Equilibrium Measures

Samuel Lam

Department of Mathematics, MiT

2020 Fall

- Junior, Department of Mathematics, MiT/Imperial
- Interests: Stochastic and Chaotic stuff. Maybe statistics and machine learning?

- Junior, Department of Mathematics, MiT/Imperial
- Interests: Stochastic and Chaotic stuff. Maybe statistics and machine learning?
- What I am going to talk about is slightly unrelated to those stuff though?

- Junior, Department of Mathematics, MiT/Imperial
- Interests: Stochastic and Chaotic stuff. Maybe statistics and machine learning?
- What I am going to talk about is slightly unrelated to those stuff though?
- I am going to derive semicircle law for GUE, starting from the joint eigenvalue density, then briefly talk about numerics.

- Junior, Department of Mathematics, MiT/Imperial
- Interests: Stochastic and Chaotic stuff. Maybe statistics and machine learning?
- What I am going to talk about is slightly unrelated to those stuff though?
- I am going to derive semicircle law for GUE, starting from the joint eigenvalue density, then briefly talk about numerics.
- The materials are mainly based on several notes by Sheehan Olver (2009).

Joint Density of GUE

$$p_A(\lambda_1, ..., \lambda_n) \propto \prod_{i<j} |\lambda_i - \lambda_j|^2 \exp\left(-\sum_i \frac{\lambda_i^2}{2}\right)$$
$$= \exp\left(-\left(\sum_{i\neq j} \ln|\lambda_i - \lambda_j|^{-1} + \sum_i V(\lambda_i)\right)\right)$$

Where $V(x) = x^2/2$.

Relation with the potential of a gas of n electrons at (fixed) position $x_1, \ldots, x_n$:

$$\sum_{i \neq j} \underbrace{\ln |x_i - x_j|^{-1}}_{\text{potential from repulsion}} + \sum_i \underbrace{V(x_i)}_{\text{exterior potential}}$$

This is Dyson's interpretation of GUE.

Relation with the potential of a gas of n electrons at (fixed) position $x_1, ..., x_n$:

$$\sum_{i \neq j} \underbrace{\ln |x_i - x_j|^{-1}}_{\text{potential from repulsion}} + \sum_i \underbrace{V(x_i)}_{\text{exterior potential}}$$

This is Dyson's interpretation of GUE. In a fancy way this term can be written as $I^V(\mu_n)$, where $I^V(\mu)$ is a functional of measures (with density) defined in $\mathbb{R}$:

$$I^V(\mu) \propto \int \left(\left(\int \ln |x - z|^{-1} \, d\mu \right) + V(z) \right) \, d\mu \qquad (1)$$

and $\mu_n(x)$ has density $w_n(x) = n^{-1} \sum_{i=1}^{n} \delta_{x_i}(x)$, the empirical density of the eigenvalues. Assume μ has density $\psi(x)$.

Fact: When $n \to \infty$, $w_n(x)\,dx$ converges weakly to $\mu = w(x)\,dx$, where μ is the minimiser of the functional $I^V(\mu)$.
We use this fact to obtain w.

Fact: When $n \to \infty$, $w_n(x)\,dx$ converges weakly to $\mu = w(x)\,dx$, where μ is the minimiser of the functional $I^V(\mu)$. **We use this fact to obtain** w**.**

Aside: How can we experimentally determine $w_n(x)$ (or the (possibly rescaled) positions of 'equilibrium' points x_i? Note that x_i satisfies the following system of equations ('balancing force' or 'Euler-Lagrange')

$$\forall i, \sum_{i=1}^{n} \frac{1}{x_i - x_j} - V'(x_i) = 0 \tag{2}$$

We can solve this system of non-linear equation to obtain x_i. We visualise μ_N by plotting a histogram. You will see as n becomes large a semicircle approximates the histogram. (Reminder for myself: Expt. 1)

Notice that:

$$\phi_n(z) := \int \frac{w_n(x)}{z-x}\,dx = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{z-x_j}$$

So we have established, at $x = x_i$

$$\phi_n^+(x_i) + \phi_n^-(x_i) = V'$$

Sending $n \to \infty$ we established

$$\phi^+(z) + \phi^-(z) = V', \quad \phi(z) := \int \frac{w(x)}{z-x} \sim O(z^{-1}) \qquad (3)$$

Remark: We hope $w(x)$ to have support a single interval (a, b). This makes life much easier!

