Research Note

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Introduction

Composite adaptive control combines *direct* and *indirect* schemes of adaptive control [6]. The purpose is

- to obtain the global asymptotic stability of *both* tracking and parameter estimation errors with proper exciting conditions and a matching condition, or
- to simply improve the tracking performance, where the matching condition or the exciting conditions are not satisfied.

Fundamental idea is to exploit a current estimation of the parameter estimation error. Consider[1]

$$\dot{e}(t) = Ae(t) + B(u(t) + \Delta(t)).$$

Convert this system into[2]

$$y(t) = W^{*T}\phi(t) + \varepsilon(t), \tag{1}$$

where $y(t)$ is measured using

$$y(t) = B^{\dagger}(\dot{e}(t) - Ae(t)) - u(t).$$

Observation 1.

1. equation (1) is merely a linear regression form, and
2. almost all composite adaptive control schemes [6, 2, 3] use the update law of standard least square regression, which are represented by

$$\dot{W}(t) = \Gamma_1\phi(t)e^T PB - \int_0^t c(t,\tau)\phi(\tau)\epsilon^T(t,\tau)\,\mathrm{d}\tau\,, \tag{2}$$

where

$$\epsilon(t,\tau) = W^T(t)\phi(\tau) - y(\tau).$$

Observation 2. Consider $c(t,\tau)$ in equation (2),

1. **Standard Least Square Update [9]:** If $c(t,\tau) = \delta(t-\tau)$ where δ is a Dirac delta function, then the update law is a standard least square form, which requires the PE condition for exponential convergence.

[1] We define $\Delta(t) = W^{*T}\phi(t) + \varepsilon(t)$.

[2] Sometimes we filter the system as

$$y_f = W^{*T}\phi_f(t) + \varepsilon_f(t),$$

2. **Concurrent Learning [3]:** If $c(t,\tau) = \sum_{i=1}^{p} \delta(t_i - \tau)$ for $0 \leq t_i \leq t$, then the update law is a concurrent learning form, which requires the exciting over finite interval condition.

3. **Y. Pan [8] and N. Cho [2]:** If $c(t,\tau) = \exp\left(-\int_{\tau}^{t_i} k(\nu)\,\mathrm{d}\nu\right)$ for $t_0 \leq t_i \leq t$, then the update law is the form suggested in, which requires the IE or FE condition.

Motivation

- **Without the PE Condition:** The standard least square update is valid only with the PE condition.
- **Time-Varying Parameters:** Concurrent learning, Y. Pan and N. Cho's algorithms are not suited for time-varying parameter estimation, as it can be stuck in the past time where the minimum singular/eigenvalue are dominant.
- **Stochastic Estimation:** The standard least square update can deal with the stochastic estimation[3] only when the PE condition is satisfied. Concurrent learning and its variations are sensitive to such noises, as the history stack algorithms are heavily dependent on the singular values.
- **Smooth Estimation**: Parameter estimation in concurrent learning, Y. Pan and N. Cho's algorithms are not smooth, as the update is piecewise constant in time.

[3] $\varepsilon(t)$ is a random variable

Preliminaries

Theorem 1 (Weyl, see [4]). *Let A and B be n-by-n Hermitian matrix and let the respective eigenvalues of A, B, and $A+B$ be $\{\lambda_i(A)\}_{i=1}^n$, $\{\lambda_i(B)\}_{i=1}^n$, and $\{\lambda_i(A+B)\}_{i=1}^n$, ordered algebraically as $\lambda_{\max} = \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_2 \geq \lambda_1 = \lambda_{\min}$. Then,*

$$\lambda_i(A+B) \leq \lambda_{i+j}(A) + \lambda_{n-j}(B), \quad j = 0, 1, \ldots, n-i$$

for each $i = 1, \ldots, n$. Also,

$$\lambda_{i-j+1}(A) + \lambda_j(B) \leq \lambda_i(A+B), \quad j = 1, \ldots, i$$

for each $i = 1, \ldots, n$.

