UROP Task 2 (Pre-Finalised)
Relaxation Enhancement

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CONTENTS

1. The problem

We here consider the Enhanced-Diffusion Equation (EDE) of the form

$$\partial_t f + u(y)\partial_x f = \nu\Delta f, \quad (x, y) \in \mathbb{T}^2, \quad \nu \in (0, 1) \tag{1.1}$$

The main goal of this article is to establish estimates for the L^2 norm of equation. We consider two particular cases - $u(y) = y$ and $u(y) = \sin y$.

For the case $u(y) = y$ Every x-Fourier coefficient f_k $(k \neq 0)$ of f decays as followed:

THEOREM 1.1. *The functional*

$$E_k = \frac{1}{2}\left(\|f_k\|^2 + \frac{\nu^{2/3}}{8|k|^{2/3}}\|\nabla f_k\|^2 + \frac{\nu^{1/3}}{4|k|^{4/3}}\langle \partial_x f_k, \partial_y f_k\rangle\right) \tag{1.2}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k + \frac{1}{12}\nu^{1/3}|k|^{2/3}E_k \leq 0. \tag{1.3}$$

In particular, for all $t \geq 0$,

$$E_k(t) \leq E_k(0)\exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}t\right) \tag{1.4}$$

As a direct consequence we obtain the

COROLLARY 1.2. *For all* $t \geq 0$

$$\|f_k(t)\|^2 \leq 4\mathrm{e}\|f_k(0)\|^2 \exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}t\right) \tag{1.5}$$

For the case $u(y) = \sin y$ Every x-Fourier coefficient f_k $(k \neq 0)$ of f decays as followed:

THEOREM 1.3. *The functional*

$$E_k = \frac{1}{2}\left(\|f\|^2 + \frac{\nu^{1/2}}{5|k|^{1/2}}\|\nabla f\|^2 + \frac{6}{25|k|}\langle \cos y \partial_x f,\, \partial_y f\rangle + \frac{8}{25\nu^{1/2}|k|^{3/2}}\|\cos y\, \partial_x f\|^2\right) \tag{1.6}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k + \frac{1}{20}\nu^{1/2}|k|^{1/2}E_k \leq 0. \tag{1.7}$$

In particular, for all $t \geq 0$,

$$E_k(t) \leq E_k(0)\exp\left(-\frac{1}{20}\nu^{1/2}|k|^{1/2}t\right) \tag{1.8}$$

As a direct consequence we obtain the

COROLLARY 1.4. *For all* $t \geq 0$

$$\|f_k(t)\|^2 \leq 12e\,\|f_k(0)\|^2 \exp\left(-\frac{\nu^{1/2}|k|^{1/2}t}{20(1 + |\ln \nu| + \ln|k|)}\right) \tag{1.9}$$

2. General Analysis

2.1. Integration by Parts. Before analysing (1.1), we first establish the following fundamental result.

LEMMA 2.1. *(Integration by Parts) For differentiable scalar field* $\phi : \mathbb{T}^2 \to \mathbb{R}$ *and vector field* $\boldsymbol{F} : \mathbb{T}^2 \to \mathbb{R}^2$ *we have*

$$\int_{\mathbb{T}^2} \nabla\phi.\boldsymbol{F}\; dx\, dy = -\int_{\mathbb{T}^2} \phi(\nabla.\boldsymbol{F})\; dx\, dy \tag{2.1}$$

PROOF. Utilise Divergence Theorem and noticing that $\nabla.(\phi\boldsymbol{F}) = \boldsymbol{F}.\nabla\phi + \phi(\nabla.\boldsymbol{F})$ $\square$

LEMMA 2.2. *For* $f, g \in C^1(\mathbb{R}_{>0} \times \mathbb{T}^2)$ *we have*

$$\langle f, \partial_x g\rangle = -\langle \partial_x f, g\rangle \tag{2.2}$$

$$\langle f, \partial_y g\rangle = -\langle \partial_y f, g\rangle \tag{2.3}$$

PROOF. We prove (2.2) by utilising Lemma 2.1 with $\boldsymbol{F} = (f, 0)$ and $\phi = g$. Similarly, we prove (2.3) by utilising Lemma 2.1 with $\boldsymbol{F} = (0, f)$ and $\phi = g$. $\square$

2.2. Norms of equation. Using the fundamental results as stated above we may conclude the following if f is the solution of (1.1). Here we think of the equation k by k (the x-Fourier coefficient), with $k \in \mathbb{Z}^d \setminus \{0\}$, and consider $g = f_k$.

The L^2 norm. Apply Lemma 2.1 with $\phi = f$ and $\boldsymbol{F} = \nabla\phi$ we have

$$\langle \nabla^2 f, f\rangle = -\|\nabla f\|^2 \tag{2.4}$$

Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 = \langle f,\, \partial_t f\rangle = \nu\langle f,\, \nabla^2 f\rangle - \langle f, u(y)\partial_x f\rangle \tag{2.5}$$

Noticing that

$$
\begin{aligned}
\langle f,\, u(y)\partial_x f\rangle &= \langle u(y) f,\, \partial_x f\rangle \\
&= -\langle \partial_x (u(y) f), f\rangle && \text{(Utilising Lemma 2.2)} \\
&= -\langle u(y)\partial_x f, f\rangle \\
&= -\langle f,\, u(y)\partial_x f\rangle && (2.6)
\end{aligned}
$$

We have

$$
\langle f,\, u(y)\partial_x f\rangle = 0 \tag{2.7}
$$

Combining with (2.4) we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 + \nu\|\nabla f\|^2 = 0 \tag{2.8}
$$

The gradient norm. We first differentiate (1.1) by x and y respectively:

$$
\partial_t\partial_x f + u(y)\partial_{xx} f = \nu\nabla^2(\partial_x f) \tag{2.9}
$$

$$
\partial_t\partial_y f + u(y)\partial_{xy} f + u'(y)\partial_x f = \nu\nabla^2(\partial_y f) \tag{2.10}
$$

(2.9) suggests that $\partial_x f$ is also a solution to the EDE in (1.1), so from (2.8) we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\partial_x f\|^2 = -\nu\,\|\nabla(\partial_x f)\|^2 \tag{2.11}
$$

Similarly, we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y f\|^2 = \langle \partial_y f,\, \partial_t\partial_y f\rangle &= \nu\langle \partial_y f,\, \nabla^2(\partial_y f)\rangle - \langle \partial_y f, u(y)\partial_x\partial_y f\rangle - \langle \partial_y f, u'(y)\partial_x f\rangle \\
&= -\nu\,\|\nabla(\partial_y f)\|^2 - \langle u'(y)\partial_x f, \partial_y f\rangle && (2.12)
\end{aligned}
$$

