Homework 2B

1. Traffic flows

(a) $t_1 = 10 \Rightarrow \begin{cases} t_2 = 10 \\ t_3 = -10 \end{cases}$

(b) $\begin{cases} t_3 + t_1 - t_4 = 0 \\ t_4 - t_5 = 0 \\ t_5 - t_3 - t_2 = 0 \\ t_2 - t_1 = 0 \end{cases}$

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad GE: \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Free variable: t_3, t_5 → it's possible to determine all traffic flows with the Berkeley student's suggestion.
It's impossible with the Stanford student's suggestion because t_3, t_4, t_5 are lin. dep.

(d) $N(B) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ $\quad \dim(N(B)) = 2$

(e) $M_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

(f) No. We have to measure the "free-variable" flows along k roads.
Counter example: Stanford student's suggestion.

(g) No. If $(\vec{u} - \vec{t})$ is in the null space of M_s, it means $M_s(\vec{u} - \vec{t}) = \vec{0}$.
In other words, $M_s\vec{u} = M_s\vec{t}$. This means that two different flows

will give us the same measurement, so the true flow can not be recovered.

2. Noisy Images

(a) $\vec{s} = H\vec{i} + \vec{w}$

$\Rightarrow \vec{i} = H^{-1}(\vec{s} - \vec{w})$

(b) We have
$$H\vec{v} = \lambda \vec{v}$$
$$H^{-1}H\vec{v} = \lambda H^{-1}\vec{v}$$
$$I\vec{v} = \lambda H^{-1}\vec{v}$$
$$\frac{1}{\lambda}\vec{v} = H^{-1}\vec{v} \quad (\lambda \neq 0 \text{ due to } H \text{ is invertible})$$

(c) Matrix H_1 performs best in reconstructing original image. H_1 is the identity matrix. As the abs value of the smallest eigenvalue of H decreases, the absolute value of largest eigen value of H^{-1} increases, hence the noise in the result increases.

(d) For eigenvectors with large eigenvalues, the noise signal will be attenuated. On the other hand, the noise signal will be amplified for eigenvectors with small eigenvalues.

3. The Dynamics of Romeo and Juliet's love affair

(a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad a+b = c+d$

(i) $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\rightarrow \lambda_1 = a+b$

$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We have: $\lambda_1 + \lambda_2 = a+d$

$a+b+\lambda_2 = a+d$

$\lambda_2 = d-b$

$[A - \lambda_2 I | 0] = \left[\begin{array}{cc|c} c & b & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v_2} = \begin{bmatrix} b \\ -c \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$

(i) $(\lambda - 0.75)^2 - 0.25^2 = 0$

$(\lambda - 1)(\lambda - 0.5) = 0$

$\lambda_1 = 1, \ \lambda_2 = 0.5$

Case 1: $\lambda_1 = 1$

$[A - \lambda_1 I \mid 0] = \left[\begin{array}{cc|c} -0.25 & 0.25 & 0 \\ 0.25 & -0.25 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

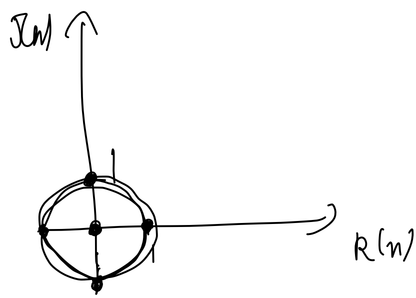
Case 2: $\lambda_2 = 0.5$.

$[A - \lambda_2 I \mid 0] = \left[\begin{array}{cc|c} 0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Ans: $\lambda_1 = 1, \ \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad \lambda_2 = 0.5, \ \vec{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(ii) $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



(iii) $\vec{s}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \vec{v_2}$

$\vec{s}(n) = -1 . (0.5)^n \vec{v_2} = \vec{0}$

(iv) $\vec{s}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 4 \vec{v_1} + 1 \vec{v_2}$

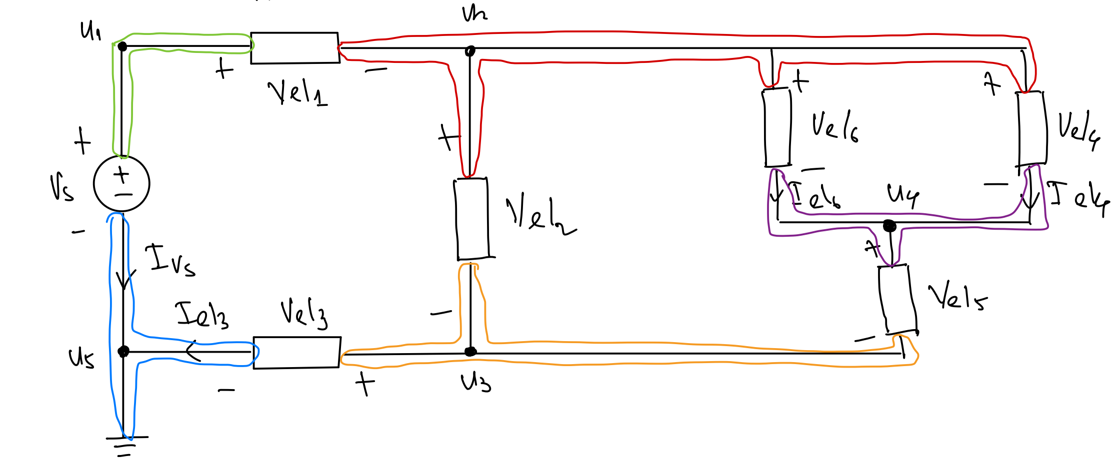
$\vec{s}(n) = 4 \lambda_1^n \vec{v_1} + \lambda_2^n \vec{v_2} = 4 \vec{v_1} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\qquad A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ $\quad \lambda^2 + 1 = 0$

$\vec{s}(n) = \frac{1}{2} i^n \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2} (-i)^n \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\vec{s}(n) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } n \geq 0 \text{ and } n \bmod 4 = 0 \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \text{if } n \geq 0 \text{ and } n \bmod 4 = 1 \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} & \text{if } n \geq 0 \text{ and } n \bmod 4 = 2 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if } n \geq 0 \text{ and } n \bmod 4 = 3 \end{cases}$$

4. Intro to Circuits



(a) 5 nodes

(c) $V_s = u_1 - 0 = u_1$

$V_{el_1} = u_1 - u_2$

$V_{el_2} = u_2 - u_3$

$V_{el_3} = u_3$

$V_{el_4} = u_2 - u_4$

$V_{el_5} = u_4 - u_3$

$V_{el_6} = u_2 - u_4$

(d) $I_{el_1} = I_{el_2} + I_{el_6} + I_{el_4}$

(e) $V_s - V_{el_1} - V_{el_2} - V_{el_3} = 0$

$V_s - V_{el_1} - V_{el_6} - V_{el_5} - V_{el_3} = 0$

$V_s - V_{el_1} - V_{el_4} - V_{el_5} - V_{el_3} = 0$