UROP Task 1
Background and Energies of Enhanced Diffusion

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Spring 2019

1 Formulation of Enhanced Diffusion

Definition 1.1: Enhanced Diffusion Equation

Let $\mathbb{T}^d$ be the d-dimensional torus. Given $\kappa \in (0,1)$ and $\boldsymbol{u} : \mathbb{T}^d \to \mathbb{R}$ with $\boldsymbol{\nabla}_{\boldsymbol{x}}.\boldsymbol{u} = 0$. The Enhanced Diffusion (Advection-Diffusion) Equations are of the form

$$\frac{\partial f}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}_{\boldsymbol{x}} f = \kappa \nabla_{\boldsymbol{x}}^2 f \tag{1}$$

where $f := f(t, \boldsymbol{x}) : \mathbb{R}^+ \times \mathbb{T}^d \to \mathbb{R}$

Without loss of generality, we assume that

$$\int_{\mathbb{T}^d} f(t, \boldsymbol{x}) = 0, \quad \forall t \geq 0$$

Such equation can describe diffusion of fluid enhanced by transportation. f may represent temperature or concentration.

2 Energy of Equation

We define the following L^2 energy for a solution:

Definition 2.1: L^2-energy

Let f be the solution to equation 1. We may define the L^2 energy of the solution as $g : \mathbb{R}^+ \to \mathbb{R}^+$,

$$g(t) = \frac{1}{2}\|f(t, \boldsymbol{x})\|_{L^2}^2 = \frac{1}{2}\int_{\mathbb{T}^d} (f(t, \boldsymbol{x}))^2 d\vec{x} \tag{2}$$

Proposition 2.2: Decay of L^2-Energy

The L^2-energy of the solution is governed by

$$g(t) = g(0) - \kappa \int_0^t \|\boldsymbol{\nabla}_{\boldsymbol{x}} f(t, s)\|_{L^2} \, ds \tag{3}$$

Remark.

$$\|\boldsymbol{\nabla}_{\boldsymbol{x}} f(s)\|_{L^2} = \int_{\mathbb{T}^d} \|\boldsymbol{\nabla}_{\boldsymbol{x}} f(t, s)\| \, d\boldsymbol{x}$$

Proof. We begin as followed:

$$\begin{aligned}\frac{dg}{dt} &= \frac{1}{2}\left(\int_{\mathbb{T}^d}(f(t,\boldsymbol{x}))^2 d\boldsymbol{x}\right)\\ &= \frac{1}{2}\int_{\mathbb{T}^d}\frac{\partial}{\partial t}\left((f(t,\boldsymbol{x}))^2\right)d\boldsymbol{x}\\ &= \int_{\mathbb{T}^d} f\frac{\partial f}{\partial t}d\boldsymbol{x}\\ &= \int_{\mathbb{T}^d}\left(\kappa f\nabla_{\boldsymbol{x}}^2 f - f(\boldsymbol{u}.\boldsymbol{\nabla}_{\boldsymbol{x}} f)\right)d\boldsymbol{x}\end{aligned}$$

Observe that

$$\begin{aligned}\int_{\mathbb{T}^d} f(\boldsymbol{u}.\boldsymbol{\nabla}_{\boldsymbol{x}} f)\,d\boldsymbol{x} &= \int_{\partial\mathbb{T}^d} f^2\boldsymbol{u}.\boldsymbol{n}\,dS - \int_{\mathbb{T}^d} f\boldsymbol{\nabla}_{\boldsymbol{x}}.(f\boldsymbol{u})d\boldsymbol{x}\\ &= -\int_{\mathbb{T}^d} f\boldsymbol{\nabla}_{\boldsymbol{x}}.(f\boldsymbol{u})\,d\boldsymbol{x}\\ &= -\left(\int_{\mathbb{T}^d} f\boldsymbol{\nabla}_{\boldsymbol{x}}.\boldsymbol{u}\,d\boldsymbol{x} + \int_{\mathbb{T}^d} f(\boldsymbol{u}.\boldsymbol{\nabla}_{\boldsymbol{x}} f)\,d\boldsymbol{x}\right)\\ &= -\int_{\mathbb{T}^d} f(\boldsymbol{u}.\boldsymbol{\nabla}_{\boldsymbol{x}} f)\,d\boldsymbol{x}\\ \implies \int_{\mathbb{T}^d} f(\boldsymbol{u}.\boldsymbol{\nabla}_{\boldsymbol{x}} f)\,d\boldsymbol{x} &= 0\end{aligned}$$

