CS 103: Mathematical Foundations of Computing
Problem Set #1

Quang Nguyen

May 22, 2022

Due Friday, October 1 at 2:30 pm Pacific

Symbols Reference

Here are some symbols that may be useful for this PSet. If you are using LaTeX, view this section in the template file (the code in `cs103-ps1-template.tex`, not the PDF) and copy-paste math code snippets from the list below into your responses, as needed. If you are typing your Pset in another program such as Microsoft Word, you should be able to copy some of the symbols below from this PDF and paste them into your program. Unfortunately the symbols with a slash through them (for "not") and font formats such as exponents don't usually copy well from PDF, but you may be able to type them in your editor using its built-in tools.

- Empty set: $\emptyset$
- Power set: $\wp(S)$
- Set of Natural numbers: $\mathbb{N}$
- Union, intersection: $\cup$, $\cap$
- Equal, not equal: $x = x$, $x \neq y$
- Element-of, not element-of: $x \in S$, $y \notin S$
- Subset-of, not subset-of: $A \subseteq B$, $A \nsubseteq C$
- Symmetric difference: $S \triangle T$
- Modular congruence: $x \equiv_k y$

LaTeX typing tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problems One and Two are to be answered by editing the appropriate files (`MuchAdoAboutNothing.sets` and `SetTheory.cpp`, respectively). Do not put your answers to Problems One and Two in this file.

Problem Three: Describing the World in Set Theory

i.

$F \subseteq (A \cap B \cap C \cap D)$

ii.

$(A - B) \neq \emptyset$

iii.

$\wp(S \cup F) - (\wp(S) \cup \wp(F))$

Problem Four: Writing Direct Proofs

i.

Pick an arbitrary odd integer n.

ii.

We want to show that n^2 is odd.

iii. Fill in the blanks to Problem Four, part iii below.

Theorem: For all integers n, if n is odd, then n^2 is odd.
Proof: Pick an arbitrary odd integer n. We want to show that n^2 is odd. Since n is odd, there is an integer k where $n = 2k + 1$. Then we see that

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Therefore, there is an integer m (namely, $2k^2 + 2k$) such that $n = 2m + 1$, so n^2 is odd, as required. ■

iv.

We chose m here instead of k because k had been used before. We could have chosen k if k had not been used before.

Problem Five: Writing Proofs by Contrapositive

i.

For all integers a, b, and c, if a, b, and c are all odd, then $a^2 + b^2 \neq c^2$.

ii.

Pick any a, b, and c that are odd integers.

iii.

We want to show that $a^2 + b^2 \neq c^2$.

iv. Fill in the blanks to Problem Five, part iv. below.

Theorem: For all integers a, b, and c, if $a^2 + b^2 = c^2$, then at least one of a, b, and c is even.
Proof: We will prove the contrapositive of this statement, namely, that for all integers a, b, and c, if a, b, and c are odd, then $a^2 + b^2 \neq c^2$. To do so, pick any a, b, and c that are odd integers. We want to show that $a^2 + b^2 \neq c^2$.
Since a, b, and c are odd integers, we know by our result from the previous problem that a^2, b^2, and c^2 are odd.
Because a^2 and b^2 are odd, there exist integers p and q such that $a^2 = 2p + 1$ and $b^2 = 2q + 1$. This means that $a^2 + b^2 = (2p + 1) + (2q + 1) = 2(p + q + 1)$, which means that $a^2 + b^2$ is even. However, as mentioned earlier we know that c^2 is odd. Therefore, we see that $a^2 + b^2 \neq c^2$ as required.■

v.

Don't Repeat Definitions; Use Them Instead

Problem Six: Writing Proofs by Contradiction

i.

For all integers m and n, if mn is even and m is odd, then n is odd.

ii. Fill in the blanks to Problem Six, part ii. below.

Theorem: For all integers m and n, if mn is even and m is odd, then n is even.
Proof: Assume for the sake of contradiction that n is odd. Since m is odd, we know that there is an integer k where $m = 2k + 1$. Similarly, since n is odd, there is an integer r where $n = 2r + 1$. Then we see that

$$\begin{aligned} mn &= (2k + 1)(2r + 1) \\ &= 4kr + 2k + 2r + 1 \\ &= 2(kr + k + r) + 1 \end{aligned}$$

which means that mn is odd, but this is impossible because mn is even.
We have reached a contradiction, so our assumption must have been wrong. Therefore, if mn is even and m is odd, then n is even.■

Problem Seven: Proving Existentially-Quantified Statements

i. Fill in the blanks to Problem Seven, part i. below.

