Contents

1 Infinite Dimensional Vector Spaces

1.3 Metric Spaces

1.3 Problem 1.

Show that a) the hamming distance, b) the Pythagorean distance and c) the Chebyshev distance each satisfy the triangle inequality.

a)

1.5 Hilber Spaces

1.5 Problem 1.

Show that

$$\langle f, g\rangle = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

with

$$\langle f, g\rangle = \int_a^b \bar{f}(x) g(x) \rho(x) \mathrm{d}x$$

Since we have that $f(x) = \sum_{k=1}^{\infty} u_k(x) c_k$ we can substitiute and get:

$$= \int_a^b \overline{\sum_{k=1}^{\infty} u_k(x) c_k} g(x) \rho(x) \mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x) \bar{c}_k g(x) \rho(x) \mathrm{d}x$$

$$= \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x) \bar{c}_k g(x) \rho(x) \mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k \int_a^b \bar{u}_k(x) g(x) \rho(x) \mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

1.5 Problem 2.

$$Tf(\omega,t)=\int_{-\infty}^{\infty}\bar{g}(x-t)e^{-i\omega x}f(x)\mathrm{d}x$$

$$\langle h_1,h_2\rangle=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\bar{h}_1(\omega,t)h_2(\omega,t)\mathrm{d}\omega\mathrm{d}t$$

Find a formula for: $\langle Tf_1,Tf_2\rangle$ in terms of

$$\int\limits_{-\infty}^{\infty}\bar{f}_1 f_2\mathrm{d}x$$

$$\langle Tf_1,Tf_2\rangle=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\overline{Tf_1}Tf_2\mathrm{d}\omega\mathrm{d}t=$$

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\overline{\left[\int_{-\infty}^{\infty}\bar{g}(x-t)e^{-i\omega x}f_1(x)\mathrm{d}x\right]}\left[\int_{-\infty}^{\infty}\bar{g}(x-t)e^{-i\omega x}f_2(x)\mathrm{d}x\right]\mathrm{d}\omega\mathrm{d}t$$

$$=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\overline{\bar{g}(x-t)e^{-i\omega x}f_1(x)}\bar{g}(y-t)e^{-i\omega y}f_2(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\omega\mathrm{d}t$$

We now return to Calc III and need to do a replacment of variables:

$$\begin{matrix}u=x-y, & v=y\\ x=u+v, & y=y\end{matrix}\ \text{which has deterimnate: } J=\left|\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\right|=1$$

$$=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\int\limits_{-\infty}^{\infty}e^{i\omega u}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

Subbing in the following identity for δ

$$\int\limits_{-\infty}^{\infty}e^{i\omega u}\mathrm{d}\omega=\delta(u)$$

we see:

$$=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\delta(u)\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\bar{g}(v-t) f_1(v)} \bar{g}(v-t) f_2(v) \mathrm{d}v \mathrm{d}t = \int_{-\infty}^{\infty} \bar{f}_1(v) f_2(v) \int_{-\infty}^{\infty} g(v-t) \bar{g}(v-t) \mathrm{d}t \mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \bar{f}_1(v) f_2(v) |g|^2 \mathrm{d}v = |g|^2 \int_{-\infty}^{\infty} \bar{f}_1 f_2 \mathrm{d}x = |g|^2 \langle f_1, f_2 \rangle$$

1.5 Problem 3.

i) Show that the set of functions

$$\left\{ \frac{\sin \pi(2\omega z - k)}{\pi(2\omega z - k)} = \operatorname{sinc}(2\omega z - k), \ k \in \mathbb{Z} \right\}$$

is an orthognal set satisfying:

$$\int_{-\infty}^{\infty} \operatorname{sinc}(2\omega z - k) \operatorname{sinc}(2\omega z - l) \mathrm{d}z = A\delta_{kl}$$

What is A? To show orthogonality follow this 3 step outline:

a)

$$\int_{-\pi}^{\pi} \delta_N \left(t - \frac{2\pi}{2N+1} k \right) \delta_N \left(t - \frac{2\pi}{2N+1} l \right) \mathrm{d}t = \frac{2N+1}{2\pi} \delta_{kl}$$

where

$$\delta_N(u) = \frac{1}{2\pi} \frac{\sin(N + \frac{1}{2})u}{\sin \frac{u}{2}} = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{inu}$$

b) Then rescaling the integration domain by using $z = \frac{N+\frac{1}{2}}{2\pi\omega} t$.

c) and finally going to the limit $N \to \infty$.

ii) This set of functions

$$\left\{ u_k = \frac{1}{\sqrt{A}} \operatorname{sinc}(2\omega z - k), \ k \in \mathbb{Z} \right\}$$

is not complete on L^2, but is complete on a specific subset. What is this subset, ie what property must a function $f(t)$ satisfy in order to be in this subset?

i) If we take $k = 0 = l$ then we can calculate A:

$$A = \int_{-\infty}^{\infty} \operatorname{sinc}(2\omega z)\operatorname{sinc}(2\omega z)\mathrm{d}z = \int_{-\infty}^{\infty} \operatorname{sinc}(2\omega z)^2 \mathrm{d}z = \int_{-\infty}^{\infty} \left(\frac{\sin \pi(2\omega z)}{\pi 2\omega z)}\right)^2 \mathrm{d}z$$

Now with $u = 2\pi\omega z$, $\mathrm{d}u = 2\pi\omega \mathrm{d}z$ we can change variables and see:

$$= \int_{-\infty}^{\infty} \left(\frac{\sin(u)}{u}\right)^2 \frac{1}{2\pi\omega}\mathrm{d}u = \frac{1}{2\pi\omega}\int_{-\infty}^{\infty} \left(\frac{\sin(u)}{u}\right)^2 \mathrm{d}u$$

Now we can apply integration by parts with

$$\begin{cases} u = \sin^2 x & \mathrm{d}u = 2\sin x\cos x = \sin(2x) \\ v = -x^{-1} & \mathrm{d}v = -x^{-2} \end{cases}$$

and arrive at:

$$= \frac{1}{2\pi\omega}\left[\frac{-\sin^2 x}{x}|_{\pm\infty} - \int_{-\infty}^{\infty} \frac{\sin(2x)}{-x}\mathrm{d}x\right] = \frac{1}{2\pi\omega}\left[\int_{-\infty}^{\infty} \frac{\sin(2x)}{x}\mathrm{d}x\right]$$

With another change of variables $2x \to x$, and noticing that $sinc(x)$ is an even function we get:

$$\frac{1}{2\pi\omega}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\mathrm{d}x = \frac{1}{\pi\omega}\int_0^{\infty} \frac{\sin(x)}{x}\mathrm{d}x = \frac{1}{\pi\omega}\frac{\pi}{2} = \frac{1}{2}\omega^{-1} = A$$

a) Let $u_k = t - \frac{2\pi}{2N+1}k$, then

$$\int_{-\pi}^{\pi} \delta_N\left(t - \frac{2\pi}{2N+1}k\right)\delta_N\left(t - \frac{2\pi}{2N+1}l\right)\mathrm{d}t = \int_{-\pi}^{\pi} \delta_N(u_k)\,\delta_N(u_l)\,\mathrm{d}t$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{n=-N}^{N} e^{inu_k}\right)\left(\frac{1}{2\pi}\sum_{n=-N}^{N} e^{inu_l}\right)\mathrm{d}t$$

$$= \int_{-\pi}^{\pi}\frac{1}{4\pi^2}\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N} e^{i(n_1u_k+n_2u_l)}\mathrm{d}t = \frac{1}{4\pi^2}\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}\int_{-\pi}^{\pi} e^{i(n_1u_k+n_2u_l)}\mathrm{d}t$$

Considering now just the integral:

$$\int_{-\pi}^{\pi} e^{i(n_1u_k+n_2u_l)}\mathrm{d}t = \int_{-\pi}^{\pi}\exp\left\{i\left(n_1\left[t - \frac{2\pi}{2N+1}k\right] + n_2\left[t - \frac{2\pi}{2N+1}l\right]\right)\right\}\mathrm{d}t$$

$$= \int_{-\pi}^{\pi}\exp\left\{i(n_1+n_2)t - i\left[n_1\frac{2\pi}{2N+1}k + n_2\frac{2\pi}{2N+1}l\right]\right\}\mathrm{d}t$$

If we let $C_{n_1n_2lk} = \exp\left\{-i\frac{2\pi}{2N+1}\left[n_1k + n_2l\right]\right\}$ then we see get:

$$= \frac{C_{n_1n_2lk}}{i(n_1+n_2)}\exp\{i(n_1+n_2)t\}\,|_{t=\pm\pi} = C_{n_1n_2lk}\delta_{-n_1,n_2}2\pi$$

Plugging this back into the above we get:

$$= \frac{1}{2\pi}\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N} C_{n_1 n_2 lk}\delta_{-n_1,n_2} = \frac{1}{2\pi}\sum_{n=-N}^{N} C_{n,-nlk} = \frac{1}{2\pi}\sum_{n=-N}^{N}\exp\left\{-in\frac{2\pi}{2N+1}\left[k-l\right]\right\}$$

$$= \frac{1}{2\pi}\sum_{n=-N}^{N}\exp\left\{-in\frac{2\pi}{2N+1}\left[k-l\right]\right\}$$

Clearly when $k = l$ then we get $\frac{1}{2\pi}\sum_{n=-N}^{N} 1 = \frac{2N+1}{2\pi}$. Now for $k \neq l$ we see that this is just

$$\delta_N(\frac{2\pi}{2N+1}(k-l)) = \frac{1}{2\pi}\frac{\sin(\pi(k-l))}{\sin\frac{\pi}{2N+1}(k-l)}$$

Notice that the denominator is never zero and that the top always is. Thus we arrive at:

$$\int_{-\pi}^{\pi}\delta_N\left(t-\frac{2\pi}{2N+1}k\right)\delta_N\left(t-\frac{2\pi}{2N+1}l\right)\mathrm{d}t = \frac{2N+1}{2\pi}\delta_{kl}$$

b) Now we wish to make the change of variable $z = \frac{N+\frac{1}{2}}{2\pi\omega}t$.

