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Research Thesis

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Chapter 1

Fourier Basics

1.1 Vector Space of Functions on Boolean Hyper-cube

Definition 1.1 (Inner Product)**.** Consider the 2^n-dimensional vector space of all functions $f : \{0,1\}^n \to \mathbb{R}$. We define an inner product on this space by

$$\langle f, g \rangle := \mathbb{E}[f \cdot g] = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x)g(x)$$

.

1.2 Characteristic Functions

Definition 1.2 (Characteristic function)**.** For each $S \subseteq [n] = \{1, 2, ..., n\}$, we define the characteristic function of S as

$$\chi_S(x) = (-1)^{S \cdot x}, \text{where } S \cdot x = \sum_{i=1}^{n} S_i \cdot x_i = \sum_{i \in S} x_i$$

.

Lemma 1.3. For every $S \subseteq [n]$,

$$\sum_{x \in \{0,1\}^n} \chi_S(x) = \begin{cases} 2^n & \text{if } S = \emptyset \\ 0 & \text{if } S \neq \emptyset \end{cases}$$

Proof. If $S = \emptyset$, then $S \cdot x = 0$. So $\sum_{x \in \{0,1\}^n} \chi_S(x) = \sum_{x \in \{0,1\}^n} 1 = 2^n$.
If $S \neq \emptyset$, then there exists k such that $S_k \neq 0$. Hence,

$$\begin{aligned}
\sum_{x\in\{0,1\}^n} \chi_S(x) &= \sum_{x\in\{0,1\}^n} (-1)^{\sum\limits_{i\in S} x_i} \\
&= \sum_{x\in\{0,1\}^n} [(-1)^{x_k}\cdot(-1)^{\sum\limits_{i\in S\setminus\{k\}} x_i}] \\
&= \sum_{x_k\in\{0,1\}} (-1)^{x_k}\cdot \sum_{x\setminus x_k\in\{0,1\}^{n-1}} (-1)^{\sum\limits_{i\in S\setminus\{k\}} x_i} \\
&= \left[(-1)^0+(-1)^1\right] \sum_{x\setminus x_k\in\{0,1\}^{n-1}} (-1)^{\sum\limits_{i\in S\setminus\{k\}} x_i} \\
&= 0
\end{aligned}$$

□

Theorem 1.4. *For every $S, T \subseteq [n]$,*

$$\langle \chi_S, \chi_T\rangle = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T \end{cases}$$

Proof.

$$\langle \chi_S, \chi_T\rangle = \frac{1}{2^n}\sum_{x\in\{0,1\}^n} (-1)^{S\cdot x+T\cdot x} = \frac{1}{2^n}\sum_{x\in\{0,1\}^n} (-1)^{(S\Delta T)\cdot x}$$

where Δ is the symmetric different between two sets S and T.
$S\Delta T = \emptyset$ if and only if $S = T$. Hence, our goal follows immediately from Lemma 1.3. □

1.3 Fourier Basis

Theorem 1.5. *The set of all χ_S defines an orthonormal basis for the space of all real-valued function on $\{0,1\}^n$*

Proof. From Theorem 1.4, the set of all χ_S is an orthonormal set. Also, there are 2^n different χ_S. Hence, the set of all χ_s must be an orthonormal basis for the space of all real-valued functions on $\{0,1\}^n$. □

The set of all χ_S is called the *the Fourier basis*.

1.4 Fourier Transform

Definition 1.6. For each $S \subseteq [n]$, we define the Fourier transform of f at S as following:

$$\widehat{f}(S) := \mathbb{E}[f\cdot\chi_S] = \langle f, \chi_S\rangle$$

Definition 1.7 (Fourier transform)**.** The mapping $\mathcal{F} : f \mapsto \widehat{f}$ is called the Fourier transform.

