CS 103: Mathematical Foundations of Computing
Problem Set #3

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Due Friday, October 15 at 2:30 pm Pacific

Problem One is autograded. You won't include your answers to that problem here.

Symbols Reference

Here are some symbols that may be useful for this problem set. If you are using LaTeX, view this section in the template file (the code in `ps3-latex-template.tex`, not the PDF) and copy-paste math code snippets from the list below into your responses, as needed. If you are typing your problem set in another program such as Microsoft Word, you should be able to copy some of the symbols below from this PDF and paste them into your program. Unfortunately the symbols with a slash through them (for "not") and font formats such as exponents don't usually copy well from PDF, but you may be able to type them in your editor using its built-in tools.

- f is a function from A to B: $f : A \to B$.
- $\clubsuit$ and $\heartsuit$ are needed for Problem Three.

LaTeXtyping tips:

- Set (curly braces need an escape character backslash): $1, 2, 3$ (incorrect), $\{1, 2, 3\}$ (correct)
- Exponents (use curly braces if exponent is more than 1 character): x^2, 2^{3x}
- Subscripts (use curly braces if subscript is more than 1 character): x_0, x_{10}

Problem Two: Iterated Functions, Part One

i.

All functions evaluate to the same value of 0.739085.

ii. Fill in the blanks to Problem Two, part ii. below.

1. $f^3(2) = 9$
2. $f^{137}(1) = 1$
3. $f^0(137) = 137$

iii.

All functions evaluate to the same value of 0.5.

iv.

All functions evaluate to the same value of 0.636364.

v.

All functions evaluate the values of either 0.495265 or 0.812427.

vi.

All functions evaluate the values of 0.382820, 0.500884, or 0.826941.

vii.

The values evaluated by functions seem random.

viii.

The values evaluated by functions seem random.

Problem Three: Infinity Times Two

i. Fill in the blank to Problem Three, part i. below.

$$f(n, x) = \begin{cases} 2n, & \text{if } x = \heartsuit \\ 2n+1, & \text{if } x = \clubsuit \end{cases}$$

ii.

Proof: Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be arbitrary monotone functions. We will prove that $g \circ f$ is monotone. To do so, pick arbitrary integers x and y where $x < y$. We want to show that $g \circ f(x) < g \circ f(y)$.
Notice that $f(x) < f(y)$ since f is monotone. Consequently, we know that $g(f(x)) < g(f(y))$ since g is monotone. This means that $g \circ f(x) < g \circ f(y)$, as required. ■

iii.

Proof: Let $f : \mathbb{Z} \to \mathbb{Z}$ be an arbitrary monotone function. We will prove that f is injective. To do so, pick arbitrary integers x and y where $x \neq y$. We want to show that $f(x) \neq f(y)$.
Assume, without loss of generality, that $x < y$. Since f is monotone, we know that $f(x) < f(y)$. This means that $f(x) \neq f(y)$, as required. ■

iv.

Proof: Assume for the sake of contradiction that $x \neq y$. Without loss of generality, assume that $x < y$. Since f is monotone, we see that

$$f(x) < f(y)$$
$$y < x.$$

This means that both $x < y$ and $y < x$ at the same time, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Therefore, if $f(x) = y$ and $f(y) = x$, then $x = y$, as required. ■

Problem Five: Idempotent Functions

i.

Functions 1, 3 and 4 are idempotent.

ii.

Functions 1, 3 and 4 are idempotent.

iii.

Proof: Pick an arbitrary $x \in A$. We need to show that $f(x) = x$. Notice that $f(f(x)) = f(x)$ since f is idempotent. Consequently, since f is injective, we know that $f(x) = x$, as required. ■

iv.

Proof: Pick an arbitrary $x \in A$. We need to show that $f(x) = x$. Since f is surjective, we know that there must be some $k \in A$ such that $f(k) = x$. Since $k \in A$ and f is idempotent, notice that $f(f(k)) = f(k)$. We see that

$$\begin{aligned} f(x) &= f(f(k)) \\ &= f(k) \\ &= x. \end{aligned}$$

This means that $f(x) = x$, as required. ■

Problem Six: Understanding Diagonalization

i.

$D = \mathbb{N}$

ii.

$S = \{1\} \neq \emptyset = f(n)$

iii.

$D = \emptyset$

iv.

$S = \{1\}$
This set isn't equal to any set produced by $f(n)$ since every set produced by $f(n)$ has a cardinality of $\aleph_0$ whereas the cardinality of S is 1.

v.

$f(n) = n$
$D \neq f(n)$ because $|D| = \aleph_0 \neq 1 = |f(n)|$.

Problem Seven: Simplifying Cantor's Theorem?

This proof has chosen a specific function $f(x) = x$ to prove, which is not an appropriate structure. Because the statement is universally-quantified, the writer must let the reader pick an arbitrary function to prove, not a particular function chosen by the writer.

Problem Eight: Iterated Functions, Part Two

i.

Proof: Pick arbitrary $x_1 \in A$ and $x_2 \in A$ such that $f(x1) = f(x2)$. We need to show that $x1 = x2$. Notice that

$$\begin{aligned} f(x1) = f(x2) &\implies f(f(x1)) = f(f(x2)) \\ &\implies f(f(f(x1))) = f(f(f(x2))). \end{aligned}$$

Since f^3 is injective, we know that $x1 = x2$, as required. ■

ii.

Proof: Pick an arbitrary $x \in A$. We need to show that there is a $k \in A$ such that $f(k) = x$. Notice that there is a $z \in A$ such that $f(f(f(z))) = x$ since f^3 is surjective. Pick $k = f(f(z))$, we see that

$$f(f(f(z))) = x \iff f(k) = x.$$

This means that f is surjective, as required. ■