Adding (2.11) and (2.12) yields

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|^2 &= -\nu\left(\|\nabla(\partial_x f)\|^2 + \|\nabla(\partial_y f)\|^2\right) - \langle u'(y)\partial_x f, \partial_y f\rangle \\
&= -\nu\left(\|\partial_{xx} f\|^2 + 2\,\|\partial_{xy} f\|^2 + \|\partial_{yy} f\|^2\right) - \langle u'(y)\partial_x f, \partial_y f\rangle && (2.13)
\end{aligned}
$$

Finally noticing that

$$
\langle \partial_{xx} f, \partial_{yy} f\rangle = -\langle \partial_x f, \partial_{xyy} f\rangle = \langle \partial_{xy} f, \partial_{xy} f\rangle = \|\partial_{xy} f\|^2 \tag{2.14}
$$

Therefore, we have

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|^2 = -\nu\,\|\partial_{xx} f + \partial_{yy} f\|^2 - \langle u'(y)\partial_x f, \partial_y f\rangle = -\nu\,\left\|\nabla^2 f\right\|^2 - \langle u'(y)\partial_x f, \partial_y f\rangle \tag{2.15}
$$

The cross term. We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle u'(y)\partial_x f, \partial_y f\rangle &= \langle u'\partial_y f, \partial_t\partial_x f\rangle + \langle u'\partial_x f, \partial_t\partial_y f\rangle \\
&= \nu\left(\langle u'\partial_y f, \nabla^2(\partial_x f)\rangle + \langle u'\partial_x f, \nabla^2(\partial_y f)\rangle\right) \\
&\quad - \left(\langle u'\partial_y, u\,\partial_{xx} f\rangle + \langle u'\partial_x, u\,\partial_{xy} f\rangle\right) - \left\|u'\partial_x f\right\|^2
\end{aligned} \tag{2.16}
$$

We note the following. Firstly,

$$
\langle u'\partial_y f, u\,\partial_{xx} f\rangle = -\langle u'\partial_{xy} f, u\,\partial_x f\rangle = -\langle u'\partial_x f, u\,\partial_{xy} f\rangle \tag{2.17}
$$

Moreover,

$$
\begin{aligned}
\langle u'\partial_y f, \nabla^2(\partial_x f)\rangle &= \langle u'\partial_y f, \partial_{xxx} f\rangle + \langle \partial_y f, \partial_{xyy} f\rangle \\
&= -\langle u'\partial_{xy} f, \partial_{xx} f\rangle - \langle \partial_{xy} f, \partial_{yy} f\rangle = -\langle u'\partial_{xy} f, \nabla^2 f\rangle && (2.18)
\end{aligned}
$$

In a similar manner,

$$\langle u'\partial_x f, \nabla^2(\partial_y f)\rangle = -\langle u'\partial_{xy} f, \nabla^2 f\rangle - \langle u''\partial_x f, \nabla^2 f\rangle \tag{2.19}$$

Substituting (2.17)-(2.19) into (2.16) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u'(y)\partial_x f, \partial_y f\rangle = -2\nu\langle u'\partial_{xy} f, \nabla^2 f\rangle - \nu\langle u''\partial_x f, \Delta f\rangle - \left\|u'\partial_x f\right\|^2 \tag{2.20}$$

Supplementary Term. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u'\partial_x f\right\|^2 = \langle (u')^2\partial_x f, \partial_t\partial_x f\rangle \tag{2.21}$$

$$= -\nu\langle (u')^2\partial_x f, \nabla^2(\partial_x f)\rangle + \langle (u')^2\partial_x, u\,\partial_{xx}\rangle \tag{2.22}$$

Again note that

$$\langle (u')^2\partial_x, u\,\partial_{xx}\rangle = -\langle (u')^2\partial_{xx}, u\,\partial_x u\rangle = -\langle (u')^2\partial_x, u\,\partial_{xx}\rangle \tag{2.23}$$

Moreover, similar to (2.18), we have

$$\langle (u')^2\partial_x f, \nabla^2(\partial_x f)\rangle = -\langle (u')^2\partial_{xx} f, \nabla^2 f\rangle \tag{2.24}$$

Therefore,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u'\partial_x f\right\|^2 = \nu\langle (u')^2\partial_{xx} f, \nabla^2 f\rangle \tag{2.25}$$

3. Special case when $u(y) = y$

For this special case we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 + \nu\|\nabla f\|^2 = 0 \tag{3.1}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|^2 + \nu\left\|\nabla^2 f\right\|^2 = -\langle \partial_x f, \partial_y f\rangle \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_x f, \partial_y f\rangle + \|\partial_x f\|^2 = -2\nu\langle \partial_{xy} f, \nabla^2 f\rangle \tag{3.3}$$

3.1. The energy functional. For $\tilde{\alpha}, \tilde{\beta} > 0$ to be fixed, the energy functional

$$E_k = \frac{1}{2}\left[\|f_k\|^2 + \tilde{\alpha}\,\|\nabla f_k\|^2 + 2\tilde{\beta}\langle \partial_x f_k, \partial_y f_k\rangle\right] \tag{3.4}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k + \nu\|\nabla f_k\|^2 + \tilde{\alpha}\nu\|\nabla^2 f_k\|^2 + \tilde{\beta}\|\partial_x f_k\|^2 = -2\nu\tilde{\beta}\langle \partial_{xy} f_k, \nabla^2 f_k\rangle - \tilde{\alpha}\langle \partial_x f_k, \partial_y f_k\rangle. \tag{3.5}$$

We consider the right-hand side above as error terms. The choice of $\tilde{\alpha}, \tilde{\beta}$ will be made k dependent as

$$\tilde{\alpha} = \alpha\frac{\nu^{2/3}}{|k|^{2/3}}, \qquad \tilde{\beta} = \beta\frac{\nu^{1/3}}{|k|^{4/3}}, \tag{3.6}$$

for α, β small enough, independent of ν, k (we can compute them explicitly as we will show). We perform estimates below that require constraints on α and β, and then we will show at the end that all the constraints can be satisfied.

