Homework 1B

1. Mechanical Inverses

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

A swaps x and y.

(b) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

A changes the sign of x and keeps y.

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

$\hookrightarrow$ Lin. dep. $\rightarrow$ No inverse

2. Quadcopter Transformations

(a) $R_{x30} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ \\ 0 & \sin 30^\circ & \cos 30^\circ \end{bmatrix}$ $\quad R_{z60} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 = R_{z60} R_{x30} = \begin{bmatrix} 0.5 & -0.75 & 0.43 \\ 0.87 & 0.43 & -0.25 \\ 0 & 0.5 & 0.87 \end{bmatrix}$

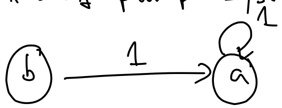
(b) $R_2 = R_{x30} R_{z60} = \begin{bmatrix} 0.5 & -0.87 & 0 \\ 0.75 & 0.43 & -0.5 \\ 0.43 & 0.25 & 0.87 \end{bmatrix}$

(c) Intention $\quad \vec{r}_{e1} = R_1 \vec{r} = \begin{bmatrix} 0.62 \\ 0.8 \\ 2.23 \end{bmatrix}$

Reality $\quad \vec{r}_{e2} = R_2 \vec{r} = \begin{bmatrix} -0.36 \\ 0.18 \\ 2.41 \end{bmatrix}$

$\Rightarrow$ Not the same!

3. Properties of pump systems



(a) $\begin{cases} x_a(n+1) = x_a(n) + x_b(n) \\ x_b(n+1) = 0 \end{cases}$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

(c) $x_a(0) = 0.5$, $x_b(0) = 0.5$

$x_a(0) = 0.3$, $x_b(0) = 0.7$

Both situations end up with $\begin{cases} x_a(1) = 1 \\ x_b(1) = 0 \end{cases}$

(d)(e) We cannot figure out what the initial $\vec{x}(0)$ water levels if we observe the reservoirs at timestep 1 because A is not inveritble.

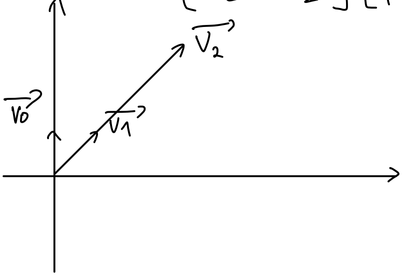
(f) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0.4 & 0.5 & 0.6 \\ 0 & 0.2 & 0.3 \end{bmatrix}$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 0.4 & 0.7 & 0.9 \end{array}$$

All the sums of entries of the cols are less than 1. This implies the amount of water is lost after each day.

4. Image stitching

(a) $\vec{V_2} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$



Rotate right 45°

Scale 2x

Shift x and y each by 1.

(b) $\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$

$= \begin{bmatrix} R_{xx} p_x + R_{xy} p_y + T_x \\ R_{yx} p_x + R_{yy} p_y + T_y \end{bmatrix}$

We have:

$$\begin{cases} q_x = R_{xx} p_x + R_{xy} p_y + T_x \\ q_y = R_{yx} p_x + R_{yy} p_y + T_y \end{cases}$$

Known values: q_x, q_y, p_x, p_y
Unknown values: $R_{xx}, R_{xy}, R_{yx}, R_{yy}, T_x, T_y$
There are 6 unknowns → Need 6 independent equations.
Each pair of $\vec{p}$ and $\vec{q}$ generates 2 equations → we need 3 pairs.

(c) $p_{x1} R_{xx} + p_{y1} R_{xy} + T_x = q_{x1}$

$p_{x1} R_{yx} + p_{y1} R_{yy} + T_y = q_{y1}$

...

$$\begin{bmatrix} p_{x1} & p_{y1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{x1} & p_{y1} & 1 \\ p_{x2} & p_{y2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{x2} & p_{y2} & 1 \\ p_{x3} & p_{y3} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{x3} & p_{y3} & 1 \end{bmatrix} \begin{bmatrix} R_{xx} \\ R_{xy} \\ T_x \\ R_{yx} \\ R_{yy} \\ T_y \end{bmatrix} = \begin{bmatrix} q_{x1} \\ q_{y1} \\ q_{x2} \\ q_{y2} \\ q_{x3} \\ q_{y3} \end{bmatrix}$$

5. Segway tours:

(a) $x(1) = A x(0) + \vec{b} u(0)$

(b) $x(2) = A x(1) + \vec{b} u(1)$

$= A(A x(0) + \vec{b} u(0)) + \vec{b} u(1)$

$= A^2 x(0) + A\vec{b} u(0) + \vec{b} u(1)$

$x(3) = A^3 x(0) + A^2 \vec{b} u(0) + A\vec{b} u(1) + \vec{b} u(2)$

$x(4) = A^4 x(0) + A^3 \vec{b} u(0) + A^2 \vec{b} u(1) + A\vec{b} u(2) + \vec{b} u(3)$

$\begin{bmatrix} A\vec{b} & \vec{b} \\ & \\ & \end{bmatrix} \underset{2\times 1}{\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}} = \underset{4\times 1}{\begin{bmatrix} 1 \\ x \\ x \\ x \end{bmatrix}}$ $\quad 42 \quad 2\times 1 \quad 4\times 1$

(c) $x(n) = A^n x(0) + \sum_{i=0}^{n-1} A^{n-1-i} \vec{b}\, u(i)$

(d) GE: $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ → No solution

(e) GE $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ → No solution

(f) GE $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -13.25 \\ 0 & 1 & 0 & 0 & 23.73 \\ 0 & 0 & 1 & 0 & -11.57 \\ 0 & 0 & 0 & 1 & 1.46 \end{array}\right]$

We can reach $\vec{x_f}$ in four steps:

$u(0) = -13.25$

$u(1) = 23.73$

$u(2) = -11.57$

$u(3) = 1.46$

(h) We can move in directions $\vec{b}$ and $A\vec{b}$. Hence we can reach all positions that are in span$\{\vec{b}, A\vec{b}\}$

(i) $\text{span}\{\vec{b}, A\vec{b}, A^2\vec{b}, \ldots, A^{n-1}\vec{b}\} + A^n x(0)$

(j) Condition: $\text{span}\{\vec{b}, A\vec{b}, A^2\vec{b}, \ldots, A^{n-1}\vec{b}\} = \mathbb{R}^4$