Research Note

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Introduction

Composite adaptive control combines *direct* and *indirect* schemes of adaptive control [5]. The purpose is

- to obtain the global asymptotic stability of *both* tracking and parameter estimation errors with proper exciting conditions and a matching condition, or
- to simply improve the tracking performance, where the matching condition or the exciting conditions are not satisfied.

FUNDAMENTAL IDEA is to exploit a current estimation of the parameter estimation error. Consider[1]

$$\dot{e}(t) = Ae(t) + B(u(t) + \Delta(t)).$$

Convert this system into[2]

$$y(t) = {W^*}^T \phi(t) + \varepsilon(t), \tag{1}$$

where $y(t)$ is measured using

$$y(t) = B^\dagger(\dot{e}(t) - Ae(t)) - u(t).$$

Observation 1.

1. equation (1) is merely a linear regression form, and
2. almost all composite adaptive control schemes [5, 2, 3] use the update law of standard least square regression, which are represented by

$$\dot{W}(t) = \Gamma_1 \phi(t) e^T PB - \int_0^t c(t,\tau)\phi(\tau)\epsilon^T(t,\tau)\,\mathrm{d}\tau\,, \tag{2}$$

   where

$$\epsilon(t,\tau) = W^T(t)\phi(\tau) - y(\tau).$$

Observation 2. Consider $c(t,\tau)$ in equation (2),

1. **Standard Least Square Update [8]:** If $c(t,\tau) = \delta(t-\tau)$ where δ is a Dirac delta function, then the update law is a standard least square form, which requires the PE condition for exponential convergence.

[1] We define $\Delta(t) = {W^*}^T\phi(t) + \varepsilon(t)$.

[2] Sometimes we filter the system as

$$y_f = {W^*}^T\phi_f(t) + \varepsilon_f(t),$$

2. **Concurrent Learning [3]:** If $c(t,\tau) = \sum_{i=1}^{p} \delta(t_i - \tau)$ for $0 \le t_i \le t$, then the update law is a concurrent learning form, which requires the exciting over finite interval condition.

3. **Y. Pan [7] and N. Cho [2]:** If $c(t,\tau) = \exp\left(-\int_{\tau}^{t_i} k(\nu)\,\mathrm{d}\nu\right)$ for $t_0 \le t_i \le t$, then the update law is the form suggested in, which requires the IE or FE condition.

Motivation

- **Without the PE Condition:** The standard least square update is valid only with the PE condition.
- **Time-Varying Parameters:** Concurrent learning, Y. Pan and N. Cho's algorithms are not suited for time-varying parameter estimation, as it can be stuck in the past time where the minimum singular/eigenvalue are dominant.
- **Stochastic Estimation:** The standard least square update can deal with the stochastic estimation[3] only when the PE condition is satisfied. Concurrent learning and its variations are sensitive to such noises, as the history stack algorithms are heavily dependent on the singular values.
- **Smooth Estimation**: Parameter estimation in concurrent learning, Y. Pan and N. Cho's algorithms are not smooth, as the update is piecewise constant in time.

[3] $\varepsilon(t)$ is a random variable

Preliminaries

Theorem 1 (Weyl, see [4])**.** *Let A and B be n-by-n Hermitian matrix and let the respective eigenvalues of A, B, and $A+B$ be $\{\lambda_i(A)\}_{i=1}^n$, $\{\lambda_i(B)\}_{i=1}^n$, and $\{\lambda_i(A+B)\}_{i=1}^n$, ordered algebraically as $\lambda_{\max} = \lambda_n \ge \lambda_{n-1} \ge \cdots \ge \lambda_2 \ge \lambda_1 = \lambda_{\min}$. Then,*

$$\lambda_i(A+B) \le \lambda_{i+j}(A) + \lambda_{n-j}(B), \quad j = 0, 1, \ldots, n-i$$

for each $i = 1, \ldots, n$. Also,

$$\lambda_{i-j+1}(A) + \lambda_j(B) \le \lambda_i(A+B), \quad j = 1, \ldots, i$$

for each $i = 1, \ldots, n$.

Definition 1 (Additive spread, see [6])**.** Let A be n-by-n matrix and let the eigenvalues of A be $\{\lambda_i\}_{i=1}^n$. The *additive spread* is defined as

$$\operatorname{ads} A = \max_{i,j} |\lambda_i - \lambda_j|.$$

Corollary 1 (Merikoski, see [6]). *Let A and B be Hermitian n-by-n matrices. Then,*

$$\text{ads}(A+B) \leq \text{ads}\, A + \text{ads}$$

Theorem 2 (Bhatia, see [1]). *Let $A,\ B \in \mathcal{M}_n(\mathbb{C})$ be compact operators. Then for $j = 1, 2, \ldots,$ we have*

$$2s_j(A^*B) \leq s_j(AA^* + BB^*)$$

where $s_j(A),\ j = 1, 2, \ldots$ denote the singular values of A in increasing order.

