CS 103: Mathematical Foundations of Computing
Final Exam

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Due Thursday, December 9 at 3:30 pm Pacific

Do not put your answers to Problems Five, Six, or Seven in this file - you'll submit your answers by downloading the starter files and submitting them separately.

Problem One: Square and Triangular Numbers (8 Points)

Theorem: For all natural numbers n, the number n is triangular if and only if $8n+1$ is a square number.

Proof: Pick an arbitrary natural number n. We will prove both directions of implication.
First, we will prove that if n is a triangular number, then $8n+1$ is a square number. Assume that n is a triangular number. We want to show that $8n+1$ is a square number. Since n is triangular, there is a natural number k such that $n = \frac{k(k+1)}{2}$. Notice that

$$\begin{aligned} 8n+1 &= \frac{8k(k+1)}{2} + 1 \\ &= 4k(k+1) + 1 \\ &= 4k^2 + 4k + 1 \\ &= (2k+1)^2. \end{aligned}$$

This means that there is a natural number m (namely, $2k+1$) such that $8n+1 = m^2$. Therefore, $8n+1$ is a square number, as required.
Next, we will prove that if $8n+1$ is a square number, then n is a triangular number. Assume that $8n+1$ is a square number. We want to show that n is a triangular number. Since $8n+1$ is a square number, there must be a natural number t such that $8n+1 = t^2$. Notice that t^2 is odd since $8n+1$ is odd. Therefore, we know, by the given theorem, that t is odd. This means that there is a natural number z such that $t = 2z+1$. We see that

$$\begin{aligned} 8n+1 &= t^2 \\ n &= \frac{t^2-1}{8} \\ &= \frac{(t-1)(t+1)}{8} \\ &= \frac{(2z+1-1)(2z+1+1)}{8} \\ &= \frac{(2z)(2z+2)}{8} \\ &= \frac{4z(z+1)}{8} \\ &= \frac{z(z+1)}{2}. \end{aligned}$$

This means that n is a triangular number, as required. Thus, the number n is triangular if and only if $8n+1$ is a square number. ■

Problem Two: Cantor's Theorem Revisited (8 Points)

Fill in the blanks to Problem Two below.

Theorem: For any set S, if $f : \wp(S) \to S$ is a function, then f is not injective.
Proof: Assume for the sake of a contradiction that f is injective. Now, define the set D as

$$D = \{x \in S \mid \exists T \in \wp(S).(x = f(T) \wedge x \notin T)\},$$

and let $y = f(D)$. We consider two cases:

- *Case 1:* $y \notin D$. Since $D \in \wp(S)$, we know, by definition of D, that $y \in D$. This contradicts the fact that $y \notin D$.
- *Case 2:* $y \in D$. By definition of D, we know that there must be a set $T \in \wp(S)$ such that $y = f(T)$ and $y \notin T$. This means that $f(D) = f(T)$ and $D \neq T$. This contradicts the fact that f is injective.

In either case, we reach a contradiction, so our assumption was wrong. Therefore, for any set S, if $f : \wp(S) \to S$ is a function, then f is not injective. ■

Problem Three: Multiplicative Graphs (8 Points)

i.

The size of the largest independent set of G_{10} is 5.

ii.

The size of the smallest dominating set of G_{137} is 1.

iii.

The smallest natural number n where G_n contains a 5-clique is 16.

iv.

Theorem: There is a natural number n where, for all natural numbers $k \geq n$, the graph G_k is not bipartite.

Proof: Pick a natural number $n = 4$.

Assume for the sake of contradiction that for all natural numbers $k \geq 4$, the graph G_k is bipartite. Let V_1 and V_2 be the two bipartite classes. Pick an arbitrary natural number k where $k \geq 4$. Since $k \geq 4$, there exist nodes $1, 2$, and 4 in the graph. Since $2 = 1 \cdot 2$, we know that $\{2, 1\} \in E$. This means that node 1 and node 2 belong to different bipartite classes. Assume, without loss of generality, that $1 \in V_1$ and $2 \in V_2$. Notice that $\{4, 1\} \in E$ and $\{4, 2\} \in E$ since $4 = 1 \cdot 4$ and $4 = 2 \cdot 2$. This means that $4 \in V_2$ and $4 \in V_1$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. This means that for all natural numbers $k \geq 4$, the graph G_k is not bipartite.

