CS 103: Mathematical Foundations of Computing
Midterm Exam II

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Due Sunday, November 7 at 2:30 pm Pacific

Do not put your answer to Problem Two, part i. in this file.

Problem One: Idempotent Contractions (8 Points)

Theorem: If $f : \mathbb{N} \to \mathbb{N}$ is an idempotent contraction, then $f(n) = 0$ for all $n \in \mathbb{N}$.
Proof: Assume $f : \mathbb{N} \to \mathbb{N}$ is an idempotent contraction. Let $P(n)$ be the statement "$f(n) = 0$". We will prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$.
As our base case, we will prove $P(0)$ and $P(1)$ are true. First, we prove $P(0)$, meaning that $f(0) = 0$. This is true because f is a contraction. Next, we prove $P(1)$, meaning that $f(1) = 0$. Since f is an idempotent contraction, we know that $f(f(1)) = f(1)$ and $f(1) \leq 0$. This means that $f(f(1)) \in \{f(0)\} = \{0\}$. Notice that $f(f(1)) = 0$ since $f(f(1))$ is in the set containing all zero elements. Therefore, $f(1) = f(f(1)) = 0$, which is what we want to show.
For our inductive step, assume for some arbitrary $k \in \mathbb{N}$ that $P(0), P(1), ..., P(k)$ holds, meaning that $f(0) = 0, f(1) = 0, ..., f(k) = 0$. We will prove $P(k+1)$, meaning that $f(k+1) = 0$. Since f is an idempotent contraction, we know that $f(f(k+1)) = f(k+1)$ and $f(k+1) \leq k$. This means that $f(f(k+1)) \in \{f(f(0)), f(f(1)), ..., f(f(k))\} = \{f(0)\} = \{0\}$. We see that $f(f(k+1)) = 0$ since $f(f(k+1))$ is in the set containing all zero elements. Therefore, $P(k+1)$ holds, completing induction. ■

Problem Two: Pantone Graphs (10 Points)

ii.

Theorem: If $G = (V, E)$ is a pantone graph with exactly $3n$ nodes, then there exists a dominating set of G containing at most n nodes.

Proof: Assume that $G = (V, E)$ is a pantone graph with exactly $3n$ nodes. We need to show that there exists a dominating set of G containing at most n nodes.

To do so, we can model this problem by representing each color as a bin. There are three possible colors for each node. By the generalized pigeonhole theorem, some bin has at most $\lfloor \frac{3n}{3} \rfloor = n$ nodes in it. Assume, without loss of generality, that this bin is ecru and the two other bins are puce and zomp. Let the set of all nodes in the ecru, puce, and zomp bins be C, P, and Z, respectively. We will show that C is a dominating set in G. Pick an arbitrary node v such that $v \in P$ or $v \in Z$. Since G is a pantone graph, we know that there is a node $u \in C$ such that $\{u, v\} \in E$.

Therefore, there exists a dominating set of G containing at most n nodes, as required. ■

Problem Three: Pushes and Pulls (12 Points)

i. Fill in the blanks to Problem Three, part i. below.

$$p(v) = \begin{cases} \{0,1\} & \text{if } v = 0 \\ \{1,2\} & \text{if } v = 1 \\ \{2,0\} & \text{if } v = 2 \\ \{3,1\} & \text{if } v = 3 \end{cases} \qquad q(e) = \begin{cases} 0 & \text{if } e = \{0,1\} \\ 1 & \text{if } e = \{1,2\} \\ 2 & \text{if } e = \{2,0\} \\ 3 & \text{if } e = \{3,1\} \end{cases}$$

ii.

Theorem: If q balances p, then p is injective.
Proof: Assume that q balances p. We want to show that p is injective. To do so, pick arbitrary $v_1 \in V$ and $v_2 \in V$ such that $p(v_1) = p(v_2)$. We will prove that $v_1 = v_2$. Since q balances p, we know that $q(p(v_1)) = v_1$ and $q(p(v_2)) = v_2$. Notice that

$$\begin{aligned} p(v_1) &= p(v_2) \\ q(p(v_1)) &= q(p(v_2)) \\ v_1 &= v_2. \end{aligned}$$

Therefore, p is injective, as required. ■

iii.

Proof: Pick arbitrary $u \in V$ and $v \in V$ such that $\{u, v\} \in E$. We want to show that $u \in C$ or $v \in C$. Assume for the sake of contradiction that $u \notin C$ and $v \notin C$. Notice that $q(\{u, v\}) \in \{u, v\}$ since q is a pull of G. Assume, without loss of generality, that $q(\{u, v\}) = u$. This means that $u \in C$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, C is a vertex cover of G, as required. ■