Cauchy Transform and Hilbert Transform

Without further specification we write f a function from $\mathbb{C}$ to $\mathbb{C}$ and a piecewise-C^1 curve in $\mathbb{C}$ as γ. We define the following

Definition 1 (Cauchy Transform – Unofficial)

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Define the Cauchy transform

$$\mathcal{C}_\gamma[f](z) = \frac{1}{2\pi i} \int_\gamma \frac{f(x)}{x - z} dz \tag{4}$$

Cauchy Transform and Hilbert Transform

Without further specification we write f a function from $\mathbb{C}$ to $\mathbb{C}$ and a piecewise-C^1 curve in $\mathbb{C}$ as γ. We define the following

Definition 1 (Cauchy Transform – Unofficial)

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Define the Cauchy transform

$$\mathcal{C}_\gamma[f](z) = \frac{1}{2\pi i} \int_\gamma \frac{f(x)}{x - z} dz \tag{4}$$

In this project γ is a (1) closed curve (e.g. circle); and (2) an interval $[a, b]$.

We also consider the Hilbert Transform

Definition 2 (Hilbert Transform)

Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Define the Hilbert Transform

$$\mathcal{H}_\gamma[f](x) = \frac{1}{\pi} \mathrm{PV} \int \frac{f(t)}{t - x}\, dt \tag{5}$$

Reminder for myself: They are probably the same... We often consider $z \in \bar{\mathbb{C}} \setminus \gamma$ for Cauchy transform and $z \in \gamma$ for Hilbert transform...

$$\mathcal{C}_\gamma[f](z) = \frac{1}{2\pi i}\int_\gamma \frac{f(x)}{x-z}dz$$

$$\mathcal{H}_\gamma[f](x) = \frac{1}{\pi}\,\mathrm{PV}\int_\gamma \frac{f(t)}{t-x}\,dt$$

Properties when *f* is 'nice' (e.g. Holder continuous) and *z* is not an endpoint/discontinuity of Γ. (The Plemelj's Lemma)

1. $\mathcal{C}_\gamma[f](z)$ is analytic in $\bar{\mathbb{C}} \setminus \gamma$
2. $\mathcal{C}_\gamma[f](\infty) = 0$
3. $\boldsymbol{\mathcal{C}_\gamma[f]^+(z) - \mathcal{C}_\gamma[f]^-(z) = f(z)}$
4. $\boldsymbol{\mathcal{C}_\gamma[f]^+(z) + \mathcal{C}_\gamma[f]^-(z) = -i\mathcal{H}_\gamma[f](z)}$

Remark: For the case when $\gamma = [a, b]$, $\mathcal{C}_\gamma[f](z)$ has weaker-than-pole singularity at *a* and *b*.

Recall our problem of finding a density of equilibrium measure.

$$\phi^{+}(z) + \phi^{-}(z) = V' \tag{6a}$$

$$\phi(z) := \int \frac{w(x)}{z - x} = \frac{2\pi}{i}\mathcal{C}_{\gamma}[w](z) \sim O(z^{-1}) \tag{6b}$$

Recall our problem of finding a density of equilibrium measure.

$$\phi^{+}(z) + \phi^{-}(z) = V' \tag{6a}$$

$$\phi(z) := \int \frac{w(x)}{z - x} = \frac{2\pi}{i}\mathcal{C}_{\gamma}[w](z) \sim O(z^{-1}) \tag{6b}$$

If we have found $\phi(x)$ satisfying (6a), then we may utilise property (3) to obtain

$$w(x) = \frac{i}{2\pi}(\phi^{+} - \phi^{-}) \tag{7}$$

The only task left is to find ϕ itself.

Recall that the following Chebyshev (Joukowski) map which maps both upper half and lower half of circle to unit interval:

$$J(z) = \frac{z + z^{-1}}{2} \tag{8}$$

(Reminder for myself: Visualisation)
Idea (?): If $\phi(z)$ has jump on $(-1, 1)$ and for all $z \in (-1, 1)$,

$$\phi^+(z) + \phi^-(z) = V' \tag{9}$$

Then $\phi(J(z))$ has jump on unit circle $\mathbb{T}$, and for all z on $\mathbb{T}$,

$$\phi^+(J(z)) + \phi^-(J(z)) = V'(J(z)) := g(z) \tag{10}$$

We know how to obtain $\phi(J(z))$.
How can we transform back to $\phi(z)$?