If we view functions $f, \widehat{f}$ as N-dimensional vectors, then we can write the Fourier transform as the product of f and some matrix F as following:

$$\begin{aligned}
\mathcal{F}(f) &= f \cdot F \\
&= \frac{1}{N}(f(0), f(1), \cdots, f(N-1)) \begin{pmatrix} \chi_0(0) & \chi_1(0) & \cdots & \chi_{N-1}(0) \\ \chi_0(1) & \chi_1(1) & \cdots & \chi_{N-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_0(N-1) & \chi_1(N-1) & \cdots & \chi_{N-1}(N-1) \end{pmatrix} \\
&= (\widehat{f}(0), \widehat{f}(1), \cdots, \widehat{f}(N-1)) \\
&= \widehat{f}
\end{aligned}$$

where $F_{ij} = \frac{\chi_i(j)}{N}$. Since $\chi_i(j) = \chi_j(i)$, F is symmetric.

Lemma 1.8. F is invertible

Proof. We have

$$(F \cdot F)_{ij} = \frac{1}{N^2} \sum_{k=0}^{N-1} \chi_i(k) \cdot \chi_j(k) = \frac{1}{N} \langle \chi_i, \chi_j \rangle$$

Based on Theorem 1.4, it is easy to see that

$$(F \cdot F)_{ij} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

So $F \cdot F = N \cdot I$, which implies that F is invertible □

Theorem 1.9. *The mapping $\mathcal{F}$ is linear.*

Proof. This follows from the properties of inner product. For any S,

$$\widehat{af + bg}(S) = \langle af + bg, \chi_S \rangle = \langle af, \chi_S \rangle + \langle bg, \chi_S \rangle = a\langle f, \chi_S \rangle + b\langle g, \chi_s \rangle = a\widehat{f}(S) + b\widehat{g}(S)$$

Thus, $\widehat{af + bg} = a\widehat{f} + b\widehat{g}$, which means $\mathcal{F}$ is linear.
Here is another way to show that.

$$\widehat{af + bg} = \mathcal{F}(af + bg) = (af + bg)F = afF + bgF = a(fF) + b(gF) = a\widehat{f} + b\widehat{g}$$

□

Theorem 1.10. *The linear map $\mathcal{F}$ is a bijection.*

Proof. It suffices to show that F is invertible, which follows immediately from Lemma 1.8. □

1.5 Parseval's Identity

Since the set of χ_S forms an orthonormal basis,

$$f = \sum_S \widehat{f}(S)\chi_S. \tag{1.1}$$

Hence,

$$\langle f, g\rangle = \sum_S \widehat{f}(S)\widehat{g}(S) \tag{1.2}$$

In particular, when f = g we get Parseval's identity:

$$\|f\|_2^2 = \sum_S \widehat{f}(S)^2 \tag{1.3}$$

This also implies:

$$\|f - g\|_2^2 = \sum_S (\widehat{f}(S) - \widehat{g}(S))^2 \tag{1.4}$$

1.6 Convolution

Definition 1.11. Given any two function f and $g : \{0,1\}^n \to \mathbb{R}$, the convolution of $f * g : \{0,1\}^n \to \mathbb{R}$ is defined as

$$(f * g)(x) := \frac{1}{2^n} \sum_{y \in \{0,1\}^n} f(x \oplus y)g(y)$$

Theorem 1.12. *If X and Y are n-bits random independent variables with probability distributions f and g, respectively, then $2^n(f * g)$ is the distribution of the random variable $Z = X \oplus Y$.*

Proof.

$$\begin{aligned}
Pr[Z = z] &= Pr[X = z \oplus Y] \\
&= \sum_{y \in \{0,1\}^n} Pr[X = z \oplus y | Y = y] \\
&= \sum_{y \in \{0,1\}^n} Pr[X = z \oplus y] \cdot Pr[Y = y] \\
&= \sum_{y \in \{0,1\}^n} f((z \oplus y) \cdot g(y) \\
&= 2^n (f * g)(z)
\end{aligned}$$

