LARES's thermal thrust

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1 Introduction

Consider a tungsten ball of radius R with a cylindrical cavity on the surface. Insert a glass cone, whose profile is an equilateral triangle of side L, in this cavity such that the base of the cone is flush with the ball's surface, and the tip of the cone just touches the bottom of the cavity. Let sunlight be incident perpendicularly to the bottom of the cone, heating up the system. Due to the high thermal conductivity of the metal, we will assume this later is at a constant temperature T_W at equilibrium, and solve for the temperature variation within the glass cone by the finite difference method.

We set up a system of cylindrical coordinates (ρ, ϕ, z) as follows: the origin is chosen to be the tip of the cone, the z-axis is the axis of the cone with the positive direction toward the sunlight. On the slanted side of the cone, pick n equally spaced points. Using these points, we can define a grid, dividing up the cone into small rings, each of which is assumed at a constant temperature. Due to azimuthal symmetry, the temperature is independent of ϕ so the temperature is indeed constant in any given ring for large n. We will label each of these rings by two indices ij, each ranging from 0 to n. The first index increases with increasing ρ, and the second index increases with increasing z. The geometry of the system and the labelling scheme is depicted in Figure 1 below for $n = 2$. There are thus $\frac{1}{2}(n+1)(n+2)+1$ different temperatures to solve for, including the metal's temperature.

2 Radiation exchange inside the cavity

First, consider radiation exchange between elements ii (the ones in contact with the cavity) with the metal (subscript W). In a first time, we will ignore scattering of radiation back and forth inside the cavity. The view factors $F_{ii,jj}$, $F_{ii,W}$ and $F_{W,ii}$ (the fraction of radiation emitted by the first element that is intercepted by the second, ignoring scattering) involved can be obtained by a few simple observations, as follows. Due to the convexity of the cone, radiation

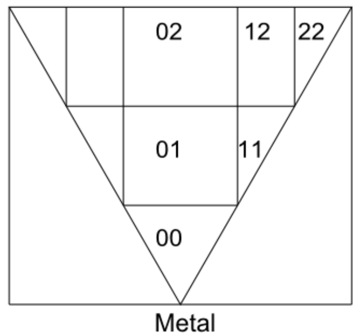

Figure 1: Approximation scheme for n=2.

emitted by element ii cannot be intercepted by any other jj, but is completely intercepted by the metal. We then have:

$$F_{ii,jj} = 0 \tag{1}$$

$$F_{ii,W} = 1 \tag{2}$$

To derive the remaining view factors $F_{W,ii}$, we use the so-called reciprocity relation between view factors:

$$A_W F_{W,ii} = A_{ii} F_{ii,W} \tag{3}$$

where A denotes the surface area of the element in question. From geometrical considerations, we find:

$$A_{ii} = \frac{\pi}{2}(\frac{L}{n+1})^2(2i+1) \tag{4}$$

$$A_W = (1+2\sqrt{3})\frac{\pi}{4}L^2 \tag{5}$$

so that

$$F_{W,ii} = (\frac{2}{1+2\sqrt{3}})\frac{2i+1}{(n+1)^2} \tag{6}$$

Finally, to compute $F_{W,W}$, notice that radiation emitted by the metal must be intercepted by either the metal or the glass (but cannot escape, since the cavity is closed). This means:

$$F_{W,W} + \sum_{k=0}^{n} F_{W,kk} = 1 \tag{7}$$

Or:

$$F_{W,W} = \frac{2\sqrt{3}-1}{2\sqrt{3}+1} \tag{8}$$

Using the blackbody radiation formula, we can now write down the amount of radiation emitted by element ii that is intercepted by element jj (denoted by $I_{ii,jj}$ as follows:

$$I_{ii,jj} = \epsilon_{ii} A_{ii} F_{ii,jj} \sigma T_{ii}^4 \tag{9}$$

where ϵ_{ii} is the emissivity of the element ii (this is a dimensionless number less than unity; it measures how badly the element fails to be an ideal blackbody), σ is the Stefan-Boltzmann constant and T_{ii} is the (constant) temperature of element ii. Since all elements ii are made of glass, they have the same emissivity, so we will replace all ϵ_{ii} by ϵ_{gl}.