Therefore, there is a natural number n where, for all natural numbers $k \geq n$, the graph G_k is not bipartite. ■

Problem Four: Graph Strain (14 Points)

i.

The minimum number of edges strained is 0.

ii.

The maximum number of edges strained is 8.

iii.

Theorem: For all natural numbers n, there is a way to pick out two out of three colors such that the nodes having these colors are composed of at least $\frac{2n}{3}$ nodes.

Proof: Assume for the sake of contradiction that there is no way to pick out two out of three colors such that the nodes having these colors are composed of at least $\frac{2n}{3}$ nodes. Let C_1, C_2, and C_3 be the numbers of nodes that have the first color, the second color and the third color, respectively, where $C_1 + C_2 + C_3 = n$. Notice that $C_1 + C_2 < \frac{2n}{3}$, $C_2 + C_3 < \frac{2n}{3}$, and $C_3 + C_1 < \frac{2n}{3}$. We see that

$$\begin{aligned}(C_1 + C_2) + (C_2 + C_3) + (C_3 + C_1) < \frac{2n}{3} + \frac{2n}{3} + \frac{2n}{3} \\ 2(C_1 + C_2 + C_3) < 2n \\ C_1 + C_2 + C_3 < n.\end{aligned}$$

We have reached a contradiction, so our assumption must have been wrong. Therefore, there is a way to pick out two out of three colors such that the nodes having these colors are composed of at least $\frac{2n}{3}$ nodes.

Theorem: Let $G = (V, E)$ be a graph with m edges. Then there is a way to color the nodes of G that strains at least $\frac{2m}{3}$ edges of G.

Proof: Let $P(n)$ be the statement "for any natural number m and for any graph $G = (V, E)$ with n nodes and m edges, there is a way to color the nodes of G that strains at least $\frac{2m}{3}$ edges". We will prove by induction that $P(n)$ holds for all natural numbers n, from which the theorem follows.

As our base case, we will prove that $P(0)$ is true, meaning that for any natural number m and for any graph $G = (V, E)$ with 0 nodes and m edges, there is a way to color the nodes of G that strains at least $\frac{2m}{3}$ edges. Pick an arbitrary empty graph $G = (V, E)$ with 0 nodes and m edges. Notice that this is vacuously true since G is empty so that $m = 0$ and $\frac{2m}{3} = 0$.

For our inductive step, assume that for an arbitrary natural number k that $P(k)$ is true, meaning that for any natural number m and for any graph $G = (V, E)$ with k nodes and m edges, there is a way to color the nodes of G that strains at least $\frac{2m}{3}$ edges. (1)

We will prove $P(k + 1)$, meaning that for any natural number m and for any graph $G = (V, E)$ with $k + 1$ nodes and m edges, there is a way to color the nodes of G that strains at least $\frac{2m}{3}$ edges. Pick an arbitrary graph $G = (V, E)$ with $k+1$ nodes and m edges. We want to show that there is a way to color the nodes of G that strains at least $\frac{2m}{3}$ edges. To do so, split G into a set V_k containing k arbitrary nodes in G and a remaining node t. Let p be the number of edges of nodes in V_k and q be the number of edges connecting node t and nodes in V_k. Notice that $m = p + q$ and $0 \leq q \leq k$. By the above lemma, we know that there are at least $\frac{2p}{3}$ nodes having two out of three colors. Color node t with the remaining color. This means that $\frac{2p}{3}$ edges are strained. (2)

Additionally, we know, by (1), that there is a way to color the nodes that strains at least $\frac{2q}{3}$ edges connecting nodes in V_k. (3)

Combining (2) and (3), we see that there is a way to color the nodes of G that strains at least $\frac{2p}{3} + \frac{2q}{3} = \frac{2m}{3}$ edges. Therefore, $P(k + 1)$ holds, completing the induction. ■

Problem Eight: Uncomputable Functions (3 Points)

Fill in the blanks to Problem Eight below.

```
string trickster(string input) {
    string me = /* source code of trickster */;
    return precompute(me, input);
}
```