2.1 Optical fibre

2.1.1 step-index fibre

A step-index fibre is schematically represented on Fig. 2.1. For such fibres, the refractive index of the core region is supposed to be constant. To remain consistent with the previous section it is called n_2. The refractive of the cladding n_1 is also constant. To allow guidance in the core the refractive index in the core has to be larger than the cladding one. For telecommunication, fibres are made of silica and the refractive index is modified by doping either the core with Germanium, in order to increase its refractive index ($n_1 > n_2$) or the cladding with Fluor or potassium in order to lower its refractive index ($n_2 < n_1$). The conventional fibre for telecommunication has

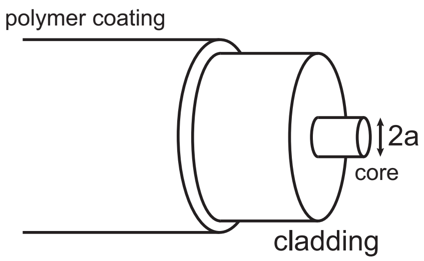

Figure 2.1: step-index fibre. The core has a radius a and a refractive index n_1. The cladding has a refractive index n_2. The polymer coating acts as a protection for the fibre as well as a way to eliminate the light leaking out of the core.

a core diameter of $\sim 8.2\,\mu$m and an external diameter of $125\,\mu$m. The *profile height parameter*

$$\Delta n = \frac{1}{2}\left(1 - \frac{n_{co}^2}{n_{cl}^2}\right) \simeq \frac{n_{co} - n_{cl}}{n_{co}} \simeq 5 \times 10^{-3} \tag{2.1}$$

Although for conventional step-index fibre that can be used for telecommunication. This is a small value it is enough to guide light very efficiently. The best transmission loss figures so far is 0.154 dB/km.

The goal here is

1. Establish the eigenvalue equation in order to evaluate the propagation constant β.
2. Distinguish the different possible modes (TE/TM/EH/HE...etc).
3. Go beyond the step-index model and discuss the possible other types of mechanisms, especially in microstructured fibres.

Finding the optical modes. The eigenvalue problem

Since we have cylindrical symmetry we need to express the electromagnetic field as[1]

$$\mathbf{E} = \mathbf{e}(r,\theta)\, e^{-i(\omega t - \beta z)} \tag{2.2a}$$

$$\mathbf{H} = \mathbf{h}(r,\theta)\, e^{-i(\omega t - \beta z)} \tag{2.2b}$$

[1] Note that the calculations follow the book from K. Okamoto [3] but with the usual sign convention of the argument in the exponential. This yields sign differences in a few equations.

and the Laplacian in the Helmholtz equation now leads to the propagation equations

$$\partial_{rr}e_z + \frac{1}{r}\partial_r e_z + \frac{1}{r^2}\partial_{\theta\theta}e_z + \left[k^2\, n(r,\theta)^2 - \beta^2\right] e_z = 0 \tag{2.3a}$$
$$\partial_{rr}h_z + \frac{1}{r}\partial_r h_z + \frac{1}{r^2}\partial_{\theta\theta}h_z + \left[k^2\, n(r,\theta)^2 - \beta^2\right] h_z = 0 \tag{2.3b}$$

Where $n(r,\theta)$ is either the core or the cladding refractive index. Note also that in that equation the refractive index can simply be written as $n(r,\theta) = n(r)$ because of the symmetry of the system. From Maxwell equations

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{2.4a}$$
$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2.4b}$$
$$\tag{2.4c}$$

where $\epsilon = \epsilon_0 n^2$ with n the refractive index of the material and using the operator ∇ in cylindrical coordinate[2] we have the different components of the electric field

$$\frac{1}{r}\frac{\partial e_z}{\partial \theta} - i\beta e_\theta = i\omega\mu_0 h_r \tag{2.5a}$$
$$i\beta e_r - \frac{\partial e_z}{\partial r} = i\omega\mu_0 h_\theta \tag{2.5b}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r e_\theta\right) - \frac{1}{r}\frac{\partial e_r}{\partial \theta} = i\omega\mu_0 h_z \tag{2.5c}$$

and the magnetic field

$$\frac{1}{r}\frac{\partial h_z}{\partial \theta} - i\beta h_\theta = -i\omega\epsilon_0 n^2 e_r \tag{2.6a}$$
$$i\beta h_r - \frac{\partial h_z}{\partial r} = -i\omega\epsilon_0 n^2 e_\theta \tag{2.6b}$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r h_\theta\right) - \frac{1}{r}\frac{\partial h_r}{\partial \theta} = -i\omega\epsilon_0 n^2 e_z \tag{2.6c}$$

From eq. (2.5b) and (2.6a) we can eliminate h_θ

$$i\beta e_r - \frac{\partial e_z}{\partial r} = i\omega\mu_0 h_\theta \qquad \times\, i\beta \tag{2.7a}$$
$$\frac{1}{r}\frac{\partial h_z}{\partial \theta} - i\beta h_\theta = -i\omega\epsilon_0 n^2 e_r \qquad \times\, (-i\omega\mu_0) \tag{2.7b}$$

$$\Rightarrow \quad i\left(\beta\frac{\partial e_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial h_z}{\partial \theta}\right) = \left(\omega^2\epsilon_0 n^2\mu_0 - \beta^2\right) e_r \tag{2.8}$$

[2]In cylindrical coordinate $\nabla = \left(\frac{\partial}{\partial r}; \frac{1}{r}\frac{\partial}{\partial \theta}; \frac{\partial}{\partial z}\right)$, and for a given vector $\mathbf{A} = (A_r, A_\theta, A_z)$

$$\nabla \times \mathbf{A} = \begin{pmatrix} \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \\ \frac{\partial A_r}{\partial_z} - \frac{\partial A_z}{\partial_r} \\ \frac{1}{r}\frac{\partial (r A_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \end{pmatrix}$$

Similarly we can express the different transverse components of the electric and the magnetic field as a function of the longitudinal part of the field e_z and h_z:

$$e_r = \frac{i}{k^2n^2 - \beta^2}\left(\beta\frac{\partial e_z}{\partial r} + \frac{\omega\mu_0}{r}\frac{\partial h_z}{\partial \theta}\right) \tag{2.9a}$$

$$e_\theta = \frac{i}{k^2n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial e_z}{\partial \theta} - \omega\mu_0\frac{\partial h_z}{\partial r}\right) \tag{2.9b}$$

$$h_r = \frac{i}{k^2n^2 - \beta^2}\left(\beta\frac{\partial h_z}{\partial r} - \frac{\omega\epsilon_0 n^2}{r}\frac{\partial e_z}{\partial \theta}\right) \tag{2.9c}$$

$$h_\theta = \frac{i}{k^2n^2 - \beta^2}\left(\frac{\beta}{r}\frac{\partial h_z}{\partial \theta} + \omega\epsilon_0 n^2\frac{\partial e_z}{\partial r}\right) \tag{2.9d}$$

Fibre modes

There exist three types of modes depending on the value of e_z and h_z:

1. if $e_z = 0$ we talk about TE mode.
2. if $h_z = 0$ we talk about TM mode.
3. if neither e_z nor h_z is null then the mode is called an *hybrid mode.*

TE modes They correspond to the case when $e_z = 0$. This obviously strongly simplify the set of eq.(2.9)

$$e_r = \frac{i\omega\mu_0}{k^2n^2 - \beta^2}\frac{1}{r}\frac{\partial h_z}{\partial \theta} \tag{2.10a}$$

$$e_\theta = \frac{-i\omega\mu_0}{k^2n^2 - \beta^2}\frac{\partial h_z}{\partial r} \tag{2.10b}$$

$$h_r = \frac{i\beta}{k^2n^2 - \beta^2}\frac{\partial h_z}{\partial r} \tag{2.10c}$$

$$h_\theta = \frac{i\beta}{k^2n^2 - \beta^2}\frac{1}{r}\frac{\partial h_z}{\partial \theta} \tag{2.10d}$$

but also the propagation equations ((2.3)). Indeed since $e_z = 0$ we only need to deal with the propagation equation applied to the magnetic field $\mathbf{H}$ (eq. (2.3b). Moreover in order to take the cylindrical symmetry of the problem the magnetic field must be expressed as a function of $\cos(p\theta + \phi)$ or $\sin(p\theta + \phi)$ where $p \in \mathbb{Z}$ and ϕ is a constant phase term.