Positivity. First, we ensure that E is a positive functional. Since

$$\begin{aligned}2\tilde{\beta}\,|\langle\partial_x f,\partial_y f\rangle| &\leq 2\,\|\partial_y f\|\,\tilde{\beta}\,\|\partial_x f\| = 2\,\|\partial_y f\|\,\tilde{\beta}|k|\,\|f\| \\ &\leq \frac{\tilde{\alpha}}{2}\,\|\nabla f\|^2 + \frac{2\tilde{\beta}^2k^2}{\alpha}\,\|f\|^2 .\end{aligned} \tag{3.7}$$

Hence, requiring

$$\frac{\tilde{\beta}^2k^2}{\tilde{\alpha}} \leq \frac{1}{4} \tag{3.8}$$

implies

$$\frac{1}{4}\left[\|f\|^2 + \tilde{\alpha}\|\nabla f\|^2\right] \leq E \leq \frac{3}{4}\left[\|f\|^2 + \tilde{\alpha}\|\nabla f\|^2\right]. \tag{3.9}$$

The first error term. For the first error term in (3.5), we have

$$\begin{aligned}2\nu\tilde{\beta}|\langle\partial_{xy}f,\nabla^2 f\rangle| \leq 2\tilde{\beta}\nu|k|\|\nabla^2 f\|\|\partial_y f\| &\leq \frac{\tilde{\alpha}\nu}{2}\|\nabla^2 f\|^2 + \frac{(2\tilde{\beta}|k|)^2\nu}{2\tilde{\alpha}}\|\partial_y f\|^2 \\ &\leq \frac{\tilde{\alpha}\nu}{2}\|\nabla^2 f\|^2 + \frac{2\tilde{\beta}^2k^2}{\tilde{\alpha}}\nu\|\nabla f\|^2\end{aligned} \tag{3.10}$$

If we impose the following condition,

$$\frac{\tilde{\beta}^2k^2}{\tilde{\alpha}} \leq \frac{1}{8}. \tag{3.11}$$

We have

$$2\nu\tilde{\beta}|\langle\partial_{xy}f,\nabla^2 f\rangle| \leq \frac{\tilde{\alpha}\nu}{2}\|\nabla^2 f\|^2 + \frac{\nu}{4}\|\nabla f\|^2, \tag{3.12}$$

One should note that if (3.11) is satisfied, then (3.8) is automatically satisfied as well.

The second error term. For the second error term in (3.5), we have

$$\tilde{\alpha}|\langle\partial_x f,\partial_y f\rangle| \leq \frac{\tilde{\alpha}^2}{2\tilde{\beta}}\|\partial_y f\|^2 + \frac{\tilde{\beta}}{2}\|\partial_x f\|^2 = \frac{\tilde{\alpha}^2}{2\tilde{\beta}}\|\partial_y f\|^2 + \frac{\tilde{\beta}}{2}k^2\,\|f\|^2 \tag{3.13}$$

If we impose the following condition,

$$\frac{\tilde{\alpha}^2}{2\tilde{\beta}} \leq \frac{\nu}{4} \iff \frac{\tilde{\alpha}^2}{\tilde{\beta}} \leq \frac{\nu}{2}, \tag{3.14}$$

we have

$$\tilde{\alpha}|\langle\partial_x f,\partial_y f\rangle| \leq \frac{\nu}{4}\|\partial_y f\|^2 + \frac{\tilde{\beta}k^2}{2}\|f\|^2 \leq \frac{\nu}{4}\|\nabla f\|^2 + \frac{\tilde{\beta}k^2}{2}\|f\|^2 \tag{3.15}$$

The inequality for E. Collecting the above estimates and using (3.5), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}E + \frac{\tilde{\beta}k^2}{2}\|f\|^2 + \frac{\nu}{2}\|\nabla f\|^2 + \frac{\tilde{\alpha}\nu}{2}\|\Delta f\|^2 \leq 0. \tag{3.16}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}E + \frac{\tilde{\beta}k^2}{2}\left(\|f\|^2 + \frac{\nu}{\tilde{\beta}k^2}\|\nabla f\|^2\right) \leq 0 \tag{3.17}$$

Assuming

$$\tilde{\alpha} \leq \frac{\nu}{\tilde{\beta}k^2} \iff \tilde{\alpha}\tilde{\beta} \leq \frac{\nu}{k^2}, \tag{3.18}$$

we end up with our required differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E + \frac{2\tilde{\beta}k^2}{3}E \leq 0 \tag{3.19}$$

Scaling of Constants. Collecting (3.11), (3.14), (3.18), we have

$$\frac{\tilde{\beta}^2 k^2}{\tilde{\alpha}} \leq \frac{1}{8}, \quad \frac{\tilde{\alpha}^2}{\tilde{\beta}} \leq \frac{\nu}{2}, \quad \tilde{\alpha}\tilde{\beta} \leq \frac{\nu}{k^2}. \tag{3.20}$$

We let $\tilde{\alpha} = \alpha\nu^{A_1}|k|^{A_2}$ and $\tilde{\beta} = \beta\nu^{B_1}|k|^{B_2}$, so that the conditions above can be reduced to conditions independent of ν and k. We have the following over-determined system:

$$\begin{cases} -A_1 + 2B_1 & = 0 \\ 2A_1 - B_1 & = 1 \\ A_1 + B_1 & = 1 \end{cases}, \qquad \begin{cases} -A_2 + 2B_2 & = -2 \\ 2A_2 - B_2 & = 0 \\ A_2 + B_2 & = -2 \end{cases} \tag{3.21}$$

The solution to this system is $(A_1, B_1) = (2/3, 1/3)$, and $(A_2, B_2) = (-2/3, 1/3)$. The choice of scaling is hence:

$$\tilde{\alpha} = \alpha\frac{\nu^{2/3}}{|k|^{2/3}}, \qquad \tilde{\beta} = \beta\frac{\nu^{1/3}}{|k|^{4/3}} \tag{3.22}$$

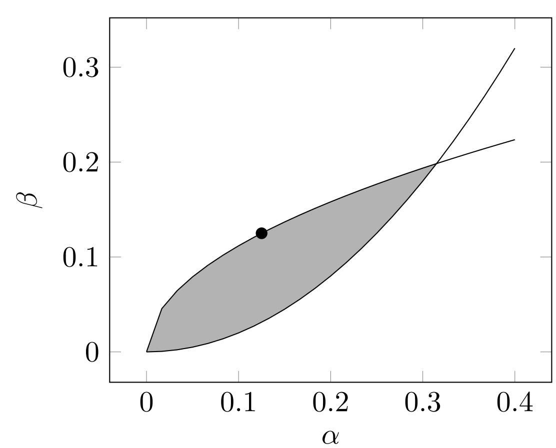
Checking if all constraints are consistent. Applying the scaling above we have

$$\frac{\beta^2}{\alpha} \leq \frac{1}{8}, \quad \frac{\alpha^2}{\beta} \leq \frac{1}{2}, \quad \alpha\beta \leq 1.$$

Multiplying the first two constraint we have $\alpha\beta \leq 1/16$, so the third constraint is redundant. In fact, if we pick α, β such that

$$\alpha = \beta = \frac{1}{8}, \tag{3.23}$$

and it is clear that all the constraints are verified. This could be illustrated by the following graph.



With these choice of constants we have the differential inequality as in Theorem 1.1.