Next, we want to take into account the contributions of scattering to $I_{ii,jj}$. To do this, consider the component $I_{ii,kk}$, i.e. radiation emitted by element ii that first hits element kk. Multiplying this amount by $(1-\epsilon_{kk})$ yields the portion that is reflected away (to see this, notice that the radiation involved is in the infrared, and both glass and metal are opaque to the IR, so that transmissivity at this wavelength is zero. On the other hand, the transmissivity, reflectivity and emissivity add up to unity). By this reasoning, equation (9) should be modified:

$$I_{ii,jj} = \epsilon_{gl} A_{ii} F_{ii,jj} \sigma T_{ii}^4 + (1-\epsilon_W) F_{W,jj} I_{ii,W} + \sum_{k=0}^{n} (1-\epsilon_{gl}) F_{kk,jj} I_{ii,kk} \tag{10}$$

Similarly, replacing either ii or jj by W yields the corresponding equations for heat exchange with the metal part of the cavity:

$$I_{W,jj} = \epsilon_W A_W F_{W,jj} \sigma T_W^4 + (1-\epsilon_W) F_{W,jj} I_{W,W} + \sum_{k=0}^{n} (1-\epsilon_{gl}) F_{kk,jj} I_{W,kk} \tag{11}$$

$$I_{ii,W} = \epsilon_{gl} A_{ii} F_{ii,W} \sigma T_{ii}^4 + (1-\epsilon_W) F_{W,W} I_{ii,W} + \sum_{k=0}^{n} (1-\epsilon_{gl}) F_{kk,W} I_{ii,kk} \tag{12}$$

$$I_{W,W} = \epsilon_W A_W F_{W,W} \sigma T_W^4 + (1-\epsilon_W) F_{W,W} I_{W,W} + \sum_{k=0}^{n} (1-\epsilon_{gl}) F_{kk,W} I_{W,kk} \tag{13}$$

Given the view factors computed above, this system of equation simplifies too:

$$I_{ii,jj} = (1-\epsilon_W) F_{W,jj} I_{ii,W} \tag{14}$$

$$I_{W,jj} = \epsilon_W A_W F_{W,jj} \sigma T_W^4 + (1-\epsilon_W) F_{W,jj} I_{W,W} \tag{15}$$

$$I_{ii,W} = \epsilon_{gl} A_{ii} \sigma T_{ii}^4 + (1-\epsilon_W) F_{W,W} I_{ii,W} + (1-\epsilon_{gl}) \sum_{k=0}^{n} I_{ii,kk} \tag{16}$$

$$I_{W,W} = \epsilon_W A_W F_{W,W} \sigma T_W^4 + (1-\epsilon_W) F_{W,W} I_{W,W} + (1-\epsilon_{gl}) \sum_{k=0}^{n} I_{W,kk} \tag{17}$$

This is a linear system in $(n+1)^2$ variables ($I_{ii,jj}$, $I_{W,jj}$, $I_{ii,W}$ and $I_{W,W}$), which we can solve for in terms of the temperature T_{ii} and T_W. Next, we form the net heat absorbed by element ii by radiation, which we denote by Q_{ii}. This is given as a function of the temperatures as follows:

$$Q_{ii}(T_{jj}, T_W) = \epsilon_{gl} I_{W,ii} - \epsilon_W I_{ii,W} + \epsilon_{gl} \sum_{k=0}^{n} (I_{kk,ii} - I_{ii,kk}) \tag{18}$$

3 The equilibrium equations

3.1 Elements 00 and nn

Since elements 00 and nn both are in contact with the metal, and since they have constant temperature, they must have the metal's temperature:

$$T_{00} = T_{nn} = T_W \tag{19}$$

The above accounts for 2 equations.