$$h_z = \left\{\begin{array}{c} f(r) \\ g(r) \end{array}\right\}\cos(p\theta + \phi) \text{ or } \; h_z = \left\{\begin{array}{c} f(r) \\ g(r) \end{array}\right\}\sin(p\theta + \phi) \tag{2.11}$$

where $f(r)$ is used to describe the field in the core region and $g(r)$ in the cladding region. The usual boundary conditions for the tangential components of the field (h_z and h_θ) must be applied. Using the cosine dependence on the magnetic field in eq. (2.10d) we obtain:

$$f(a) = g(a) \tag{2.12a}$$

$$\frac{-i\beta}{k^2n_{co}^2 - \beta^2}\frac{p}{a}f(a)\sin(p\theta + \phi) = \frac{-i\beta}{k^2n_{cl}^2 - \beta^2}\frac{p}{a}g(a)\sin(p\theta + \phi) \tag{2.12b}$$

Since $n_{co} \neq n_{cl}$, otherwise there is no waveguide of course, the eq. (2.12b) only holds if $p = 0$. This means that there is actually no dependence in θ! Since e_r and h_θ only depends on $\partial_\theta h_z$

there are both null: $e_r = 0$ and $h_\theta = 0$. As we see the original set of equations is considerably simplified:

$$\frac{d^2 h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} + \left(k^2 n(r)^2 - \beta^2\right) h_z = 0 \tag{2.13a}$$

$$e_\theta = \frac{-i\omega\mu_0}{k^2 n^2 - \beta^2}\frac{dh_z}{dr} \tag{2.13b}$$

$$h_r = \frac{i\beta}{k^2 n^2 - \beta^2}\frac{dh_z}{dr} \tag{2.13c}$$

$$e_r = h_\theta = 0 \tag{2.13d}$$

In the core we have to solve

$$\frac{d^2 h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} + \alpha^2 h_z = 0 \qquad \text{with } \alpha^2 = k^2 n_{co}^2 - \beta^2 \tag{2.14}$$

for which we can use two possible solutions: either the $0^{th}-$order Bessel function of first kind $J_p(\alpha r)$ or of 2^{nd} kind $Y_p(\alpha r)$. It is however clear (Fig. 2.3) that the only solutions that are physically acceptable are $J_p(\alpha r)$ since $\lim_{r\to 0} Y_p(\alpha r) = -\infty$.

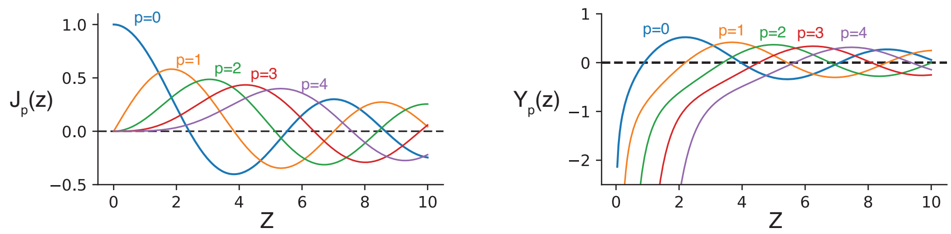

Figure 2.2: Bessel function of first and second kind $J_p(z)$ and $Y_p(z)$ for $p = 0..4$.

In the cladding on the other hand we have to solve

$$\frac{d^2 h_z}{dr^2} + \frac{1}{r}\frac{dh_z}{dr} - \gamma^2 h_z = 0 \qquad \text{with } \gamma^2 = \beta^2 - k^2 n_{cl}^2 \tag{2.15}$$

for which the solution[3] is the modified Bessel function of 2^{nd} kind $K_0(\alpha r)$.

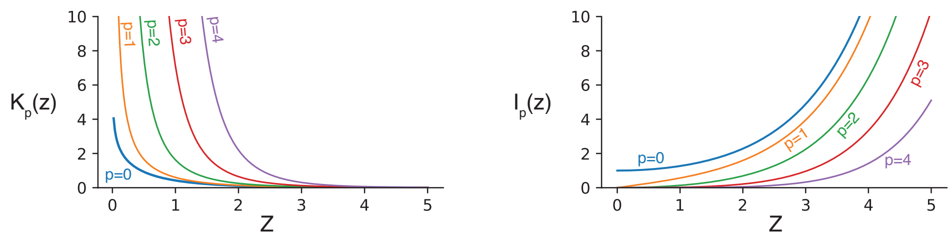

Figure 2.3: Modified Bessel function of second kind $K_p(z)$ and of first kind $I_p(z)$ for $p = 0..4$.

In conclusion, we have the longitudinal component of the magnetic field inside the core and the cladding region:

$$h_z = \begin{cases} AJ_0(\alpha r) & \forall\, |r| \leq a \\ \\ BK_0(\gamma r) & \forall r > a \end{cases} \tag{2.16}$$

[3] Mathematically the modified Bessel function of 1^{st}-order $I_0(\gamma r)$ is also a solution of this differential equation but it diverges at $r \to \infty$ and therefore cannot physically be suitable.

where A and B are two constants to be determined. The continuity of the field components h_z and e_θ at the boundary $r = a$ lead to

$$AJ_0(\alpha a) = BK_0(\gamma a) \tag{2.17a}$$
$$\frac{A}{\alpha}J_0'(\alpha a) = -\frac{B}{\gamma}K_0'(\gamma a) \tag{2.17b}$$

$$\Rightarrow \frac{J_0'(\alpha a)}{\alpha J_0(\alpha a)} = -\frac{K_0'(\gamma a)}{\gamma K_0(\gamma a)} \quad \Leftrightarrow \quad \frac{J_0'(U)}{UJ_0(U)} = -\frac{K_0'(W)}{WK_0(W)} \tag{2.18}$$

where we use the parameters $U = \alpha a$ and $W = \gamma a$ that are similar to the ones we introduced for the planar waveguide. Solving this eigenvalue problem yields the propagation constant β. Note that we can also use the relationship

$$J_0'(x) = -J_1(x) \tag{2.19a}$$
$$K_0'(x) = -K_1(x) \tag{2.19b}$$

Then we can rewrite the eigenvalue problem as

$$\boxed{\frac{J_1(U)}{UJ_0(U)} = -\frac{K_1(W)}{WK_0(W)}} \tag{2.20}$$

Strategy: The strategy[4] to obtain the modes in the fibre is the following:

1. solve the eigenvalue problem for a given set of parameter $\{a, n_{co}, n_{cl}\}$
2. Use the value of β to express the longitudinal components of the fields e_z and h_z
3. Use the longitudinal components to calculate the transverse components of the field. In the particular case of TE mode there are e_θ and h_r.

In the particular case of TE mode the field in the core and cladding are

$$\text{in the core:} \begin{cases} e_\theta &= -i\omega\mu_0 \frac{a}{U} AJ_1\left(\frac{U}{a}r\right) \\ h_r &= i\beta\frac{a}{U}AJ_1\left(\frac{U}{a}r\right) \\ h_z &= AJ_0\left(\frac{U}{a}r\right) \end{cases} \tag{2.21a}$$

$$\text{in the cladding:} \begin{cases} e_\theta &= i\omega\mu_0 \frac{a}{W} AK_1\left(\frac{W}{a}r\right) \\ h_r &= -i\beta\frac{a}{W}\frac{J_0(U)}{K_0(W)}AK_1\left(\frac{U}{a}r\right) \\ h_z &= \frac{J_0(U)}{K_0(W)}AK_0\left(\frac{W}{a}r\right) \end{cases} \tag{2.21b}$$

Note that the constant B in eq. (2.16) is replaced by $({}^{J_0(U)}/{}_{K_0(W)}\ A)$ calculated from boundary conditions at the interface core – cladding. To evaluate the constant A we use the $\mathbf{z}$-component of the Poynting vector

$$S_z = \frac{1}{2}\left(\mathbf{E}\times\mathbf{H}^*\right)\cdot\hat{\mathbf{z}} = \frac{1}{2}\left(e_r h_\theta^* - e_\theta h_r^*\right) \tag{2.22}$$

[4]Obviously this strategy is general and does not only apply to the TE mode.

and evaluate the optical power

$$P = \iint S_z \cdot rdrd\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty}(e_r h_\theta^* - e_\theta h_r^*) \cdot rdrd\theta \tag{2.23}$$

In the Fig. **??** we show the distribution of the $\hat{\mathbf{x}}-$ and $\hat{\mathbf{y}}-$component of the electric field as well as the mode profile, represented by the norm of the $\hat{\mathbf{z}}-$component of the Poynting vector $|S_z|^2$. The white arrows indicate the direction of the electric field.

