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Chapter 1

Fourier Basics

1.1 Vector Space of Functions on Boolean Hyper-cube

Definition 1.1 (Inner Product). Consider the 2^n -dimensional vector space of all functions $f : \{0, 1\}^n \rightarrow \mathbb{R}$. We define an inner product on this space by

$$\langle f, g \rangle := \mathbb{E}[f \cdot g] = \frac{1}{2^n} \sum_{x \in \{0, 1\}^n} f(x)g(x)$$

.

1.2 Characteristic Functions

Definition 1.2 (Characteristic function). For each $S \subseteq [n] = \{1, 2, \dots, n\}$, we define the characteristic function of S as

$$\chi_S(x) = (-1)^{S \cdot x}, \text{ where } S \cdot x = \sum_{i=1}^n S_i \cdot x_i = \sum_{i \in S} x_i$$

.

Lemma 1.3. For every $S \subseteq [n]$,

$$\sum_{x \in \{0, 1\}^n} \chi_S(x) = \begin{cases} 2^n & \text{if } S = \emptyset \\ 0 & \text{if } S \neq \emptyset \end{cases}$$

Proof. If $S = \emptyset$, then $S \cdot x = 0$. So $\sum_{x \in \{0, 1\}^n} \chi_S(x) = \sum_{x \in \{0, 1\}^n} 1 = 2^n$.

If $S \neq \emptyset$, then there exists k such that $S_k \neq 0$. Hence,

$$\begin{aligned}
\sum_{x \in \{0,1\}^n} \chi_S(x) &= \sum_{x \in \{0,1\}^n} (-1)^{\sum_{i \in S} x_i} \\
&= \sum_{x \in \{0,1\}^n} [(-1)^{x_k} \cdot (-1)^{\sum_{i \in S \setminus \{k\}} x_i}] \\
&= \sum_{x_k \in \{0,1\}} (-1)^{x_k} \cdot \sum_{x \setminus x_k \in \{0,1\}^{n-1}} (-1)^{\sum_{i \in S \setminus \{k\}} x_i} \\
&= [(-1)^0 + (-1)^1] \sum_{x \setminus x_k \in \{0,1\}^{n-1}} (-1)^{\sum_{i \in S \setminus \{k\}} x_i} \\
&= 0
\end{aligned}$$

□

Theorem 1.4. For every $S, T \subseteq [n]$,

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 1 & \text{if } S = T \\ 0 & \text{if } S \neq T \end{cases}$$

Proof.

$$\langle \chi_S, \chi_T \rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{S \cdot x + T \cdot x} = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{(S \Delta T) \cdot x}$$

where Δ is the symmetric different between two sets S and T .

$S \Delta T = \emptyset$ if and only if $S = T$. Hence, our goal follows immediately from Lemma 1.3. □

1.3 Fourier Basis

Theorem 1.5. The set of all χ_S defines an orthonormal basis for the space of all real-valued function on $\{0,1\}^n$

Proof. From Theorem 1.4, the set of all χ_S is an orthonormal set. Also, there are 2^n different χ_S . Hence, the set of all χ_S must be an orthonormal basis for the space of all real-valued functions on $\{0,1\}^n$. □

The set of all χ_S is called the *Fourier basis*.

1.4 Fourier Transform

Definition 1.6. For each $S \subseteq [n]$, we define the Fourier transform of f at S as following:

$$\hat{f}(S) := \mathbb{E}[f \cdot \chi_S] = \langle f, \chi_S \rangle$$

Definition 1.7 (Fourier transform). The mapping $\mathcal{F} : f \mapsto \widehat{f}$ is called the Fourier transform.

If we view functions f, \widehat{f} as N -dimensional vectors, then we can write the Fourier transform as the product of f and some matrix F as following:

$$\begin{aligned} \mathcal{F}(f) &= f \cdot F \\ &= \frac{1}{N} (f(0), f(1), \dots, f(N-1)) \begin{pmatrix} \chi_0(0) & \chi_1(0) & \cdots & \chi_{N-1}(0) \\ \chi_0(1) & \chi_1(1) & \cdots & \chi_{N-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_0(N-1) & \chi_1(N-1) & \cdots & \chi_{N-1}(N-1) \end{pmatrix} \\ &= (\widehat{f}(0), \widehat{f}(1), \dots, \widehat{f}(N-1)) \\ &= \widehat{f} \end{aligned}$$

where $F_{ij} = \frac{\chi_i(j)}{N}$. Since $\chi_i(j) = \chi_j(i)$, F is symmetric.

Lemma 1.8. F is invertible

Proof. We have

$$(F \cdot F)_{ij} = \frac{1}{N^2} \sum_{k=0}^{N-1} \chi_i(k) \cdot \chi_j(k) = \frac{1}{N} \langle \chi_i, \chi_j \rangle$$

Based on Theorem 1.4, it is easy to see that

$$(F \cdot F)_{ij} = \begin{cases} \frac{1}{N} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

So $F \cdot F = N \cdot I$, which implies that F is invertible □

Theorem 1.9. The mapping \mathcal{F} is linear.

