1 Chapter 1

1.1 Subsection 5

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1. Show that
$$\langle f, g\rangle = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

$$\langle f, g\rangle = \int_a^b \bar{f}(x)g(x)\rho(x)\mathrm{d}x$$

Since we have that $f(x) = \sum_{k=1}^{\infty} u_k(x)c_k$ we can substitiute and get:

$$= \int_a^b \overline{\sum_{k=1}^{\infty} u_k(x)c_k} g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x)\bar{c}_k g(x)\rho(x)\mathrm{d}x$$
$$= \sum_{k=1}^{\infty} \int_a^b \bar{u}_k(x)\bar{c}_k g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k \int_a^b \bar{u}_k(x)g(x)\rho(x)\mathrm{d}x = \sum_{k=1}^{\infty} \bar{c}_k d_k$$

2.
$$Tf(\omega, t) = \int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x}f(x)\mathrm{d}x$$

$$\langle h_1, h_2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{h}_1(\omega, t)h_2(\omega, t)\mathrm{d}\omega\mathrm{d}t$$

Find a formula for: $\langle Tf_1, Tf_2\rangle$ in terms of

$$\int_{-\infty}^{\infty} \bar{f}_1 f_2 \mathrm{d}x$$

$$\langle Tf_1, Tf_2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{Tf_1}Tf_2\mathrm{d}\omega\mathrm{d}t =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\left[\int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x}f_1(x)\mathrm{d}x\right]}\left[\int_{-\infty}^{\infty} \bar{g}(x-t)e^{-i\omega x}f_2(x)\mathrm{d}x\right]\mathrm{d}\omega\mathrm{d}t$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(x-t)e^{-i\omega x}f_1(x)}\bar{g}(y-t)e^{-i\omega y}f_2(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\omega\mathrm{d}t$$

We now return to Calc III and need to do a replacment of variables:

$$\begin{matrix} u = x - y, & v = y \\ x = u + v, & y = y \end{matrix} \text{ which has deterimnate: } J = \left|\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right| = 1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\int_{-\infty}^{\infty} e^{i\omega u}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

$$\int_{-\infty}^{\infty} e^{i\omega u}\mathrm{d}\omega = \delta(u)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(u+v-t)f_1(u+v)}\bar{g}(v-t)f_2(v)\delta(u)\mathrm{d}u\mathrm{d}v\mathrm{d}t$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\bar{g}(v-t)f_1(v)}\bar{g}(v-t)f_2(v)\mathrm{d}v\mathrm{d}t = \int_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)\int_{-\infty}^{\infty} g(v-t)\bar{g}(v-t)\mathrm{d}t\mathrm{d}v$$

$$= \int_{-\infty}^{\infty} \bar{f}_1(v)f_2(v)|g|^2\mathrm{d}v = |g|^2\int_{-\infty}^{\infty} \bar{f}_1 f_2\mathrm{d}x = |g|^2\langle f_1, f_2\rangle$$

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3. 1.5.3

*** [This seems tedious and maybe not worth it]

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2 Chapter 2

2.1 Subsection 1

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4. 2.1.1 Suppose $f(x) = f(x+2\pi)\ \forall x$ is periodic with period 2π. Show

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x,\ \forall a \in \mathbb{R}$$

As all great math proofs, no words are needed just equalities and beautiful integrals. Let a be given then:

$$\int_a^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_{2\pi}^{2\pi+a} f(x)\mathrm{d}x = \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x+2\pi)\mathrm{d}x$$

$$= \int_a^{2\pi} f(x)\mathrm{d}x + \int_0^{a} f(x)\mathrm{d}x = \int_0^{2\pi} f(x)\mathrm{d}x$$

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5. 2.1.2 **Dirichelet Basis**

$$W_{2N+1} = \text{span}\left\{\frac{1}{\sqrt{2\pi}}e^{ikt}\right\}_{k=\pm N}$$

Consider the set

$$g_k(t) = \frac{2\pi}{2N+1}\delta_N(t-x_k) = \frac{1}{2N+1}\sum_{n=-N}^{N} e^{in(t-k\pi/(N+\frac{1}{2})}$$

Show that

A) $B = \{g_k, k \in 1, 2, \cdots\}$ is linearly independent.