□

Theorem 1.13. *For every $S \subseteq [n]$,*

$$\widehat{f * g}(S) = \widehat{f}(S) \cdot \widehat{g}(S)$$

Proof.

$$\begin{aligned}
\widehat{f*g}(S) &= \frac{1}{2^n}\sum_x (f*g)(x)\chi_S(x) \\
&= \frac{1}{2^n}\sum_x \left(\frac{1}{2^n}\sum_y f(x\oplus y)g(y)\right)\chi_S(x) \\
&= \frac{1}{2^{2n}}\sum_x\sum_y f(x\oplus y)g(y)\chi_S(x\oplus y)\chi_S(y) \\
&= \frac{1}{2^n}\sum_x f(x\oplus y)\chi_S(x\oplus y)\left(\frac{1}{2^n}\sum_y g(y)\chi_S(y)\right) \\
&= \widehat{f}(S)\cdot\widehat{g}(S)
\end{aligned}$$

□

Intuitively, the convolution $f*g$ is the product of the Fourier transforms of f and g.

Theorem 1.14. *Let V be a subspace of dimension k of $\{0,1\}^n$ and let $V^\perp$ be the dual of V. Define*

$$f(x) = \begin{cases} \frac{1}{2^k} & \textit{if } x\in V \\ 0 & \textit{otherwise.}\end{cases}$$

.

Then

$$\widehat{f}(S) = \begin{cases} \frac{1}{N} & \textit{if } S\in V^\perp \\ 0 & \textit{otherwise.}\end{cases}$$

.

Proof. Suppose $v_1, v_2, ..., v_k$ is a basis of V.

Claim 1.15. $V = \langle v_1\rangle \oplus \langle v_2\rangle \oplus ... \oplus \langle v_k\rangle$

Claim 1.16. $\langle v_1\rangle^\perp \cap ... \cap \langle v_k\rangle^\perp = \langle v_1, ..., v_k\rangle^\perp = V^\perp$

Let $f_i = \begin{cases} \frac{1}{2} & \text{if } S\in\langle v_i\rangle \\ 0 & \text{otherwise}\end{cases}$, then $\widehat{f_i} = \begin{cases} \frac{1}{N} & \text{if } S\in\langle v_i\rangle^\perp \\ 0 & \text{otherwise.}\end{cases}$.

From above claims, we immediately obtain following result.

Claim 1.17. $f = f_1 \oplus f_2 \oplus ... \oplus f_k$

Hence,

$$\widehat{f}(S) = N^{k-1}\widehat{f}_1(S)...\widehat{f}_k(S)$$

If $S\in V^\perp$, then $S\in\langle v_i\rangle^\perp$ for every i, so $\widehat{f}(S) = N^{k-1}\cdot(\frac{1}{N})^k = \frac{1}{N}$.

If $S\notin V^\perp$, then there exits some i such that $S\notin\langle v_i\rangle^\perp$, which implies $\widehat{f}_i(S) = 0$. Hence, $\widehat{f}(S) = 0$ □

Chapter 2

Min Entropy

Let $X = (x_0, x_1, ..., x_{N-1})$ be a distribution function of a random variable over $\{0,1\}^n$, where $N := 2^n$.

Definition 2.1 (Min Entropy)**.** We define the min entropy of X as follow.

$$H_\infty(X) := \min_i(-\log x_i)$$

This implies that if $H_\infty(X) \geq k$ then $x_i \leq \frac{1}{2^k}$ for every $0 \leq i \leq N-1$

Theorem 2.2 (Collision Probability)**.** *If we sample X twice, then the probability we get the same result, denoted $Col(X)$, is $\sum\limits_{i=0}^{N-1} {x_i}^2 = N \cdot \|X\|_2^2$.*

Definition 2.3 (Flat Distribution)**.** A probability distribution function $f\colon \{0,1\}^n \to (0,1)$ is a *T-flat* if there $\exists\ S \subseteq \{0,1\}^n$ such that $|S| = T$ and $f(x) = \begin{cases} \frac{1}{T} & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

Lemma 2.4. For every $\alpha \geq \beta$, every α-flat distribution can be written as the sum of β-flat distributions.