Definition 1 (Additive spread, see [7]). Let A be n-by-n matrix and let the eigenvalues of A be $\{\lambda_i\}_{i=1}^n$. The *additive spread* is defined as

$$\operatorname{ads} A = \max_{i,j} \left|\lambda_i - \lambda_j\right|.$$

Corollary 1 (Merikoski, see [7]). *Let A and B be Hermitian n-by-n matrices. Then,*

$$\operatorname{ads}(A+B) \leq \operatorname{ads} A + \operatorname{ads}$$

Theorem 2 (Bhatia, see [1]). *Let $A,\ B \in \mathcal{M}_n(\mathbb{C})$ be compact operators. Then for $j = 1, 2, \ldots,$ we have*

$$2s_j(A^*B) \leq s_j(AA^* + BB^*)$$

where $s_j(A),\ j = 1, 2, \ldots$ denote the singular values of A in increasing order.

Problem Formulation

In (2), let

$$\mathcal{A}(t) = -\int_0^t c(t,\tau)\phi(\tau)\phi^T(\tau)\,\mathrm{d}\tau\,,$$

whose derivative is

$$\dot{\mathcal{A}}(t) = -\int_0^t \frac{\partial}{\partial t}c(t,\tau)\phi(\tau)\phi^T(\tau)\,\mathrm{d}\tau - c(t,t)\phi(t)\phi^T(t), \quad \mathcal{A}(0) = 0.$$

To deliberately consider a forgetting factor, let $c(t,\tau)$ be

$$c(t,\tau) = \beta(\tau)\exp\left(-\int_\tau^t \alpha(\nu)\,\mathrm{d}\nu\right),$$

for $\alpha : \mathbb{R}_+ \to \mathbb{R}$ and $\beta : \mathbb{R}_+ \to \mathbb{R}$, which leads to

$$\dot{\mathcal{A}}(t) = -\alpha(t)\mathcal{A}(t) - \beta(t)\phi(t)\phi^T(t), \quad \mathcal{A}(0) = 0.$$

Note that there are no constraints for α and β, if they guarantee the BIBO stability.[4]

[4] Need references

With slight abuse of notations for simplicity, the discrete counterpart can be written as

$$A^{k+1} = a_k A^k + b_k v_k v_k^T, \quad A^0 = 0, \tag{3}$$

for $a_k > 0,\ b_k \in \mathbb{R}$, where $A^k := \mathcal{A}(k\Delta t)$, and $v_k := \phi(k\Delta t)$.

THE PURPOSE is to design a_k and b_k

1. to increase the minimum eigenvalue of A^k as k increases, and
2. to bound, simultaneously, the maximum eigenvalue of A^k.

for given v_k at each step k.

Theorem 3 (Ipsen, see [5]). *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian, $y \in \mathbb{C}^n$.*

1. *(smallest eigenvalue). Let*

$$L_\pm := \begin{pmatrix} \lambda_2(A) & 0 \\ 0 & \lambda_1(A) \end{pmatrix} \pm \begin{pmatrix} \|y_{2:n}\| \\ y_1 \end{pmatrix} \begin{pmatrix} \|y_{2:n}\| & \bar{y}_1 \end{pmatrix},$$
$$U_\pm := \begin{pmatrix} \lambda_2(A) & 0 \\ 0 & \lambda_1(A) \end{pmatrix} \pm \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \begin{pmatrix} \bar{y}_2 & \bar{y}_1 \end{pmatrix}.$$

Then

$$\lambda_{\min}(L_\pm) \le \lambda_{\min}(A \pm yy^*) \le \lambda_{\min}(U_\pm),$$

where

$$\lambda_{\min}(A) \le \lambda_{\min}(L_+) \le \lambda_{\min}(U_+) \le \lambda_2(A),$$
$$\lambda_{\min}(A) - \|y\|^2 \le \lambda_{\min}(L_-) \le \lambda_{\min}(U_-) \le \lambda_{\min}(A).$$

