6.8 $P(\text{reporting heads}) = P(HT \mid \text{result reported})$

reporting a result implies that either HT or TH was the out come.

$p(\text{result reported}) = p(1-p) + (1-p)p = 2p(1-p)$

$p(HT) = p(1-p)$

$$p(\text{reporting heads}) = \frac{\cancel{p(1-p)}}{2\cancel{p(1-p)}} = \frac{1}{2}$$

b $P(\text{flipping } n \text{ rounds}) = P(d)^{n} \cdot (1 - P(d))$

where $P(d)$ denotes the probability of having a result reported ie. two coins are different.

$P(d) = 2p(1-p)$

By the property of geometric series,

$$\text{if } \sum_{n=1}^{\infty} n \cdot P(n) = \sum_{n=1}^{\infty} n \, P(A)^{n-1} P(A)$$

$$\text{Then } \sum_{n=1}^{\infty} n \cdot P(n) = \frac{1}{P(A^c)} = \frac{1}{2p(1-p)}$$

$$\text{Expected number of rounds} = \frac{1}{2p(1-p)}$$

$$E[\text{flips}] = 2 \cdot E[\text{rounds}] = \boxed{\frac{1}{p(1-p)}}$$

6.13 $P(W \geq 1) = 1 - P(W=0) = 1 - P(6\text{ men})$
$= 1 - \left(\frac{1}{2}\right)^6 = \boxed{\frac{63}{64}}$

6.21 a $f(x) = \frac{1}{\sqrt{2\pi}} \exp \frac{-x^2}{2}$

$\frac{1}{\sqrt{2\pi}}$ is a positive constant, and $\exp(x)$ for $x \in \mathbb{R}$ is nonnegative

Therefore, $f(x)$ is the product of two non-negative reals, and is necessarily non-negative for all real x.

b. by gaussian integral,

$$\left(\frac{1}{\sqrt{2\pi}}\int e^{-\frac{x^2}{2}}dx\right)^2 = \frac{1}{2\pi}\left(\int e^{-\frac{x^2}{2}}dx \int e^{-\frac{y^2}{2}}dy\right)$$

converting into 2-D polar coordinates,

$$= \frac{1}{2\pi}\left(\int_0^{2\pi}\int_0^{\infty} re^{-\frac{r^2}{2}}dr\,d\theta\right)$$

$$= \frac{1}{2\pi}\cdot\int_0^{2\pi} -e^{-\frac{r^2}{2}}\Big|_0^{\infty}$$

$$= \frac{1}{2\pi}\cdot 2\pi\cdot 1 = 1 \checkmark \quad \sqrt{1}=1$$

because the integral of $f(x)$ from $-\infty$ to ∞ is equal to 1, and the function is non-negative, $f(x)$ is a pdf.

c. $$\int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x f(x)\,dx + \int_0^{\infty} x f(x)\,dx$$

$$= -\int_0^{-\infty} x f(x)\,dx + \int_0^{\infty} x f(x)\,dx$$

$$= -\int_0^{\infty} x f(-x)\,dx + \int_0^{\infty} x f(x)\,dx$$

$$= -\int_0^{\infty} x f(x)\,dx + \int_0^{\infty} x f(x)\,dx$$

$$= 0 \checkmark$$

d. Let $y = x - \mu$

$$\int x f(x-\mu)\,dx = \int (y + \mu) f(y)\,dy$$
$$= \int y f(y)\,dy + \mu \int f(y)\,dy$$
$$= 0 + \mu \cdot 1$$
$$= \mu \checkmark$$

e. $$\int x^2 f(x)\,dx = \int \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} \int \overset{u}{x} \cdot \overset{v'}{x e^{-\frac{x^2}{2}}}$$

$$\frac{1}{\sqrt{2\pi}} \int u \cdot v' = \left(uv - \int u'v\right) \frac{1}{\sqrt{2\pi}}$$
$$= \left(x \cdot \int x \cdot f(x) - \int -e^{-\frac{x^2}{2}}\,dx\right) \frac{1}{\sqrt{2\pi}}$$
$$= \left(0 - -\int e^{-\frac{x^2}{2}}\,dx\right) \frac{1}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\int e^{-\frac{x^2}{2}}\,dx \int e^{-\frac{y^2}{2}}\,dy}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\int_0^{2\pi} \int_0^{\infty} r e^{-\frac{r^2}{2}}\,dr\,d\theta}$$
$$= \frac{1}{\sqrt{2\pi}} \sqrt{\int_0^{2\pi} e^{-\frac{r^2}{2}}\,d\theta}$$ by result of part a.
$$= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi}$$
$$= 1 \checkmark$$

6.22. $\int \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) dx$

let $y = x - \mu$

$= \int \exp\left(\frac{-(y)^2}{2\sigma^2}\right) dy$

$= \sqrt{\int \exp\left(\frac{-(y)^2}{2\sigma^2}\right) dy \cdot \int \exp\left(\frac{-(x)^2}{2\sigma^2}\right) dx}$

$= \sqrt{\int_0^{2\pi} \int_0^{\infty} r \exp\left(\frac{-r^2}{2\sigma^2}\right) dr\, d\theta}$

$= \sqrt{\int_0^{2\pi} \sigma^2 \exp\left(\frac{-r^2}{2\sigma^2}\right) d\theta}$

$= \sqrt{2\pi\sigma^2} = \sqrt{2\pi}\,\sigma$ ✓

6.24a. $p(x) = \int_1^u \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) dx$

$\sigma = \sqrt{np(1-p)} = \sqrt{10^5 \cdot .25} = 158$

$p(x) = \int_{49500}^{50500} \frac{1}{396} \exp\left(\frac{-(x-50000)^2}{2 \cdot 158^2}\right) dx = \boxed{.996}$

b $\sigma = \sqrt{10^4 \cdot \frac{1}{2}^2} = 50$

$p(\{h > 4000\}) = \int_{4000}^{10000} \cdot \frac{1}{\sqrt{2\pi}\, 50} \exp\left(\frac{-(x-5000)^2}{2 \cdot 50^2}\right) dx \approx \boxed{0}$

c $\sigma = \sqrt{10^2 \cdot \frac{1}{2}^2} = 5$. $p(\{h > 60\}) \cup p(\{h < 40\}) = 1 - p(\{h$ between 40 and $60\})$

$p(\{40 \le h \le 60\}) = \int_{40}^{60} \frac{1}{\sqrt{2\pi}\,5} \exp\left(\frac{-(x-50)^2}{2 \cdot 5^2}\right) dx$

$= .9545$

$1 - p(x^c) = \boxed{.0455}$