Theorem: There are real numbers a and b where $\lfloor a \rfloor \cdot \lceil b \rceil \neq \lfloor ab \rfloor$.
Proof: Pick $a = 1.9$ and $b = 1.9$. Then we see that

$$\lfloor a \rfloor \cdot \lceil b \rceil = \lfloor 1.9 \rfloor \cdot \lceil 1.9 \rceil = 1 \cdot 2 = 2,$$

but

$$\lfloor ab \rfloor = \lfloor 1.9 \cdot 1.9 \rfloor = \lfloor 3.61 \rfloor = 3.$$

Thus $2 \neq 3$, which means that $\lfloor a \rfloor \cdot \lceil b \rceil \neq \lfloor ab \rfloor$, as required. ■

ii. Fill in the blanks to Problem Seven, part ii. below.

Theorem: There exist natural numbers $a, b, c,$ and d such that $a > b > c > d > 0$ and $a^2 + b^2 + c^2 + d^2 = 137$.
Proof: Pick $a = 9$, $b = 6$, $c = 4$, and $d = 2$. Then we see that

$$9 > 6 > 4 > 2 > 0$$

and

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= 9^2 + 6^2 + 4^2 + 2^2 \\ &= 81 + 36 + 16 + 4 \\ &= 137, \end{aligned}$$

as required. ■

Problem Eight: Proving Mixed Universal and Existential Statements

i. Fill in the blanks to Problem Eight, part i. below.

Theorem: For all integers x and y and any integer k, if $x \equiv_k y$, then $y \equiv_k x$.
Proof: Let x, y, and k be arbitrary integers where $x \equiv_k y$. We want to show that $y \equiv_k x$. To do so, we will show that there is an integer q where $y = x + qk$. Because $x \equiv_k y$, we know there is an integer r such that $x = y + rk$.
Now, let $q = -r$. Then we see that

$$\begin{aligned} y &= x - rk \\ &= x + (-r)k \\ &= x + qk \end{aligned}$$

which is what we needed to show. ■

ii.

Theorem: For all integers x, y, z and k, if $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
Proof: Let x, y, z and k be arbitrary integers where $x \equiv_k y$ and $y \equiv_k z$. We want to show that $x \equiv_k z$.
Since $x \equiv_k y$, we know there is an integer 1 such that

$$x = y + qk. \qquad (1)$$

Similarly, because $y \equiv_k z$ there must be some integer r such that

$$y = z + rk. \qquad (2)$$

By plugging equation (2) into equation (1), we learn that:

$$\begin{aligned} x &= y + qk \\ &= z + rk + qk \\ &= z + (r + q)k. \qquad (3) \end{aligned}$$

Equation (3) tells us that there is an integer s (namely, $r + q$) such that $x = z + sk$. This means that $x \equiv_k z$, as required. ■

iii.

Theorem: For all integers x and k, $x \equiv_k x$
Proof: Pick arbitrary integers x and k. We want to show that $x \equiv_k x$. We see that:

$$\begin{aligned} x &= x \\ &= x + 0k. \end{aligned}$$

Therefore, there is an integer q (namely, 0) such that $x = x + qk$, which means that $x \equiv_k x$, as required. ■

Problem Nine: Proof Critiques

i.

This proof doesn't clearly articulate the assumption and want-to-show, which makes it harder for readers to follow. The first sentence of the proof sets the wrong goal. The purpose of the proof is to explain the theorem is true, not about adding together different kinds of numbers. The following sentences seriously violate the second rule and the third rule (Make Each Sentence Load-Bearing and Scope and Properly Introduce Variables). Because of not precisely articulating what each variable stands for and where it comes from, the writer mixes up the variable k where it's used for both variables m and n, which leads to false reasoning. Overall, this proof is not acceptable.

ii.