Theorem 2.5. *For every integer $k \geq 0$, if $H_\infty(X) \geq k$, then $X = \sum \alpha_i X_i$, where each X_i is a 2^k-flat, $\alpha_i \in [0,1]$ for every i, and $\sum\limits_i \alpha_i = 1$.*

Proof. Let S be the set of all the probability distributions X with $H_\infty(X) \geq k$, then S is a compact (closed and bounded) convex set in $\mathbb{R}^N$.

Claim 2.6. The set of all 2^k-flat distributions is the set of all extreme points of S.

First, every 2^k-flat distribution X is an extreme point of S since if $X = \alpha Y + (1-\alpha)Z$ for some $Y, Z \in S$ and $\alpha \in [0,1]$, then $X = Y = Z$. Next, we want to show that for any X, which is not a 2^k-flat distribution, X is not an extreme point of S. Since X is not a 2^k-flat distribution, there exist $i < j$ such that $0 < x_i, x_j < \frac{1}{2^k}$. So we can choose $\delta > 0$ such that $x_i + \delta, x_j + \delta \leq \frac{1}{2^k}$ and $x_i - \delta, x_j - \delta \geq 0$. Now let $y_k = z_k = x_k$ for every $k \neq i, k \neq j$, $y_i = x_i + \delta$, $y_j = x_j - \delta$, $z_i = x_i - \delta$, and $z_j = x_j + \delta$. Then $Y, Z \in S$ and $X = \frac{1}{2}Y + \frac{1}{2}Z$, which implies that X is not an extreme point.

Back to the problem, since S is a compact convex set, the set of all convex combinations of its vertices is identical to S. Hence, every distribution X with $H_\infty(X) \geq k$ can be written as a convex combination of k-flat distributions.

□

Theorem 2.7. *If $H_\infty(X) \geq k$, then $Col(X) \leq \frac{1}{2^k}$.*

Proof. By theorem 2.5, we can write X as $X = \sum_i \alpha_i X_i$, where each X_i is a 2^k-flat, $\sum \alpha_i = 1$, and $\alpha_i \in [0,1]$for every i. It is obvious that $Col(X_i) = \|X_i\|_2^2 = \frac{1}{2^k}$.
Collision functions are convex, so by Jensen's inequality,

$$Col(X) = Col\left(\sum_i \alpha_i X_i\right) \leq \sum_i \alpha_i \cdot Col(X_i) = \sum_i \alpha_i \frac{1}{2^k} = \frac{1}{2^k} \sum_i \alpha_i = \frac{1}{2^k}$$

□

Theorem 2.8. *If $H_\infty(X) \geq k$, then $\sum_S \widehat{X}(S)^2 \leq \frac{1}{N \cdot 2^k}$.*

This follow immediately from the Parseval's identity.

Definition 2.9 (Small Bias Distribution)**.** Let $\mathcal{D}$ be a probability distribution function over $\{0,1\}^n$. We say that $\mathcal{D}$ is α-bias if $\widehat{D}(S) \leq \frac{\alpha}{N}$.

Definition 2.10. *Statistical Different* between two distributions A and B is defined as follow:

$$SD(A,B) = \frac{1}{2} \sum_i |a_i - b_i|$$

Theorem 2.11. *Let $\mathcal{D}$ be a small bias distribution with $\widehat{\mathcal{D}}(S) \leq \frac{\alpha}{N}$ for all S, let $\mathcal{M}$ be a min entropy source such that $H_\infty(\mathcal{M}) \geq k$, and let $\mathcal{U}$ be the uniform distribution over n-bits string. Then*

$$SD(\mathcal{D} \oplus \mathcal{M}, \mathcal{U}) \leq \frac{\alpha\sqrt{N}}{2^{1+k/2}}$$

Proof.

