

1 Chapter 1

1.1 Subsection 5

1. ls foo.tex — entr jyour command herej

Show that

$$\langle f, g \rangle = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

$$\langle f, g \rangle = \int_a^b \bar{f}(x) g(x) \rho(x) dx$$

Since we have that $f(x) = \sum_{k=1}^{\infty} u_k(x) c_k$ we can substitiute and get:

$$\begin{aligned} &= \int_a^b \sum_{k=1}^{\infty} \overline{u_k(x) c_k} g(x) \rho(x) dx = \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x) \bar{c}_k g(x) \rho(x) dx \\ &= \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x) \bar{c}_k g(x) \rho(x) dx = \sum_{k=1}^{\infty} \bar{c}_k \int_a^b \bar{u}_k(x) g(x) \rho(x) dx = \sum_{k=1}^{\infty} \bar{c}_k d_k \end{aligned}$$

2.

$$Tf(\omega, t) = \int_{-\infty}^{\infty} \bar{g}(x - t) e^{-i\omega x} f(x) dx$$

$$\langle h_1, h_2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{h}_1(\omega, t) h_2(\omega, t) d\omega dt$$

Find a formula for: $\langle Tf_1, Tf_2 \rangle$ in terms of

$$\int_{-\infty}^{\infty} \bar{f}_1 f_2 dx$$

$$\langle Tf_1, Tf_2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{Tf_1} Tf_2 d\omega dt =$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\overline{\int_{-\infty}^{\infty} \bar{g}(x - t) e^{-i\omega x} f_1(x) dx} \right] \left[\int_{-\infty}^{\infty} \bar{g}(x - t) e^{-i\omega x} f_2(x) dx \right] d\omega dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\bar{g}(x - t) e^{-i\omega x} f_1(x)} \bar{g}(y - t) e^{-i\omega y} f_2(y) dx dy d\omega dt \end{aligned}$$

We now return to Calc III and need to do a replacment of variables:

$$\begin{aligned} u &= x - y, & v &= y \\ x &= u + v, & y &= y \end{aligned} \text{ which has deterimnate: } J = \left| \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right| = 1$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{g(u+v-t)f_1(u+v)} \bar{g}(v-t)f_2(v) \int_{-\infty}^{\infty} e^{i\omega u} d\omega du dv dt \\
&\quad \int_{-\infty}^{\infty} e^{i\omega u} d\omega = \delta(u) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{g(u+v-t)f_1(u+v)} \bar{g}(v-t)f_2(v) \delta(u) du dv dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{g(v-t)f_1(v)} \bar{g}(v-t)f_2(v) dv dt = \int_{-\infty}^{\infty} \bar{f}_1(v)f_2(v) \int_{-\infty}^{\infty} g(v-t)\bar{g}(v-t) dt dv \\
&= \int_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)|g|^2 dv = |g|^2 \int_{-\infty}^{\infty} \bar{f}_1 f_2 dx = |g|^2 \langle f_1, f_2 \rangle
\end{aligned}$$

3. 1.5.3

2 Chapter 2

2.1 Subsection 1

4. 2.1.1 Suppose $f(x) = f(x + 2\pi) \forall x$ is periodic with period 2π . Show

$$\int_a^{2\pi+a} f(x) dx = \int_0^{2\pi} f(x) dx, \quad \forall a \in \mathbb{R}$$

As all great math proofs, no words are needed just equalities and beautiful integrals. Let a be given then:

$$\begin{aligned}
\int_a^{2\pi+a} f(x) dx &= \int_a^{2\pi} f(x) dx + \int_{2\pi}^{2\pi+a} f(x) dx = \int_a^{2\pi} f(x) dx + \int_0^a f(x+2\pi) dx \\
&= \int_a^{2\pi} f(x) dx + \int_0^a f(x) dx = \int_0^{2\pi} f(x) dx
\end{aligned}$$

5. 2.1.2 *

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6. 2.1.3 Riemann-Lebesgue Lemma

$G(u)$ piecewise continuous and has left and right derivatives on $[0, 2\pi]$. Show that

$$\lim_{N \rightarrow \infty} \int_0^{2\pi} G(u) \sin(N + \frac{1}{2})u du = 0$$

WLOG $\exists a, b \in [0, \pi]$ st. $\forall x \in [a, b] \ G(x) > 0$ or $G(x) < 0$.

