CIVE 3203

Displacements in Beams & Frames using Virtual Work

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Revision History:

2. Nov 16, 2012: corrected defln due to flexure, p5 & 6
1. Nov 13, 2012: original posting.

Deflections in Beams & Frames
Method of Virtual Work (Virtual Force)

Basic relationship (from before):

$$1\,(\Delta) = \sum(u\;\delta L)$$

real

virtual

For beams & frames, where the displacements & distortions are due to flexure, the appropriate version of this is:

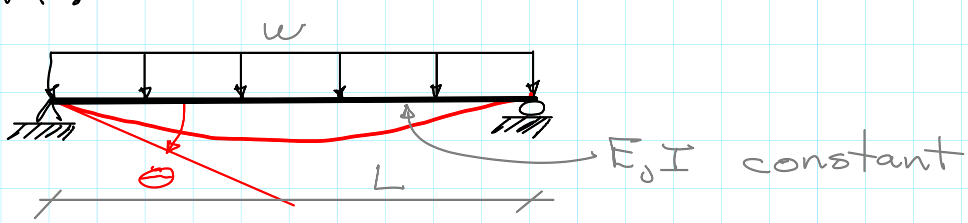
$$1\,(\Delta) = \int_{\text{length}} m\left(\frac{M}{EI}\right)\,dx$$

real

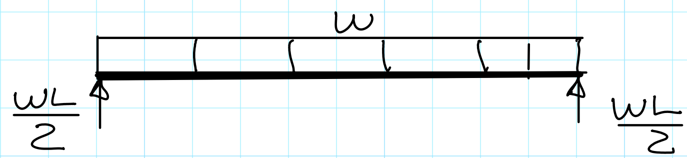
virtual

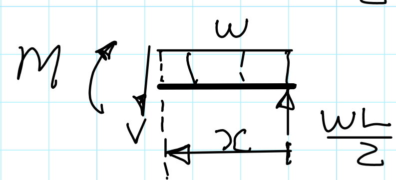
Example

Compute the rotation of the tangent at the support:

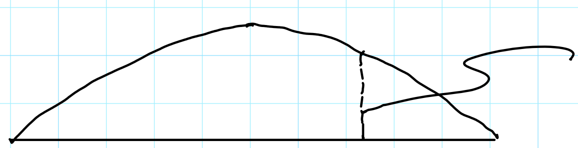


a) construct the real $\frac{M}{EI}$ 'diagram' - entire structure.





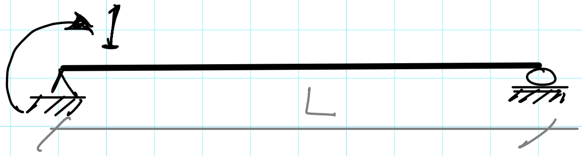
$\frac{M}{EI}$



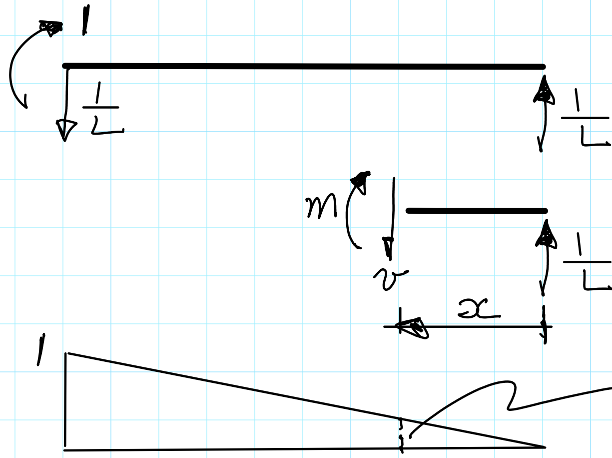
$$\frac{1}{EI}\left(\frac{wL}{2}x - \frac{wx^2}{2}\right)$$

for $0 \leq x \leq L$

b) apply a virtual force corresponding to the desired displacement (in this case, a rotation). Therefore, apply a virtual couple.



c) determine virtual moments



$m = \frac{1}{L}x$

for $0 \le x \le L$

d) compute

$$1 \cdot \theta = \int m \frac{M}{EI}\,dx$$

$$= \frac{1}{EI}\int_0^L \frac{1}{L}x\left(\frac{wL}{2}x - \frac{wx^2}{2}\right)dx$$

