Research Note

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Purpose

- To suggest a data-efficient adaptive control algorithm with stability proof.
- To consider time-varying stochastic uncertainties like wind velocity.
- Relieve the persistent / interval excitation condition by exploiting the non-single-step algorithms.

Related Works

- Yongping Pan [4, 5]
  - Composite learning control
- Girish Chowdhary [2, 3]
  - Concurrent learning
- A Review for computational optimization methods [1]

Parameter Estimation

Consider the system

$$\dot{x}(t) = Ax(t) + B_r c(t) + B\Lambda\Big(u(t) + W_0^T \phi_0(t,x)\Big),$$

and the reference system

$$\dot{x}_r(t) = Ax_r(t) + B_r c(t),$$

with the system matrix A being Hurwitz. Then, we have the following error dynamics

$$\dot{e} = Ae(t) + B\Lambda\Big(u(t) + W^T \phi(t,x,c)\Big).$$

The unknowns are a diagonal matrix Λ possessing positive elements, and a parameter matrix W.

Integrating both side of the equation yields

$$e(t) - e(t-\delta) = A\int_{t-\delta}^{t} e(\tau)\,\mathrm{d}\tau - B\Lambda v_1(t) + B\Lambda W^T v_2(t)$$

with

$$v_1(t) = \int_{t-\delta}^{t} u(\tau)\,\mathrm{d}\tau\,,$$
$$v_2(t) = \int_{t-\delta}^{t} \phi(\cdot)\,\mathrm{d}\tau\,,$$

with $\delta > 0$, and let

$$y(t) = e(t) - e(t-\delta) - A\int_{t-\delta}^{t} e(\tau)\,\mathrm{d}\tau\,.$$

Then, for a sequence $\{t_n\}$ such that $t_n > \delta$, a dataset $\mathcal{D}$ of which the input is $(v_1(t_k), v_2(t_k))$ and the output is $y(t_k)$ for $k = 0, 1, \ldots$ is generated.

We can define a regressor for $\mathcal{D}$ as follows

$$\begin{aligned} \hat{y}(t) &= -B\hat{\Lambda}(t)v_1(t) + B\hat{V}(t)v_2(t), \\ &= \begin{bmatrix} -v_1^T \otimes B & v_2^T \otimes B \end{bmatrix} \begin{bmatrix} \mathrm{vec}(\hat{\Lambda}) \\ \mathrm{vec}(\hat{V}) \end{bmatrix}, \end{aligned} \tag{1}$$

where $\hat{V} = \hat{\Lambda}\hat{W}^T$. To formulate the problem as a stochastic gradient problem, we define an error as

$$\epsilon(t) = \|\hat{y}(t) - y(t)\|^2.$$

Then, we have

$$\dot{\hat{\Lambda}} = \gamma_1 \mathrm{diag}(v_1)\mathrm{diag}\Big(B^T(\hat{y} - y)\Big),$$

and

$$\begin{aligned} \dot{\hat{V}} &= -\gamma_2\Big(v_2 \otimes B^T\Big)(\hat{y} - y) \\ &= -\gamma_2 B^T(\hat{y} - y)v_2^T. \end{aligned}$$

Since $\dot{\hat{V}} = \dot{\hat{\Lambda}}\hat{W}^T + \hat{\Lambda}\dot{\hat{W}}^T$,

$$\dot{\hat{W}} = -\gamma_1\hat{W}\mathrm{diag}(v_1)\mathrm{diag}\Big(B^T(\hat{y} - y)\Big)\hat{\Lambda}^{-1} - \gamma_2 v_2(\hat{y} - y)^T B\hat{\Lambda}^{-1}.$$

Review of Parameter Estimation Techniques

- **Time-Varying Parameter Identification Algorithm** [6]

  Consider

$$\begin{aligned} \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} &= \Theta(wt), \\ y(t) &= \Gamma^T(wt)\theta(t) + \varepsilon(t). \end{aligned}$$

In order to estimate the parameter, the following update law was introduced.

$$\dot{\hat{\theta}}(t) = -K\Gamma(wt)\left\lceil \Gamma^T(wt)\hat{\theta}(t) - y(t) \right\rfloor^{\gamma},$$

where $\lceil\cdot\rfloor^{\gamma} := |\cdot|^{\gamma}\mathrm{sign}(\cdot)$.

- **Stochastic Gradient Descent in Continuous Time**[7]

  Consider
$$\mathrm{d}X_t = f^*(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,$$
where the goal is to statistically estimate a model $f(x,\theta)$ for f^*, where $\theta \in \mathbb{R}^n$.

Note

We want to devise an exponential or similarly fast convergence of parameter estimation for dynamical system with continuous analysis.

Recursive Least Square

Concurrent Learning

Concurrent learning schemes use a fixed dataset to update the estimation for a while, so that it can guarantee the exponential convergence of the parameter estimation.

The (exponential) convergence rate to the true value of the unknown parameter is proportional to the minimum eigenvalue of constructed dataset. Hence, the dataset is updated when a new data increases the minimum eigenvalue.

However, the major drawback of concurrent learning is duplicated usages of data points, which is vulnerable to noises and changes of the parameter.

What this algorithm does can be viewed as assigning a high weight to the dataset maximizing the minimum eigenvalue. From this perspective, there arise two questions.

1. Is there a decent way to maximize the minimum eigenvalue of the dataset in continuous-time setup?
2. Does it also guarantee the exponential convergence?

LMI Formulation

If we have a history of regression vectors for each time, an weighted integral can be formulated as
$$\Phi_T^a(t) = \int_{t-T}^{t} a(t-\tau)\phi(\tau)\phi^T(\tau)\,\mathrm{d}\tau\,,$$
where $a : \mathbb{R}_+ \to \mathbb{R}$ satisfies the following condition
$$\int_0^T a(t)\,\mathrm{d}t = b,$$

with b being a positive constant.