A Canonical Example

$V(x) = x^2/2, \gamma = \operatorname{supp}\mu = (-b, b)$; We are solving the equation

$$\phi^+(x) + \phi^-(x) = x, \quad x \in (-b, b) \tag{11}$$

A Canonical Example

$V(x) = x^2/2, \gamma = \operatorname{supp} \mu = (-b, b)$; We are solving the equation

$$\phi^+(x) + \phi^-(x) = x, \quad x \in (-b, b) \tag{11}$$

If I write $\tilde{\phi}(x) = \phi(bx)$, then

$$\tilde{\phi}^+(x) + \tilde{\phi}^-(x) = bx, \quad x \in (-1, 1) \tag{12}$$

A Canonical Example

$V(x) = x^2/2, \gamma = \operatorname{supp} \mu = (-b, b)$; We are solving the equation

$$\phi^+(x) + \phi^-(x) = x, \quad x \in (-b, b) \tag{11}$$

If I write $\tilde{\phi}(x) = \phi(bx)$, then

$$\tilde{\phi}^+(x) + \tilde{\phi}^-(x) = bx, \quad x \in (-1, 1) \tag{12}$$

Again

$$\tilde{\phi}^+(J(z)) + \tilde{\phi}^-(J(z)) = \frac{b}{2}\left(z + \frac{1}{z}\right), \quad z \in \mathbb{T}$$

A Canonical Example

$V(x) = x^2/2, \gamma = \operatorname{supp} \mu = (-b, b)$; We are solving the equation

$$\phi^+(x) + \phi^-(x) = x, \quad x \in (-b, b) \tag{11}$$

If I write $\tilde{\phi}(x) = \phi(bx)$, then

$$\tilde{\phi}^+(x) + \tilde{\phi}^-(x) = bx, \quad x \in (-1, 1) \tag{12}$$

Again

$$\tilde{\phi}^+(J(z)) + \tilde{\phi}^-(J(z)) = \frac{b}{2}\left(z + \frac{1}{z}\right), \quad z \in \mathbb{T}$$

It has solution

$$\psi(z) = \tilde{\phi}(J(z)) = \begin{cases} bz/2 & |z| < 1 \\ b/2z & |z| > 1 \end{cases}$$

How can we invert back?

The Chebyshev map has two inverses

$$J_+^{-1}(x) = x - \sqrt{x-1}\sqrt{x+1}, \quad \text{(inside circle)}$$
$$J_-^{-1}(x) = x + \sqrt{x-1}\sqrt{x+1}, \quad \text{(outside circle)}$$

(these are defined when $x \notin [-1, 1]$) One can also define inverses on $[-1, 1]$

$$J_U^{-1}(x) = x + i\sqrt{1-x}\sqrt{x+1}, \quad \text{(upper circle)}$$
$$J_D^{-1}(x) = x - i\sqrt{1-x}\sqrt{x+1}, \quad \text{(lower circle)}$$

(Reminder to myself: Visualisation)

I propose a solution:

$$\begin{aligned}\tilde{\phi}(x) &= \frac{\psi(J_+^{-1}(x)) + \psi(J_-^{-1}(x))}{2} \\ &= \frac{b}{4}\left(J_+^{-1}(x) + (J_-^{-1}(x))^{-1}\right) \\ &= \frac{b}{2}(J_+^{-1}(x)) \\ &= \frac{b}{2}\left(x - \sqrt{x-1}\sqrt{x+1}\right)\end{aligned}$$

Idea of proof: Think about the jumps of $\tilde{\phi}(x)$ when $x \in (-1, 1)$:

$$\psi^+(J_+^{-1}(x)) = \psi^+(J_D^{-1}(x)), \text{ lower, inside}$$
$$\psi^-(J_+^{-1}(x)) = \psi^-(J_U^{-1}(x)), \text{ upper, inside}$$
$$\psi^+(J_-^{-1}(x)) = \psi^+(J_U^{-1}(x)), \text{ upper, outside}$$
$$\psi^-(J_-^{-1}(x)) = \psi^-(J_D^{-1}(x)), \text{ lower, outside}$$

(Ok again picture proof :/) Moreover,

$$\psi^+(J_U^{-1}(x)) + \psi^-(J_U^{-1}(x)) = bx$$
$$\psi^+(J_D^{-1}(x)) + \psi^-(J_D^{-1}(x)) = bx$$