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}}\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-k)\right)\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-l)\right)\frac{4\pi\omega}{2N+1}\mathrm{d}z = \frac{2N+1}{2\pi}\delta_{kl}$$

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}}\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-k)\right)\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-l)\right)\mathrm{d}z = \frac{1}{2}\left(\frac{2N+1}{2\pi}\right)^2\omega^{-1}\delta_{kl}$$

$$\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-k)\right) = \frac{1}{2\pi}\frac{\sin\left((N+\frac{1}{2})\frac{2\pi}{2N+1}(2z\omega-k)\right)}{\sin\left(\frac{2\pi}{2N+1}(2z\omega-k)\right)/2}$$

$$= \frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-k)\right)}{\sin\left(\frac{1}{2N+1}\pi(2z\omega-k)\right)}$$

c) Now since we are taking the limit as $N\to\infty$ we can ignore all higher terms in the sin expansion and just leave the linear factor.

$$\sim \frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-k)\right)}{\frac{1}{2N+1}\pi(2z\omega-k)}$$

Putting this back into our integral equation yields:

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}}\frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-k)\right)}{\frac{1}{2N+1}\pi(2z\omega-k)}\frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-l)\right)}{\frac{1}{2N+1}\pi(2z\omega-l)}\mathrm{d}z = \frac{1}{2}\left(\frac{2N+1}{2\pi}\right)^2\omega^{-1}\delta_{kl}$$

With some simplificiation gives us:

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}}\frac{\sin\left(\pi(2z\omega-k)\right)}{\pi(2z\omega-k)}\frac{\sin\left(\pi(2z\omega-l)\right)}{\pi(2z\omega-l)}\mathrm{d}z = \frac{1}{2}\omega^{-1}\delta_{kl}$$

Now taking limits we get the desired equality and see that A was in fact $\frac{1}{2\omega}$

$$\int_{-\infty}^{\infty}\frac{\sin\left(\pi(2z\omega-k)\right)}{\pi(2z\omega-k)}\frac{\sin\left(\pi(2z\omega-l)\right)}{\pi(2z\omega-l)}\mathrm{d}z = \frac{1}{2}\omega^{-1}\delta_{kl}$$

Thus the set is in fact an orhtogonal set satisfying the above.

ii) Limited bandwith functions???? ************************************

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2 Chapter 2

2.1 Subsection 1

2.1 Problem 1.

Suppose $f(x) = f(x + 2\pi)\ \forall x$ is periodic with period 2π. Show

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x,\ \forall a \in \mathbb{R}$$

As all great math proofs, no words are needed just equalities and beautiful integrals. Let a be given then:

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_{2\pi}^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x+2\pi)\mathrm{d}x$$
$$= \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x$$

2.1 Problem 2.

Dirichelet Basis

$$W_{2N+1} = \mathrm{span}\left\{\frac{1}{\sqrt{2\pi}}e^{ikt}\right\}_{k=\pm N}$$

Consider the set

$$g_k(t) = \frac{2\pi}{2N+1}\delta_N(t - x_k) = \frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(t-k\pi/(N+\frac{1}{2})}$$

Show that

A) $B = \{g_k, k \in 1, 2, \cdots\}$ is linearly independent.

B) B spans W_{2N+1}

A) It suffices to notice that $g_k(x_l) = \delta_{kl}$. Thus we can see for any given k that g_k is independent of all the other's as $\sum_{k' \neq k} \lambda_{k'} g_{k'}(x_k) = 0$. Thus we can not have a non trivial linear relationship between the functions.

B) It is clear that $g_k(t) \in W_{2N+1}$ since each of the elements in its sum namely $e^{in(t-k\pi/(N+\frac{1}{2})}$ is just a multiple of e^{int} a basis element of W_{2N+1}. Notice there are $2N+1$ of these independnet vectors in the vector space of dimmension $2N+1$. Thus they must be a spanning set and there must exist coefficents for any function in the space to be written as a sum of this basis.

To actually exhibit coefficents one would use $f(t) = \sum_k f(x_k) g_k(t)$.

2.1 Problem 3.

Riemann-Lebesgue Lemma
$G(u)$ piecewise continuous and has left and right derivatives on $[0, 2\pi]$. Show that

$$\lim_{N\to\infty} \int_0^{2\pi} G(u) \sin(N + \frac{1}{2}) u du = 0$$

WLOG $\exists a, b \in [0, \pi]$ st. $\forall x \in [a, b]$ $G(x) > 0$ or $G(x) < 0$.

Now it suffices to show

$$\lim_{N\to\infty} \int_a^b G(u) \sin(N + \frac{1}{2}) u du = 0$$

[since the interval $[0, 2\pi]$ can be sliced into a countable number of these intervals, and hten you can sum over them] WLOG we assume $G(x)$ is positve.

$$0 \leqslant \lim_{N\to\infty} \int_a^b G(u) \sin(N + \frac{1}{2}) u du \leqslant \lim_{N\to\infty} \int_a^b [\max_u G(u)] \sin(N + \frac{1}{2}) u du$$

Let G_m be the max above, then we have

$$0 \leqslant G_m \lim_{N\to\infty} \int_a^b \sin(N + \frac{1}{2}) u du = G_m \lim_{N\to\infty} \frac{\cos(N + \frac{1}{2}) u}{N + \frac{1}{2}} |_a^b \leqslant G_m \lim_{N\to\infty} \frac{2}{N + \frac{1}{2}} \leqslant 0$$

Thus we get the 0 value for the limit as desired.

2.1 Problem 4.

Prove or disprove:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2+a^2} = \frac{\pi}{a} \frac{\coth \pi a}{\cos^2 \pi x + \sin^2 \pi x \coth^2 \pi a}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2} = \frac{\pi^2}{\sin^2 \pi x}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m+x)^2} = \frac{1}{4\sin^2 x/2}$$

Let $f(m) = \frac{1}{(m+x)^2+a^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int\limits_{-\infty}^{\infty} e^{-ikm} f(m) \mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int\limits_{-\infty}^{\infty} e^{-ikm} \frac{1}{(m+x)^2+a^2} \mathrm{d}m = e^{ikx} \int\limits_{-\infty}^{\infty} e^{-iku} \frac{1}{u^2+a^2} \mathrm{d}u = e^{ikx} \frac{\pi}{a} e^{-|k|a}$$

Using the Poisson formula we thus see:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2+a^2} = \sum_{k=-\infty}^{\infty} e^{i2\pi kx} \frac{\pi}{a} e^{-|2\pi k|a} = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi kx} e^{-|2\pi k|a} + \sum k \leqslant 0 e^{i2\pi kx} e^{-|2\pi k|a} - 1\right]$$

$$= \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + \sum k \leqslant 0 e^{i2\pi k(x+a)} - 1\right] = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + e^{-i2\pi k(x+a)} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1}{1-e^{-i2\pi(x+a)}} + \frac{1}{1-e^{i2\pi(x-a)}} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{-i2\pi(x+a)} + 1 - e^{i2\pi(x-a)} - \left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{i2\pi(x-a)}e^{-i2\pi(x+a)}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{1-e^{-i4\pi a}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

************BELIEVE THE ABOVE DISPROVES THE SUM********

For the last 2 the answers are somewhat lack luster.

Let $f(m) = \frac{1}{(m+x)^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int\limits_{-\infty}^{\infty} e^{-ikm} f(m)\mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int\limits_{-\infty}^{\infty} e^{-ikm}\frac{1}{(m+x)^2}\mathrm{d}m = e^{ikx}\int\limits_{-\infty}^{\infty} e^{-iku}\frac{1}{u^2}\mathrm{d}u$$

This is an error function producing integral. This is bad. Notice also that the claimed value on the right hand side of the equation doesn't even make sense for $x = 0$. Thus the equation as stated is clearly false, Besides as any good student of Bergelson knows: $\sum 1/n^2 = \frac{\pi^2}{6}$

For the final one we again have issues with $\sin(0) = 0$ in the denominator.

2.1 Problem 5.

> My man Stephane G. Mallat claims the following: The family of funcitons $\phi(x-k)k = 0, \pm 1, \pm 2, \cdots$ is orthonormal iff
>
> $$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$
>
> is constant wrt ω. Prove my boy wrong or right.

Stephane is no chump and said a true thing. Lets investigate the sum:

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}(\omega + 2\pi k)}\hat{\phi}(\omega + 2\pi k)$$

Now to avoid a factor out front the rest of the analysis, the $\frac{1}{\sqrt{2\pi}}$ is suppressed when expanding the Fourier transform.

$$= \sum_{k=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{i(\omega+2\pi k)y}\bar{\phi}(y)\mathrm{d}y \int\limits_{-\infty}^{\infty} e^{-i(\omega+2\pi k)x}\phi(x)\mathrm{d}x$$

$$= \sum_{k=-\infty}^{\infty} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi(x)\bar{\phi}(y)e^{i(\omega+2\pi k)y}e^{-i(\omega+2\pi k)x}\mathrm{d}x\mathrm{d}y$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi(x)\bar{\phi}(y)\sum_{k=-\infty}^{\infty} e^{-i(\omega+2\pi k)(x-y)}\mathrm{d}x\mathrm{d}y$$

Via formula on page 62

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x)\bar{\phi}(y)e^{-i\omega(x-y)} \sum_{k=-\infty}^{\infty} \delta(x-y-k)\mathrm{d}x\mathrm{d}y$$

Now for the change of variables $u = x - y, v = y$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(u+v)\bar{\phi}(v)e^{-i\omega u} \sum_{k=-\infty}^{\infty} \delta(u-k)\mathrm{d}u\mathrm{d}v = \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega u}\phi(u+v)\delta(u-k)\mathrm{d}u\mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} e^{-i\omega k}\phi(k+v)\mathrm{d}v = \sum_{k=-\infty}^{\infty} e^{-i\omega u} \int_{-\infty}^{\infty} \bar{\phi}(v)\phi(k+v)\mathrm{d}v = \sum_{k=-\infty}^{\infty} \langle\phi(v), \phi(k+v)\rangle e^{-i\omega u}$$

Thus the sum above is in fact a Fourier series with $c_k = \langle\phi(v), \phi(k+v)\rangle$. Now this series being constnat is equivalent to $c_k = \delta_{0k}$, which is equivalent to the $\phi(v+k)$'s being an orthogonal system.

Moreover if $c_0 = 1$ then we have an orthonormal system aswell. Thus the system is orthonormal if the series is constant and equal to 1. Now in actuallity we rememeber that we have a secret factor of $\frac{1}{2\pi}$ hanging around. Thus the constants value is actually that.

2.1 Problem 6.

Prove or Disprove the following identities:

i)

$$\sum_{m=-\infty}^{\infty} f([2m+1]\pi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n F(n)$$

ii)

$$2\pi\sum_{m=-\infty}^{\infty}(-1)^m f(2\pi m) = \sum_{n=-\infty}^{\infty} F(n+\frac{1}{2})$$

iii)

$$\sum_{m=-\infty}^{\infty}\delta(u-[2m+1]\pi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n e^{inu}$$

iv) And in greater generality

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \frac{a}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n F(na)$$

v)

$$\sum_{m=-\infty}^{\infty}\frac{1}{|a|}\delta(u-\frac{[2m+1]\pi}{a}) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n e^{inau}$$

The main equation to keep in mind here is the general Poisson formula:

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{in(x-t)}f(t)\mathrm{d}t = \sum_{m=-\infty}^{\infty} f(x+2\pi m)$$

i) Begin with $x=\pi$ in the formula above and we see:

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{in(\pi-t)}f(t)\mathrm{d}t = \sum_{m=-\infty}^{\infty} f([2m+1]\pi)$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty} e^{-int}f(t)\mathrm{d}t = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n F(n)$$

ii) Begining with the right hand side:

$$\sum_{n=-\infty}^{\infty} F(n+\frac{1}{2}) = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(n+\frac{1}{2})t}f(t)\mathrm{d}t = \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-int}f(t)e^{-it/2}\mathrm{d}t$$

Now we notice this is the Fourier transform of not f but of $f(t)e^{-it/2}$, applying Poisson sum with this:

$$= 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m)e^{-i(2\pi m)/2} = 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m)e^{-i\pi m} = 2\pi \sum_{m=-\infty}^{\infty} (-1)^m f(2\pi m)$$

iii) It is straightforward to see that this is actually just 1) but in the supressed function notation. To see this we note that

$$\delta(u - [2m+1]\pi) \to f([2m+1]\pi), \quad e^{inu} \to \int_{-\infty}^{\infty} e^{inu} f(u)\mathrm{d}u = F(-n)$$

But wait we get $\sum_{n=-\infty}^{\infty}(-1)^n F(-n)$ and not the exact sum we wanted! Thankfully $(-1)^n = (-1)^{-n}$ and we just switch the order of the sum and get the identity.

iv) Let $\bar{f}(x) = f(\frac{x}{a})$, then by 1) we have:

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \sum_{m=-\infty}^{\infty} \bar{f}([2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \bar{F}(n)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} \bar{f}(t)\mathrm{d}t = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} f(t/a)\mathrm{d}t$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n a \int_{-\infty}^{\infty} e^{-inau} f(u)\mathrm{d}u = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n a F(na) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

v) Similar to 3) we note that this is just an earlier identity. A constant is shifted around but this is basicly just 4).

