Contents

1 Infinite Dimensional Vector Spaces

1.3 Metric Spaces

1.3 Problem 1.

Show that a) the hamming distance, b) the Pythagorean distance and c) the Chebyshev distance each satisfy the triangle inequality.

a) Hamming distance is defined by

$$d(x, y) = \sum_i |x_i - y_i|$$

We see that:

$$\sum_i |x_i - y_i| = \sum_i |x_i - z_i + z_i - y_i| \leqslant \sum_i |x_i - z_i| + |z_i - y_i|$$

$$\sum_i |x_i - z_i| + \sum_i |z_i - y_i| = d(x, z) + d(y, z)$$

b) Pythagorean distance

$$d(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$$

$$\sqrt{\sum_i (x_i - y_i)^2} = \sqrt{\sum_i (x_i - z_i + z_i - y_i)^2}$$

$$\leqslant \sqrt{\sum_i (x_i - z_i)^2 + (z_i - y_i)^2} = \sqrt{\sum_i (x_i - z_i)^2 + \sum_i (z_i - y_i)^2} \leqslant \sqrt{\sum_i (x_i - z_i)^2} + \sqrt{\sum_i (z_i - y_i)^2}$$

c) The Chebyshev distance

$$d(x, y) = \max\{|x_i - y_i|\}$$

$$\max\{|x_i - y_i|\} = \max\{|x_i - z_i + z_i - y_i|\} \leqslant \max\{|x_i - z_i| + |z_i - y_i|\}$$

$$\leqslant \max\{|x_i - z_i|\} + \max\{|z_i - y_i|\}$$

1.5 Hilber Spaces

1.5 Problem 1.

Show that

$$\langle f, g\rangle = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

with

$$\langle f, g\rangle = \int_a^b \bar{f}(x) g(x)\rho(x)\mathrm{d}x$$

Since we have that $f(x) = \sum_{k=1}^{\infty} u_k(x) c_k$ we can substitiute and get:

$$= \int_a^b \overline{\sum_{k=1}^{\infty} u_k(x) c_k} g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x)\bar{c}_k g(x)\rho(x)\mathrm{d}x$$

$$= \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x)\bar{c}_k g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k \int_a^b \bar{u}_k(x) g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

1.5 Problem 2.

$$Tf(\omega, t) = \int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x} f(x)\mathrm{d}x$$

$$\langle h_1, h_2\rangle = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \bar{h}_1(\omega, t) h_2(\omega, t)\mathrm{d}\omega\mathrm{d}t$$

Find a formula for: $\langle Tf_1, Tf_2\rangle$ in terms of

$$\int\limits_{-\infty}^{\infty} \bar{f}_1 f_2 \mathrm{d}x$$

$$\langle Tf_1, Tf_2\rangle = \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \overline{Tf_1} Tf_2 \mathrm{d}\omega\mathrm{d}t =$$

$$\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \overline{\left[\int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x} f_1(x)\mathrm{d}x\right]} \left[\int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x} f_2(x)\mathrm{d}x\right] \mathrm{d}\omega\mathrm{d}t$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \overline{\bar{g}(x-t)e^{-i\omega x}f_1(x)}\bar{g}(y-t)e^{-i\omega y}f_2(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\omega\mathrm{d}t$$

We now return to Calc III and need to do a replacment of variables:

$$\begin{matrix} u = x - y, & v = y \\ x = u + v, & y = y \end{matrix} \text{ which has deterimnate: } J = \left|\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right| = 1$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\int\limits_{-\infty}^{\infty} e^{i\omega u}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

Subbing in the following identity for δ

$$\int\limits_{-\infty}^{\infty} e^{i\omega u}\mathrm{d}\omega = \delta(u)$$

we see:

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\delta(u)\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \overline{\bar{g}(v-t)f_1(v)}\bar{g}(v-t)f_2(v)\mathrm{d}v\mathrm{d}t = \int\limits_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)\int\limits_{-\infty}^{\infty} g(v-t)\bar{g}(v-t)\mathrm{d}t\mathrm{d}v$$

$$= \int\limits_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)|g|^2\mathrm{d}v = |g|^2\int\limits_{-\infty}^{\infty} \bar{f}_1 f_2\mathrm{d}x = |g|^2\langle f_1, f_2\rangle$$

1.5 Problem 3.

i) Show that the set of functions

$$\left\{ \frac{\sin \pi(2\omega z - k)}{\pi(2\omega z - k)} = \text{sinc}(2\omega z - k),\ k \in \mathbb{Z} \right\}$$

is an orthognal set satisfying:

$$\int\limits_{-\infty}^{\infty} \text{sinc}(2\omega z - k)\,\text{sinc}(2\omega z - l)\mathrm{d}z = A\delta_{kl}$$

What is A? To show orthogonality follow this 3 step outline:

a)
$$\int_{-\pi}^{\pi} \delta_N\left(t - \frac{2\pi}{2N+1}k\right)\delta_N\left(t - \frac{2\pi}{2N+1}l\right)\mathrm{d}t = \frac{2N+1}{2\pi}\delta_{kl}$$

where

$$\delta_N(u) = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})u}{\sin\frac{u}{2}} = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inu}$$

b) Then rescaling the integration domain by using $z = \frac{N+\frac{1}{2}}{2\pi\omega}t$.

c) and finally going to the limit $N \to \infty$.

ii) This set of functions

$$\left\{ u_k = \frac{1}{\sqrt{A}}\,\text{sinc}(2\omega z - k),\ k \in \mathbb{Z} \right\}$$

is not complete on L^2, but is complete on a specific subset. What is this subset, ie what property must a function $f(t)$ satisfy in order to be in this subset?

i) If we take $k = 0 = l$ then we can calculate A:

$$A = \int\limits_{-\infty}^{\infty} \text{sinc}(2\omega z)\,\text{sinc}(2\omega z)\mathrm{d}z = \int\limits_{-\infty}^{\infty} \text{sinc}(2\omega z)^2\mathrm{d}z = \int\limits_{-\infty}^{\infty} \left(\frac{\sin\pi(2\omega z)}{\pi 2\omega z)}\right)^2 \mathrm{d}z$$

Now with $u = 2\pi\omega z$, $\mathrm{d}u = 2\pi\omega\mathrm{d}z$ we can change variables and see:

$$= \int\limits_{-\infty}^{\infty} \left(\frac{\sin(u)}{u}\right)^2 \frac{1}{2\pi\omega}\mathrm{d}u = \frac{1}{2\pi\omega}\int\limits_{-\infty}^{\infty} \left(\frac{\sin(u)}{u}\right)^2 \mathrm{d}u$$

Now we can apply integration by parts with

$$\begin{cases} u = \sin^2 x & \mathrm{d}u = 2\sin x\cos x = \sin(2x) \\ v = -x^{-1} & \mathrm{d}v = -x^{-2} \end{cases}$$

and arrive at:

$$= \frac{1}{2\pi\omega}\left[\frac{-\sin^2 x}{x}|_{\pm\infty} - \int\limits_{-\infty}^{\infty}\frac{\sin(2x)}{-x}\mathrm{d}x\right] = \frac{1}{2\pi\omega}\left[\int\limits_{-\infty}^{\infty}\frac{\sin(2x)}{x}\mathrm{d}x\right]$$

With another change of variables $2x \to x$, and noticing that $sinc(x)$ is an even function we get:

$$\frac{1}{2\pi\omega}\int\limits_{-\infty}^{\infty}\frac{\sin(x)}{x}\mathrm{d}x = \frac{1}{\pi\omega}\int_0^{\infty}\frac{\sin(x)}{x}\mathrm{d}x = \frac{1}{\pi\omega}\frac{\pi}{2} = \frac{1}{2}\omega^{-1} = A$$

a) Let $u_k = t - \frac{2\pi}{2N+1}k$, then

$$\int_{-\pi}^{\pi}\delta_N\left(t - \frac{2\pi}{2N+1}k\right)\delta_N\left(t - \frac{2\pi}{2N+1}l\right)\mathrm{d}t = \int_{-\pi}^{\pi}\delta_N(u_k)\,\delta_N(u_l)\,\mathrm{d}t$$

$$= \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\sum_{n=-N}^{N}e^{inu_k}\right)\left(\frac{1}{2\pi}\sum_{n=-N}^{N}e^{inu_l}\right)\mathrm{d}t$$

$$= \int_{-\pi}^{\pi}\frac{1}{4\pi^2}\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}e^{i(n_1u_k+n_2u_l)}\mathrm{d}t = \frac{1}{4\pi^2}\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}\int_{-\pi}^{\pi}e^{i(n_1u_k+n_2u_l)}\mathrm{d}t$$

Considering now just the integral:

$$\int_{-\pi}^{\pi}e^{i(n_1u_k+n_2u_l)}\mathrm{d}t = \int_{-\pi}^{\pi}\exp\left\{i\left(n_1\left[t - \frac{2\pi}{2N+1}k\right] + n_2\left[t - \frac{2\pi}{2N+1}l\right]\right)\right\}\mathrm{d}t$$

$$= \int_{-\pi}^{\pi}\exp\left\{i(n_1+n_2)t - i\left[n_1\frac{2\pi}{2N+1}k + n_2\frac{2\pi}{2N+1}l\right]\right\}\mathrm{d}t$$

If we let $C_{n_1n_2lk} = \exp\left\{-i\frac{2\pi}{2N+1}\left[n_1k + n_2l\right]\right\}$ then we see get:

$$= \frac{C_{n_1n_2lk}}{i(n_1+n_2)}\exp\left\{i(n_1+n_2)t\right\}|_{t=\pm\pi} = C_{n_1n_2lk}\delta_{-n_1,n_2}2\pi$$

Plugging this back into the above we get:

$$= \frac{1}{2\pi}\sum_{n_1=-N}^{N}\sum_{n_2=-N}^{N}C_{n_1n_2lk}\delta_{-n_1,n_2} = \frac{1}{2\pi}\sum_{n=-N}^{N}C_{n,-nlk} = \frac{1}{2\pi}\sum_{n=-N}^{N}\exp\left\{-in\frac{2\pi}{2N+1}\left[k-l\right]\right\}$$

$$= \frac{1}{2\pi}\sum_{n=-N}^{N}\exp\left\{-in\frac{2\pi}{2N+1}\left[k-l\right]\right\}$$

Clearly when $k = l$ then we get $\frac{1}{2\pi}\sum_{n=-N}^{N}1 = \frac{2N+1}{2\pi}$. Now for $k \neq l$ we see that this is just

$$\delta_N\left(\frac{2\pi}{2N+1}(k-l)\right) = \frac{1}{2\pi}\frac{\sin(\pi(k-l))}{\sin\frac{\pi}{2N+1}(k-l)}$$

Notice that the denominator is never zero and that the top always is. Thus we arrive at:

$$\int_{-\pi}^{\pi}\delta_N\left(t - \frac{2\pi}{2N+1}k\right)\delta_N\left(t - \frac{2\pi}{2N+1}l\right)\mathrm{d}t = \frac{2N+1}{2\pi}\delta_{kl}$$

b) Now we wish to make the change of variable $z = \frac{N+\frac{1}{2}}{2\pi\omega}t$.

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}} \delta_N\left(\frac{2\pi}{2N+1}(2z\omega-k)\right)\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-l)\right)\frac{4\pi\omega}{2N+1}\mathrm{d}z = \frac{2N+1}{2\pi}\delta_{kl}$$

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}} \delta_N\left(\frac{2\pi}{2N+1}(2z\omega-k)\right)\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-l)\right)\mathrm{d}z = \frac{1}{2}\left(\frac{2N+1}{2\pi}\right)^2\omega^{-1}\delta_{kl}$$

$$\delta_N\left(\frac{2\pi}{2N+1}(2z\omega-k)\right) = \frac{1}{2\pi}\frac{\sin\left((N+\frac{1}{2})\frac{2\pi}{2N+1}(2z\omega-k)\right)}{\sin\left(\frac{2\pi}{2N+1}(2z\omega-k)\right)/2}$$
$$= \frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-k)\right)}{\sin\left(\frac{1}{2N+1}\pi(2z\omega-k)\right)}$$

c) Now since we are taking the limit as $N\to\infty$ we can ignore all higher terms in the sin expansion and just leave the linear factor.

$$\sim \frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-k)\right)}{\frac{1}{2N+1}\pi(2z\omega-k)}$$

Putting this back into our integral equation yields:

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}} \frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-k)\right)}{\frac{1}{2N+1}\pi(2z\omega-k)}\frac{1}{2\pi}\frac{\sin\left(\pi(2z\omega-l)\right)}{\frac{1}{2N+1}\pi(2z\omega-l)}\mathrm{d}z = \frac{1}{2}\left(\frac{2N+1}{2\pi}\right)^2\omega^{-1}\delta_{kl}$$

With some simplificiation gives us:

$$\int_{-\frac{N+\frac{1}{2}}{2\omega}}^{\frac{N+\frac{1}{2}}{2\omega}} \frac{\sin\left(\pi(2z\omega-k)\right)}{\pi(2z\omega-k)}\frac{\sin\left(\pi(2z\omega-l)\right)}{\pi(2z\omega-l)}\mathrm{d}z = \frac{1}{2}\omega^{-1}\delta_{kl}$$

Now taking limits we get the desired equality and see that A was in fact $\frac{1}{2\omega}$

$$\int_{-\infty}^{\infty} \frac{\sin\left(\pi(2z\omega-k)\right)}{\pi(2z\omega-k)}\frac{\sin\left(\pi(2z\omega-l)\right)}{\pi(2z\omega-l)}\mathrm{d}z = \frac{1}{2}\omega^{-1}\delta_{kl}$$

Thus the set is in fact an orhtogonal set satisfying the above.

ii) Limited bandwith functions
[According to wikipedia this is bascily shannon's original line or reasoning]
If we wish to avoid concerns with infinite sums, then we simply want to consider $\sum_{-N}^{N}|c_k|^2$.
Thus we want the function f to be represnetable as:

$$f(t) = \sum_{k=-N}^{N} c_k\frac{1}{\sqrt{A}}\operatorname{sinc}(2\omega t-k)$$

Now

$$c_k = \langle f, u_k\rangle = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{A}}\operatorname{sinc}(2\omega t-k)$$

*************PACKET*****************

Wonderful derivation from stack exchange. A usefull lemma here is the fourier transform of $\operatorname{sinc} x$. To this end notice that $sin(x) = \int_{-1}^{1} e^{iyx}\mathrm{d}y$.

$$\hat{\operatorname{sinc}}[\omega] = \int_{-\infty}^{\infty} \operatorname{sinc} x e^{-i\omega x}\mathrm{d}x = \int_{-\infty}^{\infty}\int_{-1}^{1} e^{iyx}\mathrm{d}y e^{-i\omega x}\mathrm{d}x$$

$$= \int_{-\infty}^{\infty}\int_{-1}^{1} e^{-i(\omega-y)x}\mathrm{d}y\mathrm{d}x = \int_{-1}^{1}\int_{-\infty}^{\infty} e^{-i(\omega-y)x}\mathrm{d}x\mathrm{d}y$$

$$= \int_{-1}^{1} \delta(\omega - y)\mathrm{d}y = \begin{cases} 1 & |\omega| \leqslant 1 \\ 0 & |\omega| > 1 \end{cases}$$

Thus we see that the fourier transform of sinc is a rectangular function.

So if f has a finite specturm then we can simply apply the fourier inversion theorm to get:

$$f(t) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \hat{f}(\omega)e^{i\omega t}\mathrm{d}\omega$$

If we let $t = n/2B$ then we get:

$$f(n/2B) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} \hat{f}(\omega)e^{i\omega n/2B}\mathrm{d}\omega$$

Notice that this is the caluclation of the n th fourier coefficent of $\hat{f}$ (on the interval $\pm B$) , and that this is the nth 'sample' of f. We can conclude from this that f is determined as a badwidth limited fucntion with bandwith limit B, by its $\lfloor B \rfloor$ many samples. So we can rewrite f as:

$$f(t) = \sum_{n=-B}^{B} \frac{1}{2B} x(n/2B)e^{-i\pi n/Bt}$$

Now with these 4-ier coefficents of $\hat{f}$ we can rewrite it aswell:

$$\hat{f}(\omega) = \operatorname{rect}(\omega/2B)\sum_{n=-B}^{B} \frac{1}{2B} x(n/2B)e^{-i\pi n/Bt} = \sum_{n=-B}^{B} x(n/2B)\frac{1}{2B}\operatorname{rect}(\omega/2B)e^{-i\pi n/Bt}$$

Using the divine foresight we have on sinc functions we can rewrite $\hat{f}$ as:

$$\hat{f} = \sum_{n=-B}^{B} x(n/2B)\mathcal{F}[\operatorname{sinc}(t2B - n)]$$

Now taking $\mathcal{F}^{-1}$ of both sides we see:

$$f = \sum_{n=-B}^{B} x(n/2B)\operatorname{sinc}(t2B - n)$$

Thus our Parsevel relation is satisfied, since the sum is finite.

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2 Chapter 2

2.1 Subsection 1

2.1 Problem 1.

Suppose $f(x) = f(x+2\pi)\ \forall x$ is periodic with period 2π. Show

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x,\ \forall a \in \mathbb{R}$$

As all great math proofs, no words are needed just equalities and beautiful integrals. Let a be given then:

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_{2\pi}^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x+2\pi)\mathrm{d}x$$
$$= \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x$$

2.1 Problem 2.

Dirichelet Basis

$$W_{2N+1} = \mathrm{span}\left\{\frac{1}{\sqrt{2\pi}}e^{ikt}\right\}_{k=\pm N}$$

Consider the set

$$g_k(t) = \frac{2\pi}{2N+1}\delta_N(t-x_k) = \frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(t-k\pi/(N+\frac{1}{2})}$$

Show that

A) $B = \{g_k, k \in 1, 2, \cdots\}$ is linearly independent.

B) B spans W_{2N+1}

A) It suffices to notice that $g_k(x_l) = \delta_{kl}$. Thus we can see for any given k that g_k is independent of all the other's as $\sum_{k'\neq k}\lambda_{k'}g_{k'}(x_k) = 0$. Thus we can not have a non trivial linear relationship between the functions.

B) It is clear that $g_k(t) \in W_{2N+1}$ since each of the elements in its sum namely $e^{in(t-k\pi/(N+\frac{1}{2})}$ is just a multiple of e^{int} a basis element of W_{2N+1}. Notice there are $2N+1$ of these independnet vectors in the vector space of dimmension $2N+1$. Thus they must be a spanning set and there must exist coefficents for any function in the space to be written as a sum of this basis.

To actually exhibit coefficents one would use $f(t) = \sum_k f(x_k)g_k(t)$.

2.1 Problem 3.

Riemann-Lebesgue Lemma
$G(u)$ piecewise continuous and has left and right derivatives on $[0, 2\pi]$. Show that

$$\lim_{N\to\infty} \int_0^{2\pi} G(u)\sin(N+\frac{1}{2})u\mathrm{d}u = 0$$

WLOG $\exists a, b \in [0, \pi]$ st. $\forall x \in [a, b]$ $G(x) > 0$ or $G(x) < 0$.

