Research Note

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Introduction

Composite adaptive control combines *direct* and *indirect* schemes of adaptive control [6]. The purpose is

- to obtain the global asymptotic stability of *both* tracking and parameter estimation errors with proper exciting conditions and a matching condition, or
- to simply improve the tracking performance, where the matching condition or the exciting conditions are not satisfied.

FUNDAMENTAL IDEA is to exploit a current estimation of the parameter estimation error. Consider[1]

$$\dot{e}(t) = Ae(t) + B(u(t) + \Delta(t)).$$

Convert this system into[2]

$$y(t) = W^{*T}\phi(t) + \varepsilon(t), \tag{1}$$

where $y(t)$ is measured using

$$y(t) = B^{\dagger}(\dot{e}(t) - Ae(t)) - u(t).$$

Observation 1.

1. equation (1) is merely a linear regression form, and
2. almost all composite adaptive control schemes [6, 2, 3] use the update law of standard least square regression, which are represented by

$$\dot{W}(t) = \Gamma_1\phi(t)e^T PB - \int_0^t c(t,\tau)\phi(\tau)\epsilon^T(t,\tau)\,\mathrm{d}\tau\,, \tag{2}$$

   where
$$\epsilon(t,\tau) = W^T(t)\phi(\tau) - y(\tau).$$

Observation 2. Consider $c(t,\tau)$ in equation (2),

1. **Standard Least Square Update [9]:** If $c(t,\tau) = \delta(t-\tau)$ where δ is a Dirac delta function, then the update law is a standard least square form, which requires the PE condition for exponential convergence.

[1] We define $\Delta(t) = W^{*T}\phi(t) + \varepsilon(t)$.

[2] Sometimes we filter the system as
$$y_f = W^{*T}\phi_f(t) + \varepsilon_f(t),$$

2. **Concurrent Learning [3]:** If $c(t,\tau) = \sum_{i=1}^{p} \delta(t_i - \tau)$ for $0 \le t_i \le t$, then the update law is a concurrent learning form, which requires the exciting over finite interval condition.

3. **Y. Pan [8] and N. Cho [2]:** If $c(t,\tau) = \exp\left(-\int_{\tau}^{t_i} k(\nu)\,\mathrm{d}\nu\right)$ for $t_0 \le t_i \le t$, then the update law is the form suggested in, which requires the IE or FE condition.

Motivation

- **Without the PE Condition:** The standard least square update is valid only with the PE condition.
- **Time-Varying Parameters:** Concurrent learning, Y. Pan and N. Cho's algorithms are not suited for time-varying parameter estimation, as it can be stuck in the past time where the minimum singular/eigenvalue are dominant.
- **Stochastic Estimation:** The standard least square update can deal with the stochastic estimation[3] only when the PE condition is satisfied. Concurrent learning and its variations are sensitive to such noises, as the history stack algorithms are heavily dependent on the singular values.
- **Smooth Estimation**: Parameter estimation in concurrent learning, Y. Pan and N. Cho's algorithms are not smooth, as the update is piecewise constant in time.

[3] $\varepsilon(t)$ is a random variable

Preliminaries

Theorem 1 (Weyl, see [4])**.** *Let A and B be n-by-n Hermitian matrix and let the respective eigenvalues of A, B, and $A+B$ be $\{\lambda_i(A)\}_{i=1}^{n}$, $\{\lambda_i(B)\}_{i=1}^{n}$, and $\{\lambda_i(A+B)\}_{i=1}^{n}$, ordered algebraically as $\lambda_{\max} = \lambda_n \ge \lambda_{n-1} \ge \cdots \ge \lambda_2 \ge \lambda_1 = \lambda_{\min}$. Then,*

$$\lambda_i(A+B) \le \lambda_{i+j}(A) + \lambda_{n-j}(B), \quad j = 0, 1, \ldots, n-i$$

for each $i = 1, \ldots, n$. Also,

$$\lambda_{i-j+1}(A) + \lambda_j(B) \le \lambda_i(A+B), \quad j = 1, \ldots, i$$

for each $i = 1, \ldots, n$.