2.2 Dirac Delta Distribution

2.2 Problem 1.

Show that

$$\lim_{\omega \to \infty} \frac{\sin 2\pi\omega x}{\pi x}, \ \omega > 0$$

is a representation of the Dirac δ DISTRIBUTION.

This equality can only be expressed inside of an integral, thus we must apply the above to test functions and see that the answer is the same as with the delta *distribution*.

Thus if we consider f continuous on some $[-a,a]$ then we get:

$$\lim_{\omega\to\infty}\int_{-a}^{a}\delta_\omega(x)f(x)\mathrm{d}x = \lim_{\omega\to\infty}\int_{-a}^{a}\frac{\sin 2\pi\omega x}{\pi x}f(x)\mathrm{d}x$$

Now in following with the style of the Fourier series theorem we add and subtract the same term, namely a $f(0)$ (inside some paranthesis but basicly the same)

$$= \lim_{\omega\to\infty}\int_{-a}^{a}\frac{\sin 2\pi\omega x}{\pi x}\left(f(x)-f(0)+f(0)\right)\mathrm{d}x = \lim_{\omega\to\infty}\int_{-a}^{a}\frac{\sin 2\pi\omega x}{\pi}\frac{f(x)-f(0)}{x}+f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

Now we can consider WLOG just the positive side of the integral.

$$\lim_{\omega\to\infty}\int_{0}^{a}\frac{\sin 2\pi\omega x}{\pi}\frac{f(x)-f(0)}{x}+f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

Notice for the exact same resonaning as $G(u)$ on page 57 that we get $\frac{f(x)-f(0)}{x}$ is continuous at 0 and converges to $f'(0^+)$. Thus again we see that the integral:

$$\int_{0}^{a}\frac{\sin 2\pi\omega x}{\pi}\frac{f(x)-f(0)}{x}\mathrm{d}x \to 0$$

as $\omega\to\infty$. Thus we only have:

$$\lim_{\omega\to\infty}\int_{0}^{a}f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x = f(0)\lim_{\omega\to\infty}\int_{0}^{a}\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

$$y = 2\pi\omega x, \quad dy = 2\pi\omega dx$$

$$= f(0)\lim_{\omega\to\infty}\int_{0}^{a2\pi\omega}\frac{\sin y}{\pi y/(2\pi\omega)}\frac{\mathrm{d}y}{2\pi\omega} = \frac{f(0)}{\pi}\int_{0}^{\infty}\frac{\sin y}{y}\mathrm{d}y = \frac{f(0)}{\pi}\frac{\pi}{2} = \frac{f(0)}{2}$$

Using a isomorphic version of the logic above one can get the $f(0^-)$ term and complete the proof.

2.2 Problem 2.

Assuming that $f(x)$ is nearly linear, that is to say that

$$f(-a) = f(0) - af'(0) + \text{ H.O.T.}$$

Show that

$$I = \int\limits_{-\infty}^{\infty}\delta(x+a)f(x)\mathrm{d}x$$

can be evaluated by means of the formal equation:

$$\delta(x+a) = \delta(x) + a\delta'(xx)$$

By the definition of the δ funciton we have:

$$I = \int\limits_{-\infty}^{\infty} \delta(x+a)f(x)\mathrm{d}x = f(-a)$$

$$= f(0) - af'(0) + \text{ H.O.T.} = \int\limits_{-\infty}^{\infty} \delta(x)f(x) - a\delta(x)f'(x)\mathrm{d}x = \int\limits_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x - a\int\limits_{-\infty}^{\infty} \delta(x)f'(x)\mathrm{d}x$$

Via integration by parts we know that:

$$\int\limits_{-\infty}^{\infty} \delta(x)f'(x)\mathrm{d}x = \delta(x)f(x)|_{\pm\infty} - \int\limits_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = -\int\limits_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x$$

Putting stuff together:

$$= \int\limits_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x + a\int\limits_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = \int\limits_{-\infty}^{\infty} \left[\delta(x) + a\delta'(x)\right] f(x)\mathrm{d}x$$

Thus it makes some sense to claim $\delta(x+a) = \delta(x) + a\delta'(x)$

2.3 The Fourier Integral

2.3 Problem 1.

a) Consider the LInear Operator $\mathfrak{F}^2$ and its eigenvalue equatio

$$\mathfrak{F}^2 f = \lambda f$$

What are the eigenvalues and eigenfunctions of $\mathfrak{F}^2$?

b) Same with $\mathfrak{F}^4$?

c) Same with $\mathfrak{F}$?

For the sake of clarity:

$$\mathfrak{F}(f) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ikx} f(x)\mathrm{d}x$$

a) Obviously we begin with the calculation in question

$$\mathfrak{F}^2(f)[k] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky}\int_{-\infty}^{\infty} e^{-iyx}f(x)\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(ky+yx)}f(x)\mathrm{d}x\mathrm{d}y$$

This looks close to the idenity we are given on page 70, namely:

$$\delta(x-t) = \int_{-\infty}^{\infty}\frac{1}{2\pi}e^{ik(x-t)}dk$$

which carries the note that we must inegrate on the outside with dt for this to make sense. Now rearranging some integrals and swapping x with $-x$ we arive at:

$$\int_{-\infty}^{\infty} f(-x)\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-iy(k-x)}\mathrm{d}y\mathrm{d}x = \int_{-\infty}^{\infty} f(-x)\delta(k-x)\mathrm{d}x = f(-k)$$

Now we can see that the constraint: $\mathfrak{F}^2 f = \lambda f$ is really just $f(-x) = \lambda f(x)$. Two obvious cases come to mind, namely even and odd functions for the eigenvalues ± 1. For any other value of λ one could apply the relation twice to get $f(x) = \lambda^2 f(-x)$ which only has 2 roots. Thus those are the only eigenvalues of $\mathfrak{F}^2$.

b) Thanks to $\mathfrak{F}^2 f[x] = f(-x)$ from the previous problem we know that $\mathfrak{F}^4 = \mathfrak{F}^2\mathfrak{F}^2 f[x] = f(-(-x)) = f(x)$. Thus every function is an eigen function of $\mathfrak{F}^4$ =Id, with eigenvalue 1.

c) We know that $\mathfrak{F}^4$ =Id, and thus if λ is an eigenvalue of $\mathfrak{F}$ then $\lambda^4 = 1$. Thus the only possible eigenvalules of $\mathfrak{F}$ are 4th roots of unity. Thus the eigenvalues are $\pm 1, \pm i$.

2.3 Problem 2.

Let

$$W = \text{span}\{\phi, \mathfrak{F}\phi, \mathfrak{F}^2\phi, \dots\}$$

a) Show that W is finite dimensional, and what is its dimension?

b) Exhibit a basis for W.

c) It is evident that $\mathfrak{F}$ is a unitary transform of W. Find the bassis representation matrx $[\mathfrak{F}]_B$ relative to the basis B found in part b).

d) Find the secular determinant, the eigenvalues and the corresponding eigenvectors of $[\mathfrak{F}]_B$.

e) For W exhibit an alternative basis which consists entirely of eigenvectors of $\mathfrak{F}$, each one labelled by its respective eigenvalue.

f) What can you say about the eigenvalues of $\mathfrak{F}$ as a transformation on L^2 as compared to $[\mathfrak{F}]_B$ which acts ona finite dim. vector space

a) W clearly has dimension $\leqslant 4$ by the previous problem since $\mathfrak{F}^4$ =Id. In fact if ϕ is even, we have only dimension 2 and the two basis elements of the space are just ϕ and $\mathfrak{F}\phi$. Due to the limits of the roots of unity argument above we know that the only number of dimensions can be those two or 1, namely dim = 1,2, or 4.

b) The possible basis are ϕ, or $\phi, \mathfrak{F}\phi$, or all 4: $\phi, \mathfrak{F}\phi, \mathfrak{F}^2\phi, \mathfrak{F}^{-1}\phi$.

c) With the basis: $\phi, \mathfrak{F}\phi, \mathfrak{F}^2\phi, \mathfrak{F}^{-1}\phi$

$$[\mathfrak{F}]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

A classic style of shifting operator on a finite dimensional space.

d) Easily enough we see:

$$\det\left([\mathfrak{F}]_B - \lambda I\right) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}\right) = \lambda^4 - 1$$

Thus the eigenvalues are $\pm 1, \pm i$. The corresponding vectors are listed below:

$$\begin{array}{ll} \lambda = 1 & \phi + \mathfrak{F}\phi + \mathfrak{F}^2\phi + \mathfrak{F}^{-1}\phi \\ \lambda = -1 & \phi - \mathfrak{F}\phi + \mathfrak{F}^2\phi - \mathfrak{F}^{-1}\phi \\ \lambda = i & \phi + i\mathfrak{F}\phi - \mathfrak{F}^2\phi - i\mathfrak{F}^{-1}\phi \\ \lambda = -i & \phi - i\mathfrak{F}\phi - \mathfrak{F}^2\phi + i\mathfrak{F}^{-1}\phi \end{array}$$

(Something something permutation matricies)

e) The eigenvalues of $\mathfrak{F}$ are the same viewed as a finite-dimensional vecotr space and as an infinte dimensional one. This seems to have been forced by the simpliity of the characterisitc polynomial more than anything else.