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k + \frac{1}{12}\nu^{1/3}|k|^{2/3}E_k \leq 0. \tag{3.24}$$

By monotonicity of integral

$$E_k(t) \leq E_k(0)\exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}t\right) \tag{3.25}$$

3.2. The estimate from L^2 to L^2. We want to prove Corollary 1.2, that is the inequality

$$\|f_k(t)\|^2 \leq 4\mathrm{e}\|f_k(0)\|^2 \exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}t\right) \tag{3.26}$$

PROOF. First note that $\|f_k(t)\|^2$ is decreasing since it has a non-positive derivative (see (2.4)). Therefore, we have $\|f_k(t)\|^2 \leq \|f_k(0)\|^2$. Define

$$\tilde{t}_{\nu,k} = \frac{12}{\nu^{1/3}|k|^{2/3}}$$

We divide our analysis to two cases.

Case 1: $t < \tilde{t}_{\nu,k}$ Immediately follows by monotonicity.

Case 2: $t \geq \tilde{t}_{\nu,k}$ When integrating both sides of (2.4) we have

$$\left\|f_k(\tilde{t}_{\nu,k})\right\|^2 - \|f_k(0)\|^2 = -2\nu \int_0^{\tilde{t}_{\nu,k}} \|\nabla f\|^2 \, dt \tag{3.27}$$

$$\implies 2\nu \int_0^{\tilde{t}_{\nu,k}} \|\nabla f(t)\|^2 \mathrm{d}t \leq \|f_k(0)\|^2 - \left\|f_k(\tilde{t}_{\nu,k})\right\|^2 \leq \|f_k(0)\|^2 . \tag{3.28}$$

By mean value theorem, there exists $t^* \in (0, \tilde{t}_{\nu,k})$ such that

$$\|\nabla f(t^*)\|^2 \leq \frac{1}{2\nu\tilde{t}_{\nu,k}} \|f_k(0)\|^2 = \frac{1}{24}\nu^{-2/3}|k|^{2/3} \|f_k(0)\|^2 . \tag{3.29}$$

With the fact that E is decreasing, we find that

$$\begin{aligned} E(\tilde{t}_{\nu,k}) \leq E(t^*) &\leq \frac{3}{4}\left(\|f_k(t^*)\|^2 + \frac{\alpha\nu^{2/3}}{|k|^{2/3}} \|\nabla f(t^*)\|^2\right) \\ &\leq \frac{3}{4}\left(1 + \frac{1}{24 \times 8}\right) \|f_k(0)\|^2 \leq \|f_k(0)\|^2. \end{aligned} \tag{3.30}$$

$$\frac{E(t)}{E(\tilde{t}_{\nu,k})} \leq \exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}(t - \tilde{t}_{\nu,k})\right) \leq e\exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}t\right) \tag{3.31}$$

$$\implies \|f_k(t)\|^2 \leq 4E(t) \leq 4e\|f_k(0)\|^2 \exp\left(-\frac{1}{12}\nu^{1/3}|k|^{2/3}t\right) \tag{3.32}$$

This proves our corollary. □

4. Special case when $u(y) = \sin y$

For this special case we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2 + \nu\|\nabla f\|^2 = 0 \tag{4.1}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla f\|^2 + \nu \left\|\nabla^2 f\right\|^2 = -\langle \cos y \, \partial_x f, \partial_y f\rangle \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \cos y \, \partial_x f, \partial_y f\rangle + \|\cos y \, \partial_x f\|^2 = -2\nu\langle \cos y \, \partial_{xy} f, \nabla^2 f\rangle - \nu\langle \cos y \, \partial_x f, \partial_y f\rangle \tag{4.3}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\cos y \, \partial_x f\|^2 = -\nu\langle \cos^2 y \, \partial_{xx} f, \, \nabla^2 f\rangle \tag{4.4}$$

Notice that the forth equation can be re-written as followed:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\cos y\,\partial_x f\right\|^2 &= -\nu\langle\cos^2 y\,\partial_{xx} f, \nabla^2 f\rangle \\
&= -\nu\langle\nabla(\cos^2 y\partial_x f), \nabla(\partial_x f)\rangle \\
&= -\nu(\langle\nabla(\cos^2 y)\partial_x f, \nabla(\partial_x f)\rangle + \left\|\cos y\,\partial_x\nabla f\right\|^2) \\
&= \nu\langle 2\sin y\cos y\partial_x f, \partial_{xy} f\rangle - \nu\left\|\cos y\,\partial_x\nabla f\right\|^2 \\
&= \nu\langle\sin y\cos y, \partial_y((\partial_x f)^2)\rangle - \nu\left\|\cos y\,\partial_x\nabla f\right\|^2 \\
&= -\nu\langle(\sin y\cos y)', (\partial_x f)^2\rangle - \nu\left\|\cos y\,\partial_x\nabla f\right\|^2 \\
&= \nu\left\|\sin y\,\partial_x f\right\|^2 - \nu\left\|\cos y\,\partial_x f\right\|^2 \\
&= \nu\left\|\partial_x f\right\|^2 - 2\nu\left\|\cos y\,\partial_x f\right\|^2 \qquad (4.5)
\end{aligned}
$$

4.1. Energy Functional. We consider the following functional:

$$
E := E_k = \frac{1}{2}\left(\|f\|^2 + \tilde\alpha\left\|\nabla f\right\|^2 + 2\tilde\beta\langle\cos y\partial_x f,\ \partial_y f\rangle + \tilde\gamma\left\|\cos y\,\partial_x f\right\|^2\right) \tag{4.6}
$$

The energy function satisfies:

$$
\begin{aligned}
&\frac{dE_k}{dt} + \nu\left\|\nabla f\right\|^2 + \tilde\alpha\nu\left\|\nabla^2 f\right\|^2 + (\tilde\beta + 2\tilde\gamma\nu)\left\|\cos y\,\partial_x f\right\|^2 + \nu\tilde\gamma\left\|\cos y\,\partial_x\nabla f\right\|^2 \\
&= -(\tilde\alpha + 2\tilde\beta\nu)\langle\cos y\,\partial_x f,\ \partial_y f\rangle - 2\tilde\beta\nu\langle\cos y\,\partial_{xy} f, \nabla^2 f\rangle + \nu\tilde\gamma\left\|\partial_x f\right\|^2 \qquad (4.7)
\end{aligned}
$$

Again, we consider the right-hand side above as error terms. The choice of $\tilde\alpha, \tilde\beta, \tilde\gamma$ will be made ν, k dependent as

$$
\tilde\alpha = \alpha\frac{\nu^{1/2}}{|k|^{1/2}}, \qquad \tilde\beta = \beta\frac{1}{|k|}, \qquad \tilde\gamma = \gamma\frac{1}{\nu^{1/2}|k|^{3/2}}
$$

for α, β, γ small enough, independent of ν, k (we can compute them explicitly as we will show). We perform estimates below that require constraints on α, β, γ, and then we will show at the end that all the constraints can be satisfied.

