CS 103: Mathematical Foundations of Computing
Problem Set #2

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Due Friday, October 8 at 2:30 pm Pacific

Problems One through Six are to be answered by editing the appropriate files (see the Problem Set #2 instructions). You won't include your answers to those problems here.

Symbols Reference

Here are some symbols that may be useful for this PSet. If you are using LaTeX, view this section in the template file (the code in `cs103-ps2-template.tex`, not the PDF) and copy-paste math code snippets from the list below into your responses, as needed. If you are typing your Pset in another program such as Microsoft Word, you should be able to copy some of the symbols below from this PDF and paste them into your program. Unfortunately the symbols with a slash through them (for "not") and font formats such as exponents don't usually copy well from PDF, but you may be able to type them in your editor using its built-in tools.

- Logical AND: $\wedge$
- Logical OR: $\vee$
- Logical NOT: $\neg$
- Logical implies: $\to$
- Logical biconditional: $\leftrightarrow$
- Logical TRUE: $\top$
- Logical FALSE: $\bot$
- Universal quantifier: $\forall$
- Existential quantifier: $\exists$

LaTeXtyping tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem Seven: Never Never Land

i.

$\text{N}x.(A(x) \land \neg B(x))$

ii.

$\text{N}x.(\neg A(x) \lor \neg B(x))$

iii.

$\text{N}x.(A(x) \land B(x))$

iv.

$\text{N}x.(\neg A(x) \lor B(x))$

Problem Eight: All Squared Away

i. Fill in the blanks for Problem Eight, part i. below.

- When $n = 3$, we can write $n = 2 \cdot 1 + 1$, and one choice of m that works is 1.

- When $n = 5$, we can write $n = 2 \cdot 2 + 1$, and one choice of m that works is 4.

- When $n = 7$, we can write $n = 2 \cdot 3 + 1$, and one choice of m that works is 9.

- When $n = 9$, we can write $n = 2 \cdot 4 + 1$, and one choice of m that works is 16.

- When $n = 11$, we can write $n = 2 \cdot 5 + 1$, and one choice of m that works is 25.

ii. Fill in the blank for Problem Eight, part ii. below.

When $n = 2k + 1$, pick $m = k^2$.

iii.

Theorem: If n is an odd natural number and $n \geq 3$, then there is an $m \in \mathbb{N}$ where $m > 0$ and $m^2 + mn$ is a perfect square.

Proof: Pick an arbitrary odd natural number n where $n \geq 3$. We want to show that there is an $m \in \mathbb{N}$ where $m > 0$ and $m^2 + mn$ is a perfect square.

Since n is an odd natural number and $n \geq 3$, there must be an integer k such that $k \geq 1$ and $n = 2k + 1$. Notice that $k^2 \in \mathbb{N}$ and $k^2 > 0$. Therefore, choose $m = k^2$, we can see that

$$\begin{aligned} m^2 + mn &= m(m + n) \\ &= k^2(k^2 + 2k + 1) \\ &= k^2(k + 1)^2 \\ &= (k(k + 1))^2 \\ &= (k^2 + k)^2. \end{aligned}$$

This means that there is an integer t, namely, $k^2 + k$, that $m^2 + mn = t^2$. Therefore, $m^2 + mn$ is a perfect square, as required. ■

Problem Nine: Yablo's Paradox

i.

Theorem: There does not exist a natural number n where statement (S_n) is true.
Proof: Assume for the sake of contradiction that there exists a natural number n where statement (S_n) is true. Then every statement that comes after it must be false. In particular, this means that statement (S_{n+1}) must be false. Since statement (S_{n+1}) states that every statement after it is false, there must be some true statement after statement (S_{n+1}); let's call that statement (S_m) where $m > n + 1$. But since statement (S_n) is true, this means that statement (S_m) which comes after it must be false. Therefore, statement (S_m) is both true and false, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, there does not exist a natural number n where statement (S_n) is true. ■

ii.

Theorem: There does not exist a natural number n where statement (S_n) is false.
Proof: Assume for the sake of contradiction that there exists a natural number n where statement (S_n) is false. Then there exists a true statement coming after statement (S_n); let's call that statement (S_m) where $m > n$. We, nonetheless, know by our result from the previous problem that statement (S_m) can't be true. We have reached a contradiction, so our assumption must have been wrong. Therefore, there does not exist a natural number n where statement (S_n) is false. ■

iii.

Let the first 9,999,999,999 statements be false and the last statement $(T_{9,999,999,999})$ be true. Since the first 9,999,999,999 statements are false, each statement has a true statement coming after it. Since statement $(T_{9,999,999,999})$ comes after all the first 9,999,999,999 statements and is vacuously true, this choice is consistent with one another.

Problem Ten: Tournament Champions

i.

Player D is not a tournament champion since D didn't win against C, nor is there any player who D won against (A and E) who in turn won against C. In contrast, although E only won against B, B, however, won against A, C, and D. Therefore, E is a tournament champion.

ii.

Theorem: If c won more games than anyone else in T or is tied for winning the greatest number of games, then c is a tournament champion in T.

Proof: Assume for the sake of contradiction that if c won more games than anyone else in T or is tied for winning the greatest number of games, then c is not a tournament champion in T. Since c is not a tournament champion, there must be a player x that c did not win against or there doesn't exist a player where c won the game against and this player won the game against x. Therefore, x must have won against c and all the players that were defeated by c. This means that x won more games than c, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, c is a tournament champion in T. ■