Definition 1 (Additive spread, see [7])**.** Let A be n-by-n matrix and let the eigenvalues of A be $\{\lambda_i\}_{i=1}^{n}$. The *additive spread* is defined as

$$\operatorname{ads} A = \max_{i,j} \left|\lambda_i - \lambda_j\right|.$$

Corollary 1 (Merikoski, see [7]). *Let A and B be Hermitian n-by-n matrices. Then,*

$$\text{ads}(A+B) \leq \text{ads}\, A + \text{ads}$$

Theorem 2 (Bhatia, see [1]). *Let $A,\ B \in \mathcal{M}_n(\mathbb{C})$ be compact operators. Then for $j = 1, 2, \ldots,$ we have*

$$2s_j(A^*B) \leq s_j(AA^* + BB^*)$$

where $s_j(A),\ j = 1, 2, \ldots$ denote the singular values of A in increasing order.

Problem Formulation

Consider the second term of (2), which can be represented by

$$\dot{U}(t) = -p_1(t)U(t) + p_2(t)\phi(t)\phi^T(t), \quad U(0) = 0.$$

With slight abuse of notations for simplicity, the discrete counterpart can be written as

$$A^{k+1} = a_k A^k + b_k v_k v_k^T, \quad A^0 = 0, \tag{3}$$

for $a_k \in (0, 1],\ b_k \geq 0$, and $A^k := U(t_0 + k\Delta t),\ v_k := \phi(t_0 + k\Delta t)$.

THE PURPOSE is to design a_k and b_k

1. to increase the minimum eigenvalue of A^k as k increases, and
2. to bound, simultaneously, the maximum eigenvalue of A^k.

for given v_k at each step k.

Main Results

Let A be an n-by-n positive semidefinite matrix, and v be an n-dimensional real vector, and $\{\lambda_i(\cdot)\}$ be the eigenvalues of $(\cdot)$ ordered algebraically as $\lambda_{\max} = \lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_2 \geq \lambda_1 = \lambda_{\min}$.

Let Λ and X be the diagonal matrix of eigenvalues of A, and the corresponding matrix of eigenvectors, i.e.

$$A = X\Lambda X^T.$$

Finally, let

$$A' = aA + bvv^T,$$

which is an abbreviated form of (3).

Lemma 1. *If there exist $r \in [0,1]$ such that*

$$a \cdot \operatorname{ads} A + b\|v\|^2 \le r \cdot \operatorname{ads} A. \tag{4}$$

for some $a \in (0, r]$ and $b \ge 0$, then,

$$\operatorname{ads} A' \le r \cdot \operatorname{ads} A.$$

Proof. From Corollary 2 of [7],

$$\begin{aligned}
\operatorname{ads} A' &= \operatorname{ads}(aA + bvv^T) \\
&\le \operatorname{ads}(aA) + \operatorname{ads}(bvv^T) \\
&= a \operatorname{ads} A + b\|v\|^2 \\
&\le r \cdot \operatorname{ads} A
\end{aligned}$$

□

Lemma 2. *For all $a, b \ge 0$, and $j = 1, \ldots, n$,*

$$2\sqrt{ab}\lambda_j\Big(\Lambda^{1/2} \operatorname{diag}(X^T v)\Big) \le \lambda_j(A').$$

Proof. Observe that $A = X\Lambda^{1/2}(X\Lambda^{1/2})^T$ and $vv^T = XC(XC)^T$ where $C := \operatorname{diag}(X^T v)$. From Bhatia's theorem [1], we have

$$\begin{aligned}
\lambda_j(A') &= s_j\Big((\sqrt{a}X\Lambda^{1/2})(\sqrt{a}X\Lambda^{1/2})^T + (\sqrt{b}XC)(\sqrt{b}XC)^T\Big) \\
&\ge 2s_j\Big(\sqrt{ab}\Lambda^{1/2}X^T XC\Big) \\
&= 2\sqrt{ab}\lambda_j\Big(\Lambda^{1/2}C\Big).
\end{aligned}$$

