions asked in "Assignment1.pdf" respectively.

3.1 Key features

(i) **Trend**: Figure 1 shows the original time series of foreign auto sales, denoted as $fautonsa_t$. We do not observe a pronounced trend in the original time series over the full sample looking just at the graph, however we do observe a changing trend. Interesting to see is that the hypothesis of no trend on the full sample is not confirmed by the regression

$$fautonsa_t = \alpha + \beta t + \epsilon_t \ t = 1, 2, ..., T \tag{1}$$

,which gives an estimate (obtained by least squares using the complete sample period 1970M01-2019M12,T= 600) of $\beta = -0.000853$ *(std. error = 0.000348)*, which is significantly different from zero at the 5% significance level (p-value: 0.0145).

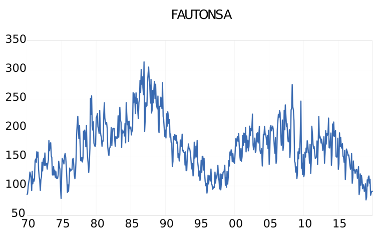

Figure 1: Foreign auto sales (1970M01-2019M12)

(ii) **Seasonality**: In the left window of figure 2 we look at the difference between the observation at that point in time compared to the observation on a previous point in time. In the right window the same metric is plotted for the different months of the year. In the right window there is no remarkable difference between the different months. We do, however, observe a 'saw-tooth' pattern in the left window. To test whether a seasonal effect exists we regress the difference between current and previous observation on twelve dummy variables, one for each month in the year. The model is shown below:

$$d(fautonsa_t) = \mu_1 M_1 + \mu_2 M_2 + ... + \mu_{12} M_{12} + \epsilon_t \tag{2}$$

Where $d(fautonsa_t)$ is the difference between the observation of $fautonsa$ on t an $t-1$ and M_i is the dummy variable for month i with $i = 1, ..., 12$. A Wald test to test joint significance of all monthly variables with the null hypothesis $H_0 : \mu_1 = ... = \mu_{12}$ yields a F-statistic with the value 47.72 and a p-value of 0.00, therefore we know that the monthly dummy variables $M_1, ..., M_{12}$ are jointly significant. The R^2 of this regression is 0.47. We thus conclude that a seasonal pattern does exist in this data.

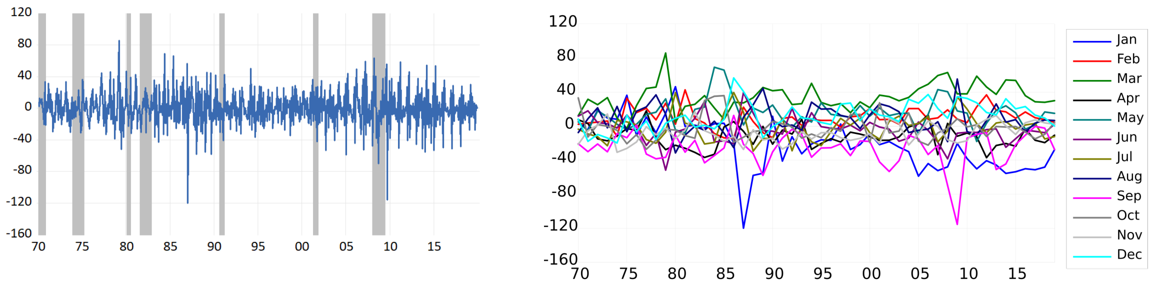

Figure 2: First differences of foreign auto sales

(iii) **Abberant observations**: From the graph of the foreign auto sales we observe no abberent observations. We do however observe that during periods of recession (1985-1990), (2008-2009) the moves are bigger than during periods of expansion. This pattern becomes clearer when we look at the left window in figure 2. In the figure the monthly change is displayed. We notice that there are three very large observations, one in 1979, one in 1987 and one in 2009. We conclude that these three observations are abberant observations and as two out of three abberant observations occurred during recessions we conclude that abberant observations are more like during periods of recession.

(iv) **Heteroskedasticity**: In the left window of figure 2 we see that it is possible that the monthly changes, and therefore the standard deviation, are (1) higher in recessions and (2) are higher in the period 2005M01-2019M12 compared to 1990M01-2004M12. For both the possibilities we will look at the sample standard deviation. First we will compare the sample standard deviation in recessions to periods of expansions. While the standard deviation during periods of expansion is 45.46, the sample standard deviation during recessions is 42.07. We conclude that there is a slight difference in standard deviation, however the difference is not big. Secondly, we look at the sample standard deviation of the period 2005M01-2019M12 compared to 1990M01-2004M12. In the period 2005M01-2019M12 the standard deviation was 38.86, compared to 33.70 in the period 1995M01-2004M12, again there is a small difference. Therefore we conclude that the data is heteroskedastic.