So

$$\phi^+ + \phi^- = bx$$

Moreover

$$\tilde{\phi}(x) = \frac{bx}{2}\left(1 - \sqrt{1 - \frac{1}{x^2}}\right) = \frac{b}{4}\frac{1}{x} + \ldots \tag{13}$$

So $\tilde{\phi}$ is our solution to equation (13):

$$\tilde{\phi}^+(x) + \tilde{\phi}^-(x) = bx, \quad x \in (-1, 1) \tag{14}$$

Rescaling yields

$$\phi(x) = \frac{x}{2}\left(1 - \sqrt{1 - \frac{b^2}{x^2}}\right) = \frac{b^2}{4}\frac{1}{x} + \ldots \tag{15}$$

$$\phi(z) := \int \frac{w(x)}{z-x} = \frac{2\pi}{i}\mathcal{C}_\gamma[w](z) \sim O(z^{-1}) \tag{16}$$

For it to be a Cauchy transform of density we need $b^2/4 = 1$, so $b = 2$.

$$\phi(z) := \int \frac{w(x)}{z-x} = \frac{2\pi}{i}\mathcal{C}_\gamma[w](z) \sim O(z^{-1}) \tag{16}$$

For it to be a Cauchy transform of density we need $b^2/4 = 1$, so $b = 2$. We therefore get

$$\phi(x) = \frac{x}{2}\left(1 - \sqrt{1 - \frac{4}{x^2}}\right) = \frac{1}{2}(x - \sqrt{x^2 - 4}) \tag{17}$$

This is the Cauchy Transform of semicircle law!

Not convinced?

Not convinced? Think about the jumps of $\sqrt{x^2-4}$ on $(-2,2)$:

$$\begin{aligned}\left(\sqrt{x^2-4}\right)_+ - \left(\sqrt{x^2-4}\right)_- &= \sqrt{x+2}\left(\left(\sqrt{x+2}\right)_+ - \left(\sqrt{x+2}\right)_-\right)\\ &= 2i\sqrt{4-x^2}\end{aligned}$$

So $\phi(x)$ has jump $-i\sqrt{4-x^2}$. Using property (3) yields

$$w(x) = \frac{1}{2\pi}\sqrt{4-x^2} \tag{18}$$

Numerics

How about this?

$$\phi^+(z) + \phi^-(z) = V' := f, \quad z \in [a, b]$$

$$\phi(z) := \int \frac{w(x)}{z - x} = \frac{2\pi}{i}\mathcal{C}_\gamma[w](z) \sim O(z^{-1})$$

Numerics

How about this?

$$\phi^+(z) + \phi^-(z) = V' := f, \quad z \in [a, b]$$

$$\phi(z) := \int \frac{w(x)}{z - x} = \frac{2\pi}{i}\mathcal{C}_\gamma[w](z) \sim O(z^{-1})$$

Simplification: there is a bijective map from (a, b) to $(-1, 1)$:

$$M_{(a,b)}(x) = \frac{2x - a - b}{b - a} \tag{20}$$

Write $\tilde{\phi}(x) = \phi(M_{(a,b)}^{-1}(x))$ and $\tilde{f}(x) = f(M_{(a,b)}^{-1}(x))$, we have

$$\tilde{\phi}^{+}(x) + \tilde{\phi}^{-}(x) = \tilde{f}(x), \quad z \in [-1, 1]$$
$$\tilde{\phi}^{+}(J(z)) + \tilde{\phi}^{-}(J(z)) = \tilde{f}(J(z)), \quad z \in \mathbb{T}$$

Write $\tilde{\phi}(x) = \phi(M^{-1}_{(a,b)}(x))$ and $\tilde{f}(x) = f(M^{-1}_{(a,b)}(x))$, we have

$$\tilde{\phi}^+(x) + \tilde{\phi}^-(x) = \tilde{f}(x), \quad z \in [-1, 1]$$
$$\tilde{\phi}^+(J(z)) + \tilde{\phi}^-(J(z)) = \tilde{f}(J(z)), \quad z \in \mathbb{T}$$

Recall in the case when $\tilde{f}(x) = z$, we have $\tilde{f}(J(z)) = z/2 + 1/2z$. You can easily split the function into two halves and obtain $\tilde{\phi}(J(z))$. Can we generalise this?

