Cauchy's Integral Formula

Introduction

Notation: Let $D(a,r) = \{z : |z-a| < r\}$ denote the open ball (disk) with centre a and radius $r \in \mathbb{R}^+$ $(r > 0)$. Then, for $a, z \in \mathbb{C}$, we denote:

$$\int_a^z f := \int_{[a \to z]} f$$

the integral of f along the line segment from a to z.

Theorem: Let f be continuous on $D(a,r)$ and let

$$F(z) = F(a) + \int_a^z f \quad ; \quad z \in D(a,r)$$

where $F(a) \in \mathbb{C}$ is an arbitrary constant. Then $F'(a) = f(a)$

Definition: A subset $A \subset \mathbb{C}$ is called a star set iff $\exists a \in A : [a \to z] \subset A \ \forall z \in A$.
We call such a point $a \in A$ a star center of A and say that A is starlike about a.

Notation: For three points $a, z, b \in \mathbb{C}$, we denote by $\Delta = \Delta(a,z,b)$ the (closed) triangular path that traverses the triangle with vertices a, z and b, going from a to z, z to b and b back to a. Formally, $\Delta = [a \to z] \cup [z \to b] \cup [b \to a]$

Theorem: Let $A \subset \mathbb{C}$ be open and starlike about $a \in A$. Let $f \in C(A)$ (f is cts on A) and assume $\int_\Delta f = 0$ for any Δ such that Δ and its interior are contained in A. Then, $F'(z) = f(z) \ \forall z \in A$ where

$$F(z) = F(a) + \int_a^z f \ ; \ z \in A, \ F(a) \in \mathbb{C}$$

Cauchy-Goursat Lemma: Let f be analytic in an open set $A \subset \mathbb{C}$. If $\Delta \subset A$ with its interior contained in A, then we have:

$$\int_{\Delta} f = 0$$

Cauchy Integral Theorem: If f is analytic in an open star set $A \subset \mathbb{C}$ and $\gamma \in A$ is a closed path, then we have:

$$\int_{\gamma} f = 0$$

Definition: A path $\gamma : [\alpha, \beta] \to \mathbb{C}$ is called simple iff it is surjective except possibly at its endpoints. In other words, if we have $s, t \in [\alpha, \beta]$ not being endpoints of γ then we have $s \neq t \Rightarrow \gamma(s) \neq \gamma(t)$

Cauchy Integral Thm Refined: Let f be analytic in an open set $A \subset \mathbb{C}$ and suppose that $\gamma_1, \gamma_2 \in A$ are simple, closed and positively oriented paths with γ_2 inside the interior of γ_1.

Suppose that it is possible to join points on γ_1 to γ_2 by line segments to obtain positively oriented, closed paths $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ each lying in an open star set $A^* \subset A$ s.t.

$$\gamma_1 \cup^{-} \gamma_2 = \bigcup_{i=1}^{n} \Gamma_i$$

Then,

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

intro over

Cauchy's Integral Formula

Theorem

Let $a, b \in \mathbb{C}$, $r > 0$. Then,

$$\int_{C(a,r)} \frac{1}{z-b}\, dz = \begin{cases} 0 ; & b \text{ outside } C(a,r) \\ 2\pi i ; & b \text{ inside } C(a,r) \end{cases}$$

Proof:

Case 1: b outside $C(a,r)$

- If b is outside $C(a,r)$, then $|b-a| \geq r + \varepsilon$ for some $\varepsilon > 0$. Hence the function

$$f(z) = \frac{1}{z-b}$$

is differentiable in $D(a, r+\varepsilon)$ and hence analytic in this disk which is an open star set. Thus, by Cauchy's Integral Theorem (CIT),

$$\int_{C(a,r)} \frac{1}{z-b}\, dz = \int_{C(a,r)} f(z)\, dz = 0$$

because $C(a,r)$ is a closed path in the star set $D(a, r+\varepsilon)$

Case 2: b inside $C(a,r)$

- If b is inside $C(a,r)$ then $|b-a| < r$ and hence we choose ε small enough so that $C(b, \varepsilon)$ lies inside $C(a,r)$. But the region between these two paths can be divided into suitable star sets on which $f(z) = (z-b)^{-1}$ is analytic and hence by Cauchy's Integral Theorem Refined (CITR), we have that

$$\int_{C(a,r)} \frac{1}{z-b}\, dz = \int_{C(b,\varepsilon)} \frac{1}{z-b}\, dz = 2\pi i$$

