1 Chapter 1

1.1 Subsection 5

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1. Show that
$$\langle f, g\rangle = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

$$\langle f, g\rangle = \int_a^b \bar{f}(x)g(x)\rho(x)\mathrm{d}x$$

Since we have that $f(x) = \sum_{k=1}^{\infty} u_k(x)c_k$ we can substitiute and get:

$$= \int_a^b \overline{\sum_{k=1}^{\infty} u_k(x)c_k} g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x)\bar{c}_k g(x)\rho(x)\mathrm{d}x$$

$$= \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x)\bar{c}_k g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k \int_a^b \bar{u}_k(x)g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

2.
$$Tf(\omega, t) = \int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x} f(x)\mathrm{d}x$$

$$\langle h_1, h_2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{h}_1(\omega, t)h_2(\omega, t)\mathrm{d}\omega\mathrm{d}t$$

Find a formula for: $\langle Tf_1, Tf_2\rangle$ in terms of

$$\int_{-\infty}^{\infty} \bar{f}_1 f_2 \mathrm{d}x$$

$$\langle Tf_1, Tf_2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{Tf_1}Tf_2\mathrm{d}\omega\mathrm{d}t =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\left[\int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x} f_1(x)\mathrm{d}x\right]} \left[\int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x} f_2(x)\mathrm{d}x\right] \mathrm{d}\omega\mathrm{d}t$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(x-t)e^{-i\omega x} f_1(x)}\bar{g}(y-t)e^{-i\omega y} f_2(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\omega\mathrm{d}t$$

We now return to Calc III and need to do a replacment of variables:

$$\begin{matrix} u = x - y, & v = y \\ x = u + v, & y = y \end{matrix} \text{ which has deterimnate: } J = \left|\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right| = 1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\int_{-\infty}^{\infty} e^{i\omega u}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

$$\int_{-\infty}^{\infty} e^{i\omega u}\mathrm{d}\omega = \delta(u)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\delta(u)\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(v-t)f_1(v)}\bar{g}(v-t)f_2(v)\mathrm{d}v\mathrm{d}t = \int_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)\int_{-\infty}^{\infty} g(v-t)\bar{g}(v-t)\mathrm{d}t\mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)|g|^2\mathrm{d}v = |g|^2\int_{-\infty}^{\infty} \bar{f}_1 f_2\mathrm{d}x = |g|^2\langle f_1, f_2\rangle$$

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3. 1.5.3

*********************************** [This seems tedious and maybe not worth it]

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2 Chapter 2

2.1 Subsection 1

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4. 2.1.1 Suppose $f(x) = f(x+2\pi)\ \forall x$ is periodic with period 2π. Show

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x,\ \forall a \in \mathbb{R}$$

As all great math proofs, no words are needed just equalities and beautiful integrals. Let a be given then:

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_{2\pi}^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x+2\pi)\mathrm{d}x$$

$$= \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x$$

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5. 2.1.2 **Dirichelet Basis**

$$W_{2N+1} = \mathrm{span}\left\{\frac{1}{\sqrt{2\pi}}e^{ikt}\right\}_{k=\pm N}$$

Consider the set

$$g_k(t) = \frac{2\pi}{2N+1}\delta_N(t-x_k) = \frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(t-k\pi/(N+\frac{1}{2})}$$

Show that

A) $B = \{g_k, k \in 1, 2, \cdots\}$ is linearly independent.

B) B spans W_{2N+1}

A) It suffices to notice that $g_k(x_l) = \delta_{kl}$. Thus we can see for any given k that g_k is independent of all the other's as $\sum_{k' \neq k} \lambda_{k'} g_{k'}(x_k) = 0$. Thus we can not have a non trivial linear relationship between the functions.

B) It is clear that $g_k(t) \in W_{2N+1}$ since each of the elements in its sum namely $e^{in(t-k\pi/(N+\frac{1}{2})}$ is just a multiple of e^{int} a basis element of W_{2N+1}. Notice there are $2N+1$ of these independnet vectors in the vector space of dimmension $2N+1$. Thus they must be a spanning set and there must exist coefficents for any function in the space to be written as a sum of this basis.

