CS 103: Mathematical Foundations of Computing
Problem Set #4

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Due Friday, October 22 at 2:30 pm Pacific

Do not put your answer to Problem Two, part iii. in this file.
(The other parts of Problem Two ***do*** go in this file.)

Problem One: Independent and Dominating Sets

i.

$D = \{a, d, g\}$
$D = \{b, c, g\}$

ii.

Proof: Pick an arbitrary node $v \in I$. We need to show that there is a node $u \in V - I$ such that $\{u, v\} \in E$. Assume for the sake of contradiction that there is no such node in $V - I$ exists. Consequently, we see that v is a standalone node since I is an independent set in G. However, notice that every node in G is adjacent to at least one other node in G. We have reached a contradiction, so our assumption was wrong. This means that if I is an independent set in G, then $V - I$ is a dominating set in G, as required. ■

iii.

$I = \{b, g, d\}$
$J = \{a, f, c, h\}$

iv.

Proof: Assume for the sake of contradiction that if I is a maximal independent set in G, then I is not a dominating set of G. Therefore, there must be a node $v \in I$ such that for every node $u \in I$, $\{u, v\} \notin E$. Consequently, there must be an independent set I' such that $I' = I \cup \{v\}$. This means that $I \subsetneq I'$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, if I is a maximal independent set in $G = (V, E)$, then I is a dominating set of G, as required. ■

Problem Two: Linkage Graphs

i.

Proof: Assume for the sake of contradiction that there is a cycle of length four in G. Let four nodes in the cycle be x, y, z, and t, respectively. Since G is a linkage graph, assume, without loss of generality, that z is the one and only node such that $\{x, z\} \in E$ and $\{y, z\} \in E$. Notice, in this case, that y and t are the two intermediate nodes between x and z since $\{x, y\} \in E$ and $\{z, y\} \in E$, and $\{x, t\} \in E$ and $\{z, t\} \in E$. We have reached a contradiction, so our assumption must have been wrong. Therefore, if $G = (V, E)$ is a linkage graph, then G does not contain any cycles of length four, as required. ■

ii.

Proof: Let v be a random node in V. We want to show that there's a triangle that contains v.
Since G has more than one node, there must be a node u, other than v, in V. Since G is a linkage graph, there is a node z such that $\{v, z\} \in E$ and $\{u, z\} \in E$. Notice, also, that there must be a node k such that $\{v, k\} \in E$ and $\{z, k\} \in E$. Consequently, we see that $\{v, z\} \in E$, $\{v, k\} \in E$, and $\{z, k\} \in E$ construct a triangle containing v, as required. ■

Problem Three: Friends, Strangers, Enemies, and Hats

i.

Proof: We need to show that there always exists three mutual friends all wearing the same color hat or three mutual strangers all wearing the same color hat.
To do so, we can model this problem by representing each hat color as a bin and each person as a node. By the generalized pigeonhole principle, we know that at least one bin contains at least $\lceil \frac{36}{7} \rceil = 6$ nodes. Consider these 6 nodes as a 6-clique, as in the Theorem on Friends and Strangers, we know that we can always find three mutual friends or mutual strangers wearing the same color hat, as required. ■

ii.

Proof: From a graph-theoretic perspective, we need to show that any 17-clique whose edges are colored red, blue, and green contains a red triangle, a blue triangle, or a green triangle.
Let x be any node in the 17-clique. It is incident to 16 edges and there are three possible colors for those edges. Therefore, by the generalized pigeonhole theorem, at least $\lceil \frac{16}{3} \rceil = 6$ of those edges must be the same color. Without loss of generality, assume that those edges are red.
Let $y1$, $y2$, ..., $y6$ be six of the nodes adjacent to x along the red edges. If any of the edges from the 6-clique constructed by the six nodes are red, the one of those edges plus the two edges connecting back to node x form a red triangle. Otherwise, we leave with a 6-clique where each edge can only be blue or green. As in the previous problem, we know that that 6-clique contains a blue triangle or green triangle. Overall, this gives us a triangle with edges of the same color, as required. ■

Problem Four: Bipartite Graphs

i.

Let V_1 be a set of nodes of black squares and V_2 be a set of nodes of white squares. Notice that each edge can only connect two nodes of different colors. This means that every edge has one endpoint in V_1 and one endpoint in V_2. Therefore, this is a bipartite graph with V_1 and V_2 as the bipartite classes.

ii.

Suppose that there is a closed walk of odd length. Because G is a bipartite graph, any two adjacent nodes must be of different classes. However, notice, no matter how we color the nodes with two different colors, that there always exist two adjacent nodes of the same color. We have reached a contradiction, so our assumption must have been wrong. Therefore, every closed walk in G has to have even length.

iii.

Proof: Suppose for the sake of contradiction that there is a node y such that $y \in V_1$ and $y \in V_2$. Since $y \in V_1$, we know that there is an odd-length walk from y to x. Similarly, we know that there also is an even-length walk from y to x. This means there are two different walks from y to x, which forms an odd-length closed walk. By the previous part, we know that this is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, V_1 and V_2 have no nodes in common. ■

iv.

Proof: We need to show that G is bipartite.
First, we will prove that every node $v \in V$ belongs to exactly one of V_1 and V_2. Since G has just one connected component, we know that every node, other than x, in G and x are reachable. This means each node is either in V_1 or in V_2.
Next, we will prove that each edge $e \in E$ has one endpoint in V_1 and the other in V_2. Suppose for the sake of contradiction that there is an edge that has two endpoints in the same class. Let these two endpoints be y and z so that $\{y, z\} \in E$. Consider two cases:
Case 1: y and z in V_1. This means that there are an odd-length walk from x to y and an odd-length walk from x to z. Combining these two walks with the edge connecting y and z, we form an odd-length closed walk.
Case 1: y and z in V_2. This means that there are an even-length walk from x to y and an even-length walk from x to z. Combining these two walks with the edge connecting y and z, we form an odd-length closed walk.
Overall, we, in both cases, all form an odd-length closed walk. By the previous part, we know that these are impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, G is bipartite. ■

Problem Five: Iterated Injections

i.

Proof: We need to show the given sequence must contain at least one duplicate value. Notice that the sequence has $k+1$ elements where each element can be one of k different values. By the generalized pigeonhole theorem, we know that there are at least $\left\lceil \frac{k+1}{k} \right\rceil = 2$ elements sharing the same value, as required. ■

ii.

Proof: Assume for the sake of contradiction that $f^{n+1}(a) \neq a$. This mean that there is a natural number m such that $0 < m < n+1$ and $f^{n+1}(a) = f^m(a)$. Since f is injective, we know that $f^n(a) = f^{m-1}(a)$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, we know that $f^{n+1}(a) = a$, as required. ■

iii.

Proof: Assume for the sake of contradiction that f is not surjective. This means that there is a $b \in A$ such that for all $a \in A$, $f(a) \neq b$. Let $k = |A|$ and $a_1, a_2, ..., a_k$ be k different values in A. We see that $f(a_1), f(a_2), ..., f(a_k)$, which are k different functions, can evaluate, except b, to any of $k-1$ different values. By the generalized pigeonhole theorem, we know that there are at least $\left\lceil \frac{k}{k-1} \right\rceil$ two functions evaluate to the same value. Assume, without loss of generality, that these two functions are $f(a_1)$ and $f(a_2)$. Since f is injective, we know that $f(a_1) = f(a_2)$ means that $a_1 = a_2$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, f is surjective. Additionally, since f is also injective, f is a bijection, as required. ■