MSA 8200 - Econometrics Homework 5

1 a) Given $X_t = \phi X_{t-1} + w_t$ and $w_t \sim N(0, \sigma_w^2)$

$$
\begin{aligned}
X_t &= \phi X_{t-1} + w_t \\
&= \phi(\phi X_{t-2} + w_{t-1}) + w_t \\
&= \phi^2 X_{t-2} + \phi w_{t-1} + w_t \\
&\vdots \\
&= \phi^k X_{t-k} + \sum_{j=0}^{k-1} \phi^j \cdot w_{k-j}
\end{aligned}
$$

If $|\phi| < 1$ and $k \to \infty$, then $\phi^k \to 0$, thus

$$X_t = \sum_{j=0}^{k-1} \phi^j w_{k-j}.$$

b) $$E(X_t) = \sum_{j=0}^{k-1} \phi^j E(w_{k-j})$$

$$= 0.$$

c) $$\mathrm{Var}(X_t) = \mathrm{Var}\left(\sum_{j=0}^{k-1} \phi^j \cdot w_{k-j}\right)$$

$$= \sum^{t} \phi^{2j} \cdot \mathrm{Var}(w_{k-j})$$

$$= \sum^{t} \phi^{2j} \cdot \sigma_w^2 = \sigma_w^2 \cdot \sum^{t} \phi^{2j}$$

$$\Rightarrow \sigma_w^2 \cdot \left(\frac{1 - \phi^{2(t+1)}}{1 - \phi^2}\right)$$

d) $cov(X_t, X_{t+h}) = E[(X_t - \mu_{X_t})(X_{t+h} - \mu_{X_{t+h}})]$

$\rightarrow$ Given $\cancel{x}$ $E(X_t) = E(X_{t+h}) = 0$,

$$cov(X_t; X_{t+h}) = E[X_t \cdot X_{t+h}]$$

$$\Rightarrow E\left[\sum_{j=0}^{t} \phi^j w_{t-j} \cdot \sum_{j=0}^{t+h} \phi^j \cdot w_{t+h-j}\right]$$

Lets set $i = j+h$ to re-index the second summation

$$\Rightarrow E\left[\sum_{j=0}^{t} \phi^j w_{t-j} \cdot \sum_{i=0} \phi^i \cdot w_{t+h-i}\right]$$

$$= E\left[\sum_{j=0}^{t} \phi^{j+i} \cdot w_{t-j} \cdot w_{t+h-i}\right]$$

$$= \sum \phi^{2j} \cdot \phi^h \cdot E(w_{t-j} \cdot w_{t-j})$$

$$= \phi^h \sigma_w^2 \cdot \sum_{j=0}^{t} \phi^{2j} = \phi^h \cdot var(X_t)$$

e) Since $var(X_t)$ depends on ~~n~~ t, X_t is not stationary.

f) As $t \to \infty$:

i) $E(X_t) \to 0$ [from part b]

ii) $Var(X_t) \to \frac{\sigma_w^2}{1-\phi^2}$ [given $|\phi|<1$]

iii) $Cov(X_t, X_{t+h}) \to \frac{\phi^h \sigma_w^2}{1-\phi^2}$

Since the $E(X_t) = 0$ and $Var(X_t)$ does not depend on t, X_t becomes stationary as $t \to \infty$.

3 a) $X_t = 0.8X_{t-1} - 0.15X_{t-2} + w_t - 0.3w_{t-1}$

$0.15X_{t-2} - 0.8X_{t-1} + X_t = w_t - 0.3w_{t-1}$

$\Rightarrow (1 - 0.8B + 0.15B^2)X_t = (1-0.3B)w_t$

$(1-0.3B)(1-0.5B)X_t = (1-0.3B)w_t$

Since $\phi(B) = (1-0.5B)\,\theta(B)$, this is an AR(1) model, not an ARMA(2,1).

<u>Causality:</u>

$\phi(z) = 1 - 0.5z = 0$

$\therefore z = 2$

As $|z| > 1$, this is causal

<u>Invertible:</u>

The model can be written as

$(1-0.5B)X_t = w_t$, thus it is invertible.

b) $X_t = X_{t-1} - 0.5X_{t-2} + w_t - w_{t-1}$

$0.5X_{t-2} - X_{t-1} + X_t = w_t - w_{t-1}$

$\Rightarrow (1 - B + 0.5B^2)X_t = (1-B)w_t$

Since $\phi(B)$ and $\theta(B)$ share no factors, this is an ARMA(2,1) model.

Causality:

(i) $\phi(z) = 1 - z + 0.5z^2 = 0$

$\therefore z = \frac{1 \pm \sqrt{1-2}}{1} = 1 \pm \sqrt{-1}$

As $|z| > 1$, this is causal.

(ii) $\theta(z) = 1 - z = 0$

$\therefore z = 1$

As $|z| \not> 1$, this is not causal.

Invertible:

The model cannot be written in the form $\pi(B)X_t = w_t$, thus it is not invertible.