CS 103: Mathematical Foundations of Computing
Midterm Exam I

Quang Nguyen

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Due Sunday, October 17 at 2:30 pm Pacific

Please submit your answers to Problem Two (Translating Into Logic) separately from this PDF.

Symbols Reference

Here are some symbols that may be useful for this midterm.

- Floor: $\lfloor x \rfloor$
- Ceiling: $\lceil x \rceil$
- Less-than-or-equal-to and greater-than-or-equal-to: $x \leq y$ and $y \geq x$
- Natural numbers: $\mathbb{N}$
- Integers: $\mathbb{Z}$

LaTeXtyping tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem One: Quarter-Squares (8 Points)

Proof: We will prove that for all natural numbers n, if $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ is odd, then n is even. To do so, we will instead prove the contrapositive, that for all natural numbers n, if n is odd, then $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor$ is even. Pick an arbitrary odd natural number x. We want to show that $\lceil \frac{x}{2} \rceil \lfloor \frac{x}{2} \rfloor$ is even. Since x is odd, there must be a natural number k such that $x = 2k + 1$. We see that

$$\begin{aligned}\left\lceil \frac{x}{2} \right\rceil \left\lfloor \frac{x}{2} \right\rfloor &= \left\lceil \frac{2k+1}{2} \right\rceil \left\lfloor \frac{2k+1}{2} \right\rfloor \\ &= \lceil k + 0.5 \rceil \lfloor k + 0.5 \rfloor \\ &= (k+1)k.\end{aligned}$$

Notice that k and $k+1$ have opposite parity, so we know that $(k+1)k$ is even. This means that $\lceil \frac{x}{2} \rceil \lfloor \frac{x}{2} \rfloor$ is even, as required. ■

Problem Three: Tower of Power (4 Points)

i.

$n = 2$

ii.

$n = 3$

iii.

$n = 5$

iv.

$n = 6$

Problem Four: A Clever Little Equation (8 Points)

i.

Proof: Assume for the sake of contradiction that m, n, and p are all greater than zero. Consequently, we see that $m \geq 1, n \geq 1$, and $p \geq 1$. Notice that

$$(1+\frac{1}{m+1})(1+\frac{1}{n+1})(1+\frac{1}{p+1}) \leq (1+\frac{1}{2})(1+\frac{1}{2})(1+\frac{1}{2})$$
$$\leq \frac{27}{8} < 5.$$

However, we know that $(1+\frac{1}{m+1})(1+\frac{1}{n+1})(1+\frac{1}{p+1}) = 5$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, at least one of m, n, and p is equal to zero. ■

ii.

Proof: Assume for the sake of contradiction that there is at most one of m, n, and p is equal to zero. We know, by the previous proof, that at least of m, n, and p is equal to zero. Assume, without loss of generality, that $m = 0$, $n \geq 1$, and $p \geq 1$. We see that

$$(1+\frac{1}{m+1})(1+\frac{1}{n+1})(1+\frac{1}{p+1}) \leq (1+\frac{1}{1})(1+\frac{1}{2})(1+\frac{1}{2})$$
$$\leq \frac{9}{2} < 5.$$

However, we know that $(1+\frac{1}{m+1})(1+\frac{1}{n+1})(1+\frac{1}{p+1}) = 5$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, at least two of m, n, and p are equal to zero. ■

iii.

$m = 0, n = 0, p = 3$
$m = 0, n = 3, p = 0$
$m = 3, n = 0, p = 0$