Proof. This follows from the properties of inner product. For any S ,

$$\widehat{af + bg}(S) = \langle af + bg, \chi_S \rangle = \langle af, \chi_S \rangle + \langle bg, \chi_S \rangle = a \langle f, \chi_S \rangle + b \langle g, \chi_S \rangle = a \widehat{f}(S) + b \widehat{g}(S)$$

Thus, $\widehat{af + bg} = a \widehat{f} + b \widehat{g}$, which means \mathcal{F} is linear.

Here is another way to show that.

$$\widehat{af + bg} = \mathcal{F}(af + bg) = (af + bg)F = afF + bgF = a(fF) + b(gF) = a \widehat{f} + b \widehat{g}$$

□

Theorem 1.10. The linear map \mathcal{F} is a bijection.

Proof. It suffices to show that F is invertible, which follows immediately from Lemma 1.8. □

1.5 Parseval's Identity

Since the set of χ_S forms an orthonormal basis,

$$f = \sum_S \widehat{f}(S) \chi_S. \quad (1.1)$$

Hence,

$$\langle f, g \rangle = \sum_S \widehat{f}(S) \widehat{g}(S) \quad (1.2)$$

In particular, when $f = g$ we get Parseval's identity:

$$\|f\|_2^2 = \sum_S \widehat{f}(S)^2 \quad (1.3)$$

This also implies:

$$\|f - g\|_2^2 = \sum_S (\widehat{f}(S) - \widehat{g}(S))^2 \quad (1.4)$$

1.6 Convolution

Definition 1.11. Given any two function f and $g : \{0, 1\}^n \rightarrow \mathbb{R}$, the convolution of $f * g : \{0, 1\}^n \rightarrow \mathbb{R}$ is defined as

$$(f * g)(x) := \frac{1}{2^n} \sum_{y \in \{0, 1\}^n} f(x \oplus y) g(y)$$

Theorem 1.12. If X and Y are n -bits random independent variables with probability distributions f and g , respectively, then $2^n(f * g)$ is the distribution of the random variable $Z = X \oplus Y$.

Proof.

$$\begin{aligned} \Pr[Z = z] &= \Pr[X = z \oplus Y] \\ &= \sum_{y \in \{0, 1\}^n} \Pr[X = z \oplus y | Y = y] \\ &= \sum_{y \in \{0, 1\}^n} \Pr[X = z \oplus y] \cdot \Pr[Y = y] \\ &= \sum_{y \in \{0, 1\}^n} f((z \oplus y)) \cdot g(y) \\ &= 2^n(f * g)(z) \end{aligned}$$

□

Theorem 1.13. For every $S \subseteq [n]$,

$$\widehat{f * g}(S) = \widehat{f}(S) \cdot \widehat{g}(S)$$

Proof.

$$\begin{aligned}
\widehat{f * g}(S) &= \frac{1}{2^n} \sum_x (f * g)(x) \chi_S(x) \\
&= \frac{1}{2^n} \sum_x \left(\frac{1}{2^n} \sum_y f(x \oplus y) g(y) \right) \chi_S(x) \\
&= \frac{1}{2^{2n}} \sum_x \sum_y f(x \oplus y) g(y) \chi_S(x \oplus y) \chi_S(y) \\
&= \frac{1}{2^n} \sum_x f(x \oplus y) \chi_S(x \oplus y) \left(\frac{1}{2^n} \sum_y g(y) \chi_S(y) \right) \\
&= \widehat{f}(S) \cdot \widehat{g}(S)
\end{aligned}$$

□

Intuitively, the convolution $f * g$ is the product of the Fourier transforms of f and g .

Theorem 1.14. *Let V be a subspace of dimension k of $\{0, 1\}^n$ and let V^\perp be the dual of V . Define*

$$f(x) = \begin{cases} \frac{1}{2^k} & \text{if } x \in V \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\widehat{f}(S) = \begin{cases} \frac{1}{N} & \text{if } S \in V^\perp \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose v_1, v_2, \dots, v_k is a basis of V .

Claim 1.15. $V = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \dots \oplus \langle v_k \rangle$

Claim 1.16. $\langle v_1 \rangle^\perp \cap \dots \cap \langle v_k \rangle^\perp = \langle v_1, \dots, v_k \rangle^\perp = V^\perp$

Let $f_i = \begin{cases} \frac{1}{2} & \text{if } S \in \langle v_i \rangle \\ 0 & \text{otherwise} \end{cases}$, then $\widehat{f}_i = \begin{cases} \frac{1}{N} & \text{if } S \in \langle v_i \rangle^\perp \\ 0 & \text{otherwise.} \end{cases}$

From above claims, we immediately obtain following result.

Claim 1.17. $f = f_1 \oplus f_2 \oplus \dots \oplus f_k$

Hence,

$$\widehat{f}(S) = N^{k-1} \widehat{f}_1(S) \dots \widehat{f}_k(S)$$

If $S \in V^\perp$, then $S \in \langle v_i \rangle^\perp$ for every i , so $\widehat{f}(S) = N^{k-1} \cdot \left(\frac{1}{N}\right)^k = \frac{1}{N}$.