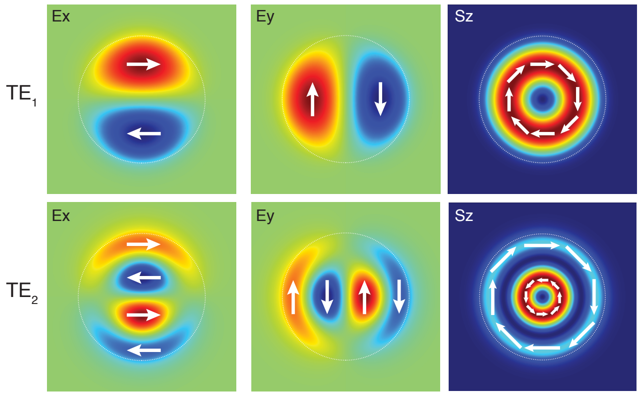

Figure 2.4: Components of the electric fields as well as the intensity profile of the mode $|S_z|^2$ for the first two TE modes. The white dashed circle indicate the limit of the core region.

TM modes This is the very same idea expect that this time $h_z = 0$. This yields that $e_\theta = h_r = h_z = 0$. In that case the eigenvalue equation is

$$\boxed{\frac{J_1(U)}{UJ_0(U)} = -\left(\frac{n_{cl}}{n_{co}}\right)^2 \frac{K_1(W)}{WK_0(W)}} \tag{2.24}$$

and various components of the electromagnetic field

$$\text{in the core:} \begin{cases} e_r &= i\beta\frac{a}{U}AJ_1\left(\frac{U}{a}r\right) \\ e_z &= AJ_0\left(\frac{U}{a}r\right) \\ h_\theta &= i\omega\epsilon_0 n_{co}^2 A\ J_1\left(\frac{U}{a}r\right) \end{cases} \tag{2.25a}$$

$$\text{in the cladding:} \begin{cases} e_r &= -i\beta\frac{a}{W}\frac{J_0(U)}{K_0(W)}AK_1\left(\frac{W}{a}r\right) \\ e_z &= \frac{J_0(U)}{K_0(W)}AK_0\left(\frac{W}{a}r\right) \\ h_\theta &= -i\omega\epsilon_0 n_{cl}^2\frac{a}{W}\frac{J_0(U)}{K_0(W)}\ K_1\left(\frac{W}{a}r\right) \end{cases} \tag{2.25b}$$

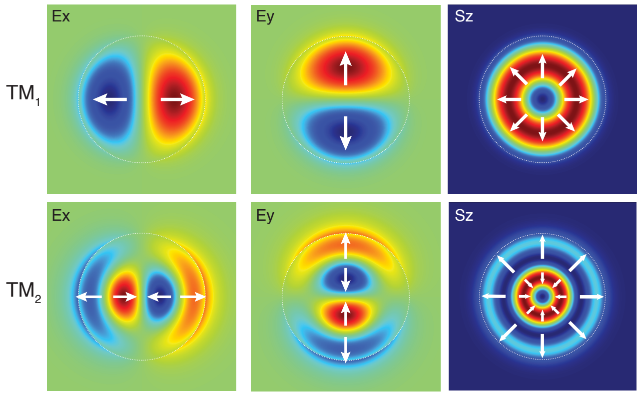

Figure 2.5: Components of the electric fields as well as the intensity profile of the mode $|S_z|^2$ for the first two TM modes. The white dashed circle indicate the limit of the core region.

Hybrid modes In the case when neither e_z nor h_z is null[5] the solutions of the propagation equation eq. (2.3a) and (2.3b) will be given as a product of the p^{th}-order Bessel function and the azimuthal dependence $\cos(p\theta + \phi)$ or $\sin(p\theta + \phi)$ where $p \in \mathbb{Z}$. Since now both propagation equations are coupled we cannot simply eliminate the angular dependence ($p \neq 0$). Note however that the azimuthal dependence has to remain when crossing the interface core–cladding. Moreover in the expression of e_r, e_θ, h_r and h_θ appear terms as $\partial_r e_z$, $\partial_\theta e_z$ ... the azimuthal dependencies for the electric and the magnetic fields must follow

$$e_z = \begin{cases} A J_p\left(\frac{U}{a} r\right) \cos(p\theta + \phi) & \text{in the core} \\ \\ A \frac{J_p(U)}{K_p(W)} K_p\left(\frac{W}{a} r\right) \cos(p\theta + \phi) & \text{in the cladding} \end{cases} \tag{2.26a}$$

$$h_z = \begin{cases} C J_p\left(\frac{U}{a} r\right) \sin(p\theta + \phi) & \text{in the core} \\ \\ C \frac{J_p(U)}{K_p(W)} K_p\left(\frac{W}{a} r\right) \sin(p\theta + \phi) & \text{in the cladding} \end{cases} \tag{2.26b}$$

which we can use to express the e_r, e_θ and h_r, h_θ:

- in the core ($|r| \leq a$):

$$e_r = \frac{-ia^2}{U^2} \left[A\beta \frac{U}{a} J_p'\left(\frac{U}{a} r\right) + C\omega\mu_0 \frac{p}{r} J_p\left(\frac{U}{a} r\right) \right] \cos\left(p\theta + \phi\right) \tag{2.27a}$$

$$e_\theta = \frac{-ia^2}{U^2} \left[-A\beta \frac{p}{r} J_p\left(\frac{U}{a} r\right) - C\omega\mu_0 \frac{U}{a} J_p'\left(\frac{U}{a} r\right) \right] \sin\left(p\theta + \phi\right) \tag{2.27b}$$

$$h_r = \frac{-ia^2}{U^2} \left[A\omega\epsilon_0 n_{co}^2 \frac{p}{r} J_p\left(\frac{U}{a} r\right) + C\beta \frac{U}{a} J_p'\left(\frac{U}{a} r\right) \right] \sin\left(p\theta + \phi\right) \tag{2.27c}$$

$$h_\theta = \frac{-ia^2}{U^2} \left[A\omega\epsilon_0 n_{co}^2 \frac{U}{r} J_p'\left(\frac{U}{a} r\right) + C\beta \frac{U}{a} J_p\left(\frac{U}{a} r\right) \right] \cos\left(p\theta + \phi\right) \tag{2.27d}$$

[5]The continuity conditions still need to hold!

- in the cladding ($r > a$):

$$e_r = \frac{ia^2}{W^2}\left[A\beta\frac{W}{a}K_p'\left(\frac{W}{a}r\right) + C\omega\mu_0\frac{p}{r}K_p\left(\frac{W}{a}r\right)\right]\frac{J_p(U)}{K_p(W)}\cos(p\theta+\phi) \tag{2.28a}$$

$$e_\theta = \frac{ia^2}{W^2}\left[-A\beta\frac{p}{r}K_p\left(\frac{W}{a}r\right) - C\omega\mu_0\frac{W}{a}K_p'\left(\frac{W}{a}r\right)\right]\frac{J_p(U)}{K_p(W)}\sin(p\theta+\phi) \tag{2.28b}$$

$$h_r = \frac{ia^2}{W^2}\left[A\omega\epsilon_0 n_{cl}^2\frac{p}{r}K_p\left(\frac{W}{a}r\right) + C\beta\frac{W}{a}K_p'\left(\frac{W}{a}r\right)\right]\frac{J_p(U)}{K_p(W)}\sin(p\theta+\phi) \tag{2.28c}$$

$$h_\theta = \frac{ia^2}{W^2}\left[A\omega\epsilon_0 n_{cl}^2\frac{W}{r}K_p'\left(\frac{W}{a}r\right) + C\beta\frac{p}{r}K_p\left(\frac{W}{a}r\right)\right]\frac{J_p(U)}{K_p(W)}\cos(p\theta+\phi) \tag{2.28d}$$

Additionally we need to have continuity of the e_θ and h_θ at $r = a$, which leads to

$$A\beta\left(\frac{1}{U^2}+\frac{1}{W^2}\right)p = -C\omega\mu_0\left[\frac{J_p'(U)}{UJ_p(U)} + \frac{K_p'(W)}{WK_p(W)}\right] \tag{2.29a}$$

$$A\omega\epsilon_0\left[n_{co}^2\frac{J_p'(U)}{UJ_p(U)} + n_{cl}^2\frac{K_p'(W)}{WK_p(W)}\right] = -C\beta\left(\frac{1}{U^2}+\frac{1}{W^2}\right)p \tag{2.29b}$$

which has a solution for $\{A, C\}$ if and only if

$$\boxed{\begin{aligned}\left[\frac{J_p'(U)}{UJ_p(U)} + \frac{K_p'(W)}{WK_p(W)}\right]&\left[n_{co}^2\frac{J_p'(U)}{UJ_p(U)} + n_{cl}^2\frac{K_p'(W)}{WK_p(W)}\right]\\ &= \frac{\beta^2}{k^2}\left(\frac{1}{U^2}+\frac{1}{W^2}\right)p^2 = \left(\frac{n_{co}^2}{U^2}+\frac{n_{cl}^2}{W^2}\right)p^2\end{aligned}} \tag{2.30}$$

which is the eigenvalue equation that we need to solve in order to find the propagation constant β. In this equation, p is a positive integer number and k is the vacuum wavenumber. As previously the equation must be solved numerically for a given value of the normalised frequency V.