3.2 The remaining elements ii

For the remaining elements ii, the neat heat absorbed by radiation must equal the heat conducted inward. By Fourier's Law, this latter is given by the product of the thermal conductivity of glass κ_{gl} and the magnitude of the temperature gradient $|\vec{\nabla} T|$. For the gradient, we will approximate first derivatives in the standard way. Thus, at equilibrium:

$$Q_{ii}(T_{jj}, T_W) = 2\kappa_{gl}(\frac{n+1}{L})\sqrt{(T_{ii} - T_{i-1,i})^2 + \frac{1}{3}(T_{ii} - T_{i,i+1})^2} \tag{20}$$

Equation (20) represents (n-1) equations (i ranges from 1 to n-1). Here we have assumed $T_{i-1,i+1} > T_{ii}$ since the temperature should decrease as we penetrate deeper into the glass.

3.3 Interior, off-axis elements.

Next, consider elements ij for which $i \neq 0$, $j \neq n+1$ and $i \neq j$. In equilibrium, the temperature inside the cone satisfies the Laplace equation:

$$\nabla^2 T(\rho, z) = \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{\partial^2 T}{\partial \rho^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{21}$$

where we used cylindrical symmetry. We will now derive a discrete version for the Laplacian in cylindrical coordinates. The radial coordinate of element ij is only a function of i, and is given by:

$$\rho_{ij} = \frac{Li}{2(n+1)} \tag{22}$$

First derivatives are approximated in the standard way:

$$\frac{\partial T}{\partial \rho}(ij) = \frac{T_{i+1,j} - T_{i-1,j}}{\rho_{i+1,j} - \rho_{i-1,j}} = \frac{(n+1)}{L}(T_{i+1,j} - T_{i-1,j}) \tag{23}$$

Second derivatives are approximated by applying this approximation scheme twice. Thus,

$$\frac{\partial^2 T}{\partial \rho^2}(ij) = \frac{\frac{\partial T}{\partial \rho}(ij) - \frac{\partial T}{\partial \rho}(i-1,j)}{\rho_{ij} - \rho_{i-1,j}} = \frac{4(n+1)^2}{L^2}(T_{i+1,j} + T_{i-1,j} - 2T_{i,j}) \tag{24}$$

Similarly,

$$\frac{\partial^2 T}{\partial z^2}(ij) = \frac{4(n+1)^2}{3L^2}(T_{i,j+1} + T_{i,j-1} - 2T_{i,j}) \tag{25}$$

Equation (21) becomes in this approximation scheme,

$$(2 + \frac{1}{i})T_{i+1,j} + (2 - \frac{1}{i})T_{i-1,j} - \frac{16}{3}T_{ij} + \frac{2}{3}T_{i,j+1} + \frac{2}{3}T_{i,j-1} = 0 \tag{26}$$

Equation (26) represents $\frac{n^2}{2} - \frac{3}{2}n + 1$ equations.

3.4 Interior, on-axis elements.

To obtain equations for the interior elements on the symmetry axis of the cone, i.e. elements of the form $0i$ where $i \neq 0$ and $i \neq n$, we switch back to Cartesian coordinates (as the expression for the Laplacian in cylindrical coordinates is ill-defined at $\rho = 0$). Laplace's equation reads:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{27}$$

where, using cylindrical symmetry:

$$\frac{\partial^2 T}{\partial x^2}(0j) = \frac{\partial^2 T}{\partial y^2}(0j) = \frac{4(n+1)}{L}\frac{\partial T}{\partial \rho}(0j) = \frac{8(n+1)^2}{L^2}(T_{1j} - T_{0j}) \tag{28}$$

The Laplace equation for on-axis elements then reads:

$$-\frac{14}{3}T_{0j} + 4T_{1j} + \frac{1}{3}T_{0,j+1} + \frac{1}{3}T_{0,j-1} = 0 \tag{29}$$

This accounts for $n - 1$ equations.