$$\begin{aligned}
SD(\mathcal{D} \oplus \mathcal{M}, \mathcal{U}) &= \frac{1}{2} \sum_i |(\mathcal{D} \oplus \mathcal{M})(i) - \mathcal{U}(i)| \\
&\leq \frac{1}{2} \sqrt{N \sum_i [(\mathcal{D} \oplus \mathcal{M})(i) - \mathcal{U}(i)]^2} \\
&= \frac{1}{2} \sqrt{N^2 \cdot \|(\mathcal{D} \oplus \mathcal{M}) - \mathcal{U}\|_2^2} \\
&= \frac{N}{2} \sqrt{\sum_S [\widehat{\mathcal{D} \oplus \mathcal{M}}(S) - \mathcal{U}(S)]^2} \\
&= \frac{N}{2} \sqrt{\sum_{S \neq \emptyset} \widehat{\mathcal{D} \oplus \mathcal{M}}(S)^2}
\end{aligned}$$

By convolution,

$$\begin{aligned}
\sum_{S\neq\emptyset}\widehat{\mathcal{D}\oplus\mathcal{M}}(S)^2 &= \sum_{S\neq\emptyset} N^2\cdot\widehat{\mathcal{D}*\mathcal{M}}(S)^2\\
&= N^2\sum_{S\neq\emptyset}\widehat{\mathcal{D}}(S)^2\cdot\widehat{\mathcal{M}}(S)^2\\
&\leq N^2\cdot\sum_{S\neq\emptyset}(\frac{\alpha}{N})^2\cdot\widehat{\mathcal{M}}(S)^2\\
&= \alpha^2\cdot\sum_{S\neq\emptyset}\widehat{\mathcal{M}}(S)^2\\
&\leq \frac{\alpha^2}{N\cdot 2^k}
\end{aligned}$$

Hence,

$$SD(\mathcal{D}\oplus\mathcal{M},\mathcal{U})\leq\frac{\alpha\sqrt{N}}{2^{1+k/2}}$$

$\square$

Theorem 2.12. *Let M be a distribution with min entropy k over $\{0,1\}^n$, let $G_0\sim 1\times\frac{n}{2}$, $G\sim\frac{n}{2}\times n$, and let X be a uniform distribution over $\{0,1\}^{\frac{n}{2}}$. Then*

$$SD\{(XG_0, XG\oplus M, G_0, G),(U, XG\oplus M, G_0, G)\}\leq\cdots$$

Proof. For convenience, let $A=(XG_0, XG\oplus M|G_0, G)$ and $B=(U, XG\oplus M|G_0, G)$.

Claim 2.13. For any distributions C, D,

$$\widehat{(C,D)}(S)=\cdots,\text{ where } S=S_CS_D$$

Proof.

$$\begin{aligned}
\widehat{(C,D)}(S) &= \langle(C,D),\chi_S\rangle\\
&= \frac{1}{2N}\sum_{(c,d)}(C,D)(c,d)\cdot\chi_S(c,d)\\
&= \frac{1}{2N}\sum_{(c,d)}C(c)\cdot(D|C=c)(d)\cdot\chi_{S_C}(c)\cdot\chi_{S_D}(d)\\
&= \frac{1}{2N}\sum_{c}\left[C(c)\cdot\chi_{S_C}(c)\cdot\sum_{d}(D|C=c)(d)\chi_{S_D}(d)\right]\\
&= \frac{1}{2}\sum_{c}\left[C(c)\cdot\chi_{S_C}(c)\cdot(\widehat{D|C=c})(S_D)\right]
\end{aligned}$$

$\square$

Claim 2.14. For any $S \subseteq [n+1]$, $\widehat{A}(S) = \widehat{B}(S)$ if $S_1 = 0$, and $\widehat{B}(S) = 0$ if $S_1 = 1$.