Postitivity. We note that

$$
\begin{aligned}
2\tilde\beta\left|\langle\cos y\,\partial_x f,\ \partial_y f\rangle\right| &\le 2\left\|\partial_y f\right\|\tilde\beta\left\|\cos y\partial_x f\right\| \\
&\le \frac{\tilde\alpha}{2}\left\|\partial_y f\right\|^2 + \frac{2\tilde\beta}{\tilde\alpha}\left\|\cos y\partial_x f\right\|^2
\end{aligned}
$$

Imposing the condition

$$
\frac{\tilde\beta^2}{\tilde\alpha\tilde\gamma} \le \frac{1}{4} \tag{4.8}
$$

we have

$$
0 \le \frac{1}{2}\left(\|f\|^2 + \frac{\tilde\alpha}{2}\left\|\nabla f\right\|^2 + \frac{\tilde\gamma}{2}\left\|\cos y\,\partial_x f\right\|^2\right) \le E \le \frac{1}{2}\left(\|f\|^2 + \frac{3\tilde\alpha}{2}\left\|\nabla f\right\|^2 + \frac{3\tilde\gamma}{2}\left\|\cos y\,\partial_x f\right\|^2\right) \tag{4.9}
$$

First error Term. We have

$$
\begin{aligned}
2\tilde\beta\nu\left|\langle\nabla^2 f, \cos y\partial_{xy} f\rangle\right| &\le 2\tilde\beta\nu\left\|\nabla^2 f\right\|\left\|\cos y\,\partial_y(\partial_x f)\right\| \\
&\le 2\tilde\beta\nu\left\|\nabla^2 f\right\|\left\|\cos y\,\nabla(\partial_x f)\right\| \\
&\le \frac{\tilde\alpha\nu}{2}\left\|\nabla^2 f\right\|^2 + \frac{2\tilde\beta^2}{\tilde\alpha}\nu\left\|\cos y\,\nabla(\partial_x f)\right\|^2
\end{aligned}
$$

So by imposing (4.8) we have

$$2\tilde{\beta}\nu \left|\langle \nabla^2 f, \cos y \partial_{xy} f\rangle\right| \leq \frac{\tilde{\alpha}\nu}{2} \left\|\nabla^2 f\right\|^2 + \frac{\tilde{\gamma}}{2}\nu \left\|\cos y\, \nabla(\partial_x f)\right\|^2 \tag{4.10}$$

Second error term. There are two terms involving the inner product $\langle \cos y\, \partial_x f, \partial_y f\rangle$:

$$\begin{aligned}
\tilde{\alpha}\left|\langle \cos y\, \partial_x f, \partial_y f\rangle\right| &\leq \left\|\cos y\, \partial_x f\right\| \tilde{\alpha} \left\|\partial_y f\right\| \\
&\leq \frac{\tilde{\beta}}{2} \left\|\cos y\, \partial_x f\right\|^2 + \frac{\tilde{\alpha}^2}{2\tilde{\beta}} \left\|\partial_y f\right\|^2 \\
&\leq \frac{\tilde{\beta}}{2} \left\|\cos y\, \partial_x f\right\|^2 + \frac{\tilde{\alpha}^2}{2\tilde{\beta}} \left\|\nabla f\right\|^2
\end{aligned}$$

Moreover,

$$\begin{aligned}
\tilde{\beta}\nu \left|\langle \cos y\, \partial_x f, \partial_y f\rangle\right| &\leq \left\|\cos y\, \partial_x f\right\| \tilde{\beta}\nu \left\|\partial_y f\right\| \\
&\leq \frac{2\tilde{\gamma}\nu}{2} \left\|\cos y\, \partial_x f\right\|^2 + \frac{(\tilde{\beta}\nu)^2}{2(2\tilde{\gamma}\nu)} \left\|\partial_y f\right\|^2 \\
&\leq \tilde{\gamma}\nu \left\|\cos y\, \partial_x f\right\|^2 + \frac{\tilde{\beta}^2}{4\tilde{\gamma}}\nu \left\|\nabla f\right\|^2
\end{aligned}$$

Therefore, by imposing the conditions:

$$\frac{\tilde{\alpha}^2}{2\tilde{\beta}} \leq \frac{\nu}{4} \iff \frac{\tilde{\alpha}^2}{\tilde{\beta}} \leq \frac{\nu}{2} \tag{4.11}$$

$$\frac{\tilde{\beta}^2}{4\tilde{\gamma}} \leq \frac{1}{4} \iff \frac{\tilde{\beta}^2}{\tilde{\gamma}} \leq 1 \tag{4.12}$$

we have

$$\tilde{\alpha}\left|\langle \cos y\, \partial_x f, \partial_y f\rangle\right| \leq \frac{\tilde{\beta}}{2} \left\|\cos y\, \partial_x f\right\|^2 + \frac{\nu}{4} \left\|\nabla f\right\|^2 \tag{4.13}$$

$$\tilde{\beta}\nu \left|\langle \cos y\, \partial_x f, \partial_y f\rangle\right| \leq \tilde{\gamma}\nu \left\|\cos y\, \partial_x f\right\|^2 + \frac{\nu}{4} \left\|\nabla f\right\|^2 \tag{4.14}$$

Combining all estimates we now have

$$\frac{dE}{dt} + \frac{\nu}{2}\left\|\nabla f\right\|^2 + \frac{\tilde{\alpha}\nu}{2}\left\|\nabla^2 f\right\| + \frac{\tilde{\gamma}\nu}{2}\left\|\cos y\, \partial_x \nabla f\right\|^2 + \left(\frac{\tilde{\beta}}{2} + \tilde{\gamma}\nu\right)\left\|\cos y\, \partial_x f\right\|^2 \leq \tilde{\gamma}\nu \left\|\partial_x f\right\|^2 \tag{4.15}$$

In particular,

$$\frac{dE}{dt} + \frac{\nu}{2}\left\|\nabla f\right\|^2 + \frac{\tilde{\beta}}{2}\left\|\cos y\, \partial_x f\right\|^2 \leq \tilde{\gamma}\nu \left\|\partial_x f\right\|^2 \tag{4.16}$$

The RHS Term. We rely on the following lemma:

LEMMA 4.1. *For $g \in L^2(\mathbb{T}^2)$, $\sigma \in (0, 1]$, we have*

$$\sigma^{1/2}\left\|g\right\|^2 \leq 3\left(\sigma\left\|\partial_y g\right\|^2 + \left\|\cos y\, g\right\|^2\right) \tag{4.17}$$

PROOF. First note that

$$\left\|g\right\|^2 = \left\|\cos y\, g\right\|^2 + \left\|\sin y\, g\right\|^2 \tag{4.18}$$