Write $\tilde{\phi}(x) = \phi(M^{-1}_{(a,b)}(x))$ and $\tilde{f}(x) = f(M^{-1}_{(a,b)}(x))$, we have

$$\tilde{\phi}^{+}(x) + \tilde{\phi}^{-}(x) = \tilde{f}(x), \quad z \in [-1, 1]$$
$$\tilde{\phi}^{+}(J(z)) + \tilde{\phi}^{-}(J(z)) = \tilde{f}(J(z)), \quad z \in \mathbb{T}$$

Recall in the case when $\tilde{f}(x) = z$, we have $\tilde{f}(J(z)) = z/2 + 1/2z$. You can easily split the function into two halves and obtain $\tilde{\phi}(J(z))$. Can we generalise this?

Hint: The Chebyshev map is $\cos\theta$ when you plug in $z = e^{i\theta}$.

Write $\tilde{\phi}(x) = \phi(M^{-1}_{(a,b)}(x))$ and $\tilde{f}(x) = f(M^{-1}_{(a,b)}(x))$, we have

$$\tilde{\phi}^{+}(x) + \tilde{\phi}^{-}(x) = \tilde{f}(x), \quad z \in [-1,1]$$
$$\tilde{\phi}^{+}(J(z)) + \tilde{\phi}^{-}(J(z)) = \tilde{f}(J(z)), \quad z \in \mathbb{T}$$

Recall in the case when $\tilde{f}(x) = z$, we have $\tilde{f}(J(z)) = z/2 + 1/2z$. You can easily split the function into two halves and obtain $\tilde{\phi}(J(z))$. Can we generalise this?

Hint: The Chebyshev map is $\cos\theta$ when you plug in $z = e^{i\theta}$.
Answer: Chebyshev expansion!

If $\tilde{f}$ can be written as a Chebyshev expansion, say

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \hat{f}_k T_k(x) \tag{21}$$

Then

$$\tilde{f}(J(e^{i\theta})) = \hat{f}_0 + \frac{1}{2} \sum_{k=-\infty}^{\infty} \hat{f}_{|k|} e^{ik\theta} \tag{22}$$

In other words, for all $z \in \mathbb{T}$, we have

$$\tilde{f}(J(z)) = \hat{f}_0 + \frac{1}{2} \sum_{k=-\infty}^{\infty} \hat{f}_{|k|} z^k \tag{23}$$

If $\tilde{f}$ can be written as a Chebyshev expansion, say

$$\tilde{f}(x) = \sum_{k=0}^{\infty} \hat{f}_k T_k(x) \tag{21}$$

Then

$$\tilde{f}(J(e^{i\theta})) = \hat{f}_0 + \frac{1}{2} \sum_{k=-\infty}^{\infty} \hat{f}_{|k|} e^{ik\theta} \tag{22}$$

In other words, for all $z \in \mathbb{T}$, we have

$$\tilde{f}(J(z)) = \hat{f}_0 + \frac{1}{2} \sum_{k=-\infty}^{\infty} \hat{f}_{|k|} z^k \tag{23}$$

Remark: $\hat{f}_{|k|}$ is the Fourier coefficients of $\tilde{f}(J(e^{i\theta}))$, and you can estimate those by (Fast) Fourier Transform.

$$\tilde{f}(J(z)) = \hat{f}_0 + \frac{1}{2}\sum_{k=0}^{\infty}\hat{f}_{|k|}z^k + \sum_{k=-\infty}^{-1}\hat{f}_{|k|}z^k$$

Define

$$F^+(z) = \hat{f}_0 + \frac{1}{2}\sum_{k=0}^{\infty}\hat{f}_{|k|}z^k, \quad F^-(z) = \frac{1}{2}\sum_{k=-\infty}^{-1}\hat{f}_{|k|}z^k$$

Then the function

$$\psi(z) = \tilde{\phi}(J(z)) = \begin{cases} F^+(z) & |z| < 1 \\ F^-(z) & |z| > 1 \end{cases}$$

satisfies $\psi^+(J(z)) + \psi^-(J(z)) = \tilde{f}(J(z))$

Using previous argument, the following function

$$\tilde{\phi}'(x) = \frac{\psi(J_+^{-1}(x)) + \psi(J_-^{-1}(x))}{2}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(z))^k$$

satisfies

$$\tilde{\phi}'^{+}(x) + \tilde{\phi}'^{-}(x) = \tilde{f}(x), \quad z \in [-1, 1]$$

Using previous argument, the following function

$$\tilde{\phi}'(x) = \frac{\psi(J_+^{-1}(x)) + \psi(J_-^{-1}(x))}{2}$$
$$= \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(z))^k$$

satisfies

$$\tilde{\phi}'^{+}(x) + \tilde{\phi}'^{-}(x) = \tilde{f}(x), \quad z \in [-1, 1]$$

BUT $\tilde{\phi}'^{+}(\infty) = \hat{f}_0/2$ so $\tilde{\phi}'^{+}(\infty) = \hat{f}_0/2$ does not decay?! What can we do?