If $S \notin V^\perp$, then there exists some i such that $S \notin \langle v_i \rangle^\perp$, which implies $\widehat{f}_i(S) = 0$. Hence, $\widehat{f}(S) = 0$

□

Chapter 2

Min Entropy

Let $X = (x_0, x_1, \dots, x_{N-1})$ be a distribution function of a random variable over $\{0, 1\}^n$, where $N := 2^n$.

Definition 2.1 (Min Entropy). We define the min entropy of X as follow.

$$H_\infty(X) := \min_i (-\log x_i)$$

This implies that if $H_\infty(X) \geq k$ then $x_i \leq \frac{1}{2^k}$ for every $0 \leq i \leq N - 1$

Theorem 2.2 (Collision Probability). *If we sample X twice, then the probability we get the same result, denoted $Col(X)$, is $\sum_{i=0}^{N-1} x_i^2 = N \cdot \|X\|_2^2$.*

Definition 2.3 (Flat Distribution). A probability distribution function $f: \{0, 1\}^n \rightarrow (0, 1)$ is a T -flat if there $\exists S \subseteq \{0, 1\}^n$ such that $|S| = T$ and $f(x) = \begin{cases} \frac{1}{T} & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$

Lemma 2.4. For every $\alpha \geq \beta$, every α -flat distribution can be written as the sum of β -flat distributions.

Theorem 2.5. *For every integer $k \geq 0$, if $H_\infty(X) \geq k$, then $X = \sum \alpha_i X_i$, where each X_i is a 2^k -flat, $\alpha_i \in [0, 1]$ for every i , and $\sum_i \alpha_i = 1$.*

Proof. Let S be the set of all the probability distributions X with $H_\infty(X) \geq k$, then S is a compact (closed and bounded) convex set in \mathbb{R}^N .

Claim 2.6. The set of all 2^k -flat distributions is the set of all extreme points of S .

First, every 2^k -flat distribution X is an extreme point of S since if $X = \alpha Y + (1 - \alpha)Z$ for some $Y, Z \in S$ and $\alpha \in [0, 1]$, then $X = Y = Z$. Next, we want to show that for any X , which is not a 2^k -flat distribution, X is not an extreme point of S . Since X is not a 2^k -flat distribution, there exist $i < j$ such that $0 < x_i, x_j < \frac{1}{2^k}$. So we can choose $\delta > 0$ such that $x_i + \delta, x_j + \delta \leq \frac{1}{2^k}$ and $x_i - \delta, x_j - \delta \geq 0$. Now let $y_k = z_k = x_k$ for every $k \neq i, k \neq j$, $y_i = x_i + \delta, y_j = x_j - \delta, z_i = x_i - \delta$, and $z_j = x_j + \delta$. Then $Y, Z \in S$ and $X = \frac{1}{2}Y + \frac{1}{2}Z$, which implies that X is not an extreme point.

Back to the problem, since S is a compact convex set, the set of all convex combinations of its vertices is identical to S . Hence, every distribution X with $H_\infty(X) \geq k$ can be written as a convex combination of k -flat distributions. \square

Theorem 2.7. *If $H_\infty(X) \geq k$, then $Col(X) \leq \frac{1}{2^k}$.*

Proof. By theorem 2.5, we can write X as $X = \sum_i \alpha_i X_i$, where each X_i is a 2^k -flat, $\sum \alpha_i = 1$, and $\alpha_i \in [0, 1]$ for every i . It is obvious that $Col(X_i) = \|X_i\|_2^2 = \frac{1}{2^k}$. Collision functions are convex, so by Jensen's inequality,

$$Col(X) = Col\left(\sum_i \alpha_i X_i\right) \leq \sum_i \alpha_i \cdot Col(X_i) = \sum_i \alpha_i \frac{1}{2^k} = \frac{1}{2^k} \sum_i \alpha_i = \frac{1}{2^k}$$

\square

Theorem 2.8. *If $H_\infty(X) \geq k$, then $\sum_S \widehat{X}(S)^2 \leq \frac{1}{N \cdot 2^k}$.*

This follows immediately from the Parseval's identity.

Definition 2.9 (Small Bias Distribution). Let \mathcal{D} be a probability distribution function over $\{0, 1\}^n$. We say that \mathcal{D} is α -bias if $\widehat{\mathcal{D}}(S) \leq \frac{\alpha}{N}$.