Moreover, note that

$$\begin{aligned}\|\sin y\, g\|^2 &= \langle g^2, \sin^2 y\rangle \\ &= -\langle g^2 \sin y\,, \partial_y(\cos y)\rangle \\ &= (\langle \partial_y(g^2\ \sin y), \cos y\rangle) \\ &= \langle 2g\,\partial_y g,\ \sin y \cos y\rangle + \langle g^2 \cos y,\ \cos y\rangle \\ &= 2\langle g\,\partial_y g,\ \sin y\cos y\rangle + \|g\ \cos y\|^2 \\ &\le 2\,|\langle \partial_y g,\ g\cos y\rangle| + \|g\ \cos y\|^2 \\ &\le 2\,\|\partial_y g\|\,\|g\cos y\| + \|g\ \cos y\|^2 \\ &\le \sigma^{1/2}\,\|\partial_y g\|^2 + \left(1+\frac{1}{\sigma^{1/2}}\right)\|g\ \cos y\|^2 \\ &\le \sigma^{1/2}\,\|\partial_y g\|^2 + \frac{2}{\sigma^{1/2}}\,\|g\ \cos y\|^2\end{aligned}$$

Therefore,

$$\|g\|^2 \le \sigma^{1/2}\,\|\partial_y g\|^2 + \frac{3}{\sigma^{1/2}}\,\|g\ \cos y\|^2 \le 3\left(\sigma^{1/2}\,\|\partial_y g\|^2 + \frac{1}{\sigma^{1/2}}\,\|\cos y\, g\|^2\right) \tag{4.19}$$

Hence result □

Utilising this lemma we have,

$$\begin{aligned}\frac{\nu}{4}\,\|\nabla f\|^2 + \frac{\tilde\beta}{4}\,\|\cos y\partial_x f\|^2 &\ge \frac{\tilde\beta}{12}\left(\frac{\nu/4}{\tilde\beta/12}\,\|\partial_y f\|^2 + 3\,\|\cos y\partial_x f\|^2\right) \\ &\ge \frac{\tilde\beta}{12}\left(\frac{\nu/4}{\tilde\beta k^2/12}\,\|\partial_{xy} f\|^2 + 3\,\|\cos y\partial_x f\|^2\right) \\ &\ge \left(\frac{\tilde\beta}{12}\right)^{1/2}\left(\frac{\nu}{4}\right)^{1/2}\frac{1}{|k|}\,\|\partial_x f\|^2 = \frac{\tilde\beta^{1/2}\nu^{1/2}}{4\sqrt{3}\,|k|}\,\|\partial_x f\|^2\end{aligned} \tag{4.20}$$

Therefore, by imposing

$$\tilde\gamma\nu \le \frac{\tilde\beta^{1/2}\nu^{1/2}}{4\sqrt{3}\,|k|} \iff \frac{\tilde\gamma^2}{\tilde\beta} \le \frac{1}{48\nu k^2} \tag{4.21}$$

we obtain

$$\frac{dE}{dt} + \frac{\nu}{4}\,\|\nabla f\|^2 + \frac{\tilde\beta}{4}\,\|\cos y\,\partial_x f\|^2 \le 0 \tag{4.22}$$

The Differential Inequality. We may go further to obtain

$$\frac{dE}{dt} + \frac{1}{2}\left(\left(\frac{\nu}{4}\,\|\partial_y f\|^2 + \frac{\tilde\beta}{4}\,\|\cos y\,\partial_x f\|^2\right) + \left(\frac{\nu}{4}\,\|\nabla f\|^2 + \frac{\tilde\beta}{4}\,\|\cos y\,\partial_x f\|^2\right)\right) \le 0 \tag{4.23}$$

Again, from (4.20), we have

$$\frac{\tilde\beta^{1/2}\nu^{1/2}}{4\sqrt{3}}\,|k|\,\|f\|^2 = \frac{\tilde\beta^{1/2}\nu^{1/2}}{4\sqrt{3}\,|k|}\,\|\partial_x f\|^2 \le \frac{\nu}{4}\,\|\partial_y f\|^2 + \frac{\tilde\beta}{4}\,\|\cos y\,\partial_x f\|^2 \tag{4.24}$$

From this we have

$$\frac{dE}{dt} + \frac{1}{2}\left(\frac{\tilde{\beta}^{1/2}\nu^{1/2}}{4\sqrt{3}}|k|\,\|f\|^2 + \left(\frac{\nu}{4}\|\nabla f\|^2 + \frac{\tilde{\beta}}{4}\|\cos y\,\partial_x f\|^2\right)\right) \leq 0 \tag{4.25}$$

$$\frac{dE}{dt} + \frac{\tilde{\beta}^{1/2}\nu^{1/2}}{8\sqrt{3}}|k|\left(\|f\|^2 + \left(\frac{\sqrt{3}\nu^{1/2}}{\tilde{\beta}^{1/2}|k|}\|\nabla f\|^2 + \frac{\sqrt{3}\tilde{\beta}^{1/2}}{\nu^{1/2}|k|}\|\cos y\,\partial_x f\|^2\right)\right) \leq 0 \tag{4.26}$$

So by imposing the following conditions

$$\frac{\sqrt{3}\nu^{1/2}}{\tilde{\beta}^{1/2}|k|} \geq \frac{3\tilde{\alpha}}{2} \iff \tilde{\alpha}^2\tilde{\beta} \leq \frac{4\nu}{3k^2} \tag{4.27}$$

$$\frac{\sqrt{3}\tilde{\beta}^{1/2}}{\nu^{1/2}|k|} \geq \frac{3\tilde{\gamma}}{2} \iff \frac{\tilde{\gamma}^2}{\tilde{\beta}} \leq \frac{4}{3\nu k^2} \tag{4.28}$$

We have the required differential inequality

$$\frac{dE}{dt} + \frac{\tilde{\beta}^{1/2}\nu^{1/2}}{4\sqrt{3}}|k|\,E \leq 0 \tag{4.29}$$

Scaling of Constants. Collecting (4.8), (4.11), (4.12), (4.21), (4.27) and (4.28), we have

$$\frac{\tilde{\beta}^2}{\tilde{\alpha}\tilde{\gamma}} \leq \frac{1}{4}, \quad \frac{\tilde{\alpha}^2}{\tilde{\beta}} \leq \frac{\nu}{2}, \quad \frac{\tilde{\beta}^2}{\tilde{\gamma}} \leq 1, \quad \tilde{\alpha}^2\tilde{\beta} \leq \frac{4\nu}{3k^2}, \quad \frac{1}{48\nu k^2} \leq \frac{\tilde{\gamma}^2}{\tilde{\beta}} \leq \frac{4}{3\nu k^2}$$