2.3 Problem 3.

Define the equivalent width as

$$\Delta_t = \left| \frac{\int\limits_{-\infty}^{\infty} f(t)\mathrm{d}t}{f(0)} \right|$$

Define the equivalent Fourier width as

$$\Delta_\omega = \left| \frac{\int\limits_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t}{\hat{f}(0)} \right|$$

a) Show that $\Delta_t\Delta_\omega$ =const, is independent of the function f, and find its value.

b) Determine the equivalent width and Fourier width of

$$e^{-x^2/2b^2}$$

and compare them with its full width as defined by its inflection points.

a)

$$\Delta_t\Delta_\omega = \left| \frac{\int\limits_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t}{\hat{f}(0)} \right| \left| \frac{\int\limits_{-\infty}^{\infty} f(t)\mathrm{d}t}{f(0)} \right| = \left| \frac{\int\limits_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t \int\limits_{-\infty}^{\infty} f(t)\mathrm{d}t}{\hat{f}(0)f(0)} \right|$$

$$= \left| \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \hat{f}(x)f(y)\mathrm{d}x\mathrm{d}y}{\hat{f}(0)f(0)} \right| = \left| \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{2\pi}}e^{-ixz}f(z)\mathrm{d}z\mathrm{d}x\mathrm{d}y}{\hat{f}(0)f(0)} \right| =$$

$$\left| \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(y)f(z)\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-ixz}\mathrm{d}x\mathrm{d}z\mathrm{d}y}{\hat{f}(0)f(0)} \right| = \left| \frac{\sqrt{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(y)f(z)\delta(z)\mathrm{d}z\mathrm{d}y}{\hat{f}(0)f(0)} \right| = \sqrt{2\pi}\left| \frac{\int\limits_{-\infty}^{\infty} f(y)\mathrm{d}y}{\hat{f}(0)} \right|$$

$$= \sqrt{2\pi}\left|\frac{\int\limits_{-\infty}^{\infty} f(y)\mathrm{d}y}{\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i*0*x}f(y)\mathrm{d}y}\right| = \sqrt{2\pi}\sqrt{2\pi} = 2\pi$$

b) With $f(x) = e^{-x^2/2b^2}$

Since I stared at completing the squares for way too long to justify not writing this down, here is the Fourier transform of the Gaussian:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-x^2/2b^2}e^{-i\omega x}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-1/2b^2\left[x^2+2b^2i\omega x\right]}\mathrm{d}x$$

We complete the square in the exponent:

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-1/2b^2\left[x^2+2b^2i\omega x-\omega^2b^4\right]-\omega^2b^2/2}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}e^{-\omega^2b^2/2}\int\limits_{-\infty}^{\infty} e^{-1/2b^2\left[x-\omega b^2\right]^2}\mathrm{d}x$$

Now with $|b|u = x - \omega b^2, |b|\mathrm{d}u = \mathrm{d}x$ we get:

$$= |b|e^{-\omega^2b^2/2}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\mathrm{d}u = |b|e^{-\omega^2b^2/2}$$

So we have: $\hat{f}(\omega) = |b|e^{-\omega^2b^2/2}$ and can now do our calculations. (we could before and actually don't need this at all but I'll be damned if I didn't spend too much time on this part to not just write something)

The Δ_ω is actually the famous Gaussian integral:

$$\Delta_\omega = \left|\frac{\int\limits_{-\infty}^{\infty} e^{-t^2/2b^2}\mathrm{d}t}{1}\right| = \left|\int\limits_{-\infty}^{\infty} e^{-t^2/2b^2}\mathrm{d}t\right| = \sqrt{\frac{\pi}{1/2b^2}} = \sqrt{\pi 2b^2} = |b|\sqrt{\pi 2}$$

Thanks to the relation $\Delta_t\Delta_\omega = 2\pi$ we see that Δ_t must be $1/|b|\sqrt{2\pi}$.

I am too stuburn to not write this after the above

$$\Delta_t = \left|\frac{\int\limits_{-\infty}^{\infty}\hat{f}(t)\mathrm{d}t}{\hat{f}(0)}\right| = \left|\frac{\int\limits_{-\infty}^{\infty}|b|e^{-\omega^2b^2/2}\mathrm{d}\omega}{|b|}\right| = \left|\int\limits_{-\infty}^{\infty} e^{-\omega^2b^2/2}\mathrm{d}\omega\right|$$

$$= \left|\int\limits_{-\infty}^{\infty} e^{-\omega^2b^2/2}\mathrm{d}\omega\right| = \sqrt{2\pi/b^2} = \frac{1}{|b|}\sqrt{2\pi}$$

The inflection points are at $\pm b$ and thus its 'inflection witdth' is $2|b|$

2.3 Problem 4.

Define the auto-correlation h of the funciton f:

$$h(y) := \int\limits_{-\infty}^{\infty} f(x)f(x-y)\mathrm{d}x$$

Compute the Fourier transform of the auto correlation funcation and show that it equals the "spectral intensity" (aka power spectrum) of f whenever f is real valued.

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-iky} \int\limits_{-\infty}^{\infty} f(x)f(x-y)\mathrm{d}x\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-iky} f(x)f(x-y)\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(x) \int\limits_{-\infty}^{\infty} e^{-ik(x-u)} f(u)\mathrm{d}u\mathrm{d}x = \int\limits_{-\infty}^{\infty} e^{-ikx} f(x)\hat{f}(-k)\mathrm{d}x = \hat{f}(-k) \int\limits_{-\infty}^{\infty} e^{-ikx} f(x)\mathrm{d}x$$

$$= \hat{f}(-k)\hat{f}(k) = |\hat{f}(k)|^2$$

The $\hat{f}(-k) = \overline{\hat{f}(k)}$ is implied by f being real valued and is the only point we make use of this fact.

2.3 Problem 5.

a) Compute the total energy

$$\int\limits_{-\infty}^{\infty} |h(T)|^2 \mathrm{d}T$$

of the cross correlation $h(T)$ in terms of the Fourier amplitudes of f_0 and f.

b) Consider

$$h_k(T) = \frac{\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_k(t)\mathrm{d}t}{\left[\int\limits_{-\infty}^{\infty} |f_k(t)|^2\mathrm{d}t\right]^{1/2}}$$

i) Show that $h_0(t)$ is the peak intensity, ie $|h_k(T)|^2 \leqslant |h_0(T)|^2$.

ii) Show that equality holds if $f_k(t) = \kappa f_0(t)$ for κ some constant.

a) For a matchd filter we have that $\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f(t)\mathrm{d}t = h(T)$, using this: Using the fact that $\mathfrak{F}$ is an isometery of L^2:

$$\int\limits_{-\infty}^{\infty} |h(T)|^2 \mathrm{d}T = \|h\|_2 = \left\|\hat{h}\right\|_2 = \int\limits_{-\infty}^{\infty} |\mathfrak{F}h(\omega)|^2 \mathrm{d}\omega$$

$$= \int\limits_{-\infty}^{\infty} |\int\limits_{-\infty}^{\infty} e^{-iT\omega} h(T)\mathrm{d}T|^2 \mathrm{d}\omega = \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-iT\omega} \int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f(t)\mathrm{d}t\mathrm{d}T|^2 \mathrm{d}\omega$$

$$= \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) \int\limits_{-\infty}^{\infty} e^{-iT\omega} \bar{f}_0(t-T)\mathrm{d}T\mathrm{d}t|^2 \mathrm{d}\omega$$

Let $u = t - T,\ \mathrm{d}u = -\mathrm{d}T$ (the negative sign is lost in the $||$).

$$= \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) \int\limits_{-\infty}^{\infty} e^{-i(t-u)\omega} \bar{f}_0(u)\mathrm{d}u\mathrm{d}t|^2 \mathrm{d}\omega = \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-it\omega} f(t) \int\limits_{-\infty}^{\infty} e^{iu\omega} \bar{f}_0(u)\mathrm{d}u\mathrm{d}t|^2 \mathrm{d}\omega$$

Thanks to the relationship between conjugates and the fourier transform, namely $\mathfrak{F}\bar{f}[k] = \mathfrak{F}f[-k]$ we get:

$$= \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \hat{f}_0(\omega) \int\limits_{-\infty}^{\infty} e^{-it\omega} f(t)\mathrm{d}t|^2 \mathrm{d}\omega = \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \hat{f}_0(\omega)\hat{f}(\omega)|^2 \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |\hat{f}_0(\omega)\hat{f}(\omega)|^2 \mathrm{d}\omega$$

b) i) This problem is actually wrong, consider $f_0(t) = \mathbb{1}_{[0,1]}(t)$ and $f_1(t) = \mathbb{1}_{[2,3]}(t)$ and $T = 2$. Then $\|f_1\|_2 = 1 = \|f_0\|$ but

$$\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_0(t)\mathrm{d}t = 0$$

$$\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_k(t)\mathrm{d}t = 1$$

Which gives $h_1(T) > h_0(T)$, contrary to the statment of the problem.
*********************MESSGE GERLACH***************************

ii) Oddly enough the equality still holds. If we have $f_k(t) = \kappa f_0(t)$ then we see:

$$|h_k(T)|^2 = \left\| \frac{\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)\kappa f_0(t)\mathrm{d}t}{\left[\int\limits_{-\infty}^{\infty} |\kappa f_0(t)|^2 \mathrm{d}t\right]^{1/2}} \right\|^2 = \left\| \frac{\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_0(t)\mathrm{d}t}{\left[\int\limits_{-\infty}^{\infty} |f_0(t)|^2 \mathrm{d}t\right]^{1/2}} \right\|^2 = |h_0(T)|^2$$

2.3 Problem 6.

What functions are eigenvectors of $\mathfrak{F}^2$ with eigenvalue $\lambda = 1$?

Already did this.

2.3 Problem 7.

Let $\hat{g}(k) = \mathfrak{F}[g(x)](k)$ and $H(k) = \mathfrak{F}[h](k)$ be the Fourier transforms of g and h. Express the following in terms of $\hat{g}$ and $\hat{f}$.

i) $\mathfrak{F}[\alpha g + \beta h]$ for some constants α, β

ii) $\mathfrak{F}[g(x - \xi)]$

iii) $\mathfrak{F}[e^{ik_0 x} g]$

iv) $\mathfrak{F}[g(ax)]$

v) $\mathfrak{F}[\frac{dg}{dx}]$

vi) $\mathfrak{F}[xg(x)]$

i) It is clear by the linearity of the integrals that we have:

$$\mathfrak{F}[\alpha g + \beta h] = \mathfrak{F}[\alpha g] + \mathfrak{F}[\beta h] = \alpha \mathfrak{F}[g] + \beta \mathfrak{F}[h]$$

ii)

$$\mathfrak{F}[g(x - \xi)] = \int_{-\infty}^{\infty} e^{-ikx} g(x - \xi) \mathrm{d}x = \int_{-\infty}^{\infty} e^{-ik(u+\xi)} g(u) \mathrm{d}u = e^{-ik(\xi)} \int_{-\infty}^{\infty} e^{-iku} g(u) \mathrm{d}u = e^{-ik(\xi)} \hat{g}$$

iii)

$$\mathfrak{F}[e^{ik_0 x} g] = \int_{-\infty}^{\infty} e^{-ikx} e^{ik_0 x} g(x) \mathrm{d}x = \int_{-\infty}^{\infty} e^{-i(k - k_0)x} g(x) \mathrm{d}x = \hat{g}(k - k_0)$$

iv)

$$\mathfrak{F}[g(ax)] = \int_{-\infty}^{\infty} e^{-ikx} g(ax) \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{a} e^{-iku/a} g(u) \mathrm{d}u = \frac{1}{a} \hat{g}(k/a)$$

v) Using integration by parts:

$$\mathfrak{F}[\frac{dg}{dx}] = \int_{-\infty}^{\infty} e^{-ikx}\frac{dg}{dx}\mathrm{d}x = -\int_{-\infty}^{\infty} g\frac{de^{-ikx}}{dx}\mathrm{d}x = \int_{-\infty}^{\infty} ikge^{-ikx}\mathrm{d}x = ik\hat{g}(k)$$

vi)

$$\mathfrak{F}[xg(x)] = \int_{-\infty}^{\infty} e^{-ikx}xg\mathrm{d}x = \int_{-\infty}^{\infty} xe^{-ikx}g\mathrm{d}x = \int_{-\infty}^{\infty}\left[\frac{1}{-i}\frac{d}{dk}e^{-ikx}\right]g\mathrm{d}x$$

$$= \frac{1}{-i}\frac{d}{dk}\int_{-\infty}^{\infty} e^{-ikx}g\mathrm{d}x = \frac{1}{-i}\frac{d}{dk}\hat{g}(k)$$

**************************DOUBLE CHECK LAST COUPLE

To make life easier to see, eveyrthin in a box gives:

i) $\mathfrak{F}[\alpha g + \beta h] = \alpha\mathfrak{F}g + \beta\mathfrak{F}h$ for some constants α, β

ii) $\mathfrak{F}[g(x-\xi)] = e^{ik\xi}\mathfrak{F}g[k]$

iii) $\mathfrak{F}[e^{ik_0x}g] = \mathfrak{F}g[k-k_0]$

iv) $\mathfrak{F}[g(ax)] = \frac{1}{a}\hat{g}(k/a)$

v) $\mathfrak{F}[\frac{dg}{dx}] = ik\mathfrak{F}g$

vi) $\mathfrak{F}[xg(x)] = \frac{1}{-i}\frac{d}{dk}\hat{g}(k)$

2.3 Problem 8.

Show that any periodic function $f(\xi) = f(\xi + a)$ is the convolution of a nonperiodic function with a train of Dirac delta DISTRIBUTIONS.

