Homework 0B

1. Counting solutions:

(a) $\begin{cases} 2x + 3y = 5 \\ x + y = 2 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

The system has a unique sol $x=1, y=1$.

(b) $\begin{cases} x + y + z = 3 \\ 2x + 2y + 2z = 5 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row implies contradiction $\to$ No solution.

(c) $\begin{cases} -y + 2z = 1 \\ 2x + z = 2 \end{cases}$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

choose $z \to$ solve for x, y Infinite sols

$\begin{cases} x = 1 - z/2 \\ y = -1 + 2z \end{cases}$

One sol: $x=1, y=-1, z=0$

(d) $\begin{cases} x + 2y = 3 \\ 2x - y = 1 \\ 3x + y = 4 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The system has a unique sol $x=1, y=1$.

(e) $\begin{cases} x + 2y = 3 \\ 2x - y = 1 \\ x - 3y = -5 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & 5 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

The last row implies contradiction $\to$ No solution.

2. Skider's Optimal Boba

(a) Let rating of: $\begin{cases} \text{Black} - x \\ \text{Oolong} - y \\ \text{Roasted} - z \\ \text{Earl Grey} - t \end{cases}$

We have

$$\begin{cases} \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}t = 7 \\ \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z = 7 \\ \frac{2}{5}y + \frac{3}{5}z = 7\frac{2}{5} \\ \frac{2}{3}x + \frac{1}{3}y = 6\frac{1}{3} \end{cases} \Rightarrow \begin{cases} x = 7 \\ y = 5 \\ z = 9 \\ t = 9 \end{cases}$$

Thus Professor Kuo's ratings for each tea are:

Black: 7, Oolong: 5, Green: 9, Earl Grey: 9

(b) From the result got from part (a), we can see the Professor Kuo's preference are Green and Earl Grey. So a mystery tea could be any combination of these two pure teas. For example: 1/2 Green and 1/2 Earl Grey.

3. Filtering out the troll

(a)
$$\begin{cases} \cos 45^\circ \vec{a} + \cos(-30^\circ)\vec{b} = \vec{m_1} \\ \sin 45^\circ \vec{a} + \sin(-30^\circ)\vec{b} = \vec{m_2} \end{cases}$$

$$\Leftrightarrow \begin{cases} \sqrt{2}/2 \, \vec{a} + \sqrt{3}/2 \, \vec{b} = \vec{m_1} \quad (1) \\ \sqrt{2}/2 \, \vec{a} + -1/2 \, \vec{b} = \vec{m_2} \quad (2) \end{cases}$$

(b) Multiply (2) with $\sqrt{3}$, we have:

$$\frac{\sqrt{6}}{2}\vec{a} - \frac{\sqrt{3}}{2}\vec{b} = \sqrt{3}\,\vec{m_2} \quad (3)$$

Add (1) and (3):

$$\frac{\sqrt{2}(\sqrt{3}+1)}{2}\vec{a} = \vec{m_1} + \sqrt{3}\,\vec{m_2}$$

$$\Rightarrow \quad \vec{a} = \frac{\sqrt{6}-\sqrt{2}}{2}\vec{m_1} + \frac{\sqrt{6}+3\sqrt{2}}{2}\vec{m_2}$$

$$u = \frac{\sqrt{6}-\sqrt{2}}{2}, \quad v = \frac{\sqrt{6}+3\sqrt{2}}{2}$$

(c) All human beings are born free and equal in dignity and rights.

4. Fountain codes

(a) Example: $\vec{r} = \begin{bmatrix} * \\ * \\ * \\ * \\ b \\ c \end{bmatrix}$

(b) $G_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ * \\ * \\ 3 \\ 4 \\ * \\ * \end{bmatrix}$

$$\begin{cases} a = 7 \\ a + b = 3 \\ a + c = 4 \end{cases} \Rightarrow \begin{cases} a = 7 \\ b = -4 \\ c = -3 \end{cases}$$

(d) Three uncorrupted symbols : Yes
Four uncorrupted symbols : No.
Example : Row 1, 2, 4 in G_F don't contain 1 in c
$\Rightarrow$ Infinte sols.

(e) G_F is better because G_F has maximum 3 uncorrupted symbols to ensure that the original information is intact meanwhile G_R just has 2.