In Fig. 2.6, we plot the eigenvalue equation for a 4 μm-core diameter rod, which can be seen as an optical fibre with a silica core and an air-cladding. In that case, the *normalised parameter* is $V = 12.395$. As we can see, this curve has several zeros, each of which corresponds to one particular value of U and therefore to one propagation constant β. As a general form, the modes are labelled as HE_{pm} and EH_{pm} where p is the integer number used in eq. (2.30), corresponding to the order of the Bessel function used, and m is the index of the considered zero. Note that the modes are alternating: the first one is HE_{11}. Then comes an EH mode and so on.

From this figure, it is obvious that, besides solving the eigenvalue for the fundamental HE_{11} mode, solving the problem will be a rather difficult task since the root corresponding to EH_{ij} and $\mathrm{HE}_{i(j+1)}$ are very close to each other, and on a curve that is extremely flat... An alternative approach is to modify[6] the eq. 2.30 so as to distinguish:

$$\text{EH modes:}\quad \frac{J_{p+1}(U)}{UJ_p(U)} = \left(\frac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\right)\frac{K_p'(W)}{WK_p(W)} + \left(\frac{p}{U^2} - R\right) \tag{2.31}$$

$$\text{HE modes:}\quad \frac{J_{p-1}(U)}{UJ_p(U)} = -\left(\frac{n_{co}^2+n_{cl}^2}{2n_{co}^2}\right)\frac{K_p'(W)}{WK_p(W)} + \left(\frac{p}{U^2} - R\right) \tag{2.32}$$

where

$$R = \sqrt{\left(\frac{\Delta\, K_p'(W)}{WK_p(W)}\right)^2 + \left(\frac{p\beta}{k_0 n_{co}}\right)^2\left(\frac{1}{U^2}+\frac{1}{W^2}\right)^2} \tag{2.33}$$

[6] see exercise

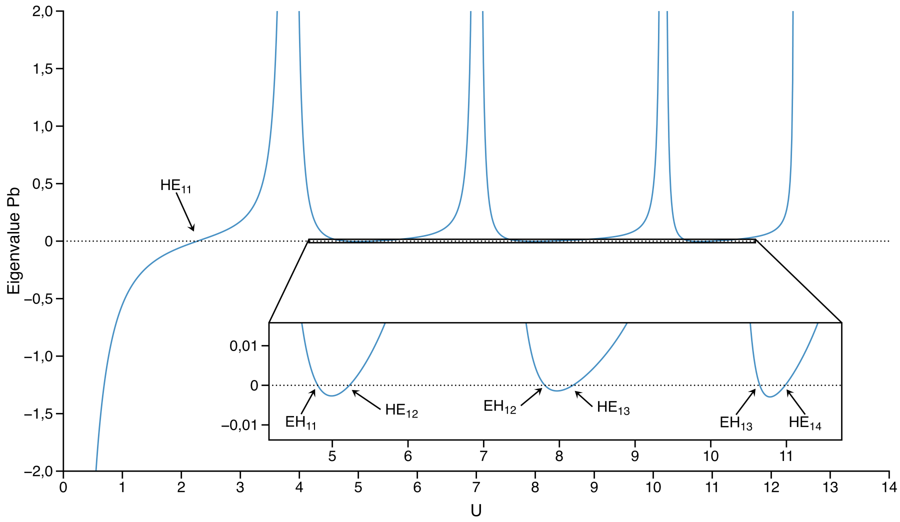

Figure 2.6: Eigenvalue problem eq. (2.30) for a fibre with a diameter $2a = 4\ \mu$m. The wavelength is 1064 nm. This corresponds to $V = 12.395$ for the considered waveguide (silica core surrounding by air). The arrow indicates the zeros and the corresponding optical modes.

We used this approach to evaluate the roots of the eigenvalue equations (2.31) and (2.32) or the same waveguide as for Fig. 2.6. In this case, the situation is much easier to solve since each root can easily be identified, and any root solver can be used to find the solutions[7].

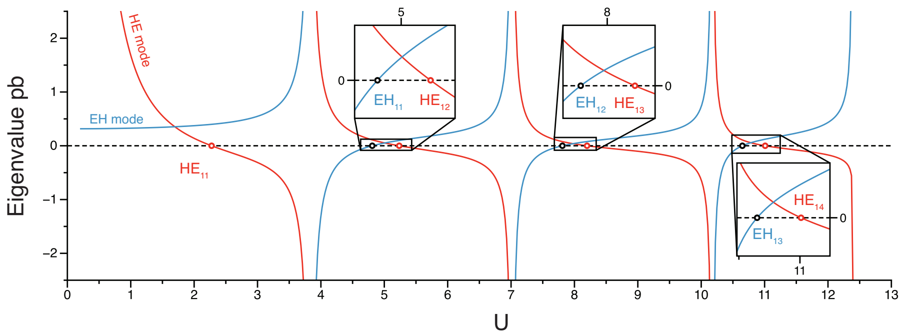

Figure 2.7: Eigenvalue problem eq. (2.31) and (2.32) for the same fibre as for Fig. 2.6 corresponding to $V = 12.395$. Zoom on the curves in the vicinity of the roots indicates the respective solutions.

As we did for the planar waveguide, we can plot the numerical solution of the eigenvalue equations for the respective modes existing in an optical fibre.

[7]The attentive reader will notice that if we take $p = 0$ there is no azimuthal dependence (Eq. 2.11) yielding radially symmetric modes. From the general equation for the hybrid modes (Eq. 2.30), this readily implies[8] that

$$\frac{J_0'(U)}{UJ_0(U)} + \frac{K_0'(W)}{WK_0(W)} = \frac{-J_1(U)}{UJ_0(U)} - \frac{K_1(W)}{WK_0(W)} = 0 \tag{2.34}$$

$$n_{co}^2\frac{J_0'(U)}{UJ_0(U)} + n_{cl}^2\frac{K_0'(W)}{WK_0(W)} = -n_{co}^2\frac{J_1(U)}{UJ_0(U)} - n_{cl}^2\frac{K_1(W)}{WK_0(W)} = 0 \tag{2.35}$$

which are the eigenvalue equations for respectively TE and TM modes that we had found starting from Maxwell's equations.

