CS 103: Mathematical Foundations of Computing
Problem Set #5

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May 28, 2022

Due Friday, October 29 at 2:30 pm Pacific

This Problem Set has no coding questions; all answers to the Problem Set go in this file.
Some notation you might find useful here:

- Subscripts can be written as a_{index}. Remember to use curly braces if you have a multicharacter expression as a subscript.
- The notation $f^{\star}(n)$ comes up in the last problem.

Problem One: Induction Proof Critiques

i.

- P isn't a predicate; the quantity $\frac{n(n-1)}{2}$ isn't something that evaluates to true or false.
- The base case $P(1)$ is not necessary.
- We need to choose a specific choice of k instead of all natural numbers k.
- The direction of induction is wrong.

ii.

- Two different variables n in the predicate $P(n)$.
- The direction of induction is wrong.

Problem Two: Recurrence Relations

Theorem: For all natural numbers n, we have $a_n = 2^n$.
Proof: Let $P(n)$ be the statement "$a_n = 2^n$." We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.
As our base case, we prove $P(0)$, that $a_0 = 2^0$. Since $a_0 = 1$ and $2^0 = 1$ as well, we see that $P(0)$ is true.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(k)$ is true, meaning that $a_k = 2^k$. We need to show $P(k+1)$, meaning that $a_{k+1} = 2^{k+1}$. To see this, note that

$$\begin{aligned} a_{k+1} &= 2a_k \\ &= 2 \cdot 2^k \\ &= 2^{k+1}. \end{aligned}$$

Therefore, we see that $a_{k+1} = 2^{k+1}$, so $P(k+1)$ is true, completing the induction. ■

Problem Three: Stacking Cans

i.

Theorem: For all natural numbers $n \geq 1$, we have $h_n = 3n(n-1)+1$.
Proof: Let $P(n)$ be a statement "$h_n = 3n(n-1)+1$". We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.
For our base case, we need to show $P(1)$ is true, meaning that $h_1 = 3 \cdot 1(1-1)+1$. Since $h_1 = 1$ and $3 \cdot 1(1-1)+1 = 1$ as well, we see that $P(1)$ is true.
For the inductive step, assume that for some arbitrary natural number $k \geq 1$ that $P(k)$ holds, meaning that

$$h_k = 3k(k-1)+1. \qquad (1)$$

We need to show that $P(k+1)$ holds, meaning that $h_{k+1} = 3k(k+1)+1$. To see this, notice that

$$\begin{aligned} h_{k+1} &= h_k + 6k \\ &= 3k(k-1)+1+6k \quad (via(1)) \\ &= 3k^2 - 3k + 1 + 6k \\ &= 3k^2 + 3k + 1 \\ &= 3k(k+1)+1. \end{aligned}$$

Therefore, $P(k+1)$ is true, completing the induction. ■

ii. Fill in the blanks to Problem Three, part ii. below.

- A 1-layer tower has 1 can in it.
- A 2-layer tower has 7 cans in it.
- A 3-layer tower has 19 cans in it.
- A 4-layer tower has 37 cans in it.
- A 5-layer tower has 61 cans in it.
- A 6-layer tower has 91 cans in it.
- A 7-layer tower has 127 cans in it.
- A 8-layer tower has 169 cans in it.
- A 9-layer tower has 217 cans in it.
- A 10-layer tower has 271 cans in it.

iii. Fill in the blank to Problem Three, part iii. below.

An n-layer tower has n^3 cans in it.

iv.

Theorem: For all $n \in \mathbb{N}$, we have $s_n = n^3$.
Proof: Let $P(n)$ be a statement "$s_n = n^3$". We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$, from which the theorem follows.
For our base case, we need to show $P(0)$ is true, meaning that $s_0 = 0^3 = 0$. This is true because there are no layers in the pyramid, meaning there are no cans in it.
For the inductive step, assume that for some arbitrary natural number $k \geq 1$ that $P(k)$ holds, meaning that

$$s_k = k^3. \qquad (1)$$

We need to show that $P(k+1)$ holds, meaning that $s_{k+1} = (k+1)^3$. To do so, notice that

$$\begin{aligned} s_{k+1} &= s_k + h_{k+1} \\ &= k^3 + 3k(k+1) + 1 \\ &= k^3 + 3k^2 + 3k + 1 \\ &= (k+1)^3. \end{aligned}$$

Therefore, $P(k+1)$ is true, completing the induction. ■

Problem Four: The Circle Game

i.

$P(n)$ is a universally-quantified statement.

ii.

Theorem: For any natural number n, if n points labeled $+1$ are placed on the boundary of the circle and n points labeled -1 are placed on the boundary of the circle, there's always some starting position on the circle from which you can start and win the circle game.

Proof: Let $P(n)$ be the statement "for any circle with n points labeled $+1$ and n points labeled -1 on its boundary, there is a starting position on the circle from which you can win the circle game." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

For our base case, we need to show that $P(0)$ is true, meaning that for any circle with no points, there is a starting position on the circle from which you can win the game. This is vacuously true because you never pass through more -1 points than $+1$ points.

For the inductive step, assume that for some arbitrary natural number k that $P(k)$ holds, meaning that for any circle with k points labeled $+1$ and k points labeled -1 on its boundary, there is a starting position on the circle from which you can win the game (1).

