SEM Algorithm

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1. Motivation

- Expectation-Maximization (EM) is one of the most popular tools to obtain maximum likelihood estimates in incomplete data problems
- Problem: EM algorithm does not provide an asymptotic variance-covariance matrix for the parameters
- Such matrix is needed for asymptotically valid inference
- Existing approaches are computationally problematic and not generically applicable
- SEM algorithm provides numerically stable asymptotic variance-covariance matrix for estimates obtained by EM

2. EM algorithm

- In latent variable problems, especially in incomplete data problems, maximum likelihood estimation is analytically not possible
- EM algorithm enables convergence to optimum, which could be a local optimum depending on the starting values
- EM algorithm consists of two iteratively applied steps:

UNTIL $\Delta\theta < \varepsilon$ REPEAT:

E-step:

Find expected complete-data log-likelihood and treat $\theta^{(t)}$ as the true parameter θ

$$Q\left(\theta \middle| \theta^{(t)}\right) = E_{Y_{mis}|Y_{obs},\theta^{(t)}}[logL(\theta; Y_{obs}, Y_{mis})]$$

M-step:

Update $\theta^{(t)}$ by the value that maximizes the expected log-likelihood from the E-step

$$\theta^{(t+1)} = \underset{\theta}{\operatorname{argmax}}\, Q\left(\theta \middle| \theta^{(t)}\right)$$

Calculate change in estimated parameters

$$\Delta\theta = |\theta^{(t+1)} - \theta^{(t)}|$$

3. Supplemented EM algorithm

- Calculation of asymptotic variance-covariance matrix V is done in three steps:

Step 1:

Evaluate adjusted information matrix I_{oc}

$$I_{oc} = I_o(\theta^* | S^*(Y_{obs})), where\ S^* = E[S(Y)|Y_{obs}, \theta]$$

Step 2:

Calculate the elements of rate-of-convergence matrix DM

$$DM = \begin{pmatrix} r_{11} & \cdots & r_{1j} \\ \vdots & \ddots & \vdots \\ r_{i1} & \cdots & r_{ij} \end{pmatrix}$$

FOR $i = 1, \dots, d$

FOR $j = 1, \dots, d$

UNTIL $\Delta r_{ij} < \sqrt{\varepsilon}$ REPEAT:

Calculate element in rate-of-convergence matrix DM

$$r_{ij}^{(t)} = \frac{[\tilde{\theta}_j^{(t+1)}(i) - \theta_j^*]}{[\theta_i^{(t)} - \theta_i^*]}, \text{ for } t = 1, \dots, T$$

Calculate change in $r_{ij}^{(t)}$

$$\Delta r_{ij} = \left|r_{ij}^{(t)} - r_{ij}^{(t-1)}\right|, \text{ for } t = 1, \dots, T$$

Step 3:

Calculate asymptotic variance-covariance matrix

$$V = I_{oc}^{-1} + \Delta V, where\ \Delta V = = I_{oc}^{-1} \mathrm{DM}(\mathrm{I} - \mathrm{DM})^{-1}$$

- Advantages of SEM:
  + Matrix V not necessarily symmetric, which can be used for diagnostics of EM algorithm
  + Computationally more stable than numerical differentiation
  + Applicable in non-iid cases and under complicated missing data patters
  + Parameters allowed to convergence with different number of steps
  + Parallel computing possible

4. Simulation study

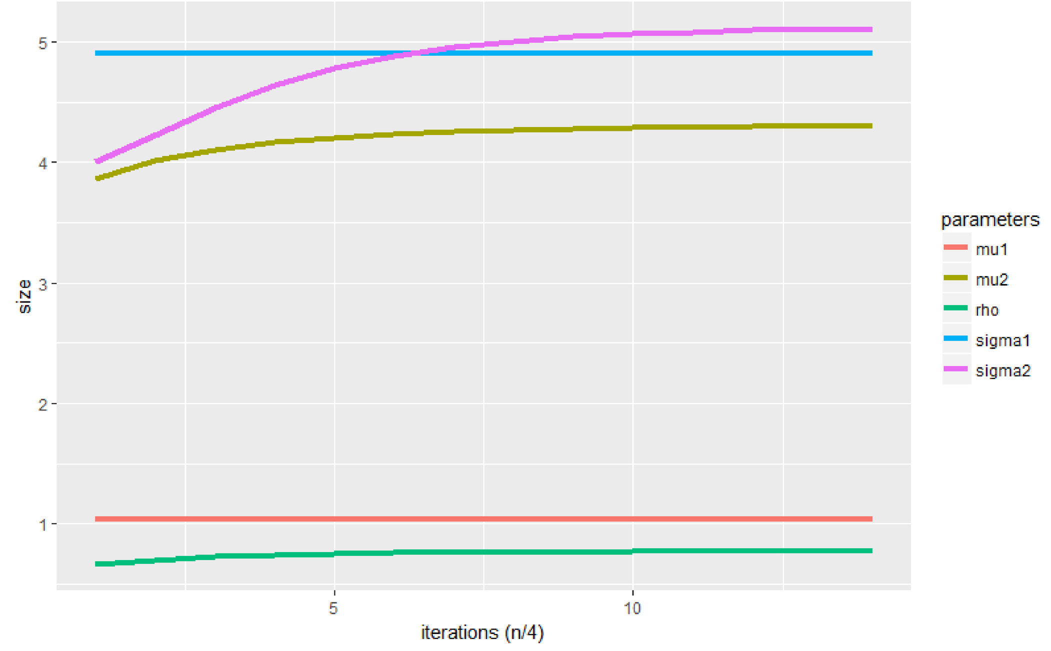
- Data simulation by random draw from bivariate normal distribution:

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 5 \end{pmatrix}\right)$$

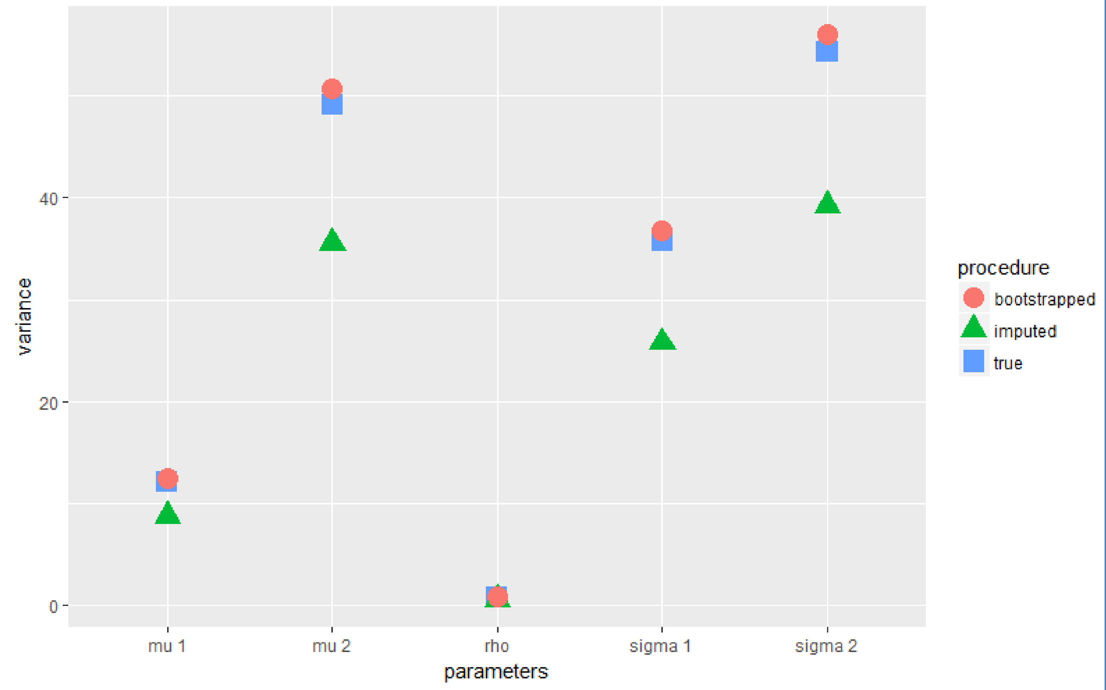
- MNAR pattern of missing data in Y_2, fraction $\tau = 0.4$
- Parameter vector $\theta = (\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \rho)$
- True variance-covariance matrix is compared with EM imputed matrix an bootstrapped matrix

5. Results

- Convergence of EM parameters



- Comparison of parameter variances by calculation procedure



6. Conclusion

- EM works well in our multi-parameter problem and algorithm converges fast
- Complexity of SEM implementation depends on type of model
- In contrast an asymptotic variance-covariance matrix can relatively easily be obtained from bootstrapping method
- Bootstrapping delivers very accurate results, as can be seen by the difference $V_c(\theta) - V(\hat{\theta})$
- Bootstrapping computationally more costly than SEM

7. Discussion

- SEM still in discussion as an alternative methods like bootstrapping and jacknife
- EM widely established in open-source and commercial statistical programs
- Nevertheless, SEM only rarely implemented
- Implementations in IRTPRO and R (openmx, coarseDataTools)
- SEM mostly used in the field of item response theory
- Refinements of SEM, like agile-SEM for item response problems
- Bootstrapping heavily used in many fields, especially when large sample and iid-structure available

References

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