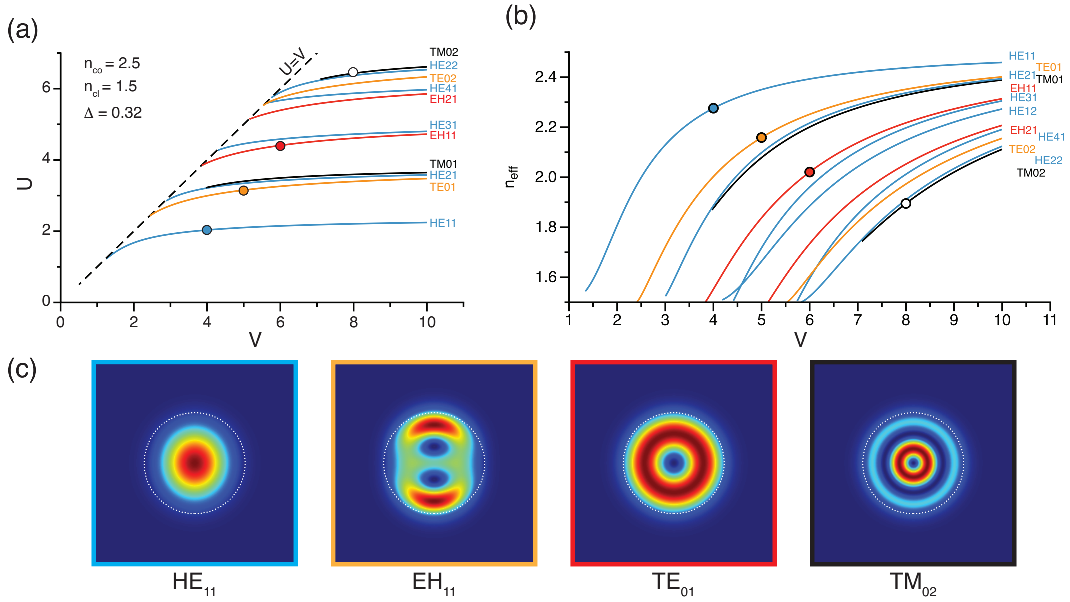

Figure 2.8: (a) numerical solutions of the eigenvalues for the different modes of an optical fibre with $n_{\rm co} = 2.5$ and $n_{\rm cl} = 1.5$, corresponding to $\Delta = 0.32$ and (b) effective index of the different spatial modes, and (c) norm of the $\mathbf{z}-$component of the Poynting vector $|S_z|^2$ for several modes. The corresponding root and its effective index are indicated on (a) and (b) by the dot with corresponding colours.

2.1.2 Weakly-guidance approximation

introduction

Let's assume that we have an EM wave propagating along $\hat{\mathbf{z}}$ in an unbounded material with a refractive index n_{co}. This would of course correspond to the *free-space* situation. We know from a basic electromagnetic lecture that the field would then be perfectly transverse and its wavenumber $\beta = k_0 n_{co}$. The Helmholtz equations are in that case simply

$$\left[\nabla^2 + k_0^2\, n^2(\mathbf{r})\right] \left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right\} = 0 \tag{2.36}$$

Besides since $e_z = h_z = 0$ we also have

$$h_T = \sqrt{\frac{\epsilon_0}{\mu_0}} n_{co}\, \hat{\mathbf{z}} \times e_T \tag{2.37}$$

where the subscript T indicates the transverse components. Of course, β is independent of the orientation of e_T and therefore such a "*waveguide*" has no polarisation properties.

In the most general case where the EM wave propagates inside a non-uniform media where $n = n(\mathbf{r})$, we saw that Maxwell's equation lead to a set of vectorial equations for the electric and the magnetic fields:

$$\nabla^2 \mathbf{E} + k_0^2\, n^2(\mathbf{r}) \mathbf{E} = -\nabla \left[\mathbf{E} \cdot \nabla \ln n^2(\mathbf{r})\right] \tag{2.38a}$$

$$\nabla^2 \mathbf{H} + k_0^2\, n^2(\mathbf{r}) \mathbf{H} = [\nabla \times \mathbf{H}] \times \nabla \ln n^2(\mathbf{r}) \tag{2.38b}$$

These not only yield *vectorial modes* where the fields are not necessarily transverse but also since the propagation constant β now depends on the orientation of the electric field we can have polarisation effects. These actually result from the presence of the $\nabla \ln n^2(\mathbf{r})$ on the RHS of the vectorial form of the Helmholtz equations.

Reminding the profile height parameter

$$\Delta = \frac{n_{co}^2 - n_{cl}^2}{2n_{co}^2} \tag{2.39}$$

we will look here as a common approximation that occurs when $\Delta \ll 1$. This is the case when the refractive index for the cladding approaches those of the core. This is perfectly justified in many cases. The corresponds to the **weak guidance approximation**. For common fibres such as telecommunication fibres for instance $\Delta \sim 0.3 - 0.8\%$. This has direct obvious consequences:

1. Since $V = \frac{2\pi}{\lambda} a \sqrt{n_{co}^2 - n_{cl}^2} = \frac{2\pi}{\lambda} a\, n_{co} \sqrt{2\Delta}$, the waveguide parameter becomes very small. The fibre will therefore support only a few modes.

2. The guidance requirement $k_0\, n_{cl} \leq \beta \leq k_0\, n_{co}$ leads to $\beta \simeq k_0\, n_{co} \simeq k_0\, n_{cl}$ or the effective index $n_{\text{eff}} \simeq n_{co} \simeq n_{cl}$.

3. If $n_{cl} \rightarrow n_{co}$ then the situation will resemble our original *free-space situation*. Not only the polarisation effect will decrease[9], but also the longitudinal contribution of the field should decrease too as for the case of homogenous material.

On Fig. 2.9, we calculated the effective refractive index $n_{\text{eff}} = ({}^{\beta}/_{k_0})$ for decreasing value of the profile height parameter Δ.

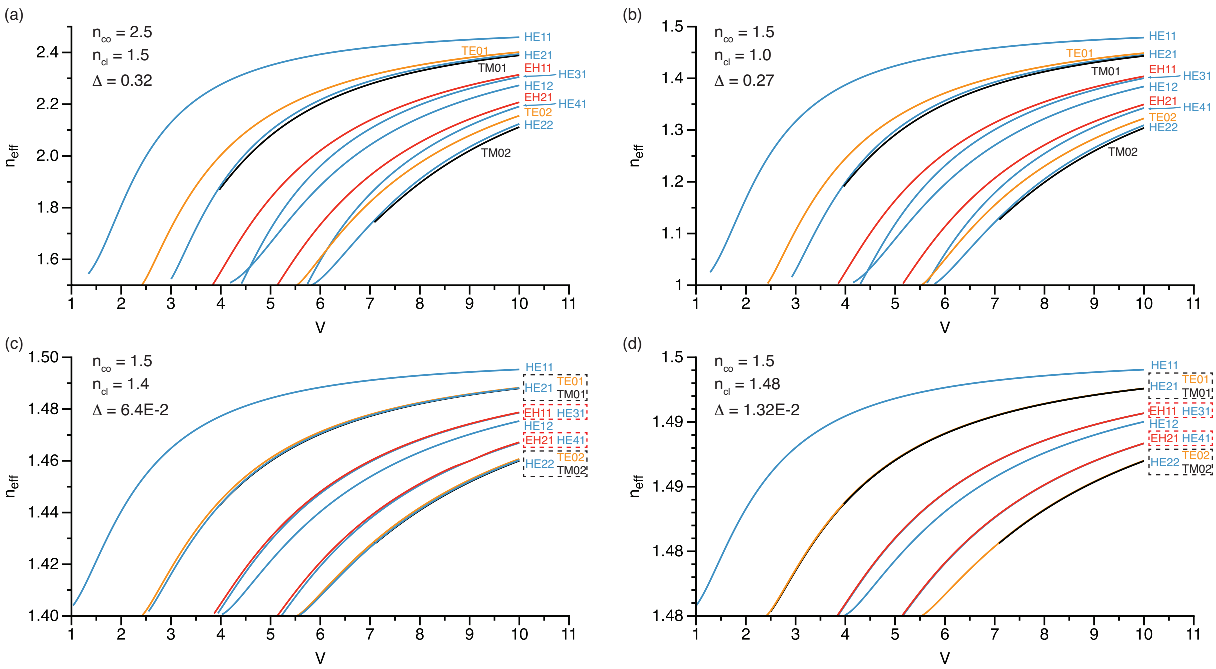

Figure 2.9: Evolution of the effective refractive index $n_{\text{eff}} = ({}^{\beta}/_{k_0})$ of various modes as the function of the $V-$parameter calculated from several step-index Δ.