Definition 2.10. *Statistical Different* between two distributions A and B is defined as follow:

$$SD(A, B) = \frac{1}{2} \sum_i |a_i - b_i|$$

Theorem 2.11. *Let \mathcal{D} be a small bias distribution with $\widehat{\mathcal{D}}(S) \leq \frac{\alpha}{N}$ for all S , let \mathcal{M} be a min entropy source such that $H_\infty(\mathcal{M}) \geq k$, and let \mathcal{U} be the uniform distribution over n -bits string. Then*

$$SD(\mathcal{D} \oplus \mathcal{M}, \mathcal{U}) \leq \frac{\alpha \sqrt{N}}{2^{1+k/2}}$$

Proof.

$$\begin{aligned} SD(\mathcal{D} \oplus \mathcal{M}, \mathcal{U}) &= \frac{1}{2} \sum_i |(\mathcal{D} \oplus \mathcal{M})(i) - \mathcal{U}(i)| \\ &\leq \frac{1}{2} \sqrt{N \sum_i [(\mathcal{D} \oplus \mathcal{M})(i) - \mathcal{U}(i)]^2} \\ &= \frac{1}{2} \sqrt{N^2 \cdot \|(\mathcal{D} \oplus \mathcal{M}) - \mathcal{U}\|_2^2} \\ &= \frac{N}{2} \sqrt{\sum_S [\widehat{\mathcal{D} \oplus \mathcal{M}}(S) - \mathcal{U}(S)]^2} \\ &= \frac{N}{2} \sqrt{\sum_{S \neq \emptyset} \widehat{\mathcal{D} \oplus \mathcal{M}}(S)^2} \end{aligned}$$

By convolution,

$$\begin{aligned}
\sum_{S \neq \emptyset} \widehat{\mathcal{D} \oplus \mathcal{M}}(S)^2 &= \sum_{S \neq \emptyset} N^2 \cdot \widehat{\mathcal{D} * \mathcal{M}}(S)^2 \\
&= N^2 \sum_{S \neq \emptyset} \widehat{\mathcal{D}}(S)^2 \cdot \widehat{\mathcal{M}}(S)^2 \\
&\leq N^2 \cdot \sum_{S \neq \emptyset} \left(\frac{\alpha}{N}\right)^2 \cdot \widehat{\mathcal{M}}(S)^2 \\
&= \alpha^2 \cdot \sum_{S \neq \emptyset} \widehat{\mathcal{M}}(S)^2 \\
&\leq \frac{\alpha^2}{N \cdot 2^k}
\end{aligned}$$

Hence,

$$SD(\mathcal{D} \oplus \mathcal{M}, \mathcal{U}) \leq \frac{\alpha \sqrt{N}}{2^{1+k/2}}$$

□

Theorem 2.12. *Let M be a distribution with min entropy k over $\{0, 1\}^n$, let $G_0 \sim 1 \times \frac{n}{2}$, $G \sim \frac{n}{2} \times n$, and let X be a uniform distribution over $\{0, 1\}^{\frac{n}{2}}$. Then*

$$SD\{(XG_0, XG \oplus M, G_0, G), (U, XG \oplus M, G_0, G)\} \leq \frac{\sqrt{N}}{2^{k/2}}$$

Proof. For convenience, let $A_{G_0, G} = (XG_0, XG \oplus M | G_0, G)$, $A'_{G_0, G} = (XG_0, XG | G_0, G)$ and $B_{G_0, G} = (U, XG \oplus M | G_0, G)$.

Claim 2.13. For any $S \subseteq [n+1]$, $\widehat{A_{G_0, G}}(S) = \widehat{B_{G_0, G}}(S)$ if $S_1 = 0$, and $\widehat{B_{G_0, G}}(S) = 0$ if $S_1 = 1$.

Claim 2.14. For any set $S \subseteq [n+1]$ with $S_1 = 1$,

$$\mathbb{E}_{G_0, G}[\widehat{A'_{G_0, G}}(S)^2] \leq \frac{1}{(2N)^2 \cdot N}$$

Proof.

$$\widehat{A'_{G_0, G}}(S) = \mathbb{E}[A'_{G_0, G}(x) \cdot \chi_S(x)] = \frac{1}{2N} \text{bias}_S(A'_{G_0, G})$$

$$\begin{aligned}
\text{where } \text{bias}_S(A'_{G_0, G}) &= \frac{1}{2N} \left| A'_{G_0, G}\{x : \bigoplus_{i \in S} x G_{i-1} = 0\} - A'_{G_0, G}\{x : \bigoplus_{i \in S} x G_{i-1} = 1\} \right| \\
&= \frac{1}{2N} \left| A'_{G_0, G}\{x : x \cdot (\bigoplus_{i \in S} G_{i-1}) = 0\} - A'_{G_0, G}\{x : x \cdot (\bigoplus_{i \in S} G_{i-1}) = 1\} \right|
\end{aligned}$$

If $G_0 = \bigoplus_{i \in S_{>1}} G_{i-1}$, then $\text{bias}_S(A'_{G_0,G}) = 1$, and if $G_0 \neq \bigoplus_{i \in S_{>1}} G_{i-1}$ then $\text{bias}_S(A'_{G_0,G}) = 0$.