[I was very stuck on this and stack exchange provided an answer.]
Let $a > 0$ be the length of the period of the function $f(x)$. Then let $g(x) = \mathbb{1}_{[0,a)}(x)f(x)$ have value in the 'first' period of f and then be 0 elsewhere. Obviously g is non periodic unless $f = 0$ (that case being triival and not relavent). Now consider:

$$g \star \left(\sum_{n=-\infty}^{\infty}\delta(x-an)\right) = \sum_{n=-\infty}^{\infty} g \star \delta(u-(x-an)) = \sum_{n=-\infty}^{\infty} g(x-an) = f(x)$$

Thus we have written f as a convlution of a non periodic function and a train of delta distributions.

2.3 Problem 9.

Find the Fourier specturm of a finite train of identical coherent pulses of the kind shown in Fig. 2.9.

The function in reference is of the form:

$$f_n(t) = e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{i\delta_n}$$

Which in our specific case is:

$$f_n(t) = e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

So our sum is:

$$\sum_{n=-N}^{N} f_n(t) = \sum_{n=-N}^{N} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

We could in theory calculate the Fourier transform of each of the elmeents of the sum and then combine. However the text has outlined a process we can just semi blindly follow with less work.

To this end we notice that with $\delta_n = 0 = n\Delta\phi$ we have the same form as page 90 with:

$$f(t) == \sum_{n=-N}^{N} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{in\Delta\phi}$$

but with finite bounds on our sum. Thankfully we can still rewrite it as a convolution:

$$f(t) = \int_{-\infty}^{\infty} e^{-(t-\xi)^2/2b^2} e^{i\omega_0(t-\xi)} \sum_{n=-N}^{N} \delta(\xi - nT) \mathrm{d}\xi$$

Our pulses have the same form as before, but our comb is much shorter this time.

$$\mathfrak{F}[\text{pulse}](\omega) = \int_{-\infty}^{\infty} e^{-t^2/2b^2} e^{i\omega_0 t} \mathrm{d}t = be^{-(\omega-\omega_0)^2/2b^2}$$

$$\mathfrak{F}[\text{small comb}](\omega) = \sum_{n=-N}^{N} \mathfrak{F}\delta(\xi - nT) = \sum_{n=-N}^{N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\xi} \delta(\xi - nT) \mathrm{d}\xi$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} e^{-in\omega T} = \sqrt{2\pi}\frac{\sin(N+\frac{1}{2})\omega T}{\sin \omega T/2}$$

In the book they use Poisson's sum formula, here we are not so lucky as the bounds are finite.

Combining now our expressions for the Fourier transform we get:

$$\mathfrak{F}[f](\omega) = \sqrt{2\pi} b e^{-(\omega-\omega_0)^2/2b^2} \frac{\sin(N+\frac{1}{2})\omega T}{\sin \omega T/2}$$

Thus in the end the spectral envelope ends up being the same as it was determined by the amplidude. The spectral lines portion though, is now just a finite approximation. Notably it is a function and has not achieved yet distribution status. The spectral lines in this case wobbles much more and has support on the whole real line and not just at integerl multilpes of 2π plus a $\Delta\phi$ factors.

2.3 Problem 10.

Verify that

$$f(t) = \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

is a periodic function of t, and that $f(t+T) = f(t)$. Find the full 4-ier representation

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}$$

of f by determing ω_m and c_m.

Verifying periodicity is straightforward:

$$f(t+T) = \sum_{n=-\infty}^{\infty} e^{-(t+T-nT)^2/2b^2} e^{i\omega_0(t+T-nT)} = \sum_{n=-\infty}^{\infty} e^{-(t-(n-1)T)^2/2b^2} e^{i\omega_0(t-(n-1)T)}$$

$$= \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} = f(t)$$

Now to find the 4-ier representation:

Similar to how $\mathfrak{F}\delta = 1 \Rightarrow \delta(x) = \sum_{n=-\infty}^{\infty} e^{inx}$, and $\mathfrak{F}1 = \delta \Rightarrow 1 = \sum_{n=-\infty}^{\infty} \delta(n) e^{inx}$, we can do the same with this series. (Note the δ distriutions outside of integrals is 'problematic' but not a problem)

Or for a more close situation: $\mathfrak{F}e^{iwx}[\omega] = \sqrt{2\pi}\delta(\omega - w) \;\Rightarrow e^{iwx} = \sum_{n=-\infty}^{\infty}\sqrt{2\pi}\delta(n-w)e^{inx}$. Now something to remmeber here is that we are no longer working with L^2 functions where $\mathfrak{F}$ gives us a bijective map to l^2. Now we know the specturm of the distribution from $\mathfrak{F}$ but it is not neccisarily true that $\sum_n \mathfrak{F}f(n)e^{inx} = f(x)$ or that the left hand side even has a meaning.

So following $e^{iwx} \to e^{iwx}$ even for $w \notin \mathbb{Z}$. We know from the work done in the book that

$$\mathfrak{F}[f](\omega) = \sqrt{2\pi}be^{-(\omega-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}\delta\left(\omega T - 2\pi m\right)$$

The Fourier 'type' series is then:

$$f(t) = \frac{1}{2\pi}\sum ne^{-i2\pi nt}\mathfrak{F}f(2\pi n) = \frac{1}{2\pi}\sum ne^{-int}\sqrt{2\pi}be^{-(n-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}\delta\left(nT - 2\pi m\right)$$

$$= b\sum_{n=-\infty}^{\infty}e^{-int-(n-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}\delta\left(nT-2\pi m\right) = b\sum_{n=-\infty}^{\infty}e^{-int-(n-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}T\delta\left(n - \frac{2}{T}\pi m\right)$$

Thus we want $n = \frac{2}{T}\pi m$ to have nonzero values from the infinite sum: **************SKETCHY************

$$= \sum_m bTe^{-(\frac{2}{T}\pi m-\omega_0)^2/2b^2}e^{-i2\pi m/Tt}$$

$\omega_m = 2\pi m/T,\; c_m = bTe^{-(m-\omega_0)^2/2b^2}$.

2.4 Orthonormal Wave Packet Representation

2.4 Problem 1.

Consider the set of functions:

$$\left\{P_{jl}(t) = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}e^{2\pi il\omega/\epsilon}\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\mathrm{d}\omega, \quad j,l = 0,\pm1,\pm2,\ldots\right\}$$

a) Show that these wave packets are orthonormal

b) Show that these wave packets form a complete set.

a)

$$\int_{-\infty}^{\infty}P_{jl}(t)P_{j'l'}(t)\mathrm{d}t = \int_{-\infty}^{\infty}\frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}e^{2\pi il\omega_1/\epsilon}\frac{1}{\sqrt{2\pi}}e^{-i\omega_1 t}\mathrm{d}\omega_1\frac{1}{\sqrt{\epsilon}}\int_{j'\epsilon}^{(j'+1)\epsilon}\frac{1}{\sqrt{2\pi}}\overline{e^{2\pi il'\omega_2/\epsilon}e^{-i\omega_2 t}}\mathrm{d}\omega_2\mathrm{d}t$$

$$= \frac{1}{\epsilon 2\pi} \int\limits_{-\infty}^{\infty} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j'\epsilon}^{(j'+1)\epsilon} e^{2\pi i l \omega_1/\epsilon} e^{-i\omega_1 t} e^{-2\pi i l' \omega_2/\epsilon} e^{i\omega_2 t} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}t$$

$$= \frac{1}{\epsilon 2\pi} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j'\epsilon}^{(j'+1)\epsilon} e^{-2\pi i l' \omega_2/\epsilon} e^{2\pi i l \omega_1/\epsilon} \int\limits_{-\infty}^{\infty} e^{-i(\omega_1-\omega_2)t} \mathrm{d}t \mathrm{d}\omega_1 \mathrm{d}\omega_2$$

$$= \frac{1}{\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j'\epsilon}^{(j'+1)\epsilon} e^{-2\pi i l' \omega_2/\epsilon} e^{2\pi i l \omega_1/\epsilon} \delta(\omega_1 - \omega_2) \mathrm{d}\omega_1 \mathrm{d}\omega_2$$

We see at this stage that we need $\omega_1 = \omega_2$ on some positive measure set, otherwise the whole endever will be 0. Thus to continue the calculation we can add in a $\delta_{jj'}$ to ensure that the integration domains coincde.

$$= \delta_{jj'} \frac{1}{\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i l' \omega_1/\epsilon} e^{2\pi i l \omega_1/\epsilon} \mathrm{d}\omega_1 = \delta_{jj'} \frac{1}{\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i (l-l')\omega_1/\epsilon} \mathrm{d}\omega_1 = \frac{\epsilon}{\epsilon} \delta_{jj'} \delta_{ll'} = \delta_{jj'} \delta_{ll'}$$

The last equality follows from considering the $1/\epsilon$ periodicity of $e^{2\pi i (l-l')\omega_1/\epsilon}$ whenever $l \neq l'$.

b) As a student I once had would say: "we write it down and bash"

$$\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} P_{jl}(t) \bar{P}_{jl}(t') = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i l \omega/\epsilon} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} \mathrm{d}\omega \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i l \omega/\epsilon} \frac{1}{\sqrt{2\pi}} e^{i\omega t'} \mathrm{d}\omega$$

$$= \frac{1}{\epsilon 2\pi} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i l \omega_1/\epsilon} e^{-i\omega_1 t} \mathrm{d}\omega_1 \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i l \omega_2/\epsilon} e^{i\omega_2 t'} \mathrm{d}\omega_2$$

$$= \frac{1}{\epsilon 2\pi} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i l (\omega_1-\omega_2)/\epsilon} e^{-i(\omega_1 t - \omega_2 t')} \mathrm{d}\omega_2 \mathrm{d}\omega_1$$

$$= \frac{1}{\epsilon 2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-i(\omega_1 t - \omega_2 t')} \sum_{l=-\infty}^{\infty} e^{2\pi i l (\omega_1-\omega_2)/\epsilon} \mathrm{d}\omega_2 \mathrm{d}\omega_1$$

$$= \frac{1}{\epsilon 2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-i(\omega_1 t - \omega_2 t')} \delta((\omega_1 - \omega_2)/\epsilon) \mathrm{d}\omega_2 \mathrm{d}\omega_1 = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-i\omega(t-t')} \mathrm{d}\omega = \delta(t-t')$$

Bashing complete, we arrive at the answer.