Now it suffices to show

$$\lim_{N\to\infty} \int_a^{b} G(u)\sin(N+\frac{1}{2})u\mathrm{d}u = 0$$

[since the interval $[0, 2\pi]$ can be sliced into a countable number of these intervals, and hten you can sum over them] WLOG we assume $G(x)$ is positve.

$$0 \leqslant \lim_{N\to\infty} \int_a^{b} G(u)\sin(N+\frac{1}{2})u\mathrm{d}u \leqslant \lim_{N\to\infty} \int_a^{b} [\max_u G(u)]\sin(N+\frac{1}{2})u\mathrm{d}u$$

Let G_m be the max above, then we have

$$0 \leqslant G_m \lim_{N\to\infty} \int_a^{b} \sin(N+\frac{1}{2})u\mathrm{d}u = G_m \lim_{N\to\infty} \frac{\cos(N+\frac{1}{2})u}{N+\frac{1}{2}}|_a^b \leqslant G_m \lim_{N\to\infty} \frac{2}{N+\frac{1}{2}} \leqslant 0$$

Thus we get the 0 value for the limit as desired.

2.1 Problem 4.

Prove or disprove:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2+a^2} = \frac{\pi}{a}\frac{\coth \pi a}{\cos^2 \pi x + \sin^2 \pi x \coth^2 \pi a}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2} = \frac{\pi^2}{\sin^2 \pi x}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m+x)^2} = \frac{1}{4\sin^2 x/2}$$

Let $f(m) = \frac{1}{(m+x)^2+a^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int_{-\infty}^{\infty} e^{-ikm} f(m) \mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int_{-\infty}^{\infty} e^{-ikm}\frac{1}{(m+x)^2+a^2}\mathrm{d}m = e^{ikx}\int_{-\infty}^{\infty} e^{-iku}\frac{1}{u^2+a^2}\mathrm{d}u = e^{ikx}\frac{\pi}{a}e^{-|k|a}$$

Using the Poisson formula we thus see:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2+a^2} = \sum_{k=-\infty}^{\infty} e^{i2\pi kx}\frac{\pi}{a}e^{-|2\pi k|a} = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi kx}e^{-|2\pi k|a} + \sum k \leqslant 0 e^{i2\pi kx}e^{-|2\pi k|a} - 1\right]$$

$$= \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + \sum k \leqslant 0 e^{i2\pi k(x+a)} - 1\right] = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + e^{-i2\pi k(x+a)} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1}{1-e^{-i2\pi(x+a)}} + \frac{1}{1-e^{i2\pi(x-a)}} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{-i2\pi(x+a)} + 1 - e^{i2\pi(x-a)} - \left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{i2\pi(x-a)}e^{-i2\pi(x+a)}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{1-e^{-i4\pi a}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

***********BELIEVE THE ABOVE DISPROVES THE SUM********

For the last 2 the answers are somewhat lack luster.

Let $f(m) = \frac{1}{(m+x)^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int\limits_{-\infty}^{\infty} e^{-ikm} f(m)\mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int\limits_{-\infty}^{\infty} e^{-ikm} \frac{1}{(m+x)^2}\mathrm{d}m = e^{ikx} \int\limits_{-\infty}^{\infty} e^{-iku} \frac{1}{u^2}\mathrm{d}u$$

This is an error function producing integral. This is bad. Notice also that the claimed value on the right hand side of the equation doesn't even make sense for $x = 0$. Thus the equation as stated is clearly false, Besides as any good student of Bergelson knows: $\sum 1/n^2 = \frac{\pi^2}{6}$

For the final one we again have issues with $\sin(0) = 0$ in the denominator.

2.1 Problem 5.

My man Stephane G. Mallat claims the following: The family of funcitons $\phi(x-k)k = 0, \pm 1, \pm 2, \cdots$ is orthonormal iff

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

is constant wrt ω. Prove my boy wrong or right.

Stephane is no chump and said a true thing. Lets investigate the sum:

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 = \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}(\omega + 2\pi k)}\hat{\phi}(\omega + 2\pi k)$$

Now to avoid a factor out front the rest of the analysis, the $\frac{1}{\sqrt{2\pi}}$ is suppressed when expanding the Fourier transform.

$$= \sum_{k=-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{i(\omega+2\pi k)y}\bar{\phi}(y)\mathrm{d}y \int\limits_{-\infty}^{\infty} e^{-i(\omega+2\pi k)x}\phi(x)\mathrm{d}x$$

$$= \sum_{k=-\infty}^{\infty} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi(x)\bar{\phi}(y)e^{i(\omega+2\pi k)y}e^{-i(\omega+2\pi k)x}\mathrm{d}x\mathrm{d}y$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \phi(x)\bar{\phi}(y) \sum_{k=-\infty}^{\infty} e^{-i(\omega+2\pi k)(x-y)}\mathrm{d}x\mathrm{d}y$$

Via formula on page 62

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x)\bar{\phi}(y) e^{-i\omega(x-y)} \sum_{k=-\infty}^{\infty} \delta(x-y-k) \mathrm{d}x\mathrm{d}y$$

Now for the change of variables $u = x - y, v = y$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(u+v)\bar{\phi}(v) e^{-i\omega u} \sum_{k=-\infty}^{\infty} \delta(u-k) \mathrm{d}u\mathrm{d}v = \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega u} \phi(u+v)\delta(u-k)\mathrm{d}u\mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} e^{-i\omega k}\phi(k+v)\mathrm{d}v = \sum_{k=-\infty}^{\infty} e^{-i\omega u} \int_{-\infty}^{\infty} \bar{\phi}(v)\phi(k+v)\mathrm{d}v = \sum_{k=-\infty}^{\infty} \langle \phi(v), \phi(k+v) \rangle e^{-i\omega u}$$

Thus the sum above is in fact a Fourier series with $c_k = \langle \phi(v), \phi(k+v) \rangle$. Now this series being constnat is equivalent to $c_k = \delta_{0k}$, which is equivalent to the $\phi(v+k)$'s being an orthogonal system.

Moreover if $c_0 = 1$ then we have an orthonormal system aswell. Thus the system is orthonormal if the series is constant and equal to 1. Now in actuallity we rememeber that we have a secret factor of $\frac{1}{2\pi}$ hanging around. Thus the constants value is actually that.

2.1 Problem 6.

Prove or Disprove the following identities:

i)

$$\sum_{m=-\infty}^{\infty} f([2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(n)$$

ii)

$$2\pi \sum_{m=-\infty}^{\infty} (-1)^m f(2\pi m) = \sum_{n=-\infty}^{\infty} F(n+\frac{1}{2})$$

iii)

$$\sum_{m=-\infty}^{\infty} \delta(u-[2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inu}$$

iv) And in greater generality

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

v)

$$\sum_{m=-\infty}^{\infty} \frac{1}{|a|}\delta(u-\frac{[2m+1]\pi}{a}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inau}$$

The main equation to keep in mind here is the general Poisson formula:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(x-t)} f(t)\mathrm{d}t = \sum_{m=-\infty}^{\infty} f(x+2\pi m)$$

i) Begin with $x=\pi$ in the formula above and we see:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(\pi-t)} f(t)\mathrm{d}t = \sum_{m=-\infty}^{\infty} f([2m+1]\pi)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} f(t)\mathrm{d}t = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(n)$$

ii) Begining with the right hand side:

$$\sum_{n=-\infty}^{\infty} F(n+\frac{1}{2}) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(n+\frac{1}{2})t} f(t)\mathrm{d}t = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-int} f(t) e^{-it/2}\mathrm{d}t$$

Now we notice this is the Fourier transform of not f but of $f(t)e^{-it/2}$, applying Poisson sum with this:

$$= 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m)e^{-i(2\pi m)/2} = 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m)e^{-i\pi m} = 2\pi \sum_{m=-\infty}^{\infty} (-1)^m f(2\pi m)$$

iii) It is straightforward to see that this is actually just 1) but in the supressed function notation. To see this we note that

$$\delta(u - [2m+1]\pi) \to f([2m+1]\pi), \quad e^{inu} \to \int_{-\infty}^{\infty} e^{inu} f(u)\mathrm{d}u = F(-n)$$

But wait we get $\sum_{n=-\infty}^{\infty}(-1)^n F(-n)$ and not the exact sum we wanted! Thankfully $(-1)^n = (-1)^{-n}$ and we just switch the order of the sum and get the identity.

iv) Let $\bar{f}(x) = f(\frac{x}{a})$, then by 1) we have:

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \sum_{m=-\infty}^{\infty} \bar{f}([2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \bar{F}(n)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} \bar{f}(t)\mathrm{d}t = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} f(t/a)\mathrm{d}t$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n a \int_{-\infty}^{\infty} e^{-inau} f(u)\mathrm{d}u = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n a F(na) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

v) Similar to 3) we note that this is just an earlier identity. A constant is shifted around but this is basicly just 4).

2.2 Dirac Delta Distribution

2.2 Problem 1.

> Show that
> $$\lim_{\omega\to\infty} \frac{\sin 2\pi\omega x}{\pi x}, \ \omega > 0$$
> is a representation of the Dirac δ DISTRIBUTION.

This equality can only be expressed inside of an integral, thus we must apply the above to test functions and see that the answer is the same as with the delta *distribution*.

Thus if we consider f continuous on some $[-a,a]$ then we get:

$$\lim_{\omega\to\infty}\int_{-a}^{a}\delta_\omega(x)f(x)\mathrm{d}x = \lim_{\omega\to\infty}\int_{-a}^{a}\frac{\sin 2\pi\omega x}{\pi x}f(x)\mathrm{d}x$$

Now in following with the style of the Fourier series theorem we add and subtract the same term, namely a $f(0)$ (inside some paranthesis but basicly the same)

$$= \lim_{\omega\to\infty}\int_{-a}^{a}\frac{\sin 2\pi\omega x}{\pi x}\left(f(x)-f(0)+f(0)\right)\mathrm{d}x = \lim_{\omega\to\infty}\int_{-a}^{a}\frac{\sin 2\pi\omega x}{\pi}\frac{f(x)-f(0)}{x}+f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

Now we can consider WLOG just the positive side of the integral.

$$\lim_{\omega\to\infty}\int_{0}^{a}\frac{\sin 2\pi\omega x}{\pi}\frac{f(x)-f(0)}{x}+f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

Notice for the exact same resonaning as $G(u)$ on page 57 that we get $\frac{f(x)-f(0)}{x}$ is continuous at 0 and converges to $f'(0^+)$. Thus again we see that the integral:

$$\int_{0}^{a}\frac{\sin 2\pi\omega x}{\pi}\frac{f(x)-f(0)}{x}\mathrm{d}x \to 0$$

as $\omega\to\infty$. Thus we only have:

$$\lim_{\omega\to\infty}\int_{0}^{a}f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x = f(0)\lim_{\omega\to\infty}\int_{0}^{a}\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

$$y = 2\pi\omega x, \quad dy = 2\pi\omega dx$$

$$= f(0)\lim_{\omega\to\infty}\int_{0}^{a2\pi\omega}\frac{\sin y}{\pi y/(2\pi\omega)}\frac{\mathrm{d}y}{2\pi\omega} = \frac{f(0)}{\pi}\int_{0}^{\infty}\frac{\sin y}{y}\mathrm{d}y = \frac{f(0)}{\pi}\frac{\pi}{2} = \frac{f(0)}{2}$$

Using a isomorphic version of the logic above one can get the $f(0^-)$ term and complete the proof.

2.2 Problem 2.

Assuming that $f(x)$ is nearly linear, that is to say that

$$f(-a) = f(0) - af'(0) + \text{ H.O.T.}$$

Show that

$$I = \int\limits_{-\infty}^{\infty}\delta(x+a)f(x)\mathrm{d}x$$

can be evaluated by means of the formal equation:

$$\delta(x+a) = \delta(x) + a\delta'(xx)$$

By the definition of the δ funciton we have:

$$I = \int_{-\infty}^{\infty} \delta(x+a) f(x) \mathrm{d}x = f(-a)$$

$$= f(0) - af'(0) + \text{ H.O.T.} = \int_{-\infty}^{\infty} \delta(x)f(x) - a\delta(x)f'(x)\mathrm{d}x = \int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x - a\int_{-\infty}^{\infty} \delta(x)f'(x)\mathrm{d}x$$

Via integration by parts we know that:

$$\int_{-\infty}^{\infty} \delta(x)f'(x)\mathrm{d}x = \delta(x)f(x)|_{\pm\infty} - \int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = -\int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x$$

Putting stuff together:

$$= \int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x + a\int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \left[\delta(x) + a\delta'(x)\right] f(x)\mathrm{d}x$$

Thus it makes some sense to claim $\delta(x+a) = \delta(x) + a\delta'(x)$

2.3 The Fourier Integral

2.3 Problem 1.

a) Consider the LInear Operator $\mathfrak{F}^2$ and its eigenvalue equatio

$$\mathfrak{F}^2 f = \lambda f$$

What are the eigenvalues and eigenfunctions of $\mathfrak{F}^2$?

b) Same with $\mathfrak{F}^4$?

c) Same with $\mathfrak{F}$?

For the sake of clarity:

$$\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\mathrm{d}x$$

a) Obviously we begin with the calculation in question

$$\mathcal{F}^2(f)[k] = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-iky}\int\limits_{-\infty}^{\infty} e^{-iyx} f(x)\mathrm{d}x\mathrm{d}y = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-i(ky+yx)} f(x)\mathrm{d}x\mathrm{d}y$$

This looks close to the idenity we are given on page 70, namely:

$$\delta(x-t) = \int\limits_{-\infty}^{\infty} \frac{1}{2\pi} e^{ik(x-t)} dk$$

which carries the note that we must inegrate on the outside with dt for this to make sense. Now rearranging some integrals and swapping x with $-x$ we arive at:

$$\int\limits_{-\infty}^{\infty} f(-x)\int\limits_{-\infty}^{\infty} \frac{1}{2\pi} e^{-iy(k-x)}\mathrm{d}y\mathrm{d}x = \int\limits_{-\infty}^{\infty} f(-x)\delta(k-x)\mathrm{d}x = f(-k)$$

Now we can see that the constraint: $\mathfrak{F}^2 f = \lambda f$ is really just $f(-x) = \lambda f(x)$. Two obvious cases come to mind, namely even and odd functions for the eigenvalues ± 1. For any other value of λ one could apply the relation twice to get $f(x) = \lambda^2 f(-x)$ which only has 2 roots. Thus those are the only eigenvalues of $\mathcal{F}^2$.

b) Thanks to $\mathfrak{F}^2 f[x] = f(-x)$ from the previous problem we know that $\mathcal{F}^4 = \mathcal{F}^2\mathcal{F}^2 f[x] = f(-(-x)) = f(x)$. Thus every function is an eigen function of $\mathcal{F}^4$ =Id, with eigenvalue 1.

c) We know that $\mathcal{F}^4$ =Id, and thus if λ is an eigenvalue of $\mathcal{F}$ then $\lambda^4 = 1$. Thus the only possible eigenvalules of $\mathcal{F}$ are 4th roots of unity. Thus the eigenvalues are $\pm 1, \pm i$.

2.3 Problem 2.

Let

$$W = \text{span}\{\phi, \mathcal{F}\phi, \mathcal{F}^2\phi, \dots\}$$

a) Show that W is finite dimensional, and what is its dimension?

b) Exhibit a basis for W.

c) It is evident that $\mathcal{F}$ is a unitary transform of W. Find the bassis representation matrx $[\mathcal{F}]_B$ relative to the basis B found in part b).

d) Find the secular determinant, the eigenvalues and the corresponding eigenvectors of $[\mathcal{F}]_B$.

e) For W exhibit an alternative basis which consists entirely of eigenvectors of $\mathcal{F}$, each one labelled by its respective eigenvalue.

f) What can you say about the eigenvalues of $\mathcal{F}$ as a transformation on L^2 as compared to $[\mathcal{F}]_B$ which acts ona finite dim. vector space

a) W clearly has dimension $\leqslant 4$ by the previous problem since $\mathcal{F}^4$ =Id. In fact if ϕ is even, we have only dimension 2 and the two basis elements of the space are just ϕ and $\mathcal{F}\phi$. Due to the limits of the roots of unity argument above we know that the only number of dimensions can be those two or 1, namely dim = 1,2, or 4.

b) The possible basis are ϕ, or $\phi, \mathcal{F}\phi$, or all 4: $\phi, \mathcal{F}\phi, \mathcal{F}^2\phi, \mathcal{F}^{-1}\phi$.

c) With the basis: $\phi, \mathcal{F}\phi, \mathcal{F}^2\phi, \mathcal{F}^{-1}\phi$

$$[\mathcal{F}]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

A classic style of shifting operator on a finite dimensional space.

d) Easily enough we see:

$$\det\left([\mathcal{F}]_B - \lambda I\right) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{bmatrix}\right) = \lambda^4 - 1$$

Thus the eigenvalues are $\pm 1, \pm i$. The corresponding vectors are listed below:

$$\begin{array}{ll} \lambda = 1 & \phi + \mathcal{F}\phi + \mathcal{F}^2\phi + \mathcal{F}^{-1}\phi \\ \lambda = -1 & \phi - \mathcal{F}\phi + \mathcal{F}^2\phi - \mathcal{F}^{-1}\phi \\ \lambda = i & \phi + i\mathcal{F}\phi - \mathcal{F}^2\phi - i\mathcal{F}^{-1}\phi \\ \lambda = -i & \phi - i\mathcal{F}\phi - \mathcal{F}^2\phi + i\mathcal{F}^{-1}\phi \end{array}$$

(Something something permutation matricies)

e) The eigenvalues of $\mathcal{F}$ are the same viewed as a finite-dimensional vecotr space and as an infinte dimensional one. This seems to have been forced by the simpliity of the characterisitc polynomial more than anything else.