□

Lemma 3. *For all $a, b \ge 0$,*

$$2\sqrt{ab}\Big\|\Lambda^{1/2}X^T v\Big\|_\infty \le \lambda_{\max}(A') \le a\lambda_{\max}(A) + b\|v\|^2. \tag{5}$$

Proof. The right inequality (5) is directly derived from Weyl's theorem [4] as

$$\lambda_n(A') \le a\lambda_n(A) + b\lambda_n(vv^T),$$

and the left inequality is from Lemma 2 for $j = n$. □

Now, we want to derive an algorithm that

1. increases the minimum eigenvalue,

$$\lambda_1(A') \ge \lambda_1(A),$$

2. and bounding the maximum eigenvalue as

$$\lambda_{\max}(A') \le \lambda_{\max}(A).$$

Given r, from Lemma 1, we have the following condition

$$(\operatorname{ads} A)a + \|V\|^2 b \leq r \cdot \operatorname{ads} A. \tag{6}$$

Moreover, from Lemma 3, we have following two conditions

$$\begin{aligned} ab &\geq \frac{\lambda_1(A) + r \cdot \operatorname{ads} A}{4\left\|\Lambda^{1/2} X^T v\right\|_\infty^2} \\ \lambda_n(A)a + \|v\|^2 b &\leq \lambda_n(A) \end{aligned} \tag{7}$$

Theorem 3. *Let A be an n-by-n positive semidefinite matrix with* $\operatorname{ads} A > 0$*, and v be an n-dimensional vector. Also, let $A' = aA + bvv^T$, for $a \in (0, 1]$ and $b \geq 0$. Suppose that there exists (a, b, r) such that*

$$a \leq r \leq 1, \tag{8a}$$

$$\operatorname{ads} A \cdot a + \|v\|^2 b \leq \operatorname{ads} A \cdot r, \tag{8b}$$

$$ab \leq \frac{(\lambda_1(A) + \operatorname{ads} A \cdot r)^2}{4\left\|\Lambda^{1/2} X^T v\right\|_\infty^2} \tag{8c}$$

Then,

$$\lambda_{\min}(A') \geq \lambda_{\min}(A), \tag{9a}$$

$$\lambda_{\max}(A') \leq \lambda_{\max}(A). \tag{9b}$$

Proof. From Lemma 1, and Lemma 3, we have

$$\begin{aligned} \lambda_1(A') &\geq \lambda_n(A') - \operatorname{ads} A \cdot r \\ &\geq 2\sqrt{ab}\left\|\Lambda^{1/2} X^T v\right\|_\infty - \operatorname{ads} A \cdot r. \end{aligned}$$

Since ab satisfies the condition (8c), $\lambda_1(A') \geq \lambda_1(A)$.

Now, from the second inequality of Lemma (3), and the condition (8a),

$$\begin{aligned} \lambda_n(A') &\leq a\lambda_n(A) + b\|v\|^2 \\ &= \operatorname{ads} A \cdot a + \|v\|^2 b + a\lambda_1(A) \\ &\leq \operatorname{ads} A \cdot r + a\lambda_1(A) \\ &= r\lambda_n(A) - (r - a)\lambda_1(A) \end{aligned}$$

From the condition (8b),

$$\lambda_n(A')$$

□

Another Approach

Theorem 4 (Ipsen, see [5]). *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian, $y \in \mathbb{C}^n$.*

