Properties of the Henkel and Bessel Functions

1. Linear superopoition of plane waves
2. They are Integration contours in the complex plane
3. The integral rep. of the two Hankel functions do not depend on any real changes in the integration limits.
4. The cylinder harmonics are eigenfunctions of the rotational gneerator $L_\theta = \frac{1}{i}\frac{\partial}{\partial\theta}$.
5. They satisfy the Helmholtz's euqation, whihc in polar coordinates beocome bessels equation
6. The domain of a cylinder hrarmonic is the r and θ coordinate transverse cross section of a cylinder. A cylinder harmonic itself is the r and θ dependent part of a cylinder wave.
7. The two Hnakel functions are distinguished by the direction and shape of their contour in the complex plane.
8. The bessel function is the average of the two types of hankel function
9. Neuman is the 'sin' average of the two hankel funcitosn (the arugnemnt of the hankels are not negated though)
10. Analog to exponential functions
11. $J_\nu(\rho)$ is real when ν is real and $J_0(0) = 1$. Moreover J_ν satisifies the reflection principal from complex analysis.
12. For integer $\nu = m$ we have
$$J_m(\rho) = \frac{1}{\pi}\int_0^\pi \cos(\rho\sin\beta - m\beta)\mathrm{d}\beta$$
13. There is a very ugly power expansion of J_ν
14. $$H_\nu^1 = \frac{e^{-i\nu\pi}J_\nu - J_{-\nu}}{-i\sin\pi\nu}$$
15. $$\overline{H_\nu^2} = H_\nu^1$$
16. $$Z_{\nu+1} + Z_{\nu-1} = \frac{2\nu}{\rho}Z_\nu$$
$$Z_{\nu+1} - Z_{\nu-1} = -2\frac{\mathrm{d}}{\mathrm{d}\rho}Z_\nu$$

17.

$$L_+ Z_\nu e^{i\nu\theta} = -Z_{\nu+1} e^{i(\nu+1)\theta}$$
$$L_- Z_\nu e^{i\nu\theta} = -Z_{\nu-1} e^{i(\nu-1)\theta}$$

18. Plane wave in the Euclidian plane can be represented as a linear combination of cylinder harmonics of integral order.

19.

Since part (ii) gives a guess we might as well say λ cooresponds with $\bar{\lambda}$ for eigenvalues between L and L^*.

We can see this with u and v as in the satement of the problem

$$\langle Lu, v\rangle = \bar{\lambda}\langle u, v\rangle$$

from the definition of eigenvalue/function.

$$= \langle u, L^* v\rangle = \lambda' \langle u, v\rangle$$

Thus we see that if $\langle u, v\rangle \neq 0$ that $\bar{\lambda} = \lambda'$.

*********8MAKE NOTE FOR LATER POST*****Problem is Done though*************
In fact it is clear from the above that $\bar{\lambda}$ being an eigenvalue of the adjoint is implied by $\exists v, \langle u, v\rangle \neq 0$. Thus if an operator is self adjoint we see that $\lambda = \bar{\lambda}$ and the eigenvalues must all be real valued.

We consider:

$$u'' + b(x)u' + c(x)u = 0$$

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Let u be some solution, let us try $v(x) = F(x)u(x)$ then:

$$v' = F'u + Fu', v'' = F''u + 2F'u' + Fu''$$

Now we plug this into $v'' + Qv$ and find:

$$F''u + 2F'u' + Fu'' + Fu = F''u + 2F'u' + F(u'' + Qu)$$

$$= F\left[u'' + 2F'/Fu' + (F''/F + Q)u\right]$$

We know that $u'' = -bu' - cu$

$$= F\left[(2F'/F - b)u' + (F''/F + Q - c)u\right]$$

If we let $Q = c - F''/F$ then all we have to do is solve $2F'/F = b$. This leads to

$$2F'/F - b = 0 \Rightarrow 2\int^x F'/F = \int^x b \Rightarrow 2\ln F = \int^x b$$

$$\Rightarrow F = \exp\{\frac{1}{2}\int^x b\}$$

Thus our subsitition ends up being: $v(x) = \exp\{\frac{1}{2}\int^x b\}u(x)$. Note that $F' = \frac{b}{2}F,\ F'' = \frac{b'+b^2/2}{2}F$ and our equation gets:

$$Q = c - F''/F = c - F''/F = c - \frac{b' + b^2/2}{2}$$

All togther we have:

$$u \rightarrow v = \exp\{\frac{1}{2}\int^x b\}u(x),$$

$$u'' + b(x)u' + c(x)u = 0 \ \rightarrow \ v'' + Q(x)v = 0,\ Q(x) = c - \frac{b' + b^2/2}{2}$$