2.3 Problem 3.

Define the equivalent width as

$$\Delta_t = \left| \frac{\int\limits_{-\infty}^{\infty} f(t)\mathrm{d}t}{f(0)} \right|$$

Define the equivalent Fourier width as

$$\Delta_\omega = \left| \frac{\int\limits_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t}{\hat{f}(0)} \right|$$

a) Show that $\Delta_t\Delta_\omega$ =const, is independent of the function f, and find its value.

b) Determine the equivalent width and Fourier width of

$$e^{-x^2/2b^2}$$

and compare them with its full width as defined by its inflection points.

a)

$$\Delta_t\Delta_\omega = \left| \frac{\int\limits_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t}{\hat{f}(0)} \right| \left| \frac{\int\limits_{-\infty}^{\infty} f(t)\mathrm{d}t}{f(0)} \right| = \left| \frac{\int\limits_{-\infty}^{\infty} \hat{f}(t)\mathrm{d}t \int\limits_{-\infty}^{\infty} f(t)\mathrm{d}t}{\hat{f}(0)f(0)} \right|$$

$$= \left| \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \hat{f}(x)f(y)\mathrm{d}x\mathrm{d}y}{\hat{f}(0)f(0)} \right| = \left| \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{2\pi}}e^{-ixz}f(z)\mathrm{d}z\mathrm{d}x\mathrm{d}y}{\hat{f}(0)f(0)} \right| =$$

$$\left| \frac{\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(y)f(z)\int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-ixz}\mathrm{d}x\mathrm{d}z\mathrm{d}y}{\hat{f}(0)f(0)} \right| = \left| \frac{\sqrt{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(y)f(z)\delta(z)\mathrm{d}z\mathrm{d}y}{\hat{f}(0)f(0)} \right| = \sqrt{2\pi} \left| \frac{\int\limits_{-\infty}^{\infty} f(y)\mathrm{d}y}{\hat{f}(0)} \right|$$

$$= \sqrt{2\pi}\left|\frac{\int\limits_{-\infty}^{\infty} f(y)\mathrm{d}y}{\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i*0*x} f(y)\mathrm{d}y}\right| = \sqrt{2\pi}\sqrt{2\pi} = 2\pi$$

b) With $f(x) = e^{-x^2/2b^2}$

Since I stared at completing the squares for way too long to justify not writing this down, here is the Fourier transform of the Gaussian:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-x^2/2b^2}e^{-i\omega x}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-1/2b^2\left[x^2+2b^2 i\omega x\right]}\mathrm{d}x$$

We complete the square in the exponent:

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-1/2b^2\left[x^2+2b^2 i\omega x-\omega^2 b^4\right]-\omega^2 b^2/2}\mathrm{d}x = \frac{1}{\sqrt{2\pi}}e^{-\omega^2 b^2/2}\int\limits_{-\infty}^{\infty} e^{-1/2b^2\left[x-\omega b^2\right]^2}\mathrm{d}x$$

Now with $|b|u = x - \omega b^2, |b|\mathrm{d}u = \mathrm{d}x$ we get:

$$= |b|e^{-\omega^2 b^2/2}\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\mathrm{d}u = |b|e^{-\omega^2 b^2/2}$$

So we have: $\hat{f}(\omega) = |b|e^{-\omega^2 b^2/2}$ and can now do our calculations. (we could before and actually don't need this at all but I'll be damned if I didn't spend too much time on this part to not just write something)

The Δ_ω is actually the famous Gaussian integral:

$$\Delta_\omega = \left|\frac{\int\limits_{-\infty}^{\infty} e^{-t^2/2b^2}\mathrm{d}t}{1}\right| = \left|\int\limits_{-\infty}^{\infty} e^{-t^2/2b^2}\mathrm{d}t\right| = \sqrt{\frac{\pi}{1/2b^2}} = \sqrt{\pi 2b^2} = |b|\sqrt{\pi 2}$$

Thanks to the relation $\Delta_t\Delta_\omega = 2\pi$ we see that Δ_t must be $1/|b|\sqrt{2\pi}$.

I am too stuburn to not write this after the above

$$\Delta_t = \left|\frac{\int\limits_{-\infty}^{\infty}\hat{f}(t)\mathrm{d}t}{\hat{f}(0)}\right| = \left|\frac{\int\limits_{-\infty}^{\infty}|b|e^{-\omega^2 b^2/2}\mathrm{d}\omega}{|b|}\right| = \left|\int\limits_{-\infty}^{\infty} e^{-\omega^2 b^2/2}\mathrm{d}\omega\right|$$

$$= \left|\int\limits_{-\infty}^{\infty} e^{-\omega^2 b^2/2}\mathrm{d}\omega\right| = \sqrt{2\pi/b^2} = \frac{1}{|b|}\sqrt{2\pi}$$

The inflection points are at $\pm b$ and thus its 'inflection witdth' is $2|b|$

2.3 Problem 4.

Define the auto-correlation h of the funciton f:

$$h(y) := \int_{-\infty}^{\infty} f(x)f(x-y)\mathrm{d}x$$

Compute the Fourier transform of the auto correlation funcation and show that it equals the "spectral intensity" (aka power spectrum) of f whenever f is real valued.

$$\hat{h}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iky} \int_{-\infty}^{\infty} f(x)f(x-y)\mathrm{d}x\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iky} f(x)f(x-y)\mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} e^{-ik(x-u)} f(u)\mathrm{d}u\mathrm{d}x = \int_{-\infty}^{\infty} e^{-ikx} f(x)\hat{f}(-k)\mathrm{d}x = \hat{f}(-k) \int_{-\infty}^{\infty} e^{-ikx} f(x)\mathrm{d}x$$

$$= \hat{f}(-k)\hat{f}(k) = |\hat{f}(k)|^2$$

The $\hat{f}(-k) = \overline{\hat{f}(k)}$ is implied by f being real valued and is the only point we make use of this fact.

2.3 Problem 5.

a) Compute the total energy

$$\int_{-\infty}^{\infty} |h(T)|^2 \mathrm{d}T$$

of the cross correlation $h(T)$ in terms of the Fourier amplitudes of f_0 and f.

b) Consider

$$h_k(T) = \frac{\int_{-\infty}^{\infty} \bar{f}_0(t-T)f_k(t)\mathrm{d}t}{\left[\int_{-\infty}^{\infty} |f_k(t)|^2\mathrm{d}t\right]^{1/2}}$$

i) Show that $h_0(t)$ is the peak intensity, ie $|h_k(T)|^2 \leqslant |h_0(T)|^2$.

ii) Show that equality holds if $f_k(t) = \kappa f_0(t)$ for κ some constant.

a) For a matchd filter we have that $\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f(t)\mathrm{d}t = h(T)$, using this: Using the fact that $\mathcal{F}$ is an isometery of L^2:

$$\int\limits_{-\infty}^{\infty} |h(T)|^2 \mathrm{d}T = \|h\|_2 = \left\|\hat{h}\right\|_2 = \int\limits_{-\infty}^{\infty} |\mathcal{F}h(\omega)|^2 \mathrm{d}\omega$$

$$= \int\limits_{-\infty}^{\infty} |\int\limits_{-\infty}^{\infty} e^{-iT\omega} h(T)\mathrm{d}T|^2 \mathrm{d}\omega = \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-iT\omega} \int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f(t)\mathrm{d}t\mathrm{d}T|^2 \mathrm{d}\omega$$

$$= \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) \int\limits_{-\infty}^{\infty} e^{-iT\omega} \bar{f}_0(t-T)\mathrm{d}T\mathrm{d}t|^2 \mathrm{d}\omega$$

Let $u = t - T,\ \mathrm{d}u = -\mathrm{d}T$ (the negative sign is lost in the $||$).

$$= \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(t) \int\limits_{-\infty}^{\infty} e^{-i(t-u)\omega} \bar{f}_0(u)\mathrm{d}u\mathrm{d}t|^2 \mathrm{d}\omega = \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-it\omega} f(t) \int\limits_{-\infty}^{\infty} e^{iu\omega} \bar{f}_0(u)\mathrm{d}u\mathrm{d}t|^2 \mathrm{d}\omega$$

Thanks to the relationship between conjugates and the fourier transform, namely $\mathcal{F}\bar{f}[k] = \mathcal{F}f[-k]$ we get:

$$= \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \hat{f}_0(\omega) \int\limits_{-\infty}^{\infty} e^{-it\omega} f(t)\mathrm{d}t|^2 \mathrm{d}\omega = \int\limits_{-\infty}^{\infty} |\frac{1}{\sqrt{2\pi}} \hat{f}_0(\omega)\hat{f}(\omega)|^2 \mathrm{d}\omega$$

$$= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |\hat{f}_0(\omega)\hat{f}(\omega)|^2 \mathrm{d}\omega$$

b) i) This problem is actually wrong, consider $f_0(t) = \mathbb{1}_{[0,1]}(t)$ and $f_1(t) = \mathbb{1}_{[2,3]}(t)$ and $T = 2$. Then $\|f_1\|_2 = 1 = \|f_0\|$ but

$$\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_0(t)\mathrm{d}t = 0$$

$$\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_k(t)\mathrm{d}t = 1$$

Which gives $h_1(T) > h_0(T)$, contrary to the statment of the problem.
**********************MESSGE GERLACH****************************

ii) Oddly enough the equality still holds. If we have $f_k(t) = \kappa f_0(t)$ then we see:

$$|h_k(T)|^2 = \left\| \frac{\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)\kappa f_0(t)\mathrm{d}t}{\left[\int\limits_{-\infty}^{\infty} |\kappa f_0(t)|^2 \mathrm{d}t\right]^{1/2}} \right\|^2 = \left\| \frac{\int\limits_{-\infty}^{\infty} \bar{f}_0(t-T)f_0(t)\mathrm{d}t}{\left[\int\limits_{-\infty}^{\infty} |f_0(t)|^2 \mathrm{d}t\right]^{1/2}} \right\|^2 = |h_0(T)|^2$$

2.3 Problem 6.

What functions are eigenvectors of $\mathcal{F}^2$ with eigenvalue $\lambda = 1$?

Already did this.

2.3 Problem 7.

Let $\hat{g}(k) = \mathcal{F}[g(x)](k)$ and $H(k) = \mathcal{F}[h](k)$ be the Fourier transforms of g and h. Express the following in terms of $\hat{g}$ and $\hat{f}$.

i) $\mathcal{F}[\alpha g + \beta h]$ for some constants α, β

ii) $\mathcal{F}[g(x-\xi)]$

iii) $\mathcal{F}[e^{ik_0 x} g]$

iv) $\mathcal{F}[g(ax)]$

v) $\mathcal{F}[\frac{dg}{dx}]$

vi) $\mathcal{F}[xg(x)]$

i) It is clear by the linearity of the integrals that we have:

$$\mathcal{F}[\alpha g + \beta h] = \mathcal{F}[\alpha g] + \mathcal{F}[\beta h] = \alpha \mathcal{F}[g] + \beta \mathcal{F}[h]$$

ii)

$$\mathcal{F}[g(x-\xi)] = \int_{-\infty}^{\infty} e^{-ikx} g(x-\xi) \mathrm{d}x = \int_{-\infty}^{\infty} e^{-ik(u+\xi)} g(u) \mathrm{d}u = e^{-ik(\xi)} \int_{-\infty}^{\infty} e^{-iku} g(u) \mathrm{d}u = e^{-ik(\xi)} \hat{g}$$

iii)

$$\mathcal{F}[e^{ik_0 x} g] = \int_{-\infty}^{\infty} e^{-ikx} e^{ik_0 x} g(x) \mathrm{d}x = \int_{-\infty}^{\infty} e^{-i(k-k_0)x} g(x) \mathrm{d}x = \hat{g}(k-k_0)$$

iv)

$$\mathcal{F}[g(ax)] = \int_{-\infty}^{\infty} e^{-ikx} g(ax) \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{a} e^{-iku/a} g(u) \mathrm{d}u = \frac{1}{a} \hat{g}(k/a)$$

v) Using integration by parts:

$$\mathcal{F}[\frac{dg}{dx}] = \int_{-\infty}^{\infty} e^{-ikx}\frac{dg}{dx}\mathrm{d}x = -\int_{-\infty}^{\infty} g\frac{de^{-ikx}}{dx}\mathrm{d}x = \int_{-\infty}^{\infty} ikge^{-ikx}\mathrm{d}x = ik\hat{g}(k)$$

vi)

$$\mathcal{F}[xg(x)] = \int_{-\infty}^{\infty} e^{-ikx}xg\mathrm{d}x = \int_{-\infty}^{\infty} xe^{-ikx}g\mathrm{d}x = \int_{-\infty}^{\infty}\left[\frac{1}{-i}\frac{d}{dk}e^{-ikx}\right]g\mathrm{d}x$$

$$= \frac{1}{-i}\frac{d}{dk}\int_{-\infty}^{\infty} e^{-ikx}g\mathrm{d}x = \frac{1}{-i}\frac{d}{dk}\hat{g}(k)$$

*************************DOUBLE CHECK LAST COUPLE

To make life easier to see, eveyrthin in a box gives:

i) $\mathcal{F}[\alpha g + \beta h] = \alpha\mathcal{F}g + \beta\mathcal{F}h$ for some constants α, β

ii) $\mathcal{F}[g(x-\xi)] = e^{ik\xi}\mathcal{F}g[k]$

iii) $\mathcal{F}[e^{ik_0x}g] = \mathcal{F}g[k-k_0]$

iv) $\mathcal{F}[g(ax)] = \frac{1}{a}\hat{g}(k/a)$

v) $\mathcal{F}[\frac{dg}{dx}] = ik\mathcal{F}g$

vi) $\mathcal{F}[xg(x)] = \frac{1}{-i}\frac{d}{dk}\hat{g}(k)$

2.3 Problem 8.

Show that any periodic function $f(\xi) = f(\xi + a)$ is the convolution of a nonperiodic function with a train of Dirac delta DISTRIBUTIONS.

[I was very stuck on this and stack exchange provided an answer.]
(This also ends up being the comb thing) Let $a > 0$ be the length of the period of the function $f(x)$. Then let $g(x) = \mathbb{1}_{[0,a)}(x)f(x)$ have value in the 'first' period of f and then be 0 elsewhere. Obviously g is non periodic unless $f = 0$ (that case being triival and not relavent). Now consider:

$$g \star \left(\sum_{n=-\infty}^{\infty}\delta(x-an)\right) = \sum_{n=-\infty}^{\infty} g\star\delta(u-(x-an)) = \sum_{n=-\infty}^{\infty} g(x-an) = f(x)$$

Thus we have written f as a convlution of a non periodic function and a train of delta distributions.

2.3 Problem 9.

Find the Fourier specturm of a finite train of identical coherent pulses of the kind shown in Fig. 2.9.

The function in reference is of the form:

$$f_n(t) = e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{i\delta_n}$$

Which in our specific case is:

$$f_n(t) = e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

So our sum is:

$$\sum_{n=-N}^{N} f_n(t) = \sum_{n=-N}^{N} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

We could in theory calculate the Fourier transform of each of the elmeents of the sum and then combine. However the text has outlined a process we can just semi blindly follow with less work.

To this end we notice that with $\delta_n = 0 = n\Delta\phi$ we have the same form as page 90 with:

$$f(t) == \sum_{n=-N}^{N} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} e^{in\Delta\phi}$$

but with finite bounds on our sum. Thankfully we can still rewrite it as a convolution:

$$f(t) = \int_{-\infty}^{\infty} e^{-(t-\xi)^2/2b^2} e^{i\omega_0(t-\xi)} \sum_{n=-N}^{N} \delta(\xi - nT) \mathrm{d}\xi$$

Our pulses have the same form as before, but our comb is much shorter this time.

$$\mathcal{F}[\text{pulse}](\omega) = \int_{-\infty}^{\infty} e^{-t^2/2b^2} e^{i\omega_0 t} \mathrm{d}t = be^{-(\omega-\omega_0)^2/2b^2}$$

$$\mathcal{F}[\text{small comb}](\omega) = \sum_{n=-N}^{N} \mathcal{F}\delta(\xi - nT) = \sum_{n=-N}^{N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega\xi} \delta(\xi - nT) \mathrm{d}\xi$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} e^{-in\omega T} = \sqrt{2\pi} \frac{\sin(N+\frac{1}{2})\omega T}{\sin \omega T/2}$$

In the book they use Poisson's sum formula, here we are not so lucky as the bounds are finite.

Combining now our expressions for the Fourier transform we get:

$$\mathcal{F}[f](\omega) = \sqrt{2\pi} b e^{-(\omega-\omega_0)^2/2b^2} \frac{\sin(N+\frac{1}{2})\omega T}{\sin \omega T/2} = \sqrt{2\pi} b e^{-(\omega-\omega_0)^2/2b^2} \delta_N(\omega T)$$

(The δ_N comes from problem 1.5.3) Thus in the end the spectral envelope ends up being the same as it was determined by the amplidude. The spectral lines portion though, is now just a finite approximation. Notably it is a function and has not achieved yet distribution status. The spectral lines in this case wobbles much more and has support on the whole real line and not just at integerl multilpes of 2π plus a $\Delta\phi$ factors.

2.3 Problem 10.

Verify that

$$f(t) = \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)}$$

is a periodic function of t, and that $f(t+T) = f(t)$. Find the full 4-ier representation

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{i\omega_m t}$$

of f by determing ω_m and c_m.

Verifying periodicity is straightforward:

$$f(t+T) = \sum_{n=-\infty}^{\infty} e^{-(t+T-nT)^2/2b^2} e^{i\omega_0(t+T-nT)} = \sum_{n=-\infty}^{\infty} e^{-(t-(n-1)T)^2/2b^2} e^{i\omega_0(t-(n-1)T)}$$

$$= \sum_{n=-\infty}^{\infty} e^{-(t-nT)^2/2b^2} e^{i\omega_0(t-nT)} = f(t)$$

Now to find the 4-ier representation:

Similar to how $\mathcal{F}\delta = 1 \Rightarrow \delta(x) = \sum_{n=-\infty}^{\infty} e^{inx}$, and $\mathcal{F}1 = \delta \Rightarrow 1 = \sum_{n=-\infty}^{\infty} \delta(n) e^{inx}$, we can do the same with this series. (Note the δ distriutions outside of integrals is 'problematic' but not a problem)

Or for a more close situation: $\mathcal{F}e^{iwx}[\omega] = \sqrt{2\pi}\delta(\omega - w) \;\Rightarrow e^{iwx} = \sum_{n=-\infty}^{\infty}\sqrt{2\pi}\delta(n-w)e^{inx}$. Now something to remmeber here is that we are no longer working with L^2 functions where $\mathcal{F}$ gives us a bijective map to l^2. Now we know the specturm of the distribution from $\mathcal{F}$ but it is not neccisarily true that $\sum_n \mathcal{F}f(n)e^{inx} = f(x)$ or that the left hand side even has a meaning.

So following $e^{iwx} \to e^{iwx}$ even for $w \notin \mathbb{Z}$. We know from the work done in the book that

$$\mathcal{F}[f](\omega) = \sqrt{2\pi}be^{-(\omega-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}\delta\left(\omega T - 2\pi m\right)$$

The Fourier 'type' series is then:

$$f(t) = \frac{1}{2\pi}\sum ne^{-i2\pi nt}\mathcal{F}f(2\pi n) = \frac{1}{2\pi}\sum ne^{-int}\sqrt{2\pi}be^{-(n-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}\delta\left(nT-2\pi m\right)$$

$$= b\sum_{n=-\infty}^{\infty}e^{-int-(n-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}\delta\left(nT-2\pi m\right) = b\sum_{n=-\infty}^{\infty}e^{-int-(n-\omega_0)^2/2b^2}\sum_{m=-\infty}^{\infty}T\delta\left(n-\frac{2}{T}\pi m\right)$$

Thus we want $n = \frac{2}{T}\pi m$ to have nonzero values from the infinite sum: **************SKETCHY***********

$$= \sum_m bTe^{-(\frac{2}{T}\pi m-\omega_0)^2/2b^2}e^{-i2\pi m/Tt}$$

$\omega_m = 2\pi m/T,\; c_m = bTe^{-(m-\omega_0)^2/2b^2}$.