B) B spans W_{2N+1}

A) It suffices to notice that $g_k(x_l) = \delta_{kl}$. Thus we can see for any given k that g_k is independent of all the other's as $\sum_{k' \neq k} \lambda_{k'} g_{k'}(x_k) = 0$. Thus we can not have a non trivial linear relationship between the functions.

B) It is clear that $g_k(t) \in W_{2N+1}$ since each of the elements in its sum namely $e^{in(t-k\pi/(N+\frac{1}{2})}$ is just a multiple of e^{int} a basis element of W_{2N+1}. Notice there are $2N+1$ of these independnet vectors in the vector space of dimmension $2N+1$. Thus they must be a spanning set and there must exist coefficents for any function in the space to be written as a sum of this basis.

To actually exhibit coefficents one would use $f(t) = \sum_k f(x_k) g_k(t)$.

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6. 2.1.3 **Riemann-Lebesgue Lemma**
$G(u)$ piecewise continuous and has left and right derivatives on $[0, 2\pi]$. Show that

$$\lim_{N\to\infty} \int_0^{2\pi} G(u) \sin(N + \frac{1}{2}) u \mathrm{d}u = 0$$

WLOG $\exists a, b \in [0, \pi]$ st. $\forall x \in [a, b]$ $G(x) > 0$ or $G(x) < 0$.

Now it suffices to show

$$\lim_{N\to\infty} \int_a^b G(u) \sin(N + \frac{1}{2}) u \mathrm{d}u = 0$$

[since the interval $[0, 2\pi]$ can be sliced into a countable number of these intervals, and hten you can sum over them] WLOG we assume $G(x)$ is positve.

$$0 \leqslant \lim_{N\to\infty} \int_a^b G(u) \sin(N + \frac{1}{2}) u \mathrm{d}u \leqslant \lim_{N\to\infty} \int_a^b [\max_u G(u)] \sin(N + \frac{1}{2}) u \mathrm{d}u$$

Let G_m be the max above, then we have

$$0 \leqslant G_m \lim_{N\to\infty} \int_a^b \sin(N + \frac{1}{2}) u \mathrm{d}u = G_m \lim_{N\to\infty} \frac{\cos(N+\frac{1}{2})u}{N+\frac{1}{2}}\Big|_a^b \leqslant G_m \lim_{N\to\infty} \frac{2}{N+\frac{1}{2}} \leqslant 0$$

Thus we get the 0 value for the limit as desired.

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7. 2.1.4 Prove or disprove:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2 + a^2} = \frac{\pi}{a} \frac{\coth \pi a}{\cos^2 \pi x + \sin^2 \pi x \coth^2 \pi a}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2} = \frac{\pi^2}{\sin^2 \pi x}$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{(2\pi m+x)^2} = \frac{1}{4 \sin^2 x/2}$$

Let $f(m) = \frac{1}{(m+x)^2+a^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int_{-\infty}^{\infty} e^{-ikm} f(m)\mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int_{-\infty}^{\infty} e^{-ikm} \frac{1}{(m+x)^2+a^2}\mathrm{d}m = e^{ikx} \int_{-\infty}^{\infty} e^{-iku} \frac{1}{u^2+a^2}\mathrm{d}u = e^{ikx}\frac{\pi}{a}e^{-|k|a}$$

Using the Poisson formula we thus see:

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+x)^2+a^2} = \sum_{k=-\infty}^{\infty} e^{i2\pi kx}\frac{\pi}{a}e^{-|2\pi k|a} = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi kx}e^{-|2\pi k|a} + \sum k \leqslant 0 e^{i2\pi kx}e^{-|2\pi k|a} - 1\right]$$

$$= \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + \sum k \leqslant 0 e^{i2\pi k(x+a)} - 1\right] = \frac{\pi}{a}\left[\sum k \geqslant 0 e^{i2\pi k(x-a)} + e^{-i2\pi k(x+a)} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1}{1-e^{-i2\pi(x+a)}} + \frac{1}{1-e^{i2\pi(x-a)}} - 1\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{-i2\pi(x+a)} + 1 - e^{i2\pi(x-a)} - \left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{1-e^{i2\pi(x-a)}e^{-i2\pi(x+a)}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{1-e^{-i4\pi a}}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

$$= \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right] = \frac{\pi}{a}\left[\frac{\left(1-e^{-i2\pi a}\right)\left(1+e^{-i2\pi a}\right)}{\left(1-e^{i2\pi(x-a)}\right)\left(1-e^{-i2\pi(x+a)}\right)}\right]$$

************BELIEVE THE ABOVE DISPROVES THE SUM********

For the last 2 the answers are somewhat lack luster.

Let $f(m) = \frac{1}{(m+x)^2}$, then we wish to find $\sum f(m)$. To this end we consider $F(k) = \int_{-\infty}^{\infty} e^{-ikm} f(m)\mathrm{d}m$. To find this we use the u sub: $u = m + x$

$$\int_{-\infty}^{\infty} e^{-ikm} \frac{1}{(m+x)^2}\mathrm{d}m = e^{ikx} \int_{-\infty}^{\infty} e^{-iku} \frac{1}{u^2}\mathrm{d}u$$

This is an error function producing integral. This is bad. Notice also that the claimed value on the right hand side of the equation doesn't even make sense for $x = 0$. Thus the equation as stated is clearly false, Besides as any good student of Bergelson knows: $\sum 1/n^2 = \frac{\pi^2}{6}$

For the final one we again have issues with $\sin(0) = 0$ in the denominator.

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8. 2.1.5 My man Stephane G. Mallat claims the following: The family of funcitons $\phi(x-k)k = 0, \pm 1, \pm 2, \cdots$ is orthonormal iff

$$\sum_{k=-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi k)|^2$$

is constant wrt ω. Prove my boy wrong or right.

Stephane is no chump and said a true thing. Lets investigate the sum:

$$\sum_{k=-\infty}^{\infty}|\hat{\phi}(\omega+2\pi k)|^2=\sum_{k=-\infty}^{\infty}\overline{\hat{\phi}(\omega+2\pi k)}\hat{\phi}(\omega+2\pi k)$$

Now to avoid a factor out front the rest of the analysis, the $\frac{1}{\sqrt{2\pi}}$ is suppressed when expanding the Fourier transform.

$$=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i(\omega+2\pi k)y}\bar{\phi}(y)\mathrm{d}y\int_{-\infty}^{\infty}e^{-i(\omega+2\pi k)x}\phi(x)\mathrm{d}x$$

$$=\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x)\bar{\phi}(y)e^{i(\omega+2\pi k)y}e^{-i(\omega+2\pi k)x}\mathrm{d}x\mathrm{d}y$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x)\bar{\phi}(y)\sum_{k=-\infty}^{\infty}e^{-i(\omega+2\pi k)(x-y)}\mathrm{d}x\mathrm{d}y$$

Via formula on page 62

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x)\bar{\phi}(y)e^{-i\omega(x-y)}\sum_{k=-\infty}^{\infty}\delta(x-y-k)\mathrm{d}x\mathrm{d}y$$

Now for the change of variables $u=x-y, v=y$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(u+v)\bar{\phi}(v)e^{-i\omega u}\sum_{k=-\infty}^{\infty}\delta(u-k)\mathrm{d}u\mathrm{d}v=\int_{-\infty}^{\infty}\bar{\phi}(v)\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\omega u}\phi(u+v)\delta(u-k)\mathrm{d}u\mathrm{d}v$$

$$=\int_{-\infty}^{\infty}\bar{\phi}(v)\sum_{k=-\infty}^{\infty}e^{-i\omega k}\phi(k+v)\mathrm{d}v=\sum_{k=-\infty}^{\infty}e^{-i\omega u}\int_{-\infty}^{\infty}\bar{\phi}(v)\phi(k+v)\mathrm{d}v=\sum_{k=-\infty}^{\infty}\langle\phi(v),\phi(k+v)\rangle e^{-i\omega u}$$

Thus the sum above is in fact a Fourier series with $c_k=\langle\phi(v),\phi(k+v)\rangle$. Now this series being constnat is equivalent to $c_k=\delta_{0k}$, which is equivalent to the $\phi(v+k)$'s being an orthogonal system.