$$\Pr[G_0 = \bigoplus_{i \in S_{>1}} G_{i-1}] \leq \Pr[G_0 \in \langle G_{i-1} : i \in S \rangle] \leq \frac{1}{2^{n/2}} = \frac{1}{\sqrt{N}}$$

$$\Pr[G_0 \neq \bigoplus_{i \in S_{>1}} G_{i-1}] \geq \Pr[G_0 \in \langle G_{i-1} : i \in S \rangle] \geq 1 - \frac{1}{\sqrt{N}}$$

$$\mathbb{E}_{G_0,G}[\widehat{A'_{G_0,G}}(S)^2] \leq \frac{1}{N} \cdot \left(\frac{1}{2N}\right)^2 = \frac{1}{(2N)^2 \cdot N}$$

Back to the problem,

$$\begin{aligned} & SD\{(XG_0, XG \oplus M, G_0, G), (U, XG \oplus M, G_0, G)\} \\ &= \mathbb{E}_{G_0,G} [SD(A_{G_0,G}, B_{G_0,G})] \\ &= \mathbb{E}_{G_0,G} \left[\frac{1}{2} \sum_{i,j} |A_{G_0,G}(i) - B_{G_0,G}(i)| \right] \\ &\leq \frac{1}{2} \mathbb{E}_{G_0,G} \left[\sqrt{2N \cdot (2N)^2 \cdot \|A_{G_0,G} - B_{G_0,G}\|_2^2} \right] \\ &\leq \frac{(2N)^{3/2}}{2} \sqrt{\mathbb{E}_{G_0,G} \left[\sum_i (A_{G_0,G}(i) - B_{G_0,G}(i))^2 \right]} \\ &= \frac{(2N)^{3/2}}{2} \sqrt{\sum_S \mathbb{E}_{G_0,G} \left[(\widehat{A_{G_0,G}}(S) - \widehat{B_{G_0,G}}(S))^2 \right]} \\ &= \frac{(2N)^{3/2}}{2} \sqrt{\sum_{S_1=1} \mathbb{E}_{G_0,G} \left[\widehat{A_{G_0,G}}(S)^2 \right]} \\ &= \frac{(2N)^{3/2}}{2} \sqrt{\sum_{S_1=1} \mathbb{E}_{G_0,G} \left[((\widehat{O, M})(S) \oplus (XG_0, \widehat{XG|G_0, G})(S))^2 \right]} \\ &= \frac{(2N)^{3/2}}{2} \sqrt{\sum_{S_1=1} \mathbb{E}_{G_0,G} \left[(2N)^2 \cdot (\widehat{O, M})(S)^2 \cdot (XG_0, \widehat{XG|G_0, G})(S)^2 \right]} \\ &= \frac{(2N)^{5/2}}{2} \sqrt{\sum_{S_1=1} (\widehat{O, M})(S)^2 \cdot \mathbb{E}_{G_0,G} \left[\widehat{A'_{G_0,G}}(S)^2 \right]} \\ &= \frac{(2N)^{5/2}}{2} \sqrt{\sum_{S_1=1} (\widehat{O, M})(S)^2 \cdot \frac{1}{(2N)^2 \cdot N}} \\ &\leq \frac{(2N)^{5/2}}{2} \sqrt{\frac{1}{2N \cdot 2^k} \cdot \frac{1}{(2N)^2 \cdot N}} \\ &= \frac{\sqrt{N}}{2^{k/2}} \end{aligned}$$

□

Chapter 3

Generalized Fourier Transform

The entire treatment for Fourier analysis in chapter 1 can be generalized to functions from any finite abelian group to the set of complex numbers.

3.1 Notation

\mathbb{C} : the set of complex numbers.

\mathbb{G} : a finite abelian group of order n .

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$: multiplicative group of complex numbers.

\mathbb{F}_q : the field of size q

For a complex number x , we use \bar{x} to denote the complex conjugate of x .

3.2 Inner Product

Let $f : \mathbb{G} \rightarrow \mathbb{C}$, and $g : \mathbb{G} \rightarrow \mathbb{C}$ be two functions from a finite abelian group to the complex numbers.

Definition 3.1. The inner product of f and g is defined as

$$\langle f, g \rangle = \mathbb{E}_x[f(x)\overline{g(x)}]$$

The L^p norm of f is defined as $\|f\|_p = (\mathbb{E}_x|f(x)|^p)^{1/p}$

Proposition 3.2 (Cauchy Schwartz Inequality). For any two functions $f, g : G \rightarrow \mathbb{C}$,

$$|\langle f, g \rangle| \leq \|f\|_2 \cdot \|g\|_2$$

3.3 Characters

Definition 3.3. A character of \mathbb{G} is a homomorphism $\chi : \mathbb{G} \rightarrow \mathbb{C}^*$.

χ is a trivial (principal) character if $\chi(a) = 1$ for every $a \in G$.

Clearly,

$$\chi(a + b) = \chi(a)\chi(b) \text{ for every } a, b \in G$$

Hence,

$$\chi(a)^n = \chi(na) = \chi(0) = 1$$

Intuitively, the values of χ are n^{th} roots of unity.