Now, the problem we consider can be stated as follows.

$$\begin{aligned} \underset{a(\cdot)}{\text{maximize}} \quad & \lambda_{\min}(\Phi_T^a(t)) \\ \text{subject to} \quad & \int_0^T a(t)\,\mathrm{d}t = b \end{aligned}$$

As $\Phi_T^a(\cdot)$ is the sum of symmetric matrices, it can be equivalently formulated as

$$\begin{aligned} \underset{a(\cdot),\, s}{\text{maximize}} \quad & s \\ \text{subject to} \quad & \Phi_T^a(t) - s \cdot I \succeq 0 \\ & \int_0^T a(t)\,\mathrm{d}t = b \end{aligned}$$

SDP Formulation

The above problem can be viewed as an SDP problem.

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & \sum_{i=1}^{N} x_i \Phi_i + s \cdot (-I) \succeq 0 \\ & \int_0^T a(t)\,\mathrm{d}t = b \end{aligned}$$

where

$$\begin{aligned} c &= [0, \ldots, 0, -1]^T \\ x &= [a_1, \ldots, a_N, s]^T \end{aligned}$$

Appendix

Lemma 1 (Some facts of matrix algebra). *All matrices we discuss here are over the real numbers.*

1. *(See [8]) If A and B are positive semi-definite matrices, then,*

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A)\operatorname{tr}(B).$$

Lemma 2. *Let A be a symmetric matrix. Let $\lambda_{\min}$, $\lambda_{\max}$ denote respectively the smallest and largest eigenvalue of A. Then,*

$$\lambda_{\min} \cdot I \preceq A \preceq \lambda_{\max} \cdot I$$

Proof. We will show only the first inequality, which is equivalent to $A - \lambda_{\min} \cdot I \succeq 0$. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of A, then, the eigenvalues of $A - \lambda_{\min} \cdot I$ are

$$\lambda_1 - \lambda_{\min}, \ldots, \lambda_d - \lambda_{\min}.$$

Observe that

$$\min_j(\lambda_j - \lambda_{\min}) = 0,$$

which completes the proof. $\square$

Lemma 3. *Let A, B and C be symmetric $d \times d$ matrices satisfying $A \succeq 0$ and $B \preceq C$. Then*

$$\operatorname{tr}(AB) \leq \operatorname{tr}(AC)$$

Proof. Let $A = \sum_{i=1}^{d} \lambda_i u_i u_i^T$, where $\{u_1, \ldots, u_d\}$ is an orthonormal basis consisting of eigenvectors of A, and λ_i is the eigenvalue corresponding to u_i. Since $A \succeq 0$, we can set $v_i = \sqrt{\lambda_i} u_i$ and write $A = \sum_{i=1}^{d} v_i v_i^T$. Then,

$$\begin{aligned}\operatorname{tr}(AB) &= \operatorname{tr}\left(\sum_i v_i v_i^T B\right) = \sum_i \operatorname{tr}\left(v_i^T B v_i\right) \\ &\leq \sum_i \operatorname{tr}\left(v_i^T C v_i\right) = \operatorname{tr}(AC).\end{aligned}$$

$\square$

Corollary 1. *Let $B = \sum_{i=1}^{d} v_i v_i^T$, and $\lambda_{\min}(B)$ be the smallest eigenvalue of B. Then,*

$$\operatorname{tr}\left(A^T B A\right) \geq \lambda_{\min}(B) \operatorname{tr}\left(A^T A\right).$$

Proof. Since $B \succeq \lambda_{\min}(B) \cdot I$, and AA^T is symmetric, we can apply Lemma 3 as

$$\operatorname{tr}\left(A^T B A\right) = \operatorname{tr}\left(B A A^T\right) \geq \operatorname{tr}\left(\lambda_{\min}(B) A A^T\right) = \lambda_{\min}(B) \operatorname{tr}\left(A^T A\right).$$

$\square$

References

[1] Léon Bottou, Frank E. Curtis, and Jorge Nocedal. Optimization Methods for Large-Scale Machine Learning. *SIAM Review*, May 2018.

[2] Girish Chowdhary, Tansel Yucelen, Maximillian Mühlegg, and Eric N. Johnson. Concurrent learning adaptive control of linear systems with exponentially convergent bounds. *International Journal of Adaptive Control and Signal Processing*, 27(4):280–301, April 2013.

[3] R. Kamalapurkar, B. Reish, G. Chowdhary, and W. E. Dixon. Concurrent Learning for Parameter Estimation Using Dynamic State-Derivative Estimators. *IEEE Transactions on Automatic Control*, 62(7):3594–3601, July 2017.

[4] Yongping Pan, Tairen Sun, and Haoyong Yu. Composite adaptive dynamic surface control using online recorded data. *International Journal of Robust and Nonlinear Control*, 26(18):3921–3936, December 2016.

[5] Yongping Pan and Haoyong Yu. Composite learning robot control with guaranteed parameter convergence. *Automatica*, 89:398–406, March 2018.

[6] H. Ríos, D. Efimov, J. A. Moreno, W. Perruquetti, and J. G. Rueda-Escobedo. Time-Varying Parameter Identification Algorithms: Finite and Fixed-Time Convergence. *IEEE Transactions on Automatic Control*, 62(7):3671–3678, July 2017.

[7] J. Sirignano and K. Spiliopoulos. Stochastic Gradient Descent in Continuous Time. *SIAM Journal on Financial Mathematics*, 8(1):933–961, January 2017.

[8] Zübeyde Ulukök and Ramazan Türkmen. On Some Matrix Trace Inequalities. *Journal of Inequalities and Applications*, 2010:1–8, 2010.