2.4 Orthonormal Wave Packet Representation

2.4 Problem 1.

Consider the set of functions:

$$\left\{P_{jl}(t) = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}e^{2\pi il\omega/\epsilon}\frac{1}{\sqrt{2\pi}}e^{-i\omega t}\mathrm{d}\omega, \quad j,l = 0,\pm1,\pm2,\ldots\right\}$$

a) Show that these wave packets are orthonormal

b) Show that these wave packets form a complete set.

a)

$$\int_{-\infty}^{\infty}P_{jl}(t)P_{j'l'}(t)\mathrm{d}t = \int_{-\infty}^{\infty}\frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon}e^{2\pi il\omega_1/\epsilon}\frac{1}{\sqrt{2\pi}}e^{-i\omega_1 t}\mathrm{d}\omega_1\frac{1}{\sqrt{\epsilon}}\int_{j'\epsilon}^{(j'+1)\epsilon}\frac{1}{\sqrt{2\pi}}\overline{e^{2\pi il'\omega_2/\epsilon}e^{-i\omega_2 t}}\mathrm{d}\omega_2\mathrm{d}t$$

$$= \frac{1}{\epsilon 2\pi} \int\limits_{-\infty}^{\infty} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j'\epsilon}^{(j'+1)\epsilon} e^{2\pi i l\omega_1/\epsilon} e^{-i\omega_1 t} e^{-2\pi i l'\omega_2/\epsilon} e^{i\omega_2 t} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \mathrm{d}t$$

$$= \frac{1}{\epsilon 2\pi} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j'\epsilon}^{(j'+1)\epsilon} e^{-2\pi i l'\omega_2/\epsilon} e^{2\pi i l\omega_1/\epsilon} \int\limits_{-\infty}^{\infty} e^{-i(\omega_1-\omega_2)t} \mathrm{d}t \mathrm{d}\omega_1 \mathrm{d}\omega_2$$

$$= \frac{1}{\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j'\epsilon}^{(j'+1)\epsilon} e^{-2\pi i l'\omega_2/\epsilon} e^{2\pi i l\omega_1/\epsilon} \delta(\omega_1 - \omega_2) \mathrm{d}\omega_1 \mathrm{d}\omega_2$$

We see at this stage that we need $\omega_1 = \omega_2$ on some positive measure set, otherwise the whole endever will be 0. Thus to continue the calculation we can add in a $\delta_{jj'}$ to ensure that the integration domains coincde.

$$= \delta_{jj'} \frac{1}{\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i l'\omega_1/\epsilon} e^{2\pi i l\omega_1/\epsilon} \mathrm{d}\omega_1 = \delta_{jj'} \frac{1}{\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i(l-l')\omega_1/\epsilon} \mathrm{d}\omega_1 = \frac{\epsilon}{\epsilon} \delta_{jj'} \delta_{ll'} = \delta_{jj'} \delta_{ll'}$$

The last equality follows from considering the $1/\epsilon$ periodicity of $e^{2\pi i(l-l')\omega_1/\epsilon}$ whenever $l \neq l'$.

b) As a student I once had would say: "we write it down and bash"

$$\sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} P_{jl}(t) \bar{P}_{jl}(t') = \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i l\omega/\epsilon} \frac{1}{\sqrt{2\pi}} e^{-i\omega t} \mathrm{d}\omega \frac{1}{\sqrt{\epsilon}} \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i l\omega/\epsilon} \frac{1}{\sqrt{2\pi}} e^{i\omega t'} \mathrm{d}\omega$$

$$= \frac{1}{\epsilon 2\pi} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i l\omega_1/\epsilon} e^{-i\omega_1 t} \mathrm{d}\omega_1 \int_{j\epsilon}^{(j+1)\epsilon} e^{-2\pi i l\omega_2/\epsilon} e^{i\omega_2 t'} \mathrm{d}\omega_2$$

$$= \frac{1}{\epsilon 2\pi} \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \int_{j\epsilon}^{(j+1)\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} e^{2\pi i l(\omega_1-\omega_2)/\epsilon} e^{-i(\omega_1 t - \omega_2 t')} \mathrm{d}\omega_2 \mathrm{d}\omega_1$$

$$= \frac{1}{\epsilon 2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-i(\omega_1 t - \omega_2 t')} \sum_{l=-\infty}^{\infty} e^{2\pi i l(\omega_1-\omega_2)/\epsilon} \mathrm{d}\omega_2 \mathrm{d}\omega_1$$

$$= \frac{1}{\epsilon 2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} e^{-i(\omega_1 t - \omega_2 t')} \delta((\omega_1 - \omega_2)/\epsilon) \mathrm{d}\omega_2 \mathrm{d}\omega_1 = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-i\omega(t-t')} \mathrm{d}\omega = \delta(t - t')$$

Bashing complete, we arrive at the answer.

2.4 Problem 2.

Consider the wave packet

$$Q_{jl}(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega$$

Express the summed wave packets:

a)

$$\sum_{j=-\infty}^{\infty} Q_{jl}(t)$$

b)

$$\sum_{l=-\infty}^{\infty} Q_{jl}(t)$$

c)

$$\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} Q_{jl}(t)$$

in terms of appropriate Dirac delta DISTRIBUTIONS if necessary.

a)

$$\sum_{j=-\infty}^{\infty} Q_{jl}(t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\sum_{j=-\infty}^{\infty}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{-\infty}^{\infty} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega$$

$$= \frac{1}{\sqrt{\epsilon}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\omega(t-2\pi l/\epsilon)}\mathrm{d}\omega = \frac{1}{\sqrt{\epsilon}}\sqrt{2\pi}\delta(t-2\pi l/\epsilon)$$

b)

$$\sum_{l=-\infty}^{\infty} Q_{jl}(t) = \sum_{l=-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}\sum_{l=-\infty}^{\infty} e^{-2\pi i l\omega/\epsilon}\mathrm{d}\omega$$

we recognize this immediatly as

$$\sum_{l=-\infty}^{\infty} e^{-2\pi i l\omega/\epsilon} = \epsilon\sum_{l=-\infty}^{\infty}\delta(\omega - l\epsilon)$$

from page 81.

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\epsilon}}\int_{(j-1/2)\epsilon}^{(j+1/2)\epsilon} e^{i\omega t}\epsilon\sum_{l=-\infty}^{\infty}\delta(\omega - l\epsilon)\mathrm{d}\omega = \sqrt{\epsilon}\frac{1}{\sqrt{2\pi}}e^{ij\epsilon t}$$

c) By the first part we immediatly see:

$$\sum_{l=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} Q_{jl}(t) = \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\epsilon}}\sqrt{2\pi}\delta(t-2\pi l/\epsilon) = \sqrt{\frac{2\pi}{\epsilon}}\sum_{l=-\infty}^{\infty}\delta(t-2\pi l/\epsilon)$$

$$= \sqrt{\frac{2\pi}{\epsilon}}\text{comb}_2\pi l/\epsilon(t)$$

To double check we can reverse the order of summation:

$$\sum_{j=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} Q_{jl}(t) = \sum_{j=-\infty}^{\infty} \sqrt{\epsilon}\frac{1}{\sqrt{2\pi}}e^{ij\epsilon t} = \sqrt{\epsilon}\frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty} e^{ij\epsilon t}$$

Which we can use 2.52 on page 81 again with $a = \frac{2\pi}{\epsilon}$

$$= \sqrt{\epsilon}\frac{1}{\sqrt{2\pi}}a\sum_{j=-\infty}^{\infty}\delta(x-ja) = \frac{\sqrt{2\pi}}{\sqrt{\epsilon}}\sum_{j=-\infty}^{\infty}\delta(x-j\frac{2\pi}{\epsilon})$$

Which is the same result just written slightly differently. Now we can be confident that we are right or atleast twice as wrong as before.

2.5 Orthonormal Wavelet Representation

2.6 Multiresolutions Analysis

2.6 Problem 1.

Show that

$$\overline{\bigcup_{k=-\infty}^{\infty} V_k} = L^2 \iff \lim_{k\to\infty}\|P_{V_k}f - f\| = 0$$

where P_{V_k} is the projection onto V_{-k} (the sign is fliped to make the limits easier to write) and the norm is the L^2 norm.

We go forward first:

$$\overline{\bigcup_{k=-\infty}^{\infty} V_k} = L^2$$

Thus given an $f \in L^2, \exists h_n,\ st.\ \lim_{n\to\infty}\|h_n - f\| = 0, h_n \in \bigcup V_k$. Now $\forall n\ \exists k\ st. h_n \in V_k$. Now either $h_n = P_{V_k}f$ or $\|h_n - f\| \geqslant \|P_{V_k} - f\|$ and we can replace h_n with the actual projection without making the approximation any worse. Obviously our new $\bar{h}_n$ still converges and is made entirly of projections onto subspaces. Thus we have constructed the desired sequence. (We may need to additonally doctor the sequence and insert terms if h_k skipped many subspaces.) **********************HATE HOW THIS IS WRITTEN***********************8

Now we go backwards: Let f again be some funciton in L^2, then $\|P_{V_k}f - f\| \to 0$ as $k \to \infty$. Thus there exists some sequence $h_k = P_{V_k}f$ where $\|h_k - f\| \to 0$ as $k \to \infty$. Additionally $h_k \in V_k \forall k$ thus $\forall f \in L^2, f \in \overline{\bigcup V_k k}$. The reverse inclusion is obvious and we are done.

2.6 Problem 2.

Show that

$$\bigcap_{k=-\infty}^{\infty} V_k = \{0\} \iff \lim_{k\to\infty} \|P_{V_k}f\| = 0$$

First we go forward:
Notice that $\|P_{V_k}f\|$ is a decreasing sequence and thus has some limit. Now suppose for controdiciton that $\|P_{V_k}f\| \to \epsilon > 0$ as $k \to \infty$. Notice now that this is a Cauchy sequence in L^2 and thus there is some function $g \in L^2$ st. $\|P_{V_k}f - g\| \to 0$. Now we wish to show that $g \in V_k, \forall k$. Suppose it is not, there is some k_0 st. g is no longer in any of the V_k's after k_0. But then (since $\overline{\bigcap V_k} = \{0\}$) there would be some nonzero gap that emerges between $P_{V_k}f$ and g, namely that: $\|P_{V_k}f - g\| \geqslant d(V_k, g) = \epsilon_g > 0$. Thus $g \in V_k, \forall k$ but then $g \in \bigcap V_k$ which then means $g = 0$ and thus $\|P_{V_k}f\| \to \|g\| = 0$.

Then we go back:
We do this by controdiciton, so suppose that $\bigcap_{k=-\infty}^{\infty} V_k \supseteq \{0, g\}$ for some nonzero functions g. Then $\|g\| > 0$ and since $g \in \bigcap_{k=-\infty}^{\infty} V_k, \to g \in V_k \forall k$. Thus $g \in P_{V_k}$ for all k and $\lim_{k\to\infty} \|P_{V_k}g\| = \|g\| > 0$. This is a controdicition and we see that there is no g.

2.6 Problem 3.

a) Show that V_0 is discrete translation invarient, ie. whenever $l \in \mathbb{Z}$ that:

$$f(t) \in V_0 \iff f(t-l) \in V_0$$

b) Show that V_k is 2^k shift invariant, ie with $l, k \in \mathbb{Z}$ that:

$$f(t) \in V_0 \iff f(t-2^k l) \in V_0$$

a) Suppose $f \in V_0$ then $\exists \alpha_l$ st. $f(t) = \sum_l \alpha_l \phi(t-l)$. By the construction of the basis of V_0. Notice that for $k \in \mathbb{Z}$

$$f(t-k) = \sum_l \alpha_l \phi(t-k-l) = \sum_{m=k+l} \alpha_{m-k}\phi(t-m) = \sum_{m=k+l} \alpha'_m \phi(t-m)$$

Thus we still have an expansion for $f(t-l)$ in terms of the original basis.

b) Similar tricks:

$$f(t) \in V_k \Rightarrow f(2^k t) \in V_0 \Rightarrow$$

Now we remember that shifting by a constant value keeps you in V_0. $f(2^k(t-j)) = f(2^k t - 2^k j) \in V_0$
Now scaling the t by 2^{-k} will get us back to V_k, that is: $f(t-2^k j) \in V_k$.

2.6 Problem 4.

a) Point out why this inner product is the (l,l')th entry of the $\sqrt{2}$- multilple of a unitary matrix, which is independent of k.

b) Show that $\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \delta_{0l'}$

a) Let U be the map from $V_{k+1} \to V_k$ that is change of basis, that is we consider soem bi infinte vector

$$u = \begin{pmatrix} \cdots \\ u_1 \\ u_0 \\ u_{-1} \\ \cdots \end{pmatrix}$$

That corespond with $f_u(t)$ via $f_u(t) = \sum_{l=-\infty}^{\infty} u_l \phi(2^{-k-1}t - l)$. We define U by $f_{U(u)}(t) = \sum_{l=-\infty}^{\infty} U(u)_l \phi(2^{-k}t - l) = f(t)$. That is, we want U to be the change of basis for a funciton inside V_{k+1} from basis elements inside its 'home' space to basis elements inside V_k.

Notice that U is an 'infinite matrix', so the most straightforward way to check unitarity is:

$$\langle Ux, Uy \rangle = \langle x, y \rangle$$

From the definition of h_l we see that

$$f_{U(u)}(t) = \sum_{l=-\infty}^{\infty} U(u)_l \phi(2^{-k}t - l) = f(t) = \sum_{l=-\infty}^{\infty} \sqrt{2} \sum_{l'} u_{l'} h_{l-2l'} \phi(2^{-k}t - l)$$

Which is to say that $U(u)_l = \sqrt{2} \sum_{l'} u_{l'} h_{l-2l'}$

So if $f, g \in V_{k+1}$ and u, v are their correpsonding vecotrs, then we have by Parsevel:

$$\langle x, y\rangle = \sum_i u_i \bar{v}_i$$

Similarly we have:

$$\langle Ux, Uy\rangle = \sum_i U(u)_i \overline{U(v)_i}$$

$$= \sum_l \sqrt{2} \sum_{l'} u_{l'} h_{l-2l'} \overline{\sqrt{2} \sum_{l'} v_{l'} h_{l-2l'}}$$

$$= 2 \sum_l \sum_{l'} u_{l'} h_{l-2l'} \overline{\sum_{l'} v_{l'} h_{l-2l'}}$$

**********************PROBLEM IS WRONG?***

b) If $l' = 0$ then we have:

$$\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \sum_{l=-\infty}^{\infty} |h_l|^2 = \sum_{l=-\infty}^{\infty} \frac{1}{4} |\int_{-\infty}^{\infty} \bar{\phi}(u-l)\phi(u/2)\mathrm{d}u|^2$$

We notice now that h_l is the coefficent of the projection of $\phi(u/2)$ onto the space V_0. Thankfully $\phi(u/2)\imath V_1 \subset V_0$. Thus we see that by Parsevel's that:

$$\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \|\phi(u/2)\|^2 \frac{1}{4} = \frac{4}{4} = 1$$

If $l \neq 0$ then:

$$\sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'} = \sum_{l=-\infty}^{\infty} \overline{\int_{-\infty}^{\infty} \bar{\phi}(u-l)\phi(u/2)\mathrm{d}u} \int_{-\infty}^{\infty} \bar{\phi}(u-(l-2l'))\phi(u/2)\mathrm{d}u$$

Now for the general case:
Notice that we have the follwoing equalities:

$$\langle \sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u - l'), f(u)\rangle = \sum_{l=-\infty}^{\infty} \langle \sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u - l'), \sqrt{2^{-k}}\phi(2^{-k}u - l)\rangle\langle \sqrt{2^{-k}}\phi(2^{-k}u - l), f(u)\rangle$$

$$= \sum_{l=-\infty}^{\infty} \langle \sqrt{2^{-k}}\phi(2^{-k}u - l), f(u)\rangle \bar{h}_{l-2l'}$$

Now if consider $f(u) = \sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u)$ in the above, then we get:

$$\langle \sqrt{2^{-(k+1)}}\phi(2^{-(k+1)}u - l'), f(u)\rangle = \delta_{0,l'} = \sum_{l=-\infty}^{\infty} h_l \bar{h}_{l-2l'}$$

Since $\delta_{0,l'} = \bar{\delta}_{0,l'}$ we see that

$$\delta_{0,l'} = \sum_{l=-\infty}^{\infty} \bar{h}_l h_{l-2l'}$$

which is the desired equality.

2.6 Problem 5.

Verify the validity of the funcitonal constatin:

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

Begin with :

$$\sum_{n=-\infty}^{\infty} |\hat{\phi}(\omega+2\pi n)|^2 = \frac{1}{2\pi}, \hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega)$$

Now plug $2W = \omega$ in the first equation:

$$\sum_{n=-\infty}^{\infty} |\hat{\phi}(2W+2\pi n)|^2 = \frac{1}{2\pi} = \sum_{n=-\infty}^{\infty} |\hat{\phi}(2(W+\pi n))|^2$$

$$= \sum_{n=-\infty}^{\infty} |H(W+\pi n)\hat{\phi}(W+\pi n))|^2 = \sum_{n=-\infty}^{\infty} |H(W+\pi n)|^2 |\hat{\phi}(W+\pi n))|^2$$

Now we can split the sum into odd and even numbers and use the periodicity of H to simplify somewhat

$$= \sum_{n=2m} |H(W)|^2 |\hat{\phi}(W+2\pi m))|^2 + \sum_{n=2m+1} |H(W+\pi)|^2 |\hat{\phi}(W+2\pi m+\pi))|^2$$

$$= |H(W)|^2 | \sum_{n=2m} \hat{\phi}(W+2\pi m))|^2 + |H(W+\pi)|^2 \sum_{n=2m+1} |\hat{\phi}(W+2\pi+\pi))|^2$$

Now notice that

$$\sum_{n=2m} \hat{\phi}(W+2\pi m))|^2 = \frac{1}{2\pi} = \sum_{n=2m+1} |\hat{\phi}(W+2\pi+\pi))|^2$$

since the W term can be exchanged for $W' = W + \pi$ and we see the oroiginal identity. Now we have arived at:

$$\frac{1}{2\pi} = |H(W)|^2 \frac{1}{2\pi} + |H(W+\pi)|^2 \frac{1}{2\pi}$$

Which is planely equivalent to

$$1 = |H(W)|^2 + |H(W+\pi)|^2$$

which was what we wanted.

2.6 Problem 6.

Consider a function $\phi(t)$ having the property

$$\left|\int_{-\infty}^{\infty} \phi(t)\mathrm{d}t\right| \neq 0$$

Find the solution to the scaling equation:

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega)$$

Answer/Hint:

$$\hat{\phi}(\omega) = \hat{\phi}(0)\prod_{k=1}^{\infty} H(\omega/2^k)$$

We are given $H(\omega) = \frac{1}{\sqrt{2}}\sum_{l=-\infty}^{\infty} h_l e^{i\omega l}$ and the scaling equation and wish to construct a ϕ that satisifes the scaling equation.

We verify the solution with:

$$\hat{\phi}(2\omega) = \hat{\phi}(0)\prod_{k=1}^{\infty} H(2\omega/2^k) = \hat{\phi}(0)\prod_{k=1}^{\infty} H(\omega/2^{k-1}) = H(\omega)\hat{\phi}(0)\prod_{k=1}^{\infty} H(\omega/2^k) = H(\omega)\phi(\omega)$$

Now to ensure a non trivial solution we obviously need $\hat{\phi}(0) \neq 0$ which is given in the statemnet of the problem. We also need that H is nonzero, which is to say we need atleas one of the h_l to be nonzero. Which is required in the construciton of these systems as otherwise the subspaces would not have inclusions in the directions we need.