2.4 Problem 2.

Consider the wave packet

$$Q_{jl}(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega$$

Express the summed wave packets:

a)

$$\sum_{j=-\infty}^{\infty} Q_{jl}(t)$$

b)

$$\sum_{l=-\infty}^{\infty} Q_{jl}(t)$$

c)

$$\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} Q_{jl}(t)$$

in terms of appropriate Dirac delta DISTRIBUTIONS if necessary.

a)

$$\sum_{j=-\infty}^{\infty} Q_{jl}(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\sum_{j=-\infty}^{\infty}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{-\infty}^{\infty} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega$$

$$= \frac{1}{\sqrt{\epsilon}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\omega(t-2\pi l/\epsilon)}\mathrm{d}\omega = \frac{1}{\sqrt{\epsilon}}\sqrt{2\pi}\delta(t-2\pi l/\epsilon)$$

b)

$$\sum_{l=-\infty}^{\infty} Q_{jl}(t) = \sum_{l=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}\sum_{l=-\infty}^{\infty} e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega$$

Now letting $y = -2\pi\omega/\epsilon, \mathrm{d}y = -2\pi/\epsilon\mathrm{d}\omega$, we can change variables and evaluate the sum:

$$= \frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}\sqrt{2\pi}\sum_{l=-\infty}^{\infty} e^{il(-2\pi\omega/\epsilon)}\mathrm{d}\omega = \frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)(-2\pi)}^{(j+1/2)(-2\pi)} e^{i\epsilon y/(-2\pi)t}\sqrt{2\pi}\sum_{l=-\infty}^{\infty} e^{ily}\mathrm{d}y\frac{\epsilon}{-2\pi}$$

$$= \frac{1}{-2\pi}\sqrt{\epsilon}\int_{(j-1/2)(-2\pi)}^{(j+1/2)(-2\pi)} e^{i\epsilon y/(-2\pi)t}\sqrt{2\pi}\delta(y)\mathrm{d}y = \frac{1}{-2\pi}\sqrt{\epsilon}\sqrt{2\pi}\sum_{|j|\leqslant 3}\delta_{j,0}$$

Since $0 \in [j-\frac{1}{2}, j+\frac{1}{2}]$ iff $j = 0, \pm 1, \pm 2, \pm 3$.

c) By the first part we immediatly see:

$$\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} Q_{jl}(t) = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\epsilon}}\sqrt{2\pi}\delta(t-2\pi l/\epsilon) = \sqrt{\frac{2\pi}{\epsilon}}\sum_{l=-\infty}^{\infty}\delta(t-2\pi l/\epsilon)$$

******************DOUBLE CHECK

2.5 Orthonormal Wavelet Representation

2.6 Multiresolutions Analysis

2.6 Problem 1.

> Show that
>
> $$\overline{\bigcup_{k=-\infty}^{\infty} V_k} = L^2 \iff \lim_{k\to\infty} \|P_{V_k} f - f\| = 0$$
>
> where P_{V_k} is the projection onto V_{-k} (the sign is fliped to make the limits easier to write) and the norm is the L^2 norm.

We go forward first:

$$\overline{\bigcup_{k=-\infty}^{\infty} V_k} = L^2$$

Thus given an $f \in L^2, \exists h_n,\ st.\ \lim_{n\to\infty}\|h_n - f\| = 0, h_n \in \bigcup V_k$. Now $\forall n\ \exists k\ st. h_n \in V_k$. Now either $h_n = P_{V_k} f$ or $\|h_n - f\| \geqslant \|P_{V_k} - f\|$ and we can replace h_n with the actual projection without making the approximation any worse. Obviously our new $\bar{h}_n$ still converges and is made entirly of projections onto subspaces. Thus we have constructed the desired sequence. (We may need to additonally doctor the sequence and insert terms if h_k skipped many subspaces.) *********************HATE HOW THIS IS WRITTEN**********************8

Now we go backwards: Let f again be some funciton in L^2, then $\|P_{V_k} f - f\| \to 0$ as $k \to \infty$. Thus there exists some sequence $h_k = P_{V_k} f$ where $\|h_k - f\| \to 0$ as $k \to \infty$. Additionally $h_k \in V_k \forall k$ thus $\forall f \in L^2, f \in \overline{\bigcup V_k k}$. The reverse inclusion is obvious and we are done.

2.6 Problem 2.

> Show that
>
> $$\bigcap_{k=-\infty}^{\infty} V_k = \{0\} \iff \lim_{k\to\infty} \|P_{V_k} f\| = 0$$

First we go forward:
Notice that $\|P_{V_k} f\|$ is a decreasing sequence and thus has some limit. Now suppose for controdiciton that $\|P_{V_k} f\| \to \epsilon > 0$ as $k \to \infty$. Notice now that this is a Cauchy sequence in L^2 and thus there is some function $g \in L^2$ st. $\|P_{V_k} f - g\| \to 0$. Now we wish to show that $g \in V_k, \forall k$. Suppose it is not, there is some k_0 st. g is no longer in any of the V_k's after k_0. But then (since $\overline{\bigcap V_k} = \{0\}$) there would be some nonzero gap that emerges between $P_{V_k} f$ and g, namely that: $\|P_{V_k} f - g\| \geqslant d(V_k, g) = \epsilon_g > 0$. Thus $g \in V_k, \forall k$ but then $g \in \bigcap V_k$ which then means $g = 0$ and thus $\|P_{V_k} f\| \to \|g\| = 0$.

Then we go back:
We do this by controdiciton, so suppose that $\bigcap_{k=-\infty}^{\infty} V_k \supseteq \{0, g\}$ for some nonzero functions g. Then $\|g\| > 0$ and since $g \in \bigcap_{k=-\infty}^{\infty} V_k, \to g \in V_k \forall k$. Thus $g \in P_{V_k}$ for all k and $\lim_{k\to\infty} \|P_{V_k} g\| = \|g\| > 0$. This is a controdicition and we see that there is no g.

2.6 Problem 3.

a) Show that V_0 is discrete translation invarient, ie. whenever $l \in \mathbb{Z}$ that:

$$f(t) \in V_0 \iff f(t-l) \in V_0$$

b) Show that V_k is 2^k shift invariant, ie with $l, k \in \mathbb{Z}$ that:

$$f(t) \in V_0 \iff f(t - 2^k l) \in V_0$$

a) Suppose $f \in V_0$ then $\exists \alpha_l$ st. $f(t) = \sum_l \alpha_l \phi(t-l)$. By the construction of the basis of V_0. Notice that for $k \in \mathbb{Z}$

$$f(t-k) = \sum_l \alpha_l \phi(t-k-l) = \sum_{m=k+l} \alpha_{m-k} \phi(t-m) = \sum_{m=k+l} \alpha'_m \phi(t-m)$$

Thus we still have an expansion for $f(t-l)$ in terms of the original basis.

b) Similar tricks:

$$f(t) \in V_k \Rightarrow f(2^k t) \in V_0 \Rightarrow$$

Now we remember that shifting by a constant value keeps you in V_0. $f(2^k(t-j)) = f(2^k t - 2^k j) \in V_0$
Now scaling the t by 2^{-k} will get us back to V_k, that is: $f(t - 2^k j) \in V_k$.

2.6 Problem 4.

a) Point out why this inner product is the (l, l')th entry of the $\sqrt{2}$- multilple of a unitary matrix, which is independent of k.

b) Show that $\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \delta_{0l'}$

a) Let M be the map from $V_k \to V_{k+1}$ that is Change of basis from two orhtonormal basis? So it sunitary and stuff? **

b) If $l' = 0$ then we have:

$$\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \sum_{l=-\infty}^{\infty} |h_l|^2 = \sum_{l=-\infty}^{\infty} \frac{1}{4} | \int_{-\infty}^{\infty} \bar{\phi}(u-l)\phi(u/2)\mathrm{d}u|^2$$

We notice now that h_l is the coefficent of the projection of $\phi(u/2)$ onto the space V_0. Thankfully $\phi(u/2)\imath V_1 \subset V_0$. Thus we see that by Parsevel's that:

$$\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \|\phi(u/2)\|^2 \frac{1}{4} = \frac{4}{4} = 1$$

If $l \neq 0$ then:

$$\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \sum_{l=-\infty}^{\infty} \overline{\int_{-\infty}^{\infty} \bar{\phi}(u-l)\phi(u/2)\mathrm{d}u} \int_{-\infty}^{\infty} \bar{\phi}(u-(l-2l'))\phi(u/2)\mathrm{d}u$$

****

2.6 Problem 5.

Verify the validity of the funcitonal constatin:

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

Begin with :

$$\sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2 = \frac{1}{2\pi}, \hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega)$$

$$H(\omega) = \frac{\sqrt{2}}{2} \sum_{l=-\infty}^{\infty} h_l e^{i\omega l}$$

2.6 Problem 6.

Consider a function $\phi(t)$ having the property

$$\left| \int_{-\infty}^{\infty} \phi(t)\mathrm{d}t \right| \neq 0$$

Find the solution to the scaling equation:

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega)$$

Answer/Hint:

$$\hat{\phi}(\omega) = \hat{\phi}(0) \prod_{k=1}^{\infty} H(\omega/2^k)$$

2.6 Problem 7.

Let $\phi^+(t)$ be soltuion to the scaling equation

$$\phi(t) = \sqrt{2} \sum_{l=-\infty}^{\infty} h_l \phi(2t - l)$$

1. Point out why

   $$\hat{\phi}^- = \begin{cases} \hat{\phi}^+(\omega) & \omega \geqslant 0 \\ -\hat{\phi}^+(\omega) & \omega \leqslant 0 \end{cases}$$

   is the Fourier transform of a second independent solution to the above scaling equaiton.

2. Show that these two solutions are orhtogonal:

   $$\int_{-\infty}^{\infty} \bar{\phi}^+ \phi^- \mathrm{d}t = 0$$

   whenever $\phi(t)$ is a real funciton or whenever its Fourier transform is an even function of ω.

2.6 Problem 8.

Validate conclusion # II of the theorem on page 145. Point out why, whenever $k \neq k'$, the funcitons in O_k are orthogonal to $O_{k'}$.