It is clear that the propagation constant for a few modes merge into one single. We see for instance that HE_{21}, TE_{01} and TM_{01} will have almost the same effective refractive index or propagation constant independently of the value of the V-parameter. We observe the same for HE_{22}, TE_{02} and TM_{02}, but also for EH_{11} and HE_{31} or for EH_{21} and HE_{41}. From the characteristic equations for TE- and TM-modes (eq. 2.24 and 2.24), it is obvious that when

[9] The phase-shift at reflection is becoming minute and therefore the distinction between TE and TM polarisation will diminish too

$n_{co} \simeq n_{cl}$ both characteristic equations merge into one:

$$\left.\begin{array}{lrcl} \textbf{TE-mode:} & \dfrac{J_1(U)}{UJ_0(U)} & = & -\dfrac{K_1(W)}{WK_0(W)} \\[2em] \textbf{TM-mode:} & \dfrac{J_1(U)}{UJ_0(U)} & = & -\left(\dfrac{n_{cl}}{n_{co}}\right)^2 \dfrac{K_1(W)}{WK_0(W)} \end{array}\right\} \xrightarrow[n_{co}\simeq n_{cl}]{} \boxed{\dfrac{J_1(U)}{UJ_0(U)} = -\dfrac{K_1(W)}{WK_0(W)}} \tag{2.40}$$

For the hybrid mode, we can easily obtain from eq. 2.30

$$\frac{J_p'(U)}{UJ_p(U)} + \frac{K_p'(W)}{WK_p(U)} = \pm p\left(\frac{1}{U^2} + \frac{1}{W^2}\right) \tag{2.41}$$

which yields[10]

$$\frac{J_{p+1}(U)}{UJ_p(U)} = -\frac{K_{p+1}(W)}{WK_p(W)} \qquad \textbf{EH-mode} \tag{2.42a}$$

$$\frac{J_{p-1}(U)}{UJ_p(U)} = -\frac{K_{p-1}(W)}{WK_p(W)} \qquad \textbf{HE-mode} \tag{2.42b}$$

As we noticed on Fig. 2.9, in the weak guidance approximation HE_{p+2} and EH_p do merge into one single characteristic equation. Mathematically we can indeed check that[11]

$$\mathbf{HE}_{p+2} \Rightarrow \frac{J_{p+1}(U)}{UJ_{p+2}(U)} = -\frac{K_{p+1}(W)}{WK_{p+2}(W)} \Leftrightarrow \frac{J_{p+1}(U)}{UJ_p(U)} = -\frac{K_{p+1}(W)}{WK_p(W)} \quad \mathbf{EH}_p \tag{2.44}$$

Finally we can unify those characteristic equation into a single one:

$$\frac{J_m(U)}{UJ_{m-1}(U)} = -\frac{K_m(W)}{WK_{m-1}(W)} \quad \text{with} \left\{\begin{array}{rcll} m & = & 1 & \text{TE and TM modes} \\ m & = & p+1 & \text{EH modes} \\ m & = & p-1 & \text{HE modes} \end{array}\right. \tag{2.45}$$

Linearly polarised modes

Starting from the vectorial characteristic equations (eq. 2.38a and 2.38b), and since the refractive index only changes radially (transversally) but not longitudinally along the propagation axis $\hat{\mathbf{z}}$ we could rewrite the Helmholtz equations while distinguishing the transverse part from the longitudinal one[12] :

$$\nabla^2 = \nabla_T^2 + \partial_{zz} \tag{2.46}$$

[10] We remind the recurrence relations for the Bessel functions:

$$\begin{aligned} 2\,J_p'(z) &= J_{p-1}(z) - J_{p+1}(z) & \qquad 2\,K_p'(z) &= -K_{p-1}(z) - K_{p+1}(z) \\ 2\,\frac{p}{z}J_p(z) &= J_{p-1}(z) + J_{p+1}(z) & \qquad 2\,\frac{p}{z}K_p(z) &= -K_{p-1}(z) + K_{p+1}(z) \end{aligned}$$

[11] We remind that

$$J_{p+2}(z) = \frac{2(p+1)}{z}J_{p+1}(z) - J_p(z) \qquad \text{and} \qquad K_{p+2}(z) = \frac{2(p+1)}{z}K_{p+1}(z) + K_p(z) \tag{2.43}$$

[12] Be attentive to the notations. Since we expressed the fields by $\left\{\begin{array}{c}\mathbf{E}\\ \mathbf{H}\end{array}\right\} = \left\{\begin{array}{c}\mathbf{E}_0\\ \mathbf{H}_0\end{array}\right\} e^{-i(\omega t - \beta z)}$, this yields the operator $\nabla = \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} = \nabla_T + i\beta\hat{\mathbf{z}}$, where ∇_T is the transverse gradient. For the Laplacian operator, there exist several symbols: $\triangle = \nabla^2 = \nabla_T^2 + \partial_{zz} = \triangle_\perp - \beta^2$.

where $\triangle_\perp$ is the transverse Laplacian operator. The Helmholtz equations for the electric and the magnetic fields become:

$$\left[\nabla_T^2 + k_0^2 n^2(\mathbf{r}) - \beta^2\right](E_T + E_z \cdot \hat{\mathbf{z}}) = -\left(\nabla_T + i\beta\hat{\mathbf{z}}\right)\left[E_T \cdot \nabla_T \ln n^2(\mathbf{r})\right] \tag{2.47a}$$

$$\left[\nabla_T^2 + k_0^2 n^2(\mathbf{r}) - \beta^2\right](H_T + H_z \cdot \hat{\mathbf{z}}) = \left[(\nabla_T + i\beta\hat{\mathbf{z}}) \times H_T\right] \times \nabla_T \ln n^2(\mathbf{r}) \tag{2.47b}$$

On the RHS, we took into account that $n(z)$ =constant, and therefore the longitudinal contribution of the Laplacian will yield zero. These can split into transverse

$$\left[\nabla_T^2 + k_0^2 n^2(\mathbf{r}) - \beta^2\right] E_T = -\nabla_T\left[E_T \cdot \nabla_T \ln n^2(\mathbf{r})\right] \tag{2.48a}$$

$$\left[\nabla_T^2 + k_0^2 n^2(\mathbf{r}) - \beta^2\right] H_T = (\nabla_T \times H_T) \times \nabla_T \ln n^2(\mathbf{r}) \tag{2.48b}$$

and longitudinal components:

$$\left[\nabla_T^2 + k_0^2 n^2(\mathbf{r}) - \beta^2\right] E_z = -i\beta\left[E_T \cdot \nabla_T \ln n^2(\mathbf{r})\right] \tag{2.49a}$$

$$\left[\nabla_T^2 + k_0^2 n^2(\mathbf{r}) - \beta^2\right] H_z = i\beta\left(\hat{\mathbf{z}} \times H_T\right) \times \nabla_T \ln n^2(\mathbf{r}) \tag{2.49b}$$

transverse contribution: It is usually convenient to represent the transverse variation of the refractive index by[13] (Fig. 2.10)

$$n^2(r) = n_{co}^2\left[1 - 2\Delta f(r)\right] \text{ with } \begin{cases} f = 0 & \text{at} \quad r = 0 \\ f = 1 & \text{at} \quad |r| = a \end{cases} \tag{2.50}$$

where the function $f(r)$ takes into account the real shape of the index profile.

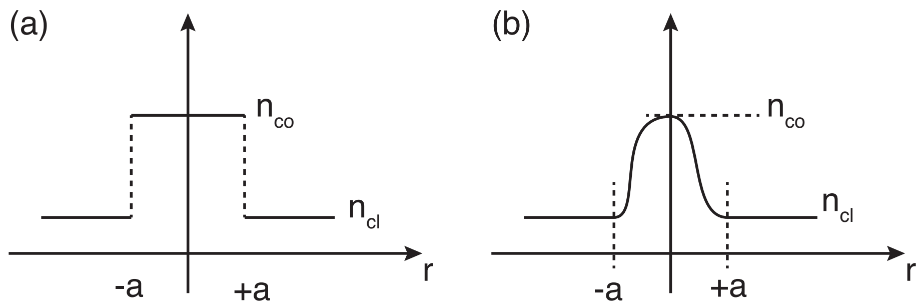

Figure 2.10: (a) step-index and (b) variation of the index across the fibre.

Taking into account the definition of the step-index profile (eq. 2.50) we can evaluate the RHS of eq. 2.48a:

$$\begin{aligned} \nabla_T \ln n^2 &= \nabla_T \ln n_{co}^2\left[1 - 2\Delta f(r)\right] \\ &= \nabla_T\left\{\ln n_{co}^2 + \ln\left[1 - 2\Delta f(r)\right]\right\} \\ \Rightarrow \nabla_T \ln n^2 &\underset{\Delta \ll 1}{\sim} -2\Delta\,\nabla_T f(r) \end{aligned} \tag{2.51}$$

yielding

$$\left(\nabla_T^2 + k^2 n^2(\mathbf{r}) - \beta^2\right) E_T = 2\Delta\, E_T \cdot \nabla_T f(r) \sim 0 \tag{2.52}$$

and we could obtain a very similar equation for the magnetic field.