3.5 Off-axis elements facing the vacuum.

Next, we want to write down the equilibrium equations for the off-axis elements upon which the sunlight is incident, i.e. elements in such that $i \neq 0$ and $i \neq n$. We will assume that elements $i, n-1$ and $i+1, n$ are hotter than in, since they are closer to the metal. With this assumption, the heat flowing into in from

$i, n-1$ and $i+1, n$ must equal the heat flowing out into $i-1, n$ and radiated into space. The heat flowing into in is:

$$\kappa_{gl} \frac{T_{i,n-1} - T_{in}}{\frac{\sqrt{3}}{2}(\frac{L}{n+1})} A_{i+1} + \kappa_{gl} \frac{T_{i+1,n} - T_{in}}{\frac{1}{2}(\frac{L}{n+1})} S_{i+1} \tag{30}$$

whereas the heat flowing out of in is:

$$\kappa_{gl} \frac{T_{in} - T_{i-1,n}}{\frac{1}{2}(\frac{L}{n+1})} S_i + \epsilon_{gl} \sigma T_{in}^4 A_{i+1} \tag{31}$$

where the temperature gradient is approximated in the same way as in equation (20), and A_i and S_i are the following surface areas:

$$A_i = \frac{\pi}{4} (\frac{L}{n+1})^2 (2i-1) \tag{32}$$

$$S_i = (\frac{L}{n+1})^2 \frac{\sqrt{3}}{2} \pi i \tag{33}$$

Equating the two yields the equilibrium equation:

$$\frac{1}{2\sqrt{3}} (T_{i,n-1} - T_{in})(2i+1) + (T_{i+1,n} - T_{in})\sqrt{3}(i+1) = (T_{in} - T_{i-1,n})\sqrt{3}i + \frac{\epsilon_{gl}}{\kappa_{gl}} \sigma T_{in}^4 (2i+1)(\frac{L}{n+1}) \tag{34}$$

This equation accounts for $n-1$ equations.

3.6 Element 0n.

As for the element $0n$ (the one furthest from the metal), it absorbs heat from elements $1n$ and $0, n-1$. The quantity absorbed is obtained by setting $i = 0$ in equation (30):

$$\kappa_{gl} \frac{T_{0,n-1} - T_{0n}}{\frac{\sqrt{3}}{2}(\frac{L}{n+1})} A_1 + \kappa_{gl} \frac{T_{1n} - T_{0n}}{\frac{1}{2}(\frac{L}{n+1})} S_1 \tag{35}$$

On the other hand, the heat radiated by $0n$ into space is:

$$\epsilon_{gl} A_1 \sigma T_{0n}^4 \tag{36}$$

Again, the two quantities above balance each other in equilibrium, so that:

$$\frac{1}{2\sqrt{3}} (T_{0,n-1} - T_{0,n}) + \sqrt{3}(T_{1n} - T_{0n}) = \frac{\epsilon_{gl}}{\kappa_{gl}} \frac{1}{4} (\frac{L}{n+1}) \sigma T_{0n}^4 \tag{37}$$

3.7 The metal.

Finally, we need an equation for the metal's temperature. For a solid metal ball, the metal's temperature is found to be around 406 K. We can simply reuse this value as an approximation, but we can also compute it from first principles. Consider the net heat absorbed by the whole system from sunlight. This quantity is given by:

$$\Phi_{solar}\alpha_{vis}\pi R^2 \tag{38}$$

where Φ is the solar constant, and α_{vis} is the absorptivity of tungsten with respect to sunlight (notice that only tungsten absorbs in the visible, glass is transparent to sunlight). On the other hand, the heat radiated into space by the system as a whole has two contributions, one from the glass and one from the metal. Suppose the line from the satellite's center to any point on the edge of the cone's base makes an angle θ_0 with the z-axis, then the area of tungsten exposed to vacuum can be found to be $2\pi R^2(1+\cos\theta_0)$, and the total heat radiated into space is:

$$2\pi R^2(1+\cos\theta_0)\epsilon_W\sigma T_W^4 + \epsilon_{gl}\sigma\sum_{i=0}^{n} A_{i+1}T_{in}^4 \tag{39}$$