Proposition 3.4. For any non-trivial character χ ,

$$\sum_{a \in \mathbb{G}} \chi(a) = 0$$

Proof. Since χ is a non-trivial character, there exists $b \in \mathbb{G}$ such that $\chi(b) \neq 1$. We have

$$\chi(b) \cdot \sum_{a \in \mathbb{G}} \chi(a) = \sum_{a \in \mathbb{G}} \chi(a)\chi(b) = \sum_{a \in \mathbb{G}} \chi(a + b) = \sum_{a \in \mathbb{G}} \chi(a)$$

Hence, $\sum_{a \in \mathbb{G}} \chi(a) = 0$ □

Lemma 3.5. For any two characters χ and ψ of \mathbb{G} ,

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } \chi = \psi \\ 0 & \text{otherwise.} \end{cases}$$

3.4 Bi-linear Maps

Definition 3.6. We say that a map $e : G \times G \rightarrow \mathbb{C}$ is a bilinear map if it is a homomorphism in each variable. It is a non-degenerate if for every x , both $e(\cdot, x)$ and $e(x, \cdot)$ are non-trivial. It is symmetric if $e(x, y) = e(y, x)$ for every x, y .

Lemma 3.7. Let e be the map that maps $(x, y) \mapsto \exp(2\pi xyi/r)$, then e is a symmetric non-degenerate bilinear map.

Let G^\wedge denote the set of all characters of G .

Lemma 3.8. G^\wedge is an abelian group under pointwise multiplication.

Lemma 3.9. For every abelian group G , there exists a symmetric non-degenerate bilinear $e : G \times G \rightarrow \mathbb{C}$

Now fix any symmetric, non-degenerate, bilinear map e . For every $x \in G$, let e_x denote the character $e(x, \cdot)$. The map $x \mapsto e_x$ can be shown to be an isomorphism from G to G^\wedge . Thus, we have following lemma.

Lemma 3.10. Every finite abelian group G is isomorphic to its character group G^\wedge

Theorem 3.11. G^\wedge forms an orthonormal basis in \mathbb{C}^n

This follows immediately from Lemmas 3.4 and 3.9.

3.5 Fourier Transform

Definition 3.12. We define $\widehat{f} : G^\wedge \rightarrow \mathbb{C}$ as

$$\widehat{f}(\chi) := \langle f, \chi \rangle$$

\widehat{f} is called the Fourier transform of f . From Theorem 3.10, every function $f \in \mathbb{C}^n$ can be written as linear combinations of characters.

$$f = \sum_{\chi \in G^\wedge} \widehat{f} \chi$$

Definition 3.13. The mapping $\mathcal{F} : f \mapsto \widehat{f}$ is called the Fourier transform.

Proposition 3.14. \mathcal{F} is a linear, bijective mapping.

Chapter 4

Bourgain's Extractor

4.1 Introduction and Motivation

4.2 Bourgain's Extractor

Lemma 4.1. Let X be a distribution on a finite abelian group G such that $|\mathbb{E}[\psi(X)]| \leq \epsilon$ for every nontrivial character ψ . Then X is $\epsilon/\sqrt{|G|}$ close to the uniform distribution: $\|X - U\|_1 \leq \epsilon/\sqrt{|G|}$

4.2.1 The Hadamard Extractor

Appendix A

A.1 Dual of a Vector Space

Definition A.1 (Dual space). Let V be a subspace of $\{0, 1\}^n$. We define the dual of V as $V^\perp = \{x \in \{0, 1\}^n \mid x \cdot v = 0 \forall v \in V\}$.

Theorem A.2. V^\perp is a subspace of $\{0, 1\}^n$.

Proof. For any $x, y \in V^\perp, a \in \{0, 1\}, (a \cdot x + y) \cdot v = a \cdot (x \cdot v) + y \cdot v = 0 + 0 = 0$. □

Lemma A.3. $\sum_{i:\text{even}}^t \binom{n}{i} = \sum_{i:\text{odd}}^t \binom{n}{i} = 2^{t-1}$.

Theorem A.4. For any subspace V of dimension k of $\{0, 1\}^n$, there exists a unique dual space V^\perp of dimension $(n - k)$.

Proof. We will show that $|V^\perp| = 2^{n-k}$ by induction on k .

If $k = 0$, then $V = \{\mathbf{0}\}$. Clearly, $V^\perp = \{0, 1\}^n$.

If $k = 1$, let $V = \{\mathbf{0}, v\}$. Suppose the number of $v_i = 1$ is t , then the number of x such that $x \cdot v = 0$ is $\sum_{i:2|t-i} \binom{n}{i} 2^{n-t} = 2^{t-1} \cdot 2^{n-t} = 2^{n-1}$ by Lemma A.3.

Suppose that there exists a unique orthogonal subspace V^\perp of dimension $(n - k + 1)$ for any subspace V of dimension $k - 1$ of $\{0, 1\}^n$, where $k \geq 2$.

Let $V = \langle v_1, v_2, \dots, v_k \rangle$, $S_1 = \langle v_1, v_2, \dots, v_{k-1} \rangle$, and $S_2 = \langle v_k \rangle$. Then, $V^\perp = S_1^\perp \cap S_2^\perp$.