1. *(smallest eigenvalue). Let*

$$L_\pm := \begin{pmatrix} \lambda_2(A) & 0 \\ 0 & \lambda_1(A) \end{pmatrix} \pm \begin{pmatrix} \|y_{2:n}\| \\ y_1 \end{pmatrix} \begin{pmatrix} \|y_{2:n}\| & \bar{y}_1 \end{pmatrix},$$
$$U_\pm := \begin{pmatrix} \lambda_2(A) & 0 \\ 0 & \lambda_1(A) \end{pmatrix} \pm \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \begin{pmatrix} \bar{y}_2 & \bar{y}_1 \end{pmatrix}.$$

Then

$$\lambda_{\min}(L_\pm) \le \lambda_{\min}(A \pm yy^*) \le \lambda_{\min}(U_\pm),$$

where

$$\lambda_{\min}(A) \le \lambda_{\min}(L_+) \le \lambda_{\min}(U_+) \le \lambda_2(A),$$
$$\lambda_{\min}(A) - \|y\|^2 \le \lambda_{\min}(L_-) \le \lambda_{\min}(U_-) \le \lambda_{\min}(A).$$

2. *(largest eigenvalue). Let*

$$L_\pm := \begin{pmatrix} \lambda_n(A) & 0 \\ 0 & \lambda_{n-1}(A) \end{pmatrix} \pm \begin{pmatrix} \|y_n\| \\ y_{n-1} \end{pmatrix} \begin{pmatrix} \bar{y}_n & \bar{y}_{n-1} \end{pmatrix},$$
$$U_\pm := \begin{pmatrix} \lambda_n(A) & 0 \\ 0 & \lambda_{n-1}(A) \end{pmatrix} \pm \begin{pmatrix} y_n \\ \|y_{1:n-1}\| \end{pmatrix} \begin{pmatrix} \bar{y}_n & \|y_{1:n-1}\| \end{pmatrix}.$$

Then

$$\lambda_{\max}(L_\pm) \le \lambda_{\max}(A \pm yy^*) \le \lambda_{\max}(U_\pm),$$

where

$$\lambda_{\max}(A) \le \lambda_{\max}(L_+) \le \lambda_{\max}(U_+) \le \lambda_{\max}(A) + \|y\|^2,$$
$$\lambda_{n-1}(A) \le \lambda_{\max}(L_-) \le \lambda_{\max}(U_-) \le \lambda_{\max}(A).$$

Lemma 4. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian, $v \in \mathbb{C}^n$, and $A' := aA + bvv^T$ for $a, b \ge 0$. Then,*

$$\lambda_{\min}(A') \ge a\lambda_1(A) + \frac{1}{2}\left(a\,\mathrm{gap}_1(A) + b\|v\|^2 - \sqrt{\left(a\,\mathrm{gap}_1(A) + b\|v\|^2\right)^2 - 4ab\,\mathrm{gap}_1(A)|v_1|^2} \right), \quad (10)$$

and

$$\lambda_{\max}(A') \le a\lambda_n(A) + \frac{1}{2}\left(-a\,\mathrm{gap}_n(A) + b\|v\|^2 + \sqrt{\left(a\,\mathrm{gap}_n(A) + b\|v\|^2\right)^2 - 4ab\,\mathrm{gap}_n(A)|v_{1:n-1}|^2} \right), \quad (11)$$

Proof. With the fact that $\lambda_i(aA) = a\lambda_i(A)$, and $\mathrm{gap}_i(aA) = a\,\mathrm{gap}_i(A)$, the proof directly follows Theorem 2.1, and Corollary 2.2 of [5]. □

For simplicity, we abbreviate $\lambda_i := \lambda_i(A)$, and $\mathrm{gap}_i := \mathrm{gap}_i(A)$. Define a function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$f(a,b) := a\lambda_1 + \frac{1}{2}\left(a\,\mathrm{gap}_1 + b\|v\|^2 - \sqrt{\left(a\,\mathrm{gap}_1 + b\|v\|^2\right)^2 - 4ab\,\mathrm{gap}_1\,|v_1|^2} \right). \quad (12)$$

Lemma 5. *$f(a,b)$ is a monotonically increasing function for each $a \geq 0$ and $b \geq 0$.*