2. *(largest eigenvalue). Let*

$$L_\pm := \begin{pmatrix} \lambda_n(A) & 0 \\ 0 & \lambda_{n-1}(A) \end{pmatrix} \pm \begin{pmatrix} \|y_n\| \\ y_{n-1} \end{pmatrix} \begin{pmatrix} \bar{y}_n & \bar{y}_{n-1} \end{pmatrix},$$
$$U_\pm := \begin{pmatrix} \lambda_n(A) & 0 \\ 0 & \lambda_{n-1}(A) \end{pmatrix} \pm \begin{pmatrix} y_n \\ \|y_{1:n-1}\| \end{pmatrix} \begin{pmatrix} \bar{y}_n & \|y_{1:n-1}\| \end{pmatrix}.$$

Then

$$\lambda_{\max}(L_\pm) \le \lambda_{\max}(A \pm yy^*) \le \lambda_{\max}(U_\pm),$$

where

$$\lambda_{\max}(A) \le \lambda_{\max}(L_+) \le \lambda_{\max}(U_+) \le \lambda_{\max}(A) + \|y\|^2,$$
$$\lambda_{n-1}(A) \le \lambda_{\max}(L_-) \le \lambda_{\max}(U_-) \le \lambda_{\max}(A).$$

Main Results

Let A be an n-by-n positive semidefinite matrix, and v be an n-dimensional real vector, and $\{\lambda_i(\cdot)\}$ be the eigenvalues of $(\cdot)$ ordered algebraically as $\lambda_{\max} = \lambda_n \ge \lambda_{n-1} \ge \cdots \ge \lambda_2 \ge \lambda_1 = \lambda_{\min}$.

Let Λ and V be the diagonal matrix of eigenvalues of A, and the corresponding matrix of eigenvectors, i.e.

$$A = V \Lambda V^T.$$

Also, let

$$A' = aA + bvv^T,$$

which is an abbreviated form of (3).

Now, we want to derive an algorithm that

1. increases the minimum eigenvalue,

$$\lambda_1(A') \ge \lambda_1(A),$$

2. and bounding the maximum eigenvalue as

$$\lambda_{\max}(A') \le \lambda_{\max}(A).$$

Main Results

Lemma 1. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian, $v \in \mathbb{C}^n$, and $A' := aA + bvv^T$ for $a > 0$ and $b \in \mathbb{R}$. Then,*

$$\lambda_{\min}(A') \geq a\lambda_1(A) + \frac{1}{2}\left(a\,\mathrm{gap}_1(A) + b\|v\|^2 - \sqrt{\left(a\,\mathrm{gap}_1(A) + b\|v\|^2\right)^2 - 4ab\,\mathrm{gap}_1(A)|v_1|^2}\right), \tag{4}$$

and

$$\lambda_{\max}(A') \leq a\lambda_n(A) + \frac{1}{2}\left(-a\,\mathrm{gap}_n(A) + b\|v\|^2 + \sqrt{\left(a\,\mathrm{gap}_n(A) + b\|v\|^2\right)^2 - 4ab\,\mathrm{gap}_n(A)|v_{1:n-1}|^2}\right), \tag{5}$$

Proof. With the fact that $\lambda_i(aA) = a\lambda_i(A)$, and $\mathrm{gap}_i(aA) = a\,\mathrm{gap}_i(A)$, the proof directly follows Theorem 2.1, and Corollary 2.2 of [5]. $\square$

For simplicity, we abbreviate $\lambda_i := \lambda_i(A)$, and $\mathrm{gap}_i := \mathrm{gap}_i(A)$. Define a function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$f(a,b) := a\lambda_1 + \frac{1}{2}\left(a\,\mathrm{gap}_1 + b\|v\|^2 - \sqrt{\left(a\,\mathrm{gap}_1 + b\|v\|^2\right)^2 - 4ab\,\mathrm{gap}_1\,|v_1|^2}\right). \tag{6}$$