Back to the problem,

$$
\begin{aligned}
& SD\{(XG_0, XG \oplus M, G_0, G), (U, XG \oplus M, G_0, G)\} \\
=& \mathop{\mathbb{E}}_{G_0,G}[SD(A,B)] \\
=& \mathop{\mathbb{E}}_{G_0,G}\left[\frac{1}{2}\sum_{i,j}|A(i) - B(i)|\right] \\
\leq& \frac{1}{2}\mathop{\mathbb{E}}_{G_0,G}\left[\sqrt{2N \cdot \sum_i ((A(i) - B(i))^2}\right] \\
\leq& \frac{\sqrt{2N}}{2}\sqrt{\mathop{\mathbb{E}}_{G_0,G}\left[\sum_i (A(i) - B(i))^2\right]} \\
=& \frac{\sqrt{2N}}{2}\sqrt{\sum_S \mathop{\mathbb{E}}_{G_0,G}\left[(\widehat{A}(S) - \widehat{B}(S))^2\right]} \\
=& \frac{\sqrt{2N}}{2}\sqrt{\sum_{S_1=1} \mathop{\mathbb{E}}_{G_0,G}\left[\widehat{A}(S)^2\right]}
\end{aligned}
$$

□

Chapter 3

Bourgain's Extractor

Appendix A

A.1 Dual of a Vector Space

Definition A.1 (Dual space)**.** Let V be a subspace of $\{0,1\}^n$. We define the dual of V as $V^\perp = \{x \in \{0,1\}^n | x \cdot v = 0\ \forall v \in V\}$.

Theorem A.2. *$V^\perp$ is a subspace of $\{0,1\}^n$.*

Proof. For any $x, y \in V^\perp, a \in \{0,1\}, (a \cdot x + y) \cdot v = a \cdot (x \cdot v) + y \cdot v = 0 + 0 = 0$. □

Lemma A.3. $\sum\limits_{i:\text{even}}^{t} \binom{n}{i} = \sum\limits_{i:\text{odd}}^{t} \binom{n}{i} = 2^{t-1}$.

Theorem A.4. *For any subspace V of dimension k of $\{0,1\}^n$, there exists a unique dual space $V^\perp$ of dimension $(n-k)$.*

Proof. We will show that $|V^\perp| = 2^{n-k}$ by induction on k.
If $k = 0$, then $V = \{\mathbf{0}\}$. Clearly, $V^\perp = \{0,1\}^n$.
If $k = 1$, let $V = \{\vec{0}, v\}$. Suppose the number of $v_i = 1$ is t, then the number of x such that $x \cdot v = 0$ is $\sum\limits_{i:2|t-i} \binom{n}{i} 2^{n-t} = 2^{t-1} \cdot 2^{n-t} = 2^{n-1}$ by Lemma A.3.
Suppose that there exists a unique orthogonal subspace $V^\perp$ of dimension $(n-k+1)$ for any subspace V of dimension $k-1$ of $\{0,1\}^n$, where $k \geq 2$.
Let $V = \langle v_1, v_2, ..., v_k \rangle$, $S_1 = \langle v_1, v_2, ..., v_{k-1} \rangle$, and $S_2 = \langle v_k \rangle$. Then, $V^\perp = S_1^\perp \cap S_2^\perp$.
Suppose $dim(V^\perp) = t$. We want to show $t = n-k$.
By induction hypothesis, $dim(S_1^\perp) = n-k+1$ and $dim(S_2^\perp) = n-1$.
If $t \leq n-k-1$, then we need $[(n-k+1)-t]$ independent vectors to cover $S_1^\perp$ from extending $V^\perp$, and we need $[(n-1)-t]$ independent vectors to cover $S_2^\perp$ from extending $V^\perp$. Since $S_1^\perp \cup S_2^\perp \subseteq \{0,1\}^n$, we must have $[(n-k+1)-t] + [(n-1)-t] + t \leq n$, which is equivalent to $t \geq n-k$, contradiction.
If $t \geq n-k+1$, then $S_1^\perp \subseteq S_2^\perp$, this is impossible since v_k is independent from $v_1, v_2, ..., v_{k-1}$.
Thus, $t = n-k$. So $|V^\perp| = 2^{n-k}$. □