Moreover, given a,

$$\lim_{b\to\infty} f(a,b) = a\left(\lambda_1(A) + \mathrm{gap}_1(A)\frac{|v_1|^2}{\|v\|^2}\right).$$

Lemma 6. *Suppose that there exist $a, b \geq 0$, such that $f(a,b) \geq \lambda_1(A)$. Then*

$$a \geq \frac{\lambda_1(A)}{\lambda_2(A)}. \tag{13}$$

Theorem 5. *Suppose that there exist $a \in (0,1]$ and $b \geq 0$ satisfying the following two hyperbolic inequalities*

$$a^2 + \frac{k_1\|v\|^2}{\lambda_1}ab - \frac{2\lambda_1 + \mathrm{gap}_1}{\lambda_1 + \mathrm{gap}_1}a - \frac{\|v\|^2}{\lambda_1 + \mathrm{gap}_1}b + \frac{\lambda_1}{\lambda_1 + \mathrm{gap}_1} \geq 0, \tag{14}$$

$$a^2 + \frac{k_n\|v\|^2}{\lambda_n}ab - \frac{2\lambda_n - \mathrm{gap}_n}{\lambda_n - \mathrm{gap}_n}a - \frac{\|v\|^2}{\lambda_n - \mathrm{gap}_n}b + \frac{\lambda_n}{\lambda_n - \mathrm{gap}_n} \geq 0, \tag{15}$$

where

$$\frac{\lambda_1}{\lambda_1 + \mathrm{gap}_1} \leq k_1 := \frac{\lambda_1 + \frac{|v_1|^2}{\|v\|^2}\mathrm{gap}_1}{\lambda_1 + \mathrm{gap}_1} \leq 1,$$

$$1 \leq k_n := \frac{\lambda_n - \frac{|v_n|^2}{\|v\|^2}\mathrm{gap}_n}{\lambda_n - \mathrm{gap}_n} \leq \frac{\lambda_n}{\lambda_n - \mathrm{gap}_n}.$$

Then,

$$\lambda_{\min}(aA + bvv^T) \geq \lambda_{\min}(A), \tag{16}$$

$$\lambda_{\max}(aA + bvv^T) \leq \lambda_{\max}(A). \tag{17}$$

Proof. The proof is direct result from Lemma 4, □

Corollary 2. *Suppose that $v \in \mathbb{R}^n$ satisfies*

$$\frac{|v_1|^2}{\lambda_{\min}(A)} > \frac{|v_n|^2}{\lambda_{\max}(A)}. \tag{18}$$

Then, there exist $a \in (0,1)$ and $b > 0$ such that

$$\lambda_{\min}(aA + bvv^T) > \lambda_{\min}(A), \tag{19}$$

$$\lambda_{\max}(aA + bvv^T) < \lambda_{\max}(A). \tag{20}$$

Proof. Let $f_1,\ f_n : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ be

$$f_1(a,b) := a^2 + \frac{k_1\|v\|^2}{\lambda_1}ab - \frac{2\lambda_1 + \mathrm{gap}_1}{\lambda_1 + \mathrm{gap}_1}a - \frac{\|v\|^2}{\lambda_1 + \mathrm{gap}_1}b + \frac{\lambda_1}{\lambda_1 + \mathrm{gap}_1},$$

$$f_n(a,b) := a^2 + \frac{k_n\|v\|^2}{\lambda_n}ab - \frac{2\lambda_n - \mathrm{gap}_n}{\lambda_n - \mathrm{gap}_n}a - \frac{\|v\|^2}{\lambda_n - \mathrm{gap}_n}b + \frac{\lambda_n}{\lambda_n - \mathrm{gap}_n}.$$