2.6 Problem 7.

Let $\phi^+(t)$ be soltuion to the scaling equation

$$\phi(t) = \sqrt{2} \sum_{l=-\infty}^{\infty} h_l \phi(2t - l)$$

1. Point out why
$$\hat{\phi}^- = \begin{cases} \hat{\phi}^+(\omega) & \omega \geqslant 0 \\ -\hat{\phi}^+(\omega) & \omega \leqslant 0 \end{cases}$$
is the Fourier transform of a second independent solution to the above scaling equaiton.

2. Show that these two solutions are orhtogonal:
$$\int_{-\infty}^{\infty} \bar{\phi}^+ \phi^- \mathrm{d}t = 0$$
whenever $\phi(t)$ is a real funciton or whenever its Fourier transform is an even function of ω.

1. The functions are indepent as if $\phi^+ = \lambda\phi^-$ then taking the Fourier transform of both sides gives: $\hat{\phi}^+ = \lambda\hat{\phi}^-$, whihc is not the case unless $\hat{\phi}^+$ is zero for all poisitive or negative values of ω. Now since $\hat{\phi}^+(\omega)$ is a solution to the wave equation we have for $\omega < 0$:
$$-\hat{\phi}^+(2\omega) = \hat{\phi}^-(2\omega) = -H(\omega)\hat{\phi}^+(\omega) = H(\omega)(-\hat{\phi}^+(\omega)) = H(\omega)\hat{\phi}^-(\omega)$$
For $\omega \geqslant 0$ we just use the original equation and replace ϕ^+ with ϕ^- to get the proof.

2.
$$\langle \phi^+, \phi^- \rangle = \langle \hat{\phi}^+, \hat{\phi}^- \rangle = \int_{-\infty}^{\infty} \overline{\hat{\phi}^+}(\omega)\hat{\phi}^-(\omega)\mathrm{d}\omega$$
Using the relationship of the funcitons we get:
$$= \int_0^\infty |\hat{\phi}^+|^2(\omega)\mathrm{d}\omega - \int_0^\infty |\hat{\phi}^+|^2(-\omega)\mathrm{d}\omega = \int_0^\infty |\hat{\phi}^+|^2(\omega) - |\hat{\phi}^+|^2(-\omega)\mathrm{d}\omega$$
Thus clearly if $\hat{\phi}^+$ is an even funciton than the integrand is clearly zero $\forall\omega$.

Now if $\phi^+(t)$ is a real valued function than $Re\hat{\phi}^+$ is an even function and $Im\hat{\phi}^+$ is odd. With both components we ge that their modulus cancels inside the integrand above and we see again that the integral evalueates to 0.

2.6 Problem 8.

Validate conclusion # II of the theorem on page 145. Point out why, whenever $k \neq k'$, the funcitons in O_k are orthogonal to $O_{k'}$.

We have the chain of incluisions:

$$V_{k+1} \subset V_k \subset V_{k-1}$$

We also have the following equlaities:

$$V_{k+1} \oplus O_{k+1} = V_k, \quad V_k \oplus O_k = V_{k-1}$$

Where $O_k \perp V_k$ by construction (constraint equation 2.128).

Combing these equalities we get:

$$V_{k+1} \oplus O_{k+1} \oplus O_k = V_{k-1}$$

Thus $O_{k+1} \perp O_k$ as V_{k-1} is the direct sum of the two (and V_{k+1}). Obviously we can repeate this $k - k'$ times to see that $O_{k'} \perp O_k$.

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3 Strum-Liouville Theory

3.3 Strum-Liouville Systems

3.3 Problem 1.

a) Show that any equation of the form

$$u'' + b(x)u' + c(x)u = 0$$

can always be brought into the Shrodinger form:

$$v'' + Q(x)v = 0$$

Apply this result to obtain the Schrodinger form for:

b)

$$u'' - 2xu' + \lambda u = 0$$

c)

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0$$

d)

$$xu'' + (1 - x)u' + \lambda u = 0$$

e)

$$(1 - x^2)u'' - xu' + \alpha^2 u = 0$$

f)

$$(pu')' + (q + \lambda r)u = 0$$

g)

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + l(l+1) - \frac{m^2}{\sin^2\theta}\right]u = 0$$

****

a) We consider:

$$u'' + b(x)u' + c(x)u = 0$$

********************PACKT*****************

Let u be some solution, let us try $v(x) = F(x)u(x)$ then:

$$v' = F'u + Fu', v'' = F''u + 2F'u' + Fu''$$

Now we plug this into $v'' + Qv$ and find:

$$F''u + 2F'u' + Fu'' + Fu = F''u + 2F'u' + F(u'' + Qu)$$

$$= F\left[u'' + 2F'/Fu' + (F''/F + Q)u\right]$$

We know that $u'' = -bu' - cu$

$$= F\left[(2F'/F - b)u' + (F''/F + Q - c)u\right]$$

If we let $Q = c - F''/F$ then all we have to do is solve $2F'/F = b$. This leads to

$$2F'/F - b = 0 \Rightarrow 2\int^x F'/F = \int^x b \Rightarrow 2\ln F = \int^x b$$

$$\Rightarrow F = \exp\{\frac{1}{2}\int^x b\}$$

Thus our subsitition ends up being: $v(x) = \exp\{\frac{1}{2}\int^x b\}u(x)$. Note that $F' = \frac{b}{2}F,\ F'' = \frac{b'+b^2/2}{2}F$ and our equation gets:

$$Q = c - F''/F = c - F''/F = c - \frac{b' + b^2/2}{2}$$

All togther we have:

$$u \to v = \exp\{\frac{1}{2}\int^x b\}u(x),$$

$$u'' + b(x)u' + c(x)u = 0 \ \rightarrow \ v'' + Q(x)v = 0,\ Q(x) = c - \frac{b' + b^2/2}{2}$$

b)

$$u'' - 2xu' + \lambda u = 0$$

Thus the things we have to calcuate are:

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

$$F = \exp\{\frac{1}{2}\int^x -2y\mathrm{d}y\}, \quad Q = \lambda - \frac{-2 + 4x^2/2}{2}$$

And our equations become:

$$v = \exp\{-x^2/2\}u, \quad v'' + \left(\lambda + 1 - x^2\right)v = 0$$

c)

$$x^2u'' + xu' + (x^2 - \nu^2)u = 0 = u'' + u'/x + (1 - \frac{\nu^2}{x^2})u = 0$$

We divide by x^2 here to get ride of the coefficent on u". Now following the formeioli

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

$$F = \exp\{\frac{1}{2}\int^x \frac{1}{y}\mathrm{d}y\}, \quad Q = 1 - \frac{\nu^2}{x^2} - \frac{-1/x^2 + \frac{1}{x^2 2}}{2}$$

$$Q = 1 - \frac{\nu^2}{x^2} + \frac{1}{4x^2} = 1 - \frac{4\nu^2 - 1}{4x^2}$$

And our equations become:

$$v = \exp\{\ln(x)/2\}u = \sqrt{x}u, \quad v'' + \left(1 - \frac{4\nu^2 - 1}{4x^2}\right)v = 0$$

d)

$$xu'' + (1-x)u' + \lambda u = 0 = u'' + (1/x - 1)u' + \frac{\lambda}{x}u$$

Again we do the divide by the whole something that could be zero trick to deal with a coefficent.

$$F = \exp\{\frac{1}{2}\int^x 1/y - 1\mathrm{d}y\}, \quad Q = \frac{\lambda}{x} - \frac{-\frac{1}{x^2} + (1/x-1)^2/2}{2}$$

$$F = \exp\{\frac{1}{2}[\ln x - x]\}, \quad Q = \frac{\lambda}{x} - \frac{-\frac{1}{2x^2} + -1/x + 1/2}{2}$$

$$F = \sqrt{x}e^{-x/2}, \quad Q = \frac{2\lambda+1}{2x} + \frac{1}{4x^2} - 1/4$$

And our equation becomes:

$$v = \sqrt{x}e^{-x/2}u(x), \quad v'' + \left(\frac{2\lambda+1}{2x} + \frac{1}{4x^2} - 1/4\right)v = 0$$

e)

$$(1-x^2)u'' - xu' + \alpha^2 u = 0, \quad u'' - \frac{x}{1-x^2}u' + \frac{\alpha^2}{1-x^2}u = 0$$

$$F = \exp\{\frac{1}{2}\int^x b\}, \quad Q = c - \frac{b' + b^2/2}{2}$$

$$F = \exp\{-\frac{1}{2}\int^x \frac{y}{1-y^2}\mathrm{d}y\}, \quad Q = \frac{\alpha^2}{1-x^2} - \frac{\frac{1+x^2}{(1-x^2)^2} + \left(\frac{x}{1-x^2}\right)^2\frac{1}{2}}{2}$$

$$F = \exp\{\frac{1}{4}\ln|1-x^2|\}, \quad Q = \frac{\alpha^2}{1-x^2} - \frac{1+x^2}{2(1-x^2)^2} + \frac{1}{4}\frac{x}{(1-x^2)^2}$$

$$F = |1-x^2|^{\frac{1}{4}}, \quad Q = \frac{\alpha^2(1-x^2) - \frac{1}{2}(1+x^2) + \frac{x}{4}}{(1-x^2)^2}$$

And our equation becomes:

$$v = |1-x^2|^{\frac{1}{4}}u(x), \quad v'' + \left(\frac{\alpha^2(1-x^2) - \frac{1}{2}(1+x^2) + \frac{x}{4}}{(1-x^2)^2}\right)v = 0$$

f)

$$(pu')' + (q+\lambda r)u = 0u'' + \frac{p'}{p}u' + \frac{1}{p}(q+\lambda r)u = 0$$

$$F = \exp\{\frac{1}{2}\int^x \frac{p'}{p}\}, \quad Q = \frac{1}{p}(q+\lambda r) - \frac{\frac{p'}{p}' + (\frac{p'}{p})^2/2}{2}$$

$$F = \exp\{\frac{1}{2}\ln(p(x))\}, \quad Q = \frac{1}{p}(q+\lambda r) - \frac{\frac{p''p+(p')^2}{p} + (\frac{p'}{p})^2/2}{2}$$

$$F = \sqrt{p(x)}\}, \quad Q = \frac{1}{p}(q+\lambda r) - \frac{p''p + 3/2(p')^2}{2p}$$

$$F = \sqrt{p(x)}\}, \quad Q = \frac{2(q+\lambda r) - p''p - 3/2(p')^2}{2p}$$

And our equation becomes:

$$v = \sqrt{p(x)}u(x), \quad v'' + \left(\frac{2(q+\lambda r) - p''p - 3/2(p')^2}{2p}\right) v = 0$$

g)

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + l(l+1) - \frac{m^2}{\sin^2\theta}\right] u = 0$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \cot\theta\frac{\mathrm{d}}{\mathrm{d}\theta} + l(l+1) - \frac{m^2}{\sin^2\theta}\right] u = 0$$

$$F = \exp\{\frac{1}{2}\int^x \cot\theta \mathrm{d}\theta\}, \quad Q = l(l+1) - \frac{m^2}{\sin^2\theta} - \frac{\cot\theta' + \cot\theta^2/2}{2}$$

$$F = \exp\{\frac{1}{2}\ln[\sin x]\}, \quad Q = l(l+1) - m^2\csc^2\theta - \frac{\csc^2\theta + \cot\theta^2/2}{2}$$

$$F = \sqrt{\sin x}\}, \quad Q = l(l+1) + \csc^2\theta\left[-m^2 - \frac{1+\cos^2\theta/2}{2}\right]$$

And our equation becomes:

$$v = \sqrt{\sin x}u(x), \quad v'' + \left(l(l+1) + \csc^2\theta\left[-m^2 - \frac{1+\cos^2\theta/2}{2}\right]\right) v = 0$$

3.3 Problem 2.

Consider the S-L eigenvalue problem:

$$[Lu_n](x) = \left(-\frac{d^2}{dx^2} + x^2\right) u_n(x) = \lambda_n u_n(x), \quad \lim_{x\to\pm\infty} u(x) = 0$$

on the infinite interval $(-\infty, \infty)$
Show that the eigenvalues λ_n are nondegenerate, ie. show that, except for a constant multiple, the correpsonding eigenfunctions are unique.

By Abel's theorem we have that if two solutions to the above have the same eigenvalue then (since p(x) =1 here)

$$u_m u_n' - u_m' u_n = const.$$

From here it suffices to show that this constant is zero. Once there the same logic as in the end of theorem 3 applies and we would see that $\frac{u_m'}{u_m} = \frac{u_n'}{u_n} \Rightarrow u_m = ku_n$. To this end notice that

$$\lim_{x\to\infty} u_m u_n' - u_m' u_n = C \Rightarrow \lim_{x\to\infty} u_m u_n' - \lim_{x\to\infty} u_m' u_n$$

Now since u_n', u_m' are bounded and continuous as $x \to \infty$ then we are done and get:

$$\lim_{x\to\infty} u_m u_n' - u_m' u_n = 0$$

************Message? GERLACH******************************

3.3 Problem 3.

Consider the "parity" operator $P : L^2 \to L^2$ ($L^2 = L^2(-\infty, \infty)$) defined by

$$P\psi(x) = \psi(-x)$$

1. For a given function ψ what are the eigenvlaues and eigenfunction of P?
2. Show that the eigenfunctions of the operator L defined in problem 3.3.2 are eigenfunctions of P. Do this by computing

$$P^{-1}LP\psi(x)$$

for $\psi \in L^2$ and the pointing out how $P^{-1}LP$ is related to L. Next point out how this relationship applied to an eigenfunction u_n of the previous problem leads ot the result $Pu_n = \mu u_n$.

i) The given funciton part of the question is a typeo. The eigenvalues are ± 1 ($\psi(x) = \lambda\psi(-x)$) and the eigenfuncitons are even and odd functions. (The logic here is the same when we found the eigenvalues of $\mathcal{F}^2$, basiclly $P^2 = I$d and we see $\lambda^2 = 1$)

ii)

$$P^{-1}LP\psi(x) = P^{-1}L\psi(-x) = P^{-1}\left[-\frac{d^2}{dx^2} + x^2\right]\psi(-x)$$

$$= P^{-1}\left[-\frac{d^2\psi(-x)}{dx^2} + x^2\psi(-x)\right] = \left[-P^{-1}\frac{d^2\psi(-x)}{dx^2} + P^{-1}x^2\psi(-x)\right]$$

$$= \left[-P^{-1}\frac{-d\psi'(-x)}{dx} + x^2\psi(x)\right] = \left[-P^{-1}\psi''(-x) + x^2\psi(x)\right] = -\psi''(x) + x^2\psi(x) = L\psi$$

Thus $P^{-1}LP = L$. $LP = PL$, Notice that L and P are self adjoint and therfore Hermetian operators. Since they commute we realie that they share an eigenbases. We can see this with the following: if u, λ are eigenvalues of P, then

$$LPu = \lambda Lu = PLu$$

Thus Lu is also an eigenvector of P with eigenvalue λ. Now P in this paticular case is degenerate and we see that L maps u back into W_λ the space of eigenvectors to P with value λ. Now thinking along similar lines but swapping L with P we see that. Pv is an eigenvector of L with value λ'. We know however now that L is nondegenerate and only has unique eigenvectors. Thus we see that Pv is forced to be a multiple of v the original eigenvector. Thus for some μ we have $Pv = \mu v$ and we see that this hold for any eigenvector.

3.3 Problem 4.

Consider the S-L eigenvalue problem:

$$[Lu_n](x) = \left(-\frac{d^2}{dx^2} + x^2\right) u_n(x) = \lambda_n u_n(x), \qquad \lim_{x\to\pm\infty} u(x) = 0$$

on the infinite interval $(-\infty, \infty)$
We are now blessed with the knowledge that these eigenvalues are nondegenerate and are $\lambda_n = 2n + 1$. Consider now the Fourier transform on L^2

$$\mathcal{F}u = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u \mathrm{d}x$$

a) By computing $\mathcal{F}L\mathcal{F}^{-1}\hat{\psi}$ for an aribitary $\hat{\psi} \in L^2$, determine the Fourier represnetation of $\mathcal{F}L\mathcal{F}^{-1} = \hat{L}$, of the operator $L = -\frac{d^2}{dx^2} + x^2$.

b) By viewing $\mathcal{F}$ as a map from L^2 to itself, compare the operators $\hat{L}$ and L. State in english senence and in math equation.

c) Us the result of b to show that each eigenfunction u_n of the S-L operator L is also an eigenfunction of $\mathcal{F}$.

$$\mathcal{F}u_n = \mu u_n$$

By applying the result (e) of the exercise on page 75, determine the only allowed values for μ. What *is* the 4-ier transform of a Hermite-Gauss polynomial?

1.

$$\mathcal{F}L\mathcal{F}^{-1}\hat{\psi} = \mathcal{F}L\psi = \mathcal{F}\left(-\frac{d^2}{dx^2} + x^2\right)\psi(x)$$

$$= \mathcal{F}\left(-\frac{d^2}{dx^2} + x^2\right)\psi(x) = -\mathcal{F}\frac{d^2\psi(x)}{dx^2} + \mathcal{F}x^2\psi(x)$$

$$= -(i\omega)^2\mathcal{F}\psi[\omega] + i^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathcal{F}\psi[\omega] = \omega^2\mathcal{F}\psi[\omega] - \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\mathcal{F}\psi[\omega] = \left[\omega^2 - \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\right]\hat{\psi}[\omega]$$

Thus

$$\hat{L} = \omega^2 - \frac{\mathrm{d}^2}{\mathrm{d}\omega^2}$$

2. They do the same thing, one in frequency space and the other in real space.

$$\hat{L}\hat{\psi}(\omega) = L\phi(\omega)$$

3. Using the same logic as in part 2 of the above problem we see that since $\mathcal{F}$ is self adjoint and L is non degenerate and self adjoint, that $\mathcal{F}u_n = \mu u_n$ for eigefunctions u_n of L. Now we know from the problem on page 75 that $\mathcal{F}u_n = \mu u_n$ forces μ to be a root of unity, namely a root of $x^4 = 1$. Thus for any Hermite-Gauss polynomial, its fourier transform is itself times some root of unity. Thus we can see that $\{\mathcal{F}^k u_n | n \in \mathbb{N}, k \in \{0,1,2,3\}\}$ is an orthogonal system of functions and is an eigenbassi for the Fourier transform in terms of the eigenbasis of the S-L operator L. ****************THINK ABOUT A LITTLE MORE???**************************

3.3 Problem 5.

Consider the S-L System:

$$\left[\frac{d}{dx}p\frac{d}{dx} + q + \lambda\rho\right]u = 0, \quad a < x < b$$

$$\alpha u(a) + \alpha' u'(a) = 0; \quad \beta u(a) + \beta' u'(a) = 0$$

Let $\omega(x,\lambda)$ be that unqiue soltuion to the above with boundary conditions satisified. Then $\omega_n(x) = \omega(x,\lambda_n)$ is an eigenfunction with eigenvalue λ_n. Calculate $\int_a^b \omega_n^2 \rho \mathrm{d}x$ as follows:

1. $$(\lambda - \lambda_n)\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = p(x)W(\omega,\omega_n)|_a^{x=b}$$

2. By taking the limit $\lambda \to \lambda_n$ show that:

$$\int_a^b \omega_n^2 = p(b)\left[w_n'(b)\frac{d\omega(b,\lambda)}{d\lambda}|_{\lambda=\lambda_n} - \omega(b)\frac{d}{d\lambda}w_n'(b,\lambda)|_{\lambda=\lambda_n}\right]$$

primes here refering to $\frac{d}{dx}$.