A.2 Statistical Distance between Two Joint Distributions

Theorem A.5. *Let A, B be some probability distributions over the same sample space, and let C be a probability distribution. Then*

$$SD\{(A,C),(B,C)\} = \mathop{\mathbb{E}}_{c \sim C}[SD\{(A|C=c),(B|C=c)\}]$$

Proof.

$$\begin{aligned}
SD\{(A,C),(B,C)\} &= \frac{1}{2}\sum_{i,c} |(A,C)(i,c) - (B,C)(i,c)| \\
&= \frac{1}{2}\sum_{i,c} |Pr(C=c)\cdot Pr(A=i|C=c) - Pr(C=c)\cdot Pr(B=i|C=c)| \\
&= \sum_{c\sim C}\left(Pr(C=c)\cdot \frac{1}{2}\sum_{i} |Pr(A=i|C=c) - Pr(B=i|C=c)|\right) \\
&= \sum_{c\sim C} Pr(C=c)\cdot SD\{(A|C=c),(B|C=c)\} \\
&= \mathop{\mathbb{E}}_{c\sim C}[SD\{(A|C=c),(B|C=c)\}]
\end{aligned}$$

□

A.3 Group Basics

A.3.1 Notation

We reverse the variable p to denote primes.

$\mathbb{F}_p$ denotes the field of size p.

G denotes a finite abelian group.

$\mathbb{C}$ denotes the set of complex numbers.

Definition A.6. We say $\psi : G \to \mathbb{C}$ is a character if ψ is a homomorphism.

Definition A.7. We say a map $e : G \times G \to \mathbb{C}$ is a bilinear map if it is a homomorphism in each variable.

Theorem A.8. *For every abelian group G, there exists a symmetric non-degenerate bilinear $e : G \times G \to \mathcal{C}$*

A.3.2 Dual of a finite Abelian Group

Theorem A.9. *Every nite abelian group G is isomorphic to its character group $G^\wedge$*

A.4 Product Graph

Definition A.10. $D = (a, b, c, d)$ is a 2×2 distribution graph if and only if

1. $a + b + c + d = 1$,
2. $a, b, c, d \in [0, 1]$.

Definition A.11. $G = (x, y, z, t)$ is a 2×2 product graph if and only if

1. G is a distribution graph,
2. $xt = yz$, or $x = t = 0$, or $y = z = 0$.

Let $\mathbb{G}$ be the space of all 2×2 product graphs and let $\mathbb{D}$ be the space of all 2×2 distribution graphs. We want to find

$$D^* = \underset{D \in \mathbb{D}}{argmax}\ dist(G, \mathbb{G})$$

$$m = \max_{D \in \mathbb{D}} dist(G, \mathbb{G})$$

Let $D = (a, b, c, d)$ be any 2×2 distribution graph. Without lost of generality, assume $a \geq d$, $b \geq c$, and $ad \geq bc$

Claim A.12. *$dist(D, \mathbb{G}) \leq f(a, b, c, d)$,*
where $f(a, b, c, d) = \min\{(b + c), \frac{(ad-bc)}{a+b}, \frac{1}{2}(|\sqrt{a} - a - b| + |\sqrt{a} - a - c|, |(1 - \sqrt{a})^2 - d|)\}$