Now it suffices to show

$$\lim_{N \rightarrow \infty} \int_a^b G(u) \sin(N + \frac{1}{2})u du = 0$$

[since the interval $[0, 2\pi]$ can be sliced into a countable number of these intervals, and then you can sum over them] WLOG we assume $G(x)$ is positive.

$$0 \leq \lim_{N \rightarrow \infty} \int_a^b G(u) \sin(N + \frac{1}{2})u du \leq \lim_{N \rightarrow \infty} \int_a^b [\max_u G(u)] \sin(N + \frac{1}{2})u du$$

Let G_m be the max above, then we have

$$0 \leq G_m \lim_{N \rightarrow \infty} \int_a^b \sin(N + \frac{1}{2})u du = G_m \lim_{N \rightarrow \infty} \frac{\cos(N + \frac{1}{2})u}{N + \frac{1}{2}} \Big|_a^b \leq G_m \lim_{N \rightarrow \infty} \frac{2}{N + \frac{1}{2}} \leq 0$$

Thus we get the 0 value for the limit as desired.

7. 2.1.4 Prove or disprove:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2 + a^2} = \frac{\pi}{a} \frac{\coth \pi a}{\cos^2 \pi x + \sin^2 \pi x \coth^2 \pi a}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2} = \frac{\pi^2}{\sin^2 \pi x}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m + x)^2} = \frac{1}{4 \sin^2 x/2}$$

8. 2.1.5 My man Stephane G. Mallat claims the following: The family of functions $\phi(x-k)$ $k = 0, \pm 1, \pm 2, \dots$ is orthonormal iff

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

is constant wrt ω . Prove my boy wrong or right.

Stephane is no chump and said a true thing. Lets investigate the sum:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2 &= \sum_{k=-\infty}^{\infty} \overline{\hat{\phi}(\omega + 2\pi k)} \hat{\phi}(\omega + 2\pi k) \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\omega + 2\pi k)y} \bar{\phi}(y) dy \int_{-\infty}^{\infty} e^{-i(\omega + 2\pi k)x} \phi(x) dx \\ &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x) \bar{\phi}(y) e^{i(\omega + 2\pi k)y} e^{-i(\omega + 2\pi k)x} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x) \bar{\phi}(y) \sum_{k=-\infty}^{\infty} e^{-i(\omega + 2\pi k)(x-y)} dx dy \end{aligned}$$

Via formula on page 62

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x) \bar{\phi}(y) e^{-i\omega(x-y)} \sum_{k=-\infty}^{\infty} \delta(x-y-k) dx dy$$

Now for the change of variables $u = x - y, v = y$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(u+v) \bar{\phi}(v) e^{-i\omega u} \sum_{k=-\infty}^{\infty} \delta(u-k) du dv = \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega u} \phi(u+v) \delta(u-k) du dv \\
&= \int_{-\infty}^{\infty} \bar{\phi}(v) \sum_{k=-\infty}^{\infty} e^{-i\omega k} \phi(k+v) dv = \sum_{k=-\infty}^{\infty} e^{-i\omega k} \int_{-\infty}^{\infty} \bar{\phi}(v) \phi(k+v) dv = \sum_{k=-\infty}^{\infty} \langle \phi(v), \phi(k+v) \rangle e^{-i\omega k}
\end{aligned}$$

Thus the sum above is in fact a Fourier series with $c_k = \langle \phi(v), \phi(k+v) \rangle$. Now this series being constant is equivalent to $c_k = \delta_{0k}$, which is equivalent to the $\phi(v+k)$'s being an orthogonal system.

Moreover if $c_0 = 1$ then we have an orthonormal system as well. Thus the system is orthonormal if the series is constant and equal to 1.

9. 2.1.6