1. Thanks to the first 2 steps of the 3 step proof on page 168 of the orthogonality of eigenvalues we see:
$$(\lambda - \lambda_n)\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = p(x)W(\omega,\omega_n)|_a^{x=b}$$
In fact we realize this is true for any S-L system with any boundary condiotns. With that aside aside, we consider the case at hand.
$$p(x)W(\omega,\omega_n)|_a^{x=b} = p(b)W(\omega,\omega_n)(b) - p(a)W(\omega,\omega_n)(a)$$
note that the boundary condition gives: $\omega(a,\lambda) = -\alpha'/\alpha\omega'(a,\lambda)$ Thus we have:
$$\begin{aligned} p(a)W(\omega,\omega_n)(a) &= p(a)(\omega(a)\omega_n'(a) - \omega'(a)\omega_n(a)) \\ &= p(a)(-\omega'(a)\alpha'/\alpha\omega_n'(a) + \omega'(a)\omega_n'(a)\alpha'/\alpha) = 0 \end{aligned}$$
Thus we get:
$$(\lambda - \lambda_n)\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = p(x)W(\omega,\omega_n)|^{x=b}$$
2. We first divide by $\lambda - \lambda_n$:
$$\int_a^b \omega_n(x)\omega(x,\lambda)\rho(x)\mathrm{d}x = \frac{1}{\lambda-\lambda_n}p(x)W(\omega,\omega_n)|^{x=b}$$
Now we take the limit $\lambda \to \lambda_n$: the left hand side clearly does nothing funky and becomes:
$$\int_a^b \omega_n(x)^2\rho(x)\mathrm{d}x$$
Now for the right hand side:
$$\frac{1}{\lambda-\lambda_n}p(x)W(\omega,\omega_n)|^{x=b} = \frac{1}{\lambda-\lambda_n}p(b)(\omega(b)\omega_n'(b) - \omega'(b)\omega_n(b))$$
$$= p(b)(\frac{\omega(b)}{\lambda-\lambda_n}\omega_n'(b) - \frac{\omega'(b)}{\lambda-\lambda_n}\omega_n(b))$$
We add and subtract the same term:
$$= p(b)(\frac{\omega(b)-\omega_n(b)}{\lambda-\lambda_n}\omega_n'(b) - \frac{\omega'(b)-\omega_n'(b)}{\lambda-\lambda_n}\omega_n(b))$$
Pasing to the limit we get:
$$= p(b)\left[w_n'(b)\frac{d\omega(b,\lambda)}{d\lambda}|_{\lambda=\lambda_n} - \omega(b)\frac{d}{d\lambda}w_n'(b,\lambda)|_{\lambda=\lambda_n}\right]$$
which is the desired result.

3.3 Problem 6.

Consider the S-L problem

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\nu^2}{x}\right]u=\lambda xu$$

Here u, $\frac{\mathrm{d}u}{\mathrm{d}x}$ bounded as $x\to 0, u(1)=0$ and $\nu\in\mathbb{R}$.

1. Using the sub $t=\sqrt{\lambda x}$ show that the above differential equation reduces to Bessel's equation of order ν. One solution which is bouned as $t\to 0$ is $J_\nu(t)$; a second linearly indep. solution, denoted by $Y_\nu(t)$ is unbounded as $t\to 0$.
2. SHow that the eigenvalues of the given problem are the squares fo the positive zeros of $J_\nu(\sqrt{\lambda})$ and that the corresponding eigenfunctions are
$$u_n(x)=J_\nu(\sqrt{\lambda_n}x)$$
3. Show that the eigenfunctions u_n satisfy the orhtogonality relaiton:
$$\int_0^1 xu_mu_n\mathrm{d}x=0,\quad m\neq n$$
4. For the case $\nu=0$, apply the method of the previous problem to exhibit the set of orhtonormalized eiqgenfuncitons.
5. Determine the coeffiencts of the Fourier-Bessel sereis expnaison:
$$f(x)=\sum_{n=1}^{\infty}c_nu_n(x)$$

1. The bessel equation is:
$$\left[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}}{\mathrm{d}x}+x^2-\alpha^2\right]u=0$$
So begining with: Let $t=\sqrt{\lambda}x$, then $t/\sqrt{\lambda}=x$, and $\frac{\mathrm{d}t}{\mathrm{d}x}=\sqrt{\lambda}$, $\frac{\mathrm{d}}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\lambda}$ Our equation is:
$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{\nu^2}{x}\right]u=\lambda xu$$
We replace with $\frac{\mathrm{d}}{\mathrm{d}t}$ to get:
$$\left[-\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}t/\sqrt{\lambda}\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}+\frac{\nu^2}{t/\sqrt{\lambda}}\right]u=\lambda t/\sqrt{\lambda}u$$
$$\left[-\sqrt{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}+\sqrt{\lambda}\frac{\nu^2}{t}\right]u=t\sqrt{\lambda}u$$
Now multiplying across by $t/\sqrt{\lambda}$ we get:
$$\left[-t\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}+\nu^2\right]u=t^2u$$

$$\left[-t\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}+\nu^2-t^2\right]u=0$$

$$\left[t\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{\mathrm{d}}{\mathrm{d}t}-\nu^2+t^2\right]u=0$$

Which is plainly the Bessel equation of order ν.

2. Since $J_\nu(\sqrt{\lambda}x)$ is a solution to the eigenvalue problem and we get non degenerate eigenvalues by the regularity of the SL set up, we see that for whatever eigenvalues λ_n there are that

$$u_n(x)=J_\nu(\sqrt{\lambda_n}x)$$

is the eigenfunction.

It just remains to show that the eigenvalues are zeros of $J_\nu(\sqrt{\lambda})$. Despite the amount of time I thought about this, this is easily done. Namely the boundary conditions are $u(1)=0=J_\nu(\sqrt{\lambda})$ thus $\sqrt{\lambda}$ must be a positive zero of the bessel function J_ν. Thus the eigenvalues are squares of the positive zeros of J_ν.

3. We know that in genral for a S-L problem with eigenvectors u_n, u_m that we have:

$$(\lambda_m-\lambda_n)\int_a^b\omega_n(x)\omega_m(x)\rho(x)\mathrm{d}x=p(x)W(\omega_m,\omega_n)|_a^{x=b}$$

In our specific case we have:

$$(\lambda_m-\lambda_n)\int_0^1\omega_n(x)\omega_m(x)x\mathrm{d}x=p(x)W(\omega_m,\omega_n)|_0^{x=1}$$

Since the eigenvalues are disitinct it suffices to show from here that

$$p(x)W(\omega_m,\omega_n)|_0^{x=1}=0$$

Since $u(1)=0$ we see that $p(x)W(\omega_m,\omega_n)|^{x=1}=0$. and since $p(x)=x, p(0)=0$ and we get our result.

4. We have that

$$\omega(x,\lambda)=J(\sqrt{\lambda}x),\quad \omega'=\sqrt{\lambda}J'(\sqrt{\lambda}x),\quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\omega(x,\lambda)=\frac{x}{2\sqrt{\lambda}}J'(\sqrt{\lambda}x)$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\omega'(x,\lambda)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\sqrt{\lambda}J'(\sqrt{\lambda}x)=\frac{1}{2\sqrt{\lambda}}J'(\sqrt{\lambda}x)+\sqrt{\lambda}\frac{x}{2\sqrt{\lambda}}J''(\sqrt{\lambda}x)$$

$$=\frac{1}{2\sqrt{\lambda}}J'(\sqrt{\lambda}x)+\frac{x}{2}J''(\sqrt{\lambda}x)$$

And the normalization integral from the previous problem

$$\int_0^1\omega_n^2\mathrm{d}x=p(1)\left[w_n'(1)\frac{d\omega(1,\lambda)}{d\lambda}|_{\lambda=\lambda_n}-\omega(1)\frac{d}{d\lambda}w_n'(1,\lambda)|_{\lambda=\lambda_n}\right]$$

$$= \sqrt{\lambda} J'(\sqrt{\lambda} 1) \frac{1}{2\sqrt{\lambda}} J'(\sqrt{\lambda}) - J(\sqrt{\lambda}_n) \left(\frac{1}{2\sqrt{\lambda}} J'(\sqrt{\lambda}) + \frac{1}{2} J''(\sqrt{\lambda}) \right)$$

$$= \frac{1}{2} \left[J'(\sqrt{\lambda})^2 - J(\sqrt{\lambda}) \left(\frac{1}{\sqrt{\lambda}} J'(\sqrt{\lambda}) + J''(\sqrt{\lambda}) \right) \right]$$

******************MESSAGE GERLACH ABOUT***************

We have that

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z)$$

And differentiating the above

$$J''_\nu(z) = J'_{\nu-1}(z) + \frac{\nu}{z^2} J_\nu(z) - \frac{\nu}{z} J'_\nu(z)$$

Now subbing the *above* into the above

$$= J_{\nu-2}(z) - \frac{\nu-1}{z} J_{\nu-1}(z) + \frac{\nu}{z^2} J_\nu(z) - \frac{\nu}{z} \left(J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z) \right)$$

Using this we can rewrite the above as:

$$= \frac{1}{2} \left[J'_\nu(\sqrt{\lambda})^2 - J_\nu(\sqrt{\lambda}) \left(\frac{1}{\sqrt{\lambda}} J'_\nu(\sqrt{\lambda}) + J'_{\nu-1}(\sqrt{\lambda}) - \frac{\nu}{\sqrt{\lambda}} J'_\nu(\sqrt{\lambda}) \right) \right]$$

5.

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x)$$

What we need to do is determine c_n above

$$c_n = \langle f, u_n \rangle = \int_0^1 f(x) u_n(x) x \mathrm{d}x$$

Now we can write this expansion in terms of the original un normalized bessel functions.

$$= \frac{1}{\int \omega_n^2 \mathrm{d}x} \int_0^1 f(x) J_\nu(\sqrt{\lambda}_n x) x \mathrm{d}x$$

*******************FINISH SIMPLIFYING NORMALZIATION THING******************

3.3 Problem 7.

Consider the S-L problem

$$\left[-\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}+\frac{m^2}{1-x^2}\right]u=\lambda u$$

Here u, $\frac{\mathrm{d}u}{\mathrm{d}x}$ bounded as $x\to\pm1$. Here m is an integr. The solutions to this S-L problem are $u_n=P_n^m(x)$ the "asosciated Legendre polynomials", corresponding to $\lambda_n=n(n+1)$ n being an integer. SHow that:

$$\int_{-1}^{1}P_n^mP_{n'}^m\mathrm{d}x=0\quad\lambda_n\neq\lambda_{n'}$$

For $\lambda_n\neq\lambda_{n'}$ we have (via the same Green's idneity derived integral as before)

$$(\lambda_n-\lambda_{n'})\int_{-1}^{1}P_n^mP_{n'}^m\mathrm{d}x=p(x)W(\lambda_n,\lambda_{n'})|_{-1}^{x=1}$$

Here we notice that $p(x)=1-x^2$ and that $p(\pm1)=0$, (also the boundedness of W as $x\to\pm1$ helps) thus we get:

$$p(x)W(\lambda_n,\lambda_{n'})|_{-1}^{x=1}=0$$

and the problem is done.

---

4 Green's Function Theory

4.3 Pictorial Definition of a Green's Function

4.3 Problem 1.

Find the adjoint L^* and the space on which it acts:

a)

$$Lu=u''+a(x)u'+b(x)u$$

with $u(0)=u'(1),\quad u(1)=u'(0)$

b)

$$Lu=-(p(x)u')'+q(x)u$$

with $u(0)=u(1),\quad u'(1)=u'(0)$

a)

$$\langle Lu, v\rangle = \int_0^1 [u'' + a(x)u' + b(x)u]\, v\mathrm{d}x$$

Now we use integration by parts to get:

$$= [u'v + a(x)uv]_0^1 - \int_0^1 u'v' + a(x)uv' - b(x)uv\mathrm{d}x = [u'v + a(x)uv - uv']_0^1 + \int_0^1 uv'' - a(x)uv' + b(x)uv\mathrm{d}x$$

Thus we see that $L^*v = v'' - a(x)v' + b(x)v$. Now considering the boundary conditions we see that the boundary terms are:

$$u'(1)v(1) + u(1)\left(a(1)v(1) - v'(1)\right) - \left[u'(0)v(0) + u(0)\left(a(0)v(0) - v'(0)\right)\right]$$

$$= u'(1)v(1) + u(1)\left(a(1)v(1) - v'(1)\right) - \left[u(1)v(0) + u'(1)\left(a(0)v(0) - v'(0)\right)\right]$$
$$= u'(1)(v(1) - v(0)) + u(1)\left(a(1)v(1) - v'(1) - (a(0)v(0) - v'(0))\right)$$

From the $u'(1)$ coefficent we see we need $v(1) = v(0)$, this then leaves

$$u(1)\left(-v'(1) + v'(0)\right)$$

which gives us $v'(1) = v'(0)$.

Overall we see $L^*v = v'' - a(x)v' + b(x)v$ and the adjoint domain being $\{v | v'(1) = v'(0), v(1) = v(0), v \in C^2\}$.

***************DOUBLE CHECK a(x) IS WIERD******************

b)

$$\langle Lu, v\rangle = \int_0^1 \left[-(p(x)u')' + q(x)u\right] v\mathrm{d}x = \int_0^1 -(p(x)u')'v + q(x)uv\mathrm{d}x$$

Again we apply integration by parts:

$$= -(p(x)u')'v|_0^1 + \int_0^1 p(x)u'v'\mathrm{d}x + \int_0^1 q(x)uv\mathrm{d}x$$

$$= -p(x)u'v|_0^1 + up(x)v'|_0^1 - \int_0^1 u(p(x)v')'\mathrm{d}x + \int_0^1 q(x)uv\mathrm{d}x$$
$$= -p(x)u'v|_0^1 + up(x)v'|_0^1 + \int_0^1 -u(p(x)v')' + q(x)uv\mathrm{d}x$$

$L^*v = (p(x)v')' + q(x)v$

****

4.3 Problem 2.

Let L be a operator defined on S and L^*, S^* the adjoint and its domain satisifying $B_1(u) = 0 = B_2(u),\ B_1^*(v) = 0 = B_2^*(v)$ respectivly. Let $u, \lambda,\ v, \lambda'$ be eigenvalues, eigenvectors of L and L^*

a) Make a guess as to the relationship between the eigenvalue of L and L^*.

b) Prove: If $\lambda \neq \bar{\lambda}'$ then $\langle u, v\rangle = 0$.

Since part (ii) gives a guess we might as well say λ cooresponds with $\bar{\lambda}$ for eigenvalues between L and L^*.

We can see this with u and v as in the satement of the problem

$$\langle Lu, v\rangle = \bar{\lambda}\langle u, v\rangle$$

from the definition of eigenvalue/function.

$$= \langle u, L^*v\rangle = \lambda'\langle u, v\rangle$$

Thus we see that if $\langle u, v\rangle \neq 0$ that $\bar{\lambda} = \lambda'$.

*********8MAKE NOTE FOR LATER POST*****Problem is Done though*************
In fact it is clear from the above that $\bar{\lambda}$ being an eigenvalue of the adjoint is implied by $\exists v, \langle u, v\rangle \neq 0$. Thus if an operator is self adjoint we see that $\lambda = \bar{\lambda}$ and the eigenvalues must all be real valued.

************************MESSAGE GERLACH AOBUT ORDER ISSUE WITH THESE PROBLEMMS******************
Also no boundary conditions ***

4.3 Problem 3.

Find the Green's function for the Bessel operators:

a)

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}u(x)}{\mathrm{d}x}$$

b)

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}u(x)}{\mathrm{d}x} - \frac{n^2}{x} u(x)$$

with $y(0)$ finite and $y(1) = 0$.

ie. solve the equations $Lu = -\delta(x - \xi)$ with the given boundary conditions.

To construct the Green's function we begin by finding two functions (u_1, u_2) that satisfy:

$$Lu_1 = 0, \quad \alpha u_1(a) + \alpha' u_1'(a) = 0$$

$$Lu_2 = 0, \quad \beta u_2(b) + \beta' u_2'(b) = 0$$

From this we construct the Green's fucntion via

$$G(x,\zeta) = \frac{-1}{c}\begin{cases} u_1(x)u_2(\zeta) \text{ for } x < \zeta \\ u_1(\zeta)u_2(x) \text{ for } \zeta < x \end{cases}$$

with $c = p(x)[u_1(x)u_2'(x) - u_1'(x)u_2(x)]$

a) We don't have boundary conditions here $(\alpha, \alpha' = 0)$.

$$Lu_1 = \frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}u_1}{\mathrm{d}x} = 0$$

Integrating we get:

$$x\frac{\mathrm{d}u_1}{\mathrm{d}x} = C \quad \Rightarrow \quad \frac{\mathrm{d}u_1}{\mathrm{d}x} = \frac{C}{x}$$

Which then gives (since u_2 has the same conditions)

$$u_1 = C_1 \ln x, \quad u_2 = C_2 \ln x$$

Thus

$$G(x,\zeta) = \frac{-1}{c}\begin{cases} C_1C_2 \ln x \ln \zeta \text{ for } x < \zeta \\ C_1C_2 \ln x \ln \zeta \text{ for } \zeta < x \end{cases}$$

b)

$$Lu(x) = \frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}u(x)}{\mathrm{d}x} - \frac{n^2}{x}u(x)$$

with $y(0)$ finite and $y(1) = 0$.

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{n^2}{x}\right]u_1 = 0, \quad u_1(0) < \infty$$

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x} - \frac{n^2}{x}\right]u_1 = 0$$

This is planely bessel's equation of order n and thus its solutions is $J_n(x)$.

Now we have the same ODE for u_2 but with only th econdiiton that $u_2(1) = 0$ Thus we can have $Y_n(x)/Y_n(1)$ so long as $Y_n(1) \neq 0$.

So we get:

$$G(x,\zeta) = \frac{1}{Y_n(1)}\frac{-1}{c}\begin{cases} J_n(x)Y_n(\zeta) \text{ for } x < \zeta \\ J_n(\zeta)Y_n(x) \text{ for } \zeta < x \end{cases}$$

with $c = \frac{x}{(Y_n(1))^2}[J_n(x)Y_n'(x) - J_n'(x)Y_n(x)]$

***************DESIRED ANSWER?************************

4.3 Problem 4.

1. Find Green's function for the operator

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2$$

with $u(a) = u(b) = 0,\ a < b$ and ω^2 some fixed cosntant.

2. Does this Green's function exist $\forall\omega$? If not what values fail?

3. Having found the Green's function in (1), go find the Green's function for the same operator but different boudnary conditions namely $u(a) = u'(a) = 0$. Do this with minimal work.

1.