[13]Be careful not to mix the index height profile Δ with the transverse gradient operator ∇_T and the Laplacian $\nabla_T^2 = \triangle_\perp$.

longitudinal contribution: In order to evaluate the strength of the longitudinal component we use

$$\nabla \cdot \mathbf{D} = \ldots \text{ see exercise} \tag{2.53}$$

$$\Rightarrow \; E_z \simeq \frac{i}{\beta} \nabla_T E_T \tag{2.54}$$

And since $\beta \simeq k_0 n_{co}$ and $V = k_0 n_{co} a \sqrt{2\Delta}$ then

$$E_z \simeq \frac{i\sqrt{2\Delta}}{V} a \nabla_T E_T \Rightarrow \left| \frac{E_z}{E_T} \right| \simeq \frac{\sqrt{2\Delta}}{V} \ll 1 \tag{2.55}$$

since $|\nabla_T \cdot E_T| \sim {}^{|E_T|}/_{a}$. We could follow the same calculation for the magnetic field and find that $|H_z| \ll |H_T|$. As we guessed, the longitudinal contribution is much smaller than the transverse ones in the case of weak-guidance approximation. Note that this does not necessarily mean that this component is small by itself. From the Faraday equation $\nabla \times \mathbf{E} = i\omega\mu_0 \mathbf{H}$ we have

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_T = \frac{\beta}{k_0} \hat{\mathbf{z}} E_T - \frac{i}{\beta} \hat{\mathbf{z}} \times \nabla_T E_z \tag{2.56}$$

The magnitude of the first term is $n_{co} \left| \hat{\mathbf{z}} \times E_T \right|$ and the magnitude of the second is

$$\left(\frac{1}{k_0 a} \right)^2 |E_T| \simeq \Delta \, |E_T| \ll 1 \text{ since } |E_z| \simeq \frac{\sqrt{2\Delta}}{V} |E_T| \tag{2.57}$$

and therefore we can neglect the second term. Then we have

$$\left| \frac{E_z}{E_T} \right| \ll 1; \; \left| \frac{\mathrm{H_z}}{\mathrm{H_T}} \right| \ll 1 \tag{2.58}$$

$$\sqrt{\frac{\mu_0}{\epsilon_0}} H_T \simeq n_{co} \hat{\mathbf{z}} \times E_T \tag{2.59}$$

which means that we have quasi-TEM modes in the weak approximation.

Finally we see that

$$\left(\nabla_T^2 + k^2 n^2 - \beta^2 \right) E_T \simeq 0 \Rightarrow \begin{cases} \left(\nabla_T^2 + k^2 n^2 - \beta^2 \right) E_x & = & 0 \\ \left(\nabla_T^2 + k^2 n^2 - \beta^2 \right) E_y & = & 0 \end{cases} \tag{2.60}$$

We have now two identical **scalar equations**. If β is a solution, then $-\beta$ is also a solution. We can therefore express the electric field as

$$\hat{\mathbf{E}}_T = \psi(x,y) e^{\pm i\beta z} \hat{\mathbf{u}}_T \tag{2.61}$$

where $\hat{\mathbf{u}}_T$ is the transverse unit vector, which can have <u>any</u> direction but remains fixed. From eq. 2.59 we have the transverse magnetic field:

$$H_T = \psi(x,y) e^{\pm i\beta z} \left(\hat{\mathbf{z}} \times \hat{\mathbf{u}_\mathbf{T}} \right) n \sqrt{\frac{\epsilon_0}{\mu_0}} \tag{2.62}$$

Moreover, the solution β is the same for both E_x and E_y: the mode is *linearly polarised*. And since E_x and E_y can be exchanged in the equations 2.60, then each solution is two-fold degenerate, each solution $\psi(x,y)$ corresponding to two orthogonal polarisations (Fig. 2.11).

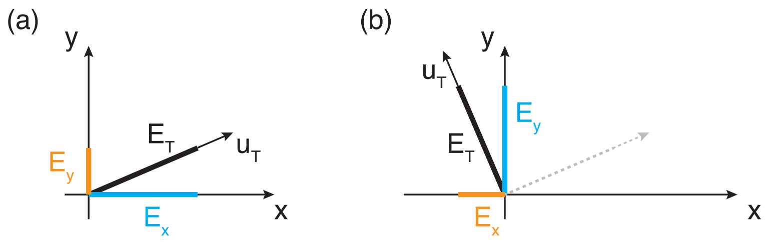

Figure 2.11: (a) and (b) the two orthogonally polarised possibilities for the electric field.

Continuity of the fields Since the problem is now reduced to the transverse components only, it is important to look at the continuity conditions at the boundary $r = a$ between the core and the cladding region. Of course, as usual, the tangential component E_t and H_t are continuous. Regarding the normal components of the electric field E_N, since the index difference is small the continuity of $n\,E_N$ leads to the continuity of E_N. This also applies to the continuity of the normal component of the magnetic field H_N.

Components of the field

As we saw, we have now to deal with a scalar equation for the transverse component of either the electric or the magnetic field. As we already did we use the transverse Laplacian in cylindrical coordinates so as to obtain

$$\partial_{rr}\psi(r,\theta) + \frac{1}{r}\partial_r\psi(r,\theta) + \frac{1}{r^2}\partial_{\theta\theta}\psi(r,\theta) + \left(k_0^2 n^2(\mathbf{r}) - \beta^2\right)\psi(r,\theta) = 0 \tag{2.63}$$

which we need to solve. As usual, we can try separating the variables by expressing $\psi(r,\theta) = F(r)\Theta(\theta)$ where $F(r)$ takes the radial dependence into account and $\Theta(\theta)$ the azimuthal one. Inserting inside eq. 2.63 we obtain:

$$\begin{aligned}
&\Theta\frac{d^2F}{dr^2} + \Theta\frac{1}{r}\frac{dF}{dr} + F\frac{1}{r^2}\frac{d^2\Theta(\theta)}{d\theta^2} + \left[k_0^2 n^2(r) - \beta^2\right]F\,\Theta = 0 \qquad \Big/\frac{F\Theta}{r^2} \\
&\Rightarrow \frac{r^2}{F}\frac{d^2F}{dr^2} + \frac{r}{F}\frac{dF}{dr} + r^2\left[k_0^2 n^2(r) - \beta^2\right] = -\frac{1}{\theta}\frac{d^2\Theta}{d\theta^2}
\end{aligned} \tag{2.64}$$

And since both side only depends on one of the variables, both sides must be equal to the same constant which we choose as ℓ^2. Using this we readily obtain:

$$\frac{d^2\Theta}{d\theta^2} + \ell^2\Theta = 0 \Rightarrow \Theta = \left\{\begin{array}{l} \cos\ell\theta \\ \\ \sin\ell\theta \end{array}\right. \quad \text{for} \quad \ell \geq 1 \tag{2.65}$$

Since we are dealing with optical fibres, which are cylindrically symmetric, then Θ must be $({}^{2\pi}/_N)$ symmetric imposing that $\ell \in \mathbb{N}$. Not only do we have two possible solutions (cos or sin) but we should also consider the available orthogonal polarisations. For the electric field, we, therefore, have the following possibilities:

$$\left.\begin{array}{rcl}
E_T &=& F_l(r)\left[\cos\ell\theta\,\hat{\mathbf{x}} - \sin\ell\theta\,\hat{\mathbf{y}}\right] \\
E_T &=& F_l(r)\left[\cos\ell\theta\,\hat{\mathbf{x}} + \sin\ell\theta\,\hat{\mathbf{y}}\right] \\
E_T &=& F_l(r)\left[\sin\ell\theta\,\hat{\mathbf{x}} - \cos\ell\theta\,\hat{\mathbf{y}}\right] \\
E_T &=& F_l(r)\left[\sin\ell\theta\,\hat{\mathbf{x}} - \cos\ell\theta\,\hat{\mathbf{y}}\right]
\end{array}\right\} \text{for} \quad \ell \geq 1 \tag{2.66}$$

So for $\ell \geq 1$ the solution is actually four-fold degenerate but is only two-fold degenerate when $\ell = 0$.