Suppose $\dim(V^\perp) = t$. We want to show $t = n - k$.

By induction hypothesis, $\dim(S_1^\perp) = n - k + 1$ and $\dim(S_2^\perp) = n - 1$.

If $t \leq n - k - 1$, then we need $[(n - k + 1) - t]$ independent vectors to cover S_1^\perp from extending V^\perp , and we need $[(n - 1) - t]$ independent vectors to cover S_2^\perp from extending V^\perp . Since $S_1^\perp \cup S_2^\perp \subseteq \{0, 1\}^n$, we must have $[(n - k + 1) - t] + [(n - 1) - t] + t \leq n$, which is equivalent to $t \geq n - k$, contradiction.

If $t \geq n - k + 1$, then $S_1^\perp \subseteq S_2^\perp$, this is impossible since v_k is independent from v_1, v_2, \dots, v_{k-1} . Thus, $t = n - k$. So $|V^\perp| = 2^{n-k}$. □

A.2 Statistical Distance between Two Joint Distributions

Theorem A.5. *Let A, B be some probability distributions over the same sample space, and let C be a probability distribution. Then*

$$SD\{(A, C), (B, C)\} = \mathbb{E}_{c \sim C} [SD\{(A|C = c), (B|C = c)\}]$$

Proof.

$$\begin{aligned} SD\{(A, C), (B, C)\} &= \frac{1}{2} \sum_{i,c} |(A, C)(i, c) - (B, C)(i, c)| \\ &= \frac{1}{2} \sum_{i,c} |Pr(C = c) \cdot Pr(A = i|C = c) - Pr(C = c) \cdot Pr(B = i|C = c)| \\ &= \sum_{c \sim C} \left(Pr(C = c) \cdot \frac{1}{2} \sum_i |Pr(A = i|C = c) - Pr(B = i|C = c)| \right) \\ &= \sum_{c \sim C} Pr(C = c) \cdot SD\{(A|C = c), (B|C = c)\} \\ &= \mathbb{E}_{c \sim C} [SD\{(A|C = c), (B|C = c)\}] \end{aligned}$$

□

Claim A.6. For any distributions C, D , for any $S = S_C S_D \subseteq [n + 1]$, if $S_C = 0$, then

$$\widehat{(C, D)}(S) = \mathbb{E}_c[(\widehat{D|C = c})(S_D)]$$

Proof.

$$\begin{aligned} \widehat{(C, D)}(S) &= \langle (C, D), \chi_S \rangle \\ &= \frac{1}{2N} \sum_{(c,d)} (C, D)(c, d) \cdot \chi_S(c, d) \\ &= \frac{1}{2N} \sum_{(c,d)} C(c) \cdot (D|C = c)(d) \cdot \chi_{S_C}(c) \cdot \chi_{S_D}(d) \\ &= \frac{1}{2N} \sum_c \left[C(c) \cdot \chi_{S_C}(c) \cdot \sum_d (D|C = c)(d) \chi_{S_D}(d) \right] \\ &= \frac{1}{2} \sum_c \left[C(c) \cdot \chi_{S_C}(c) \cdot \widehat{(D|C = c)}(S_D) \right] \\ &= \frac{1}{2} \sum_c \left[C(c) \cdot \widehat{(D|C = c)}(S_D) \right] \\ &= \mathbb{E}_c[\widehat{(D|C = c)}(S_D)] \end{aligned}$$

□

A.3 Group Basics

A.3.1 Notation

We reverse the variable p to denote primes.

\mathbb{F}_p denotes the field of size p .

G denotes a finite abelian group.

\mathbb{C} denotes the set of complex numbers.

Definition A.7. We say $\psi : G \rightarrow \mathbb{C}$ is a character if ψ is a homomorphism.

Definition A.8. We say a map $e : G \times G \rightarrow \mathbb{C}$ is a bilinear map if it is a homomorphism in each variable.

Theorem A.9. *For every abelian group G , there exists a symmetric non-degenerate bilinear $e : G \times G \rightarrow \mathbb{C}$*

A.3.2 Dual of a finite Abelian Group

Let G^\wedge denote the set of all characters of G .

Lemma A.10. G^\wedge is an abelian group under point-wise product operation

Theorem A.11. *Every finite abelian group G is isomorphic to its character group G^\wedge*

A.4 Product Graph

Definition A.12. $D = (a, b, c, d)$ is a 2×2 distribution graph if and only if

1. $a + b + c + d = 1$,
2. $a, b, c, d \in [0, 1]$.