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2$$

Following the same method as last problem we find u_1 that satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_1 + \omega^2 u_1 = 0, \quad u_1(a) = 0$$

Re arranging terms we see:

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}x^2} = -\omega^2 u_1$$

We see $e^{i\omega x}$ is a solution, and noting the boundary condiitons we see that:

$$u_1(x) = C\left(e^{i\omega(x-a)} - e^{-i\omega(x-a)}\right)$$

is a solution. Using the exact same reason we arrive at

$$u_2(x) = C\left(e^{i\omega(x-b)} - e^{-i\omega(x-b)}\right)$$

Realizing some facts about these functions we can write the greens function as:

$$G(x,\zeta) = \frac{1}{c}\begin{cases} \sin(x-a)\sin(\zeta-b) & \text{for } x < \zeta \\ \sin(\zeta-a)\sin(x-b) & \text{for } \zeta < x \end{cases}$$

with $c = \sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b) = \sin(b-a)$

2. From the exponential represnetation of the solution we see that $\omega = 0$ is a problematic value. This is becasue $\frac{\mathrm{d}^2}{\mathrm{d}x^2}u = 0$ has only the solutio $c_1 x + c_2$ which has issues being zero at both end points. All other values will have a solution.

3. **

4.3 Problem 5.

> Let $Lu = u''$
> $$a_1 u(0) + b_1 u'(0) + c_1 u(1) + d_1 u'(1) = 0$$
> $$a_2 u(0) + b_2 u'(0) + c_2 u(1) + d_2 u'(1) = 0$$
>
> 1. Find L^* and the space on which it acts with $\langle u, v\rangle = \int_0^1 uv\mathrm{d}x$
> 2. For what values of the cosntants is the operator self adjoint?

1. As is standard we apply integration by parts.

$$\langle Lu, v\rangle = \int_0^1 u''v\mathrm{d}x = u'v|_0^1 - \int_0^1 u'v'\mathrm{d}x = u'v|_0^1 - uv'|_0^1 + \int_0^1 uv''\mathrm{d}x$$

Thus $L^* = L$ which makes sense since the problem is asking about the operator being self adjoint. Now the boundary terms are:

$$[u'v - uv']_0^1 = u'(1)v(1) - u(1)v'(1) - u'(0)v(0) + u(0)v'(0)$$

2. ********************************

4.7 The Totally Inhomogeneous Boundary Value Problem

4.7 Problem 1.

Let $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with boundary contions $u(0) = 0, u'(0) = u(1)$, so that $S = \{u | Lu$ is square integral and satisfies b.c.$\}$.

1. Find S^* with
$$\langle u, v\rangle = \int_0^1 \bar{u} v \mathrm{d}x$$
and compare S with its cooler twin S^*.

2. Compare the eigenvalues of L and L^*. IF the two sequences are different point out the distinction, if the are the same justify the result.

3. Exhibit the corresponding eigenfunctions.

4. Is $\lambda = 0$ an eigenvalue? Why or why not?

5. Verify that $\int_0^1 \bar{v}_n u_m \mathrm{d}x = 0$ for $n \neq m$.

1. I have never not started a problem with integration by parts
$$\langle Lu, v\rangle = \int_0^1 \overline{Lu} v \mathrm{d}x = \int_0^1 \overline{-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}} v \mathrm{d}x$$
$$= -u'v|_0^1 + \int_0^1 \overline{\frac{\mathrm{d}u}{\mathrm{d}x}} \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = -u'v|_0^1 + uv'|_0^1 - \int_0^1 \overline{u} \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \mathrm{d}x$$
Thus $L^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Our boundary terms are (with accounting for the boundary contiions on u):
$$u'v|_0^1 - uv'|_0^1 = u'v|_0^1 - uv'|^1 u'(1)v(1) - u(1)(v(0) + v'(1))$$
Thus we see the adjoint boundary conditions are $v(1) = 0, v(0) = -v'(1)$. Thus the problem is not self adjoint even thought the operator itself satisfies $L^* = L$.

2. We notice that the eigenvalue problem is the same as 4.3.4.
****

3.

4.

5.

4.7 Problem 2.

Attack the eigenvalue problem:

$$-u''(x) = \lambda u(x),\ 0 < x < 1, u'(1) = \lambda u(1),\ u(0) = 0$$

as follows:
Let $U = \begin{pmatrix} u(x) \\ u_1 \end{pmatrix}$ be a two-component vector whose first component is a twice differentiable function $u(x)$, and whose second componnent is a real number u_1. Consdier the corresponding vector space $\mathfrak{H}$ with inner product

$$\langle U, V\rangle = \int_0^1 u(x)v(x)\mathrm{d}x + u_1 v_1$$

Let $S \subset \mathfrak{H}$ be the subspace

$$S = \{U : U = \begin{pmatrix} u(x) \\ u(1) \end{pmatrix};\ u(0) = 0\}$$

and let:

$$LU = \begin{pmatrix} -u''(x) \\ u'(1) \end{pmatrix}$$

The above eigenvalue problem can now be rewritten in standard form

$$LU = \lambda U, \quad U \in S$$

1. Prove or disprove that L is self adjoint.
2. Prove or disprove that L is postiive-definite, ie that $\langle U, LU\rangle > 0 \forall U \neq \vec{0}$.
3. Find the (transcendental) equation for the eigenvalues of L.
4. Denoting these eigenvalues by $\lambda_1, \lambda_2, \lambda_3, \cdots$ exhibit the orthonrmalized eigenvectors U_n associated with these eigenvalues.

1.

$$\langle -LU, V\rangle = \int_0^1 u''(x)v(x)\mathrm{d}x - u'(1)v_1 = u'v|_0^1 - \int_0^1 u'(x)v'(x)\mathrm{d}x - u'(1)v_1 + u'(1)v'(1) - u'(1)v'(1)$$

$$= u'v|_0^1 - uv'|_0^1 - u'(1)\left(v_1 + v'(1)\right) + \int_0^1 u(x)v''(x)\mathrm{d}x - u'(1)v'(1)$$

We have that $u(0) = 0$ and thus:

$$= u'v|_0^1 - uv'|^1 - u'(1)\left(v_1 - v'(1)\right) + \langle U, -LV\rangle$$

Thus we can see from here that the 'extra' terms are

$$u'v|_0^1 - u(1)v'(1) - u'(1)\left(v_1 - v'(1)\right)$$

We want to know if L is self adjoint on S, thus we have also that $v_1 = v(1)$ and $v(0) = 0$. With this we get

$$-u(1)v'(1) + u'(1)\left(v'(1)\right)$$

So we get

$$\langle LU, V\rangle = \langle U, LV\rangle + \det \begin{bmatrix} u(1) & v(1) \\ u'(1) & v'(1) \end{bmatrix}$$

Thus L is self adjoint whenever the determinat = 0. Thus the most obvious space is the eigenspace where $u(1) = \lambda u'(1)$.

All in all the operator is NOT self adjoint on S but it is self adjoint on $S\cap$ the eigenspace.

2. Just calculating We get:

$$\langle LU, U\rangle = \int_0^1 -u''(x)u(x)\mathrm{d}x + u(1)u'(1)$$

Integration by parts and with $u(0) = 0$ we get:

$$= -u'u|_0^1 + \int_0^1 u'(x)u'(x)\mathrm{d}x + u(1)u'(1) = \int_0^1 u'(x)u'(x)\mathrm{d}x$$

which is then just $\int_0^1 u'(x)^2\mathrm{d}x = \|u'\|_2^2$

Now is $\langle LU, U\rangle = \|u'\|_2^2 > 0\ \forall U \neq 0$. Now if u' is ever non zero then the abouve would be positive, so if there is a counter exapmle it would have to have $u' = 0$. This is doabled, we have a whole host of constant functions to chose from! However we also have the condition that $u(0) = 0$. With this we see that any function that satisfies: $\langle LU, U\rangle = 0$ and is in S must be zero.

All together we see that L is positive semi definite.

3.

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = \lambda u(x)$$

We know that there has to be a fourier series for this function by the set up of the this problem. Thus we try and see if what the fourier basis elements do under this transform.

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}e^{i2\pi kx} = \lambda e^{i2\pi kx} = 4\pi^2k^2e^{i2\pi kx}$$

We arrive the eigenvalues being:

$$\lambda = 4\pi^2k^2$$

Thus we see that $\pm k$ gets maped to the same eigenvalues. Since these eigenvalues are non degenerate we realize we have to combine them to get the actual function for our problem.

4. We notice that

$$\frac{e^{i2\pi kx} - e^{i2\pi(-k)x}}{2}$$

satisfies our boundary conditions, for normality we need to divide by 2. Now we notice that this is in fact just $u_k(x) = \sin(2\pi x)$.

*************************CHECK NORMALIZZATION***********************

So our actual eigenvetors are:

$$U_n = \begin{pmatrix} u_k(x) \\ u(1) \end{pmatrix} = \begin{pmatrix} \sin(k\pi x) \\ u(1) \end{pmatrix}$$

****

4.7 Problem 3.

The eigenvalue equaion for 4.7.1 is

$$\sin \lambda^{\frac{1}{2}} = \lambda^{\frac{1}{2}}$$

Prove or disprove that an asymptotic formula for the roots is

$$\lambda^{\frac{1}{2}} \sim (2m + \frac{1}{2})\pi - \frac{2\log(2m1)\pi}{(4m+1)\pi} \pm i\log(4m+1)\pi$$

*********************HAS HINT************************

Let $\lambda^{\frac{1}{2}} = \alpha + i\beta$ so that

$$\sin\alpha\cosh\beta = \alpha \quad \cos\alpha\sinh\beta = \beta$$

asdflkfj

4.7 Problem 4.

Consider the eigenvalue problem

$$Lu = \lambda u \quad L = \alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta\frac{\mathrm{d}}{\mathrm{d}x} + \gamma$$

$$B_1(u) = B_2(u) = 0$$

and its ajdoint

$$L^* v = \bar{\lambda} v \quad B_1^*(u) = B_2^*(u) = 0$$

with respect to the inner products $\langle v, u \rangle = \int_a^b \bar{v} u \mathrm{d}x$. One can show and you may safely assume, that the eigenvalue spectra of these two problems are complex conjugates of each other (this in factfollows from previous exercises).

1. Prove that the solution $u(x, \lambda)$ for the problem

$$Lu - \lambda u = -f(x)$$

$$B_1(u) = B_2(u) = 0$$

   is given by

$$u(x, \lambda) = \sum_n \frac{\langle v_n, f \rangle}{\lambda - \lambda_n} u_n(x)$$

   where u_n, v_n are the eigenfunctions of L and L^* and have been normalized to satisfy:

$$\langle v_n, u_m = \delta_{nm}$$

2. Show that the Green's function is

$$G_\lambda(x|\zeta) = \sum_n \frac{u_n(x)\bar{v}_n(x)}{\lambda - \lambda_n}$$

****

1.

2.

4.7 Problem 5.

Obtain the o.n. set of eigen functions for the S-L problem

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \omega^2 u$$

$$u(a) = u(b) = 0$$

by applying the complex integration technique to the Green's function $G_\omega(x, \zeta)$.

$$(L^2 - \omega^2)G = -\frac{\mathrm{d}^2 G_\omega}{\mathrm{d}x^2} - \omega^2 G_\omega = \delta(x - \zeta) \quad a < x, \zeta < b$$

$$G_\omega(a|\zeta) = 0 \quad G_\omega(b|\zeta) = 0, \quad a < \zeta < b$$

****

4.9 Boundary Value Problem via Green's function: Integral Equations

4.9 Problem 1.

Consider the inhomogenous Fredholm equation of the second kind:

$$u(x) = \lambda \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x; \zeta) u(\zeta) \mathrm{d}\zeta + \phi(x)$$

Here λ is a paramter and ϕ is a known and given function. Also the integration kernal K, which in this problem is given to be translation invarient. ie. you should assume that $K(x; \zeta) = v(x - \zeta)$, where v is a given function whose Fourier transform exists. Solve the integral equation by finding the function u in terms of what is given.

We notice that this is a convolutuion:

$$u(x) = \lambda \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x - \zeta) u(\zeta) \mathrm{d}\zeta + \phi(x) = \lambda \frac{1}{\sqrt{2\pi}} K \star u + \phi(x)$$

Now if we apply the 4-ier convolution thoerem we get:

$$\hat{u}(\omega) = \lambda \frac{1}{\sqrt{2\pi}} \hat{K}(\omega) \hat{u}(\omega) + \hat{\phi}(\omega)$$

Now we subtract over the $\hat{u}$ terms and then divide to get:

$$\hat{u}(\omega) = \frac{\hat{\phi}(\omega)}{1 - \lambda \frac{1}{\sqrt{2\pi}} \hat{K}(\omega)}$$

So our actual function u ends up being

$$u(x) = \mathcal{F}^{-1}\left[\frac{\hat{\phi}(\omega)}{1 - \lambda \frac{1}{\sqrt{2\pi}} \hat{K}(\omega)}\right]$$

4.9 Problem 2.

> Look up an integral equation of the 2nd kind, either of the Volterra or of the Fredholm type. Submit it and its solution.

Problem:

$$\phi(x) = x^2 - x^4 + \lambda \int_{-1}^{1} (x^4 + 5x^3 y)\phi(y)\mathrm{d}y$$

Solution:
This is calle dthe 'direct computation' approach from Stack Exchange

$$\phi(x) = x^2 - x^4 + \lambda \int_{-1}^{1} (x^4 + 5x^3 y)\phi(y)\mathrm{d}y = x^2 - x^4 + \lambda \left[x^4 \int_{-1}^{1} \phi(y)\mathrm{d}y + 5x^3 \int_{-1}^{1} y)\phi(y)\mathrm{d}y\right]$$

Now let

$$c_1 = \int_{-1}^{1} \phi(y)\mathrm{d}y \quad c_2 = \int_{-1}^{1} y)\phi(y)\mathrm{d}y$$

and we have

$$\phi(x) = x^2 - x^4 + \lambda\left[x^4 c_1 + 5x^3 c_2\right] = x^2 + 5\lambda x^3 c_2 + (\lambda c_1 - 1)x^4$$

Now if we plug this expression back into the first equaiton for $\phi(y)$ we get:

$$\phi(x) = x^2 - x^4 + \lambda \int_{-1}^{1} (x^4 + 5x^3 y)\left(y^2 + 5\lambda y^3 c_2 + (\lambda c_1 - 1)y^4\right)\mathrm{d}y$$

$$= x^2 - x^4 + \lambda x^4 \int_{-1}^{1} y^2 + 5\lambda y^3 c_2 + (\lambda c_1 - 1)y^4 \mathrm{d}y + \lambda 5x^3 \int_{-1}^{1} y^3 + 5\lambda y^4 c_2 + (\lambda c_1 - 1)y^5 \mathrm{d}y$$

The region is symetric around 0 so all the odd powers of y die. Thus the only integrals we actually have to deal with are:

$$= x^2 - x^4 + \lambda x^4 \int_{-1}^{1} y^2 + (\lambda c_1 - 1)y^4 \mathrm{d}y + \lambda 5x^3 \int_{-1}^{1} 5\lambda y^4 c_2 \mathrm{d}y$$

$$= x^2 - x^4 + 2\lambda x^4\left[\frac{1}{3} + (\lambda c_1 - 1)\frac{1}{5}\right] + \lambda 10x^3\lambda c_2$$

Now we remember this is equal to $\phi(x)$ and get

$$x^2 + 5\lambda x^3 c_2 + (\lambda c_1 - 1)x^4 = x^2 - x^4 + 2\lambda x^4\left[\frac{1}{3} + (\lambda c_1 - 1)\frac{1}{5}\right] + \lambda 10x^3\lambda c_2$$

Now equaint coefficents of the powers of x gives:

$$1 = 1 \tag{1}$$
$$5\lambda c_2 = \lambda^2 10c_2 \tag{2}$$
$$\lambda c_1 - 1 = 2\lambda\left[\frac{1}{3} + (\lambda c_1 - 1)\frac{1}{5}\right] - 1 \tag{3}$$

So from this we see that

$$c_2 = 0$$

Now with some effort we go after c_1:

$$\lambda c_1 - 1 = \frac{2}{5}\lambda^2 c_1 + 2\lambda\left[\frac{1}{3} - \frac{1}{5}\right] - 1$$

$$c_1(-\frac{2}{5}\lambda^2 + \lambda) = +2\lambda\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$c_1 = \frac{\frac{4}{15}}{-\frac{2}{5}\lambda + 1} = \frac{4}{3}\frac{1}{5 - 2\lambda}$$

So big deal we calculated some coefficents, what about ϕ?

$$\phi(x) = x^2 + 5\lambda x^3 c_2 + (\lambda c_1 - 1)x^4 = x^2 + \left(\frac{4}{3}\frac{\lambda}{5 - 2\lambda} - 1\right)x^4$$

4.11 Spectral Represensation of the Dirac Delta Function

4.11 Problem 1.

Over the interval $-\infty < x < \infty$ consider

$$\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} + \lambda G = -\delta(x-\zeta)$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \lambda u = -f(x) \quad \text{and} \quad \frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} + \lambda \phi = 0$$

We are looking for solutions in L^2 assume that f is in L^2.

1. Show that there are tow candidates for G, namely

$$G = G^{out}(x|\zeta;\lambda) = \frac{i}{2\sqrt{\lambda}} \exp(i\sqrt{\lambda}|x-\zeta|)$$

   and

$$G^{in}(x|\zeta;\lambda) = -\frac{i}{2\sqrt{\lambda}} \exp(i\sqrt{\lambda}|x-\zeta|)$$

2. Given the fact that $\sqrt{\lambda} = \alpha + i\beta$ with $\beta > 0$, point out why only one of them is square- integrable.

3. Consider the contour integral $\oint G \mathrm{d}\lambda$ over a large circle of radius R. Demonstrate that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint G \mathrm{d}\lambda = -\delta(x-\zeta)$$

4. Next deform the contour until it fits snugly around the branch cut of $\sqrt{\lambda}$, and show that

$$\delta(x-\zeta) = \int_0^\infty \cdots \mathrm{d}\lambda$$

   and then show that the above can be rewriteten as

$$\delta(x-\zeta) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{i\omega(x-\zeta)} \mathrm{d}\omega$$

   for $x < \zeta$, and $\zeta < x$.

5. Express $u(x)$ as a Fourier integral in terms of f.

6. Express G in the same way, ie obtain a bilinear expansion for G.

1. ***************************************

2.

3.

4.

5.

6.

4.11 Problem 2.

Again consider

$$\frac{\mathrm{d}^2G}{\mathrm{d}x^2} + \lambda G = -\delta(x-\zeta)$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \lambda u = -f(x), \text{ and } \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \lambda\phi = 0$$

over the interval $-\infty < x < \infty$, but leav the boundary conditons as yet to be specified.

1. Express $u(x)$ as a Fourier integral in terms of f.
2. Express $G(x, \zeta, \lambda)$ in the same way, ie obtain a bilinear expansion for G.
3. How, do you think, should on incorporate boundary conditions into these expressions?

1.

2.

3.

---

5 Special Function Theory

5.1 The Helmholtz Equation

5.1 Problem 1.

Evaluate the integral:

$$\int_C e^{i\rho\cos\alpha + i\nu\alpha}\mathrm{d}\alpha$$

along the curve C (in the complex α plane below) in terms of the two kinds of Hankel functions.
************Include a Tixz picutre?*********************

****

5.1 Problem 2.

In the complex β plane, determine those semi-infinte strip regions where the line integral

$$\int_C e^{i\rho\cos\beta - i\nu\beta}\mathrm{d}\beta$$

converges if the integration limits of the path C are extended to infinity in each of a pair of such strips.