We let $\tilde{\alpha} = \alpha\nu^{A_1}|k|^{A_2}$, $\tilde{\beta} = \beta\nu^{B_1}|k|^{B_2}$, $\tilde{\gamma} = \gamma\nu^{C_1}|k|^{C_2}$, so that the conditions above can be reduced to conditions independent of ν and k. We have the following over-determined system:

$$\begin{cases} -A_1 + 2B_1 - C_1 & = 0 \\ 2A_1 - B_1 & = 1 \\ 2B_1 - C_1 & = 0 \\ 2A_1 + B_1 & = 1 \\ -B_1 + 2C_1 & = -1 \end{cases}, \qquad \begin{cases} -A_2 + 2B_2 - C_2 & = 0 \\ 2A_2 - B_2 & = 0 \\ 2B_2 - C_2 & = 0 \\ 2A_2 + B_2 & = -2 \\ -B_2 + 2C_2 & = -2 \end{cases} \tag{4.30}$$

The aim now is to seek a 'psuedo-solution' to the system so that it satisfies most equations. For the first system, such 'psuedo-solution' is $(A_1, B_1, C_1) = (1/2, 0, -1/2)$; while for the second solution, it is $(A_2, B_2, C_2) = (-1/2, -1, -3/2)$. These solution satisfies all equations in the system except the one related to the condition $\tilde{\beta}^2/\tilde{\gamma} \leq 1$. We need to check that such condition is satisfied even if we apply the scaling. Note that $\nu < 1$ and $|k| \geq 1$. Therefore:

$$\frac{\beta^2}{\gamma} \leq 1 \implies \frac{\tilde{\beta}^2}{\tilde{\gamma}} = \frac{\beta^2\nu^{1/2}}{\gamma|k|^{1/2}} \leq \frac{\beta^2}{\gamma} \leq 1$$

The choice of scaling is hence:

$$\tilde{\alpha} = \alpha\frac{\nu^{1/2}}{|k|^{1/2}}, \qquad \tilde{\beta} = \beta\frac{1}{|k|}, \qquad \tilde{\gamma} = \gamma\frac{1}{\nu^{1/2}|k|^{3/2}}. \tag{4.31}$$

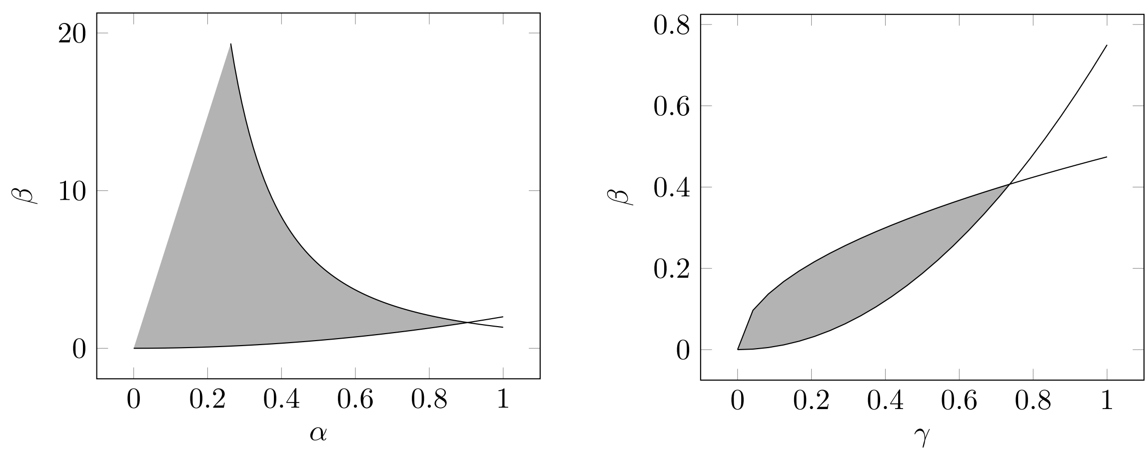
Checking if all constraints are consistent. Applying the scaling above we have

$$\frac{\beta^2}{\alpha\gamma} \leq \frac{1}{4}, \quad \frac{\alpha^2}{\beta} \leq \frac{1}{2}, \quad \frac{\beta^2}{\gamma} \leq 1, \quad \alpha^2\beta \leq \frac{4}{3}, \quad \frac{1}{48} \leq \frac{\gamma^2}{\beta} \leq \frac{4}{3}$$

From this, we have

$$\alpha^4 \leq \frac{\alpha^2\beta}{2} \leq \frac{2}{3} \implies \alpha \leq \sqrt[4]{\frac{2}{3}} \approx 0.903 < 1$$

Therefore
$$\frac{\beta^2}{\gamma} \leq \frac{1}{4}\alpha < \frac{1}{4} < 1$$
So the third condition $\beta^2/\gamma \leq 1$ is redundant. We now seek a solution to the constraints with $\alpha \leq 0.9$. With this assumption we have $\beta^2/\gamma \leq 9/40$. Together with $\gamma^2/\beta \leq 4/3$, we have $\beta\gamma \leq 3/10$. Thus:
$$\beta^3 \leq \beta^2 \frac{3}{10\gamma} \leq \frac{27}{400} \implies \beta \leq \frac{3}{20^{2/3}} \approx 0.407$$
$$\gamma^3 \leq \gamma^2 \frac{3}{10\beta} \leq \frac{2}{5} \implies \gamma \leq \sqrt[3]{\frac{2}{5}} \approx 0.736$$
These conditions could be illustrated by the following two graphs.



With these bounds, the forth condition has also made redundant since $\alpha^2\beta < 1 \leq 4/3$.
Put the constants as followed, then all conditions have been satisfied:
$$\alpha = \frac{1}{5}, \quad \beta = \frac{3}{25}, \quad \gamma = \frac{8}{25} \tag{4.32}$$
with these choice of constants, we have the differential inequality
$$\frac{dE}{dt} + \frac{1}{20}\nu^{1/2}|k|^{1/2}E \leq 0 \tag{4.33}$$
By monotonicity of integral,
$$E(t) := E_k(t) \leq E_k(0)\exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20}t\right) \tag{4.34}$$

4.2. Estimates from L^2 to L^2. We aim to prove Corollary 1.4, that is the inequality
$$\|f_k(t)\|^2 \leq 12e\,\|f_k(0)\|^2 \exp\left(-\frac{\nu^{1/2}|k|^{1/2}t}{20(1+|\ln\nu| + \ln|k|)}\right) \tag{4.35}$$

PROOF. First note that $\|f_k(t)\|^2$ is decreasing since it has a non-positive derivative (see (2.4)). Therefore, we have $\|f_k(t)\|^2 \leq \|f_k(0)\|^2$. Define
$$T_{\nu,k} = \frac{20(1+|\log\nu| + \log|k|)}{\nu^{1/2}|k|^{1/2}}, \qquad \tilde{t}_{\nu,k} = \frac{20}{\nu^{1/2}|k|^{1/2}}$$
We divide our analysis to two cases.