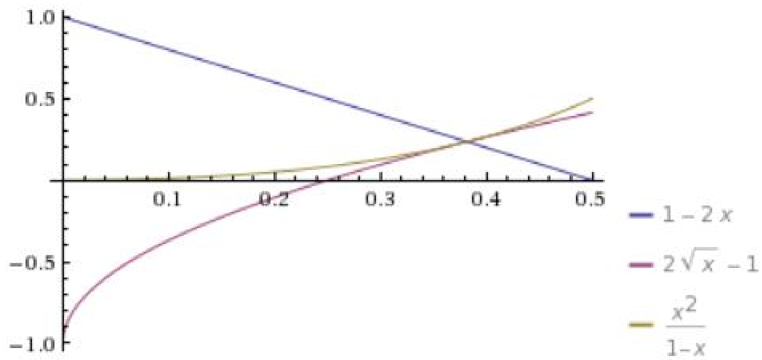
Proof. Let $G = (x, y, z, t)$ be a product graph.

$$dist(D, \mathbb{G}) \leq dist(D, G) = \frac{1}{2}(|a - x| + |b - y| + |c - z| + |d - t|)$$

So if we can find some graphs $G's$ such that $dist(D, G)$ equal to the three values above respectively, we are done. From the second property of distribution graph, it suggests the way to choose such $G's$.

1. Choose $y = z = 0$, $x = a + b$, and $t = c + d$, then $dist(D, G) = b + c$.
2. Choose $xt = yz$, $x = a$, $y = b$, $z = \frac{(c+d)a}{a+b}$, and $\frac{(c+d)b}{a+b}$, then
$$dist(D, G) = \frac{1}{2}(|c - \frac{(c + d)a}{a + b}| + |d - \frac{(c + d)b}{a + b}| = \frac{(ad - bc)}{a + b}$$
3. Choose $x = a$, $y = z = \sqrt{a} - a$, and $t = (1 - \sqrt{a})^2$, then
$$dist(D, G) = \frac{1}{2}(|\sqrt{a} - a - b| + |\sqrt{a} - a - c|, |(1 - \sqrt{a})^2 - d|)$$

$\square$



Claim A.13. $m \leq \max_{a,b,c,d} f(a,b,c,d)$

Claim A.14. $\max_{a,b,c,d} f(a,b,c,d) = \sqrt{5}-2$

Proof. 1. If $b(1-d) \geq ad$, then $\frac{(ad-bc)}{a+b} = \frac{(ad-bc)}{1-c-d} \leq \frac{ad}{1-d} \leq \frac{(\frac{a+d}{2})^2}{1-(\frac{a+d}{2})}$. Then

$$\max_{a,b,c,d} f(a,b,c,d) \leq \max\min\{(1-2h), \frac{h^2}{1-h}\}$$

where $h = (a+d)/2$

2. If $b(1-d) = ba + b^2 + bc \leq ad$, suppose $\max\min\{(1-2h), \frac{h^2}{1-h}\} > \sqrt{5}-2$. Then $\sqrt{a} \geq a+b \geq a+c$ since $a \geq (a+b)^2 \Leftrightarrow a(a+b+c+d) \geq a^2+2ab+b^2 \Leftrightarrow ac+ad \geq ab+b^2$, which is true. So $d \geq (1-\sqrt{a})^2$. Thus,

$$\begin{aligned}
&\frac{1}{2}(|\sqrt{a}-a-b| + |\sqrt{a}-a-c|, |(1-\sqrt{a})^2-d|) \\
&= \frac{1}{2}(2\sqrt{a}-2a-(1-a-d)+d-(1-\sqrt{a})^2 \\
&= d-(1-\sqrt{a})^2 \\
&= (\sqrt{d}+\sqrt{a}-1)(1+\sqrt{d}-\sqrt{a}) \\
&\leq \sqrt{d}+\sqrt{a}-1 \\
&\leq \sqrt{2(a+d)}-1
\end{aligned}$$

Hence,

$$\max_{a,b,c,d} f(a,b,c,d) \leq \max\min\{1-2h, 2\sqrt{h}-1\}\}$$

From the graph, we can see that

$$\max\min\{1-2h, 2\sqrt{h}-1, \frac{h^2}{1-h}\} = \sqrt{5}-2$$

when $a = d = \frac{3-\sqrt{5}}{2}$, $b = \sqrt{5}-2$, and $c = 0$ $\square$