To solve the radial part of the problem (LHS of eq. 2.64), we need to take into account the shape of the refractive index $n^2(r) = n_{co}\left[1 - 2\Delta f(r)\right]$. Using the waveguide parameters

$$V = k_0 n_{co} a\sqrt{2\Delta}$$
$$U^2 = a^2\left(k_0^2 n_{co}^2 - \beta^2\right)$$
$$W^2 = a^2\left(\beta^2 - k_0^2 n_{cl}^2\right)$$

in le LHS of eq. 2.64 we obtain

$$\text{in the core:}\quad r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} + \left(\frac{r^2U^2}{a^2} - \ell^2\right) = 0 \tag{2.67a}$$

$$\text{in the cladding:}\quad r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - \left(\frac{r^2W^2}{a^2} + \ell^2\right) = 0 \tag{2.67b}$$

radial components: As previously the solution of the radial dependence can either be the Bessel function of the first kind $J_\ell(x)$ or the Bessel function of the second kind $Y_\ell(x)$ but for physical reasons, only $J_\ell(x)$ is suitable. Similarly, the solution in the cladding can either be described by the modified Bessel function of the first kind $I_\ell(x)$ or the modified Bessel function of the second kind $K_\ell(x)$. However, as for the hybrid modes, only $K_\ell(x)$ is physically possible. Finally using the continuity equation at the core/cladding interfaceăwe can express the mode as

$$\psi_\ell(r,\theta) = \begin{cases} AJ_\ell\left(U_m\dfrac{r}{a}\right)\begin{cases}\cos\ell\theta \\ \\ \sin\ell\theta\end{cases} & \text{for } |r| \leq a \\ \\ A\dfrac{J_\ell(U_m)}{K_\ell(W_n)}K_\ell\left(W_n\dfrac{r}{a}\right)\begin{cases}\cos\ell\theta \\ \\ \sin\ell\theta\end{cases} & \text{for}|r| > a \end{cases} \tag{2.68}$$

ăă

Nomenclature

It is clear from eq. 2.68 that the $\mathrm{LP}_{\ell m}$ vary radially according to $F(r)$, following the oscillations of the Bessel function of first kind in the core, and the rapid decrease of the Hankel function $K(r)$ in the cladding. The number of extrema is governed by the index m (Fig. 2.12).

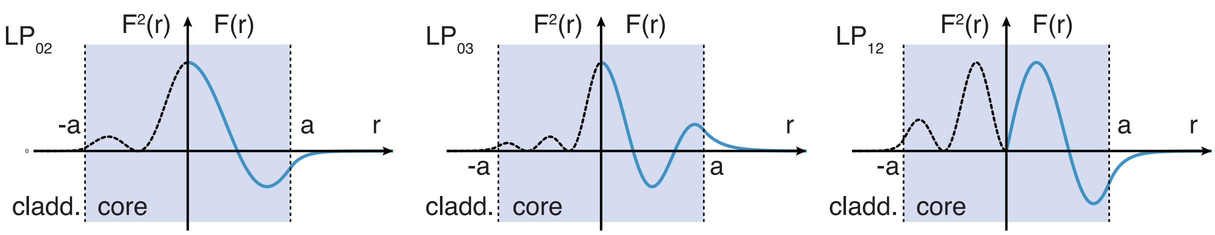

Figure 2.12: Radial contribution (amplitude $F(r)$ and intensity $F^2(r)$) for different $\mathrm{LP}_{\ell m}$ modes.

The other index takes into account the azimuthal dependence $\Theta(\theta)$, ℓ representing $2\times$ number of holes along 2π.

We should now remind that when the profile height parameter Δ was decreased, several modes were presenting a very similar propagation constant β yielding to the possible superposition of the exact modes so as to "produce" the LP modes (Fig. 2.14).

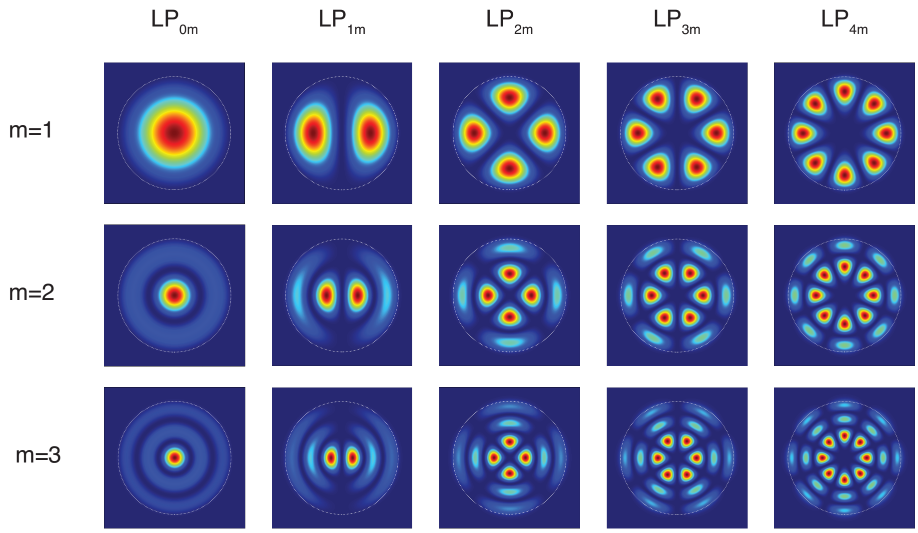

Figure 2.13: Intensity distribution for several $\mathrm{LP}_{\ell m}$ modes.

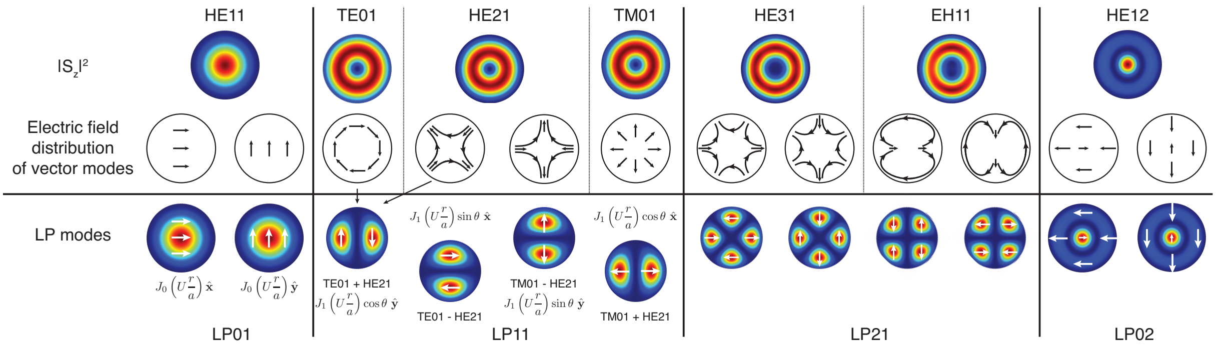

Figure 2.14: Exact modes with the distribution of the electric field and intensity profile of the corresponding LP modes with the respective polarisation.

Fundamental LP01 mode

As we already mentioned the step-index for telecommunications fibres is very small and therefore the LP modes, and in particular, the fundamental LP_{01} mode, are very important. This mode, described by a J_0 Bessel function in the core and a K_0 Hankel function in the cladding, has a rather similar shape as the Gaussian mode that characterises a propagating laser beam. Such a beam is represented as:

$$\psi_g(r) = \exp\left(-\frac{r^2}{w_0^2}\right) \tag{2.69}$$

where w_0 is the waist of the laser beam. The waist yielding the best fit of the LP_{01} fundamental mode by the Gaussian beam corresponds to the one optimising the integral overlap

$$\eta = \frac{\left|\displaystyle\int_0^\infty E(r)\psi_g\, rdr\right|^2}{\displaystyle\int_0^\infty |E(r)|^2\, rdr \cdot \int_0^\infty |\psi_g|^2\, rdr} \Rightarrow \frac{d\eta}{dw_0} = 0 \tag{2.70}$$

A very good approximation, within less than 1%, was given by D. Marcuse [2]:

$$\frac{w_0}{a} = 0.65 + \frac{1.619}{V^{3/2}} + \frac{2.879}{V^6} \qquad \text{for } 1.2 < V < 4 \tag{2.71}$$

Of course, such an empirical equation can be directly used for the excitation of the single-mode fibres by a Gaussian-shaped laser beam.

Bibliography

[1] D. Gloge. Dispersion in weakly guiding fibers. *Appl. Opt.*, 10(11):2442–2445, Nov 1971.

[2] D. Marcuse. Loss analysis of single-mode fiber splices. *Bell System Technical Journal*, 56(5):703–718, 1977.

[3] K. Okamoto. *Fundamentals of Optical Waveguides*. ACADEMIC PressINC, 2005.

[4] A .W. Snyder and J. Love. *Optical Waveguide Theory*. Springer, 1 edition, 1983.