To actually exhibit coefficents one would use $f(t) = \sum_k f(x_k) g_k(t)$.

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6. 2.1.3 **Riemann-Lebesgue Lemma**
   $G(u)$ piecewise continuous and has left and right derivatives on $[0, 2\pi]$. Show that

$$\lim_{N\to\infty} \int_0^{2\pi} G(u) \sin(N + \frac{1}{2}) u \mathrm{d}u = 0$$

WLOG $\exists a, b \in [0, \pi]$ st. $\forall x \in [a, b]$ $G(x) > 0$ or $G(x) < 0$.

Now it suffices to show

$$\lim_{N\to\infty} \int_a^b G(u) \sin(N + \frac{1}{2}) u \mathrm{d}u = 0$$

[since the interval $[0, 2\pi]$ can be sliced into a countable number of these intervals, and hten you can sum over them] WLOG we assume $G(x)$ is positve.

$$0 \leqslant \lim_{N\to\infty} \int_a^b G(u) \sin(N + \frac{1}{2}) u \mathrm{d}u \leqslant \lim_{N\to\infty} \int_a^b [\max_u G(u)] \sin(N + \frac{1}{2}) u \mathrm{d}u$$

Let G_m be the max above, then we have

$$0 \leqslant G_m \lim_{N\to\infty} \int_a^b \sin(N + \frac{1}{2}) u \mathrm{d}u = G_m \lim_{N\to\infty} \frac{\cos(N + \frac{1}{2})u}{N + \frac{1}{2}} \Big|_a^b \leqslant G_m \lim_{N\to\infty} \frac{2}{N + \frac{1}{2}} \leqslant 0$$

Thus we get the 0 value for the limit as desired.

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7. 2.1.4 Prove or disprove:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m + x)^2 + a^2} = \frac{\pi}{a} \frac{\coth \pi a}{\cos^2 \pi x + \sin^2 \pi x \coth^2 \pi a}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m + x)^2} = \frac{\pi^2}{\sin^2 \pi x}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m + x)^2} = \frac{1}{4 \sin^2 x/2}$$

Let $f(m) = \frac{1}{(m+x)^2+a^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int_{-\infty}^{\infty} e^{-ikm} f(m)\mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int_{-\infty}^{\infty} e^{-ikm}\frac{1}{(m+x)^2+a^2}\mathrm{d}m = e^{ikx}\int_{-\infty}^{\infty} e^{-iku}\frac{1}{u^2+a^2}\mathrm{d}u = e^{ikx}\frac{\pi}{a}e^{-|k|a}$$

Using the Poisson formula we thus see:

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+x)^2+a^2} = \sum_{k=-\infty}^{\infty} e^{i2\pi kx}\frac{\pi}{a}e^{-|2\pi k|a} = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi kx}e^{-|2\pi k|a} + \sum k \leqslant 0 e^{i2\pi kx}e^{-|2\pi k|a} - 1\right]$$

$$= \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + \sum k \leqslant 0 e^{i2\pi k(x+a)} - 1\right] = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + e^{-i2\pi k(x+a)} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1}{1-e^{-i2\pi(x+a)}} + \frac{1}{1-e^{i2\pi(x-a)}} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{-i2\pi(x+a)} + 1 - e^{i2\pi(x-a)} - \left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{i2\pi(x-a)}e^{-i2\pi(x+a)}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{1-e^{-i4\pi a}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

************BELIEVE THE ABOVE DISPROVES THE SUM********

For the last 2 the answers are somewhat lack luster.

Let $f(m) = \frac{1}{(m+x)^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int_{-\infty}^{\infty} e^{-ikm} f(m)\mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int_{-\infty}^{\infty} e^{-ikm}\frac{1}{(m+x)^2}\mathrm{d}m = e^{ikx}\int_{-\infty}^{\infty} e^{-iku}\frac{1}{u^2}\mathrm{d}u$$

This is an error function producing integral. This is bad. Notice also that the claimed value on the right hand side of the equation doesn't even make sense for $x = 0$. Thus the equation as stated is clearly false, Besides as any good student of Bergelson knows: $\sum 1/n^2 = \frac{\pi^2}{6}$

For the final one we again have issues with $\sin(0) = 0$ in the denominator.