Moreover if $c_0=1$ then we have an orthonormal system aswell. Thus the system is orthonormal if the series is constant and equal to 1. Now in actuallity we rememeber that we have a secret factor of $\frac{1}{\sqrt{2\pi}}$ hanging around. Thus the constants value is actually that.

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9. 2.1.6 Prove or Disprove the following identities:

   i)

$$\sum_{m=-\infty}^{\infty}f([2m+1]\pi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^nF(n)$$

   ii)

$$2\pi\sum_{m=-\infty}^{\infty}(-1)^mf(2\pi m)=\sum_{n=-\infty}^{\infty}F(n+\frac{1}{2})$$

iii)

$$\sum_{m=-\infty}^{\infty} \delta(u - [2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inu}$$

iv) And in greater generality

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

v)

$$\sum_{m=-\infty}^{\infty} \frac{1}{|a|} \delta(u - \frac{[2m+1]\pi}{a}) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n e^{inau}$$

The main equation to keep in mind here is the general Poisson formula:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(x-t)} f(t) \mathrm{d}t = \sum_{m=-\infty}^{\infty} f(x + 2\pi m)$$

i) Begin with $x = \pi$ in the formula above and we see:

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{in(\pi-t)} f(t) \mathrm{d}t = \sum_{m=-\infty}^{\infty} f([2m+1]\pi)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} f(t) \mathrm{d}t = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(n)$$

ii) Begining with the right hand side:

$$\sum_{n=-\infty}^{\infty} F(n + \frac{1}{2}) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(n+\frac{1}{2})t} f(t) \mathrm{d}t = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-int} f(t) e^{-it/2} \mathrm{d}t$$

Now we notice this is the Fourier transform of not f but of $f(t)e^{-it/2}$, applying Poisson sum with this:

$$= 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m) e^{-i(2\pi m)/2} = 2\pi \sum_{m=-\infty}^{\infty} f(2\pi m) e^{-i\pi m} = 2\pi \sum_{m=-\infty}^{\infty} (-1)^m f(2\pi m)$$

iii) It is straightforward to see that this is actually just 1) but in the supressed function notation. To see this we note that

$$\delta(u - [2m+1]\pi) \to f([2m+1]\pi), \quad e^{inu} \to \int_{-\infty}^{\infty} e^{inu} f(u) \mathrm{d}u = F(-n)$$

But wait we get $\sum_{n=-\infty}^{\infty} (-1)^n F(-n)$ and not the exact sum we wanted! Thankfully $(-1)^n = (-1)^{-n}$ and we just switch the order of the sum and get the identity.

iv) Let $\bar{f}(x) = f(\frac{x}{a})$, then by 1) we have:

$$\sum_{m=-\infty}^{\infty} f(\frac{[2m+1]\pi}{a}) = \sum_{m=-\infty}^{\infty} \bar{f}([2m+1]\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \bar{F}(n)$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} \bar{f}(t)\mathrm{d}t = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} e^{-int} f(t/a)\mathrm{d}t$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n a \int_{-\infty}^{\infty} e^{-inau} f(u)\mathrm{d}u = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n a F(na) = \frac{a}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n F(na)$$

v) Similar to 3) we note that this is just an earlier identity. A constant is shifted around but this is basicly just 4).

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10. 2.2.1 Show that

$$\lim_{\omega\to\infty} \frac{\sin 2\pi\omega x}{\pi x}, \ \omega > 0$$

is a representation of the Dirac δ DISTRIBUTION.