(v) **Nonlinearity**: From figure 1 in appears that the foreign auto sales experience a decline during recession periods and a increase during periods of expansion. We test this possible non-linearity using the following regression:

$$d(fautonsa_t) = \mu_1 DREC_t + \mu_2(1 - DREC_t) + \epsilon_t \tag{3}$$

Where DREC is a binary variable, 1 during recession periods and 0 during periods of expansion. This regression yields a R^2 of 0.00 and the variables DREC and (1-DREC) have a p-value of respectively 0.76 and 0.89, therefore we conclude that there is no non-linearity in the foreign auto sales.

We found that the time series for foreign car sales has the following characteristics: trend, seasonality, abberent observations and heteroskedasticity. To come to this conclusion we used multiple graphical techniques as well as a number of regressions to test whether the time series has these characteristics.

3.2 Normality foreign auto sales

The kurtosis of $fautosa$ is 3.02 which is close to the kurtosis of a normal distribution which is 3.00. However, the skewness of $fautosa$ is 0.45, the data is thus positively skewed. Using the Jarcque-Bera test we find a test value of 20.50 with a p-value of 0.00. Therefore, we conclude that over the entire sample period the time series of foreign car sales is not normally distributed. Most of the (positive) outliers are in the period from 1985 to 1990. When we exclude this period from the sample the Jarcque-Bera test yields a test value of 14.15 which still leads to the conclusion that $fautosa$ is not normally distributed, however its value is much lower than previously.

3.3 Autocorrelation foreign auto sales

In the correlogram with 60 lags (see figure 3) we see that the auto-correlation decreases monotonically, but slowly, toward zero. The partial autocorrelation decreases very fast, only the first five are significant. This pattern is inherent to autoregressive models. The measurement that only the first five partial autocorrelations are significant indicates that the appropriate model could be a autoregressive model with $p = 5$ (thus, $AR(5)$).

3.4 Estimation AR(p) and MA(q) models

In table 1 we see the information criteria for autoregressive models with $p = 0, ..., 6$ and moving average models with $q = 1, ...6$. We select the model with the lowest criteria. The Akaike information criterion as well as the Schwarz information criterion select AR(5) as the best model, with AIC 7.80 and SIC 7.85. The conclusion that AR(5) is the most best model is in line with the result in question 1c.

Model	**AIC**	**SIC**
AR(0)	10.29	10.30
AR(1)	7.87	7.90
AR(2)	7.83	7.86
AR(3)	7.82	7.86
AR(4)	7.82	7.86
AR(5)	7.80	7.85
AR(6)	7.81	7.86
MA(1)	9.36	9.38
MA(2)	8.87	8.90
MA(3)	8.55	8.59
MA(4)	8.40	8.44
MA(5)	8.29	8.34
MA(6)	8.22	8.28

Table 1: Akaike and Schwarz information criterion for AR, MA and ARMA models

3.5 Estimation of the AR(2) model

After estimating the parameters of the AR(2) model, the results in the following table are found:

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	166.3732	14.33511	11.606	0.0000
AR(1)	0.747688	0.040023	18.6814	0.0000
AR(2)	0.217961	0.040035	5.444314	0.0000

Table 2: Estimated parameters of the AR(2) model

We define the characteristic polynomial of AR(p) as the lag polynomial $\phi_p(L) = 1 - \phi_1 L - ... - \phi_p L^p$, but considered as a function of z:

$$\phi_p(z) = 1 - \phi_1(z) - ... - \phi_p z^p \tag{4}$$

such that $\phi_2(z) = 1 - \phi_1 z - \phi_2 z^2$ for the AR(2) model. To find the roots of this equation, simply set $\phi_2(z) = 0$. If z is a root of the previous characteristic equation and setting $z^{-1} = \lambda$, we can rewrite the equation such that:

$$\begin{aligned} 1 - \phi_1(z) - \phi_2 z^2 &= 0 \\ \Leftrightarrow z^{-2} - \phi_1 z^{-1} - \phi_2 &= 0 \\ \Leftrightarrow \lambda^2 - \phi_1 \lambda - \phi_2 &= 0 \end{aligned}$$

We now recognize this as a quadratic equation and can solve it's roots using:

$$\lambda_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \tag{5}$$

Using the results from (2), we derive $\lambda_1 = 0.972$ and $\lambda_2 = -0.224$. Recall $z^{-1} = \lambda$ such that $z_1 = 1.003$ and $z_2 = -4.464$. As all roots lie inside the unit circle, as $\|z\| > 1$ we can recall from the theory that the shocks have transitory effects and thus the AR(2) model is stationary.