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8. 2.1.5 My man Stephane G. Mallat claims the following: The family of funcitons $\phi(x-k)k = 0, \pm 1, \pm 2, \cdots$ is orthonormal iff
$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$
is constant wrt ω. Prove my boy wrong or right.

Stephane is no chump and said a true thing. Lets investigate the sum:

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}(\omega + 2\pi k)}\hat{\phi}(\omega + 2\pi k)$$

Now to avoid a factor out front the rest of the analysis, the $\frac{1}{\sqrt{2\pi}}$ is suppressed when expanding the Fourier transform.

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega+2\pi k)y}\bar{\phi}(y)\mathrm{d}y \int_{-\infty}^{\infty} e^{-i(\omega+2\pi k)x}\phi(x)\mathrm{d}x$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\bar{\phi}(y)e^{i(\omega+2\pi k)y}e^{-i(\omega+2\pi k)x}\mathrm{d}x\mathrm{d}y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\bar{\phi}(y) \sum_{k=-\infty}^{\infty} e^{-i(\omega+2\pi k)(x-y)}\mathrm{d}x\mathrm{d}y$$

Via formula on page 62

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x)\bar{\phi}(y)e^{-i\omega(x-y)} \sum_{k=-\infty}^{\infty} \delta(x-y-k)\mathrm{d}x\mathrm{d}y$$

Now for the change of variables $u = x - y, v = y$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(u+v)\bar{\phi}(v)e^{-i\omega u} \sum_{k=-\infty}^{\infty} \delta(u-k)\mathrm{d}u\mathrm{d}v = \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega u}\phi(u+v)\delta(u-k)\mathrm{d}u\mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} e^{-i\omega k}\phi(k+v)\mathrm{d}v = \sum_{k=-\infty}^{\infty} e^{-i\omega u} \int_{-\infty}^{\infty} \bar{\phi}(v)\phi(k+v)\mathrm{d}v = \sum_{k=-\infty}^{\infty} \langle \phi(v), \phi(k+v)\rangle e^{-i\omega u}$$

Thus the sum above is in fact a Fourier series with $c_k = \langle \phi(v), \phi(k+v)\rangle$. Now this series being constnat is equivalent to $c_k = \delta_{0k}$, which is equivalent to the $\phi(v+k)$'s being an orthogonal system.

Moreover if $c_0 = 1$ then we have an orthonormal system aswell. Thus the system is orthonormal if the series is constant and equal to 1. Now in actuallity we rememeber that we have a secret factor of $\frac{1}{\sqrt{2\pi}}$ hanging around. Thus the constants value is actually that.

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9. 2.1.6 Prove or Disprove the following identities:

   i)

$$\sum_{m=-\infty}^{\infty} f([2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(n)$$

   ii)

$$2\pi \sum_{m=-\infty}^{\infty} (-1)^m f(2\pi m) = \sum_{n=-\infty}^{\infty} F(n + \frac{1}{2})$$

iii)

$$\sum_{m=-\infty}^{\infty} \delta(u - [2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inu}$$

iv) And in greater generality

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

v)

$$\sum_{m=-\infty}^{\infty} \frac{1}{|a|}\delta(u - \frac{[2m+1]\pi}{a}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inau}$$

The main equation to keep in mind here is the general Poisson formula:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(x-t)} f(t) \mathrm{d}t = \sum_{m=-\infty}^{\infty} f(x + 2\pi m)$$

i) Begin with $x = \pi$ in the formula above and we see:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(\pi-t)} f(t) \mathrm{d}t = \sum_{m=-\infty}^{\infty} f([2m+1]\pi)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} f(t) \mathrm{d}t = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(n)$$

ii)

iii)

iv)

v)

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10. 2.2.1