We need to show that $P(k+1)$ holds, meaning that for any circle with $k+1$ points labeled $+1$ and $k+1$ points labeled -1 on its boundary, there is a starting position on the circle from which you can win the game. To do so, pick a random circle A with $k+1$ points labeled $+1$ and $k+1$ points labeled -1 on its boundary. Notice that circle A can be constructed by inserting a $+1$ point and a -1 point into a circle B which has k points labeled $+1$ and k points labeled -1 on its boundary. We know, by (1), that there is a starting position in circle B from which you can win the game. By inserting a $+1$ point and a -1 point into circle B, consider two cases:

Case 1: Point $+1$ inserted before point -1. If point $+1$ is inserted at the beginning of the chain, choose this point as a starting position. Otherwise, start as normal. This is true because the total point on the pass-through is never less than zero because the point $+1$ is inserted before the point -1.

Case 2: Point -1 inserted before point $+1$. If point -1 is inserted at the beginning of the chain, start as normal. Otherwise, pick a starting point as the first $+1$ point after the point where the total point passing through is less than or equal to zero. This is true because this means that the total point passing through the right is always greater than or equal to zero.

Therefore, $P(k+1)$ holds, completing the induction. ■

Problem Five: Regular Graphs

Theorem: For all natural numbers n, there exists an n-regular graph containing exactly 2^n nodes.
Proof: Let $P(n)$ be a statement "there exists an n-regular graph containing exactly 2^n nodes." We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.
For our base case, we need to show that $P(0)$ is true, meaning that there exists an 0-regular graph containing exactly $2^0 = 1$ node. This is true because a one-node graph contains only one node to which has no nodes adjacent.
For our inductive step, assume for some arbitrary natural number k that $P(k)$ is true, meaning that "there exists an k-regular graph containing exactly 2^k nodes." (1)
We need to prove $P(k+1)$, meaning that "there exists an $k+1$-regular graph containing exactly 2^{k+1} nodes." To do this, split 2^{k+1} nodes into two groups G_1 and G_2; each group has 2^k nodes. We know, by (1), that there is a way to connect the nodes in G_1 and G_2 such that G_1 and G_2 are k-regular graphs. This means that each node in both graphs has k adjacent nodes. Let $n_{11}, n_{12}, ..., n_{1k}$ be the nodes of G_1 and $n_{21}, n_{22}, ..., n_{2k}$ be the nodes of G_2. Consider a graph G by connecting n_{11} with n_{21}, n_{12} with n_{22}, ..., n_{1k} with n_{2k}. Notice that, since each node has one additional adjacent node, each node in graph G has $k+1$ adjacent nodes. Consequently, G is a $k+1$-regular graph. Therefore, $P(k+1)$ holds, completing the induction. ■

Problem Six: It'll All Even Out

i.

$P(n)$ is a valid predicate. However, the proof does not pass the rule: Choose the Simplest Base Cases Possible, and Avoid Redundant Base Cases. The proof misses $P(1)$ as the base case since $P(1)$ can't be built up from $P(0)$. It's actually the case that if $P(n)$ is true for all $n \in \mathbb{N}$, then the theorem in question is true. It is, nonetheless, not the case in this proof because the predicate is only true for $n = 0$. Therefore, the structure of this problem must be a claim and a disproof.

ii.

Yes, it's correct.

iii.

$P(1)$ is false. For example, choose $\{1\}$ as a collection, which has an odd sum, of one real number.

Problem Seven: Contractions

i.

Functions $2, 3$, and 4 are contractions.

ii.

2. $f^\star(137) = 136$
3. $f^\star(137) = 0$
4. $f^\star(137) = 7$

iii.

Theorem: If f is a contraction, then $f^\star$ is also a contraction.
Proof: Assume that f is a contraction, we need to show that $f^\star$ is also a contraction.
First, we will prove that $f^\star(0) = 0$. This is true because $f(0) = 0$.
Next, we will prove that for all $n \in \mathbb{N}$, if $n \geq 1$, then $f^\star(n) \leq n - 1$. Let $P(n)$ be the statement "$f^\star(n) \leq n - 1$". We will prove by induction that $P(n)$ holds for all $n \geq 1$.
For our base case we need to show that $P(1)$ is true. Notice, since f is a contraction, that $f(1) \leq 1 - 1 = 0$. We see that $f(1) = 0$ since the codomain of f is $\mathbb{N}$. By the definition of $f^\star$, we know that $f^\star(1) = 0 \leq 0$.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ such that $k \geq 1$ and $P(1), P(2), ..., P(k)$ are true, meaning that $f^\star(1) <= 0, f^\star(2) <= 1, ..., f^\star(k) <= k - 1$. We need to prove $P(k+1)$, meaning that $f^\star(k+1) <= k$. We know that $f^\star(k+1) = 1 + f^\star(f(k+1))$. Since $f(k+1) \leq k$, notice that

$$\begin{aligned} f^\star(k+1) &\leq \max\{1 + f^\star(0), 1 + f^\star(1), 1 + f^\star(2) ..., 1 + f^\star(k)\} \\ &\leq \max\{1 + 0, 1 + 0, 1 + 1, ..., 1 + k - 1\} \\ &= \max\{1, 1, 2, ..., k\} = k. \end{aligned}$$

Therefore, $P(k+1)$ holds, completing induction.