The writer has a problem with distinguishing between proofs and disproofs. The given argument is a claim, not a theorem so we have to write a disproof, not a proof. Furthermore, the reasoning part also has problems. The number 137 comes out of nowhere, which violates the rule Scope and Properly Introduce Variables. Overall, the reasoning is cumbersome and not straight to the point.

iii.

This proof has several problems. First, the contrapositive is wrongly constructed. The right contrapositive is that if n is an integer and $n \not\equiv_4 0$, then $n \not\equiv_2 0$. Next, the given argument is a claim, not a theorem so the writer has to lay out a disproof, not a proof.

Problem Ten: Multiples of Three

Theorem: For all integers n, n is a multiple of three if and only if n^2 is a multiple of three.
Proof: Pick an arbitrary integer n. We want to show that n is a multiple of three if and only if n^2 is a multiple of three. To do so, we will prove both directions of implication.
First, we will prove that if n is a multiple of three, then n^2 is a multiple of three. Since n is a multiple of three, there must be an integer k such that $n = 3k$. We see that

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \\ &= 3(3k^2). \end{aligned}$$

Therefore, there is an integer m (namely, $3k^2$) such that $n^2 = 3m$, which means that n^2 is a multiple of three, as required.
Next, we will prove that if n^2 is a multiple of three, then n is a multiple of three. To do so, we will instead prove the contrapositive, that if n is not a multiple of three, then n^2 is not a multiple of three. We consider two cases:
Case 1: n is congruent to one modulo three. This means that there is an integer r such that $n = 3r + 1$. We see that

$$\begin{aligned} n^2 &= (3r + 1)^2 \\ &= 9r^2 + 6r + 1 \\ &= 3(3r^2 + 2r) + 1 \end{aligned}$$

which means there is an integer z (namely, $3r^2 + 2r$) that $n^2 = 3r + 1$. Therefore, n^2 is not a multiple of three.
Case 2: n is congruent to two modulo three. In that case, there must be an integer q such that $n = 3q + 2$. We see that

$$\begin{aligned} n^2 &= (3q + 2)^2 \\ &= 9q^2 + 12q + 4 \\ &= 3(3q^2 + 6q + 1) + 1 \end{aligned}$$

which means that there is an integer u (namely, $3q^2 + 6q + 1$) that $n^2 = 3u + 1$. Therefore, n^2 is not a multiple of three.
In both cases, we see that n^2 is not a multiple of three, which is what we need to show.
In conclusion, we see that n is a multiple of three if and only if n^2 is a multiple of three, as required. ■

Problem Eleven: Tiling a Checkerboard

i.

Theorem: It is impossible to tile an 8 x 8 checkerboard missing two opposite corners with right triominoes.
Proof: Assume for the sake of contradiction that it is possible to tile an 8 x 8 checkerboard missing two opposite corners with right triominoes.
Consider n is the number of squares of the checkerboard. Since the checkerboard misses two opposite corners, we know that $n = 8*8-2 = 62$ squares. Because the checkerboard is tiled with triominoes which have three squares each, we see that $n \equiv_3 0$. However, notice that $62 \equiv_3 2$, which means that $n \not\equiv_3 0$.
We have reached a contradiction, so our assumption must have been wrong. Therefore, it is impossible to tile an 8 x 8 checkerboard missing two opposite corners with right triominoes. ■

ii.

Theorem: For an arbitrary integer n where $n \geq 3$, it is impossible to tile an n x n checkerboard missing two opposite corners with right triominoes.
Proof: Assume for the sake of contradiction that for an arbitrary integer n where $n \geq 3$, it is possible to tile an n x n checkerboard missing two opposite corners with right triominoes.
Consider m is the number of squares of the checkerboard. Since the checkerboard misses two opposite corners, we know that $m = n^2 - 2$ squares. Because the checkerboard is tiled with triominoes which have three squares each, we see that $m \equiv_3 0$. Notice that $n^2 \equiv_3 0$ or $n^2 \equiv_3 1$, which means that $n^2 - 2 \equiv_3 1$ or $n^2 - 2 \equiv_3 2$. Therefore, we can see that $n^2 - 2 \not\equiv_3 0$, which means that $m \not\equiv_3 0$.
We have reached a contradiction, so our assumption must have been wrong. Therefore, it is impossible to tile an n x n checkerboard missing two opposite corners with right triominoes, as required. ■