Equating the two quantities above yields the final equilibrium equation we are looking for:

$$\Phi_{solar}\alpha_{vis}\pi R^2 = 2\pi R^2(1+\cos\theta_0)\epsilon_W\sigma T_W^4 + \epsilon_{gl}\sigma\frac{\pi}{4}(\frac{L}{n+1})^2\sum_{i=0}^{n}(2i+1)T_{in}^4 \tag{40}$$

Equations (19), (20), (26), (29), (34), (37) and (40) form a system of equations that can now be solved for the temperatures T_{ij} and T_W.

4 Example: the case n=2

Solving the system of equations above for n=2, we find that the only real and positive solution is:

$$T_W = T_{00} = T_{22} \approx 403\text{K} \tag{41}$$

$$T_{01} \approx 392\text{K} \tag{42}$$

$$T_{02} \approx 389\text{K} \tag{43}$$

$$T_{11} \approx 391\text{K} \tag{44}$$

$$T_{12} \approx 391\text{K} \tag{45}$$

Thus, we find that the CCR is colder than the metal by at most 14 K. If we exclude elements 00 and 22 (intermediary elements between the CCR and the metal), the temperature variation within the glass is negligible (of the order of 3 K). This latter is in agreement with the value given in Slabinski.

5 Thermal thrust

We assume the surface of the satellite to be Lambertian: the angular dependence of the intensity emitted by any surface element $I(\theta)$ is proportional to the cosine of the angle between the normal vector to the surface element and the observer's line of sight:

$$I(\theta) = I_0 \cos\theta \tag{46}$$

where I_0 is the intensity emitted in the normal direction. Integrating over all angles yields the total radiated intensity by the surface element, given by the blackbody radiation formula:

$$\epsilon\sigma T^4 = \int I_0 \cos\theta d\Omega = \pi I_0 \tag{47}$$

where ϵ is the emissivity of the surface element, σ is the Stefan-Boltzmann constant, and T is the temperature of the surface element. The total momentum emitted in a unit time by the surface element into an angular ring $d\Omega$ is:

$$dP(\theta) = \frac{I(\theta)}{c} d\Omega \tag{48}$$

The total momentum emitted per unit time by the surface element into any direction (i.e. the force on the surface element) is then:

$$F = \int \cos\theta dP(\theta) = \frac{2}{3}\frac{\epsilon\sigma T^4}{c} \tag{49}$$

where, by symmetry, it is enough to integrate over the normal component of dP. Finally, integrating over the sphere gives the total force on the satellite. We will denote the spherical coordinates on the sphere's surface by ϑ and φ, and assume that there is axial symmetry (T is only a function of ϑ). It is then enough to integrate the component of the force along the axis of symmetry (taken to be the z-axis):

$$F_{total} = \int \frac{2}{3}\frac{\epsilon\sigma T(\vartheta)^4}{c}(R^2 \sin\vartheta d\vartheta d\varphi)\cos\vartheta = \frac{4}{3}\frac{\sigma}{c}\pi R^2 \int_0^\pi \epsilon T_\vartheta^4 \cos\vartheta \sin\vartheta d\vartheta \tag{50}$$

For the case at hand, we split the integral into two contributions: one for the glass region and one for the tungsten region. Let θ_0 be the angle subtended by the CCR as viewed from the satellite's center, and ϵ_{gl} and ϵ_W are glass's and tungsten's emissivity in the infrared, respectively. The metal is at 403 K, and for simplicity, we approximate the temperature of the glass interface as uniform at 389 K. Then:

$$F_{total} = -\frac{4}{3}\frac{\sigma}{c}\pi R^2 \left(\epsilon_{gl}(389^4)\int_0^{\theta_0} \cos\vartheta \sin\vartheta d\vartheta + \epsilon_w(403^4)\int_{\theta_0}^\pi \cos\vartheta \sin\vartheta d\vartheta\right) \tag{51}$$