Case 1: $t < T_{\nu,k}$ Immediately follows from monotonicity.

Case 2: $t \geq T_{\nu,k}$ When integrating both sides of (2.4) we have

$$\left\|f_k(\tilde{t}_{\nu,k})\right\|^2 - \|f_k(0)\|^2 = -2\nu \int_0^{\tilde{t}_{\nu,k}} \|\nabla f\|^2 \, dt \tag{4.36}$$

$$\implies 2\nu \int_0^{\tilde{t}_{\nu,k}} \|\nabla f_k\|^2 \, dt = \|f_k(0)\|^2 - \left\|f_k(\tilde{t}_{\nu,k})\right\| \leq \|f_k(0)\|^2 \tag{4.37}$$

By mean value theorem, there is $t^* \in (0, \tilde{t}_{\nu,k})$ such that

$$2\nu\tilde{t}_{\nu,k} \|\nabla f_k(t^*)\|^2 \leq \|f_k(0)\|^2 \tag{4.38}$$

$$\implies \|\nabla f_k(t^*)\|^2 \leq \frac{1}{2\nu\tilde{t}_{\nu,k}} \|f_k(0)\|^2 = \frac{|k|^{1/2}}{40\nu^{1/2}} \|f_k(0)\|^2 \tag{4.39}$$

Now we look at the term involving $\|\cos y\, \partial_x f_k\|^2$. Re-arranging (4.5) we have

$$\frac{d}{dt} \|\cos y\, \partial_x f_k\|^2 + 4\nu \|\cos y\, \partial_x f_k\|^2 = 2\nu \|\partial_x f_k\|^2 = 2\nu k^2 \|f_k\|^2 \tag{4.40}$$

$$\implies \frac{d}{dt}\left(e^{4\nu t} \|\cos y\, \partial_x f_k\|^2\right) = 2\nu k^2 e^{4\nu t} \|f_k\|^2 \tag{4.41}$$

Integrating both sides yield

$$e^{4\nu t^*} \|\cos y\, \partial_x f_k(t^*)\|^2 - \|\cos y\, \partial_x f_k(0)\|^2 = \int_0^{t^*} 2\nu k^2 e^{4\nu t} \|f_k(t)\|^2 \, dt \tag{4.42}$$

We look at the integral at the RHS. Notice that when $t \in (0, t^*)$, we have $\|f_k(t)\|^2 \leq \|f_k(0)\|^2$. Moreover, $1 \leq e^{4\nu t} \leq e^{4\nu t^*}$. Therefore, the integral in RHS is bounded as followed:

$$\int_0^{t^*} 2\nu k^2 e^{4\nu t} \|f_k(t)\|^2 \, dt \leq t^* \nu k^2 e^{4\nu t^*} \|f_k(0)\|^2 \tag{4.43}$$

Therefore,

$$\begin{aligned} \|\cos y\, \partial_x f_k(t^*)\|^2 &\leq t^* \nu k^2 \|f_k(0)\|^2 + \|\cos y \partial_x f_k(0)\|^2 e^{-4\nu t^*} \\ &\leq \tilde{t}_{\nu,k} \nu k^2 \|f_k(0)\|^2 + k^2 \|f_k(0)\|^2 < 21k^2 \|f_k(0)\|^2 \end{aligned}$$

With the fact that E is decreasing, we have

$$\begin{aligned} E_k(\tilde{t}_{\nu,k}) \leq E_k(t^*) &\leq \frac{1}{2}\left(\|f_k(t^*)\|^2 + \frac{3\alpha}{2}\frac{\nu^{1/2}}{|k|^{1/2}} \|\nabla f_k(t^*)\|^2 + \frac{3\gamma}{2}\frac{1}{\nu^{1/2}|k|^{3/2}} \|\cos y\, \partial_x f_k(t^*)\|^2\right) \\ &\leq \frac{1}{2}\left(\|f_k(0)\|^2 + \frac{3}{10}\frac{\nu^{1/2}}{|k|^{1/2}}\frac{|k|^{1/2}}{40\nu^{1/2}} \|f_k(0)\|^2 + \frac{12}{25}\frac{21k^2}{\nu^{1/2}|k|^{3/2}} \|f_k(0)\|^2\right) \\ &\leq \frac{1}{2}\left(\left(1 + \frac{3}{400}\right) \|f_k(0)\|^2 + \frac{252}{25}\frac{|k|^{1/2}}{\nu^{1/2}} \|f_k(0)\|^2\right) \leq 6\frac{|k|^{1/2}}{\nu^{1/2}} \|f_k(0)\|^2 \end{aligned} \tag{4.44}$$

Again, by the differential inequality (4.33), for our case when $t > T_{\nu,k} > \tilde{t}_{\nu,k} > t^*$

$$\frac{E(t)}{E(\tilde{t}_{\nu,k})} \leq \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20}(t - \tilde{t}_{\nu,k})\right) \leq e \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20} t\right) \tag{4.45}$$

$$\implies \|f_k(t)\|^2 \leq 2E(\tilde{t}_{\nu,k}) \leq 12e \|f_k(0)\|^2 \frac{|k|^{1/2}}{\nu^{1/2}} \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20} t\right) \tag{4.46}$$

Finally, note that in our case, we have

$$
\begin{aligned}
\frac{|k|^{1/2}}{\nu^{1/2}} \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20} t\right) &= \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20} t + \frac{1}{2}\ln|k| + \frac{1}{2}|\ln\nu|\right) \\
&= \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20} t \left(1 - \frac{1}{\frac{\nu^{1/2}|k|^{1/2}}{20} t}\left(\frac{1}{2}\ln|k| + \frac{1}{2}|\ln\nu|\right)\right)\right) \\
&\leq \exp\left(-\frac{\nu^{1/2}|k|^{1/2}}{20} t \left(1 - \frac{\frac{1}{2}\ln|k| + \frac{1}{2}|\ln\nu|}{1 + |\ln\nu| + \ln|k|}\right)\right) \\
&\leq \exp\left(-\frac{\nu^{1/2}|k|^{1/2} t}{20(1 + |\ln\nu| + \ln|k|)}\right)
\end{aligned}
$$

This proves our corollary. □

5. More general problem

In fact, for any problem in the form of (1.1), we may analyse the decay of $\|f_k(0)\|^2$ using the following energy functional:

$$
E = \frac{1}{2}\left[\|f\|^2 + \tilde{\alpha}\|\nabla f\|^2 + 2\tilde{\beta}\langle u'\partial_x f, \partial_y f\rangle + \tilde{\gamma}\|u'\partial_x f\|^2\right]. \tag{5.1}
$$

The scaling of $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ depends on the choice of $u(y)$, and we will not do the general case in this task.

One final note is that, we do not know whether our bound is sharp. This would require tools from SDE, which we will study in the next task.