CS 103: Mathematical Foundations of Computing
Problem Set #6

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Due Friday, November 5 at 2:30 pm Pacific

Do not put your answers to Problems 1, 2, and 7 in this file. You'll submit those separately on Gradescope. Here's a quick reference of symbols you may want to use in this problem set.

- Alphabets are written as Σ.
- The set of all strings over Σ is denoted $\Sigma^\star$
- The empty string is written as ε. The "var" here refers to a "variant" of the letter epsilon; that's the one we use in this class.
- Subscripts are done as is q_{137}; superscripts are done as a^{137}.
- You can make text render `like a typewriter` in text mode or in $\mathtt{math}$ mode.
- The Greek letter ρ appears in Q6.
- The Greek letter δ appears in Q7.

Problem Three: $\wp(\Sigma^*)$

The set of all subsets of the set of all strings composed from letters in Σ.

Problem Four: Much Ado About Nothing, Part II

i.

Yes. For example, L is the set of palindromes over the alphabet.

ii.

Yes. For example, L is the set of strings s ending in a.

iii.

No. This is not defined because ε is a string, whereas L is a set.

iv.

No. This is not defined because ε is a string, whereas L is a set.

v.

No. This is not defined because ε is a string, whereas $\emptyset$ is a set.

vi.

No. Because $\emptyset$ is a set containing no element, whereas $\{\varepsilon\}$ is a set containing one element which is an empty string ε.

Problem Five: Hard Resets

Fill in the table and blank to Problem Five below.

state	a	b
$^\star\{q_s, q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
$^\star\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1, q_2\}$
$\{q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_2\}$
$^\star\{q_0, q_2\}$	$\{q_1, q_2\}$	$\{q_0, q_1\}$
$^\star\{q_0, q_1\}$	$\{q_1\}$	$\{q_1, q_2\}$
$\{q_1\}$	$\{q_1\}$	$\{q_2\}$
$\{q_2\}$	$\{q_2\}$	$\{q_0\}$
$^\star\{q_0\}$	$\{q_1\}$	$\{q_1\}$

Sample hard reset string: abbabb.

Problem Six: Induction on Strings

i. Fill in the blanks to Problem Six, part i. below.

1. $\rho(\mathsf{r}) = \mathsf{r}$.
2. $\rho(\mathsf{sr}) = \mathsf{rs}$.
3. $\rho(\mathsf{rrs}) = \mathsf{srr}$.
4. $\rho(\mathsf{r}^{100}\mathsf{s}^{100}) = \mathsf{s}^{100}\mathsf{r}^{100}$.

ii. Fill in the blanks to Problem Six, part ii. below.

Theorem: For any string $w \in \Sigma^*$, we have $|\rho(w)| = |w|$.
Proof: Let $P(w)$ be the statement "$|\rho(w)| = |w|$." We will prove by induction that $P(w)$ holds for all $w \in \Sigma^\star$, from which the theorem follows.
As a base case, we prove $P(\varepsilon)$, namely, that $|\rho(\varepsilon)| = |\varepsilon|$. This is true because we know, by definition, that $\rho(\varepsilon) = \varepsilon$.
For our inductive step, pick some $x \in \Sigma^*$ and $a \in \Sigma$ and assume $P(x)$ is true, meaning that $|\rho(x)| = |x|$. We need to show $P(xa)$ is true, that $|\rho(xa)| = |xa|$. To see this, note that

$$\begin{aligned} |\rho(xa)| &= |a\rho(x)| \\ &= |\rho(x)| + 1 \\ &= |x| + 1 && \text{(by our IH)} \\ &= |xa|. \end{aligned}$$

This means that $|\rho(xa)| = |xa|$, so $P(xa)$ holds, completing the induction. ■

iii.

Because they're equivalent. We've put the universal quantifier for all $w \in \Sigma^\star$ to make a predicate.

iv.

Theorem: For all strings $w, z \in \Sigma^\star$, we have $\rho(zw) = \rho(w)\rho(z)$.
Proof: Let $P(w)$ be the statement "for any $z \in \Sigma^\star$, we have $\rho(zw) = \rho(w)\rho(z)$" We will prove by induction that $P(w)$ holds for all $w \in \Sigma^\star$, from which the theorem follows.
As a base case, we prove $P(\varepsilon)$, namely, that for any $z \in \Sigma^\star$, we have $\rho(z\varepsilon) = \rho(\varepsilon)\rho(z)$. Pick an arbitrary $t \in \Sigma^\star$, we have to show that $\rho(t\varepsilon) = \rho(\varepsilon)\rho(t)$. This is true because $\rho(t\varepsilon) = \rho(t)$ and $\rho(\varepsilon)\rho(t) = \varepsilon\rho(t) = \rho(t)$.
For our inductive step, assume for some arbitrary $k \in \Sigma^\star$ that $P(k)$ is true, meaning that for any $z \in \Sigma^\star$, we have $\rho(zk) = \rho(k)\rho(z)$. Pick an arbitrary $a \in \Sigma$. We need to show $P(ka)$, meaning that for any $z \in \Sigma^\star$, we have $\rho(zka) = \rho(ka)\rho(z)$. Pick an arbitrary $u \in \Sigma^\star$. We have to show that $\rho(uka) = \rho(ka)\rho(u)$. Notice that

$$\begin{aligned} \rho(uka) &= a\rho(uk) \\ &= a\rho(k)\rho(u) \\ &= \rho(ka)\rho(u). \end{aligned}$$

Therefore, $P(ka)$ holds, completing the induction. ■

Problem Eight: Concatenation, Kleene Stars, and Complements

i.

Lemma: If L_1 and L_2 are nonempty, finite languages, then $|L_1 L_2| = |L_1||L_2|$.
Theorem: If L is nonempty, finite language and k is a positive natural number, then $|L|^k = |L^k|$
Proof: Pick an arbitrary nonempty, finite language L. Let $P(k)$ be a statement $|L|^k = |L^k|$. We will prove by induction that $P(k)$ holds for all natural numbers k, from which the theorem follows.
As our base case, we prove $P(0)$, meaning that $|L|^0 = |L^0|$. This is true because $|L|^0 = 1$ and $|L^0| = |\{\varepsilon\}| = 1$.
For our inductive step, assume for some arbitrary natural number t that $P(t)$ is true, meaning that $|L|^t = |L^t|$. We need to show $P(t+1)$, meaning that $|L|^{t+1} = |L^{t+1}|$. Notice that

$$\begin{aligned} |L|^{t+1} &= |L||L|^t \\ &= |L||L^t| \\ &= |LL^t| \\ &= |L^{t+1}|. \end{aligned}$$

Therefore, $P(t+1)$ holds, completing the induction. ■