Research Note

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Theorem 1 (Weyl, see [?]). *Let A and B be n-by-n Hermitian matrix and let the respective eigenvalues of A, B, and $A+B$ be $\{\lambda_i(A)\}_{i=1}^n$, $\{\lambda_i(B)\}_{i=1}^n$, and $\{\lambda_i(A+B)\}_{i=1}^n$, ordered algebraically as $\lambda_{\max} = \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_2 \geq \lambda_1 = \lambda_{\min}$. Then,*

$$\lambda_i(A+B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B), \quad j = 0, 1, \ldots, n-i$$

for each $i = 1, \ldots, n$. Also,

$$\lambda_{i-j+1}(A) + \lambda_j(B) \leq \lambda_i(A+B), \quad j = 1, \ldots, i$$

for each $i = 1, \ldots, n$.

Definition 1 (Additive spread, see [?]). *Let A be n-by-n matrix and let the eigenvalues of A be $\{\lambda_i\}_{i=1}^n$. The additive spread is defined as*

$$\operatorname{ads} A = \max_{i,j} |\lambda_i - \lambda_j|.$$

Lemma 1. *Let A be a real symmetric matrix, and v be a real vector. Then,*

$$\operatorname{ads}(aA + bvv^T) \leq a \operatorname{ads}(A) + b\|v\|^2$$

for all $a,\ b \geq 0$.

Proof. See Corollary 2 of [?]. □

Lemma 2.

$$A^{k+1} = a_k A^k + b_k v_k v_k^T$$

$$b_k = \frac{(r_1 - a_k) \operatorname{ads} A^k}{\|v_k\|^2}$$

Proof.

$$\begin{aligned} \operatorname{ads} A^{k+1} &\leq a_k \cdot \operatorname{ads} A^k + b_k \|v_k\|^2 \\ &\leq r_1 \cdot \operatorname{ads} A^k \end{aligned}$$

□

Lemma 3.

$$a_k \lambda_{\max}(A^k) \leq \lambda_{\max}(A^{k+1}) \leq a_k \lambda_{\max}(A^k) + b_k \|v_k\|^2$$

$$b_k = \frac{r_2 \lambda_n(A^k) - a_k \lambda_1(A^k)}{\|v_k\|^2}$$

Proof.

$$
\begin{aligned}
\lambda_n(A^{k+1}) &\geq a_k\lambda_1(A^k) + b_k\lambda_n(v_k v_k^T) \\
&= r_2\lambda_n(A^k) \\
&= a_k\lambda_1(A^k) + (r - a_k)(\lambda_n(A^k) - \lambda_1(A^k))
\end{aligned}
$$

□

$$
\frac{(r_1 - a_k)(\lambda_n(A^k) - \lambda_1(A^k))}{\|v_k\|^2} = \frac{r_2\lambda_n(A^k) - a_k\lambda_1(A^k)}{\|v_k\|^2}
$$

$$
a_k = \frac{r_1\lambda_n(A^k) - r_1\lambda_1(A^k) - r_2\lambda_n(A^k)}{\lambda_n(A^k) - 2\lambda_1(A^k)}
$$

Let $v = \sum_{i=1}^n c_i x_i,\ c_i \in \mathbb{R}$, where $\{x_i\}_{i=1}^n$ are normalized eigenvectors of A. Then,

$$
vv^T = Xcc^T X^T = XC(XC)^T
$$

where $X = [x_1, \ldots, x_n]$, $c = [c_1, \ldots, c_n]^T$, and $C = \mathrm{diag}(c)$.

Let $A = XSS^T X^T$, where $SS^T = \Lambda$ is the diagonal matrix of eigenvalues.

Theorem 2 (See **[?]**). *Let A, B be compact operators. Then for $j = 1, 2, \ldots,$ we have*

$$
2s_j(A^*B) \leq s_j(AA^* + BB^*)
$$

where $s_j(A),\ j = 1, 2, \ldots$ denote the singular values of A in increasing order.

Now, we have

$$
2s_j(\sqrt{ab}SC) \leq s_j(aA + bvv^T)
$$

and

$$
2\sqrt{ab}\max_i(c_i\sqrt{\lambda_i}) \leq \lambda_{\max}(aA + bvv^T)
$$