Numerically, we find:

$$F_{total} = -2.57 \times 10^{-9} N \tag{52}$$

Since the force is negative, it is a drag. The mass of the system is about $450kg$, so this force corresponds to a deceleration of about 5.73 pm/s^2. For LAGEOS, this drag has been experimentally determined to be about 3.4 pm/s^2 (given in Slabinski). Our theoretical values for the acceleration and the thrust are then in the right order of magnitude.

A Temperature variation within the metal sphere

In this appendix, we ignore the presence of the CCRs and consider a static tungsten sphere illuminated by the Sun. In the end, we will show that the temperature variation with the metal is negligible. The temperature obeys the heat equation:

$$\kappa^2 \nabla^2 T = \rho C_p \frac{\partial T}{\partial t} \tag{53}$$

Here κ is the thermal conductivity of tungsten (173 W/K m), ρ is the mass density of tungsten (19.25 g/cm^3), and C_p is the specific heat of tungsten (24.27 J/(K mol)). As the sphere is static, the right-hand side above vanishes in the steady state, and the temperature is solution to Laplace's equation.
Orient the z-axis in the direction of the sunlight, and consider a patch of surface dA at angle θ, where θ is the polar angle of the spherical coordinates. For $\theta = 0$, the patch is perpendicular to the incident light, and the intensity there is the solar constant ϕ (1370 W/m^2) at the Earth's location. For $0 < \theta < \pi/2$, the intensity is $\phi \cos(\theta)$, since the normal to the patch makes an angle θ with the incident radiation. As the sunlight only illuminates the upper hemisphere, the intensity is zero for $\pi/2 < \theta < \pi$. At the surface, the incident energy is either conducted into the sphere's interior, or remmitted as thermal radiation. The boundary condition then reads:

$$\kappa \frac{\partial T}{\partial r} + \epsilon_{IR} \sigma T^4 = \alpha_{vis} \phi \cos \theta \Theta(\frac{\pi}{2} - \theta) \tag{54}$$

where ϵ_{IR} is the emissivity of tungsten in the infrared (0.10), σ is the Stefan-Boltzmann constant, α_{vis} is the absorptivity of tungsten in the visible (0.45). The Heaviside step function Θ is 1 when its argument is positive and 0 otherwise, and all quantities are evaluated at the surface. Next, we linearize the boundary condition by letting $T = T_0 + \Delta T$, where T_0 is the average temperature of the sphere, and $\Delta T << T_0$ is function of position. As the sphere is made of metal (high conductivity), and its radius is small, we expect little variation across the sphere. Then $T^4 = (T_0 + \Delta T)^4 \approx T_0^4 + 4T_0^3 \Delta T$. Equations (1) and (2) become:

$$\nabla^2 (\Delta T) = 0 \tag{55}$$

$$\kappa \frac{\partial T}{\partial r} + \epsilon_{IR} \sigma T_0^4 + 4\epsilon_{IR} \sigma T_0^3 \Delta T = \alpha_{vis} \phi \cos \theta \Theta(\frac{\pi}{2} - \theta) \tag{56}$$

Note that the problem has azimuthal symmetry (no ϕ dependence). The general solution to Laplace's equation with this symmetry is given by:

$$\Delta T(r,\theta,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta) \tag{57}$$

But ΔT must be finite at the origin, so all B_l's vanish. Substituting the general solution into the boundary condition, we obtain:

$$\kappa\sum_{l=1}^{\infty} A_l l R^{l-1}P_l(\cos\theta)+4\epsilon_{IR}\sigma T_0^3\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -\epsilon_{IR}\sigma T_0^4 P_0(\cos\theta)+\alpha_{vis}\phi\cos\theta\Theta(\frac{\pi}{2}-\theta) \tag{58}$$