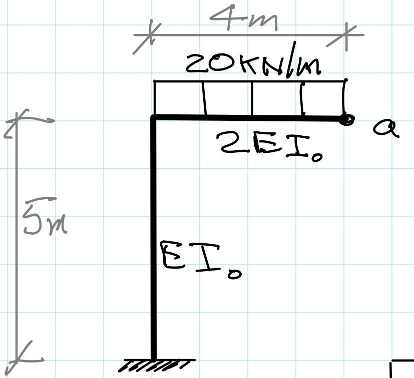
$$= \frac{1}{EI}\int_0^L \left(\frac{wx^2}{2} - \frac{wx^3}{2L}\right)dx$$

$$= \frac{1}{EI}\left[\frac{wx^3}{6} - \frac{wx^4}{8L}\right]_0^L$$

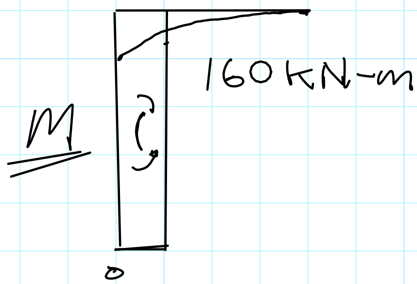
$$= \frac{1}{EI}\left[\frac{wL^3}{6} - \frac{wL^3}{8}\right]$$

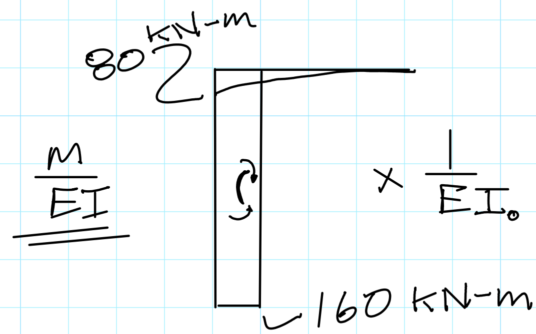
$$\boxed{\theta = \frac{wL^3}{24EI}}$$

Examples using the integral tables.

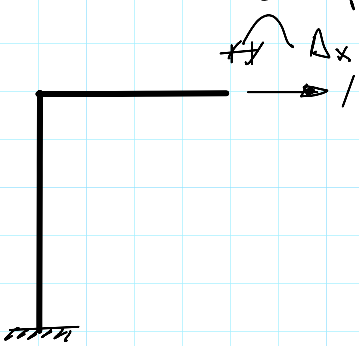


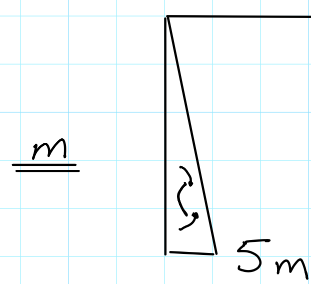
Find horizontal displacement, vertical displacement, and rotation of pt. a.

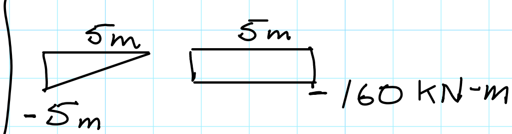




Horizontal Displacement:





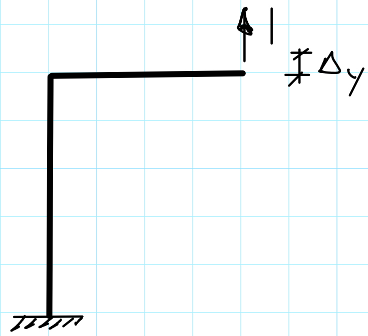
$1 \times \Delta_x = \dfrac{1}{EI_o} \int$ 

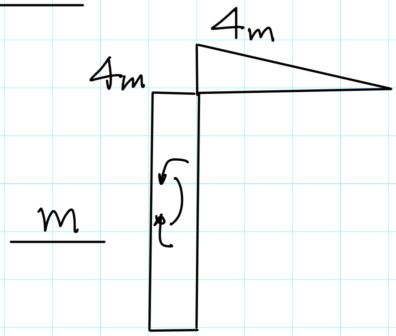
(row 2, col 1 in table)

$$= \frac{1}{EI_o} \quad \frac{5}{2} \times -5 \times -160 \text{ kN-m}^3$$

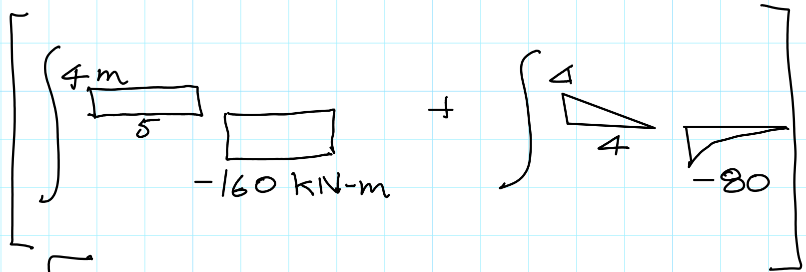
$$\Delta_x = \frac{2000 \text{ kN-m}^3}{EI_o}$$