Definition A.13. $G = (x, y, z, t)$ is a 2×2 product graph if and only if

1. G is a distribution graph,
2. $xt = yz$, or $x = t = 0$, or $y = z = 0$.

Let \mathbb{G} be the space of all 2×2 product graphs and let \mathbb{D} be the space of all 2×2 distribution graphs. We want to find

$$D^* = \underset{D \in \mathbb{D}}{\operatorname{argmax}} \operatorname{dist}(G, \mathbb{G})$$

$$m = \max_{D \in \mathbb{D}} \operatorname{dist}(G, \mathbb{G})$$

Let $D = (a, b, c, d)$ be any 2×2 distribution graph. Without loss of generality, assume $a \geq d$, $b \geq c$, and $ad \geq bc$

Claim A.14. $\text{dist}(D, \mathbb{G}) \leq f(a, b, c, d)$,

where $f(a, b, c, d) = \min\{(b+c), \frac{(ad-bc)}{a+b}, \frac{1}{2}(|\sqrt{a}-a-b| + |\sqrt{a}-a-c|, |(1-\sqrt{a})^2-d|)\}$

Proof. Let $G = (x, y, z, t)$ be a product graph.

$$\text{dist}(D, \mathbb{G}) \leq \text{dist}(D, G) = \frac{1}{2}(|a-x| + |b-y| + |c-z| + |d-t|)$$

So if we can find some graphs G 's such that $\text{dist}(D, G)$ equal to the three values above respectively, we are done. From the second property of distribution graph, it suggests the way to choose such G 's.

1. Choose $y = z = 0$, $x = a + b$, and $t = c + d$, then $\text{dist}(D, G) = b + c$.

2. Choose $xt = yz$, $x = a$, $y = b$, $z = \frac{(c+d)a}{a+b}$, and $\frac{(c+d)b}{a+b}$, then

$$\text{dist}(D, G) = \frac{1}{2}(|c - \frac{(c+d)a}{a+b}| + |d - \frac{(c+d)b}{a+b}|) = \frac{(ad-bc)}{a+b}$$

3. Choose $x = a$, $y = z = \sqrt{a} - a$, and $t = (1 - \sqrt{a})^2$, then

$$\text{dist}(D, G) = \frac{1}{2}(|\sqrt{a} - a - b| + |\sqrt{a} - a - c|, |(1 - \sqrt{a})^2 - d|)$$

□

Claim A.15. $m \leq \max_{a,b,c,d} f(a, b, c, d)$

Claim A.16. $\max_{a,b,c,d} f(a, b, c, d) = \sqrt{5} - 2$

Proof. 1. If $b(1-d) \geq ad$, then $\frac{(ad-bc)}{a+b} = \frac{(ad-bc)}{1-c-d} \leq \frac{ad}{1-d} \leq \frac{(\frac{a+d}{2})^2}{1-(\frac{a+d}{2})}$. Then

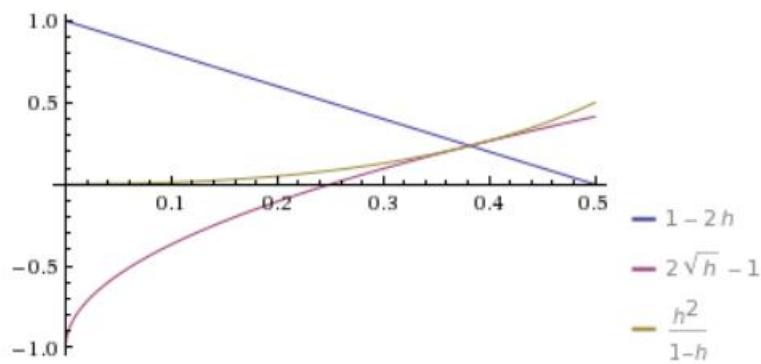
$$\max_{a,b,c,d} f(a, b, c, d) \leq \max \min\{(1-2h), \frac{h^2}{1-h}\}$$

where $h = (a+d)/2$

2. If $b(1-d) = ba + b^2 + bc \leq ad$, suppose $\max \min\{(1-2h), \frac{h^2}{1-h}\} > \sqrt{5} - 2$.

Then $\sqrt{a} \geq a+b \geq a+c$ since $a \geq (a+b)^2 \Leftrightarrow a(a+b+c+d) \geq a^2 + 2ab + b^2 \Leftrightarrow ac + ad \geq ab + b^2$, which is true. So $d \geq (1 - \sqrt{a})^2$. Thus,

$$\begin{aligned} & \frac{1}{2}(|\sqrt{a} - a - b| + |\sqrt{a} - a - c|, |(1 - \sqrt{a})^2 - d|) \\ &= \frac{1}{2}(2\sqrt{a} - 2a - (1 - a - d) + d - (1 - \sqrt{a})^2) \\ &= d - (1 - \sqrt{a})^2 \\ &= (\sqrt{d} + \sqrt{a} - 1)(1 + \sqrt{d} - \sqrt{a}) \\ &\leq \sqrt{d} + \sqrt{a} - 1 \\ &\leq \sqrt{2(a+d)} - 1 \end{aligned}$$



Hence,

$$\max_{a,b,c,d} f(a,b,c,d) \leq \max \min\{1 - 2h, 2\sqrt{h} - 1\}$$

From the graph, we can see that

$$\max \min\{1 - 2h, 2\sqrt{h} - 1, \frac{h^2}{1-h}\} = \sqrt{5} - 2$$

when $a = d = \frac{3-\sqrt{5}}{2}$, $b = \sqrt{5} - 2$, and $c = 0$

□