Lemma 2. *$f(a,b)$ is a monotonically increasing function for each $a > 0$ and $b \in \mathbb{R}$.*

Moreover, given a,

$$\lim_{b\to\infty} f(a,b) = a\left(\lambda_1(A) + \mathrm{gap}_1(A)\frac{|v_1|^2}{\|v\|^2}\right).$$

Lemma 3. *Suppose that there exist $a, b \geq 0$, such that $f(a,b) \geq \lambda_1(A)$. Then*

$$a \geq \frac{\lambda_1(A)}{\lambda_2(A)}. \tag{7}$$

Given $A \in \mathbb{S}^{n\times n}$, and $v \in \mathbb{R}^n$, let $f_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be

$$f_1(a,b;c) := a^2 + \frac{k_1\|v\|^2}{\lambda_1}ab - \frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}ac - \frac{\|v\|^2}{\lambda_1\lambda_2}bc + \frac{1}{\lambda_1\lambda_2}c^2,$$

where

$$a + \frac{\|v\|^2}{\lambda_1+\lambda_2}b - \frac{2c}{\lambda_1+\lambda_2} \geq 0,$$
$$\frac{\lambda_1}{\lambda_2} \leq k_1 := \frac{\lambda_1}{\lambda_2} + \frac{|v_1|^2}{\|v\|^2}\left(1 - \frac{\lambda_1}{\lambda_2}\right) \leq 1.$$

Also, let $f_n : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$f_n(a,b;c) := a^2 + \frac{k_n\|v\|^2}{\lambda_n}ab - \frac{\lambda_n+\lambda_{n-1}}{\lambda_n\lambda_{n-1}}ac - \frac{\|v\|^2}{\lambda_n\lambda_{n-1}}bc + \frac{1}{\lambda_n\lambda_{n-1}}c^2,$$

where

$$a + \frac{\|v\|^2}{\lambda_n + \lambda_{n-1}} b - \frac{2c}{\lambda_n + \lambda_{n-1}} \leq 0,$$
$$1 \leq k_n := \frac{\lambda_n}{\lambda_{n-1}} - \frac{|v_n|^2}{\|v\|^2} \left(\frac{\lambda_n}{\lambda_{n-1}} - 1 \right) \leq \frac{\lambda_n}{\lambda_{n-1}}.$$

Theorem 4. *1. Suppose that there exist $a > 0, b, c \in \mathbb{R}$ satisfying $f_1(a, b; c) \geq 0$. Then, $\lambda_{\min}(aA + bvv^T) \geq c$.*

2. Also, if $f_n(a, b; c) \geq 0$, then, $\lambda_{\max}(aA + bvv^T) \leq c$.

Proof. The proof is direct result from Lemma 1, □

Remark 1. 1. If $\text{gap}_1 = 0$, then $k_1 = 1$, and $f_1(a, b, c) \geq 0$ reads

$$f_1(a, b; c) = \left(a - \frac{c}{\lambda_1} \right) \left(a + \frac{\|v\|^2}{\lambda_1} b - \frac{c}{\lambda_1} \right) \geq 0,$$
$$a + \frac{\|v\|^2}{2\lambda_1} b - \frac{c}{\lambda_1} \geq 0.$$

2. If $\text{gap}_n = 0$, then $k_n = 1$, and $f_n(a, b, c) \geq 0$ reads

$$f_n(a, b; c) = \left(a - \frac{c}{\lambda_n} \right) \left(a + \frac{\|v\|^2}{\lambda_n} b - \frac{c}{\lambda_n} \right) \geq 0,$$
$$a + \frac{\|v\|^2}{2\lambda_n} b - \frac{c}{\lambda_n} \leq 0.$$

Remark 2. Note that $f_1(c_1/\lambda_1, 0; c_1) = f_n(c_n/\lambda_n, 0; c_n) = 0$ for all $c_1, c_2 \in \mathbb{R}$.