Vertical Displacement





$1(\Delta_y) = \frac{1}{EI_o}$



$= \frac{1}{EI_o}\left[5 \times 4 \times -160 \ \text{kN-m}^3 \right.$ (row1, col 1)

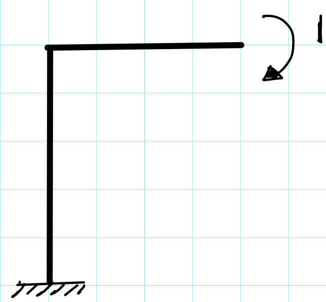
$\left. + \frac{4}{4} \times 4 \times -80 \ \text{kN-m}^3 \right]$ (row3, col 6)

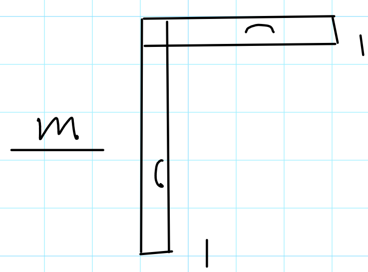
Note col contributes 10x as much as beam

$$\boxed{\Delta_y = \frac{-3520 \ \text{kN-m}^3}{EI_o}}$$

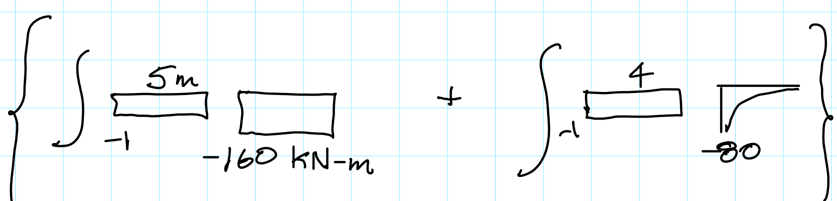
$(\therefore \downarrow)$

Rotation





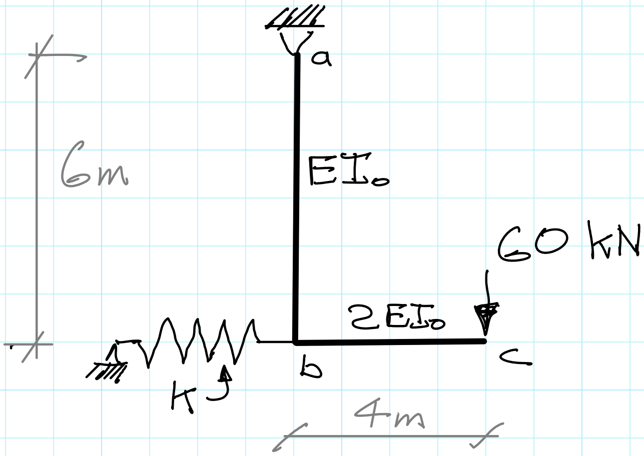
$1 \times \theta = \frac{1}{EI_o}$



$= \frac{1}{EI_o}\left\{ 5 \times -1 \times -160 \ \text{kN-m}^2 + \frac{4}{3} \times -1 \times -80 \ \text{kN-m}^2 \right\}$