3 Strum-Liouville Theory

3.3 Strum-Liouville Systems

3.3 Problem 1.

a) Show that any equation of the form

$$u'' + b(x)u' + c(x)u = 0$$

can always be brought into the Shrodinger form:

$$v'' + Q(x)v = 0$$

Apply this result to obtain the Schrodinger form for:

b)

$$u'' - 2xu' + \lambda u = 0$$

c)

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0$$

d)

$$xu'' + (1 - x)u' + \lambda u = 0$$

e)

$$(1 - x^2)u'' - xu' + \alpha^2 u = 0$$

f)

$$(pu')' + (q + \lambda r)u = 0$$

g)

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + l(l+1) - \frac{m^2}{\sin^2\theta}\right]u = 0$$

****

a) We consider:

$$u'' + b(x)u' + c(x)u = 0$$

********************PACKT*****************

Let u be some solution, let us try $v(x) = F(x)u(x)$ then:

$$v' = F'u + Fu', v'' = F''u + 2F'u' + Fu''$$

Now we plug this into $v'' + Qv$ and find:

$$F''u + 2F'u' + Fu'' + Fu = F''u + 2F'u' + F(u'' + Qu)$$

$$= F\left[u'' + 2F'/Fu' + (F''/F + Q)u\right]$$

We know that $u'' = -bu' - cu$

$$= F\left[(2F'/F - b)u' + (F''/F + Q - c)u\right]$$

If we let $Q = c - F''/F$ then all we have to do is solve $2F'/F = b$. This leads to

$$2F'/F - b = 0 \Rightarrow 2\int^x F'/F = \int^x b \Rightarrow 2\ln F = \int^x b$$

$$\Rightarrow F = \exp\{\frac{1}{2}\int^x b\}$$

Thus our subsitition ends up being: $v(x) = \exp\{\frac{1}{2}\int^x b\}u(x)$. Note that $F' = \frac{b}{2}F,\ F'' = \frac{b'+b^2/2}{2}F$ and our equation gets:

$$Q = c - F''/F = c - F''/F = c - \frac{b' + b^2/2}{2}$$

All togther we have:

$$u \to v = \exp\{\frac{1}{2}\int^x b\}u(x),$$

$$u'' + b(x)u' + c(x)u = 0 \ \to \ v'' + Q(x)v = 0,\ Q(x) = c - \frac{b' + b^2/2}{2}$$

b)

$$u'' - 2xu' + \lambda u = 0$$

Thus the things we have to calcuate are:

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

$$F = \exp\{\frac{1}{2}\int^x -2y\mathrm{d}y\}, \quad Q = \lambda - \frac{-2 + 4x^2/2}{2}$$

And our equations become:

$$v = \exp\{-x^2/2\}u, \quad v'' + \left(\lambda + 1 - x^2\right)v = 0$$

c)

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0 = u'' + u'/x + (1 - \frac{\nu^2}{x^2})u = 0$$

We divide by x^2 here to get ride of the coefficent on u". Now following the formeioli

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

$$F = \exp\{\frac{1}{2}\int^x \frac{1}{y}\mathrm{d}y\}, \quad Q = 1 - \frac{\nu^2}{x^2} - \frac{-1/x^2 + \frac{1}{x^2 2}}{2}$$

$$Q = 1 - \frac{\nu^2}{x^2} + \frac{1}{4x^2} = 1 - \frac{4\nu^2 - 1}{4x^2}$$

And our equations become:

$$v = \exp\{\ln(x)/2\}u = \sqrt{x}u, \quad v'' + \left(1 - \frac{4\nu^2 - 1}{4x^2}\right)v = 0$$

d)

$$xu'' + (1 - x)u' + \lambda u = 0 = u'' + (1/x - 1)u' + \frac{\lambda}{x}u$$

Again we do the divide by the whole something that could be zero trick to deal with a coefficent.

$$F = \exp\{\frac{1}{2}\int^x 1/y - 1\mathrm{d}y\}, \quad Q = \frac{\lambda}{x} - \frac{-\frac{1}{x^2} + (1/x - 1)^2/2}{2}$$

$$F = \exp\{\frac{1}{2}[\ln x - x]\}, \quad Q = \frac{\lambda}{x} - \frac{-\frac{1}{2x^2} + -1/x + 1/2}{2}$$

$$F = \exp\{\frac{1}{2}[\ln x - x]\}, \quad Q = \frac{\lambda}{x} + \frac{1}{4x^2} + 1/2x - 1/4$$

****

e)

$$(1 - x^2)u'' - xu' + \alpha^2 u = 0$$

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

f)

$$(pu')' + (q + \lambda r)u = 0$$

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

g)

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + l(l+1) - \frac{m^2}{\sin^2\theta}\right] u = 0$$

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

3.3 Problem 2.

Consider the S-L eigenvalue problem:

$$[Lu_n](x) = \left(-\frac{d^2}{dx^2} + x^2\right) u_n(x) = \lambda_n u_n(x), \quad \lim_{x\to\pm\infty} u(x) = 0$$

on the infinite interval $(-\infty, \infty)$
Show that the eigenvalues λ_n are nondegenerate, ie. show that, except for a constant multiple, the correpsonding eigenfunctions are unique.

By Abel's theorem we have that if two solutions to the above have the same eigenvalue then (since p(x) =1 here)

$$u_m u_n' - u_m' u_n = const.$$

From here it suffices to show that this constant is zero. Once there the same logic as in the end of theorem 3 applies and we would see that $\frac{u_m'}{u_m} = \frac{u_n'}{u_n} \Rightarrow u_m = ku_n$. To this end notice that

$$\lim_{x\to\infty} u_m u_n' - u_m' u_n = C \Rightarrow$$

****

*******************PACKET NOTE.

3.3 Problem 3.

Consider the "parity" operator $P : L^2 \to L^2$ ($L^2 = L^2(-\infty, \infty)$) defined by

$$P\psi(x) = \psi(-x)$$

1. For a given function ψ what are the eigenvlaues and eigenfunction of P?
2. Show that the eigenfunctions of the operator L defined in problem 3.3.2 are eigenfunctions of P. Do this by computing

   $$P^{-1}LP\psi(x)$$

   for $\psi \in L^2$ and the pointing out how $P^{-1}LP$ is related to L. Next point out how this relationship applied to an eigenfunction u_n of the previous problem leads ot the result $Pu_n = \mu u_n$.

i) The given funciton part of the question is a typeo. The eigenvalues are ± 1 ($\psi(x) = \lambda\psi(-x)$) and the eigenfuncitons are even and odd functions.

ii)

$$P^{-1}LP\psi(x) = P^{-1}L\psi(-x) = P^{-1}\left[-\frac{d^2}{dx^2} + x^2\right]\psi(-x)$$

$$= P^{-1}\left[-\frac{d^2\psi(-x)}{dx^2} + x^2\psi(-x)\right] = \left[-P^{-1}\frac{d^2\psi(-x)}{dx^2} + P^{-1}x^2\psi(-x)\right]$$

$$= \left[-P^{-1}\frac{-d\psi'(-x)}{dx} + x^2\psi(x)\right] = \left[-P^{-1}\psi''(-x) + x^2\psi(x)\right] = -\psi''(x) + x^2\psi(x) = L\psi$$

Thus $P^{-1}LP = L$. **

Thus if μ, u_n are eigenvector

3.3 Problem 4.

Consider the S-L eigenvalue problem:

$$[Lu_n](x) = \left(-\frac{d^2}{dx^2} + x^2\right) u_n(x) = \lambda_n u_n(x), \qquad \lim_{x\to\pm\infty} u(x) = 0$$

on the infinite interval $(-\infty, \infty)$
We are now blessed with the knowledge that these eigenvalues are nondegenerate and are $\lambda_n = 2n + 1$. Consider now the Fourier transform on L^2

$$\mathfrak{F}u = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-ikx} u \mathrm{d}x$$

a) By computing $\mathfrak{F}L\mathfrak{F}^{-1}\hat{\psi}$ for an aribitary $\hat{\psi} \in L^2$, determine the Fourier represnetation of $\mathfrak{F}L\mathfrak{F}^{-1} = \hat{L}$, of the operator $L = -\frac{d^2}{dx^2} + x^2$.

b) By viewing $\mathfrak{F}$ as a map from L^2 to itself, compare the operators $\hat{L}$ and L. State in english senence and in math equation.

c) Us the result of b to show that each eigenfunction u_n of the S-L operator L is also an eigenfunction of $\mathfrak{F}$.

$$\mathfrak{F}u_n = \mu u_n$$

By applying the result (e) of the exercise on page 75, determine the only allowed values for μ. What *is* the 4-ier transform of a Hermite-Gauss polynomial?

1.

$$\mathfrak{F}L\mathfrak{F}^{-1}\hat{\psi} = \mathfrak{F}L\psi = \mathfrak{F}\left(-\frac{d^2}{dx^2} + x^2\right)\psi(x)$$

$$= \mathfrak{F}\left(-\frac{d^2}{dx^2} + x^2\right)\psi(x) = -\mathfrak{F}\frac{d^2\psi(x)}{dx^2} + \mathfrak{F}x^2\psi(x)$$

$$= -(i\omega)^2\mathfrak{F}\psi[\omega] + i^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathfrak{F}\psi[\omega] = \omega^2\mathfrak{F}\psi[\omega] - \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathfrak{F}\psi[\omega] = \left[\omega^2 - \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\right]\hat{\psi}[\omega]$$

Thus

$$\hat{L} = \omega^2 - \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}$$

2. They do the same thing, one in frequency space and the other in real space.

$$\hat{L}\hat{\psi}(\omega) = L\phi(\omega)$$

*****************DOUBLE CHECK*******************

3. *************************

3.3 Problem 5.

Consider the S-L System:

$$\left[\frac{d}{dx}p\frac{d}{dx} + q + \lambda\rho\right]u = 0, \quad a < x < b$$

$$\alpha u(a) + \alpha' u'(a) = 0; \quad \beta u(a) + \beta' u'(a) = 0$$

Let $\omega(x, \lambda)$ be that unqiue soltuion to the above with boundary conditions satisified. Then $\omega_n(x) = \omega(x, \lambda_n)$ is an eigenfunction with eigenvalue λ_n. Calculate $\int_a^b \omega_n^2 \rho \mathrm{d}x$ as follows:

1. $$(\lambda - \lambda_n)\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = p(x)W(\omega, \omega_n)|_a^{x=b}$$

2. By taking the limit $\lambda \to \lambda_n$ show that:

   $$\int_a^b \omega_n^2 = p(b)\left[w_n'(b)\frac{d\omega(b,\lambda)}{d\lambda}|_{\lambda=\lambda_n} - \omega(b)\frac{d}{d\lambda}w_n'(b,\lambda)|_{\lambda=\lambda_n}\right]$$

   primes here refering to $\frac{d}{dx}$.