We define the theoretical autocorrelations in the AR(2) model using the following equation:

$$\rho(t) = \phi_1 \rho(t-1) + \phi_2 \rho(t-2) \; for \; t = 3, 4, ... \tag{6}$$

such that the values of the first 24 'implied' autorcorrelations are obtained, these are shown in the the following table. The figure shows the comparison with the empirical (actual) autocorrelations.

t	$\rho(t)$	t	$\rho(t)$
1	0.956	13	0.682
2	0.933	14	0.662
3	0.906	15	0.644
4	0.881	16	0.626
5	0.856	17	0.608
6	0.832	18	0.591
7	0.809	19	0.575
8	0.786	20	0.558
9	0.764	21	0.543
10	0.742	22	0.528
11	0.722	23	0.513
12	0.701	24	0.498

Table 3: Theoretical autocorrelations as computed using equation (6)

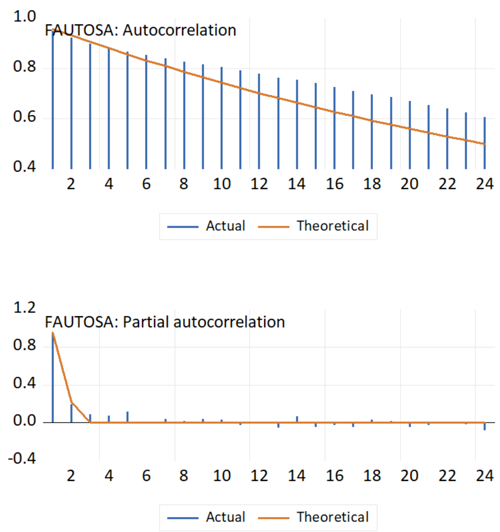

Figure 3: Comparison of theoretical and empirical (actual) autocorrelations/partial auto correlations

As we see in figure 3, we notice that the actual and empirical autocorrelations follow each other closely up to $t = 6$, after this the actual and theoretical autocorrelations diverge. While both are still converging to 0, because of stationarity, we notice that the actual autocorrelations converge much slower than the theoretical autocorrelations.

3.6 Properties of residuals

(i) We find in the correlogram of the residuals that for lags up to 5, the residuals have no significant autocorrelation (at the 5% signicance level). This is confirmed by the Breusch-Godfrey Serial Correlation LM test for up to 5 lags. However residuals seem to be significantly autocorrelated after more than 5 lags. Testing this using the previously mentioned test with a full year lag (12 lags), this hypothesis is confirmed. We find $LM = 15.954 > F(12, 223)_{\alpha=0.05} = 1.752$, where LM is the test statistic and F is the F-distribution. As the null hypothesis of no serial correlation at up to 12 lags is rejected, we find that the white noise property of $E[\epsilon_s \epsilon_t] = 0$ for $s, t = 1, 2, ..., T$ and $s \neq t$ does not hold for this time series.

(ii) We find in the correlogram of the squared residuals that for lags up to 4, the squared residuals again have no significant autocorrelation. This again is confirmed by the Breusch-Godfrey Serial Correlation LM test for up to 4 lags. As the squared residuals do have signicant autocorrelation after 4 lags, which are mostly positve for up to 9 lags and mostly negative for up to at least 36 lags, this implies conditional heteroskedasticity. This is confirmed by the White Heteroskedasticity test, which rejects the null hypothesis of homoskedasticity with a p-value of 0.0000. As we find heteroskedasticity in our model, the white noise property of $E[\epsilon_t^2] = \sigma^2$ for $t = 1, 2, ..., T$ does not hold for this time series.

(iii): The residuals are shown in figure 4. One can observe that the kurtosis differs significantly from 3, which is the kurtosis of the normal distribution. Furthermore the Jarque-Bera null-hypothesis is rejected. Thus we conclude that the residuals are not normally distributed.