This equality can only be expressed inside of an integral, thus we must apply the above to test functions and see that the answer is the same as with the delta *distribution*.

Thus if we consider f continuous on some $[-a, a]$ then we get:

$$\lim_{\omega\to\infty} \int_{-a}^{a} \delta_\omega(x) f(x)\mathrm{d}x = \lim_{\omega\to\infty} \int_{-a}^{a} \frac{\sin 2\pi\omega x}{\pi x} f(x)\mathrm{d}x$$

Now in following with the style of the Fourier series theorem we add and subtract the same term, namely a $f(0)$ (inside some paranthesis but basicly the same)

$$= \lim_{\omega\to\infty} \int_{-a}^{a} \frac{\sin 2\pi\omega x}{\pi x} \left(f(x) - f(0) + f(0)\right)\mathrm{d}x = \lim_{\omega\to\infty} \int_{-a}^{a} \frac{\sin 2\pi\omega x}{\pi} \frac{f(x) - f(0)}{x} + f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

Now we can consider WLOG just the positive side of the integral.

$$\lim_{\omega\to\infty} \int_{0}^{a} \frac{\sin 2\pi\omega x}{\pi} \frac{f(x) - f(0)}{x} + f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

Notice for the exact same resonaning as $G(u)$ on page 57 that we get $\frac{f(x)-f(0)}{x}$ is continuous at 0 and converges to $f'(0^+)$. Thus again we see that the integral:

$$\int_{0}^{a} \frac{\sin 2\pi\omega x}{\pi} \frac{f(x) - f(0)}{x}\mathrm{d}x \to 0$$

as $\omega \to \infty$. Thus we only have:

$$\lim_{\omega\to\infty} \int_{0}^{a} f(0)\frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x = f(0) \lim_{\omega\to\infty} \int_{0}^{a} \frac{\sin 2\pi\omega x}{\pi x}\mathrm{d}x$$

$$y = 2\pi\omega x, \quad dy = 2\pi\omega dx$$

$$\frac{y}{2\pi\omega} = x, \quad \frac{dy}{2\pi\omega} = dx$$

$$= f(0) \lim_{\omega\to\infty} \int_0^{a2\pi\omega} \frac{\sin y}{\pi y/(2\pi\omega)} \frac{\mathrm{d}y}{2\pi\omega} = \frac{f(0)}{\pi} \int_0^{\infty} \frac{\sin y}{y} \mathrm{d}y = \frac{f(0)}{\pi}\frac{\pi}{2} = \frac{f(0)}{2}$$

Using a isomorphic version of the logic above one can get the $f(0^-)$ term and complete the proof.

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11. 2.2.2 Assuming that $f(x)$ is nearlly linear, that is to say that

$$f(-a) = f(0) - af'(0) + \text{ H.O.T.}$$

Show that

$$I = \int_{-\infty}^{\infty} \delta(x+a) f(x) \mathrm{d}x$$

can be evaluated by means of the formal equation:

$$\delta(x+a) = \delta(x) + a\delta'(xx)$$

By the definition of the δ funciton we have:

$$I = \int_{-\infty}^{\infty} \delta(x+a) f(x) \mathrm{d}x = f(-a)$$

$$= f(0) - af'(0) + \text{ H.O.T.} = \int_{-\infty}^{\infty} \delta(x)f(x) - a\delta(x)f'(x)\mathrm{d}x = \int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x - a\int_{-\infty}^{\infty} \delta(x)f'(x)\mathrm{d}x$$

Via integration by parts we know that:

$$\int_{-\infty}^{\infty} \delta(x)f'(x)\mathrm{d}x = \delta(x)f(x)|_{\pm\infty} - \int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = -\int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x$$

Putting stuff together:

$$= \int_{-\infty}^{\infty} \delta(x)f(x)\mathrm{d}x + a\int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \delta(x) + a\delta'(x)f(x)\mathrm{d}x$$

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12. 8.8.8