Theorem: Cauchy's Integral Formula

Let f be analytic in the open disk $D(a,r)$ and let $s\in(0,r)$. Then for $z\in D(a,s)$,

$$f(z) = \frac{1}{2\pi i}\int_{C(a,s)} \frac{f(t)}{t-z}\,dt$$

Proof:

- We have f analytic in $D(a,r)$ and $s\in(0,r)$. Fix $z\in D(a,s)$ and let $\varepsilon>0$. Then,

$$t\in C(z,\delta) \qquad \exists\,\delta>0 : |t-z|<\delta \Rightarrow t\in D(a,s) \text{ and } |f(t)-f(z)|<\varepsilon$$

- Since $C(z,\delta)$ lies inside $C(a,s)$ and the function $g(t)=f(t)\cdot(t-z)^{-1}$ is analytic in the region between these circles, using CITR, we have:

$$\begin{aligned}
\left|\int_{C(a,s)}\frac{f(t)}{t-z}\,dt-2\pi i f(z)\right| &= \left|\int_{C(z,\delta)}\frac{f(t)}{t-z}\,dt-2\pi i f(z)\right| \\
&= \left|\int_{C(z,\delta)}\frac{f(t)}{t-z}\,dt-f(z)\int_{C(z,\delta)}\frac{1}{t-z}\,dt\right| \\
&= \left|\int_{C(z,\delta)}\frac{f(t)-f(z)}{t-z}\,dt\right| \\
&\leq \text{length}(C(z,\delta))\cdot \varepsilon/\delta \\
&= 2\pi\delta\cdot\varepsilon/\delta \\
&= 2\pi\varepsilon
\end{aligned}$$

- Finally, since $\varepsilon>0$ was arbitrary, it follows from the above that:

$$\int_{C(a,s)}\frac{f(t)}{t-z}\,dt = 2\pi i\cdot f(z)$$

as required

Corollary: Mean Value Property

If f is analytic in the disk $D(a,r)$ and $s \in (0,r)$ then we have that

$$f(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a+se^{it})\, dt$$

Proof:

Let $\gamma(t) = a + se^{it}$, $t \in [-\pi, \pi]$. As this is a parametrisation of $C(a,s)$, by CIF we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz$$
$$= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(\gamma(t))}{\gamma(t)-a} \gamma'(t)\, dt$$
$$= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(a+se^{it})}{se^{it}} ise^{it}\, dt$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a+se^{it})\, dt$$

as required.

Example

Evaluate

$$I = \int_{C(0,1)} \frac{1}{(z-b)^k (z-c)}\, dz$$

where $b, c \in \mathbb{C}$, $|b| > 1$ and $|c| < 1$

First, write $f(z) = (z-b)^{-k}$ so that

$$I = \int_{C(0,1)} \frac{f(z)}{z-c}\, dz$$

Then, since $|b-0| = |b| > 1$, f is analytic in $D(0, 1+\varepsilon)$ for some

$\varepsilon > 0$ and $1 \in (0, 1+\varepsilon)$ so by CIF:

$$f(c) = \frac{1}{2\pi i} \cdot I$$

$$\therefore I = 2\pi i \cdot f(c)$$

$$= \frac{2\pi i}{(c-b)^k}$$

Evaluate

$$I = \int_{C(1,1)} \frac{1}{z^2-1}\, dz$$

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{-1/2}{z-1} + \frac{1/2}{z+1}$$

$$\therefore I = \frac{1}{2}\int_{C(1,1)} \frac{1}{z+1}\, dz - \frac{1}{2}\int_{C(1,1)} \frac{1}{z-1}\, dz$$

since $C(1,1) = \{1 + e^{-it} : t \in [-\pi, \pi]\} = \{z : |z-1| \leq 1\}$ then we have 1 inside $C(1,1)$ $(|1-1| = |0| \leq 1)$ and -1 outside $C(1,1)$ $(|-1-1| = |-2| = |2| > 1)$,

$$I = 0 - \frac{1}{2}(2\pi i)$$

$$= -\pi i$$

$f = \frac{1}{z-1}$ $\qquad$ f analytic in $D(1, 1+\varepsilon)$

$1 \in (0, 1+\varepsilon)$

$\frac{f(z)}{z+1}$ $\qquad$ $I = 2\pi i f(-1)$

$= 2\pi i \cdot \frac{-1}{2}$

$= -\pi i$

$\leftarrow$ alt approach