Note that $f_1(1,0) = f_n(c/\lambda_n, 0) = 0$, and

$$\nabla f_1(1,0) = \frac{\text{gap}_1}{\lambda_1 + \text{gap}_1}\begin{bmatrix}1 & \frac{|v_1|^2}{\lambda_1}\end{bmatrix}^T,$$
$$\nabla f_n(1,0) = -\frac{\text{gap}_n}{\lambda_n - \text{gap}_n}\begin{bmatrix}1 & \frac{|v_n|^2}{\lambda_n}\end{bmatrix}^T.$$

There exists a region $\mathcal{D} \subset [0,1] \times \mathbb{R}_+$ satisfying both $f_1(a,b) > 0$ and $f_n(a,b) > 0$ for $(a,b) \in \mathcal{D}$, if only

$$\nabla f_1(1,0) \times \nabla f_n(1,0) = \frac{\text{gap}_1 \, \text{gap}_n}{(\lambda_1 + \text{gap}_1)(\lambda_n - \text{gap}_n)}\left(\frac{|v_1|^2}{\lambda_1} - \frac{|v_n|^2}{\lambda_n}\right) > 0,$$

which completes the proof. □

Analytic Solution

For $\lambda_{\min}(A') \geq \lambda_1$ for all v, the following conditions are necessary:

$$\Big((a\lambda_2 - \lambda_1)|v_1|^2 - (1-a)\lambda_1\|v_{2:n}\|^2\Big)b - (1-a)(a\lambda_2 - \lambda_1)\lambda_1 \geq 0, \tag{21}$$
$$a = 1, \quad \forall v \quad \text{s.t.} \quad v_1 = 0. \tag{22}$$

asf

$$\Big(a\,\text{ads}_1 + b\|v\|^2\Big)^2 - 4ab\,\text{ads}_1\,|v_1|^2 \leq \Big(\Big(a\,\text{ads}_1 + b\|v\|^2\Big) - 2(1-a)\lambda_1\Big)^2$$
$$-4ab\,\text{ads}_1\,|v_1|^2 \leq -4(1-a)\lambda_1\Big(a\,\text{ads}_1 + b\|v\|^2\Big) + 4(1-a)^2\lambda_1^2$$
$$\Big(a\,\text{ads}_1\,|v_1|^2 - (1-a)\lambda_1\|v\|^2\Big)b - (1-a)\lambda_1\,\text{ads}_1\,a + (1-a)^2\lambda_1^2 \geq 0$$

References

[1] Rajendra Bhatia and Fuad Kittaneh. On the Singular Values of a Product of Operators. *SIAM Journal on Matrix Analysis and Applications*, 11(2):272–277, April 1990.

[2] N. Cho, H. Shin, Y. Kim, and A. Tsourdos. Composite Model Reference Adaptive Control with Parameter Convergence Under Finite Excitation. *IEEE Transactions on Automatic Control*, 63(3):811–818, March 2018.

[3] Girish Chowdhary, Maximilian Mühlegg, and Eric Johnson. Exponential parameter and tracking error convergence guarantees for adaptive controllers without persistency of excitation. *International Journal of Control*, 87(8):1583–1603, August 2014.

[4] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, NY, 2 edition edition, October 2012.

[5] I. C. F. Ipsen and B. Nadler. Refined Perturbation Bounds for Eigenvalues of Hermitian and Non-Hermitian Matrices. *SIAM Journal on Matrix Analysis and Applications*, 31(1):40–53, January 2009.

[6] E. Lavretsky. Combined/Composite Model Reference Adaptive Control. *IEEE Transactions on Automatic Control*, 54(11):2692–2697, November 2009.

[7] Jorma K Merikoski and Ravinder Kumar. Inequalities For Spreads Of Matrix Sums And Products. *Applied Mathematics E-Notes*, 4:150–159, 2004.

[8] Yongping Pan and Haoyong Yu. Composite learning robot control with guaranteed parameter convergence. *Automatica*, 89:398–406, March 2018.

[9] Jean-Jacques Slotine and Weiping Li. *Applied Nonlinear Control*. Pearson, Englewood Cliffs, N.J, 1991.