We can test if the mean is signicantly different from 0 using the t-test. We find that $t = \frac{-3.92*10^{-16}}{\frac{12.012}{598}} <$ 1.645 (the 5% significance), which implies that we do not reject the null hypothesis and the mean does not differ significantly from zero. This implies that the white noise property of $E[\epsilon_t] = 0$ for $t = 1, 2, ..., T$ holds for this time series.

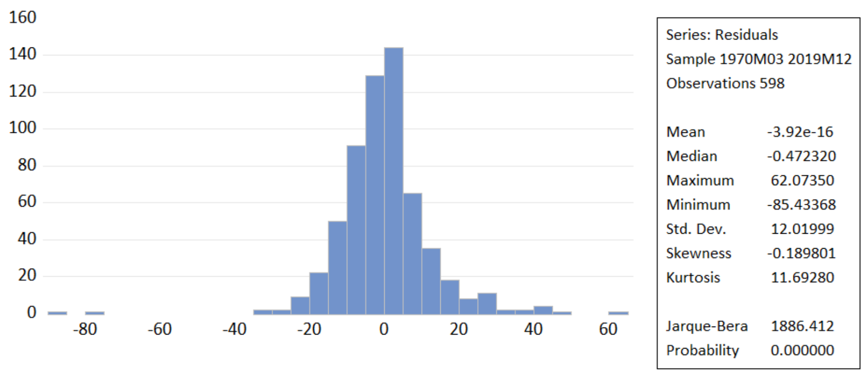

Figure 4: Residuals

As two of the three white noise properties do not hold, the AR(2) model does not suffice the minimum requirement of being an ”adequate” model.

3.7 Forecast evaluation

We re-estimate the AR(2) model for the sample period of January 1970 - December 1989 (T=240 observations). We construct one-step ahead forecasts for the sample period January 1990 - December 2019 (360 observations). The forecasts are shown in the following figure:

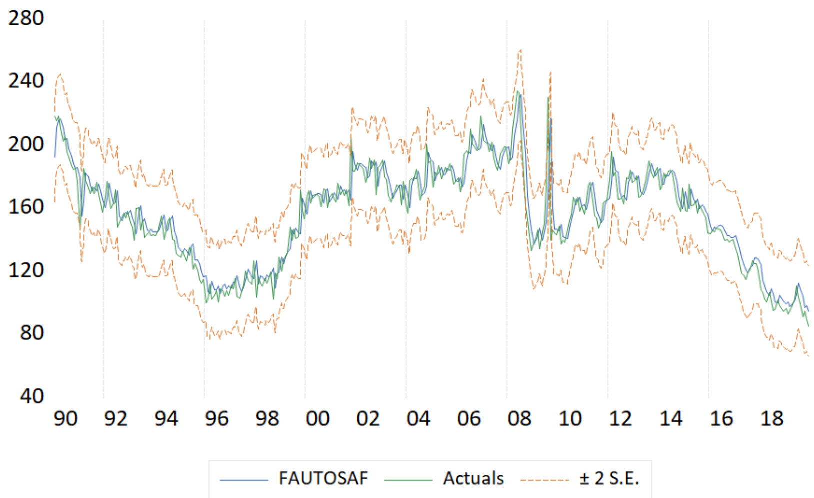

Figure 5: Forecast of January 1990 - December 2019 using the AR(2) model

We wish to evaluate the accuracy of these point forecasts in 'absolute' terms, we do this by considering their unbiasedness, accuracy and efficiency.

The unbiasedness is shown if the residuals have a mean not significant different of zero, which implies that on average the one-step ahead point forecasts are on the same level as the actual value. To test whether the mean is significantly different from zero, we use the t-test. As we can use the regression of $e_t = \alpha + \epsilon_t$ (where $e_t = fautosa_t - fautosaf_t$), we find $t = \frac{-1.915912}{0.536804} = -3.569$ and find a p-value of 0.0004. As the mean is significantly different from zero and negative, our forecasts overestimate the forecast values and hence we reject the null hypothesis that our AR(2) forecasts are unbiased.

To show if the estimators are efficient, we can use of the regression $fautosa_t = \alpha + \beta fautosaf_t + \epsilon_t$. We test if β is significantly different from 1 using the t-test. We find $t = \frac{1.008576-1}{0.012522} = 0.685$, as $0.685 < 1.645$ which implies that at the 5% significance level we reject the null hypothesis of β equal to 1. This also implies that our forecasts are not efficient (at the 5% significance), however the $R^2 = 0.916$ which implies that our forecasts explain quite a lot of the variation in the original time series.