We introduce a correction term:

$$\kappa(z) = \frac{1}{\sqrt{z+1}\sqrt{z-1}}$$

which does not have jump over $[-1, 1]$ and is in $O(z^{-1})$. Therefore the function

$$\tilde{\phi}(x) = \tilde{\phi}'(x) - \frac{z\hat{f}_0 + C}{2}\kappa(x) \tag{24}$$

has same jump as $\tilde{\phi}'$ but decay like $O(z^{-1})$ (here C is a free parameter). :) We therefore have $\phi(x) = \tilde{\phi}(M_{(a,b)}(x))$, and we can look at its jump and find $w(x)$...

Technical Problem

$$\phi(x) = \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(M_{(a,b)}(x)))^k - \frac{M_{(a,b)}(x)\hat{f}_0 + C}{2}\kappa(M_{(a,b)}(x))$$

There is a free parameter C.

Technical Problem

$$\phi(x) = \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(M_{(a,b)}(x)))^k - \frac{M_{(a,b)}(x)\hat{f}_0 + C}{2}\kappa(M_{(a,b)}(x))$$

There is a free parameter C. **We don't even know what is a and b!!!!!!!!!**

We should bear in mind that $\phi(x)$ is a Cauchy transform of our density in some sense... We do want $\phi(x)$ to be bounded AND $\phi(x) = 1/x + ...$

Technical Problem

$$\phi(x) = \frac{1}{2}\sum_{k=0}^{\infty} \hat{f}_k (J_+^{-1}(M_{(a,b)}(x)))^k - \frac{M_{(a,b)}(x)\hat{f}_0 + C}{2}\kappa(M_{(a,b)}(x))$$

There is a free parameter C. **We don't even know what is *a* and *b*!!!!!!!!!**

We should bear in mind that $\phi(x)$ is a Cauchy transform of our density in some sense... We do want $\phi(x)$ to be bounded AND $\phi(x) = 1/x + ...$. We therefore choose a and b such that $\hat{f}_0 = C = 0$.

In addition, notice that

$$J_+^{-1}(x) = \frac{1}{2x} + \ldots, \quad M_{(a,b)}(x) = \frac{2x}{b-a} + \ldots$$

Therefore

$$\phi(x) = \frac{1}{2}\hat{f}_1\left(\frac{1}{2\left(\frac{2x}{b-a}\right)}\right) + \ldots = \frac{b-a}{8}\hat{f}_1\frac{1}{x} + \ldots$$

We want to choose *a* and *b* such that

$$\frac{b-a}{8}\hat{f}_1 = 1$$

$$\hat{f}_0 = 0, \quad \frac{b-a}{8}\hat{f}_1 = 1$$

We have a system of 'equations' to be solved. This can be solved by numerical schemes, e.g. Newton Method.

$$\hat{f}_0 = 0, \quad \frac{b-a}{8}\hat{f}_1 = 1$$

We have a system of 'equations' to be solved. This can be solved by numerical schemes, e.g. Newton Method. (**Remark:** As pointed out last week, Julia is specialised for doing Newton Method via automatic differentiation.)

$$\hat{f}_0 = 0, \quad \frac{b-a}{8}\hat{f}_1 = 1$$

We have a system of 'equations' to be solved. This can be solved by numerical schemes, e.g. Newton Method. (**Remark:** As pointed out last week, Julia is specialised for doing Newton Method via automatic differentiation.)

From this we obtain a, b and $\phi(x)$. We may then use property (3) to obtain $w(x)$. The final formula is stated without proof

$$w(x) = \frac{\sqrt{1 - M_{(a,b)}}}{\pi} \sum_{k=1}^{\infty} \hat{f}_k U_{k-1}(M_{(a,b)}(x)) \tag{25}$$

Felina

With this equilibrium measure, you can:

- obtain a set of orthonormal polynomials (w.r.t. $V(x)$) using a Riemann Hilbert approach.
- compute gap value statistics (e.g. Airy Kernel)
- predict the distribution of Unitary Ensemble with density $V(x)$ and think about universality.

Thank you!