Lemma 4. *Let $A \in \mathbb{S}^{n \times n}$ and $v \in \mathbb{R}^n$. Then, for all $c_1, c_n \in \mathbb{R}$ such that*

$$\frac{c_1}{\lambda_1} \leq \frac{c_n}{\lambda_n},$$

there exist $a > 0, b \in \mathbb{R}$ satisfying $c_1 \leq \lambda_i(aA + bvv^T) \leq c_2$, for $i = 1, \ldots, n$.

Theorem 5. *Let $A \in \mathbb{S}^{n \times n}$ and $v \in \mathbb{R}^n$. Suppose that*

$$\frac{|v_1|^2}{\lambda_{\min}(A)} \neq \frac{|v_n|^2}{\lambda_{\max}(A)}, \tag{8}$$
$$\text{gap}_1, \text{gap}_n \neq 0. \tag{9}$$

Then, there exist $a > 0$ and $b \in \mathbb{R}$ such that

$$c_1 < \lambda_i(aA + bvv^T) < c_n,$$

for all c_1, c_2 such that $c_1/\lambda_1 \leq c_n/\lambda_n$.

Proof. Note that $f_1(c_1/\lambda_1, 0; c_1) = f_n(c_n/\lambda_n, 0; c_n) = 0$, and

$$\begin{aligned}
\boldsymbol{\nabla} f_1(c_1/\lambda_1, 0; c_1) &= \frac{c_1 \operatorname{gap}_1}{\lambda_1 \lambda_2} \begin{bmatrix} 1 & \frac{|v_1|^2}{\lambda_1} \end{bmatrix}^T, \\
\boldsymbol{\nabla} f_n(c_n/\lambda_n, 0; c_n) &= -\frac{c_n \operatorname{gap}_n}{\lambda_n \lambda_{n-1}} \begin{bmatrix} 1 & \frac{|v_n|^2}{\lambda_n} \end{bmatrix}^T.
\end{aligned}$$

Since $f_1(a, b; c_1) = 0$ and $f_n(a, b; c_n) = 0$ intersect the point $(a, b) = (c_1/\lambda_1, 0)$, and $(a, b) = (c_n/\lambda_n, 0)$, respectively, there exists a region $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}$ satisfying both $f_1(a, b; c_1) > 0$ and $f_n(a, b; c_n) > 0$ for $(a, b) \in \mathcal{D}$, if only

$$\boldsymbol{\nabla} f_1(c_1/\lambda_1, 0; c_1) \times \boldsymbol{\nabla} f_n(c_n/\lambda_n, 0; c_n) \neq 0,$$

which completes the proof. □

Analytic Solution

For $\lambda_{\min}(A') \geq \lambda_1$ for all v, the following conditions are necessary:

$$\Big((a\lambda_2 - \lambda_1)|v_1|^2 - (1-a)\lambda_1 \|v_{2:n}\|^2\Big) b - (1-a)(a\lambda_2 - \lambda_1)\lambda_1 \geq 0, \tag{10}$$

$$a = 1, \quad \forall v \quad \text{s.t.} \quad v_1 = 0. \tag{11}$$

asf

$$\begin{aligned}
\Big(a \operatorname{ads}_1 + b\|v\|^2\Big)^2 - 4ab \operatorname{ads}_1 |v_1|^2 &\leq \Big(\Big(a \operatorname{ads}_1 + b\|v\|^2\Big) - 2(1-a)\lambda_1\Big)^2 \\
-4ab \operatorname{ads}_1 |v_1|^2 &\leq -4(1-a)\lambda_1 \Big(a \operatorname{ads}_1 + b\|v\|^2\Big) + 4(1-a)^2\lambda_1^2 \\
\Big(a \operatorname{ads}_1 |v_1|^2 - (1-a)\lambda_1 \|v\|^2\Big) b - (1-a)\lambda_1 \operatorname{ads}_1 a &+ (1-a)^2 \lambda_1^2 \geq 0
\end{aligned}$$

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