To show if the estimators are accurate, we use the RMSE (Root Mean Squared error) of the forecast sample (January 1980 - December 2019) and the residuals over the re-estimated AR(2) model on the sample period January 1980 - December 2019. We find that the RMSE on the forecast sample is 10.35 while the standard deviation of the dependant variable of the re-estimated AR(2) model (our forecast target) is 31.38. As the RMSE is about $frac13$ of the standard deviation of the dependant variable indicating that the forecast caputres quite a bit of the variation of the original time series. Furthermore the standard deviation of the residuals in the re-estimated AR(2) model is 10.18, we find that the difference between the RMSE and the standard deviation of these residuals are not that large. We thus conclude that our estimators are accurate. *From hence on forth we will use the estimated AR(2) model on the sample January 1970 - December 1990 again, as shown in table 2.*

Comparing the point forecasts obtained from the AR(2) model with so-called 'random walk' forecasts, that is $\hat{y}_{T+1|T} = y_T$, we obtain the following figure.

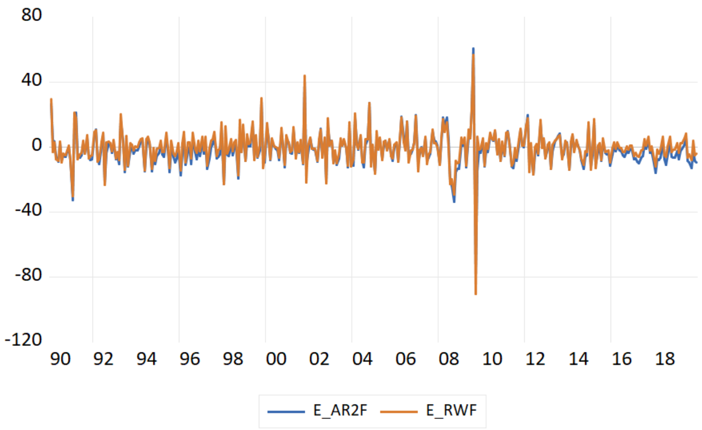

Figure 6: Comparison of residuals obtained by forecasts using AR(2) model and random walk forecasts

As one can see the random walk forecasts and the AR(2) forecasts follow each other quite closely. Furthermore summary statistics show that the mean of the random walk forecasts is quite close to 0 (-0.289) as opposed to the mean of the AR(2) model (-1.916). The standard deviation of both models are quite similar, as the standard deviation of the AR(2) model and random walk forecasts are 10.185 and 10.595 respectively.

After defining the loss function (d_t) as the difference between the squared residuals of the AR(2) model and the random walk forecasts, we obtain the following figure:

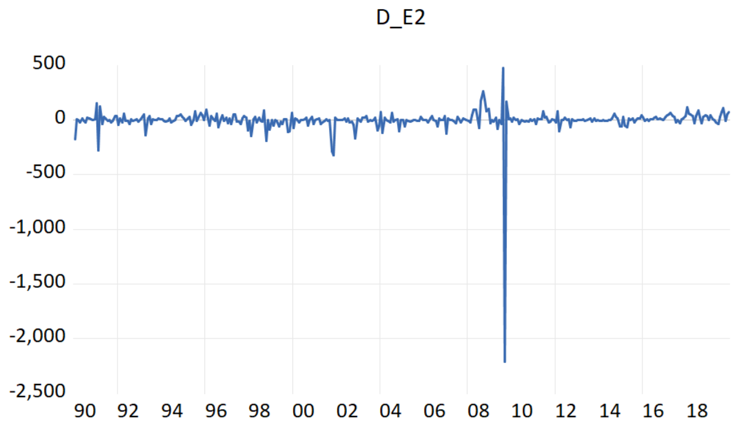

Figure 7: Difference of squared residuals of the AR(2) model and the random walk forecasts

As one can see, in the observations related to the Great Financial Crisis, the AR(2) model is superior over the random walk forecasts. We wish to test if the mean this series differs significantly from 0 to reject the null hypothesis of either model being equally informative. We test this using again the regression $d_t = \alpha + \epsilon_t$ and find that α is not significantly different from 0. Hence we do not reject the null hypothesis of either model being equally informative and thus the AR(2) model is not superior over the random walk forecasts.