Where R is the radius of LARES (18.2 cm), and we used the fact that $P_0(x) = 1$. Next, we expand the source term in the Legendre polynomials:

$$\cos\theta\Theta(\frac{\pi}{2}-\theta) = \sum_{l=0}^{\infty} C_l P_l(\cos\theta) \tag{59}$$

Using the orthogonality of the Legendre polynomials,

$$\int_{-1}^{1} P_m(x)P_n(x)\,\mathrm{d}x = \frac{2}{2n+1}\delta_{nm} \tag{60}$$

the coefficients are found to be:

$$C_l = \frac{2l+1}{2}\int_{-1}^{1} P_l(\cos\theta)\cos\theta\Theta(\frac{\pi}{2}-\theta)d(\cos\theta) \tag{61}$$

Only the range $\theta < \frac{\pi}{2}$ contributes:

$$C_l = \frac{2l+1}{2}\int_{0}^{1} P_l(x)P_1(x)dx \tag{62}$$

where we used the fact that $P_1(x) = x$. The value of this integral can be found in tables. The first few coefficients are $C_0 = \frac{1}{4}$, $C_1 = \frac{1}{2}$, and $C_2 = \frac{5}{16}$, and all coefficients with odd l vanish, except $l = 1$. The boundary condition can be rewritten as:

$$\kappa\sum_{l=1}^{\infty} A_l l R^{l-1}P_l(\cos\theta)+4\epsilon_{IR}\sigma T_0^3\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -\epsilon_{IR}\sigma T_0^4 P_0(\cos\theta)+\alpha_{vis}\phi\sum_{l=0,1,even}^{\infty} C_l P_l(\cos\theta) \tag{63}$$

Equating the coefficients of boths sides yields:

$$A_0 = -\frac{1}{4}T_0 + \frac{\alpha_{vis}\phi}{16\epsilon_{IR}\sigma T_0^3} \tag{64}$$

$$A_1 = \frac{\frac{1}{2}\alpha_{vis}\phi}{\kappa + 4\epsilon_{IR}\sigma T_0^3 R} \tag{65}$$

$$A_2 = \frac{\frac{5}{16}\alpha_{vis}\phi}{2\kappa R + 4\epsilon_{IR}\sigma T_0^3 R^2} \tag{66}$$

It remains to find T_0. If we regard the sphere as at a single temperature T_0, then the statement of thermal equilibrium reads:

$$\epsilon_{IR}\sigma T_0^4(4\pi R^2) = \alpha_{vis}\phi(\pi R^2) \tag{67}$$

Solving for T_0, we find an average temperature of 406 K. Also, the equation above implies that A_0 vanishes. Substituting into the expressions for A_1 and A_2, we find the final solution:

$$\Delta T(r,\theta) = 0.323(\frac{r}{R})\cos\theta + 0.101(\frac{r}{R})^2\frac{1}{2}(3\cos^2\theta - 1) \tag{68}$$

In particular, we find that $\Delta T = 0.424$ K at the point closest to the Sun ($\theta = 0$, $r = R$) and $\Delta T = -0.222$ K at the antipodal point ($\theta = \pi$, $r = R$). The maximum temperature difference across the metal sphere is thus 0.646 K.

B Numerical values of the constants

Solar constant	Φ	1367.6 W/m^2
Stefan-Boltzmann constant	σ	5.67 $\times 10^{-8} W/m^2K^4$
Absorptivity of tungsten to sunlight	α_{vis}	0.45
Emissivity of tungsten in the IR	ϵ_W	0.1
Emissivity of glass in the IR	ϵ_{gl}	0.9
Thermal conductivity of glass	κ_{gl}	1.1 W/(m K)
Radius of the metal sphere	R	0.182 m
Size of CCR	L	0.0381 m
Angle subtended by the CCR	θ_0	0.1048 rad

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