Therefore,

$$\frac{dg}{dt} = \kappa\int_{\mathbb{T}^d} f\nabla_{\boldsymbol{x}}^2 f\,d\boldsymbol{x} = \kappa\left(\int_{\partial\mathbb{T}^d} f\boldsymbol{\nabla}_{\boldsymbol{x}} f.\boldsymbol{n}\,dS - \int_{\mathbb{T}^d}\|\boldsymbol{\nabla}_{\boldsymbol{x}} f\|d\boldsymbol{x}\right) = -\kappa\|\boldsymbol{\nabla}_{\boldsymbol{x}} f\|_{L^2}^2 \tag{4}$$

Integrating with respect to t yield the result required. □

We would like to find the minimum time $\tau(\chi)$ such that $g(t) < \chi$. Note the following theorem:

Theorem 2.3: Poincare Inequality

For every function $f \in W_0^{1,p}(\Omega)$, the Sobolev space, we have

$$\|f\|_{L^2}^2 \leq C\|\boldsymbol{\nabla}_{\boldsymbol{x}} f\|_{L^2}^2 \tag{5}$$

for some constant $C > 0$.

The inequality will be proved in latter task. We may utilise Poincare Inequality to equation (4) to obtain

$$\frac{dg}{dt} \leq -\frac{\kappa}{C}\|f\|_{L^2}^2 = -\frac{2\kappa}{C}g \tag{6}$$

This yield the following

Theorem 2.4: Time of Decay

$$g(t) \leq g(0)\exp\left(-\frac{2\kappa}{C}t\right) \tag{7}$$

and hence $\tau \leq \chi \sim O(\kappa^{-1})$

Proof. Note that

$$\frac{d}{dt}\left(g\exp\left(\frac{2\kappa}{C}t\right)\right) \leq 0$$

By monotonicity of integral one have

$$\int_0^t \frac{d}{ds}\left(g \exp\left(\frac{2\kappa}{C}s\right)\right) ds \leq 0 \implies g(t)\exp\left(\frac{2\kappa}{C}t\right) - g(0) \leq 0$$

Therefore

$$g(t) \leq g(0)\exp\left(-\frac{2\kappa}{C}t\right) \tag{8}$$

Finally,

$$g(0)\exp\left(-\frac{2\kappa}{C}\tau\right) = \chi \implies \tau = -\frac{C}{2\kappa}\ln\left(\frac{\chi}{g(0)}\right) \sim O(\kappa^{-1})$$

And thus if $t > \tau(\chi) = -\frac{C}{2\kappa}\ln\left(\frac{\chi}{g(0)}\right)$ one have

$$g(t) \leq g(0)\exp\left(-\frac{2\kappa}{C}t\right) \leq g(0)\exp\left(-\frac{2\kappa}{C}\tau(\chi)\right) = \chi$$

□

3 Poincare Inequality

We do not offer a formal proof for the inequality at this stage. Note that f is an element of Sobolev Space then we may write f in Fourier series as followed (here $c_n := c_n(t) \in \mathbb{C}$)

$$f(t, \boldsymbol{x}) = \sum_{n=-\infty}^{\infty} c_n e^{i\boldsymbol{n}.\boldsymbol{x}} \tag{9}$$

where $\boldsymbol{n} = (n_1, ..., n_d)$ and $n = n_1 + ... + n_d$. Differentiating yields

$$\boldsymbol{\nabla}_x f(t, \boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \boldsymbol{n} c_n e^{i\boldsymbol{n}.\boldsymbol{x}} \tag{10}$$

By Parseval's Theorem we have

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{11a}$$

$$\|\boldsymbol{\nabla}_x f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} n^2 |c_n|^2 \tag{11b}$$

and it is apparent that

$$\|f\|_{L^2}^2 \leq \|\boldsymbol{\nabla}_{\boldsymbol{x}} f\|_{L^2}^2$$

4 Aim of Project

The UROP Project is to obtain a tighter bound for $\tau(\chi)$ in different scenario of $\boldsymbol{u}$.