****

5.1 Problem 3.

By slightly deforming the integration path prove or disprove that the integral In the complex β plane, determine those semi-infinte strip regions where the line integral

$$\int\limits_{-\infty}^{\infty} e^{i\rho\cos\beta - i\nu\beta}\mathrm{d}\beta$$

can be expressed in terms of a Hankel function, what kind and which order?

****

5.1 Problem 4.

Apply

$$t = \zeta \cosh \tau, \;\; z = \zeta \sinh \tau, \quad 0 < \zeta < \infty, -\infty < \tau < \infty$$

to the wave equation

$$-\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial z^2} - k^2 \psi = 0$$

in order to obtain the wave equation relative to the coordinates ζ, τ. TO do this take advantage of the fact that letting

$$r = \zeta, \quad \theta = i\tau$$

and

$$x = t, \quad y = iz$$

yields the hyperbolic transformation of the wave euqation.

1. Write down the wave equation in terms of the "pseudo" polar coordinates ζ, τ.
2. Consider a solution whihc is a "psuedo" rotation eigenfunction ψ_ω:

   $$\frac{\partial \psi_\omega}{\partial \tau} = -i\omega\psi_\omega$$

   and determine the differential euqation:

   $$\left[\alpha(\zeta)\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \beta(\zeta)\frac{\mathrm{d}}{\mathrm{d}\zeta} + \gamma(\zeta)\right]\psi = 0$$

   it satisfies.
3. Verify that the translation (in the t,z plane) eigenfunction

   $$\psi = e^{-i(k_0 t - k_z z)}$$

   is a solution ot the wave equation, whenever the two k's satisfy $k_0^2 - k_z^2 = k^2$. Then using $k_0 = k\cosh\alpha, k_z = k\sinh\alpha$ and $t = \zeta\cosh\tau, z = \zeta\sinh\tau$, and the hiyperbolic angle additon formula, rewrite the phase and hence the wave funciton ψ in terms of ζ, τ.
4. Construct a superpositoin (as an integral over α) of waves ψ whihc is a "pseudo" ortation eigenfunction ie satsifies

   $$\frac{\partial \psi_\omega}{\partial \tau} = -i\omega\psi_\omega$$

   where ψ_ω is that superposition.
5. Exhibit two indendpent soutions ψ_ω to the wave equation corresponding to two different integration contours. WHat are they? If your solutions are porooprtional to Hankel funcitons, specifiy what kind and identify their order.

1.

2.

3.

5.2 Properties of Hankel and Bessel Functions

5.2 Problem 1.

Show that:

$$H^{(1)}_{-n}(\rho) = (-1)^n H^{(1)}_n(\rho)$$
$$H^{(2)}_{-n}(\rho) = (-1)^n H^{(2)}_n(\rho)$$
$$N_{-n}(\rho) = (-1)^n N_n(\rho)$$

$$H^{(1)}_{-n}(\rho) = c_1 \int_{\alpha_1}^{\alpha_2} e^{i\rho\cos\alpha + i(-n)\alpha} \mathrm{d}\alpha$$

Now we introduce $\alpha = \pi - \bar{\alpha}, \to \alpha - \pi = -\bar{\alpha}$ which shifts the bounds of the integrals by θ but this does not matter as shown on page 302 under the no angular dependence property. With thi ssub we see

$$c_1 \int_{\alpha_2+\pi}^{\alpha_1+\pi} e^{i\rho\cos(\pi-\bar{\alpha})+i(-n)(\pi-\bar{\alpha})} \mathrm{d}\alpha = c_1 \int_{\alpha_2+\pi}^{\alpha_1+\pi} e^{i\rho\cos(\pi-\bar{\alpha})+i(-n)(\pi-\bar{\alpha})} (-1)\mathrm{d}\bar{\alpha}$$

Since $\cos(\pi - x) = \cos(x)$ (think of the unit circle, or idk trig identities or something) and we swap the limits and switch the sign of the integral.

$$= c_1(-1)^n \int_{\alpha_1+\pi}^{\alpha_2+\pi} e^{i\rho\cos(\bar{\alpha})+i(-n)(\pi-\bar{\alpha})} \mathrm{d}\bar{\alpha} = c_1(-1)^n \int_{\alpha_1+\pi}^{\alpha_2+\pi} e^{i\rho\cos(\bar{\alpha})+in\bar{\alpha}} \mathrm{d}\bar{\alpha} = (-1)^n H^{(1)}_n(\rho)$$

Now for the 2nd identity we remmeber that

$$J_\nu = \frac{1}{2}\left[H^1_\nu + H^2_\nu\right] \Rightarrow 2J_\nu - H^1_\nu = H^2_\nu$$

and thus just from the last 2 identities we have:

$$2J_{-n} - H^1_{-n} = (-1)^n \left(2J_n - H^1_n\right) = (-1)^n H^2_\nu$$

Similarly we have

$$N_\nu = \frac{1}{2i}\left[H^1_\nu - H^2_\nu\right]$$

and thus:

$$N_{-n} = (-1)^n \frac{1}{2i}\left[H^1_n - H^2_n\right] = (-1)^n N_{-n}$$

as was deeply desired.

5.3 Applications of Hankel and Bessel Functions

5.3 Problem 1.

The transvers amplitude of an axially symmetric wave propagating in a cylindrical pipe of radius a is determined by the following eigenvalue problem:

$$-\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}u}{\mathrm{d}r} = k^2 ru \quad 0 \leqslant r \leqslant a$$

$$u(0) = \text{finite} \quad u(a) = 0$$

The eigenfunctions are $u_m(r) = J_0(rk_m)$ where the boundary condiiton $J_0(ak_m) = 0$ determines the eigenvalues $K_m^2, m \in \mathbb{N}$.

1. Show that $\{J_0(rk_m)\}$ is an orthogonal set of eigenfunctions on $(0, a)$.
2. Using the problem 3.3.5 find the squared norm of $J_0(rk_m)$.
3. Exhibit the set of orthonormalized eigenfunctions.
4. Find the Green's function for the above boundary value problem.

1. Let $u = J_0(k_m r)$ then with $x = k_m r$ then we notice that the Bessel equation gives: (we have 1 less factor of r than the standard Bessel form)

   $$\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}u}{\mathrm{d}x} = x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}u}{\mathrm{d}x} = xu$$

   Now we see that $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{1}{k_m}$. Thus our equaiton becomes:

   $$\frac{k_m}{k_m^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{k_m}\frac{\mathrm{d}u}{\mathrm{d}r} = k_m ru$$

   Which moving around some constnats yields:

   $$\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\mathrm{d}u}{\mathrm{d}r} = k_m^2 ru$$

   Which is the ODE we have.

   By theorem 1 on page 166 we see that these eigenvalues are nondegenertae and that they are orthogonal. (Have one endpoint set to zero of a S-L system).

2. Problem 3.3.5 tells us

   $$\int_0^a J_0(rk_m)^2 \mathrm{d}r = J_0'(ak_m)\frac{dJ_0(a\lambda)}{d\lambda}|_{\lambda=k_m} - J_0(ak_m)\frac{d}{d\lambda}J_0'(a\lambda)|_{\lambda=k_m}$$

By construction we have that $J_0(ak_m) = 0$, thus we only need to figure out $J_0'(x)$.

$$J_0'(ak_m)\frac{dJ_0(a\lambda)}{d\lambda}|_{\lambda=k_m} - J_0(ak_m)\frac{d}{d\lambda}J_0'(a\lambda)|_{\lambda=k_m} = aJ_0'(ak_m)^2$$

*****************DERIVE IDENTITY LATER?**********

3. Thus to normalize the eigenfucntions we would simply normalize by the norm above:

$$\frac{J_0(ak_m)}{\sqrt{a}J_0'(ak_m)} = ????$$

4. *****************Greens FUnction?**********

5.3 Problem 2.

> On a circula disc of radius a find an orthonormal set of eigenfunctions for the system defined by the eigenvalue problem
>
> $$-\nabla^2\psi = k^2\psi$$
>
> $$\frac{\partial\psi}{\partial r}(r = a, \theta) = 0$$
>
> $$\psi(r = 0, 0) = \text{finite},\ 0 \leqslant \theta \leqslant 2\pi$$
>
> Here $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ and exhibit these eigenfunctions in their optimally simple form, ie without refering to any derivatives.

****

5.3 Problem 3.

Consider a wave disturbance ψ which is governed by the wave equatioin.

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right]\psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Let this wave propogate inside an infintely long cylinder; in other words, it satisfies

$$\frac{\partial \psi}{\partial z} = ik_z\psi$$

where k_z is some real number not equal to zero. Assume that the boundary coditions satisfied by ψ is

$$\psi(r=a) = 0, \psi(r=0) = \text{finite}, \qquad \text{a = radius of cylinder}$$

1. Find the 'cut off' frequencey ie the frequency belwo which no propogation in the infite cylinder is possible.
2. Note that this frequency dpeends on the angular integer m and the radial integr j. For fixed j give an argument which supports the result that smallr m means msaller critical frequency
3. What is the smallest critical frequency in terms of a and c to an accuracy of 2% or better?

****

1.
2.
3.

5.3 Problem 4.

Consider the sector $S = \{(r,\theta)|0 \leqslant r \leqslant a, 0 \leqslant \theta \leqslant \alpha\}$.

1. Exhibit the set of those normalized eigenfunctions for this sector whic satisfy

$$(\nabla^2 + k^2)\psi = 0, \quad \psi = 0 \text{ on } \partial S$$

2. Compare the set of normal modes of a circular drum with the set of nromal modes in part a) when $\alpha = 2\pi$.

1. ***

2.

5.3 Problem 5.

Consdier

1. a circular membarne of radius a.
2. a square membrane
3. a rectangular membrane which is twice as long as it is wide.

Assume the two membranes

1. have the same area
2. obey the same wave equation $\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$
3. Have the same boundary conditons, ie $\psi = 0$ on the boundary.

A) i) the 3 lowest frequencies for each of the two membranes

ii) all the concomitant normal modes

B) FOr each of the normal modes of the circular mebrane draw a picutre of the nodes, ie the locus of points where $\psi = 0$ Label each of the pictures

C) Do the same for the other membrane (watch out for degeneracies!)

****

5.4 More Properties of Hankel and Bessel Functions

5.4 Problem 1.

Express $J_n(x_1 + x_2)$ as a sum of products of Bessel functions of x_1 and x_2 respectively.

5.5 The Method of Steepest Descent and Stationary Phase

5.5 Problem 1.

1. Using the method of steepest descent find anasymptotic expression for H_ν^2 and for J_ν where $\nu << \rho$.
2. THe gamma function Γ for which $Re\omega > -1$ is represented by

$$\Gamma(\omega + 1) = \int_0^\infty e^{-\tau}\tau^\omega \mathrm{d}\tau$$

   Using the steepest descent approach, find and asymptotic expression for $\Gamma(\omega + 1)$ when $Re\omega >> 1$. Why doesn't it work? Try again by susbstituting ωz for τ and obtaining:

$$\Gamma(\omega + 1) = \omega^{\omega+1} \int_0^\infty e^{-\omega z} z^\omega \mathrm{d}z = \omega^{\omega+1} \int_0^\infty e^{\omega(\ln z - z)} \mathrm{d}z$$

1. ********************************

2.

---

6 Partial Differential Equations

6.2 System of Partial Differntial Equations: How to solve Maxwell's equations using Linear Algebra

6.2 Problem 1.

Consider the current-charge density to an isolated moving charge:

$$\vec{J}(x,y,z,t) = q\int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{X}(\tau)}{\mathrm{d}\tau}\delta(x-X(\tau))\delta(z-Z(\tau))\delta(t-T(\tau))\mathrm{d}\tau$$

$$\rho(x,y,z,t) = q\int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{T}(\tau)}{\mathrm{d}\tau}\delta(x-X(\tau))\delta(z-Z(\tau))\delta(t-T(\tau))\mathrm{d}\tau$$

1. Show that this current-charge density satisifies

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

2. By taking advantge of the fact $\frac{\mathrm{d}T(\tau)}{\mathrm{d}\tau} > 0$, evaluate the τ-integrals, and obtain explicit expressions for the components $\vec{J}$ and ρ.

   Answer:

$$\rho(x,y,z,t) = q\delta(x-X(\tau))\delta(y-Y(\tau))\delta(z-Z(\tau))\delta(t-T(\tau))$$

$$\vec{J}(x,y,z,t) = q\frac{\mathrm{d}\vec{X}}{\mathrm{d}t}\delta(x-X(\tau))\delta(y-Y(\tau))\delta(z-Z(\tau))\delta(t-T(\tau))\mathrm{d}\tau$$

   where $\vec{X}(t) = \vec{X}(\tau)$ evaluated at τ as determined by $\delta(t-T(\tau))$.

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

1.

$$\nabla\cdot\vec{J} + \frac{\partial\rho}{\partial t} =$$

$$\nabla\cdot\vec{J} = \sum_i \frac{\partial}{\partial x_i}\vec{J} = q\int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{X}(\tau)}{\mathrm{d}\tau}\sum_i \frac{\partial}{\partial x_i}\delta(x-X(\tau))\delta(z-Z(\tau))\delta(t-T(\tau))\mathrm{d}\tau$$

$$= q\int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{X}(\tau)}{\mathrm{d}\tau}\sum_i \delta'(x_i-X_i(\tau))\prod_{j\neq i}\delta(x_i-X_i(\tau))\delta(t-T(\tau))\mathrm{d}\tau$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}q\int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{T}(\tau)}{\mathrm{d}\tau}\delta(x-X(\tau))\delta(z-Z(\tau))\delta(t-T(\tau))\mathrm{d}\tau$$

$$= q \int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{T}(\tau)}{\mathrm{d}\tau} \frac{\partial}{\partial t} \delta(x - X(\tau))\delta(z - Z(\tau))\delta(t - T(\tau))\mathrm{d}\tau$$

$$= q \int_{-\infty}^{\infty} \frac{\mathrm{d}\vec{T}(\tau)}{\mathrm{d}\tau} \sum_i -\delta'(x_i - X_i(\tau)) \frac{\mathrm{d}X_i(\tau)}{\mathrm{d}t} \prod_{j \neq i} \delta(x_i - X_i(\tau))\mathrm{d}\tau$$

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2. Since $\frac{\mathrm{d}T}{\mathrm{d}\tau} > 0$ we see that T is an injective function of τ and thus there is only one specific value where $t = T(\tau)$ (if one exists at all).

6.2 Problem 2.

a) Exhibit the pde which each of the scalars $\Phi, \cdots, \Phi^{TM}$, satifies, point out why each solution is unique and why

$$(\phi, A_z, A_x, A_y)^T \leftrightarrow \left(\Phi, \Phi^{TE}, \Psi, \Phi^{TM}\right)$$

is a one to one maping. ***************************SOLUTION HINT IN PROBLEM***************

b) Point out why the four vectors

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix}, \begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\partial_t \\ \partial_z \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \partial_x \\ \partial_y \end{bmatrix} \right\}$$

form a linearly indepente set.

c) SHow that the set of vecotrs

$$\left\{ \vec{\nu}^{(1)} = \begin{bmatrix} 0 \\ 0 \\ \partial_y \\ -\partial_x \end{bmatrix}, \vec{\nu}^{(2)} = \begin{bmatrix} -\partial_z \\ \partial_t \\ 0 \\ 0 \end{bmatrix}, \vec{\nu}^{(3)} = \begin{bmatrix} \partial_t c \\ -\partial_z c \\ \partial_x d \\ \partial_y d \end{bmatrix}, \vec{v}_r = \begin{bmatrix} -\partial_t \\ \partial_z \\ \partial_x \\ \partial_y \end{bmatrix} \right\}$$

where $c = \partial_x^2 + \partial_y^2$, and $d = \partial_z^2 - \partial_t^2$ also form a linearly independent set.

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6.2 Problem 3.

Consider a TE e.m. potential and its source:

$$[\phi, A_z, A_x, A_y] = [0, 0, \partial_y \Phi^{TE}, -\partial_x \Phi^{TE}]$$

$$[\rho, J_z, J_x, J_y] = [0, 0, \partial_y S^{TE}, -\partial_x S^{TE}]$$

1. Which two of the Maxwell field equtions

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$$

are satisfied trivially, and which imply the nontrivial result:

$$\frac{\partial}{\partial y}\{\cdots\}^{TE} = 0, \quad \frac{\partial}{\partial x}\{\cdots\}^{TE} = 0$$

2. What is $\{\cdots\}^{TE}$?

****HAS HINT********************************

6.2 Problem 4.

Consider a TM e.m. potential and its source:

$$[\phi, A_z, A_x, A_y] = [0, 0, \partial_y \Phi^{TM}, -\partial_x \Phi^{TM}]$$

$$[\rho, J_z, J_x, J_y] = [0, 0, \partial_y S^{TM}, -\partial_x S^{TM}]$$

1. Which two of the Maxwell field equtions

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$$

are satisfied trivially, and which imply the nontrivial result:

$$\frac{\partial}{\partial y}\{\cdots\}^{TM} = 0, \quad \frac{\partial}{\partial x}\{\cdots\}^{TM} = 0$$

2. What is $\{\cdots\}^{TM}$?

****HAS HINT*******************************

6.2 Problem 5.

Consider a TEM e.m. potential and its source:

$$[\phi, A_z, A_x, A_y] = [-\partial_t \Phi, \partial_z \Phi, \partial_x \Phi, \partial_y \Phi]$$

$$[\rho, J_z, J_x, J_y] = [-\partial_t J, \partial_z J, \partial_x J, \partial_y J]$$

1. Which two of the Maxwell field equtions

$$\nabla \cdot \vec{E} = 4\pi\rho$$

$$\nabla \times \vec{B} - \partial_t \vec{E} = 4\pi \vec{J}$$

   which imply the result:

$$\frac{\partial}{\partial t}\{\cdots\}^{TEM} = 0, \quad \frac{\partial}{\partial z}\{\cdots\}^{TEM} = 0$$

   and which imply the result:

$$\frac{\partial}{\partial x}[\cdots]^{TEM} = 0, \quad \frac{\partial}{\partial y}[\cdots]^{TEM} = 0$$

2. What is $\{\cdots\}^{TEM}$? What is $[\cdots]^{TEM}$?

****HAS HINT*******************************

6.2 Problem 6.

Point out why the previous three exercises imply:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi^{TE} = -4\pi S^{TE}$$

$$(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)\Phi^{TM} = -4\pi S^{TM}$$

and

$$(\partial_x^2 + \partial_y^2)(\Phi - \Psi) = -4\pi J$$

$$(\partial_z^2 - \partial_t^2)(\Phi - \Psi) = 4\pi I$$

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6.2 Problem 7.

Show that if I and J satisfy

$$0 = (\partial_z^2 - \partial_t^2)J + (\partial_x^2 + \partial_y^2)I$$

Then there exists a scalar, all it $\Phi - \Psi$ such that

$$(\partial_x^2 + \partial_y^2)(\Phi - \Psi) = -4\pi J$$

$$(\partial_z^2 - \partial_t^2)(\Phi - \Psi) = 4\pi I$$

are satisfied. Hint: Use Green's function

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6.2 Problem 8.

zt

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6.2 Problem 9.

zt

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