$$\boxed{\theta = \frac{2720 \ \text{kN-m}^2}{3\ EI_o}}$$

Combining axial & flexural effects



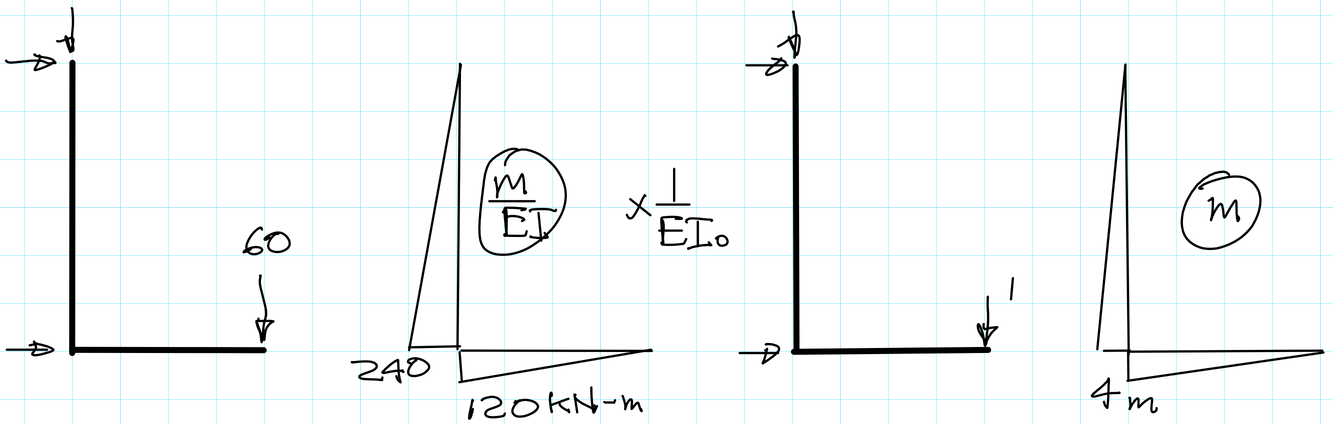
$E = 200\,000$ MPa

$I_o = 500 \times 10^6\ \text{mm}^4$

$K = 1333$ kN/m

Compute vertical displacement of pt. c.

Contribution of flexure:



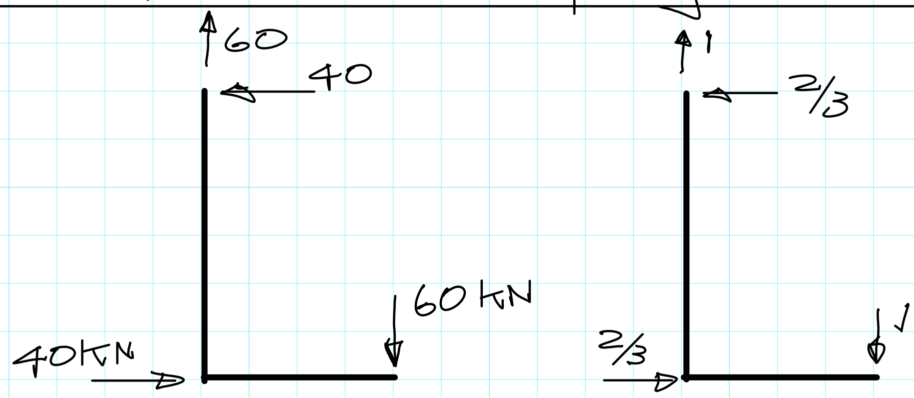
$$1 \times \Delta = \frac{1}{EI_o}\left\{ \int \left(4\text{ m},\ 6\text{ m}\right)\left(240\text{ kN-m}\right) + \int \left(4\text{ m},\ -4\text{ m}\right)\left(-120\text{ kN-m}\right) \right\}$$

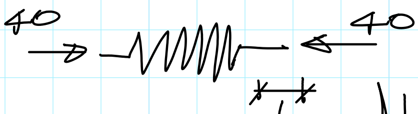
$$= \frac{1}{EI_o}\left\{ \frac{6}{3} \times \overset{4}{\cancel{6}} \times 240\ \text{kN-m}^3 + \frac{4}{3} \times -4 \times -120\ \text{kN-m}^3 \right\}$$

$$= \frac{\overset{2560}{\cancel{3520}}\ \text{kN-m}^3}{EI_o}$$

$$= \frac{\overset{2560}{\cancel{3520}} \times 10^{12}\ \text{N-mm}^3}{200 \times 10^3 \frac{\text{N}}{\text{mm}^2} \times 500 \times 10^6\ \text{mm}^4} = \overset{25.6}{\cancel{35.2}}\ \text{mm}$$

Contribution due to spring deformation





$$dL = \frac{F}{k} = \frac{40\ \text{kN}}{1333\ \text{kN/m}}$$

$$= 30 \times 10^{-3}\ \text{m} = 30\ \text{mm}$$

$$1(\Delta) = \sum u\, dL$$

$$= \frac{2}{3} \times 30\ \text{mm}$$

$$= 20\ \text{mm}$$

$\therefore$ Total Defln $= \overset{25.6}{\cancel{35.2}}\ \text{mm} + 20\ \text{mm}$

$$= \overset{45.6}{\cancel{55.2}}\ \text{mm}$$