Discussion 0C

1. Linear or Nonlinear:

(a) $f(x_1, x_2) = 3x_1 + 4x_2$

To check for linearity, check for superposition (additivity) and homogeneity (multiplicative scaling). In other words, we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2)$$
$$\forall x_1, x_2, y_1, y_2 \in \mathbb{R}$$
$$\alpha, \beta$$

Linear

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= 3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2) \\ &= \alpha(3x_1 + 4x_2) + \beta(3y_1 + 4y_2) \\ &= \alpha f(x_1, x_2) + \beta f(y_1, y_2) \end{aligned}$$

Alternatively you can state that this function is linear because it is of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constant.

(b) $f(x_1, x_2) = e^{x_2} + x_1^2$

To check for linearity, check for superposition and homogenity. In other words, we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2)$$
$$\forall x_1, x_2, y_1, y_2, \alpha, \beta \in \mathbb{R}$$

Nonlinear

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = e^{\alpha x_2 + \beta y_2} + (\alpha x_1 + \beta y_1)^2 \quad (1)$$
$$\alpha f(x_1, x_2) = \alpha e^{x_2} + \alpha x_1^2 \quad (2)$$
$$\beta f(y_1, y_2) = \alpha e^{y_2} + \alpha y_1^2 \quad (3)$$

From (1)(2)(3) $\Rightarrow f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \neq \alpha f(x_1, x_2) + \beta f(y_1, y_2)$

Alternatively you can state that this function is nonlinear because it is not of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1, a_2 are constants.

(c) $f(x_1, x_2) = x_2 - x_1 + 3$

To check for linearity, we have to check for additivity and multiplicative scaling. In other words, we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1, x_2) + \beta f(y_1, y_2)$$

$\alpha = 1, \beta = 1$ $\qquad \forall x_1, x_2, y_1, y_2, \alpha, \beta \in \mathbb{R}$

Choose $x_1 = x_2 = y_1 = y_2 = 0$, we have:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = f(0,0) = 0 - 0 + 3 = 3 \quad (1)$$

$$\alpha f(x_1, x_2) + \beta f(\cancel{y}y_1, y_2) = f(0,0) + f(0,0) = 6 \quad (2)$$

(1) and (2) imply contradiction.

$\Rightarrow$ This function is nonlinear.

Alternatively we can state that this function is nonlinear because it is not of the form:

$$f(x_1, x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

2. Gaussian Elimination

(a)
$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 6 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

$R_1 \leftarrow 1/2 R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$R_3 \leftarrow -R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 6 & -6 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow 1/6\, R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$x_1 = 5,\ x_2 = -1,\ x_3 = -1$ $\boxed{\text{Unique solution}}$

(b) $$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 3 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow 2R_1 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_2 - R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

choose $x_3 \rightarrow$ solve for x_1, x_2

$$\begin{cases} x_1 = 2 - x_3 \\ x_2 = -x_3 \end{cases}$$

$\boxed{\text{Infinite solution}}$

(c) $\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 1 & 2 & 8 & 0 \\ 1 & 3 & 5 & 3 \end{array}\right] \xrightarrow{R_2 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 2 & -6 & 2 \\ 1 & 3 & 5 & 3 \end{array}\right] \xrightarrow{R_3 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 2 & -6 & 2 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow 1/2 R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & -3 & -1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last row implies $0 = 2 \Rightarrow$ contradiction
The equation has $\boxed{\text{no solution}}$.

(d) $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 7 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftarrow 1/2\, R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \end{array}\right]$$

$R_3 \leftarrow 2R_1 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \end{array}\right]$$

$R_3 \leftarrow 2R_2 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3/2 & 7/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

choose $x_3 \rightarrow$ solve for x_1, x_2

$$\begin{cases} x_1 = 1/2 - 1/2\, x_3 \\ x_2 = 3 - x_3 \end{cases}$$

$\boxed{\text{Infinte solutions}}$

(e)

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 6 \\ 1 & 1 & 1 & 3 \\ 2 & 0 & -1 & 2 \end{array}\right]$$

Swap R_1 and R_2

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 1 & 0 & 6 \\ 2 & 0 & -1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 3 & 3 \\ 2 & 0 & -1 & 2 \end{array}\right] \quad R_2 \leftarrow 3R_1 - R_2$$

$$R_3 \leftarrow 2R_1 - R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 3 & 4 \end{array}\right]$$

$$R_3 \leftarrow R_2 - R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The last row is equivalent to $0 = -1 \Rightarrow$ contradiction.
Thus, the equation has no solution.

3. Finding the bright cave:

(a)

$$\begin{aligned} x_2 + x_4 &= m_1 \\ x_3 + x_4 &= m_2 \\ x_1 + x_3 &= m_3 \\ x_1 + x_2 &= m_4 \end{aligned}$$

(b) We have:

$$m_1 + m_3 = x_1 + x_2 + x_3 + x_4 = m_2 + m_4 \Rightarrow m_4 = m_1 + m_3 - m_2$$

Hence, we could get one equation from another. In other words, there exists measurement is redundant (because we can get it from a linear combination of other measurements), which means that the equation doesn't have a unique solution.

(c) $1/2\, x_1 + x_2 + x_3 + 1/2\, x_4 = m_5$

$$\begin{aligned} x_1 &= m_1 - m_5 + 1/2\, m_2 + 1/2\, m_3 \\ x_2 &= -m_1 + m_5 + 1/2\, m_2 - 1/2\, m_3 \\ x_3 &= m_5 - 1/2\, m_2 - 1/2\, m_3 \\ x_4 &= m_4 - m_5 + 1/2\, m_2 + 1/2\, m_3 \end{aligned}$$