Problem Formulation

Consider the second term of (2), which can be represented by

$$\dot{U}(t) = -p_1(t)U(t) + p_2(t)\phi(t)\phi^T(t), \quad U(0) = 0,$$

and its discrete counterpart as

$$A^{k+1} = a_k A^k + b_k v_k v_k^T, \quad A^0 = 0 \tag{3}$$

with slight abuse of notations for simplicity, and denote $A^k = U(t_0 + k\Delta t),\ v_k = \phi(t_0 + k\Delta t)$.

The purpose is to design a_k and b_k

1. to increase the minimum eigenvalue of A^k as k increases, and
2. to bound, simultaneously, the maximum eigenvalue of A^k.

for given v_k at each step k.

Main Results

Let A be an n-by-n positive semidefinite matrix, and v be an n-dimensional real vector, and $\{\lambda_i(\cdot)\}$ be the eigenvalues of $(\cdot)$ ordered algebraically as $\lambda_{\max} = \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_2 \geq \lambda_1 = \lambda_{\min}$.

Let Λ and X be the diagonal matrix of eigenvalues of A, and the corresponding matrix of eigenvectors, i.e.

$$A = X\Lambda X^T.$$

Finally, let

$$A' = aA + bvv^T,$$

which is an abbreviated form of (3).

Lemma 1. *If there exist $r_1 \in [0,1]$, $a \in [0, r_1]$ and $b \in [0, \infty)$ satisfying*

$$a \cdot \operatorname{ads} A + b\|v\|^2 \le r_1 \operatorname{ads} A,$$

and let $A' = aA + bvv^T$. Then,

$$\operatorname{ads} A' \le \operatorname{ads} A.$$

Proof. From Corollary 2 of [6],

$$\begin{aligned}
\operatorname{ads} A' &= \operatorname{ads}(aA + bvv^T) \\
&\le \operatorname{ads}(aA) + \operatorname{ads}(bvv^T) \\
&= a \operatorname{ads}(A) + b\|v\|^2 \\
&\le r_1 \operatorname{ads} A \\
&\le \operatorname{ads} A
\end{aligned}$$

□

Lemma 2. *For all $a, b \ge 0$, and $j = 1, \ldots, n$,*

$$2\sqrt{ab}\lambda_j\Big(\Lambda^{1/2} \operatorname{diag}(X^T v)\Big) \le \lambda_j\big(A'\big).$$

Proof. Observe that $A = X\Lambda^{1/2}(X\Lambda^{1/2})^T$ and $vv^T = XC(XC)^T$ where $C := \operatorname{diag}(X^T v)$. From Bhatia's theorem [1], we have

$$\begin{aligned}
\lambda_j(A') &= s_j\Big((\sqrt{a}X\Lambda^{1/2})(\sqrt{a}X\Lambda^{1/2})^T + (\sqrt{b}XC)(\sqrt{b}XC)^T\Big) \\
&\ge 2s_j\Big(\sqrt{ab}\Lambda^{1/2}X^T XC\Big) \\
&= 2\sqrt{ab}\lambda_j\Big(\Lambda^{1/2}C\Big).
\end{aligned}$$

□

Lemma 3. *For all $a, b \ge 0$,*

$$2\sqrt{ab}\Big\|\Lambda^{1/2}X^T v\Big\|_\infty \le \lambda_{\max}(A') \le a\lambda_{\max}(A) + b\|v\|^2. \tag{4}$$

Proof. The right inequality (4) is directly derived from Weyl's theorem [4] as

$$\lambda_n(A') \le a\lambda_n(A) + b\lambda_n(vv^T),$$

and the left inequality is from Lemma 2 for $j = n$. □

Now, we want to derive an algorithm that

1. increases the minimum eigenvalue,

$$\lambda_1(A') \ge \lambda_1(A),$$

2. and bounding the maximum eigenvalue as

$$\lambda_{\max}(A') \le \lambda_{\max}(A).$$

Given r_1, from Lemma 1, we have the following condition

$$(\operatorname{ads} A)a + \|V\|^2 b \leq r_1 \operatorname{ads} A. \tag{5}$$

Moreover, from Lemma 3, we have following two conditions

$$\begin{aligned} ab &\geq \frac{\lambda_1(A) + r_1 \operatorname{ads} A}{4\|\Lambda^{1/2} X^T v\|_\infty^2} \\ \lambda_n(A)a + \|v\|^2 b &\leq \lambda_n(A) \end{aligned} \tag{6}$$

References

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