1. Thanks to the first 2 steps of the 3 step proof on page 168 of the orthogonality of eigenvalues we see:

   $$(\lambda - \lambda_n)\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = p(x)W(\omega, \omega_n)|_a^{x=b}$$

   In fact we realize this is true for any S-L system with any boundary condiotns. With that aside aside, we consider the case at hand.

   $$p(x)W(\omega, \omega_n)|_a^{x=b} = p(b)W(\omega, \omega_n)(b) - p(a)W(\omega, \omega_n)(a)$$

   note that the boundary condition gives: $\omega(a, \lambda) = -\alpha'/\alpha\omega'(a, \lambda)$ Thus we have:

   $$\begin{aligned} p(a)W(\omega, \omega_n)(a) &= p(a)(\omega(a)\omega_n'(a) - \omega'(a)\omega_n(a)) \\ &= p(a)(-\omega'(a)\alpha'/\alpha\omega_n'(a) + \omega'(a)\omega_n'(a)\alpha'/\alpha) = 0 \end{aligned}$$

   Thus we get:

   $$(\lambda - \lambda_n)\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = p(x)W(\omega, \omega_n)|^{x=b}$$

2. We first divide by $\lambda - \lambda_n$:

$$\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = \frac{1}{\lambda-\lambda_n}p(x)W(\omega,\omega_n)|^{x=b}$$

Now we take the limit $\lambda \to \lambda_n$: the left hand side clearly does nothing funky and becomes:

$$\int_a^b \omega_n(x)^2\rho(x)\mathrm{d}x$$

Now for the right hand side:

$$\frac{1}{\lambda-\lambda_n}p(x)W(\omega,\omega_n)|^{x=b} = \frac{1}{\lambda-\lambda_n}p(b)(\omega(b)\omega_n'(b) - \omega'(b)\omega_n(b))$$

$$= p(b)(\frac{\omega(b)}{\lambda-\lambda_n}\omega_n'(b) - \frac{\omega'(b)}{\lambda-\lambda_n}\omega_n(b))$$

We add and subtract the same term:

$$= p(b)(\frac{\omega(b)-\omega_n(b)}{\lambda-\lambda_n}\omega_n'(b) - \frac{\omega'(b)+\omega_n'(b)}{\lambda-\lambda_n}\omega_n(b))$$

Pasing to the limit we get:

$$= p(b)\left[w_n'(b)\frac{d\omega(b,\lambda)}{d\lambda}|_{\lambda=\lambda_n} - \omega(b)\frac{d}{d\lambda}w_n'(b,\lambda)|_{\lambda=\lambda_n}\right]$$

which is the desired result. ********************CHECK OVER ONE LAST TIME*************

3.3 Problem 6.

Consider the S-L problem

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\nu^2}{x}\right]u=\lambda x u$$

Here u, $\frac{\mathrm{d}u}{\mathrm{d}x}$ bounded as $x\to 0, u(1)=0$ and $\nu\in\mathbb{R}$.

1. Using the sub $t=\sqrt{\lambda x}$ show that the above differential equation reduces to Bessel's equation of order ν. One solution which is bouned as $t\to 0$ is $J_\nu(t)$; a second linearly indep. solution, denoted by $Y_\nu(t)$ is unbounded as $t\to 0$.

2. SHow that the eigenvalues of the given problem are the squares fo the positive zeros of $J_\nu(\sqrt{\lambda})$ and that the corresponding eigenfunctions are

$$u_n(x)=J_\nu(\sqrt{\lambda_n}x)$$

3. Show that the eigenfunctions u_n satisfy the orhtogonality relaiton:

$$\int_0^1 x u_m u_n \mathrm{d}x=0, \quad m\neq n$$

4. For the case $\nu=0$, apply the method of the previous problem to exhibit the set of orhtonormalized eiqgenfuncitons.

5. Determine the coeffiencts of the Fourier-Bessel sereis expnaison:

$$f(x)=\sum_{n=1}^{\infty}c_n u_n(x)$$

******************TYPO IN BOOK****************

1. The bessel equation is:

$$\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}x}{\mathrm{d}+}x^2-\alpha^2\right]u=0$$

So begining with: Let $t=\sqrt{\lambda}x$, then $t/\sqrt{\lambda}=x$, and $\frac{\mathrm{d}t}{\mathrm{d}x}=\sqrt{\lambda}$, $\frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\lambda}$ Our equation is:

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\nu^2}{x}\right]u=\lambda x u$$

We replace with $\frac{\mathrm{d}}{\mathrm{d}t}$ to get:

$$\left[-\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}t/\sqrt{\lambda}\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\nu^2}{t/\sqrt{\lambda}}\right]u=\lambda t/\sqrt{\lambda}u$$

$$\left[-\sqrt{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}+\sqrt{\lambda}\frac{\nu^2}{t}\right]u=t\sqrt{\lambda}u$$

Now multiplying across by $t/\sqrt{\lambda}$ we get:

$$\left[-t\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}+\nu^2\right]u=t^2u$$

$$\left[-t\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}+\nu^2-t^2\right]u=0$$

2.
$$u_n(x)=J_\nu(\sqrt{\lambda_n}x)$$

3.
$$\int_0^1 xu_mu_n\mathrm{d}x=0,\quad m\neq n$$

4.

5.
$$f(x)=\sum_{n=1}^{\infty}c_nu_n(x)$$

3.3 Problem 7.

Consider the S-L problem

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}+\frac{m^2}{1-x^2}\right]u=\lambda u$$

Here u, $\frac{\mathrm{d}u}{\mathrm{d}x}$ bounded as $x\to\pm1$. Here m is an integr. The solutions to this S-L problem are $u_n=P_n^m(x)$ the "asosciated Legendre polynomials", corresponding to $\lambda_n=n(n+1)$ n being an integer. SHow that:

$$\int_{-1}^{1}P_n^mP_{n'}^m\mathrm{d}x=0\quad \lambda_n\neq\lambda_{n'}$$

---

4 Green's Function Theory

4.3 Pictorial Definition of a Green's Function

4.3 Problem 1.

Find the adjoint L^* and the space on which it acts:

a)

$$Lu = u'' + a(x)u' + b(x)u$$

with $u(0) = u'(1), \quad u(1) = u'(0)$

b)

$$Lu = -(p(x)u')' + q(x)u$$

with $u(0) = u(1), \quad u'(1) = u'(0)$

a)

$$\langle Lu, v \rangle = \int_0^1 [u'' + a(x)u' + b(x)u]\, v \mathrm{d}x$$

Now we use integration by parts to get:

$$= [u'v + a(x)uv]_0^1 - \int_0^1 u'v' + a(x)uv' - b(x)uv \mathrm{d}x = [u'v + a(x)uv - uv']_0^1 + \int_0^1 uv'' - a(x)uv' + b(x)uv \mathrm{d}x$$

Thus we see that $L^*v = v'' - a(x)v' + b(x)v$. Now considering the boundary conditions we see that the boundary terms are:

$$u'(1)v(1) + u(1)\left(a(1)v(1) - v'(1)\right) - \left[u'(0)v(0) + u(0)\left(a(0)v(0) - v'(0)\right)\right]$$

$$= u'(1)v(1) + u(1)\left(a(1)v(1) - v'(1)\right) - \left[u(1)v(0) + u'(1)\left(a(0)v(0) - v'(0)\right)\right]$$

$$= u'(1)(v(1) - v(0)) + u(1)\left(a(1)v(1) - v'(1) - (a(0)v(0) - v'(0))\right)$$

From the $u'(1)$ coefficent we see we need $v(1) = v(0)$, this then leaves

$$u(1)\left(-v'(1) + v'(0)\right)$$

which gives us $v'(1) = v'(0)$.

Overall we see $L^*v = v'' - a(x)v' + b(x)v$ and the adjoint domain being $\{v | v'(1) = v'(0), v(1) = v(0), v \in C^2\}$.

***************DOUBLE CHECK a(x) IS WIERD*******************

b)

$$\langle Lu, v \rangle = \int_0^1 [-(p(x)u')' + q(x)u]\, v \mathrm{d}x = \int_0^1 -(p(x)u')'v + q(x)uv \mathrm{d}x$$

Again we apply integration by parts:

$$= -(p(x)u')'v|_0^1 + \int_0^1 p(x)u'v' \mathrm{d}x + \int_0^1 q(x)uv \mathrm{d}x$$

$$= -p(x)u'v|_0^1 + up(x)v'|_0^1 - \int_0^1 u(p(x)v')'\mathrm{d}x + \int_0^1 q(x)uv\mathrm{d}x$$

$$= -p(x)u'v|_0^1 + up(x)v'|_0^1 + \int_0^1 -u(p(x)v')' + q(x)uv\mathrm{d}x$$

$L^*v = (p(x)v')' + q(x)v$
****

4.3 Problem 2.

> Let L be a operator defined on S and L^*, S^* the adjoint and its domain satisifying $B_1(u) = 0 = B_2(u),\ B_1^*(v) = 0 = B_2^*(v)$ respectivly. Let $u, \lambda,\ v, \lambda'$ be eigenvalues, eigenvectors of L and L^*
>
> a) Make a guess as to the relationship between the eigenvalue of L and L^*.
>
> b) Prove: If $\lambda \neq \bar{\lambda}'$ then $\langle u, v\rangle = 0$.

Since part (ii) gives a guess we might as well say λ cooresponds with $\bar{\lambda}$ for eigenvalues between L and L^*.

We can see this with u and v as in the satement of the problem

$$\langle Lu, v\rangle = \bar{\lambda}\langle u, v\rangle$$

from the definition of eigenvalue/function.

$$= \langle u, L^*v\rangle = \lambda'\langle u, v\rangle$$

Thus we see that if $\langle u, v\rangle \neq 0$ that $\bar{\lambda} = \lambda'$.

*********8MAKE NOTE FOR LATER POST*****Problem is Done though*************
In fact it is clear from the above that $\bar{\lambda}$ being an eigenvalue of the adjoint is implied by $\exists v, \langle u, v\rangle \neq 0$. Thus if an operator is self adjoint we see that $\lambda = \bar{\lambda}$ and the eigenvalues must all be real valued.

4.3 Problem 3.

Find the Green's function for the Bessel operators:

a)

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}u(x)}{\mathrm{d}x}$$

b)

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}u(x)}{\mathrm{d}x} - \frac{n^2}{x} u(x)$$

with $y(0)$ finite and $y(1) = 0$.

ie. solve the equations $Lu = -\delta(x - \xi)$ with the given boundary conditions.

a)

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}u(x)}{\mathrm{d}x}$$

****

b) ***

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}u(x)}{\mathrm{d}x} - \frac{n^2}{x} u(x)$$

with $y(0)$ finite and $y(1) = 0$.

4.3 Problem 4.

zt

4.3 Problem 5.

zt

4.7 The Totally Inhomogeneous Boundary Value Problem

4.7 Problem 1.

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4.7 Problem 2.

zt

4.7 Problem 3.

zt

4.7 Problem 4.

zt

4.7 Problem 5.

zt

4.9

4.9 Problem 1.

zt

4.11

4.11 Problem 1.

zt

4.11 Problem 2.

zt

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5 Special Function Theory

5.1 The Helmholtz Equation

5.1 Problem 1.

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5.1 Problem 2.

zt

5.1 Problem 3.

zt

5.1 Problem 4.

zt

5.2 Properties of Hankel and Bessel Functions

5.2 Problem 1.

zt

5.3 More Properties of Hankel and Bessel Functions

5.3 Problem 1.

zt

5.4 Applications of Hankel and Bessel Functions

5.4 Problem 1.

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5.4 Problem 2.

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5.4 Problem 3.

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5.4 Problem 4.

zt

5.4 Problem 5.

zt

5.5 More Properties of Hankel and Bessel Functions

5.5 Problem 1.

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5.6 The Method of Steepest Descent and Stationary Phase

5.6 Problem 1.

zt

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6 Partial Differential Equations

6.2 System of Partial Differntial Equations: How to solve Maxwell's equations using Linear Algebra

6.2 Problem 1.

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6.2 Problem 2.

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6.2 Problem 3.

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6.2 Problem 4.

zt

6.2 Problem 5.

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6.2 Problem 6.

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6.2 Problem 7.

zt

6.2 Problem 8.

zt

6.2 Problem 9.

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